CSC236 Assignment 2

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Question 1

(a)

① 1 element has 0 left parentheses

since $* \in \mathcal{T}$ is the only one.

② 1 element has 1 left parentheses

which is $(**) \in \mathcal{T}$.

③ 2 elements with 2 left parentheses

which are $\big((**)*\big), \big(*(**)\big) \in \mathcal{T}$.

④ 5 elements with 3 left parentheses

since if $(t_1 t_2) \in \mathcal{T}$ has 3 left parentheses, then there are 3 cases: (t_1 has 0 left parentheses and t_2 has 2 left parentheses) or (t_1 has 1 left parentheses and t_2 has 1 left parentheses) or (t_1 has 2 left parentheses and t_2 has 0 left parentheses). Add them up we can get 5 elements.

⑤ 14 elements with 4 left parentheses

since if $(t_1 t_2) \in \mathcal{T}$ has 4 left parentheses, then there are 4 cases: (t_1 has 0 left parentheses and t_2 has 3 left parentheses) or (t_1 has 3 left parentheses and t_2 has 0 left parentheses) or (t_1 has 2 left parentheses and t_2 has 1 left parentheses) or (t_1 has 1 left parentheses and t_2 has 2 left parentheses). Add them up we can get 14 elements.

(b)

$$c(n) = \begin{cases} 1 & , n = 0 \\ \sum_{k=0}^{n-1} c(k) \cdot c(n-1-k) & , n > 0 \end{cases}$$

Explanation:

Let $a(n)$ be list of elements with n parentheses with the number of left parentheses equals to $c(n)$

For each element in $a(n)$, consider two lists: $a(k)$ and $a(n-1-k)$, $0 \le k < n$. We can simply take one element x from one of the lists and take another y from the other list. By concatenating x and y together and add one pair of parentheses outside to create that element in $a(n)$, according to the definition of set $\mathcal{T}$.

Hence to calculate $c(n)$ we can separate elements in $a(n)$ in several cases according to k.

As mentioned above, we know that $k \in \{0,1,2,\cdots,n-1\}$. For each k, we have number of elements with k left parentheses, i.e. $c(k)$ multiplies number of elements with $n-1-k$ left parentheses, i.e. $c(n-1-k)$ as the total number.

In conclusion, $c(n)=1$ when n = 0 as a base case,

and $c(n)=\sum_{k=0}^{n-1} c(k)\cdot c(n-1-k), n>0$

Question 2

(a)

$p(n)$

$$= \begin{cases} 0 & ,1 \le n \le 2 \\ 1 & ,n=1 \; or \; 3 \le n \le 7 \\ 2 & ,8 \le n \le 11 \\ p(n-3)+p(n-4)+p(n-5)-p(n-7)-p(n-8)-p(n-9)+p(n-12), & n \ge 12 \end{cases}$$

Explanation:

Since amount of money cannot be negative, $n \ge 0$

For $n < 12$, these base cases can be verified by enumeration.

For $n \ge 12$, consider $p(n-3)$ which must exists since $n \ge 12 \Rightarrow n-3 \ge 0$, and combinations in $p(n-3)$ add a 3-cent stamp is one part of combinations in $p(n)$.

Similar for $p(n-4)$ and $p(n-5)$.

But when we add up all combinations in $p(n-3)$, $p(n-4)$ and $p(n-5)$, there must be overlapping.

Since combinations in $p(n-3)$ may also contain 4-cent, 5-cent stamp and the amount is $p(n-4-5)=p(n-9)$. ($p(n-9)$ exists since $n \ge 12 \Rightarrow n-9 \ge 0$)

Similarly, combinations in $p(n-4)$ may also contain 3-cent, 5-cent stamp and the amount is $p(n-3-5)=p(n-8)$. ($p(n-8)$ exists since $n \ge 12 \Rightarrow n-8 \ge 0$)

And combinations in $p(n-5)$ may also contain 4-cent, 3-cent stamp and the amount is $p(n-4-3)=p(n-7)$. ($p(n-7)$ exists since $n \ge 12 \Rightarrow n-7 \ge 0$)

So the total overlapping is $p(n-7)+p(n-8)+p(n-9)$ minus the combinations in $p(n)$

which contains 3-cent, 4-cent and 5-cent stamps, which is "overlapping when we calculate overlapping using $p(n-7)$, $p(n-8)$, $p(n-9)$", $i.e. p(n-3-4-5) = p(n-12)$.

