THE RICCI ITERATION AND DISCRETIZATION OF THE PSEUDO-CALABI FLOW

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ABSTRACT. Motivated by the problem of finding constant scalar curvature Kähler metrics, we investigate a Ricci iteration sequence of Rubinstein that discretizes the pseudo-Calabi flow. While the long time existence of the flow is still an open question, we show, using Chen–Cheng's result, that the iteration sequence does exist for all steps, along which the K-energy decreases. We further show that the iteration sequence, modulo automorphisms, converges smoothly to a constant scalar curvature Kähler metric if there is one, thus confirming a conjecture of Rubinstein from 2007 and generalizing results of Darvas–Rubinstein to arbitrary Kähler classes.

CONTENTS

1. INTRODUCTION

A long standing problem in Kähler geometry is to find a constant scalar curvature Kähler (cscK) metric in a given Kähler class. Namely, for a compact Kähler manifold (X,ω) of dimension n, we want to find $\omega^*\in\{\omega\}$ such that

$$\operatorname{tr}_{\omega^*}\operatorname{Ric}(\omega^*)=\bar{R},$$

where $\bar{R}:=n\frac{-K_X\cdot\{\omega\}^{n-1}}{\{\omega\}^n}$ is the average of the scalar curvature.

The following elementary result is well known in Kähler geometry.

Lemma 1.1. *A closed $(1,1)$-form θ on (X,ω) satisfies $\operatorname{tr}_\omega\theta=\mathrm{Const.}$ if and only if θ is harmonic with respect to the Kähler metric ω.*

Therefore, ω^* is cscK if and only if $\operatorname{Ric}(\omega^*)$ is a harmonic form with respect to ω^*. This viewpoint is by no means new, which was explored in early works of Calabi, Futaki, Bando and Mabuchi; see, e.g., [Ban06]. When combined with the framework of geometric flows, this motivates one to consider the following variant of the Kähler Ricc flow:

$$\partial_t\omega_t=-\operatorname{Ric}(\omega_t)+\operatorname{HRic}(\omega_t),\ \omega_0=\omega. \tag{1.1}$$

Here, given any Kähler form α, $\mathrm{HRic}(\alpha)$ denotes the harmonic part of $\mathrm{Ric}(\alpha)$ with respect to α. If the flow (1.1) smoothly converges to a limit ω_∞, then one has

$$\mathrm{Ric}(\omega_\infty) = \mathrm{HRic}(\omega_\infty),$$

which means that ω_∞ is a cscK metric.

Remark 1.2. *If $c_1(X) = \lambda\{\omega\}$, then the flow (1.1) reduces to*

$$\partial_t \omega_t = -\,\mathrm{Ric}(\omega_t) + \lambda\omega_t,$$

which is exactly the classical normalized Kähler Ricci flow in the study of Kähler–Einstein and Kähler Ricci soliton metrics. By Cao [Cao85] we know that such flow has long time existence.

The flow (1.1) first appeared in the work of Guan (see Guan [Gua07]; see also Simanca [Sim05] for a related flow). It was also briefly studied by Rubinstein (see [Rub07; Rub08]). Later in the work of Chen–Zheng [CZ13], this flow was studied in much greater depth, and the authors called it the *pseudo-Calabi flow* (since it can be viewed as the square root of the Calabi flow [Cal82]). This flow is well-posed, having short time existence (by [CZ13, Theorem 1.1]). Moreover, it looks simpler than the Calabi flow, since it can be reduced to a coupled system of parabolic equations. Indeed, it is easy to show that (1.1) is equivalent to the following coupled equations:

$$\begin{cases} \omega_t^n = e^{F_t}\omega_n, \\ \dot{F}_t = \Delta_t F_t + \bar{R} - \mathrm{tr}_{\omega_t}\mathrm{Ric}(\omega). \end{cases}$$

So we obtain a parabolic version of the coupled equations for cscK metrics that are studied in Chen–Cheng [CC21a; CC21b]. It is therefore likely that Chen–Cheng's techniques can be applied to show the long time existence and convergence of the flow (1.1).

However, in this paper we adopt a somewhat different approach. We consider the discretization of the pseudo-Calabi flow (1.1), which was first proposed by Rubinstein [Rub07; Rub08]. More precisely, given $\tau > 0$, we investigate the following *Ricci iteration* that first appeared in [Rub07, Definition 2.1] and [Rub08, (41)]:

$$\frac{\omega_{i+1} - \omega_i}{\tau} = -\,\mathrm{Ric}(\omega_{i+1}) + \mathrm{HRic}(\omega_{i+1}),\ i \in \mathbb{N},\ \omega_0 = \omega. \tag{1.2}$$

Part of the interest in this Ricci iteration is that, clearly, cscK metrics are fixed points. Therefore (1.2) aims to provide a natural theoretical and numerical approach to uniformization in the challenging case of cscK metrics. In [Rub07, Conjecture 2.1], Rubinstein proposed the following.

Conjecture 1.3. *Let X be a compact Kähler manifold, and assume that there exists a constant scalar curvature Kähler metric in a Kähler class Ω. Then for any $\omega \in \Omega$ the Ricci iteration (1.2) exists for all $i \in \mathbb{N}$ and converges in an appropriate sense to a constant scalar curvature metric.*

In additional, the Ricci iteration could be a source of new insights for the study of the psudo-Calabi flow, which is known to be a rather difficult problem in the field of canonical metrics. For instance, just as in the case of Calabi flow, the long time existence of the pseudo-Calabi flow is still open (see [CZ13, Conjecture 8.3]). However, after discretization, we can prove the following long time existence result for the sequence (1.2).

Theorem 1.4. *There exists a uniform constant $\tau_0 \in (0, \infty]$, depending only on X and the Kähler class $\{\omega\}$, such that for any $\tau \in (0, \tau_0)$, the iteration process (1.2) exists for all $i \in \mathbb{N}$, with each ω_i being uniquely determined by ω_0.*

This result follows from the work of Chen–Cheng [CC21b] and gives a strong evidence for the long time existence of the pseudo-Calabi flow. Indeed, sending $\tau \to 0$, the iteration sequence $\{\omega_i\}$ is expected to converge to the flow (1.1) (this is interestingly still a conjecture even for the usual Ricci iteration for the Kähler Ricci flow; compare the classical Rothe's method for parabolic equations). The fact that the sequence $\{\omega_i\}$ exists for all i should imply that the flow (1.1) exists for all t. This is of course a heuristic viewpoint, which hopefully can be made more rigorous in the future study.

It is proved in [CZ13, Theorem 3.1] that Mabuchi's K-energy decreases along the pseudo-Calabi flow. We show that this is also the case for the Ricci iteration (1.2), which was previously only known in the case where $c_1(X) = \lambda\{\omega\}$ (see [Rub08, Proposition 4.2]).

Theorem 1.5. *Along the iteration sequence $\{\omega_i\}_{i\in\mathbb{N}}$, the K-energy K_ω satisfies*

$$K_\omega(\omega_{i+1}) \leq K_\omega(\omega_i) \text{ for all } i \in \mathbb{N}.$$

The equality holds for some $i \in \mathbb{N}$ if and only if $\omega_i = \omega_0$ is cscK for all $i \in \mathbb{N}$.

Since a cscK metric, if exists, minimizes the K-energy. The above result suggests that the iteration sequence (1.2) has the tendency to be attracted by a cscK metric in a suitable sense. We show that this is indeed the case, thus confirming Conjecture 1.3.

Theorem 1.6. *Assume that there exists a cscK metric in $\{\omega\}$. Then for any $\tau > 0$ the iteration sequence ω_i exists, and, up to biholomorphic automorphisms, converges to a cscK metric smoothly.*

Our results extend those in the previous works [Rub07; Rub08; Kel09; BBEGZ19; DR19; Hun19], where $c_1(X)$ is assumed to be proportional to $\{\omega\}$. Moreover, Theorem 1.6 also gives strong evidence that the pseudo-Calabi flow shall converge to a cscK metric, if there is one in $\{\omega\}$ (see [Rub08, Conjecture 7.4] and [CZ13, Question 8.5]).

