CSC236 Assignment1

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Question 1

To proof this question by simple induction, we need to use one already-known claim.

Claim 1: For any arbitrary graph G = (V, E), I can always choose a vertex with $\leq \left\lfloor \frac{|V|}{2} \right\rfloor$ edges incident with it.

(a)

Answer: Yes.

Proof: Assume P(234)

That is, for any arbitrary bipartite graph G = (V, E) with |V| = 234, G has no more than $\frac{234^2}{4}$ edges.

WTS P(235) follows.

Let $G_1 = (V_1, E_1)$ be an arbitrary bipartite graph with $|V_1|$ = 235.

By Claim 1, I can always choose a vertex in G_1 with $\leq \left\lfloor \frac{235}{2} \right\rfloor = 117$ edges incident with it.

Without losing generality, choose this vertex, call it v_1.

Let $G_2 = (V_2, E_2)$ be a bipartite graph s.t. $V_2 = V_1 \backslash \{v_1\}$ and $E_2 = E_1 \backslash \{\text{all edges incident with } v_1\}$

Hence $|V_2| = |V_1| - 1 = 234$

By hypothesis P(234), we know that G_2 has no more than $\frac{234^2}{4}$ edges.

Hence $|E_1| = |E_2| + \#\text{edges incident with } v_1$

$$\leq \frac{234^2}{4} + 117$$
$$\leq \frac{235^2}{4} = \frac{|V_1|^2}{4}$$

We have proved that P(235) follows. ∎

(b)

Answer: No.

Reason: Assume P(235)

That is, for any arbitrary bipartite graph G = (V, E) with |V| = 235, G has no more than $\frac{235^2}{4}$ edges.

If we want to show P(236) follows:

Let $G_1 = (V_1, E_1)$ be an arbitrary bipartite graph with $|V_1|$ = 236.

By Claim 1, I can always choose a vertex in G_1 with $\leq \left\lfloor\frac{236}{2}\right\rfloor = 118$ edges incident with it.

Without losing generality, choose this vertex, call it v_1.

Let $G_2 = (V_2, E_2)$ be a bipartite graph s.t. $V_2 = V_1 \backslash \{v_1\}$ and $E_2 = E_1 \backslash \{\text{all edges incident with } v_1\}$

Hence $|V_2| = |V_1| - 1 = 235$

By hypothesis P(235), we know that G_2 has no more than $\frac{235^2}{4}$ edges.

Hence $|E_1| = |E_2| + \#\text{edges incident with } v_1$

$$\leq \frac{235^2}{4} + 118$$

However, we CANNOT get $\frac{235^2}{4} + 118 \leq \frac{236^2}{4} = \frac{|V_1|^2}{4}$

Hence we cannot prove P(236) follows by assuming P(235) holds.

(c)

Strengthen P(n), call it R(n): Every bipartite graph on n vertices has no more than $\left\lfloor\frac{n^2}{4}\right\rfloor$ edges

Proof of R(n): (Simple Induction)

Base case: n = 0

Graph with o vertex has 0 edges

$0 \leq 0 = \left\lfloor\frac{0^2}{4}\right\rfloor$

P(0) holds.

Inductive step: Let $n \in \mathbb{N}$

Assume R(n) holds, i.e. for any arbitrary bipartite graph G = (V, E) with |V| = n, G has no more than $\left\lfloor\frac{n^2}{4}\right\rfloor$ edges.

WTS that R(n+1) also holds

Let $G_1 = (V_1, E_1)$ be an arbitrary bipartite graph with $|V_1|$ = n+1.

By Claim 1, I can always choose a vertex in G_1 with $\leq \left\lfloor\frac{n+1}{2}\right\rfloor$ edges incident with it.

Without losing generality, choose one of this vertex, call it v_1.

Let $G_2 = (V_2, E_2)$ be a bipartite graph s.t. $V_2 = V_1 \backslash \{v_1\}$

and $E_2 = E_1 \backslash \{\text{all edges incident with } v_1\}$

Hence $|V_2| = |V_1| - 1 = n$

By hypothesis P(n), we know that G_2 has no more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

Then we need to show: $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \le \left\lfloor \frac{n+1^2}{4} \right\rfloor$

Let $k \in \mathbb{N}$, we have two cases to consider: n = 2k and n = 2k+1, namely n is even

Or n is odd.