($p(n-12)$ exists since $n \geq 12 \Rightarrow n-12 \geq 0$)

Hence the total overlapping is $p(n-7)+p(n-8)+p(n-9)-p(n-12)$

And $p(n) = (sum\ of\ 3\ cases) - (total\ overlapping)$

$$= p(n-3)+p(n-4)+p(n-5)-p(n-7)-p(n-8)-p(n-9)+p(n-12)$$

(b)

Define $P(n)$: $\Lambda_{m=1}^{m=n} p(m) \leq p(n)$

Want to show: $\forall n \in \mathbb{N}^+, P(n)$ holds

Proof: (Complete induction)

Let $n \in \mathbb{N}^+$, assume $P(k)$ holds for $0 \leq k < n$

Want to show $P(k)$ follows

Case 1: $n \leq 12$

$p(0) = 0 \geq p(0) = 0$

$p(1) = 0 \geq p(0), p(1) = 0$

$p(2) = 0 \geq p(0), p(1), p(2) = 0$

$p(3) = 1 \geq p(3) = 1$ and $\geq p(0), p(1), p(2) = 0$

$p(4) = 1 \geq p(3) = 1$ and $\geq p(0), p(1), p(2) = 0$

$p(5) = 1 \geq p(3), p(4) = 1$ and $\geq p(0), p(1), p(2) = 0$

$p(6) = 1 \geq p(3), p(4), p(5) = 1$ and $\geq p(0), p(1), p(2) = 0)$

$p(7) = 1 \geq p(3), p(4), p(5), p(6) = 1$ and $\geq p(0), p(1), p(2) = 0$

$p(8) = 2 \geq p(8) = 2$ and $\geq p(3), p(4), p(5), p(6) = 1$

and $\geq p(0), p(1), p(2) = 0$

$p(9) = 2 \geq p(8), p(9) = 2$ and $\geq p(3), p(4), p(5), p(6) = 1$

and $\geq p(0), p(1), p(2) = 0$

$p(10) = 2 \geq p(8), p(9), p(10) = 2$ and $\geq p(3), p(4), p(5), p(6) = 1$

and $\geq p(0), p(1), p(2) = 0$

$p(11) = 2 \geq p(8), p(9), p(10), p(11) = 2$

and $\geq p(3), p(4), p(5), p(6) = 1$ and $\geq p(0), p(1), p(2) = 0$

$p(12) = 3 \geq p(12) = 3$ and $\geq p(8), p(9), p(10), p(11), p(12) = 2$

and $\geq p(3), p(4), p(5), p(6) = 1$ and $\geq p(0), p(1), p(2) = 0$

$P(0) \sim P(12)$ hold

Case 2: $n \geq 13$

Since $0 \leq 12 \leq n - 11 < n$

$\mathrm{P}(\mathrm{n}-1)$ holds

Hence we only need to show: $\mathrm{p(n)} \geq \mathrm{p(n-1)}$, i.e. $\mathrm{p(n)} - \mathrm{p(n-1)} \geq 0$

$$\mathrm{p(n)} = p(n-3) + p(n-4) + p(n-5)$$
$$-p(n-7) - p(n-8) - p(n-9) + p(n-12)$$

$$\mathrm{p(n-1)} = p(n-4) + p(n-5) + p(n-6)$$
$$-p(n-8) - p(n-9) - p(n-10) + p(n-13)$$

$$\mathrm{p(n)} - \mathrm{p(n-1)} = \mathrm{p(n-3)} - \mathrm{p(n-6)} - \mathrm{p(n-7)} + \mathrm{p(n-10)}$$
$$+\mathrm{p(n-12)} - \mathrm{p(n-13)}$$

Since $0 \leq n - 12 < n$ and $0 \leq n - 12 < n$ and $n - 13 < n - 12$

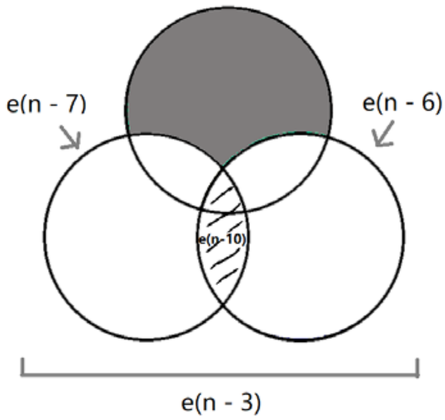
By induction hypothesis $\mathrm{p(n-12)} - \mathrm{p(n-13)} \geq 0$ ①