Compared to the recent work of Darvas–Rubinstein [DR19] in the Fano case, the main difficulty we are faced with is the lack of Ding functional in our general setting. As we shall see, this technicality can be circumvented with the help of the estimates in Chen–Cheng [CC21b, §3], which are crucially used in order to obtain the smooth convergence in Theorem 1.6.

For the direction of Ricci iteration in the real case, we refer the reader to [PR19; BPRZ21]. See also [GKY13] for a Ricci iteration in the local setting.

Organization. After recalling some standard notions and facts in §2 and §3, we prove Theorem 1.4 and Theorem 1.5 in §4. Relying on Chen–Cheng [CC21b, §3], we will derive some a priori estimates for the Ricci iteration in §5, which allows us to prove Theorem 1.6 is proved in §6. Finally in §7 we point out that our work can be extended to the setting of twisted cscK metrics.

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2. Energy functionals

We recall several standard functionals that will be used throughout this paper.

Let (X,ω) be a compact Kähler manifold of dimension n, and set

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X,\mathbb{R}) | \omega_\varphi := \omega + dd^c\varphi > 0\},$$

where $dd^c := \sqrt{-1}\partial\bar\partial/2\pi$. Put $V := \int_X \omega^n$. Also, let

$$\mathrm{Ric}(\omega_\varphi) := -dd^c \log\det\omega_\varphi \in c_1(X),\ \ R(\omega_\varphi) := \mathrm{tr}_{\omega_\varphi}\mathrm{Ric}(\omega_\varphi),$$

$$\bar R := \frac{1}{V}\int_X R(\omega)\omega^n = \frac{n}{V}\int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1} = n\frac{c_1(X)\cdot\{\omega\}^{n-1}}{\{\omega\}^n}.$$

For any $u,v\in\mathcal{H}_\omega$, define

$$I(u,v) = I(\omega_u,\omega_v) := \frac{1}{V}\int_X (v-u)(\omega_u^n - \omega_v^n).$$

$$E(u,v) := \frac{1}{(n+1)V}\int_X (v-u)\sum_{i=0}^n \omega_u^i\wedge\omega_v^{n-i}.$$

$$J(u,v) = J(\omega_u,\omega_v) := \frac{1}{V}\int_X (v-u)\omega_u^n - E(u,v).$$

$$Ent(u,v) = Ent(\omega_u,\omega_v) := \frac{1}{V}\int_X \log\frac{\omega_v^n}{\omega_u^n}\omega_v^n.$$

Note that by Jensen's inequality, it always holds that $Ent(u,v)\geq 0$.

For any closed $(1,1)$ form χ, define

$$\mathcal{J}^\chi(u,v) := \frac{1}{V}\int_X (v-u)\chi\wedge\sum_{i=0}^{n-1}\omega_u^i\wedge\omega_v^{n-1-i} - \bar\chi E(u,v),$$

where

$$\bar\chi := \frac{n}{V}\int_X \chi\wedge\omega^{n-1} = n\frac{\{\chi\}\cdot\{\omega\}^{n-1}}{\{\omega\}^n}.$$

The K-energy is defined by

$$K(u,v) = K(\omega_u,\omega_v) := Ent(u,v) + \mathcal{J}^{-\mathrm{Ric}(\omega_u)}(u,v).$$

The χ-twisted K-energy is

$$K^\chi(u,v) = K^\chi(\omega_u,\omega_v) := K(u,v) + \mathcal{J}^\chi(u,v).$$

If we choose $\chi := \omega_u$, then integration by parts gives

$$\mathcal{J}^{\omega_u}(u,v) = (I-J)(u,v).$$

More generally, if $\chi := \omega_w$ for some $w\in\mathcal{H}_\omega$, then

$$\mathcal{J}^{\omega_w}(u,v) = (I-J)(u,v) + \frac{1}{V}\int_X (w-u)(\omega_v^n - \omega_u^n),$$

so that

$$\mathcal{J}^{\omega_w}(u,w) = -J(u,w).$$

One has the following variational formulas (for any $u,v\in\mathcal{H}_\omega$ and $f\in C^\infty(X,\mathbb{R})$):

$$\begin{cases} \frac{d}{dt}\Big|_{t=0} E(u,v+tf) = \frac{1}{V}\int_X f\omega_v^n, \\ \frac{d}{dt}\Big|_{t=0} \mathcal{J}^\chi(u,v+tf) = \frac{1}{V}\int_X f(\mathrm{tr}_{\omega_v}\chi - \bar\chi)\omega_v^n, \\ \frac{d}{dt}\Big|_{t=0} K^\chi(u,v+tf) = \frac{1}{V}\int_X f(\bar R - \bar\chi + \mathrm{tr}_{\omega_v}\chi - R(\omega_v))\omega_v^n. \end{cases}$$

They imply the well known cocycle relations (for $u,v,w\in\mathcal{H}_\omega$):

$$E(u,v) + E(v,w) = E(u,w).$$

$$\mathcal{J}^\chi(u,v)+\mathcal{J}^\chi(v,w)=\mathcal{J}^\chi(u,w)$$
$$K^\chi(u,v)+K^\chi(v,w)=K^\chi(u,w).$$

One can then further deduce the following cocycle relations:

$$J(u,v)+J(v,w)=J(u,w)+\frac{1}{V}\int_X (v-w)(\omega_u^n-\omega_v^n).$$

$$(I-J)(u,v)+(I-J)(v,w)=(I-J)(u,w)+\frac{1}{V}\int_X (v-u)(\omega_w^n-\omega_v^n).$$

The following fact proved in [Tia87] will be used repeatedly.

Lemma 2.1. *For any $u,v\in\mathcal{H}_\omega$, it holds that*

$$\frac{1}{n}J(u,v)\le (I-J)(u,v)\le nJ(u,v).$$

Also, one has

$$I(u,v)\ge 0,\ \ J(u,v)\ge 0,\ \ (I-J)(u,v)\ge 0.$$

If one of them takes equality, then they all do, in which case $u=v$.

Lemma 2.2. *For any $u,v,w\in\mathcal{H}_\omega$ one has*

$$\mathcal{J}^{\omega_w}(u,v)-\mathcal{J}^{\omega_w}(u,w)=J(v,w).$$

So in particular, $\mathcal{J}^{\omega_w}(u,v)\ge\mathcal{J}^{\omega_w}(u,w)$, and the equality holds if and only if $v=w$.

Proof. Using cocycle relation, we can write

$$\mathcal{J}^{\omega_w}(u,v)-\mathcal{J}^{\omega_w}(u,w)=-\mathcal{J}^{\omega_w}(v,w)=J(v,w)$$

to conclude. □

Convention. Given an energy functional $F\in\{I,J,\mathcal{J}^\chi,K,K^\chi\}$ and $u\in\mathcal{H}_\omega$, we also use the notaion

$$F_\omega(u)=F_\omega(\omega_u):=F(0,u) \text{ and } E_\omega(u):=E(0,u)$$

in the circumstances where ω is viewed as a background metric.

Definition 2.3. *The twisted K-energy K^χ_ω is said to be proper if there exist $\gamma>0$ and $C>0$ such that*

$$K^\chi_\omega(u)\ge\gamma(I_\omega-J_\omega)(u)-C \text{ for all } u\in\mathcal{H}_\omega.$$

3. The metric completion

We will work with the finite energy space $\mathcal{E}^1_\omega$ introduced in [GZ07] and use the d_1-distance on it introduced by Darvas [Dar15].

Let us now recall several basic facts that will be used in this paper.

Lemma 3.1. *([Dar15, Theorem 3]) For any $u,v\in\mathcal{E}^1_\omega$, one has*

$$\frac{1}{C}d_1(u,v)\le\int_X |u-v|(\omega_u^n+\omega_v^n)\le Cd_1(u,v)$$

for some dimensional constant $C>1$.

