BOUNDING THE VOLUME OF KÄHLER MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Abstract. Using Calabi ansatz, Fano index and δ-invariant, we give several optimal volume upper bounds for Kähler manifolds with positive Ricci curvature, from which we get new characterizations of the complex projective space. This is a preliminary version of the author's published work [57].

1. Introduction

Let (X, g) be an m-dimensional Riemannian manifold such that

$$\mathrm{Ric}\,(g) \geq (m-1)g.$$

Then the famous Bishop–Gromov volume comparison says that

$$\mathrm{Vol}(X, g) \leq \mathrm{Vol}(S^m, g_{S^m}),$$

and the equality holds if and only if (X, g) is isometric to the standard m-sphere S^m. However, suppose in addition that X has a complex structure J such that (X, g, J) is Kähler, then Liu [40] shows that this volume upper is *never sharp* (unless $X = \mathbb{P}^1$), in the sense that there exists a dimensional gap $\epsilon(n) > 0$ such that

$$\mathrm{Vol}(X, g) \leq \mathrm{Vol}(S^m, g_{S^m}) - \epsilon(n).$$

This distinguishes the Kählerian geometry from the Riemannian case (see also Li–Wang [38] for related discussions using holomorphic bisectional curvatures). So it is natural to ask what the optimal volume upper bound should be. The purpose of this paper is to answer this question by using Calabi ansatz and some new input from algebraic geometry.

Notice that, Kähler manifolds with positive Ricci curvature are automatically Fano. These are very special projective manifolds with many additional algebriac properties. So in what follows, unless otherwise specified, we will always assume that X is an n-dimensional Fano manifold (with $n \geq 2$). Note that the Picard group $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ is a finitely generated torsion free Abelian group, hence a lattice. So the isomorphism classes of line bundles on X are in one-to-one correspondence with the lattice points in $H^2(X, \mathbb{Z})$. Also note that the Kähler cone $\mathcal{K}(X)$ of X coincides with the ample cone as $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})$ by Kodaira vanishing and Hodge decomposition. So any Kähler class in $\mathcal{K}(X)$ can be approximated by a sequence of rational classes (which correspond to ample $\mathbb{Q}$-line bundles). Therefore, instead of studying Kähler classes, we will be mainly focusing on line bundles on X.

1.1. **The greatest Ricci lower bound.**
For any line bundle L one can define its index $I(L)$ to be

$$I(L) := \max\{k \in \mathbb{N} \mid \exists \text{ line bundle } H \text{ on } X \text{ s.t. } L = kH\} \tag{1.1}$$

When $L = -K_X$, we put

$$I(X) := I(-K_X), \tag{1.2}$$

which is called the *Fano index* of X.

In the following, L is always assumed to be *ample*. In this case, one can naturally define the greatest Ricci lower bound $\beta(X, L)$ by

$$\beta(X, L) := \sup\{\mu > 0 \mid \exists \text{ Kähler form } \omega \in 2\pi c_1(L) \text{ s.t. } \mathrm{Ric}\,(\omega) \geq \mu\omega\}. \tag{1.3}$$

Note that, by the Calabi–Yau theorem, given any Kähler form $\alpha \in 2\pi c_1(X)$, one can always find $\omega_0 \in 2\pi c_1(L)$ such that $\mathrm{Ric}\,(\omega_0) = \alpha > 0$. By compactness of X we see $\mathrm{Ric}\,(\omega_0) \geq \epsilon\omega_0$ for some $\epsilon > 0$. So $\beta(X, L)$ is always a positive number. On the other hand, $\beta(X, L)$ is naturally bounded from above by the Seshadri constant

$$\epsilon(X, L) := \sup\{\mu > 0 \mid -K_X - \mu L \text{ is nef}\}. \tag{1.4}$$

Thus we always have

$$0 < \beta(X, L) \leq \epsilon(X, L). \tag{1.5}$$

Remark 1.1. *In the special case $L = -K_X$,*

$$\beta(X) := \beta(X, -K_X)$$

is the usual greatest Ricci lower bound. This invariant was the topic of Tian's article [55] *although it was not explicitly defined there, but was first explicitly defined by Rubinstein in* [44, (32)], [45, Problem 3.1] *and was later further studied by Székelyhidi* [52]*, Li* [33]*, Song–Wang* [49]*, Cable* [6]*, et al.*

Remark 1.2. *The notions $I(L)$, $\beta(X, L)$ and $\epsilon(X, L)$ extend naturally to $\mathbb{Q}$-line bundles. The greatest Ricci lower bound is also well defined for general Kähler classes.*

Our goal is to bound the volume $\mathrm{Vol}(L)$. Here $\mathrm{Vol}(L) := (c_1(L))^n$ denotes the volume of the Kähler class $c_1(L)$. Deriving volume upper bounds for ample line bundles is of interest by itself, as it is closely related to the boundedness problem in algebraic geometry. For instance, by bounding $(-K_X)^n$ from above, it was shown by Kollár–Miyaoka–Mori [29] that n-dimensional Fano manifolds form a bounded family. However to the author's knowledge, a *sharp* volume upper bound for $-K_X$ is still missing in the literature. It was once believed that one always has

$$(-K_X)^n \leq (n + 1)^n \tag{1.6}$$

with equality if and only if $X = \mathbb{P}^n$. But it turns out that one can easily disprove this by looking at projective bundles over Fano manifolds (see [2, 13]). However when $\mathrm{rk}\,\mathrm{Pic}(X) = 1$, it is still an open problem whether (1.6) holds or not (cf. [28, 21]). Note that when X is K-semistable, (1.6) has been recently estabilished by K. Fujita [16] (see also Liu [36] for volume upper bounds of $\mathbb{Q}$-Fano varieties).

For general ample line bundles, one can easily bound $\mathrm{Vol}(L)$ in terms of the greatest Ricci lower bound. Indeed, pick $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega) \geq \mu\omega.$$

Applying Bishop–Gromov theorem to (X, ω), one can quickly derive (by letting $\mu \to \beta(X, L)$)

$$\beta(X, L)^n \mathrm{Vol}(L) \leq \frac{2^{n+1}(n!)^2(2n-1)^n}{(2n)!}. \tag{1.7}$$

However, as alluded to at the very beginning, this bound is not sharp in the Kählerian world. To get sharp bound, one needs to exploit the Kähler condition. Our first main result is the following

Theorem 1.3. *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L) \leq (n+1)^{n+1}. \tag{1.8}$$

Remark 1.4. *Notice that $I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L)$ is scaling invariant, so the conclusion of Theorem 1.3 also holds for any ample $\mathbb{Q}$-line bundle.*

The bound (1.8) is sharp, in the sense that $X = \mathbb{P}^n$ achieves the equality. An easy consequence of Theorem 1.3 is the following

Corollary 1.5. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X, L) \leq n + 1.$$

Let us also say a few words about the lower bound of $I(X)\beta(X, L)$. When $L = -K_X$, by utilizing the boundedness of Fano manifolds [29] and the α-invariant of Tian [54], it follows that $I(X)\beta(X) \geq c(n) > 0$ for some dimensional constant $c(n)$. However for general ample line bundles, there is no universal positive lower bound for $I(L)\beta(X, L)$, as one can easily find a sequence L_i in the ample cone with $I(L_i) = 1$ and $\epsilon(X, L_i) \to 0$, so that $\beta(X, L_i) \to 0$.

When $L = -K_X$ and $\beta(X) = 1$ (i.e., X is K-semistable [34]), (1.8) reduces to the inequality:

$$I(X)(-K_X)^n \leq (n+1)^{n+1},$$

which was derived in [19, (2.22)], under the assumption that X is Kähler–Einstein, using Sasakian geometry. In fact we will use a similar method to prove (1.8). More precisely, the strategy is to look at the affine cone

$$V := \mathrm{Spec} \bigoplus_{m \geq 0} H^0(X, mL).$$

Using Calabi ansatz, one can construct a cone metric on V with non-negative Ricci curvature. Then we apply the comparison theorem of Bishop–Gromov to the link of this metric cone to get (1.8).

1.2. The δ-invariant.

Our next goal is to further enhance Theorem 1.3. To this end, we need more input from algebraic geometry. It turns out that the δ-invariant introduced recently in the literature is the right notion for us, which we now describe.

Following [17, 4], the δ-invariant of L is defined by

$$\delta(X, L) := \inf_{v \in \mathrm{Val}_X} \frac{A_X(v)}{S_L(v)}. \tag{1.9}$$

Here Val_X denotes the space of valuations over X, $A_X(v)$ denotes the log discrepancy of v, and $S_L(v)$ denotes the expected vanishing order of L with respect to

v. Note that δ-invariant is also called *stability threshold* in the literature, which plays important roles in the study of K-stability and has attracted intensive research attentions. The following result is essentially contained in [3], generalizing [11, Theorem 5.7], which gives an geometric interpretation of the δ-invariants on Fano manifolds.