Hence we only need to show $\mathrm{p(n-3)} - \mathrm{p(n-6)} - \mathrm{p(n-7)} + \mathrm{p(n-10)} \geq 0$

For convenice, let $\mathrm{e(n)}$ be the set of combinations of stamps related to number $\mathrm{p(n)}$

s.t. $|\mathrm{e(n)}| = \mathrm{p(n)}$

Consider the diagram below



The union of 3 circles is $\mathrm{e(n-3)}$

The left circle is $\mathrm{e(n-7)}$, representing the elements in $\mathrm{e(n-3)}$ which have at least one “4” and remove that “4”

The right circle is $\mathrm{e(n-6)}$, representing the elements in $\mathrm{e(n-3)}$ which have at least one “3” and remove that “3”

The intersection of them is $\mathrm{e(n-10)}$, representing the elements in $\mathrm{e(n-3)}$ which have at least one “3” and at least one “4” and then remove those “3” and “4”

Hence the number of elements in gray part cannot be negative, we have:

$$(|\mathrm{e(n-3)}| - (|\mathrm{e(n-7)}| + |\mathrm{e(n-6)}| - |\mathrm{e(n-10)}|)) \geq 0$$

i.e. $\mathrm{p}(\mathrm{n}-3)-\mathrm{p}(\mathrm{n}-6)-\mathrm{p}(\mathrm{n}-7)+\mathrm{p}(\mathrm{n}-10) \geq 0$ ②

By ①② we know that $\mathrm{p}(\mathrm{n})-\mathrm{p}(\mathrm{n}-1) \geq 0$

Hence $\mathrm{P}(\mathrm{n})$ follows

Conclusion: by complete induction we showed that $p(n)$ is non-decreasing ∎

Question 3

(a)

Define $P(n): \wedge_{m=1}^{m=n} T(m) \leq T(n)$

Want to show: $\forall n \in \mathbb{N}^{+}, P(n)$ holds

Proof: (Complete Induction)

Let $n \in \mathbb{N}^{+}$, assume $P(j)$ holds for $1 \leq j < n$

We will prove that $P(n)$ follows

Base Case: $n = 1 \quad T(1) = c' \leq c' = T(1),\ P(1)\ holds.$

$n = 2 \quad T(2) = 1 + c' \leq 1 + c' = T(2),$

$T(1) = c' \leq 1 + c' = T(2),\ \ P(2)\ holds.$

Inductive step: $n > 2$

$1 \leq n-1 \leq n$

By inductive hypothesis, $P(n-1)$ holds

i.e. $\wedge_{m=1}^{m=n-2} T(m) \leq T(n-1)$

So we only need to prove $T(n-1) \leq T(n)$

$T(n-1) = 1 + T(\left\lceil\frac{n-1}{2}\right\rceil)$ (by def of T(n) since $n > 2 \Rightarrow n-1 > 1$)

$\leq 1 + T\left(\left\lceil\frac{n}{2}\right\rceil\right)$ (by IH $P\left(\left\lceil\frac{n}{2}\right\rceil\right)$ since $1 \leq \left\lceil\frac{n}{2}\right\rceil < n$
and $\left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{n}{2}\right\rceil$)

$= T(n)$

Hence $P(n)$ follows in this case

Conclusion: by complete induction we showed that T is non-decreasing. ∎

(b)

Define $P(k): n = 2^k \Rightarrow T(n) = \lg(n) + c'$

Want to show: $\forall k \in \mathbb{N}, P(k)$

Proof: (Simple induction)

Base Case: $k = 0,\ n = 1$

$T(1) = c' = \lg(1) + c'$

$\mathrm{P}(0)\ holds$

Inductive step: $k \geq 1$

Assume $P(k-1)$ holds

i.e. $T(2^{k-1}) = \lg(2^{k-1}) + c'$

Want to show: $P(k)$ follows

$T(2^k) = 1 + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$ (by def of T(n) since $k \geq 1 \Rightarrow 2^k > 1$)

$= 1 + T\left(\left\lceil \frac{2^k}{2} \right\rceil\right)$

$= 1 + T(2^{k-1})$

$= 1 + \lg(2^{k-1}) + c'$ (by IH)

$= \lg(2^{k-1} \cdot 2) + c'$

$= \lg(2^k) + c'$

Hence $P(k)\ follows$

Conclusion: by simple induction with base case we can conclude that

$\forall k, n \in \mathbb{N}, n = 2^k \Rightarrow T(n) = \lg(n) + c'$ ∎

(c)

