CSC236 tutorial exercises, Week #3
sample solutions

Work on small examples until you believe the claim. Try writing out the inductive hypothesis, without completing the inductive step, and discuss whether it is reasonable. Then write a complete proof. For the first question, the solution should be in terms of an inequality, rather than proving and using an equality.

1. Full binary trees are binary trees where all internal nodes have 2 children (see page 34 of csc236 notes). Prove that any full binary tree with at least 1 node has more leaves than internal nodes. Use complete induction on the total number of nodes.

   Sample solution: Define $C(n)$: Every full binary tree with n nodes has more leaves than children. Proof by complete induction that $\forall n \in \mathbb{N} - \{0\}, C(n)$

   inductive step: Let $n \in \mathbb{N} - \{0\}$. Assume $H(n)$: $\bigwedge_{i=1}^{i=n-1} C(i)$. I will prove that $C(n)$ follows, that is, every full binary tree with n nodes has more leaves than children

   There are three cases to consider

   base case $n = 1$: There is exactly one full binary tree with one node, consisting of one leaf and zero internal nodes, so $C(n)$ follows in this case.

   base case $n > 1$ and there are no full binary trees with n nodes: $C(n)$ is vacuously true (e.g. $n = 2$ or other even numbers greater than 0). # For the purposes of this proof, there is no need to prove the absence of all full binary trees with an even number of nodes greater than 0, although the proof is not too hard. So $C(n)$ follows in this case.

   case $n > 1$ and one or more full binary trees with n nodes exist: Let $\mathcal{T}$ be a full binary tree with n nodes. Since $n > 1$, the root is an internal node with 2 subtrees rooted at its children, let's call them $\mathcal{T}_1$ and $\mathcal{T}_2$. Since they are children of $\mathcal{T}$, these subtrees have a positive number of nodes, we can denote these by $|\mathcal{T}_1| = n_1 > 0$ and $|\mathcal{T}_2| = n_2 > 0$. Also, since neither subtree includes the root node, we know $n > n_1, n_2$. Taking both inequalities together, we may use the inductive hypotheses $C(n_1)$ and $C(n_2)$. Denote the number of internal nodes, and the number of leaves, of $\mathcal{T}_1$ by (respectively) i_1 and l_1. Similarly, denote the leaves and internal nodes of $\mathcal{T}_2$ by i_2 and l_2. By $C(n_1)$ and $C(n_2)$, and the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are full binary trees (the degree of their internal nodes isn't changed by removing the original root), we know that $l_1 \geq i_1 + 1$ and $l_2 \geq i_2 + 1$. The number of leaves of $\mathcal{T}$, which I'll denote l, is simply the sum of those in $\mathcal{T}_1$ and $\mathcal{T}_2$, whereas the number of internal nodes of $\mathcal{T}$, which I'll denote i, is simply the sum of those in $\mathcal{T}_1$ and $\mathcal{T}_2$, plus one more (the root). Taken together I have:

   $$l = l_1 + l_2 \geq i_1 + 1 + i_2 + 1 > i_1 + i_2 + 1 = i$$

   So $C(n)$ follows in this case.

   In every possible case $C(n)$ follows from $H(n)$ ∎

2. Use Complete Induction to show that postage of exactly n cents can be made using only 3-cent and 4-cent stamps, for every natural number n greater than k (you will have to discover the value of k).

sample solution: Define $C(n)$: "Postage of n cents can be made using only 3- and 4-cent stamps. Let $k = 5$. Proof by complete induction that $\forall n \in \mathbb{N}, n \geq 6 \Rightarrow C(n)$.

inductive step: Let $n \in \mathbb{N}$ and assume $n \geq 6$. Assume $H(n) : \bigwedge_{i=6}^{n-1} C(i)$. I will show that $C(n)$ follows, that postage of n cents can be made using only 3- and 4- cent stamps.

base case $n = 6$: Use two 3-cent stamps. So $C(n)$ follows in this case.

base case $n = 7$: Use one 3-cent and one 4-cent stamps. So $C(n)$ follows in this case.

base case $n = 8$: Use two 4-cent stamps. So $C(n)$ follows in this case.

$n \geq 9$: Since $9 \leq n$, $6 \leq n - 3 < n$, so we know $C(n-3)$, postage of $n-3$ cents can be made using 3- and 4-cent stamps. Let k and j be integers such that $n - 3 = 3k + 4j$. Adding 3 to both sides yields $n = 3(k+1) + 4j$, so $C(n)$ follows in this case.
So $C(n)$ follows from $H(n)$ in all possible cases ∎

3. Define function f of the natural numbers by:

$$
f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2\left(f(n-2) + f(n-1)\right) + 1 & \text{if } n > 1 \end{cases}
$$

Use Complete Induction to prove that $f(n) \leq 3^n$ for all $n \in \mathbb{N}$.

sample solution: Define $C(n) : f(n) \leq 3^n$. I will use complete induction to prove $\forall n \in \mathbb{N}, C(n)$.

inductive step: Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{i=n-1} C(i)$. I will show that $C(n)$ follows, that is $f(n) \leq 3^n$.

base case $n = 0$: Then $f(n) = 1 \leq 3^0$, so $C(n)$ follows in this case.

base case $n = 1$: Then $f(n) = 3 \leq 3^1$, so $C(n)$ follows in this case.

case $n > 1$: Then

$$
\begin{aligned}
f(n) &= 2\left(f(n-2) + f(n-1)\right) + 1 && \text{\# since } n \geq 2 \\
&\leq 2\left(3^{n-2} + 3^{n-1}\right) + 1 && \text{\# by } C(n-2), C(n-1) \text{ since } n \geq 2 \Rightarrow n - 2 \geq 0 \\
&= 2 \times 3^{n-2} + 2 \times 3^{n-1} + 1 \\
&= 3 \times 3^{n-2} - 3^{n-2} + 2 \times 3^{n-1} + 1 < 3^{n-1} - 1 + 2 \times 3^{n-1} + 1 && \text{\# since } 3^{n-1} > 1 \text{ when } n > 1 \\
&= 3^n
\end{aligned}
$$

So $C(n)$ follows from $H(n)$ in this case ∎