CSC236 tutorial exercises, Week #4
sample solutions

These exercises are intended to give you practice with structural and induction and well-ordering.

1. Use the Principle of Well Ordering and contradiction to prove that there are no positive integers x, y, z such $x^3 + 3y^3 = 9z^3$.

   sample solution: Proof by contradiction.

   Assume, for the sake of contradiction, the negation of what we are proving, that is $\exists x, y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3$.

   Define $X = \{x \in \mathbb{N}^+ \mid \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3$. By our assumption X is non-empty, so by the Principle of Well-Ordering it has a smallest element. Let $x_0 \in X$ be its smallest element, and let $y_0, z_0 \in \mathbb{N}^+$ such that $x_0^3 + 3y_0^3 = 9z_0^3$.

   Then

$$\begin{array}{rcll}
x_0^3 + 3y_0^3 = 9z_0^3 & \Rightarrow & x_0^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3 \mid x_0^3 \Rightarrow 3 \mid x_0 & \text{\# by clue for A1 Q3} \\
\text{let } x_1 \in \mathbb{N}^+, 3x_1 = x_0 & \Rightarrow & 3^3x_1^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3^2x_1^3 = 3z_0^3 - y_0^3 & \text{\# divide through by 3} \\
& \Rightarrow & y_0^3 = 3z_0^3 - 3^2x_1^3 \Rightarrow 3 \mid y_0^3 \Rightarrow 3 \mid y_0 & \\
\text{let } y_1 \in \mathbb{N}^+, 3y_1 = y_0 & \Rightarrow & 3^3y_1^3 = 3z_0^3 - 3^2x_1^3 \Rightarrow 3^2y_1^3 = z_0^3 - 3x_1^3 & \text{\# divide through by 3} \\
& \Rightarrow & 3x_1^3 + 3^2y_1^3 = z_0^3 \Rightarrow 3 \mid z_0^3 \Rightarrow 3 \mid z_0 & \\
\text{let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 & \Rightarrow & 3x_1^3 + 3^2y_1^3 = 3^3z_1^3 \Rightarrow x_1^3 + 3y_1^3 = 9z_0^3 & \text{\# divide through by 3} \\
& \Rightarrow & x_1 \in X &
\end{array}$$

   $\longrightarrow\!\longleftarrow$ contradiction! $x_1 < x_0$, but x_0 is the smallest element of X. Since assuming that $\exists x, y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3$ leads to a contradiction, the assumption is false ■

2. Define the set of expressions $\mathcal{E}$ as the smallest set such that:

   (a) $x, y, z \in \mathcal{E}$.

   (b) If $e_1, e_2 \in \mathcal{E}$, then so are $(e_1 + e_2)$ and $(e_1 \times e_2)$.

   Define $s_1(e)$: Number of symbols from $\{(,), +, \times\}$ in e, counting duplicates.

   Define $s_2(e)$: Number of symbols from $\{x, y, z\}$ in e, counting duplicates.

   Use structural induction to prove that for all $e \in \mathcal{E}$, $s_1(e) = 3(s_2(e) - 1)$.

   sample solution: Proof by structural induction. Define $P(e) : s_1(e) = 3(s_2(e) - 1)$.

   basis: Let $e \in \{x, y, z\}$. Then e has zero symbols from the set $\{(,), +, \times\}$ and one symbol (itself) from $\{x, y, z\}$. So $s_1(e) = 0 = 3(0) = 3(1 - 1) = 3(s_2(e) - 1)$, so $P(e)$ holds.

inductive step: Let $e_1, e_2 \in \mathcal{E}$. Assume $P(e_1)$ and $P(e_2)$. Let $@ \in \{+, \times\}$. I will show that $P((e_1 @ e_2))$ follows.

$$\begin{aligned} s_1((e_1 @ e_2)) &= 3 + s_1(e_1) + s_1(e_2) \qquad \text{\# added two parentheses and @} \\ &= 3 + 3(s_2(e_1) - 1) + 3(s_2(e_2) - 1) \qquad \text{\# by } P(e_1) \text{ and } P(e_2) \\ &= 3((s_2(e_1) + s_2(e_2) - 1) = 3(s_2((e_1 @ e_2)) - 1) \\ &\qquad\qquad \text{\# } ((e_1 @ e_2)) \text{ has same basis symbols as } e_1 \text{ and } e_2 \text{ combined} \end{aligned}$$

So $P((e_1 @ e_2))$ follows ■

3. Define the set of non-empty full binary trees, $\mathcal{T}$, as the smallest set such that:

   (a) Any single node is an element of $\mathcal{T}$.

   (b) If $t_1, t_2 \in \mathcal{T}$, n is a node that belongs to neither t_1 nor t_2, and t_1, t_2 have no nodes in common, then n together with edges to **the root nodes** t_1 and t_2 is also an element of $\mathcal{T}$.

   Use structural induction to prove that any non-empty full binary tree has exactly one more leaf than internal nodes.

   sample solution: Proof by structural induction. Define $P(t)$: t has exactly one more leaf than internal nodes.

   basis: Let $t \in \mathcal{T}$ be a single node. Then t is itself a leaf, and has no internal nodes, and 1 is exactly one more than 0. So $P(t)$ holds.

   inductive step: Let $t_1, t_2 \in \mathcal{T}$. Assume $P(t_1)$ and $P(t_2)$, and that t_1 and t_2 have no nodes in common. Let n be an arbitrary node that belongs to neither t_1 nor t_2, and t be the tree formed by n with edges to the roots of t_1 and t_2. I will show that $P(t)$ follows, i.e. that t has exactly one more leaf than internal nodes.

   Denote the number of internal nodes and leaf nodes of t_1 by i_1 and l_1, respectively. Similarly, denote the number of internal nodes and leaf nodes of t_2 by i_2 and l_2, respectively. Notice that adding edges from n to the root nodes of these two trees does not change the status of any internal or leaf nodes, it simply adds one new internal node. Denote the number of internal and leaf nodes of t by i_t and l_t respectively, and we have:

$$\begin{aligned} i_t &= 1 + i_1 + i_2 = 1 + l_1 - 1 + l_2 - 1 \qquad \text{\# by } P(t_1) \text{ and } P(t_2) \\ &= (l_1 + l_2) - 1 = l_t - 1 \qquad \text{\# } t\text{'s leaves are exactly those of } t_1 \text{ and } t_2 \text{ combined} \end{aligned}$$

   So $P(t)$ follows ■