Lemma 3.2. *([DR17, Proposition 5.5]) There exists a constant $C>1$ depending only on (X,ω) such that*

$$\frac{1}{C}J_\omega(\varphi)-C\le d_1(0,\varphi)\le CJ_\omega(\varphi)+C \text{ for any } \varphi\in\mathcal{H}_0.$$

One can extend the functionals $Ent_\omega, I_\omega, J_\omega, E_\omega$ and $\mathcal{J}^\chi_\omega$ to the space $\mathcal{E}^1_\omega$.

Lemma 3.3. *([BDL17]) All the functionals $I_\omega, J_\omega, E_\omega, \mathcal{J}^\chi_\omega$ are d_1-continuous (lsc). The entropy Ent_ω is d_1-lower semi-continuous. Moreover, for any $u \in \mathcal{E}^1_\omega$, there exists $\mathcal{H} \ni u_i \xrightarrow{d_1} u$ such that $Ent_\omega(u_i) \to Ent_\omega(u)$.*

We need the following compactness result going back to [BBEGZ19] (see [DR17, Theorem 5.6] for a convenient formulation for our context).

Lemma 3.4. *For any $A > 0$, the set*

$$\{\varphi \in \mathcal{E}^1_\omega | d_1(0,\varphi) \leq A \text{ and } Ent_\omega(\varphi) \leq A\}$$

is compact in $\mathcal{E}^1_\omega$ with respect to the d_1-topology.

For any $u, v \in \mathcal{E}^1_\omega$, one can also define

$$I(u,v) := \frac{1}{V}\int_X (v-u)(\omega_u^n - \omega_v^n),$$
$$E(u,v) := E_\omega(v) - E_\omega(u),$$

and

$$J(u,v) := \frac{1}{V}\int_X (v-u)\omega_u^n - E(u,v).$$

Recall that (see [Dar19, Proposition 3.40])

$$|E(u,v)| \leq d_1(u,v).$$

Lemma 3.5. *Given $u_i, u, v_i, v \in \mathcal{E}^1_\omega$ such that $u_i \xrightarrow{d_1} u$ and $v_i \xrightarrow{d_1} v$, then*

$$\lim_{i\to\infty} I(u_i,v_i) = I(u,v), \ \lim_{i\to\infty} J(u_i,v_i) = J(u,v)$$

Proof. We only deal with the J-functional, since the proof for the I-functional is similar. One has

$$|J(u_i,v_i) - J(u,v)| \leq |\frac{1}{V}\int_X (v_i - u_i)\omega_{u_i}^n - \frac{1}{V}\int_X (v-u)\omega_u^n| + |E(u,u_i)| + |E(v,v_i)|$$

It is thus enough to estimate $|\int_X (v_i-u_i)\omega^n_{u_i} - \int_X (v-u)\omega^n_u|$, which can be bounded from above by

$$|\int_X (v_i - v)(\omega^n_{u_i} - \omega^n_u)| + |\int_X (u_i-u)(\omega^n_{u_i} - \omega^n_u)| + \int_X (|v_i - v| + |u_i - u|)\omega_u^n$$
$$+ |\int_X v(\omega^n_{u_i} - \omega^n_u)| + |\int_X u(\omega^n_{u_i} - \omega^n_u)|.$$

All of these terms go to zero, thanks to [Dar19, Proposition 3.48 and Corollary 3.51] (see also [BBGZ13, Lemma 3.13, Lemma 5.8]). ☐

The next quasi-triangle inquality is proved in [BBEGZ19, Theorem 1.8].

Lemma 3.6. *There exists a dimensional constant $C_n > 0$ such that for any $u, v, w \in \mathcal{E}^1_\omega$*

$$I(u,v) \leq C_n(I(u,w) + I(w,v)).$$

We will need the following convergence criterion, which is a simple consequence of [BBEGZ19, Proposition 2.3].

Lemma 3.7. *Assume that $u_i, u \in \mathcal{E}^1_\omega$ such that $E_\omega(u_i) = E_\omega(u) = 0$ and $d_1(0,u_i) \leq A$ for some $A > 0$ (independent of i). Then*

$$u_i \xrightarrow{d_1} u \Leftrightarrow I(u_i,u) \to 0.$$

Proof. That $u_i \xrightarrow{d_1} u$ implies $I(u_i, u) \to 0$ follows from Lemma 3.1.

Now assume that $I(u_i, u) \to 0$. By Lemma 3.1 the bound on $d_1(0, u_i)$ implies the L^1 bound for u_i, which in turn implies a bound on $|\sup_X u_i|$. Put $u_i' := u_i - \sup_X u_i$ and $u' := u - \sup_X u$. Then $I(u_i', u') \to 0$ as well. By [BBEGZ19, Proposition 2.3] we know that $u_i' \xrightarrow{d_1} u'$ and hence $E_\omega(u_i') = -\sup_X u_i \to E_\omega(u') = -\sup_X u$. Then from $\sup_X u_i \to \sup_X u$ and $u_i' \xrightarrow{d_1} u'$ we deduce that $u_i \xrightarrow{d_1} u$ (using Lemma 3.1). So we conclude. □

Corollary 3.8. *Given two sequences $u_i, v_i \in \mathcal{E}^1_\omega$ with $E_\omega(u_i) = E_\omega(v_i) = 0$. Assume that $u_i \xrightarrow{d_1} u$ and $I(u_i, v_i) \to 0$. Then $v_i \xrightarrow{d_1} u$ as well.*

Proof. First, that $u_i \xrightarrow{d_1} u$ implies that $I(0, u_i) \le C$ by Lemma 3.1. So one has

$$J(0, v_i) \le I(0, v_i) \le C_n(I(0, u_i) + I(u_i, v_i)) \le C'.$$

This implies that $d_1(0, u_i) \le C''$ by Lemma 3.2. Moreover, Lemma 3.6 implies that

$$I(v_i, u) \le C_n(I(v_i, u_i) + I(u_i, u)) \to 0.$$

So we conclude from the previous lemma that $v_i \xrightarrow{d_1} u$. □

We will frequently use the space

$$\mathcal{H}_0 := \{\varphi \in \mathcal{H}_\omega | E_\omega(\varphi) = 0\}.$$

Recall that

$$G := Aut_0(X),$$

the connected component of complex Lie group of holomorphic automorphisms of X, naturally acts on $\mathcal{H}_0$.

Lemma 3.9. *([DR17, Lemma 5.8]) For any $\varphi \in \mathcal{H}_0$ and $f \in G$, let $f.\varphi \in \mathcal{H}_0$ be the unique potential such that $f^*\omega_\varphi = \omega_{f.\varphi}$. Then*

$$f.\varphi = f.0 + f^*\varphi.$$

Lemma 3.10. *For any $u, v \in \mathcal{H}_0$ and $f \in G$, one has*

$$I(u, v) = I(f.u, f.v), \ \ J(u, v) = J(f.u, f.v).$$

Proof. One has

$$\begin{aligned} I(u, v) &= \frac{1}{V}\int_X (v - u)(\omega_u^n - \omega_v^n) \\ &= \frac{1}{V}\int_X (f^*v - f^*u)(f^*\omega_u^n - f^*\omega_v^n) \\ &= I(f^*\omega_u, f^*\omega_v) = I(\omega_{f.u}, \omega_{f.v}) = I(f.u, f.v). \end{aligned}$$

For J-functional, we can write

$$\begin{aligned} J(u, v) &= \frac{1}{V}\int_X (v - u)\omega_u^n = \frac{1}{V}\int_X (f^*v - f^*u) f^*\omega_u^n \\ &= \frac{1}{V}(f^*v + f.0 - f^*u - f.0)\omega_{f.u}^n \\ &= \frac{1}{V}\int_X (f.v - f.u)\omega_{f.u}^n = J(f.u, f.v). \end{aligned}$$

□

Finally, we recall that G acts on $\mathcal{H}_0$ isometrically.

