Fundamentals of UQ (AM 238)

Homework 5

Question 1 (gPC basis). Consider a random variable ξ with PDF

$$p_\xi(x) = \begin{cases} \dfrac{e^{-x}}{e - e^{-1}} & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

a) Use the Stieltjes algorithm to compute the sixth-order generalized polynomial chaos basis $\{P_0(x), P_1(x), \ldots, P_6(x)\}$ for ξ, i.e., a set of polynomials up to degree 6 that are orthogonal relative to the PDF ξ given in (1).

b) Verify that the polynomial basis you obtained in a) is orthogonal, i.e., that the matrix

$$\mathbb{E}\{P_k(\xi)P_j(\xi)\} = \int_{-1}^{1} P_k(x)P_j(x)p_\xi(x)dx \tag{2}$$

is diagonal.

c) Plot $P_k(x)$ for $k = 0, \ldots, 6$.

Question 2 (gPC convergence). Consider the following nonlinear function of the random variable $\xi(\omega)$ with PDF defined in (1)

$$\eta(\omega) = \frac{\xi(\omega) - 1}{2 + 1\sin(2\xi(\omega))}. \tag{3}$$

a) Compute the PDF of η using the relative frequency approach. To this end, sample 50000 realizations of ξ using the inverse CDF approach applied to (1), and use such samples to compute samples of $\eta(\omega)$.

b) Show numerically that the gPC expansion[1]

$$\eta_M(\omega) = \sum_{k=0}^{M} a_k P_k(\xi(\omega)), \qquad a_k = \frac{\mathbb{E}\{\eta(\xi(\omega))P_k(\xi(\omega))\}}{\mathbb{E}\{P_k^2(\xi(\omega))\}}, \tag{5}$$

converges to $\eta(\omega)$ in distribution as M increases. To this end, plot the PDF of the random variables $\eta_M(\xi(\omega))$ for $M = 1, 2, 4, 6$ using the method of relative frequencies and compare such PDFs with the PDF of η you computed in a).

[1] Note that $\mathbb{E}\{\eta(\xi(\omega))P_k(\xi(\omega))\}$ can be compute with MC, or with quadrature as

$$\mathbb{E}\{\eta(\xi)P_k(\xi)\} \int_{-1}^{1} \frac{x-1}{2 + 1\sin(2x)} P_k(x)p_\xi(x)dx. \tag{4}$$

c) Compute the mean and variance of $\eta_6(\omega)$ and compare it with the mean and variance of η. Note that such means and variances can be computed in multiple ways, e.g., by using MC, or by approximating the integral defining the moments of the random variable η using quadrature, e.g., via the trapezoidal rule applied to the integral

$$\mathbb{E}\{\eta^k\} = \int_{-1}^{1} \left(\frac{x-1}{2+\sin(2x)} \right)^k p_\xi(x)dx. \tag{6}$$

Question 3 (Stochastic Galerkin method). Compute the solution of the following random initial value problem

$$\begin{cases} \dfrac{dx}{dt} = -\xi(\omega)x + \cos(4t) \\ x(0) = 1 \end{cases} \tag{7}$$

using the stochastic Galerkin method with the gPC basis you obtained in question 1. In particular, use the following gPC expansion of degree 6 for the solution of (7)

$$x(t;\omega) = \sum_{k=0}^{6} \widehat{x}_k(t) P_k(\xi(\omega)), \tag{8}$$

where the gPC modes $\widehat{x}_k(t)$ are to be determined from (7).

a) Compute the mean and the variance of (8) for $t \in [0, 3]$.

b) Compute the PDF of (8) at times $\{0.5, 1, 2, 3\}$ (use relative frequencies).

Note: you can debug your gPC results by either computing the analytic solution of (7) and then computing moments/PDFs of such solution as a function of t, or by randomly sampling many solution paths of (7) and then computing ensemble averages.