First prove $T \in \mathrm{O}(lg)$

Define $n^* = 2^{\lceil \log_2 n \rceil}$

$$\lceil \log_2 n \rceil - 1 < \log_2 n \leq \lceil \log_2 n \rceil \Rightarrow \frac{n^*}{2} < n \leq n^*$$

Proof: Let $d = 3, d \in \mathbb{R}^+$

Let $B = \max\{2, 2^{c'}\}, B \in \mathbb{R}^+$

Let $n \in \mathbb{N}^+$, assume $n \geq B$

Then $T(n) \leq T(n^*)$ (since by(b) $T(n)$ is non-decreasing and $n \leq n^*$)

$= \lg(n^*) + c'$ (by(a) since $n^* = 2^{\lceil \log_2 n \rceil}$)

$\leq \lg(2n) + c'$ (since $\frac{n^*}{2} < n \Rightarrow n^* < 2n$)

$= \lg(n) + \lg(2) + c'$

$= \lg(n) + 1 + c'$

$\leq \lg(n) + \lg(n) + \lg(n)$ (since $n \geq B = \max\{2, 2^{c'}\}$

$\Rightarrow \lg(n) \geq 1 \text{ and } \lg(n) \geq c'$)

$= 3\lg n$

$\leq d \lg n$

Hence we have proved that $T \in O(lg)$ ■

Then prove $T \in \Omega(lg)$

Define $n^* = 2^{\lceil \log_2 n \rceil}$

$$\lceil \log_2 n \rceil - 1 < \log_2 n \leq \lceil \log_2 n \rceil \Rightarrow \frac{n^*}{2} < n \leq n^*$$

Proof: Let $d = \frac{1}{2}, d \in \mathbb{R}^+$

Let $B = 4^{1-c'}, B \in \mathbb{R}^+$

Let $n \in \mathbb{N}^+$, assume $n \geq B$

Then $T(n) \geq T(\frac{n^*}{2})$ (since by(b) $T(n)$ is non-decreasing and $n \geq \frac{n^*}{2}$)

$= \lg(\frac{n^*}{2}) + c'$ (by (a) since $n^* = 2^{\lceil \log_2 n \rceil}$)

$\geq \lg\left(\frac{n}{2}\right) + c'$ (since $n^* \geq n$)

$= \lg(n) - 1 + c'$

$= \lg(n) - (1 - c')$

$= \frac{1}{2}\lg(n) + \frac{1}{2}\lg(n) - (1 - c')$

$\geq \frac{1}{2}\lg n$ (since $n \geq B = 4^{1-c'} \Rightarrow \frac{1}{2}\lg(n) - (1 - c') \geq 0$)

Hence we have proved that $T \in \Omega(lg)$ ■

Now we have $T \in O(lg)$ and $T \in \Omega(lg)$, then we can conclude that $T \in \Theta(lg)$.

Question 4

For clear notes we write function $count\_subsequence()$ as $cs()$

Define $P(j)$: $cs(s1, s2, i, j)$ returns number of times s1[: i] occurs as a subsequence of s2[: j]

for $i \in [0, len(s1)]$

Want to show: $\forall j \in \mathbb{N}\ s.t.\ 0 \leq j \leq \text{len(s2)}, P(j)$

Proof: (Complete induction on j)

Let $\text{i} \in \mathbb{N}\ s.t.\ 0 \leq i \leq \text{len(s1)}$

Base Case: $j = 0$ then either $i = j = 0 \Rightarrow return\ 1$

(any sequence has exactly one subsequence of empty string)

or $i > j \Rightarrow return\ 0$

(any sequence has no subsequence of an string which is longer than it)

Both of them returns number of times s1[: i] occurs as a subsequence of s2[: j]

$P(0)$ holds.

Inductive step: Assume $j \geq 1$ and post-condition is satisfied for inputs of size $0 \leq k < j$ that satisfy the pre-condition.

Want to show $P(j)$ follows

When we call $cs()$, there are 4 cases to consider.

Case 1: $i = 0$,

$cs(s1, s2, i, j)\ returns\ 1$

Since any sequence has exactly one subsequence of empty string

Post-condition is satisfied in this case

Case 2: $i > j$

$cs(s1, s2, i, j)\ returns\ 0$

Since any sequence has no subsequence of an string which is longer than it

Post-condition is satisfied in this case

Case 3: $0 < i \leq j\ and\ s1[i-1]! = s2[j-1]$

$cs(s1, s2, i, j)\ returns\ cs(s1, s2, i, j-1)$

① Show $cs(s1, s2, i, j-1)$ satisfies inductive hypothesis

$0 \leq j - 1 < j$, since $j \geq 1$

② Translate post-condition to $cs(s1, s2, i, j-1)$

$cs(s1, s2, i, j-1)$ returns number of times $s1[:i]$ occurs as a subsequence of $s2[:j-1]$.