Lemma 3.11. *([DR17, Lemma 5.9]) For any $u, v \in \mathcal{H}_0$ and $f \in G$ one has*

$$d_1(u, v) = d_1(f.u, f.v).$$

Then define (as in [DR17])

$$d_{1,G}(u, v) := \inf_{f \in G} d_1(u, f.v).$$

Definition 3.12. *The K-energy K_ω is said to be proper modulo G if there exist $\gamma > 0$ and $C > 0$ such that*

$$K_\omega(u) \geq \gamma d_{1,G}(u) - C \text{ for all } u \in \mathcal{H}_0.$$

4. Basic properties of the Ricci iteration

In this part, we prove Theorem 1.4 and Theorem 1.5.

We begin by introducing the following analytic threshold.

$$\gamma(X, \{\omega\}) := \sup\left\{\gamma \in \mathbb{R} \middle| \inf_{\varphi \in \mathcal{H}_\omega} (K_\omega(\varphi) - \gamma(I_\omega - J_\omega)(\varphi)) > -\infty\right\}, \tag{4.1}$$

Lemma 4.1. *The threshold $\gamma(X, \{\omega\})$ is finite and is independent of the choice of background metric ω in its cohomology class, hence the notation.*

Proof. The finiteness of $\gamma(X, \{\omega\})$ follows from

$$K_\omega(\varphi) \geq \mathcal{J}^{-\operatorname{Ric}(\omega)}(\varphi) \geq -C d_1(0, \varphi),\ \varphi \in \mathcal{H}_0,$$

where we used [CC21b, (4.2)]. So one can find $C \gg 0$ such that $K_\omega(\varphi) + C(I_\omega - J_\omega)(\varphi) \geq 0$ for all $\varphi \in \mathcal{H}_\omega$.

To show that $\gamma(X, \{\omega\})$ does not depend on the choice of ω, we use the cocycle relations recalled in §2. It suffices to note the estimate

$$|(I - J)(u, w) - (I - J)(v, w)| \leq |(I - J)(u, v)| + \frac{1}{V}\int_X |u - v|(\omega_w^n + \omega_v^n) \leq C||u - v||_{C^0},$$

for any $u, v, w \in \mathcal{H}_\omega$, where $C > 0$ is a dimensional constant. □

Proof of Theorem 1.4. Taking trace of (1.2) we see that

$$R(\omega_{i+1}) = \bar{R} + \frac{\operatorname{tr}_{\omega_{i+1}}\omega_i - n}{\tau}, \tag{4.2}$$

which is a twisted cscK equation. As we now argue, given any $\omega_i \in \{\omega\}$, the existence of $\omega_{i+1} \in \{\omega\}$ solving (4.2) is guaranteed by Chen–Cheng's result[CC21b] (see also [Has19]), once τ is chosen to be small enough.

Indeed, for any $\gamma < \gamma(X, \{\omega\})$ and any Kähler form $\alpha \in \{\omega\}$, the twisted K-energy $K_\alpha - \gamma(I_\alpha - J_\alpha)$ is proper by Lemma 4.1. Now choosing $\tau_0 \in (0, \infty]$ so that $-1/\tau_0 \leq \gamma(X, \{\omega\})$, we see that for any $\tau \in (0, \tau_0)$ and $\omega_i \in \{\omega\}$, the $\frac{\omega_i}{\tau}$-twisted K-energy

$$K_{\omega_i}^{\frac{\omega_i}{\tau}} = K_{\omega_i} + \mathcal{J}_{\omega_i}^{\frac{\omega_i}{\tau}} = K_{\omega_i} + \frac{1}{\tau}(I_{\omega_i} - J_{\omega_i})$$

is proper, which implies the solvability of (4.2) (see [CC21b, Theorem 4.1]). Moreover, such ω_{i+1} is uniquely determined, by [BDL17, Theorem 4.13]. This completes the proof of Theorem 1.4. It is also clear that one can take $\tau_0 = \infty$ once $\gamma(X, \{\omega\}) \geq 0$.

□

Proof of Theorem 1.5. Notice that ω_{i+1} minimizes the twisted K-energy $K^{\frac{\omega_i}{\tau}}_{\omega_i}$ (see [CC21b, Corollary 4.5]), so that

$$K_{\omega_i}(\omega_{i+1}) + \frac{1}{\tau}(I_{\omega_i} - J_{\omega_i})(\omega_{i+1}) = K^{\frac{\omega_i}{\tau}}_{\omega_i}(\omega_{i+1}) \leq K^{\frac{\omega_i}{\tau}}_{\omega_i}(\omega_i) = 0.$$

This implies that

$$K_\omega(\omega_{i+1}) - K_\omega(\omega_i) = K_{\omega_i}(\omega_{i+1}) \leq -\frac{1}{\tau}(I_{\omega_i} - J_{\omega_i})(\omega_{i+1}) \leq 0,$$

thanks to the cocycle property of the K-energy.

When the equality holds for some i, one has $(I_{\omega_i} - J_{\omega_i})(\omega_{i+1}) = 0$, which means that $\omega_i = \omega_{i+1}$. Then (4.2) shows that $\omega_i = \omega_{i+1}$ are both cscK metrics. Moreover, from

$$R(\omega_i) = \bar{R} + \frac{\mathrm{tr}_{\omega_i}\omega_{i-1} - n}{\tau}$$

we get $\mathrm{tr}_{\omega_i}\omega_{i-1} = n$, and hence ω_{i-1} is harmonic with respect to ω_i. This forces that $\omega_{i-1} = \omega_i$, by the uniqueness of harmonic forms. So we eventually see that $\omega_0 = \omega_i$ for all i, which is a fixed cscK metric. Thus we conclude Theorem 1.5. □

5. A priori estimates of the Ricci iteration

In this part we derive some a priori estimates for the iteration sequence (1.2). By Theorem 1.4 we can choose some $\tau > 0$ so that the iteration carries on forever. Up to scaling the Kähler class, we assume without loss of generality that $\tau = 1$. Taking trace of (1.2) we then have

$$R(\omega_{i+1}) = \bar{R} - n + \mathrm{tr}_{\omega_{i+1}}\omega_i,\ \omega_0 = \omega.$$

Write

$$\omega_i = \omega + dd^c u_i, u_i \in \mathcal{H}_0.$$

Also, let $F_i \in C^\infty(X, \mathbb{R})$ be such that

$$(\omega + dd^c u_i)^n = e^{F_i}\omega^n. \tag{5.1}$$

Then

$$\Delta_{\omega_i} F_i = \mathrm{tr}_{\omega_i}(\mathrm{Ric}(\omega) - \omega_{i-1}) + n - \bar{R}. \tag{5.2}$$

Namely,

$$\Delta_{\omega_i}(F_i + u_{i-1}) = \mathrm{tr}_{\omega_i}(\mathrm{Ric}(\omega) - \omega) + n - \bar{R}.$$

We first derive the C^0 bound for u_i and F_i.