Case n = 2k: $\left\lfloor \frac{(2k)^2}{4} \right\rfloor + \left\lfloor \frac{2k+1}{2} \right\rfloor = k^2 + k \le k^2 + k = \left\lfloor \frac{(2k+1)^2}{4} \right\rfloor$

Case n = 2k+1: $\left\lfloor \frac{(2k+1)^2}{4} \right\rfloor + \left\lfloor \frac{2k+2}{2} \right\rfloor = k^2 + 2k + 1 \le k^2 + 2k + 1 = \left\lfloor \frac{(2k+2)^2}{4} \right\rfloor$

In both cases we have proved that $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \le \left\lfloor \frac{n+1^2}{4} \right\rfloor$, Hence R(n+1) follows. ■

We use simple induction proved R(n), since $\left\lfloor \frac{n^2}{4} \right\rfloor \le \frac{n^2}{4}$, we know that R(n) implies P(n).

Hence P(n) is also True for all natural number n.

Question 2

(a)

Answer: Yes.

Proof: Assume P(3), that means f(3) is a multiple of 4 i.e. f(3) = 4k for some $k \in \mathbb{Z}$.

Let k be this number.

We want to show that P(29) follows.

$f(29) = \left(f(\lfloor \log_3 29 \rfloor)\right)^2 + f(\lfloor \log_3 29 \rfloor)$

$= \left(f(3)\right)^2 + f(3)$ (since $\lfloor \log_3 29 \rfloor = 3$)

$= (4k)^2 + 4k$ (since P(3) and for some $k \in \mathbb{Z}$, f(3) = 4k)

$= 16k^2 + 4$

$= 4(4k^2 + k)$

$= 4m$ (namely, $m = 4k^2 + k, m \in \mathbb{Z}$)

Hence f(29) is a multiple of 4 and P(29) follows. ■

(b)

Answer: No.

Reason: Assume P(4) holds, i.e. f(4) is a multiple of 4

For P(29)

$f(29) = \left(f(\lfloor\log_3 29\rfloor)\right)^2 + f(\lfloor\log_3 29\rfloor)$

$= \left(f(3)\right)^2 + f(3)$ (since $\lfloor\log_3 29\rfloor = 3$)

There is no direct connection between f(29) and f(4), so we cannot prove P(29) follows by assuming P(4) holds.

(c)

Proof: (Complete induction)

Base Case: ① n = 1

$f(1) = \left(f(0)\right)^2 + f(0) = 12$ (since f(0) = 3 and $\lfloor\log_3 1\rfloor = 0$)

$12 = 4 \times 3$

P(1) holds.

② n = 2

$f(2) = \left(f(0)\right)^2 + f(0) = 12$ (since f(0) = 3 and $\lfloor\log_3 2\rfloor = 0$)

$12 = 4 \times 3$

P(2) holds.

Inductive Step: Assume P(i) holds for $\forall\, i \in \mathbb{N}, 2 < i \leq n-1$.

We want to show: P(n) holds for $\mathrm{n} \geq 3$.

Since $\mathrm{n} \geq 3, \lfloor\log_3 \mathrm{n}\rfloor \geq 1$.

Also, since $3^{\mathrm{n}} > n\ for\ all\ natural\ number\ n, \lfloor\log_3 \mathrm{n}\rfloor < n$

$f(\mathrm{n}) = \left(f(\lfloor\log_3 \mathrm{n}\rfloor)\right)^2 + f(\lfloor\log_3 \mathrm{n}\rfloor)$

$= \left(f(\mathrm{s})\right)^2 + f(\mathrm{s})$ $(\mathrm{s} = \lfloor\log_3 \mathrm{n}\rfloor)$

$= (4\mathrm{m})^2 + 4\mathrm{m}$ ($\mathrm{m} \in \mathbb{Z}$, since $1 \leq \mathrm{s} < n$ and by induction hypothesis)

$= 4(4\mathrm{m}^2 + \mathrm{m})$

Hence *f*(n) is a multiple of 4

P(n) follows ∎

Question 3

To prove this question, we need to use an already-known claim.