③ Show $cs(s1, s2, i, j)$ satisfies post-condition

Since $s2[j-1]$ is the last string in $s2[:j]$, $s1[i-1]$ is last string in $s1[:i]$

If $s2[j-1]! = s1[i-1]$

Then any sequence contains $s2[j-1]$ cannot be the same as $s1[:i]$.

Hence the number of $s1[:i]$ occurs as a subsequence in $s2[:j]$ is equals to it occurs as a subsequence in $s2[:j-1]$, and $cs(s1, s2, i, j-1)$ should return the the same result as $cs(s1, s2, i, j)$

Hence $cs(s1, s2, i, j-1)$ returns number of times

s1[: i] occurs as a subsequence of s2[: j]

Post-condition is satisfied in this case

Case 4: $0 < i \leq j \; and \; s1[i-1] == s2[j-1]$

$cs(s1, s2, i, j) \; returns \; cs(s1, s2, i, j-1) + cs(s1, s2, i-1, j-1)$

① Show both above satisfies inductive hypothesis

$0 \leq j-1 < j$, since $j \geq 1$

② Translate post-condition to $cs(s1, s2, i, j-1)$ and $cs(s1, s2, i-1, j-1)$

$cs(s1, s2, i, j-1)$ returns number of times $s1[:i]$ occurs as a subsequence of $s2[:j-1]$

$cs(s1, s2, i-1, j-1)$ returns number of times $s1[:i-1]$ occurs as a subsequence of $s2[:j-1]$. (notice that i – 1 satisfies pre-condition since $i > 0 \Rightarrow i-1 \geq 0$ in this case)

③ Show $cs(s1, s2, i, j)$ satisfies post-condition

Since $s2[j-1]$ is the last string in $s2[:j]$, $s1[i-1]$ is last string in $s1[:i]$ and $s2[j-1] == s1[i-1]$.

We have 2 ways to construct a subsequence in $s2$ which equals to $s1$, one is using $s2[j-1]$, the other is not.

Separate all subsequences of $s2$ which equals to $s1$ in 2 parts:

Substrings in part 1 contains $s2[j-1]$, substrings in part 2 does not.

For part 1, we need to find whole s1 in $s2[:j-1]$, which is $cs(s1, s2, i, j-1)$.

For part 2, we need to find $s1[:i-1]$ in $s2[:j-1]$, then plus $s2[j-1]$, which is $cs(s1, s2, i-1, j-1)$.

Hence $cs(s1, s2, i, j-1) + cs(s1, s2, i-1, j-1)$ returns number of times

s1[: i] occurs as a subsequence of s2[: j]

Post-condition is satisfied in this case.

So $P(j)$ follows in all 4 cases.

Conclusion: We have showed all 4 cases are true in inductive step.

With base case, we conclude that pre-condition plus execution implies its post-condition. ■

Question 5

First we prove partial correctness by proving loop invariant

Define $\mathrm{P(i)}$: after the i th iteration of the loop (if occurs):

① $0 \leq \mathrm{blue_i} \leq \mathrm{green_i} \leq \mathrm{red_i} \leq$ len(colour_list)

② colour_list$_\mathrm{i}$[0 : $\mathrm{green_i}$] + colour_list$_\mathrm{i}$[$\mathrm{red_i}$:] same colours as before

③ all([c == "b" for c in colour_list$_\mathrm{i}$[0 : $\mathrm{blue_i}$]])

④ all([c == "g" for c in colour_list$_\mathrm{i}$[$\mathrm{blue_i}$: $\mathrm{green_i}$]])

⑤ all([c == "r" for c in colour_list$_\mathrm{i}$[$\mathrm{red_i}$:]])

Want to show: $\forall \mathrm{i} \in \mathbb{N}, \mathrm{P(i)}$

Proof: (Simple Induction)

Base case: i = 0

$\mathrm{blue_0} = \mathrm{green_0} = 0$, $\mathrm{red_0} = 6$

$0 \leq \mathrm{blue_0} \leq \mathrm{green_0} \leq \mathrm{red_0} \leq$ len(colour_list) = 6, ① satisfied

colour_list$_0$[0 : $\mathrm{green_0}$] + colour_list$_0$[$\mathrm{red_0}$:] = [], ② satisfied

colour_list$_0$[0 : $\mathrm{blue_0}$] = colour_list$_0$[$\mathrm{blue_0}$: $\mathrm{green_0}$] = colour_list$_0$[$\mathrm{red_0}$:] = [],

③④⑤ are vacuously true

$\mathrm{P(0)}$ holds

Inductive step: $i \geq 1$

Assume $\mathrm{P(i)}$ holds, i.e.