Theorem 1.6 ([3])**.** *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$\beta(X, L) = \min\{\epsilon(X, L), \delta(X, L)\}.$$

Proof. For any smooth semi-positive $(1,1)$-form θ on X, θ-twisted β-, ϵ- and δ-invariants were introduced in [3] (in fact they allow θ to be currents, but we do not need this here). More precisely, put

$$\beta_\theta(X, L) := \sup\{\mu > 0 \mid \exists\, \omega \in 2\pi c_1(L) \text{ s.t. } \mathrm{Ric}\,(\omega) \geq \mu\omega + \theta\}, \tag{1.10}$$

$$\epsilon_\theta(X, L) := \sup\{\mu > 0 \mid c_1(X) - \mu c_1(L) - [\theta]/2\pi > 0\}, \tag{1.11}$$

and

$$\delta_\theta(X, L) := \inf_{v \in \mathrm{Val}_X} \frac{A_\theta(v)}{S(v)}. \tag{1.12}$$

It follows clearly from the definition that

$$\beta(X, L) = \sup_\theta \beta_\theta(X, L).$$

Moreover, as θ is smooth, one has [3, Example 3.2]

$$\delta(X, L) = \delta_\theta(X, L).$$

Now using [3, Theorem C]

$$\beta_\theta(X, L) = \min\{\epsilon_\theta(X, L), \delta_\theta(X, L)\} \text{ for any smooth } \theta,$$

we obtain

$$\beta(X, L) = \sup_\theta \min\{\epsilon_\theta(X, L), \delta(X, L)\}.$$

Using the fact $\epsilon_\theta(X, L) \leq \epsilon(X, L)$, we finish the proof. $\square$

Regarding the δ-invariant, [4, Theorem D] implies the following volume upper bound :

$$\delta(X, L)^n \mathrm{Vol}(L) \leq (n+1)^n. \tag{1.13}$$

This inequality reveals the deep relationhip between *singularities* and *volumes* of linear systems.

With the help of δ-invariant, the statement of Theorem 1.3 can now be enhanced as follows.

Theorem 1.7. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X. Then one has*

$$\begin{cases} I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L) = (n+1)^{n+1}, & X \cong \mathbb{P}^n; \\ I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L) = 2n^{n+1}, & X \cong Q \subset \mathbb{P}^{n+1} \textit{ is a smooth quadric;} \\ I(L)\beta(X, L)^{n+1}\mathrm{Vol}(L) < n(n+1)^n, & \textit{otherwise.} \end{cases}$$

The proof of this result is in fact a combination of Theorem 1.6, (1.13) and some classical results in algebraic geometry (cf. [18, 23, 26]). As we shall see, Theorem 1.7 is also related to the ordinary double point (ODP) conjecture [48, Conjecture 1.2].

1.3. Kähler manifolds with positive Ricci curvature.

Finally, let us attend to general Kähler classes in $\mathcal{K}(X)$. Using δ-invariant, we have the following sharp volume upper bound for Kähler manifolds with positive Ricci curvature, which substantially improves the bound given by the Bishop–Gromov Theorem.

Theorem 1.8. *Let (X,ω) be an n-dimensional Kähler manifold with*

$$\mathrm{Ric}\,(\omega) \geq (n+1)\omega.$$

Then one has

$$\int_X \omega^n \leq (2\pi)^n,$$

and the equality holds if and only if (X,ω) is holomorphically isometric to $(\mathbb{P}^n,\omega_{FS})$.

This gives a new characterization of the complex projective space in terms of the Ricci curvature and volume. Note that the volume upper bound follows easily from (1.13) by approximation. The difficult part of Theorem 1.8 lies in the investigation of the equality, which relies heavily on the volume function on the Néron–Severi space of X. Moreover, Newton–Okounkov bodies and the positivity criterion of Küronya–Lozovanu [30] will be crucially needed as well.

The rest of this paper is organized as follows. In Section 2 we review a classical setting where one can apply the Calabi ansatz and then construct a family of metric cones to prove Theorem 1.3 and Corollary 1.5. In Section 3, the equality case of (1.8) is investigated and Theorem 1.7 is proved. In section 4 we use δ-invariant and Newton–Okounkov bodies to prove Theorem 1.8. We end this paper by proposing a conjecture in Section 5.

Acknowledgments. The author would like to thank Chuyu Zhou, Wenshuai Jiang and Kento Fujita for many helpful discussions. Special thanks go to Yanir Rubinstein for his interest in this work and for fruitful discussions on the Calabi ansatz. The author is sponsored by the China post-doctoral grant BX20190014.

2. Calabi ansatz and metric cones

In this section we use Calabi ansatz to prove Theorem 1.3.

2.1. Calabi ansatz on the total space of line bundles.

For readers' convenience, let us review a well-studied and powerful construction, pioneered by Calabi [7, 8], which can effectively produce various explicit examples of canonical metrics in Kähler geometry. The idea is to work on complex manifolds with certain symmetries so that one can reduce geometric PDEs to simple ODEs. This approach is often referred to as the Calabi ansatz in the literature, which has been studied and generalzed to different extent by many authors; see e.g., [20] for some general discussions and historical overviews.

For our purpose, we will work on the total space of line bundles over Kähler manifolds. The goal is to construct canonical metrics on this space. Our computation will follow the exposition of [53, Section 4.4]. See also [27, 47, 56] and the references therein for similar treatment.

Let (X,ω) be an n-dimensional compact Kähler manifold, where ω is a Kähler form on X. Let $L \to X$ be a holomorphic line bundle equipped with a smooth Hermitian metric h such that its curvature form R_h satisfies

$$R_h := \sqrt{-1}\partial\bar{\partial}\log h^{-1} = \lambda\omega \tag{2.1}$$

for some constant $\lambda \neq 0$. Let

$$L^{-1} \xrightarrow{\pi} X$$

be the dual bundle of L. whose zero section will be denoted by E_0 (so E_0 is a copy of X sitting inside the total space L^{-1}). In the following we will construct a Kähler metric on $L^{-1}\backslash\{E_0\}$.

The idea is to make use of the fiberwise norm on L^{-1} induced by h^{-1}. We put

$$s(t) := \log||t||^2 = \log h^{-1}(t,t), \text{ for } t \in L^{-1}\backslash\{E_0\}.$$

So s is a globally defined function on $L^{-1}\backslash\{E_0\}$. The goal is to construct a Kähler metric η on $L^{-1}\backslash\{E_0\}$ of the form

$$\eta = \sqrt{-1}\partial\bar{\partial}f(s), \tag{2.2}$$

where f is a function to be determined.

We will carry out the computation locally. Choose $p \in X$ and let $\big(U, z = (z_1, ..., z_n)\big)$ be a local coordinate system around p such that ω can be expressed by a Kähler potential:

$$\omega = \sqrt{-1}\partial\bar{\partial}(P(z)), \tag{2.3}$$

where

$$P(z) = |z|^2 + O(|z|^4).$$

Moreover we may assume that L^{-1} is trivialized over U by a nowhere vanishing holomorphic section $\sigma \in \Gamma(U, L^{-1})$ such that

$$||\sigma||^2_{h^{-1}} = h^{-1}(\sigma,\sigma) = e^{\lambda P(z)}.$$

Under this trivialization, we have an identification:

$$\pi^{-1}(U) \cong U \times \mathbb{C}. \tag{2.4}$$

Let w be the holomorphic coordinate function in the fiber direction. So we have

$$s = \log(|w|^2 e^{\lambda P(z)}) \text{ on } U \times \mathbb{C}^*. \tag{2.5}$$

Such a choice of coordinates has the advantage that, on the fiber $\pi^{-1}(p)$ over p, one has

$$\partial P(z) = \bar{\partial}P(z) = 0. \tag{2.6}$$

So direct computation gives

$$\eta = \sqrt{-1}\partial\bar{\partial}f(s) = \lambda f'\pi^*\omega + f''\frac{\sqrt{-1}dw \wedge d\overline{w}}{|w|^2}. \tag{2.7}$$

over p. Thus we get

$$\eta^{n+1} = \frac{(n+1)\lambda^n(f')^n f''}{|w|^2}(\pi^*\omega)^n \wedge \sqrt{-1}dw \wedge d\overline{w}. \tag{2.8}$$

Now observe that this expression of volume form is true not just over p. Indeed, if we choose a different trivialization $w' = q(z)w$, the expression (2.8) remains the same. So (2.8) holds everywhere on $U \times \mathbb{C}^*$.

Expression (2.7) indicates that, to make η positively definite, f should be a strictly convex function with $f' > 0$. So let us introduce

$$\tau = f'(s), \ \varphi(\tau) = f''(s). \tag{2.9}$$

Then, over p, the Ricci form is given by

$$\begin{aligned} \mathrm{Ric}\,(\eta) &= -\sqrt{-1}\partial\bar{\partial}\log\det(\eta) \\ &= \pi^*\mathrm{Ric}\,(\omega) - \left(n\lambda\frac{\varphi}{\tau} + \lambda\varphi'\right)\pi^*\omega \\ &\quad - \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)' \frac{\sqrt{-1}dw \wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.10}$$

To build a metric η with special geometric features, it is natural to impose some conditions on the base metric ω. It turns out that there are several ways to do this.

(1) The most commonly used condition for ω is the Kähler–Einstein condition. Namely, we assume

$$\mathrm{Ric}\,(\omega) = \mu\omega \tag{2.11}$$

for some constant μ. In this setting, (2.10) becomes

$$\begin{aligned} \mathrm{Ric}\,(\eta) &= \left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega \\ &\quad - \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)' \frac{\sqrt{-1}dw \wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.12}$$

From this we easily get the expression of the scalar curvature

$$S(\eta) = \frac{1}{\tau^n}\left(\frac{\mu\tau^{n+1}}{(n+1)\lambda} - \tau^n\varphi\right)'', \tag{2.13}$$

which now holds everywhere on $L^{-1}\backslash\{E_0\}$.

(2) A less common condition for ω is the Kähler–Einstein edge condition studied in [42], by which we mean

$$\mathrm{Ric}\,(\omega) = \mu\omega + 2\pi(1-\beta)[D] \tag{2.14}$$

for some constant μ, cone angle $\beta \in (0, 1]$ and a smooth divisor D on X. We put

$$\overline{D} := \pi^* D.$$

In this case, the corresponding Hermitian metric h satisfying (2.1) is not supposed to smooth, but one can still derive (2.10) in the current sense:

$$\begin{aligned} \mathrm{Ric}\,(\eta) &= \left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega + 2\pi(1-\beta)[\overline{D}] \\ &\quad - \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)' \frac{\sqrt{-1}dw \wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.15}$$

This allows one to construct Kähler–Einstein edge metrics, which will be explored in a forthcoming paper [46].

