Homework 4

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1 Question 1

Consider the system of SDEs [1]

$$\begin{cases} dX(t;\omega) & = -X(t;\omega)^3 dt + dY(t;\omega) \\ dY(t;\omega) & = -\tau Y(t;\omega)dt + \sigma dW(t;\omega) \end{cases} \tag{1}$$

where $\sigma, \tau \leq 0$ are given parameters and $W(t;\omega)$ is a Wiener process. The initial condition $(X(0;\omega), Y(0;\omega))$ has i.i.d. components both of which are uniformly distributed in $[0,1]$, i.e., $X(0;\omega)$ and $Y(0;\omega)$ are independent random variables with uniform PDF in $[0,1]$.

[1] The system (1) is a prototype IVP where $X(t;\omega)$ is driven by the Ornstein-Uhlenbeck process $Y(t;\omega)$, which is a colored (non-white) random noise with exponential correlation function.

1.1 Part A

Plot a few sample paths of $X(t;\omega)$ for $\sigma = 0.1$ and $\tau = \{0.01, 1, 10\}$.

1.1.1 Solution

We convert (1) into a numerical simulation by first converting the SDE into a discrete time form via the Euler-Maruyama discretization.

We descritize a grid within the interval [0, T] into N equal parts, where $\Delta t = \frac{T}{N}$

Let us recursively define Y_n for $0 \leq n \leq N-1$

$$Y_{n+1} = Y_n - \tau Y_n \Delta t + \sigma \Delta W \tag{1}$$

and X_n

$$X_{n+1} = X_n - X_n^3 \Delta t + \Delta Y_n \tag{2}$$

where $\Delta Y_n = Y_{n+1} - Y_n = -\tau Y_n \Delta t + \sigma \Delta W_n$

where ΔW_n are i.i.d Gaussian random variables with zero mean and variance Δt.

```
using GLMakie
using Distributions

function partA(τ::Float64, σ::Float64)
    # define the length of subintervals
    Δt = 1e-4
    ts = 0.0:Δt:5.0
    # number of samples
    N = 5
    # Weiner processs
    W = Normal(0, sqrt(Δt))
    # initialize mesh
    X = Matrix{Float64}(undef, length(ts), N)
    Y = Matrix{Float64}(undef, length(ts), N)

    # apply initial conditions
    for i in 1:N
        X[1, i] = rand()
        Y[1, i] = rand()
    end
    # propagate the process
```

```
    for i in 1:N
        for j in 2:length(ts)
            ΔW = rand(W)
            X[j, i] = X[j-1, i] - (X[j-1, i]^3)*Δt -τ*Y[j-1, i]*Δt + σ*ΔW
            Y[j, i] = Y[j-1, i] - τ*Y[j-1, i]*Δt + σ*ΔW
        end
    end
    # make the figure
    fig = Figure()
    ax = Axis(
        fig[1, 1],
        title = "σ = $σ τ = $τ",
        xlabel = "t",
        ylabel = L"$X_{n+1}$"
    )
    # set y axis limits
    ylims!(ax, -1.0, 1.0)

    # Plot the samples
    for i in 1:N
        lines!(ax, ts, X[:, i])
    end
    return fig
end
save("parta01.png", partA(0.01, 0.1))
save("parta1.png", partA(1.0, 0.1))
save("parta10.png", partA(10.0, 0.1))
```