① $0 \leq \mathrm{blue_i} \leq \mathrm{green_i} \leq \mathrm{red_i} \leq$ len(colour_list)

② colour_list$_\mathrm{i}$[0 : $\mathrm{green_i}$] + colour_list$_\mathrm{i}$[$\mathrm{red_i}$:] same colours as before

③ all([c == "b" for c in colour_list$_\mathrm{i}$[0 : $\mathrm{blue_i}$]])

④ all([c == "g" for c in colour_list$_\mathrm{i}$[$\mathrm{blue_i}$: $\mathrm{green_i}$]])

⑤ all([c == "r" for c in colour_list$_\mathrm{i}$[$\mathrm{red_i}$:]])

Want to show $\mathrm{P(i+1)}$ follows

There are 5 cases to consider:

Case 1: colour_list$_\mathrm{i}$[0 : $\mathrm{green_i}$] = "g"

Then $\mathrm{green_{i+1}} = \mathrm{green_i} + 1$, $\mathrm{blue_{i+1}} = \mathrm{blue_i}$, $\mathrm{red_{i+1}} = \mathrm{red_i}$

Then we have $0 \leq \mathrm{blue_{i+1}} \leq \mathrm{green_{i+1}} \leq \mathrm{red_{i+1}} \leq$ len(colour_list)

(By IH and loop condition)

① is satisfied

Since colour_list did not change in this loop

colour_list$_\mathrm{i+1}$[0 : $\mathrm{green_{i+1}}$] + colour_list$_\mathrm{i+1}$[$\mathrm{red_{i+1}}$:]

$=$ $\text{colour\_list}_i[0:\ \text{green}_i]$ + $\text{colour\_list}_i[\text{red}_i\ :]$

Hence by IH $\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$ has same colours as before, ② is satisfied

Also since $\text{colour\_list}_{i+1}[0:\ \text{blue}_{i+1}]$= $\text{colour\_list}_i[0:\ \text{blue}_i]$,

$\text{colour\_list}_{i+1}[\text{blue}_{i+1}\ :\ \text{green}_{i+1}]$ = $\text{colour\_list}_i[\text{blue}_i\ :\ \text{green}_i]$ + ["g"],

$\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$ = $\text{colour\_list}_i[\text{red}_i\ :]$

By IH, ③④⑤ are satisfied

Case 2: $\text{colour\_list}_i[0:\ \text{green}_i]$ = "b"

Then $\text{green}_{i+1} = \text{green}_i + 1$, $\text{blue}_{i+1} = \text{blue}_i + 1$, $\text{red}_{i+1} = \text{red}_i$

Then we have $0 \le \text{blue}_{i+1} \le \text{green}_{i+1} \le \text{red}_{i+1} \le$ len(colour_list)

(By IH and loop condition)

① is satisfied

Since $\text{colour\_list}_i[\text{green}_i]$, $\text{colour\_list}_i[\text{blue}_i]$ are switched in this loop, and both of them are contained in $\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$, while other elements remain unchanged

$\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$ has same colours as before, ② is satisfied

Also since $\text{colour\_list}_{i+1}[0:\ \text{blue}_{i+1}]$= $\text{colour\_list}_i[0:\ \text{blue}_i]$ + ["b"],

$\text{colour\_list}_{i+1}[\text{blue}_{i+1}\ :\ \text{green}_{i+1}]$ = $\text{colour\_list}_i[\text{blue}_i\ :\ \text{green}_i]$,

$\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$ = $\text{colour\_list}_i[\text{red}_i\ :]$

By IH, ③④⑤ are satisfied

Case 3: $\text{colour\_list}_i[0:\ \text{green}_i]$ = "r" and $\text{colour\_list}_i[\text{red}_i - 1\ :]$ = "r"

Then $\text{green}_{i+1} = \text{green}_i$, $\text{blue}_{i+1} = \text{blue}_i$, $\text{red}_{i+1} = \text{red}_i - 1$

Then we have $0 \le \text{blue}_{i+1} \le \text{green}_{i+1} \le \text{red}_{i+1} \le$ len(colour_list)