(3) One can also consider the twisted Kähler–Einstein condition. More precisely, we assume

$$\mathrm{Ric}\,(\omega) = \mu\omega + \alpha \tag{2.16}$$

for some constant μ and some non-negative $(1,1)$-form α on X, in which case, (2.10) reads

$$\begin{aligned} \mathrm{Ric}\,(\eta) = & \Big(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\Big)\pi^*\omega + \pi^*\alpha \\ & - \varphi\Big(n\frac{\varphi}{\tau} + \varphi'\Big)'\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2}. \end{aligned} \tag{2.17}$$

This case will be particularly useful for the proof of Theorem 1.3.

2.2. **Metric cone and normalized volume.**

In what follows we will additionally assume that L is *ample* (so that $\lambda > 0$). The goal of this part is review a standard construction of a family of metric cones over the circle bundle of X (see also [42]). The resulting metric cone will be denoted by (V, o^*), where o^* is the vertex.

Recall that when X is Fano and ω is Kähler–Einstein, a classical result of Kobayashi [25] gurantees the existence of Sasaki–Einstein metrics on certain circle bundles over X, which turns out can be constructed using Calabi unsatz (we refer the reader to the survey [50] for more information on this subject). For completeness we will include the details of this construction and then relate it to the comparison geometry and as a byproduct of these general discussions, we prove Theorem 1.3. See also [37] for a closely related discussion of this subject from an algebro-geometric viewpoint.

To construct a metric cone (V, o^*), we use the following observation. Geometrically we want E_0 to correspond to the vertex o^* of metric cone (namely, we want E_0 to shrink to a point). To make this happen, we go back to the local expression (2.7). As $|w| \searrow 0$, we want $\tau \searrow 0$, so that η degenerates along E_0. Moreover, to make the metric complete near the vertex (where $|w| \ll 1$), one should have

$$\lim_{\tau\to 0^+} \varphi(\tau) = 0. \tag{2.18}$$

On the other hand, it is tempting to make the Ricci form $\mathrm{Ric}\,(\eta)$ as simple as possible. So let us look at its expression (2.10). A natural candidate of φ that can simplify (2.10) is supposed to satisfy the following ODE:

$$n\frac{\varphi}{\tau} + \varphi' = \mathrm{Const}. \tag{2.19}$$

Using the boundary condition (2.18), we get

$$\varphi(\tau) = a\tau, \tag{2.20}$$

for some $a > 0$, in which case, (2.10) reduces to (see also [42, Lemma 1])

$$\mathrm{Ric}\,(\eta) = \pi^*\big(\mathrm{Ric}\,(\omega) - a(n+1)\lambda\omega\big), \tag{2.21}$$

so that $\mathrm{Ric}\,(\eta)$ is captured by the Ricci curvature of base manifold (X, ω) in a simple manner. Now using (2.9) and (2.20), we easily recover

$$f(s) = Ce^{as},$$

for some constant $C > 0$. By rescaling f, we may assume $C = \frac{1}{2}$ so that

$$f(s) = e^{as}/2 \tag{2.22}$$

and

$$\eta = \frac{\sqrt{-1}}{2}\partial\bar{\partial}e^{as}, \tag{2.23}$$

where $a > 0$ is a parameter.

In the following, we show that such η indeed gives rise to a metric cone (V, o^*). To see this, we introduce a new variable

$$r \in (0, \infty)$$

such that

$$r^2 = e^{as}. \tag{2.24}$$

Then r is also a globally defined function on $L^{-1}\backslash\{E_0\}$. In our chosen coordinate system, we have

$$r^2 = |w|^{2a} e^{a\lambda P(z)}. \tag{2.25}$$

We shall show the following

Proposition 2.1 ([5, 50])**.** *Let η be the Kähler form defined by (2.23). For any $a > 0$, the Riemannian metric g_η associated to η is a warped product:*

$$g_\eta = dr \otimes dr + r^2 g_{_M}, \tag{2.26}$$

where $g_{_M}$ is a Riemannian metric on the circle bundle $M := \{r = 1\}$ of X, so that $(L^{-1}\backslash\{E_0\}, g_\eta)$ indeed defines a metric cone.

Proof. We will prove this by explicitly calculating g_η in local coordinates (recall (2.4)). Indeed, over U we have

$$\begin{aligned}
\eta &= \frac{\sqrt{-1}}{2}\partial\bar\partial r^2 = \frac{\sqrt{-1}}{2}\partial\bar\partial\big(|w|^{2a}e^{a\lambda P(z)}\big) \\
&= \frac{\sqrt{-1}r^2}{2}\Bigg(a^2\frac{dw\wedge d\overline{w}}{|w|^2} + a\lambda\frac{\partial^2 P}{\partial z_i\partial\overline{z_j}}dz_i\wedge d\overline{z_j} \\
&\qquad + a^2\lambda^2\frac{\partial P}{\partial z_i}\frac{\partial P}{\partial\overline{z_j}}dz_i\wedge d\overline{z_j} + a^2\lambda\bigg(\frac{\partial P}{\partial z_i}dz_i\wedge\frac{d\overline{w}}{\overline{w}} + \frac{\partial P}{\partial\overline{z_j}}\frac{dw}{w}\wedge d\overline{z_j}\bigg)\Bigg) \\
&= \frac{a\lambda r^2}{2}\pi^*\omega + \frac{a^2r^2}{2}\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2} + \frac{\sqrt{-1}a^2\lambda^2r^2}{2}P_iP_{\overline{j}}dz_i\wedge d\overline{z_j} \\
&\qquad + \frac{\sqrt{-1}a^2\lambda r^2}{2}\bigg(P_idz_i\wedge\frac{d\overline{w}}{\overline{w}} + P_{\overline{j}}\frac{dw}{w}\wedge d\overline{z_j}\bigg).
\end{aligned} \tag{2.27}$$

Then the corresponding Riemannian metric tensor g_η is given by

$$\begin{aligned}
g_\eta = \frac{a\lambda r^2}{2}\pi^*g_\omega &+ \frac{a^2r^2}{2}\frac{dw\otimes d\overline{w} + d\overline{w}\otimes dw}{|w|^2} + \frac{a^2\lambda^2r^2}{2}P_iP_{\overline{j}}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big) \\
&+ \frac{a^2\lambda r^2}{2}\bigg(P_i\frac{dz_i\otimes d\overline{w} + d\overline{w}\otimes dz_i}{\overline{w}} + P_{\overline{j}}\frac{d\overline{z_j}\otimes dw + dw\otimes d\overline{z_j}}{w}\bigg).
\end{aligned} \tag{2.28}$$

Now we write w using polar coordinates:

$$\begin{aligned}
w &= |w|e^{\sqrt{-1}\theta} \\
&= r^{1/a}e^{-\lambda P/2+\sqrt{-1}\theta},\ \theta\in[0,2\pi).
\end{aligned} \tag{2.29}$$

Here we used (2.25). From (2.29) we deduce

$$\begin{cases} \frac{dw}{w} = \frac{1}{ar}dr - \frac{\lambda}{2}dP + \sqrt{-1}d\theta, \\ \frac{d\overline{w}}{\overline{w}} = \frac{1}{ar}dr - \frac{\lambda}{2}dP - \sqrt{-1}d\theta. \end{cases} \tag{2.30}$$

Plugging (2.30) into (2.28), we obtain

(2.31)

$$\begin{aligned}
g_\eta = {} & \frac{a\lambda r^2}{2}\pi^* g_\omega + \frac{a^2r^2}{2}\Big(\frac{2}{a^2r^2}dr\otimes dr - \frac{\lambda}{ar}(dr\otimes dP + dP\otimes dr) + \frac{\lambda^2}{2}dP\otimes dP + 2d\theta\otimes d\theta\Big)\\
& + \frac{a^2\lambda^2 r^2}{2}P_iP_{\bar j}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big) + \frac{a^2\lambda r^2}{2}\Big(\frac{1}{ar}(dr\otimes dP + dP\otimes dr) - \lambda dP\otimes dP\\
& + \sqrt{-1}\Big(P_{\bar j}(d\overline{z_j}\otimes d\theta + d\theta\otimes d\overline{z_j}) - P_i(dz_i\otimes d\theta + d\theta\otimes dz_i)\Big)\Big)\\
= {} & dr\otimes dr + r^2\Big[\frac{a\lambda}{2}\pi^* g_\omega + a^2 d\theta\otimes d\theta - \frac{a^2\lambda^2}{4}dP\otimes dP + \frac{a^2\lambda^2}{2}P_iP_{\bar j}\Big(dz_i\otimes d\overline{z_j} + d\overline{z_j}\otimes dz_i\Big)\\
& + \frac{\sqrt{-1}a^2\lambda}{2}\Big(P_{\bar j}(d\overline{z_j}\otimes d\theta + d\theta\otimes d\overline{z_j}) - P_i(dz_i\otimes d\theta + d\theta\otimes dz_i)\Big)\Big]\\
= {} & dr\otimes dr + r^2\Big[\frac{a\lambda}{2}\pi^* g_\omega + a^2\Big(d\theta + \lambda d^{\mathbb{C}}P\Big)\otimes\Big(d\theta + \lambda d^{\mathbb{C}}P\Big)\Big],
\end{aligned}$$

where

$$d^{\mathbb{C}} := \frac{\sqrt{-1}}{2}(\bar\partial - \partial).$$

One can easily verify that, the real 1-form

$$\phi := d\theta + \lambda d^{\mathbb{C}}P \tag{2.32}$$

is globally defined on $L^{-1}\backslash\{E_0\}$. Indeed, given a different trivialization $\sigma' := q(z)\sigma$, where $q(z)$ is some locally defined nowhere vanishing holomorphic function on X, one has $w' = q^{-1}w$ and $P' = P + \frac{1}{\lambda}\log|q|^2$. The argument θ' of w' can be expressed by $\theta' = \theta + \frac{\sqrt{-1}}{2}(\log q - \log\overline{q})$. Then one has

$$\begin{aligned}
d\theta' + \lambda d^{\mathbb{C}}P' &= d\theta + \lambda d^{\mathbb{C}}P + \frac{\sqrt{-1}}{2}d(\log q - \log\overline{q}) + d^{\mathbb{C}}\log|q|^2\\
&= d\theta + \lambda d^{\mathbb{C}}P + \frac{\sqrt{-1}}{2}\Big(\frac{\partial q}{q} - \frac{\overline{\partial q}}{\overline{q}} + \frac{\overline{\partial q}}{\overline{q}} - \frac{\partial q}{q}\Big)\\
&= d\theta + \lambda d^{\mathbb{C}}P.
\end{aligned}$$