Proposition 5.1. *Assume that there is some constant $A > 0$ such that*

$$Ent_\omega(u_i) + d_1(0, u_i) \leq A \text{ for all } i \in \mathbb{N}.$$

Then there exists some constant B_1 depending only on X, ω and A such that

$$|F_i| + |u_i| \leq B_1 \text{ for all } i \in \mathbb{N}.$$

Proof. First, using Lemma 3.1, the bound $d_1(0, u_i) \leq A$ implies that the u_i has uniform L^1 bound, which in turn gives that

$$|\sup_X u_i| \leq C_1 \text{ for all } i \in \mathbb{N},$$

where $C_1 = C_1(X, \omega, A) > 0$. Moreover, for any $p > 0$, Zeriahi's version of the Skoda integrability estimate [Zer01] (see [Dar19, Corollary 4.16] for a formulation that fits our context most) implies that there exists $C_2 = C_2(X, \omega, A, p) > 0$ such that

$$\int_X e^{-pu_i} \leq C_2 \text{ for all } i \in \mathbb{N}. \tag{5.3}$$

Then one can apply [CC21b, Corollary 3.2] to find a constant $C_3 = C_3(X, \omega, A) > 0$ such that

$$F_i + u_{i-1} \leq C_3,\ |u_i| \leq C_3 \text{ for all } i \in \mathbb{N}.$$

Since $u_0 = 0$, we conclude by induction that there exists some $C_4 = C_4(X, \omega, A) > 0$ such that

$$F_i \leq C_4,\ |u_i| \leq C_4 \text{ for all } i \in \mathbb{N}.$$

Then [CC21b, Lemma 3.3] further implies that there exists some $B_1 = B_1(X, \omega, A) > 0$ such that

$$|F_i| + |u_i| \leq B_1 \text{ for all } i \in \mathbb{N}.$$

□

Corollary 5.2. *Assume that there is some constant $A > 0$ such that*

$$Ent_\omega(u_i) + d_1(0, u_i) \leq A \textit{ for all } i \in \mathbb{N}.$$

Then for any $q > 1$ there exists some constant $B_2 \geq 1$ depending only on X, ω, q and A such that

$$n + \Delta_\omega u_i \geq \frac{1}{B_2} \textit{ for all } i \in \mathbb{N}.$$

and

$$\int_X (n + \Delta_\omega u_i)^q \omega^n \leq B_2 \textit{ for all } i \in \mathbb{N}. \tag{5.4}$$

Proof. The first inequality follows from

$$n + \Delta_\omega u_i \geq n e^{F/n} \geq n e^{-B_1/n}.$$

The second inequality follows from (5.3), Proposition 5.1 and [CC21b, Corollary 3.4]. □

Proposition 5.3. *Assume that there is some constant $A > 0$ such that*

$$Ent_\omega(u_i) + d_1(0, u_i) \leq A \textit{ for all } i \in \mathbb{N}.$$

Then there exists some constant B_4 depending only on X, ω and A such that

$$\max_X |\nabla_{\omega_i}(F_i + u_{i-1})|^2_{\omega_i} + \max_X (n + \Delta_\omega u_i) \leq B_4 \textit{ for all } i \in \mathbb{N}.$$

Proof. The proof follows closely the one of [CC21b, Theorem 3.3]. The basic idea is to estimate

$$\Delta_{\omega_i}(e^{\frac{1}{2}(F_i+u_{i-1})}|\nabla_{\omega_i}(F_i + u_{i-1}))|^2_{\omega_i} + (n + \Delta_\omega u_i))$$

and then apply Nash–Moser iteration.

To simplify the notation, we put $\Delta := \Delta_\omega$, and use the subscrpit i to denote the operators associated with the metric ω_i, e.g., $\mathrm{tr}_i := \mathrm{tr}_{\omega_i}$, $\Delta_i := \Delta_{\omega_i}$. Also, put

$$w_i := F_i + u_{i-1}.$$

So one has

$$\Delta_i w_i = \mathrm{tr}_i(\mathrm{Ric}(\omega) - \omega) + n - \bar{R}.$$

In what follows, the constants $C > 0$ will change from line to line, which are all uniform (may depend on X, ω, A, but are independent of i).

Now we compute $\Delta_i(n+\Delta u_i)$. As in [CC21b, (3.25)], we have

$$\begin{aligned}\Delta_i(n+\Delta u_i) &\geq -C\mathrm{tr}_i\omega(n+\Delta u_i)+\Delta F_i - R(\omega)\\ &\geq -C\mathrm{tr}_i\omega(n+\Delta u_i)+\Delta w_i-(n+\Delta u_{i-1})-C.\end{aligned}$$

Using (5.1) and Proposition 5.1 one can estimate

$$\mathrm{tr}_i\omega \leq ne^{-F_i}(n+\Delta u_i)^{n-1}\leq C(n+\Delta u_i)^{n-1}.$$

Also, one can estimate Δw_i as in [CC21a, (4.11)]:

$$|\Delta w_i|\leq \frac{1}{2C}\frac{|(w_i)_{k\bar{k}}|^2}{(1+(u_i)_{k\bar{k}})^2}+\frac{C}{2}(1+(u_i)_{k\bar{k}})^2\leq \frac{1}{2C}\frac{|(w_i)_{k\bar{k}}|^2}{(1+(u_i)_{k\bar{k}})^2}+\frac{C}{2}(n+\Delta u_i)^2.$$

So we get

$$\Delta_i(n+\Delta u_i)\geq -C(n+\Delta u_i)^n-\frac{1}{2C}\frac{|(w_i)_{k\bar{k}}|^2}{(1+(u_i)_{k\bar{k}})^2}-\frac{C}{2}(n+\Delta u_i)^2-(n+\Delta u_{i-1})-C. \tag{5.5}$$

Next, we compute $\Delta_i(e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2)$. As in [CC21b, (3.43) and (3.49)], we have

$$\begin{aligned}\Delta_i(e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2) &\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i+e^{\frac{1}{2}w_i}\frac{|(w_i)_{k\bar{l}}|^2}{(1+(u_i)_{k\bar{k}})(1+(u_i)_{l\bar{l}})}\\ &\quad -Ce^{\frac{1}{2}w_i}|\nabla_i w_i|^2\big((n+\Delta u_i)^{2n-1}+(n+\Delta u_i)+1\big).\\ &\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i+\frac{1}{C}\frac{|(w_i)_{k\bar{l}}|^2}{(1+(u_i)_{k\bar{k}})(1+(u_i)_{l\bar{l}})}\\ &\quad -Ce^{\frac{1}{2}w_i}|\nabla_i w_i|^2\big((n+\Delta u_i)^{2n-1}+(n+\Delta u_i)+1\big).\end{aligned}$$

Note that, in the above estimate, the term involving $|(w_i)_{k\bar{l}}|^2$ is dropped in [CC21b, (3.49)] since it plays no role in *loc. cit.*, but for our purpose we do need it to dominate the bad term $|(w_i)_{k\bar{k}}|^2$ in (5.5).

Putting these estimates together, we then arrive at

$$\begin{aligned}\Delta_i(e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2+(n+\Delta u_i)) &\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i - C\big((n+\Delta u_i)^n+(n+\Delta u_i)^2+1\big)\\ &\quad -Ce^{\frac{1}{2}w_i}|\nabla_i w_i|^2\big((n+\Delta u_i)^{2n-1}+(n+\Delta u_i)+1\big)-(n+\Delta u_{i-1})\end{aligned}$$

Putting

$$U_i := e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2+(n+\Delta u_i)+1$$

and using $n+\Delta u_i\geq B_2^{-1}$ we can further simplify to get

$$\begin{aligned}\Delta_i U_i &\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i - CU_i\big((n+\Delta u_i)^{2n-1}+1\big)-(n+\Delta u_{i-1})\\ &\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i - CU_i\big((n+\Delta u_i)^{2n-1}+(n+\Delta u_{i-1})+1\big),\end{aligned}$$

where we used that $U_i\geq 1$. Put

$$\tilde{G}_i := C\big((n+\Delta u_i)^{2n-1}+(n+\Delta u_{i-1})+1\big),$$

thus we have the following key estimate:

$$\Delta_i U_i\geq 2e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i - U_i\tilde{G}_i.$$

Then for any $p>1$, we obtain

$$\begin{aligned}\int_X (p-1)U_i^{p-2}|\nabla_i U_i|_i^2\omega^n &= \int_X U_i^{p-1}(-\Delta_i U_i)\omega_i^n\\ &\leq \int_X U_i^p\tilde{G}_i\omega_i^n - \int_X 2U_i^{p-1}e^{\frac{1}{2}w_i}\nabla_i w_i\cdot_i\nabla_i\Delta_i w_i\omega_i^n.\end{aligned}$$