(By IH and loop condition)

① is satisfied

Since colour_list did not change in this loop

$\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$

$=$ $\text{colour\_list}_i[0:\ \text{green}_i]$ + $\text{colour\_list}_i[\text{red}_i\ :]$

Hence by IH $\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}\ :]$ has same colours as before, ② is satisfied

Also since $\text{colour\_list}_{i+1}[0:\ \text{blue}_{i+1}]$= $\text{colour\_list}_i[0:\ \text{blue}_i]$,

$\text{colour\_list}_{i+1}[\text{blue}_{i+1}\ :\ \text{green}_{i+1}]$ = $\text{colour\_list}_i[\text{blue}_i\ :\ \text{green}_i]$,

$\text{colour\_list}_{i+1}[\text{red}_{i+1}$:] = [“r”] + $\text{colour\_list}_{i}[\text{red}_{i}$:]

By IH, ③④⑤ are satisfied

Case 4: $\text{colour\_list}_{i}[0:\ \text{green}_{i}]$ = "r" and $\text{colour\_list}_{i}[\text{red}_{i}-1$:] = "g"

Then $\text{green}_{i+1}=\text{green}_{i}+1,\ \text{blue}_{i+1}=\text{blue}_{i},\ \text{red}_{i+1}=\text{red}_{i}-1$

Then we have $0\ \le \text{blue}_{i+1}\le \text{green}_{i+1}\ \le \text{red}_{i+1}\le$ len(colour_list)

(By IH and loop condition)

① is satisfied

Since $\text{colour\_list}_{i}[\text{green}_{i}]$, $\text{colour\_list}_{i}[\text{red}_{i}-1]$ are switched in this loop, and both of them are contained in $\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}$:], while other elements remain unchanged

$\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}$:] has same colours as before, ② is satisfied

Also since $\text{colour\_list}_{i+1}[0:\ \text{blue}_{i+1}]$= $\text{colour\_list}_{i}[0:\ \text{blue}_{i}]$,

$\text{colour\_list}_{i+1}[\text{blue}_{i+1}\ :\ \text{green}_{i+1}]$ = $\text{colour\_list}_{i}[\text{blue}_{i}\ :\ \text{green}_{i}]$ + [“g”],

$\text{colour\_list}_{i+1}[\text{red}_{i+1}$:] = [“r”] + $\text{colour\_list}_{i}[\text{red}_{i}$:]

By IH, ③④⑤ are satisfied

Case 5: $\text{colour\_list}_{i}[0:\ \text{green}_{i}]$ = "r" and $\text{colour\_list}_{i}[\text{red}_{i}-1$:] = "b"

Then $\text{green}_{i+1}=\text{green}_{i}+1,\ \text{blue}_{i+1}=\text{blue}_{i}+1,\ \text{red}_{i+1}=\text{red}_{i}-1$

Then we have $0\ \le \text{blue}_{i+1}\le \text{green}_{i+1}\ \le \text{red}_{i+1}\le$ len(colour_list)

(By IH and loop condition)

① is satisfied

Since $\text{colour\_list}_{i}[\text{green}_{i}]$, $\text{colour\_list}_{i}[\text{blue}_{i}]$, $\text{colour\_list}_{i}[\text{red}_{i}-1]$ are switched in this loop, and all of them are contained in $\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}$:], while other elements remain unchanged

$\text{colour\_list}_{i+1}[0:\ \text{green}_{i+1}]$ + $\text{colour\_list}_{i+1}[\text{red}_{i+1}$:] has same colours as before, ② is satisfied

Also since $\text{colour\_list}_{i+1}[0:\ \text{blue}_{i+1}]$= $\text{colour\_list}_{i}[0:\ \text{blue}_{i}]$ + [“b”],

$\text{colour\_list}_{i+1}[\text{blue}_{i+1}\ :\ \text{green}_{i+1}]$ = $\text{colour\_list}_{i}[\text{blue}_{i}\ :\ \text{green}_{i}]$,

$\text{colour\_list}_{i+1}[\text{red}_{i+1}$:] = [“r”] + $\text{colour\_list}_{i}[\text{red}_{i}$:]

By IH, ③④⑤ are satisfied

Hence $\mathrm{P}(i+1)$ follows in all 5 cases

Conclusion: We have showed all 5 cases are true in inductive step.