Claim: if a prime number p divides a perfect cube n^3, then p also decides n.

Proof: (Contradiction)

Assume (for the sake of contradiction) $\exists x, y, z \in \mathbb{N}^{+} \text{s.t.}\, 5x^3 + 50y^3 = 3z^3$

Let set $S = \{\, z \in \mathbb{Z}^{+} \mid 3z^3 = 5x^3 + 50y^3\}$,then $S \subset \mathbb{N}$,

and S is non-empty (by assumption).

By Well-Ordering Principle, S has a smallest element, call it z_0.

By the definition of S, $\exists\, x_0, y_0 \in \mathbb{Z}^{+}\ such\ that\ 3{z_0}^3 = 5{x_0}^3 + 50{y_0}^3$

Notice that $5 | 5{x_0}^3 + 50{y_0}^3$

$\Rightarrow 5 | 3{z_0}^3 \qquad (\text{since } 3{z_0}^3 = 5{x_0}^3 + 50{y_0}^3)$

$\Rightarrow 5 | {z_0}^3 \qquad (\text{since } 5 \nmid 3)$

$\Rightarrow 5 | z_0 \qquad (\text{by hint})$

$\Rightarrow \exists z_1 \in \mathbb{Z}^{+} \text{s.t.}\, z_0 = 5z_1, z_1 < z_0$

Take $5z_1$ back to the original equation, we get $3(5z_1)^3 = 5{x_0}^3 + 50{y_0}^3$

$\Rightarrow 5{x_0}^3 = 3(5z_1)^3 - 50{y_0}^3$

$\Rightarrow {x_0}^3 = 75{z_1}^3 - 50{y_0}^3$

Notice that $5 | 75{z_1}^3 - 50{y_0}^3$

$\Rightarrow 5 | {x_0}^3 \qquad (\text{since } {x_0}^3 = 75{z_1}^3 - 50{y_0}^3)$

$\Rightarrow 5 | x_0 \qquad (\text{by hint})$

$\Rightarrow \exists x_1 \in \mathbb{Z}^{+} \text{s.t.}\, x_0 = 5x_1, x_1 < x_0$

Take $5x_1$ back to the original equation, we get $3(5z_1)^3 = 5(5x_1)^3 + 50{y_0}^3$

$\Rightarrow 50{y_0}^3 = 5(5x_1)^3 - 3(5z_1)^3$

$\Rightarrow 2{y_0}^3 = 25{x_1}^3 - 15{z_1}^3$

Notice that $5 | 25{x_1}^3 - 15{z_1}^3$

$\Rightarrow 5 | 2{y_0}^3 \qquad (\text{since } 2{y_0}^3 = 25{x_1}^3 - 15{z_1}^3)$

$\Rightarrow 5 | {y_0}^3 \qquad (\text{since } 5 \nmid 2)$

$\Rightarrow 5 | y_0 \qquad (\text{by hint})$

$\Rightarrow \exists y_1 \in \mathbb{Z}^{+} \text{s.t.}\, y_0 = 5y_1, y_1 < y_0$

Take $5y_1$ back to the original equation, we get $3(5z_1)^3 = 5(5x_1)^3 + 50(5y_1)^3$

$\Rightarrow 3{z_1}^3 = 5{x_1}^3 + 50{y_1}^3$

It is obvious that z_1 should be in set S, and $z_1 < z_0$, which gives us a contradiction

Conclusion: no such integers x, y, z satisfy $3z^3 = 5x^3 + 50y^3$. ■

Question 4

(a)

Define P(t): $\text{left\_count}(t) \leq 2^{\max\text{\_left\_surplus}(t)} - 1$

Claim: $\forall t \in \mathcal{T}, \mathrm{P}(t)$

Proof: (Structural Induction)

Base Case: $t = "*"$

$\text{left\_count}(t) = 0$

$\max\text{\_left\_surplus}(t) = 0$

$0 \leq 0 = 2^0 - 1$

So base case P("$*$") holds.

