Fundamentals of UQ (AM 238)

Homework 3

Question 1: Consider a Gaussian random process $X(t;\omega)$ defined on the time interval $[0,5]$. The process has mean

$$\mu(t) = te^{\sin(3t)}, \tag{1}$$

and covariance function

$$\mathrm{cov}(t,s) = \exp(-|t-s|/\tau), \tag{2}$$

where $\tau > 0$ represents the temporal "correlation length" of the Gaussian process.

a) Compute the stardard deviation of $X(t;\omega)$ at time t.

b) Compute the covariance matrix of the random variables $X(1;\omega)$ and $X(2;\omega)$ as a function of τ. What happens when $\tau \to 0$?

c) Plot a few samples of $X(t;\omega)$ for $\tau = 0.02$ and $\tau = 1$ on a temporal with 5000 points in $[0,5]$ (two different figures). Show that such sample paths are approximately within $\mu(t) \pm 2\sigma(t)$, where $\sigma(t)$ is the standard deviation of the process.

Question 2: Show that the sequence of of numbers $\{U_k\}$ defined as

$$U_k = \frac{(\widehat{X}_k - \widehat{Y}_k) \mod m_1}{m_1 + 1}. \tag{3}$$

where

$$\widehat{X}_k = \begin{cases} X_k & \text{if} \quad X_k \geq 0 \\ X_k - m_1 X_k & \text{if} \quad X_k < 0 \end{cases} \qquad \widehat{Y}_k = \begin{cases} Y_k & \text{if} \quad Y_k \geq 0 \\ Y_k - m_2 Y_k & \text{if} \quad Y_k < 0 \end{cases} \tag{4}$$

$$X_k = (1403580 X_{k-2} - 810728 X_{k-3}) \mod m_1 \tag{5}$$

$$Y_k = (527612 Y_{k-1} - 1370589 Y_{k-3}) \mod m_2 \tag{6}$$

$m_1 = 2^{32} - 209$, $m_2 = 2^{32} - 22853$, and

$$X_{-3} = X_{-2} = X_{-1} = Y_{-3} = Y_{-2} = Y_{-1} = 111. \tag{7}$$

is is approximately uniformly distributed in $[0,1]$. To this end, generate $N = 10^6$ numbers U_j ($j = 1, \ldots, N$) and plot the histogram of relative frequencies approximating the PDF in $[0,1]$.

Question 3: Consider the stochastic differential equation (SDE)

$$dX(t;\omega) = -X(t;\omega)^3 dt + \frac{1}{2} dW(t;\omega) \qquad X(0;\omega) = X_0(\omega), \tag{8}$$

where $W(t;\omega)$ is a Wiener process, and $X_0(\omega)$ is a uniformly distributed random variable in $[1,2]$. Let us discretize (8) with the Euler-Maruyama scheme

$$X_{k+1} = X_k - X_k^3 \Delta t + \frac{1}{2} \Delta W_k, \tag{9}$$

where $X_k = X(t_k, \omega)$, $\Delta t = t_{k+1} - t_k$ and $\{\Delta W_k\}$ are i.i.d. Gaussian random variables with zero mean and variance Δt.

a) Write the Fokker-Planck (FKP) equation corresponding to the SDE (8).

b) Using the FKP equation show that there exists a statistically stationary solution[1] and compute the PDF $p^*(x)$ of such stationary solution analytically. Is the equilibrium distribution $p^*(x)$ Gaussian?

c) Write the conditional transition density $p(x_{k+1}|x_k)$ defined by discrete Markov process (9). Does the functional form of the transition density depend on the particular time t_k? Or is it the same for all times?

d) By using numerical integration show that the PDF $p^*(x)$ of the statistical steady state you computed in b) is a solution to the fixed point problem (Volterra integral equation)

$$p^*(x) = \int_{-\infty}^{\infty} p(x|y) p^*(y) dy \tag{10}$$

where $p(x|y)$ is the transition density you computed in c). Given that $p^*(y)$ decays quite fast, for numerical purposes it is sufficient to approximate the infinite domain in the integral (10) to $[-5, 5]$.

e) Plot a few sample paths of the SDE for $\Delta = 10^{-4}$ for $t \in [0, 5]$.

f) By computing a sufficiently large number of sample paths, Estimate the PDF of $X(t;\omega)$ numerically (e.g. by using a kernel density PDF estimator or method of relative frequencies) at different times and show that it converges to the steady state PDF you computed in b).

[1] To compute a stationary solution to the Fokker Planck equation, set the time derivative $\partial p/\partial t$ equal to zero.