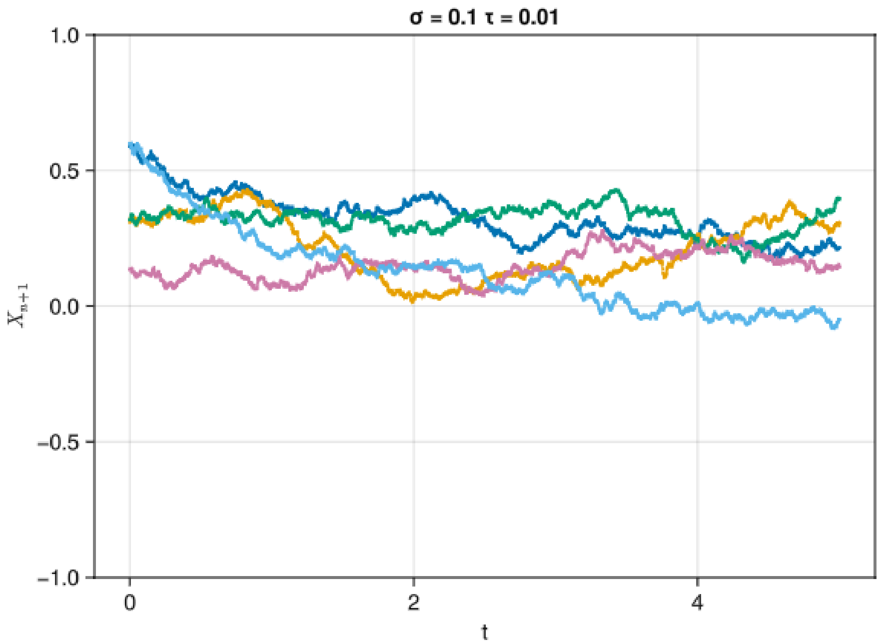

Figure 1: $\tau = 0.01$

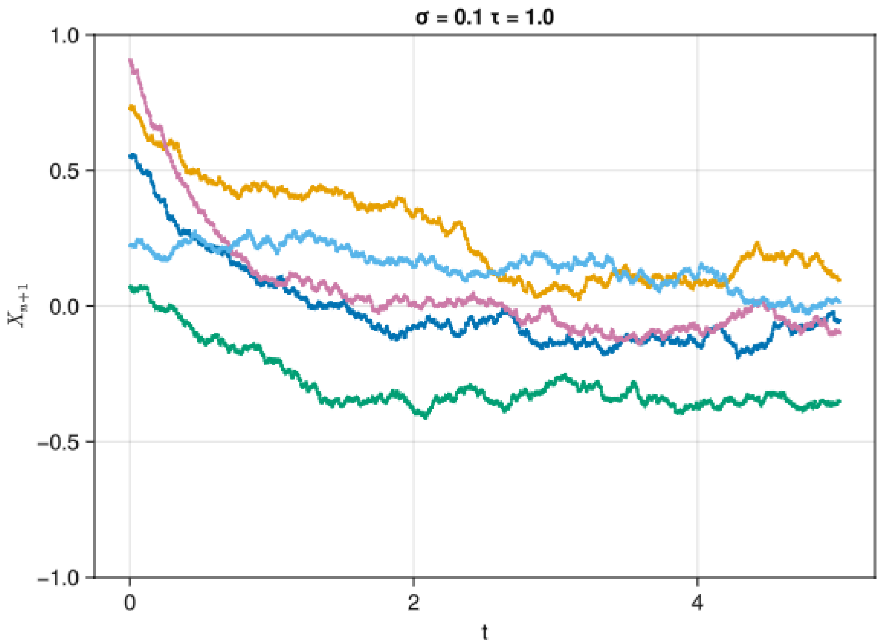

Figure 2: $\tau = 1.0$

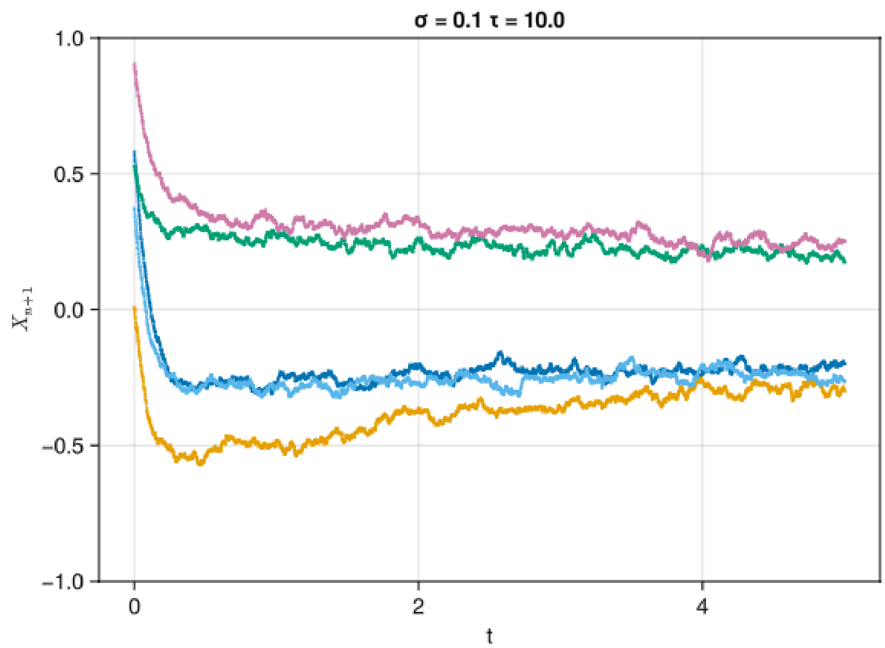

Figure 3: $\tau = 10.0$

1.2 Part B

Do you expect the system (1) to have a statistically stationary state? Justify your answer.