So (2.32) defines a global 1-form on $L^{-1}\backslash\{E_0\}$. Thus we obtain

$$g_\eta = dr\otimes dr + r^2\Big(\frac{a\lambda}{2}\pi^* g_\omega + a^2\phi\otimes\phi\Big). \tag{2.33}$$

Now let

$$M := \{r = 1\} \tag{2.34}$$

be the circle bundle over X and we define a 2-form

$$g_{_M} := \frac{a\lambda}{2}\pi^* g_\omega + a^2\phi\otimes\phi. \tag{2.35}$$

By (2.31), $g_{_M}$ can be locally expressed as

$$g_{_M} = \frac{a\lambda}{2}\pi^* g_\omega + a^2\Big(d\theta + \lambda d^{\mathbb{C}}P\Big)\otimes\Big(d\theta + \lambda d^{\mathbb{C}}P\Big) \tag{2.36}$$

This implies that $g_{_M}$ is a well defined symmetric 2-tensor when restricted to the circle bundle M since $(z_1, ..., z_n, \theta)$ forms a local coordinate system of M over $U \subset X$. Moreover $g_{_M}$ is clearly positively definite when restricted to M. Thus $g_{_M}$ induces a Riemannian metric on M, also denoted $g_{_M}$ by abuse of notation.

In summery, the Kähler metric g_η we deduced from the ODE (2.19) defines a cone metric on $L^{-1}\backslash\{E_0\}$, which degenerates E_0 to the vertex of the metric cone. Moreover g_η takes the form

$$g_\eta = dr \otimes dr + r^2 g_{_M},$$

where $g_{_M}$, given by (2.35), is a Riemannian metric defined on the circle bundle M of X. The proof is complete.

□

In particular, in the language of Sasakian geometry, $(M, g_{_M})$ is a regular Sasakian manifold. Note that in the work of Kobayashi [25], the 1-form ϕ is interpreted as a connection 1-form on M.

For any $a > 0$, the metric cone constructed above will be denoted by (V_a, o^*), where o^* the vertex. The manifold M is called the *link* of (V_a, o^*). Namely we have

$$V_a = C(M, g_{_M}) := (\mathbb{R}_+ \times M, dr^2 + r^2 g_{_M}) \cup \{o^*\}. \tag{2.37}$$

By the Gauss–Codazzi equations, one has

$$\mathrm{Ric}\,(g_\eta) = \mathrm{Ric}\,(g_{_M}) - 2n g_{_M}. \tag{2.38}$$

This relates the Ricci curvature of (V_a, o^*) to that of M.

As we shall see below, the geometry of (V_a, o^*) depends on the parameter a, since one can easily calculate the volume of the unit ball $B_1(o^*)$ of the metric cone.

Proposition 2.2 ([42])**.** *The volume of the unit ball $B_1(o^*)$ centered at the vertex of (V_a, o^*) is given by*

$$\mathrm{Vol}(B_1(o^*)) = \frac{(a\pi)^{n+1}\mathrm{Vol}(L)}{(n+1)!}. \tag{2.39}$$

Proof. We use η^{n+1} to compute the volume. Plugging (2.22) into (2.8) and using (2.5), we obtain

$$\eta^{n+1} = \frac{(n+1)e^{a\lambda(n+1)P}\lambda^n a^{n+2}}{2^{n+1}}(\pi^*\omega)^n \wedge \sqrt{-1}|w|^{2a(n+1)-2}dw \wedge d\overline{w}$$

Thus we get

$$\begin{aligned}\mathrm{Vol}(B_1(o^*)) &= \int_{r\le 1}\frac{\eta^{n+1}}{(n+1)!}\\ &\overset{(2.25)}{=} \int_X\int_{|w|\le e^{-\lambda P/2}}\frac{(n+1)e^{a\lambda(n+1)P}\lambda^n a^{n+2}}{2^{n+1}(n+1)!}(\pi^*\omega)^n \wedge \sqrt{-1}|w|^{2a(n+1)-2}dw \wedge d\overline{w}\\ &= \int_X \frac{2\pi a^{n+1}\lambda^n}{2^{n+1}(n+1)!}\omega^n\\ &\overset{(2.1)}{=} \frac{(a\pi)^{n+1}\mathrm{Vol}(L)}{(n+1)!},\end{aligned}$$

Here we used the fact that $\omega \in 2\pi c_1(L)/\lambda$.

□

From Proposition 2.2 we deduce the *nomalized volume* $\kappa(V_a, o^*)$ (the ratio between the cone volume and the Euclidean volume; see [43, (5.25)]) of the metric cone:

$$\kappa(V_a, o^*) := \frac{\mathrm{Vol}(B_1(o^*))}{\mathrm{Vol}(B_1(0^{2n+2}))} = a^{n+1}\mathrm{Vol}(L). \tag{2.40}$$

Here $\mathrm{Vol}(B_1(0^{2n+2})) = \pi^{n+1}/(n+1)!$ denotes the volume of the Euclidean unit $(2n+2)$-ball. In particular, the normalized volume $\kappa(V_a, o^*)$ only depends on $\mathrm{Vol}(L)$ and the parameter a.

Example 2.3.

(1) *The simplest example that fits into this framework is* $X := \mathbb{P}^n$. *We take* $\omega := \omega_{FS} \in \mathcal{O}_{\mathbb{P}^n}(1)$. *Let* $L := \mathcal{O}_{\mathbb{P}^n}(1)$ *and* h^{-1} *be the standard Hermitian metric on the tautological line bundle* $L^{-1} = \mathcal{O}_{\mathbb{P}^n}(-1)$. *Furthermore, we choose* $a = 1$. *In this case,* $L^{-1}\backslash\{E_0\} \cong \mathbb{C}^{n+1}\backslash\{0\}$ *and* $M = S^{2n+1}$ *(since in this case* r *is the distance function to the origin* $0 \in \mathbb{C}^{n+1}$*). Moreover* $\eta = \sqrt{-1}\partial\bar{\partial}r^2/2$ *is simply the standard flat Kähler form on* $\mathbb{C}^{n+1}$ *and* $g_{_M}$ *is the standard round metric on* S^{2n+1}. *One also has* $\mathrm{Vol}(L) = 1$. *Plugging this into* (2.39), *we obtain*

$$\mathrm{Vol}(B_1(o^*)) = \frac{\pi^{n+1}}{(n+1)!},$$

which is, of course, the volume of the unit $(2n+2)$*-ball in the Euclidean space.*

(2) *A more interesting example is the Stenzel cone* [51]. *We take*

$$X := \{X_0^2 + \ldots + X_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$$

to be a smooth quadric in $\mathbb{P}^{n+1}$. *Then* X *is a homogeneous Kähler–Einstein Fano manifold. Let* $L := \mathcal{O}_X(1)$. *Note that, by adjunction, one has* $-K_X = nL$. *We may choose* $\omega \in 2\pi c_1(L)$ *such that* $\mathrm{Ric}\,(\omega) = n\omega$. *Choose a Hermitian metric* h *on* L *such that* $R_h = \omega$. *Now we pick* $a = \frac{n}{n+1}$. *Then* (2.23) *yeilds a Calabi–Yau cone* (V, o^*), *as in this case* (2.21) *gives* $\mathrm{Ric}\,(\eta) = 0$. *This cone metric* η *on* $V\backslash\{o^*\}$ *can be thought of as a Ricci flat metric defined on the germ of an ordinary double point:*

$$\{X_0^2 + \ldots + X_{n+1}^2 = 0\} \subset \mathbb{C}^{n+2}.$$

Note that $\mathrm{Vol}(L) = 2$. *Using* (2.40), *we obtain the normalized volume:*

$$\kappa(V, o^*) = 2\Big(\frac{n}{n+1}\Big)^{n+1}.$$

(3) *More generally, let* X *be a Fano manifold admitting a Kähler–Einstein metric* $\omega \in 2\pi c_1(X)$ *such that* $\mathrm{Ric}\,(\omega) = \omega$. *Let* L *be an ample line bundle on* X *equipped with a Hermitian metric* h *such that* $R_h = \lambda\omega$ *for some* $\lambda > 0$. *We choose* $a = 1/((n+1)\lambda)$. *Then* (2.23) *yeilds a Calabi–Yau cone* (V, o^*) *with* $\mathrm{Ric}\,(\eta) = 0$.

An interesting consequence of (2.40) is the the following result (see also [19, (2.22)]), which can be thought of as a weak version of the volume upper bound for Kähler–Einstein Fano varieties studied by Fujita [16] and Liu [41].