With base case, we conclude that loop invariant is true. ■

Then we show precondition + execution + termination implies postcondition

If the loop terminates after iteration f, then the following must be true:

① $0 \le \mathrm{blue_f} \le \mathrm{green_f} \le \mathrm{red_f} \le$ len(colour_list)

② colour_list$_f$[0 : $\mathrm{green_f}$] + colour_list$_f$[$\mathrm{red_f}$:] same colours as before

③ all([c == "b" for c in colour_list$_f$[0 : $\mathrm{blue_f}$]])

④ all([c == "g" for c in colour_list$_f$[$\mathrm{blue_f}$: $\mathrm{green_f}$]])

⑤ all([c == "r" for c in colour_list$_f$[$\mathrm{red_f}$:]])

(by $\mathrm{P(f)}$)

$\mathrm{green_f} \ge \mathrm{red_f}$

(by loop condition)

Thus $\mathrm{green_f} = \mathrm{red_f}$

Since we know that after terminated all “b”, “g”, “r” are separated in 3 parts, and by previous line the union of these parts is the whole colour_list, we can conclude that after terminated colour_list has same strings as before, ordered "b" < "g" < "r".

Postcondition is satisfied.

At last we prove termination

Let k= red – green

Want to show that k is strictly decreasing

Proof: Let $\mathrm{k_i} = \mathrm{red_i} - \mathrm{green_i}$ be k at i th iteration

Suppose there is an i + 1 iteration, there are 5 cases to consider

Case 1: colour_list$_i$[0 : $\mathrm{green_i}$] = "g"

$\mathrm{green_{i+1}} = \mathrm{green_i} + 1,\ \mathrm{blue_{i+1}} = \mathrm{blue_i},\ \mathrm{red_{i+1}} = \mathrm{red_i}$

$\mathrm{k_{i+1}} = \mathrm{red_{i+1}} - \mathrm{green_{i+1}} = \mathrm{red_i} - \mathrm{green_i} - 1 = \mathrm{k_i} - 1$

$\Rightarrow \mathrm{k_{i+1}} < \mathrm{k_i}$

Case 2: colour_list$_i$[0 : $\mathrm{green_i}$] = "b"

$\mathrm{green_{i+1}} = \mathrm{green_i} + 1,\ \mathrm{red_{i+1}} = \mathrm{red_i}$

$\mathrm{k_{i+1}} = \mathrm{red_{i+1}} - \mathrm{green_{i+1}} = \mathrm{red_i} - \mathrm{green_i} - 1 = \mathrm{k_i} - 1$

$\Rightarrow \mathrm{k_{i+1}} < \mathrm{k_i}$

Case 3: colour_list$_i$[0 : $\mathrm{green_i}$] = "r" and colour_list$_i$[$\mathrm{red_i} - 1$:] = "r"

$\mathrm{green_{i+1}} = \mathrm{green_i},\ \mathrm{red_{i+1}} = \mathrm{red_i} - 1$

$\mathrm{k_{i+1}} = \mathrm{red_{i+1}} - \mathrm{green_{i+1}} = \mathrm{red_i} - \mathrm{green_i} - 1 = \mathrm{k_i} - 1$

$\Rightarrow \mathrm{k_{i+1}} < \mathrm{k_i}$

Case 4: $\text{colour\_list}_i[0: \text{green}_i]$ = "r" and $\text{colour\_list}_i[\text{red}_i - 1 :]$ = "g"

$\text{green}_{i+1} = \text{green}_i + 1, \ \text{red}_{i+1} = \text{red}_i - 1$

$k_{i+1} = \text{red}_{i+1} - \text{green}_{i+1} = \text{red}_i - \text{green}_i - 2 = k_i - 2$

$\Rightarrow k_{i+1} < k_i$

Case 5: $\text{colour\_list}_i[0: \text{green}_i]$ = "r" and $\text{colour\_list}_i[\text{red}_i - 1 :]$ = "b"

$\text{green}_{i+1} = \text{green}_i + 1, \ \text{red}_{i+1} = \text{red}_i - 1$

$k_{i+1} = \text{red}_{i+1} - \text{green}_{i+1} = \text{red}_i - \text{green}_i - 2 = k_i - 2$

$\Rightarrow k_{i+1} < k_i$

In all 5 cases $k_{i+1} < k_i$, hence we can conclude that k is strictly decreasing ■

Thus we have exhibited a decreasing sequence of natural numbers linked to loop iterations.

The last element of this sequence has the index of the last loop iteration, so the loop terminates.

By proving ① partial correctness

② precondition + execution + termination implies postcondition

③ termination

We have eventually proved the correctness of this function.