1.2.1 Solution

From part A, we can see that as we increase τ, the system rapidly converges to a statistically steady-state process. We can prove this by the Kernel Density Estimation for 1000 sample paths for varying τ and comparing the estimated PDF for later times.

The solution approaches a statistically steady state as $t \to \infty$ and approaches the statisically steady state at the rate τ. It seems though that the pdf for varying τ may not be the same, as $\tau = 10.0$ seems to have a bimodal distribution for later times.

```
using KernelDensity

function simulateB(τ::Float64)
    σ = 0.1
    # define the length of subintervals
```

```
    Δt = 1e-4
    ts = 0.0:Δt:20.0
    # number of samples
    N = 1000
    # Weiner processs
    W = Normal(0, sqrt(Δt))
    # initialize mesh
    X = Matrix{Float64}(undef, length(ts), N)
    Y = Matrix{Float64}(undef, length(ts), N)

    # apply initial conditions
    for i in 1:N
        X[1, i] = rand()
        Y[1, i] = rand()
    end
    # propagate the process
    for i in 1:N
        for j in 2:length(ts)
            ΔW = rand(W)
            X[j, i] = X[j-1, i] - (X[j-1, i]^3)*Δt -τ*Y[j-1, i]*Δt + σ*ΔW
            Y[j, i] = Y[j-1, i] - τ*Y[j-1, i]*Δt + σ*ΔW
        end
    end
    return X, Y, length(ts)
end

function partB()
    X1, Y1, ts = simulateB(0.01)
    X2, Y2, _ = simulateB(1.0)
    X3, Y3, _ = simulateB(10.0)

    fig = Figure()
    ax = Axis(
        fig[1, 1]
    )
    ylims!(ax, 0.0, 3.0)
    xlims!(ax, -2.0, 2.0)

    d1 = kde(X1[1, :])
    d2 = kde(X2[1, :])
    d3 = kde(X3[1, :])
    kde_data1 = Observable((d1.x, d1.density))
    kde_data2 = Observable((d2.x, d2.density))
    kde_data3 = Observable((d3.x, d3.density))
```

```
    kde_line1 = lines!(ax, [0.0], [0.0], color = :red, label = "τ = 0.01")
    kde_line2 = lines!(ax, [0.0], [0.0], color = :blue, label = "τ = 1.0")
    kde_line3 = lines!(ax, [0.0], [0.0], color = :green, label = "τ = 10.0")

    kde_plot1 = lift(kde_data1) do (x, density)
        kde_line1[1] = x
        kde_line1[2] = density
    end
    kde_plot2 = lift(kde_data2) do (x, density)
        kde_line2[1] = x
        kde_line2[2] = density
    end
    kde_plot3 = lift(kde_data3) do (x, density)
        kde_line3[1] = x
        kde_line3[2] = density
    end

    Legend(fig[1, 2], ax)
    record(fig, "partb.mp4", 2:400:ts; framerate = 30) do k
        d1 = kde(X1[k, :])
        d2 = kde(X2[k, :])
        d3 = kde(X3[k, :])
        kde_data1[] = (d1.x, d1.density)
        kde_data2[] = (d2.x, d2.density)
        kde_data3[] = (d3.x, d3.density)
    end
    return fig
end
partB()
```