Induction step: Let $t_1, t_2 \in \mathcal{T}$

Assume $\mathrm{P}(t_1), \mathrm{P}(t_2)$ holds, $\text{i.e.}\, \text{left\_count}(t_1) \leq 2^{\max\text{\_left\_surplus}(t_1)} - 1$

$\text{left\_count}(t_2) \leq 2^{\max\text{\_left\_surplus}(t_2)} - 1$

Then $\text{left}_{\text{count}((t_1,t_2))} = \text{left}_{\text{count}(t_1)} + \text{left}_{\text{count}(t_2)} + 1$

$\leq 2^{\max\text{\_left\_surplus}(t_1)} - 1 + 2^{\max\text{\_left\_surplus}(t_2)} - 1 + 1$

(by induction hypothesis)

$= 2^{\max\text{\_left\_surplus}(t_1)} + 2^{\max\text{\_left\_surplus}(t_2)} - 1$

$\leq 2 \cdot 2^{\max(\text{max\_left\_surplus}(t_1),\ \text{max\_left\_surplus}(t_2))} - 1$

$= 2^{\max(\text{max\_left\_surplus}(t_1), \text{max\_left\_surplus}(t_2))+1} - 1$

$= 2^{\max\text{\_left\_surplus}((t_1, t_2))} - 1$ (by Hint)

Hence $\mathrm{P}((t_1, t_2))$ holds.

Then by Structural Induction we get the conclusion. ■

b)

Define P(t): $\text{double\_count}(t) = \begin{cases} 0 & , t = "*" \\ \text{left\_count}(t) - 1 & , \text{otherwise} \end{cases}$

Claim: $\forall t \in \mathcal{T}, \mathrm{P}(t)$

Proof: (Structural Induction)

Base Case: $t = "*"$

$\text{double\_count}(t) = 0$

$\text{left\_count}(t) = 0$

So base case P("$*$") holds.

Induction Step: Let $t_1, t_2 \in \mathcal{T}$

Assume $\mathrm{P}(t_1), \mathrm{P}(t_2)$ holds,

$$\text{i.e.}\, \text{double\_count}(t_1) = \begin{cases} 0 & , t_1 = " * " \\ \text{left\_count}(t_1) - 1, & \text{otherwise} \end{cases}$$

$$\text{double\_count}(t_2) = \begin{cases} 0 & , t_2 = " * " \\ \text{left\_count}(t_2) - 1, & \text{otherwise} \end{cases}$$

Want to show that $\mathrm{P}((t_1, t_2))$ holds, and we have three cases to discuss.

Case 1: $t_1 = t_2 = " * "$

Then $(t_1, t_2) = "(**)"$

$\text{double\_count}((t_1, t_2)) = 0 = 0 = 1 - 1 = \text{left\_count}((t_1, t_2)) - 1$

Case 1 holds.

Case 2: one of $t_1, t_2 = " * "$ with another $\neq " * "$

$\text{double\_count}((t_1, t_2)) = \text{double\_count}(t_1) + \text{double\_count}(t_2) + 1$

(the added 2 represents left-most '((' OR right-most '))')

(it depends on which of t_1, t_2 is not $" * "$)

$= 0 + \text{left\_count}(t_1)\ (\text{OR left\_count}(t_2)) - 1 + 1$

(by induction hypothesis)

$= \text{left\_count}(t_1)\ (\text{OR left\_count}(t_2))$

$= \text{left\_count}((t_1, t_2)) - 1$

(the reduced 1 represents the left-most '(')

Case 2 holds.

Case 3: both $t_1, t_2 \neq " * "$

Then $\text{double\_count}((t_1, t_2)) = \text{double\_count}(t_1) + \text{double\_count}(t_2) + 2$

(the added 2 represents left-most '((' and right-most '))')

$= \text{left\_count}(t_1) - 1 + \text{left\_count}(t_2) - 1 + 2$

(by induction hypothesis)

$= \text{left\_count}(t_1) + \text{left\_count}(t_2)$

$= \text{left\_count}((t_1, t_2)) - 1$

(the reduced 1 represents the left-most '(')

Case 3 holds.

In all cases $\mathrm{P}((t_1, t_2))$ holds, then by Structural Induction, we conclude:

$$\forall \mathrm{t} \in \mathrm{T}, \text{double\_count}(\mathrm{t}) = \begin{cases} 0 & , \mathrm{t} = " * " \\ \text{left\_count}(\mathrm{t}) - 1, & \text{otherwise} \end{cases}$$ ∎