Proposition 2.4 ([19]). *There exists a dimensional constant $\epsilon(n) \in (0,1)$ such that the following holds. Let X be an n-dimensional Kähler–Einstein Fano manifold. Let $I(X)$ be its Fano index. Then one has*

$$\begin{cases} I(X)(-K_X)^n = (n+1)^{n+1}, & \text{when } X = \mathbb{P}^n \\ I(X)(-K_X)^n \leq \big(1-\epsilon(n)\big)(n+1)^{n+1}, & \text{otherwise.} \end{cases}$$

Proof. By definition of the Fano index, we may find an ample line bundle L such that

$$-K_X = I(X)L.$$

By the Kähler–Einstein condition, we may choose a Kähler form $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega) = I(X)\omega.$$

Pick a Hermitian metric h on L such that

$$R_h = \omega.$$

Given these data, one can run the above ODE construction to obtain a family of metric cones (V_a, o^*). Now we choose

$$a = I(X)/(n+1).$$

Then (2.21) gives

$$\mathrm{Ric}\,(\eta) = 0,$$

i.e., (V, o^*) is a Calabi–Yau cone. So (2.38) implies that, the link M is a $(2n+1)$-dimensional Einstein manifold with Enstein constant $2n$

Now applying the Bishop–Gromov theorem to M, one has

$$\kappa(V_a, o^*) \leq 1,$$

and the equality holds if and only if (V_a, o^*) is *isometric* to the Euclidean space $\mathbb{C}^n$, in which case, X must be $\mathbb{P}^n$. Indeed, as we have seen in Example 2.3.1, $X = \mathbb{P}^n$ implies $\kappa(V_a, o^*) = 1$. Conversely, when $\kappa(V_a, o^*) = 1$, the S^1-bundle $(M, g_{_M})$ is isometric to the round sphere S^{2n+1}. Then one can compute the Reeb vector field, which generates the Hopf S^1-action on S^{2n+1}, so that the orbit space is simply $\mathbb{P}^{n+1}$.

Now let us turn to the case where $\kappa(V_a, o^*) < 1$. In this case the link M of the Calabi–Yau cone (V_a, o^*) is an $(2n+1)$-dimensional Einstein manifold with Enstein constant $2n$, but *not isometric* to the round sphere S^{2n+1}. Then the well-known sphere gap theorem (see e.g., [9, Theorem 4.3]) guarantees that there is an *Anderson constant* $\epsilon(n) \in (0,1)$ such that

$$\mathrm{Vol}(M, g_{_M}) \leq \big(1-\epsilon(n)\big)\mathrm{Vol}(S^{2n+1}, g_{_{S^{2n+1}}}),$$

so that

$$\kappa \leq 1 - \epsilon(n).$$

This completes the proof. □

Remark 2.5. *In fact the gap $\epsilon(n)$ can be specified; see Corollary 3.4.*

Similar strategy shows the following

Theorem 2.6 (=Theorem 1.3). *Let L be an ample line bundle on a Fano manifold X. Then one has*

$$I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) \leq (n+1)^{n+1}.$$

Proof. As pointed out in Remark 1.4, by rescaling, we might as well assume

$$I(L)=1.$$

Our goal is to prove

$$\beta(X,L)^{n+1}\mathrm{Vol}(L)\le(n+1)^{n+1}.$$

Given any $\mu\in(0,\beta(X,L))$, choose $\omega\in 2\pi c_1(L)$ such that

$$\mathrm{Ric}\,(\omega)\ge\mu\omega.$$

Pick a Hermitian metric on L with $R_h=\omega$. Then the ODE construction gives us a family of metric cones (V_a,o^*) depending on the parameter $a>0$. Let us choose

$$a=\frac{\mu}{n+1},$$

in which case, (2.21) implies

$$\mathrm{Ric}\,(\eta)=\pi^*\big(\mathrm{Ric}\,(\omega)-\mu\omega\big)\ge 0.$$

So from (2.38) we deduce

$$\mathrm{Ric}\,(g_{_M})\ge 2ng_{_M}.$$

Applying Bishop–Gromov theorem to M, one has (recall (2.40))

$$\kappa(V_a,o^*)=(\mu/n+1)^{n+1}\mathrm{Vol}(L)\le 1,$$

so that

$$\mu^{n+1}\mathrm{Vol}(L)\le(n+1)^{n+1}.$$

Letting $\mu\to\beta(X,L)$, we get

$$\beta(X,L)^{n+1}\mathrm{Vol}(L)\le(n+1)^{n+1},$$

as desired. □

As an immediate consequence, we have

Corollary 2.7 (=Corollary 1.5)**.** *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$I(L)\beta(X,L)\le n+1.$$

Proof. By rescaling, we assume

$$I(L)=1.$$

In this case L is an integral ample line bundle, so

$$\mathrm{Vol}(L)=L^n\ge 1$$

is a positive integer. Then Theorem 1.3 implies that

$$\beta(X,L)\le n+1,$$

so that

$$I(L)\beta(X,L)\le n+1$$

holds for arbitrary ample $\mathbb{Q}$-line bundle L by scailing invariance. □

3. Proof of Theorem 1.7

In contrast to Section 2, the discussion in this part will be purely algebraic. The main goal is to prove Theorem 1.7. We begin with the following result, which characterizes the equality of (1.8).

Proposition 3.1. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X with*

$$I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = (n+1)^{n+1},$$

then X is biholomorphic to $\mathbb{P}^n$.

Proof. By rescaling, we might as well assume [1]

$$\beta(X,L) = 1.$$

By Theorem 1.6, we deduce that

$$\delta(X,L) \geq 1.$$

Thus (1.13) implies

$$\mathrm{Vol}(L) \leq (n+1)^n.$$

On the other hand, by assumption, we have

$$I(L)\mathrm{Vol}(L) = (n+1)^{n+1}.$$

So we find that

$$I(L) \geq n+1.$$

Thus Corollary 1.5 forces that

$$I(L) = n+1,\ \mathrm{Vol}(L) = (n+1)^n \text{ and } \delta(X,L) = \beta(X,L) = 1.$$

In particular, L is an integral ample line bundle with index $n+1$. We put

$$L = (n+1)H \text{ and } N := -K_X - L.$$

Then H is ample and N is nef (as $\epsilon(X,L) \geq \beta(X,L) = 1$).

We claim that N is a trivial line bundle. Suppose otherwise, then one has

$$H^0\big(K_X + (n+1)H\big) = H^0(-N) = 0.$$

Meanwhile, by Kodaira vanishing, we have

$$H^i(K_X + kH) = 0, \text{ for any } i \geq 1 \text{ and } k \geq 1.$$

Thus for $k \in \{1, 2, ..., n+1\}$, the Euler characteristic of $K_X + kH$ satisfies

$$\chi(K_X + kH) = h^0(K_X + kH) = h^0\big((k-n-1)H - N\big) = 0.$$

Then Riemann–Roch implies that $\chi(K_X + kH)$ is a degree n polynomial in k with $n+1$ roots, which forces that

$$\chi(K_X + kH) \equiv 0$$

for any integer k, contradicting the ampleness of H. So N is trivial as claimed.

Thus we have

$$L = -K_X,$$

so that the Fano index satisfies

$$I(X) = I(L) = n+1.$$

[1] It is possible that $\beta(X,L)$ is irrational before rescaling, but this will not cause issues for our argument. In fact we believe that one always has $\beta(X,L) \in \mathbb{Q}$.

Then the criterion of Kobayashi–Ochiai [26] guarantees that X is biholomorphic to the complex projective space. $\square$

Note that the above proof crucially used Theorem 1.6, (1.13) and the index $I(L)$. In fact, regarding the index $I(L)$, one has the following result, which refines Corollary 1.5.

Lemma 3.2 ([18, 23])**.** *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$\begin{cases} I(L)\epsilon(X,L) = n+1, & X \cong \mathbb{P}^n; \\ I(L)\epsilon(X,L) = n, & X \cong Q \subset \mathbb{P}^{n+1} \textit{ is a smooth quadric;} \\ I(L)\epsilon(X,L) < n, & \textit{otherwise.} \end{cases}$$

Proof. We sketh the proof for readers' convenience. By rescaling, we assume

$$\epsilon(X,L) = 1.$$

Let $L = I(L)H$ for some ample line bundle H. It was shown in [18, 23] that, if $K_X + nH$ is not nef, then $X \cong \mathbb{P}^n$. On the other hand, the assumption $\epsilon(X,L) = 1$ says that $-K_X - I(L)H$ is nef.

If $I(L) > n$, then $-K_X - nH$ is ample so $K_X + nH$ cannot be nef and hence $X \cong \mathbb{P}^n$, in which case $L = -K_X$ and $I(L) = n+1$.

If $I(L) = n$, then $-K_X - nH$ is nef. Meanwhile, in this case $X \neq \mathbb{P}^n$ so $K+nH$ is also nef. This implies that $-K_X - nH$ is numerically trivial and hence $-K_X \sim nH$ (numerical equivalence is the same as linear equivalence on Fano manifolds), so that [26] guarantees that X is a smooth quadric in $\mathbb{P}^{n+1}$. $\square$

Now we are ready to prove

Theorem 3.3 (=Theorem 1.7)**.** *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X, then one has*

$$\begin{cases} I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = (n+1)^{n+1}, & X \cong \mathbb{P}^n; \\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) = 2n^{n+1}, & X \cong Q \subset \mathbb{P}^{n+1} \textit{ is a smooth quadric;} \\ I(L)\beta(X,L)^{n+1}\mathrm{Vol}(L) < n(n+1)^n, & \textit{otherwise.} \end{cases}$$

Proof. By rescaling we assume

$$\beta(X,L) = 1.$$

When $X \cong \mathbb{P}^n$ or $X \cong Q \subset \mathbb{P}^{n+1}$, we have $\epsilon(X,L) = \beta(X,L) = 1$ by the existence of KE metrics. Otherwise, one has $\epsilon(X,L) \geq 1$. So Lemma 3.2 implies

$$\begin{cases} I(X) = n+1, & X \cong \mathbb{P}^n; \\ I(X) = n, & X \cong Q \subset \mathbb{P}^{n+1} \text{ is a smooth quadric;} \\ I(X) < n, & \text{otherwise.} \end{cases}$$

On the other hand, Theorem 1.6 gives $\delta(X,L) \geq 1$, then (1.13) implies

$$\mathrm{Vol}(L) \leq (n+1)^n.$$

So the result follows. $\square$

As a simple consequence, one can specify the Anderson gap $\epsilon(n)$ appearing in Proposition 2.4.