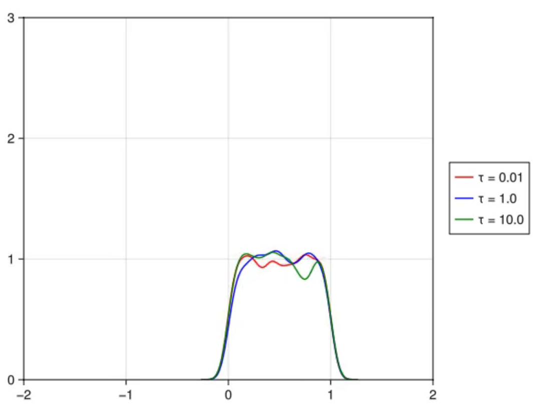

Figure 4: Image 1

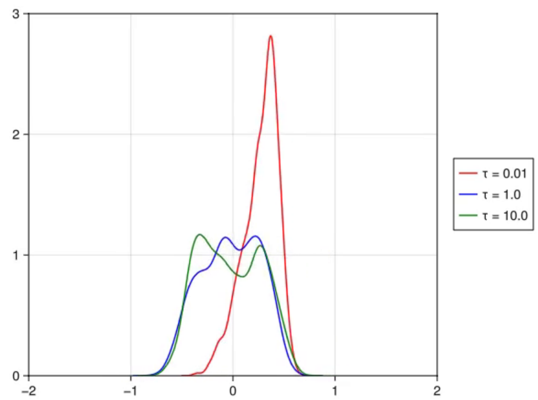

Figure 5: Image 2

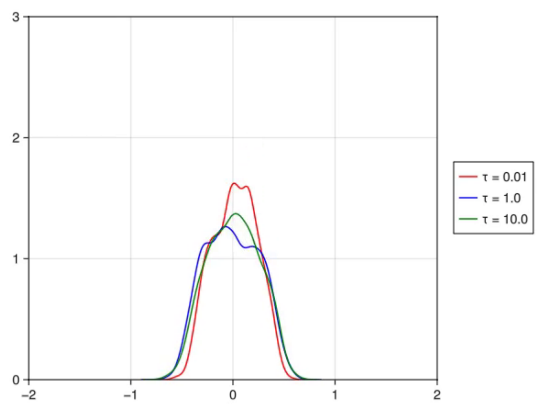

Figure 6: Image 3

Watch the full video: here.

1.3 Part C

Write the Fokker-Planck equation for the joint PDF of $X(t;\omega)$ and $Y(t;\omega)$.

1.3.1 Solution

Let us write (1) in state-space form

$$dX = -X^3 dt - \tau Y dt + \sigma dW \tag{3}$$
$$dY = -\tau Y dt + \sigma dW \tag{4}$$

The vector $\mathbf{G}(\mathbf{X}, t)$ can be written as

$$\mathbf{G}(\mathbf{X}, t) = \begin{bmatrix} -X^3 - \tau Y \\ -\tau Y \end{bmatrix} \tag{5}$$

The matrix $\mathbf{S}$ can be written as

$$\mathbf{S} = \begin{bmatrix} \sigma \\ \sigma \end{bmatrix} \tag{6}$$

We can thus write the Fokker-Planck equation according equation 59 in course notes 4 as follows:

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(G_k(\mathbf{x}, t) p(\mathbf{x})\right) + \frac{1}{2} \sum_{i,k=1}^{2} \frac{\partial^2}{\partial x_i \partial x_k} \left(\sum_{j=1}^{2} S_{ij}(\mathbf{x}, t) S_{kj}(\mathbf{x}, t) p(\mathbf{x}, t) \right) \tag{7}$$

$$\frac{\partial p}{\partial t} = -\left(\frac{\partial}{\partial x} \left((-x^3 - \tau y)p\right) + \frac{\partial}{\partial y} \left(-\tau y p\right) \right) + \frac{\sigma^2}{2} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) \tag{8}$$

$$= \frac{\partial}{\partial x}(x^3 p) + \tau p + \tau y \left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \right) + \frac{\sigma^2}{2} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) \tag{9}$$

The Fokker-Planck equation is thus

$$\frac{\partial p(x, y, t)}{\partial t} = \frac{\partial}{\partial x}(x^3 p(x, y, t)) + \tau p(x, y, t) + \tau y \left(\frac{\partial p(x, y, t)}{\partial x} + \frac{\partial p(x, y, t)}{\partial y} \right) + \frac{\sigma^2}{2} \left(\frac{\partial^2 p(x, y, t)}{\partial x^2} + \frac{\partial^2 p(x, y, t)}{\partial y^2} \right) \tag{2}$$

where $p(x, y, t)$ is the joint PDF of X and Y.

1.4 Part D

Write the reduced-order equation for the joint PDF of $X(t; \omega)$ in terms of the conditional expectation $\mathbb{E}\{Y(t; \omega) | X(t; \omega)\}$.

