Fundamentals of UQ (AM 238)

Homework 4

Question 1. Consider the system of SDEs[1]

$$\begin{cases} dX(t;\omega) = -X(t;\omega)^3 dt + dY(t;\omega) \\ \\ dY(t;\omega) = -\tau Y(t;\omega)dt + \sigma dW(t;\omega) \end{cases} \tag{1}$$

where $\sigma, \tau \geq 0$ are given parameters and $W(t;\omega)$ is a Wiener process. The initial condition $(X(0;\omega), Y(0;\omega))$ has i.i.d. components both of which are uniformly distributed in $[0,1]$, i.e., $X(0;\omega)$ and $Y(0,\omega)$ are independent random variables with uniform PDF in $[0,1]$.

a) Plot a few sample paths of $X(t;\omega)$ for $\sigma = 0.1$ and $\tau = \{0.01, 1, 10\}$.

b) Do you expect the system (1) to have a statistically stationary state? Justify your answer.

c) Write the Fokker-Planck equation for the joint PDF of $X(t;\omega)$ and $Y(t;\omega)$.

d) Write the reduced-order equation for the PDF of $X(t;\omega)$ in terms of the conditional expectation $\mathbb{E}\{Y(t;\omega)|X(t;\omega)\}$. (Hint: Integrate the Fokker-Planck equation with respect to y and use the definition of conditional PDF).

e) Set $\sigma = 0$. Compute the conditional expectation $\mathbb{E}\{Y(t;\omega)|X(t;\omega)\}$ explicitly as a function of t and substitute it in the reduced order equation you obtained in d) (with $\sigma = 0$) to obtain an exact (and closed) equation for the PDF of $X(t;\omega)$.

f) Write the PDF equation you obtained in e) as an evolution equation for the cumulative distribution function (CDF) of $X(t;\omega)$.

[1] The system (1) is a prototype initial value problem where $X(t;\omega)$ is driven by the Ornstein–Uhlenbeck process $Y(t;\omega)$, which is a colored (non-white) random noise with exponential correlation function.