Corollary 3.4. *One can choose the gap $\epsilon(n)$ in Proposition 2.4 to be*

$$\epsilon(n) = 1 - \frac{n}{n+1}$$

Note that this also follows easily from the volume upper bounds derived in [16].

Remark 3.5. *The ordinary double point conjecture* [48, Conjecture 1.2] *expects that one can even take*

$$\epsilon(n) = 1 - 2(\frac{n}{n+1})^{n+1}.$$

4. Optimal volume upper bounds for Kähler classes

This section is devoted to the proof of Theorem 1.8.

Let X be a Fano manifold and $\mathcal{K}(X)$ denote its Kähler cone. For any Kähler class $\xi \in \mathcal{K}(X)$, one can naturally define its greatest Ricci lower bound $\beta(X, \xi)$ to be

$$\beta(X, \xi) := \sup\{\mu > 0 \mid \exists \text{ Kähler form } \omega \in 2\pi\xi \text{ s.t. } \mathrm{Ric}\,(\omega) \geq \mu\omega\}. \tag{4.1}$$

In particular, we have $\beta(X, L) = \beta(X, 2\pi c_1(L))$ for ample line bundles.

Lemma 4.1. *The greatest Ricci lower bound $\beta(X, \cdot)$ is a lower semi-continuous function on $\mathcal{K}(X)$.*

Proof. Let $\{e_1, ..., e_\rho\}$ be a basis of $H^{1,1}(X, \mathbb{R})$, then any Kähler class $\xi \in \mathcal{K}(X)$ can be written as

$$\xi = \sum_{i=1}^{\rho} a_i e_i$$

for some $a_i \in \mathbb{R}$. For each $i \in \{1, ..., \rho\}$ choose a smooth real $(1,1)$-form $\eta_i \in e_i$.

Now assume that there exists $\omega \in \xi$ such that

$$\mathrm{Ric}\,(\omega) > \mu\omega$$

for some $\mu > 0$. For any $\vec{\epsilon} = (\epsilon_1, ..., \epsilon_\rho) \in \mathbb{R}^\rho$ with $||\vec{\epsilon}|| \ll 1$, we put

$$\omega_{\vec{\epsilon}} := \omega + \sum_{i=1}^{\rho} \epsilon_i \eta_i.$$

Then for $||\vec{\epsilon}|| \ll 1$ one also has

$$\mathrm{Ric}\,(\omega_{\vec{\epsilon}}) > \mu\omega_{\vec{\epsilon}}.$$

So the lower semi-continuity of $\beta(X, \cdot)$ follows. □

Note that, for Fano manifolds, $H^{1,1}(X, \mathbb{R})$ can be identified with the Névon–Severi space $N^1(X)_\mathbb{R}$, on which one can define a *continuous* volume function $\mathrm{Vol}(\cdot)$. When restricted to $\mathcal{K}(X)$, $\mathrm{Vol}(\cdot)$ is the usual volume for Kähler classes (which will be treated as ample $\mathbb{R}$-divisors in what follows). For nef classes ξ, $\mathrm{Vol}(\xi)$ is simply the top self-intersection number ξ^n. A class $\xi \in N^1(X)$ is called *big* if and only if $\mathrm{Vol}(\xi) > 0$. We refer the reader to the standard reference [31] for more details.

Proposition 4.2. *Let ξ be a Kähler class on an n-dimensional Fano manifold X. Then one has*

$$\beta(X, 2\pi\xi)^n \mathrm{Vol}(\xi) \leq (n+1)^n. \tag{4.2}$$

Proof. Choose a sequence of ample $\mathbb{Q}$-line bundles L_i such that

$$L_i \to \xi \text{ in } N^1(X)_{\mathbb{R}}.$$

By Lemma 4.1 we have

$$\beta(X, 2\pi\xi) \leq \liminf_i \beta(X, L_i).$$

So for any $\epsilon > 0$ and $i \gg 1$, one has

$$\beta(X, L_i) \geq \beta(X, 2\pi\xi) - \epsilon.$$

Now Theorem 1.6 and (1.13) implies that

$$(\beta(X, 2\pi\xi) - \epsilon)^n \mathrm{Vol}(L_i) \leq (n+1)^n.$$

Using the continuity $\mathrm{Vol}(L_i) \to \mathrm{Vol}(\xi)$ and sending $\epsilon \to 0$, we find that

$$\beta(X, 2\pi\xi)^n \mathrm{Vol}(\xi) \leq (n+1)^n.$$

□

Therefore, to finish the proof of Theorem 1.8, it remains to show that the equality of (4.2) is exactly obtained by $\mathbb{P}^n$. We first deal with rational classes.

Proposition 4.3. *Let L be an ample $\mathbb{Q}$-line bundle on a Fano manifold X with*

$$\beta(X, L)^n \mathrm{Vol}(L) = (n+1)^n,$$

then X is biholomorphic to $\mathbb{P}^n$.

Proof. We follow the argument of K. Fujita [16]. By rescaling, we put

$$\beta(X, L) = 1.$$

Then Theorem 1.6 implies that

$$\delta(X, L) \geq 1 \text{ and } \epsilon(X, L) \geq 1.$$

Pick any point $p \in X$ and let $\hat{X} \xrightarrow{\sigma} X$ be the blow-up at p. Let E be the exceptional divisor of σ. Then one has

$$A_X(E) \geq S_L(E),$$

namely,

$$\begin{aligned} n = A_X(E) &\geq S_L(E) \\ &= \frac{1}{\mathrm{Vol}(L)} \int_0^\infty \mathrm{Vol}(\sigma^* L - xE) dx \\ &\geq \frac{1}{\mathrm{Vol}(L)} \int_0^{\sqrt[n]{\mathrm{Vol}(L)}} (\mathrm{Vol}(L) - x^n) dx \\ &= \frac{n}{n+1} \sqrt[n]{\mathrm{Vol}(L)}. \end{aligned}$$

Here we used [16, Theorem 2.3(1)]. Now the condition $\mathrm{Vol}(L) = (n+1)^n$ implies that

$$\mathrm{Vol}(\sigma^* L - xE) = \mathrm{Vol}(L) - x^n$$

for any $x \in [0, n+1]$. So [16, Theorem 2.3(2)] implies that

$$\sigma^* L - (n+1)E \text{ is nef.}$$

Now using $\epsilon(X, L) \geq 1$, we find that

$$\sigma^*(-K_X) - (n+1)E = \sigma^*(-K_X - L) + \sigma^* L - (n+1)E$$

is nef as well. Thus $X \cong \mathbb{P}^n$ by [12, 24]. □

So Theorem 1.8 follows immediately when $\omega \in 2\pi c_1(L)$ for some ample $\mathbb{Q}$-line bundle L. Now for general Kähler classes, we prepare the following

Lemma 4.4. *Let X be a Fano manifold. Pick any point $p \in X$ and let $\hat{X} \xrightarrow{\sigma} X$ be the blow-up at p. Let E be the exceptional divisor of σ. Let $\xi \in \mathcal{K}(X)$ be a Kähler class.*

(1) *For any $x \in \mathbb{R}_{\geq 0}$, one has*

$$\mathrm{Vol}(\sigma^*\xi - xE) \geq \mathrm{Vol}(\xi) - x^n.$$

(2) *Suppose that ξ satisfies*

$$\beta(X, 2\pi\xi)^n \mathrm{Vol}(\xi) = (n+1)^n,$$

then for any $x \in [0, \mathrm{Vol}(\xi)^{1/n}]$,

$$\mathrm{Vol}(\sigma^*\xi - xE) = \mathrm{Vol}(\xi) - x^n.$$

Proof. This first part follows from [16, Theorem 2.3(1)] by approximation. Indeed, let L_i be a sequence of ample $\mathbb{Q}$-line bundles such that $L_i \to \xi$ in $N^1(X)_{\mathbb{R}}$. Then for any $x \in \mathbb{R}_{\geq 0}$, [16, Theorem 2.3(1)] says that

$$\mathrm{Vol}(\sigma^*(L_i) - xE) \geq \mathrm{Vol}(L_i) - x^n,$$

so the assertion follows by the continuity of $\mathrm{Vol}(\cdot)$.