One can then deal with the bad term $e^{\frac{1}{2}w_i}\nabla_i w_i \cdot_i \nabla_i \Delta_i w_i$ using integration by parts:

$$-\int_X 2U_i^{p-1}e^{\frac{1}{2}w_i}\nabla_i w_i \cdot_i \nabla_i\Delta_i w_i\omega_i^n = \int_X 2U_i^{p-1}e^{\frac{1}{2}w_i}(\Delta_i w_i)^2\omega_i^n$$
$$+\int_X U_i^{p-1}e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2\Delta_i w_i\omega_i^n + \int_X 2(p-1)U_i^{p-2}e^{\frac{1}{2}w_i}\nabla_i U_i \cdot_i \nabla_i w_i\Delta_i w_i\omega_i^n.$$

Using the simple fact that $e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2 \le U_i$, we can estimate

$$\int_X U_i^{p-1}e^{\frac{1}{2}w_i}|\nabla_i w_i|_i^2\Delta_i w_i\omega_i^n \le \int_X U_i^p|\Delta_i w_i|\omega_i^n \le \int_X U_i^p((\Delta_i w_i)^2+1)\omega_i^n$$

and

$$\int_X 2U_i^{p-2}e^{\frac{1}{2}w_i}\nabla_i U_i \cdot_i \nabla_i w_i\Delta_i w_i\omega_i^n \le \int_X \frac{1}{2}U_i^{p-2}|\nabla_i U_i|_i^2\omega_i^n + \int_X 2U_i^{p-1}e^{\frac{1}{2}w_i}(\Delta_i w_i)^2\omega_i^n.$$

Putting these together and using $U_i \ge 1$, one can derive that (as in [CC21b, (3.54)])

$$\int_X \frac{p-1}{2}U_i^{p-2}|\nabla_i U_i|_i^2\omega_i^n \le \int_X pU_i^p G_i e^{F_i}\omega^n,$$

where

$$G_i := \tilde{G}_i + (\Delta_i w_i)^2 + 2e^{\frac{1}{2}w_i}(\Delta_i w)^2 + 1.$$

The rest of the proof uses Nash–Moser iteration, which goes through in exactly the same way as in [CC21b]. The only difference lies in the L^{4n}-bound for G_i (compare [CC21b, (3.63)]), since in our definition of G_i there is an additional term $(n+\Delta u_{i-1})$. But thanks to (5.4), this term has uniform L^q-bound (which only depends on X, ω, A, q and is independent of i). So the estimates in [CC21b] goes through for us as well.

Thus we finally arrive at $||U_i||_{L^\infty} \le C = C(X,\omega,A)$, finishing the proof. □

Corollary 5.4. *Assume that there is some constant $A > 0$ such that*

$$Ent_\omega(u_i) + d_1(0,u_i) \le A \text{ for all } i \in \mathbb{N}.$$

Then there exists some constant $B_5 > 1$ depending only on X, ω and A such that

$$B_5^{-1}\omega \le \omega_i \le B_5\omega \text{ for all } i \in \mathbb{N}.$$

Proof. This follows immediately from $\omega_i^n \ge e^{-B_1}\omega^n$ and $\mathrm{tr}_\omega\omega_i \le B_4$. □

By classical elliptic estimates and bootstrapping, we then have the following uniform estimates.

Corollary 5.5. *Assume that there is some constant $A > 0$ such that*

$$Ent_\omega(u_i) + d_1(0,u_i) \le A \text{ for all } i \in \mathbb{N}.$$

Then for any $\alpha \in (0,1)$ and $k \ge 1$, there exists some constant $B_{k,\alpha} > 1$ depending only on X, ω, α, k and A such that

$$||u_i||_{C^{k,\alpha}} \le B_{k,\alpha} \text{ for all } i \in \mathbb{N}.$$

Proof. First, the estimate $B_5^{-1}\omega \le \omega_i \le B_5\omega$ implies that (5.2) is uniformly elliptic with bounded right hand side. Then arguing as in the proof of [CC21a, Proposition 4.2], one has $u_i \in C^{3,\alpha}$ and $F_i \in C^{1,\alpha}$ for all $i \in \mathbb{N}$. This implies that the equation (5.2) has $C^{1,\alpha}$-coefficients and right hand (since we already know that u_{i-1} has $C^{3,\alpha}$ bound). This gives $C^{3,\alpha}$ bound for F_i. Differentiating the equation (5.1) twice one then gets a linear ellipitc equation for the second derivatives of u_i with C^α coeficitients and right hand side. So we get the $C^{4,\alpha}$-bound for u_i. Continuing in this way we get all the $C^{k,\alpha}$ bounds for u_i. □

6. Smooth convergence of the Ricci iteration

Assume that (X, ω) admits a cscK metric ω^* in $\{\omega\}$. Then by [BDL20, Theorem 1.5] we know that the K-energy is proper modulo $G := Aut_0(X)$. Hence one can choose $\tau_0 = \infty$ in Theorem 1.4. Then for any $\tau > 0$, we wish to show that the iteration sequence $\{\omega_i\}_{i\in\mathbb{N}}$ defined by (1.2) converges in a suitable sense to a cscK metric. Up to scaling the Kähler class, we will assume without loss of generality that $\tau = 1$. To make further simplification, we will first deal with the case where the cscK metric is unique, in which case the K-energy is proper (by [DR17; BB17; BDL20]), i.e., $\gamma(X, \{\omega\}) > 0$ (recall (4.1)).

Therefore, we have that

- There exists $\gamma > 0$ and $C_0 > 0$ such that
$$K_\omega(\varphi) \geq \gamma(I_\omega - J_\omega)(\varphi) - C_0 \text{ for any } \varphi \in \mathcal{H}_\omega.$$
- There exists a sequence $\{\omega_i\}_{i\in\mathbb{N}}$ satisfying
$$\omega_{i+1} - \omega_i = -\operatorname{Ric}(\omega_{i+1}) + \operatorname{HRic}(\omega_{i+1}),\ \omega_0 = \omega.$$
Equivalently, one has
$$R(\omega_{i+1}) = \bar{R} - n + \operatorname{tr}_{\omega_{i+1}}\omega_i,\ \omega_0 = \omega. \tag{6.1}$$
- Write $\omega^* = \omega + dd^c u^*$ and $\omega_i = \omega + dd^c u_i$ for $u^*, u_i \in \mathcal{H}_0$, where
$$\mathcal{H}_0 := \{\varphi \in \mathcal{H}_\omega | E_\omega(\varphi) = 0\}.$$

We wish to show the following.

Theorem 6.1. *Assume that there exists a unique cscK potential $u^* \in \mathcal{H}_0$, then the sequence $\{u_i\}_{i\in\mathbb{N}}$ converges smoothly to u^*.*

To prove this, we need some preparations.

Lemma 6.2. *One has*
(1) ω^* minimizes K_ω over $\mathcal{E}^1_\omega$.
(2) ω_i minimizes $\mathcal{J}^{\omega_i}_\omega$ over $\mathcal{E}^1_\omega$.
(3) ω_{i+1} minimizes $K_\omega + \mathcal{J}^{\omega_i}_\omega$ over $\mathcal{E}^1_\omega$.

Proof. The second one follows from Lemmas 2.2 and 3.3. The other two are contained in [CC21b, Corollary 4.5]. □

Lemma 6.3. *One can find $A > 0$ such that for all $i \in \mathbb{N}$*
$$Ent_\omega(u_i) + d_1(0, u_i) \leq A. \tag{6.2}$$

Proof. By Theorem 1.5 we have that
$$0 = K_\omega(\omega_0) \geq K_\omega(\omega_i) = Ent_\omega(u_i) + \mathcal{J}_\omega(u_i) \geq \gamma(I_\omega - J_\omega)(u_i) - C_0.$$
This implies that (using Lemma 2.1)
$$J_\omega(u_i) \leq n(I_\omega - J_\omega)(u_i) \leq \frac{nC_0}{\gamma}.$$
So Lemma 3.2 gives that
$$d_1(0, u_i) \leq C_1.$$
On the other hand, by [CC21b, Lemma 4.4], one has
$$0 \geq K_\omega(\omega_i) = Ent_\omega(u_i) + \mathcal{J}_\omega(u_i) \geq Ent_\omega(u_i) - C_2 d_1(0, u_i).$$
So we obtain that
$$Ent_\omega(u_i) \leq C_3,$$

finishing the proof. $\square$

Lemma 6.4. *One has for all $i \in \mathbb{N}$*

$$J(u_{i+1}, u_i) \leq K_\omega(u_i) - K_\omega(u_{i+1}).$$

So one has $I(u_{i+1}, u_i) \to 0$.