HINT: Integrate the Fokker-Planck equation with respect to y and use the definition of conditional PDF.

1.4.1 Solution

To obtain the reduced-order equation, let us integrate (2) with respect to y and use the definition of conditional PDF.

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} p dy = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\left((x^3+\tau y)p\right)+\frac{\partial}{\partial y}\left(\tau y p\right)+\frac{\sigma^2}{2}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right) dy \tag{10}$$

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty}(x^3+\tau y)p dy + \int_{-\infty}^{\infty}\frac{\partial}{\partial y}(\tau y p)dy + \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} p dy + \cancelto{0}{\int_{-\infty}^{\infty}\frac{\partial^2 p}{\partial y^2}dy}\right) \tag{11}$$

$$= \frac{\partial}{\partial x}\left(x^3 p(x,t) + \tau p(x,t)\mathbb{E}\{Y|X\}\right) + \cancelto{0}{\tau y p\Big|_{-\infty}^{\infty}} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2} \tag{12}$$

$$\tag{13}$$

Thus, the reduced order equation for the joint PDF can be written as,

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}(x^3 p(x,t)) + \tau\frac{\partial}{\partial x}\left(p(x,t)\mathbb{E}\{y|x\}\right) + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2} \tag{3}$$

1.5 Part E

Set $\sigma = 0$. Compute the conditional expectation $\mathbb{E}\{Y(t;\omega)|X(t;\omega)\}$ explicity as a function of t and substitute it in the reduced order equation you obtained in part d (with $\sigma = 0$) to obtain an exact (and closed) equation for the PDF of $X(t;\omega)$.

1.5.1 Solution

Recall the orginal SDE (1) where

$$dY = -\tau Y dt + \sigma dW \tag{14}$$

setting $\sigma = 0$ we have the following ODE:

$$\begin{cases} \frac{dY}{dt} = -\tau Y \\ Y(0) \sim \text{Uniform}(0,1) \end{cases} \tag{15}$$

solving for $Y(t)$ we have:

$$Y(t) = Y(0)e^{-\tau t} \tag{16}$$

where $Y(0)$ can be expressed as the expectation of a uniform random variable in [0, 1], which we know is $\frac{1}{2}$

therefore the conditional expectation

$$\mathbb{E}\{Y|X\} = \mathbb{E}\{Y\} = \frac{1}{2}e^{-\tau t}$$

we can write a closed equation for the reduced order equation as follows:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}(x^3 p(x,t)) + \tau \frac{\partial}{\partial x}(p(x,t)\frac{1}{2}e^{-\tau t}) \tag{4}$$

1.6 Part F

Write the PDF equation you obtained in part e as an evolution equation for the cumulative distribution function (CDF) of $X(t;\omega)$.

1.6.1 Solution

Recall the definition of the cumulative distribution function:

$$F(x,y,t) = \int_{-\infty}^{x} p(y,t)dy \tag{17}$$

if we take the derivative on both sides with respect to x we have

$$p(x,t) = \frac{\partial F(x,t)}{\partial x} \tag{18}$$

plugging this into (4) we have

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^3\frac{\partial F}{\partial x}\right) + \tau\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\frac{1}{2}e^{-\tau t}\right) \tag{19}$$

taking the integral of both sides:

$$\int_{-\infty}^{x} \frac{\partial^2 F}{\partial t \partial x'} dx' = \int_{-\infty}^{x} \left(\frac{\partial}{\partial x'} \left(x'^3 \frac{\partial F}{\partial x'} \right) + \frac{\tau}{2} \frac{\partial}{\partial x'} \left(\frac{\partial F}{\partial x'} e^{-\tau t} \right) \right) dx' \tag{20}$$

$$\frac{\partial F(x,t)}{\partial t} = x'^3 \frac{\partial F}{\partial x'} \bigg|_{-\infty}^{x} + \frac{\tau}{2} \frac{\partial F}{\partial x'} e^{-\tau t} \bigg|_{-\infty}^{x} \tag{21}$$

$$\tag{22}$$

thus we are left with the evolution equation for $F(x,t)$

$$\frac{\partial F(x,t)}{\partial t} = x^3 \frac{\partial F(x,t)}{\partial x} + \frac{\tau}{2} e^{-\tau t} \frac{\partial F(x,t)}{\partial x} \tag{5}$$