For the second part, we rescale ξ such that

$$\beta(X, 2\pi\xi) = 1.$$

Let L_i be a sequence of ample $\mathbb{Q}$-line bundles such that $L_i \to \xi$ in $N^1(X)_{\mathbb{R}}$. For any $\epsilon > 0$ and $i \gg 1$, Theorem 1.6 and Lemma 4.1 implies that

$$\delta(X, L_i) \geq \beta(X, L_i) \geq 1 - \epsilon.$$

Thus we get

$$\begin{aligned} n = A_X(E) &\geq (1-\epsilon) S_{L_i}(E) \\ &= \frac{1-\epsilon}{\mathrm{Vol}(L_i)} \int_0^\infty \mathrm{Vol}(\sigma^* L_i - xE) dx. \end{aligned}$$

Letting $i \to \infty$, by dominated convergence theorem and by sending $\epsilon \to 0$, we get

$$n \geq \frac{1}{\mathrm{Vol}(\xi)} \int_0^\infty \mathrm{Vol}(\sigma^*\xi - xE) dx,$$

so that (using the assumption $\mathrm{Vol}(\xi) = (n+1)^n$)

$$\begin{aligned} n &\geq \frac{1}{\mathrm{Vol}(\xi)} \int_0^{\mathrm{Vol}(\xi)^{1/n}} (\mathrm{Vol}(\xi) - x^n) dx \\ &= \frac{n}{n+1} \mathrm{Vol}(\xi)^{1/n} = n. \end{aligned}$$

This gives

$$\mathrm{Vol}(\sigma^*\xi - xE) = \mathrm{Vol}(\xi) - x^n$$

for $x \in [0, \mathrm{Vol}(\xi)^{1/n}]$ as claimed. □

As we have seen in the proof of Proposition 4.3, [16, Theorem 2.3(2)] says that, for ample $\mathbb{Q}$-line bundles, the condition

$$\mathrm{Vol}(\sigma^*L - xE) = \mathrm{Vol}(L) - x^n,\ x \in [0, a]$$

implies that $\sigma^*L - xE$ is nef for $x \in [0, a]$, whose proof however heavily relies on the rationality of L and the ampleness criterion of [14]. To prove the same assertion for general ample $\mathbb{R}$-line bundles, there is some subtlety involved if we follow Fujita's original argument. To avoid potential issues, we take an alternative approach, using Newton–Okounkov bodies.

4.1. Newton–Okounkov bodies and positivity of $\mathbb{R}$-line bundles.

Let Y be an n-dimensional projective manifold. Choose an admissible flag of subvarieties

$$Y_\bullet :\ Y = Y_0 \supseteq Y_1 \supseteq ... \supseteq Y_{n-1} \supseteq Y_n = \{pt.\},$$

such that each Y_i is an irreducible subvariety of codimension i and smooth at the point Y_n. Then any big class $\xi \in N^1(Y)_{\mathbb{R}}$ can be associated with a convex body $\Delta_{Y_\bullet}(\xi)$ in $\mathbb{R}^n$, which is called the Newton–Okounkov body of ξ with respect to the flag $Y_\bullet$. This generalizes the classical polytope construction for divisors on toric varieties. A crucial fact is that

$$\mathrm{Vol}(\xi) = n!\mathrm{Vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\xi)).$$

In this way one can study the volume function $\mathrm{Vol}(\cdot)$ on $N^1(Y)_{\mathbb{R}}$ using convex geometry. For more details we refer the reader to [32].

It turns out that Newton–Okounkov bodies can also help us visualize the positivity of $\mathbb{R}$-line bundles. More precisely, for any big $\mathbb{R}$-divisor D, one can define its *restricted base loci* by

$$B_-(D) := \bigcup_A B(D + A),$$

where the union is over all ample $\mathbb{Q}$-divisors A on Y and $B(\cdot)$ denotes the stable base loci (cf. [15]). Then $B_-(D)$ captures the non-nef locus of D (see [15, Example 1.18]). Indeed, suppose that there exists some curve C intersecting negatively with D, then by adding a small amount of ample $\mathbb{Q}$-divisor A, one still has

$$(D + A) \cdot C < 0,$$

which implies that $C \subset B(D + A)$ and hence

$$C \subset B_-(D).$$

The recent result of Küronya–Lozovanu [30, Theorem A] says that one can characterize the restricted base loci using Newton–Okounkov bodies.

Theorem 4.5 ([30])**.** *Let D be a big $\mathbb{R}$-divisor. Then the following are equivalent.*

(1) $q \notin B_-(D)$.
(2) *There exists an admissible flag $Y_\bullet$ with $Y_n = \{q\}$ such that the origin $0 \in \Delta_{Y_\bullet}(D) \subset \mathbb{R}^n$.*
(3) *For any admissible flag $Y_\bullet$ with $Y_n = \{q\}$, one has $0 \in \Delta_{Y_\bullet}(D) \subset \mathbb{R}^n$.*

Let us also record the following useful translation property of Newton–Okounkov bodies.

Proposition 4.6. [30, Proposition 1.6]. *Let ξ be a big $\mathbb{R}$-divisor and $Y_\bullet$ an admissible flag on Y. Then for any $t \in [0, \tau(\xi, Y_1))$ we have*

$$\Delta_{Y_\bullet}(\xi)_{\nu_1 \geq t} = \Delta_{Y_\bullet}(\xi - tY_1) + te_1.$$

where $\tau(\xi, F) := \sup\{\mu > 0 | \xi - \mu Y_1 \text{ is big}\}$ denotes the pseudo-effective threshold, ν_1 denotes the first coordinate of $\mathbb{R}^n$ and $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$.

Now we are ready to show the following result, extending [16, Theorem 2.3] to $\mathbb{R}$-divisors.

Proposition 4.7. *Let X be a projective manifold. Pick any point $p \in X$ and let $\hat{X} \xrightarrow{\sigma} X$ be the blow-up at p. Let E be the exceptional divisor of σ. Let $\xi \in N^1(X)_{\mathbb{R}}$ be an ample $\mathbb{R}$-divisor. Suppose that*

$$\mathrm{Vol}(\sigma^*\xi - xF) = \mathrm{Vol}(\xi) - x^n \text{ for any } x \in [0, a].$$

Then $\sigma^*\xi - xF$ is nef for any $x \in [0, a]$.

Proof. We argue by contradiction. Suppose that $\sigma^*\xi - x_0E$ is not nef for some $x_0 \in (0, a)$. Then there exists some curve C intersecting $\sigma^*\xi - x_0E$ negatively. Note that such C necessarily intersects with E. Thus $B_-(\sigma^*\xi - x_0E) \cap E \neq \emptyset$. So we can pick a point

$$q \in B_-(\sigma^*\xi - x_0E) \cap E$$

and build an admissible flag $Y_\bullet$ on $\hat{X}$ with

$$Y_1 = E \text{ and } Y_n = \{q\}.$$

Then we get the Newton–Okounkov body $\Delta_{Y_\bullet}(\sigma^*\xi)$ and by Proposition 4.6, $\Delta_{Y\bullet}(\sigma^*\xi - xE)$ can be obtained from $\Delta_{Y_\bullet}(\sigma^*\xi)$ by truncating and translating in the first coordinate ν_1 of $\mathbb{R}^n$. Now using the condition

$$\mathrm{Vol}(\sigma^*\xi - xF) = \mathrm{Vol}(\xi) - x^n \text{ for } x \in [0, a]$$

and the Brunn–Minkowski inequality in convex geometry, we see that, in the region $\{0 \leq \nu_1 \leq a\}$, $\Delta_{Y_\bullet}(\sigma^*\xi)$ has to be a convex cone over an $(n-1)$-dimensional convex set $\Sigma := \Delta_{Y_\bullet}(\sigma^*\xi) \cap \{v_1 = a\}$. Note that the cone $\Delta_{Y_\bullet}(\sigma^*\xi) \cap \{0 \leq \nu_1 \leq a\}$ necessarily contains the line segment

$$\{(t, 0, ..., 0) | t \in [0, a]\}$$

by Theorem 4.5 and Proposition 4.6, as $\sigma^*\xi - \epsilon E$ is nef at least for some $\epsilon > 0$. Intuitively, one has the following picture.

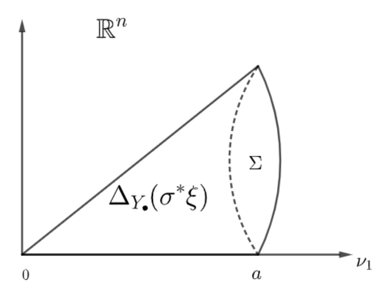

FIGURE 1. $\Delta_{Y_\bullet}(\sigma^*\xi) \cap \{0 \leq \nu_1 \leq a\}$

.

In particular, this implies that $0 \in \Delta_{Y_\bullet}(\sigma^*\xi - x_0 E)$ and hence $q \notin B_-(\sigma^*\xi - x_0 E)$ by Theorem 4.5, which contradicts our choice of q.

□

Now the following result is clear.

Proposition 4.8. *Let ξ be a Kähler class on a Fano manifold X with*

$$\beta(X, 2\pi\xi)^n \mathrm{Vol}(\xi) = (n+1)^n,$$

then X is biholomorphic to $\mathbb{P}^n$.

Proof. We set

$$\beta(X, 2\pi\xi) = 1$$

Then by Lemma 4.4(2) and Proposition 4.7,

$$\sigma^*\xi - (n+1)E \text{ is nef.}$$

Thus as in the proof of Proposition 4.3,

$$\sigma^*(-K_X) - (n+1)E \text{ is also nef,}$$

so we get $X \cong \mathbb{P}^n$ by [12, 24]. □

Finally, we arrive at our main result.

Theorem 4.9 (=Theorem 1.8)**.** *Let (X, ω) be an n-dimensional Kähler manifold with*

$$\mathrm{Ric}\,(\omega) \geq (n+1)\omega.$$

Then one has

$$\int_X \omega^n \leq (2\pi)^n,$$

and the equality holds if and only if (X, ω) is holomorphically isometric to $(\mathbb{P}^n, \omega_{FS})$.