Proof. Using that ω_{i+1} minimizes $K_\omega + \mathcal{J}_\omega^{\omega_i}$, we have

$$K_\omega(u_{i+1}) + \mathcal{J}_\omega^{\omega_i}(u_{i+1}) \leq K_\omega(u_i) + \mathcal{J}_\omega^{\omega_i}(u_i).$$

Then using $\mathcal{J}_\omega^{\omega_i}(u_{i+1}) - \mathcal{J}^{\omega_i}(u_i) = J(u_{i+1}, u_i)$ (recall Lemma 2.2) we conclude the first assertion.

For the second statement, note that the K-energy is bounded from below in our setting, so Theorem 1.5 implies that $\{K_\omega(u_i)\}_{i\in\mathbb{N}}$ is a convergent sequence. So we conclude from Lemma 2.1. $\square$

Corollary 6.5. *If $\{u_{i_k}\}_{k\in\mathbb{N}}$ is a d_1-convergent subsequen, say $u_{i_k} \xrightarrow{d_1} u$, then $u_{i_k-1} \xrightarrow{d_1} u$ as well.*

Proof. This follows from the previous lemma and Corollary 3.8. $\square$

Now we are ready to prove Theorem 6.1

Proof of Theorem 6.1. We first argue that any convergent subsequence $\{u_{i_k}\}_{k\in\mathbb{N}}$ has to converge to u^* in the d_1-topology. By (6.2) and Lemma 3.4 this will imply that $u_i \xrightarrow{d_1} u^*$.

So assume that there exists a subsequence $\{u_{i_k}\}_{k\in\mathbb{N}}$, converging in d_1 to a limit $u_\infty \in \mathcal{E}^1_\omega$. Then for any $u \in \mathcal{E}^1_\omega$ we deduce that

$$\begin{aligned}
K_\omega(u_\infty) &\leq \lim_{k\to\infty} K_\omega(u_{i_k}) \\
&= \lim_{k\to\infty} \left(K_\omega(u_{i_k}) + \mathcal{J}_\omega^{\omega_{i_k-1}}(u_{i_k}) - \mathcal{J}_\omega^{\omega_{i_k-1}}(u_{i_k})\right) \\
&\leq \lim_{k\to\infty} \left(K_\omega(u) + \mathcal{J}_\omega^{\omega_{i_k-1}}(u) - \mathcal{J}_\omega^{\omega_{i_k-1}}(u_{i_k-1})\right) \\
&= \lim_{k\to\infty} \left(K_\omega(u) + J(u, u_{i_k-1})\right) = K_\omega(u) + J(u, u_\infty).
\end{aligned}$$

Here we used that K_ω is d_1-lsc, u_{i_k} minimizes $K_\omega + \mathcal{J}_\omega^{\omega_{i_k-1}}$, u_{i_k-1} minimizes $\mathcal{J}_\omega^{\omega_{i_k-1}}$, Lemma 2.2, Corollary 6.5 and Lemma 3.5. We notice from above that u_∞ is a minimizer of the functional

$$F_\infty(u) := K_\omega(u) + J(u, u_\infty), u \in \mathcal{E}^1_\omega.$$

We now argue that u_∞ must be a cscK potential and hence $u_\infty = u^*$.

By Corollary 5.5 and Arzelà–Ascoli, we know that $u_\infty \in \mathcal{H}_\omega$. So by Lemma 2.2 we can write

$$F_\infty(u) = K_\omega(u) + \mathcal{J}_\omega^{\omega_{u_\infty}}(u) - \mathcal{J}_\omega^{\omega_{u_\infty}}(u_\infty) = K_\omega^{\omega_{u_\infty}}(u) - \mathcal{J}_\omega^{\omega_{u_\infty}}(u_\infty).$$

So u_∞ minimizes the twisted K-energy $K_\omega^{\omega_{u_\infty}}$. The variational formula of $K_\omega^{\omega_{u_\infty}}$ then implies that

$$R(\omega_{u_\infty}) = \bar{R} - n + \mathrm{tr}_{\omega_{u_\infty}} \omega_{u_\infty} = \bar{R}.$$

So ω_{u_∞} is a cscK metric. By uniqueness assumption we have that $u_\infty = u^*$.

Thus we have shown that $u_{i_k} \xrightarrow{d_1} u^*$ for any convergent subsequence. So $u_i \xrightarrow{d_1} u^*$ follows. By Corollary 5.5 and Arzelà–Ascoli we then know that $u_i \to u^*$ smoothly. $\square$

If we do not assume the uniqueness of the cscK metric ω^*, then the K-energy is proper modulo the action of biholomorpihc automorphisms of X (see [BDL20, Theorem 1.5]). Modifying our previous proofs and incorporating the ideas from [DR19], one can actually prove the following result, improving Theorem 6.1.

Theorem 6.6. *(=Theorem 1.6) Let (X, ω) be a compact Kähler manifold admitting a cscK metric in $\{\omega\}$. Then for any $\tau > 0$ the iteration sequence (1.2) sequence exists and there exist holomorphic diffeomorphsims g_i such that $g_i^*\omega_i$ converges smoothly to a cscK metric.*

Proof. We give the necessary details for the reader's convenience. First, using that the K-energy decreases along u_i and is proper modulo $G = Aut_0(X)$, we have that

$$d_{1,G}(0, u_i) \leq A_0 \text{ for all } i \in \mathbb{N}$$

Fix a cscK metric $\omega^* \in \{\omega\}$ with $\omega^* = \omega + dd^c u^*$ and $u^* \in \mathcal{H}_0$. Then pick $g_i \in G$ such that

$$d_1(u^*, g_i.u_i) \leq d_{1,G}(u^*, u_i) + \frac{1}{i} \leq d_1(0, u^*) + d_{1,G}(0, u_i) \leq A_1.$$

Thus we deduce that

$$d_1(0, g_i.u_i) \leq A_1 + d_1(0, u^*).$$

Then using that the K-energy is G-invariant (see e.g. [CC21b, Lemma 4.11]), one can argue as in the proof of Lemma 6.3 to show that

$$Ent_\omega(g_i.u_i) \leq A_2 \text{ for all } i \in \mathbb{N}.$$

So the sequence $\{g_i.u_i\}_{i\in\mathbb{N}}$ is d_1-precompact. We wish to show that it converges to u^* smoothly. To this end, we need some uniform estimates for the sequence.

