Fundamentals of UQ (AM 238)

Homework 2

Question 1: Consider the random variable

$$Y = \sum_{j=1}^{n} b_j X_j, \tag{1}$$

where at least one b_j is non-zero, and $(X_1, \ldots, X_n)$ are jointly Gaussian random variables with mean $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and covariance matrix $\boldsymbol{\Sigma}$.

a) Show that Y is a Gaussian random variable.

b) Compute the mean and the variance of Y as a function of $\{b_j\}$, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

Question 2: Consider the two-dimensional random vector $\boldsymbol{X} = [X_1, X_2]$ with joint PDF

$$p(x_1, x_2) = \begin{cases} K \cos^2(10 x_1 x_2) & (x_1, x_2) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where[1]

$$K = \frac{40}{\mathrm{Si}(20) + 20}, \qquad \mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt. \tag{3}$$

a) Compute the conditional PDFs $p(x_1|x_2)$ and $p(x_2|x_1)$.

b) Write a computer code that samples the PDF (2) using the Gibbs' sampling algorithm. Plot the PDF (2) and the samples you obtain from the Gibbs' algorithm on a 2D contour plot, similarly to Figure 3 in the course note 2.

c) Write a program to calculate the sample mean and sample standard deviation of the random function

$$f(y; X_1, X_2) = \sin(4\pi X_1 y) + \cos(4\pi X_2 y) \qquad y \in [0,1], \tag{4}$$

where X_1 and X_2 are random variables with joint PDF given by equations (2)-(3). To this end, generate $N = 5 \times 10^4$ samples of the function (4) over a grid of 500 evenly-spaced spatial points in $y \in [0,1]$, and compute the required statistics[2]. Plot the mean and standard deviation you obtain as a function of y.

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[1] $\mathrm{Si}(x)$ is a special function known as sine integral.

[2] It may be convenient to store each sample of (4) as a row vector in a 50000×500 matrix.

Question 3: Write a computer code that estimates the PDF of the random variable

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N} \qquad \text{(sample mean)} \tag{5}$$

where $\{X_j\}$ are independent identically distributed Cauchy random variables with PDF

$$p_{X_j}(x) = \frac{1}{\pi(1+x^2)} \qquad j = 1, \ldots, N \tag{6}$$

using the inverse CDF function and relative frequency approach you developed in Homework 1. Plot your results for $N = 10^3$ an $N = 10^5$.