Proof. The volume upper bound is a direct consequence of Proposition 4.2. For the equality case, Proposition 4.8 implies that $X \cong \mathbb{P}^n$. In this case the equality $\int_X \omega = (2\pi)^n$ implies that

$$[\omega] = 2\pi c_1(\mathcal{O}_{\mathbb{P}^n}(1)).$$

So $\partial\bar\partial$-lemma gives some $f \in C^\infty(X, \mathbb{R})$ such that

$$\sqrt{-1}\partial\bar\partial f = \mathrm{Ric}\,(\omega) - (n+1)\omega \geq 0,$$

which forces f to be a constant. Thus ω satisfies the Kähler–Einstein equation

$$\mathrm{Ric}\,(\omega) = (n+1)\omega.$$

Now by the uniqueness of KE metrics [1], we obtain $\omega = \omega_{FS}$ up to automorphisms.

□

5. Final remark

Let us recall the following classical result in Cheeger–Colding theory.

Theorem 5.1. [10, Theorem A.1.10]. *There exists $\epsilon(m) > 0$, such that if (M, g) is an m-dimensional Riemannian manifold with $\mathrm{Ric}\,(g) \geq (m-1)g$ and $\mathrm{Vol}(M, g) \geq \mathrm{Vol}(S^m) - \epsilon(m)$, then M is diffeomorphic to S^m.*

So it is reasonable to believe that the following holds in Kähler geometry.

Conjecture 5.2. *There exists $\epsilon(n) > 0$, such that if (X, ω) is an n-dimensional Kähler manifold with $\mathrm{Ric}\,(\omega) \geq (n+1)\omega$ and $\int_X \omega^n \geq (2\pi)^n - \epsilon(n)$, then X is biholomorphic to $\mathbb{P}^n$.*

This conjecture has been verified by Yuchen Liu recently; see [57, Appendix].

References

[1] S. Bando, T. Mabuchi, Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[2] V. Batyrev, Boundedness of the degree of multidimensional toric Fano varieties (in russian with English abstract), Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1982) 22-27, 76-77; English transl. : Moscow Univ. Math. Bull. **37** (1982), 28-33.

[3] R. J. Berman, S. Boucksom, M. Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture, arXiv:1509.04561v2 (2018).

[4] H. Blum, M. Jonsson, Thresholds, valuations, and K-stability, arXiv:1706.04548 (2017).

[5] C.P. Boyer, K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[6] J. Cable, Greatest Lower Bounds on Ricci Curvature for Fano T-manifolds of Complexity 1, Bull. Lond. Math. Soc. **51** (2019), 34-42.

[7] E. Calabi, Métriques Kählériennes et fibrés holomorphes, Ann. scient. Éc. Norm. Sup. 12 (1979), 269–294.

[8] E. Calabi, Extremal Kähler metrics, in Seminars on differential geometry (S. T. Yau, ed.), Annals of Math. Studies, Princeton Univ. Press, 1982, 259-290.

[9] G. Carron, Some old and new results about rigidity of critical metrics. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), 1091-1113.

[10] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below. I. J. Differential Geom. **46** (1997), 406-480.

[11] I. Cheltsov, Y.A. Rubinstein, K. Zhang, Basis log canonical thresholds, local intersection estimates, and asymptotically log del Pezzo surfaces, Selecta Math. (N.S.) 25 (2019), Art. 34, 36 pp.

[12] K. Cho, Y. Miyaoka and N. I. Shepherd-Barron, Characterizations of projective spaces and applications to complex symplectic manifolds, Higher dimensional birational geometry (Kyoto, 1997), 1-88, Adv. Stud. Pure Math. **35**, Math. Soc. Japan, Tokyo, 2002.

[13] O. Debarre, Fano varieties of high degree, arXiv:math/9904036 (1999).

[14] T. de Fernex, A. Küronya and R. Lazarsfeld, Higher cohomology of divisors on a projective variety, Math. Ann. **337** (2007), 443-455.

[15] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa, Asymptotic invariants of base loci. Ann. Inst. Fourier (Grenoble) **56** (2006), 1701-1734.

[16] K. Fujita, Optimal bounds for the volumes of Kähler–Einstein Fano manifolds. Amer. J. Math. **140** (2018), 391-414.

[17] K. Fujita, Y. Odaka, On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2) **70** (2018), 511-521.

[18] T. Fujita, On polarized manifolds whose adjoint bundles are not semipositive. Algebraic geometry, Sendai, 1985, 167-178, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[19] J.P. Gauntlett, D. Martelli, J. Sparks, S.-T.Yau, Obstructions to the Existence of Sasaki–Einstein Metrics, Commun. Math. Phys. **273** (2007), 803-827.

[20] A.D. Hwang, M.A. Singer, A momentum construction for circle-invariant Kähler metrics, Trans. Amer. Math. Soc. **354** (2002), 2285-2325.

[21] J.M. Hwang, On the degrees of Fano four-folds of Picard number 1. J. Reine Angew. Math. **556** (2003), 225-235.

[22] H.-J. Hein, S. Sun, Calabi-Yau manifolds with isolated conical singularities, Publ. Math. IHES **126** (2017), 73-130.

[23] P. Ionescu, Generalized adjunction and applications. Math. Proc. Camb. Philos. Soc. **99** (1986), 457-472.

[24] S. Kebekus, Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron, Complex geometry (Göttingen, 2000), 147-155, Springer, Berlin, 2002.

[25] S. Kobayashi, Principal fibre bundles with the 1-dimensional toroidal group, Tôhoku Math. J. (2) **8** (1956), 29-45.

[26] S. Kobayashi, T. Ochiai, Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ. **13** (1973), 31-47.

[27] N. Koiso, Y. Sakane, Nonhomogeneous Kähler–Einstein metrics on compact complex manifolds. Curvature and topology of Riemannian manifolds, 165-179, Lecture Notes in Math., **1201**, Springer, Berlin, 1986.

[28] J. Kollár, Rational curves on algebraic varieties, Ergeb. Math. Grenzgeb. 3. Folge **32**, Springer Verlag, 1996.

[29] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds. J. Differential Geom. **36** (1992), 765-779.

[30] A. Küronya, V. Lozovanu, Positivity of line bundles and Newton–Okounkov bodies. Doc. Math. **22** (2017), 1285-1302.

[31] R. Lazarsfeld, Positivity in Algebraic Geometry, I,II, Springer, 2004.

[32] R. Lazarsfeld, M. Mustaţă, Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 783-835.

[33] C. Li, Greatest lower bounds on Ricci curvature for toric Fano manifolds, Adv. Math. 226 (2011), 4921–4932.

[34] C. Li, Yau–Tian–Donaldson correspondence for K-semistable Fano manifolds. J. Reine Angew. Math. 733 (2017), 55-85.

[35] C. Li, K-semistability is equivariant volume minimization, Duke Math. J. **166** (2017), 3147-3218.

[36] C. Li, Minimizing normalized volumes of valuations, Math. Zeit. *289* (2018), 491-513.

[37] C. Li, Y. Liu, C. Xu, A guided tour to normalized volumes, arXiv:1806.07112 (2018).

[38] P. Li, J. Wang, Comparison theorem for Kähler manifolds and positivity of spectrum. J. Differential Geom. **69** (2005), 43-74.

[39] C. Li, C. Xu, Stability of Valuations: Higher Rank, arXiv:1707.05561, to appear in Peking Math. J. (2017).

[40] G. Liu, Kähler manifolds with Ricci curvature lower bound. Asian J. Math. **18** (2014), 69-99.

[41] Y. Liu, The volume of singular Kähler–Einstein Fano varieties. Compos. Math. **154** (2018), 1131-1158.

[42] D.B. Martin, C. Spotti, Local models for conical Kähler-Einstein metrics. Proc. Amer. Math. Soc. **147** (2019), 1217-1230.

[43] D.B. Martelli, J. Sparks, S. Yau, Sasaki–Einstein manifolds and volume minimisation, Comm. Math. Phys. **280** (2008), 611-673.

[44] Y.A. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics, Adv. Math. **218** (2008), 1526–1565.

[45] Y.A. Rubinstein, On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow, Trans. Amer. Math. Soc. 361 (2009), 5839–5850.

[46] Y.A. Rubinstein, K. Zhang, Small angle limits of canonical Kähler edge metrics, in preparation.

[47] J. Song, G. Székelyhidi, B. Weinkove, The Kähler-Ricci flow on projective bundles. Int. Math. Res. Not. IMRN 2013, 243-257.

[48] C. Spotti, S. Sun, Explicit Gromov–Hausdorff compactifications of moduli spaces of Kähler–Einstein Fano manifolds. Pure Appl. Math. Q. **13** (2017), 477-515.

[49] J. Song, X. Wang, The greatest Ricci lower bound, conical Einstein metrics and Chern number inequality, Geom. Topol. 20 (2016), 49–102.

[50] J. Sparks, Sasaki–Einstein manifolds. Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, 265-324, Surv. Differ. Geom., **16**, Int. Press, Somerville, MA, 2011.

[51] M. Stenzel, Ricci-flat metrics on the complexification of a compact rank one symmetric space. (English summary) Manuscripta Math. **80** (1993), 151-163.

[52] G. Székelyhidi, Greatest lower bounds on the Ricci curvature of Fano manifolds, Compositio Math. 147 (2011), 319–331.

[53] G. Székelyhidi, An introduction to extremal Kähler metrics. Graduate Studies in Mathematics, **152**. American Mathematical Society, Providence, RI, 2014. xvi+192 pp. ISBN: 978-1-4704-1047-6.

[54] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, Invent. Math. **89** (1987), 225–246.

[55] G. Tian, On stability of the tangent bundles of Fano varieties, Internat. J. Math. 3 (1992), 401–413.

[56] B. Yang, F. Zhang, Hirzebruch manifolds and positive holomorphic sectional curvature, arXiv:1611.06571, (2016).

[57] K. Zhang, On the optimal volume upper bound for Kähler manifolds with positive Ricci curvature, arXiv:2003.06839, (2020).

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