By Lemmas 3.10, 2.1, 6.4 and Theorem 1.5, we know that

$$\begin{aligned} I(g_i.u_i, g_i.u_{i-1}) = I(u_i, u_{i-1}) &\leq (n+1)J(u_i, u_{i-1}) \\ &\leq (n+1)(K_\omega(u_{i-1}) - K_\omega(u_i)) \to 0. \end{aligned}$$

And also, one has (by Lemma 3.6)

$$J(0, g_i.u_{i-1}) \leq I(0, g_i.u_{i-1}) \leq C_n(I(0, g_i.u_i) + I(g_i.u_i, g_i.u_{i-1})).$$

So we derive that (using Lemma 3.2)

$$d_1(0, g_i.u_{i-1}) \leq A_3.$$

The upshot is that, there exists some $A > 0$ such that

$$Ent_\omega(g_i.u_i) + d_1(0, g_i.u_{i-1}) \leq A \text{ for all } i \in \mathbb{N}.$$

For simplicity let us put

$$v_i := g_i.u_i \text{ and } h_{i-1} := g_i.u_{i-1}.$$

Then from (6.1) we deduce that

$$R(\omega_{v_i}) = \bar{R} - n + \mathrm{tr}_{\omega_{v_i}}(\omega + dd^c h_{i-1}).$$

This is equivalent to

$$\begin{cases} (\omega + dd^c v_i)^n = e^{F_i}\omega^n. \\ \Delta_{\omega_{v_i}}(F_i + h_{i-1}) = \mathrm{tr}_{\omega_{v_i}}(\mathrm{Ric}(\omega) - \omega) + n - \bar{R}. \end{cases} \tag{6.3}$$

And we have that

$$Ent_\omega(v_i) + d_1(0, h_{i-1}) \leq A \text{ for all } i \geq 1.$$

Then as in Proposition 5.1 we can obtain the C^0 estimate:

$$|v_i| \leq B_1 \text{ for all } i \geq 1.$$

This implies that (by Lemma 3.1)

$$d_1(0, v_i) = d_1(0, g_i.u_i) = d_1(g_i^{-1}.0, u_i) = d_1(g_{i+1}.(g_i^{-1}.0), h_{i+1}) \leq B_2.$$

Put

$$f_i := g_i^{-1} \circ g_{i+1},$$

Then we have

$$d_1(f_i.0, 0) \leq d_1(f_i.0, h_{i+1}) + d_1(0, h_{i+1}) \leq B_2 + A_3 \text{ for all } i \geq 1.$$

By the proof of [DR17, Proposition 6.8], $\{f_i\}_{i \geq 1}$ is contained in a bounded set of G. In particular, all derivatives of f_i up to order m, say, are bounded by some C_m independently of i. Since one has

$$h_i = g_{i+1}.u_i = f_i.v_i,$$

then v_i and h_i enjoy the same a priori estimates. So the same arguments as in §5 apply to the system of equations (6.3) as well. We conclude that there are uniform $C^{k,\alpha}$ estimates (independent of i) for v_i and h_i.

Now we are ready to show that $v_i \to u^*$ smoothly.

By Arzelà–Ascoli it suffices to argue that $v_i \xrightarrow{d_1} u^*$. We prove by contradiction. Assume that there exists a subsequence such that $v_{i_k} \xrightarrow{d_1} v_\infty$ for some $v_\infty \in \mathcal{E}^1_\omega$ with $d_1(u^*, v_\infty) > \varepsilon > 0$. By our uniform estimates for v_i and Arzelà–Ascoli we know that $v_\infty \in \mathcal{H}_0$.

For any $u \in \mathcal{H}_0$, one has (as in the proof of Theorem 6.1)

$$\begin{aligned}
K_\omega(v_\infty) &\leq \lim_{k\to\infty} K_\omega(v_{i_k}) = \lim_{k\to\infty} K_\omega(u_{i_k}) \\
&= \lim_{k\to\infty} (K_\omega(u_{i_k}) + \mathcal{J}^{\omega_{u_{i_k-1}}}(u_{i_k}) - \mathcal{J}^{\omega_{u_{i_k-1}}}(u_{i_k})) \\
&\leq \lim_{k\to\infty} (K_\omega(g_{i_k}^{-1}.u) + \mathcal{J}^{\omega_{u_{i_k-1}}}(g_{i_k}^{-1}.u) - \mathcal{J}^{\omega_{u_{i_k-1}}}(u_{i_k-1})) \\
&= \lim_{k\to\infty} (K_\omega(u) + J(g_{i_k}^{-1}.u, u_{i_k-1})) \\
&= \lim_{k\to\infty} (K_\omega(u) + J(u, h_{i_k-1})) = K_\omega(u) + J(u, v_\infty).
\end{aligned}$$

Here we used that K_ω and J are G-invariant (recall Lemma 3.10). Moreover, in the last equality we used that $h_{i_k-1} \xrightarrow{d_1} v_\infty$. Indeed, Lemma 6.4 implies that $I(v_i, h_{i-1}) = I(u_i, u_{i-1}) \to 0$, as $K_\omega(u_i)$ is decreasingly convergent (recall that the K-energy is bounded from below). So Corollary 3.8 implies that $\lim_k h_{i_k-1} = \lim_k v_{i_k} = v_\infty$, as claimed.

From above we observe that v_∞ is a minimizer for the functional

$$F_\infty(u) := K_\omega(u) + J(u, v_\infty),\ u \in \mathcal{H}_0.$$

This further implies that v_∞ is a minimizer of F_∞ over $\mathcal{H}_\omega$. Then as in the proof of Theorem 6.1, we conclude that v_∞ is a cscK potential.

By [BB17, Theorem 1.3] there exists $f \in G$ such that $v_\infty = f.u^*$. So we obtain that

$$d_1(v_{i_k}, u^*) - \frac{1}{i_k} \leq d_{1,G}(v_{i_k}, u^*) \leq d_1(f^{-1}.v_{i_k}, u^*) = d_1(v_{i_k}, v_\infty).$$

By choice the right hand goes to zero, while the left hand side is strictly bigger than $\frac{\varepsilon}{2} > 0$ for any $k \gg 1$. This is a contradiction. So we finish the proof. $\square$

But as in [DR19], one expects that the appearance of g_i in the above theorem is actually redundant, which might require substantial new ideas (see [Hun19] for progress in the toric Fano case); compare also [CC21b, Proposition 4.17] in the setting of continuity method. In the case of Kähler Ricci flow, this problem is studied in [TZ07; TZ13; TZZZ13; CS16].

7. Discretizing the twisted pseudo-Calabi flow

In the study of cscK metrics, it is often beneficial to allow for some twisted terms. More precisely, given a closed smooth real $(1,1)$ form, one can study the following χ-twisted cscK equation:

$$R(\omega_u) = \bar{R} - \bar{\chi} + \mathrm{tr}_{\omega_u}\chi. \tag{7.1}$$

This is equivalent to saying that $\mathrm{Ric}(\omega_u) - \chi$ is harmonic with respect to ω_u. Therefore, to search for χ-twisted cscK metrics, we are led to the following twisted pseudo-Calabi flow:

$$\partial_t \omega_t = -\,\mathrm{Ric}(\omega_t) + \mathrm{H}_{\omega_t}(\mathrm{Ric}(\omega_t) - \chi) + \chi,\ \omega_0 = \omega. \tag{7.2}$$

Here H_{ω_t} denotes the harmonic projection operator of the metric ω_t. When $c_1(X) = \lambda\{\omega\} + \{\chi\}$, this flow becomes

$$\partial_t \omega_t = -\,\mathrm{Ric}(\omega_t) + \lambda\omega_t + \chi,\ \omega_0 = \omega,$$

which is the twisted Kähler Ricci flow studied in [Liu13; CS16].

Discretizing the flow (7.2), we get (for some given $\tau > 0$)

$$\frac{\omega_{i+1} - \omega_i}{\tau} = -\,\mathrm{Ric}(\omega_{i+1}) + \mathrm{H}_{\omega_{i+1}}(\mathrm{Ric}(\omega_{i+1}) - \chi) + \chi,\ i \in \mathbb{N},\ \omega_0 = \omega. \tag{7.3}$$

The next result can be proved following exactly the same strategy as we did for the untwisted case. Hence we omit the details.

Theorem 7.1. *Assume that $\chi \geq 0$. There exists a constant $\tau_0 \in (0, \infty]$ depending only on $X, \{\omega\}$ and $\{\chi\}$ such that for any $\tau \in (0, \tau_0)$, the iteration sequence (7.3) exists for all $i \in \mathbb{N}$, with each ω_i being uniquely determined by ω_0, along which the χ-twisted K-energy K^χ_ω decreases. Moreover, if there exists a unique χ-twisted cscK metric $\omega^* \in \{\omega\}$, then for any $\tau > 0$ the sequence ω_i converges to ω^* smoothly.*

Note that, if $\chi > 0$, the uniqueness of χ-twisted cscK metric is automatic by [BDL17, Theorem 4.13].

One can try to extend our work further to the case of conical cscK metrics, extremal metrics and other canonical metrics. We leave this for future reserach.

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