VALUATIVE INVARIANTS WITH HIGHER MOMENTS

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Abstract. In this article we introduce a family of valuative invariants defined in terms of the p-th moment of the expected vanishing order. These invariants lie between α and δ-invariants. They vary continuously in the big cone and semi-continuously in families. Most importantly, they can detect the K-stability of Fano varieties, which generalizes the α and δ-criterions in the literature. They are also related to the d_p-geometry of Darvas.

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1. Introduction

1.1. **Fujita–Odaka invariant.** δ-invariant was introduced by Fujita-Odaka [18] in 2016 to study the K-stability of Fano varieties, which turns out to be a very powerful tool and now there is a large literature on it; see especially the work of Blum–Jonsson [2] and the references therein.

The definition is as follows. Let X be an n-dimensional complex normal projective variety with at worst klt singularities and L is an ample line bundle. Up to a multiple of L, we assume throughout that $H^0(X, mL) \neq 0$ for any $m \in \mathbb{Z}_{>0}$. Put

$$d_m := h^0(X, mL),\ m \in \mathbb{N}.$$

Consider a basis $s_1, \cdots, s_{d_m}$ of the vector space $H^0(X, mL)$, which induces an effective $\mathbb{Q}$-divisor

$$D := \frac{1}{md_m}\sum_{i=1}^{d_m}\Big\{s_i = 0\Big\} \sim_{\mathbb{Q}} L.$$

Any $\mathbb{Q}$-divisor D obtained in this way is called an m-basis type divisor of L. Let

$$\delta_m(L) := \inf\left\{\mathrm{lct}(X, D)\middle| D \text{ is } m\text{-basis type of } L\right\}.$$

Then let

$$\delta(L) = \limsup_m \delta_m(L).$$

This limsup is in fact a limit by [2]. So roughly speaking, $\delta(X, L)$ measures the singularities of basis type divisors of L.

The following result demonstrates the importance of the δ-invariant.

Theorem 1.1 ([18, 2]). *Let X be a $\mathbb{Q}$-Fano variety. The following assertions hold:*

(1) X is K-semistable if and only if $\delta(-K_X) \geqslant 1$;
(2) X is uniformly K-stable if and only if $\delta(-K_X) > 1$.

Thus by [1, 24], δ-invariant serves as a criterion for the existence of KE metrics on $\mathbb{Q}$-Fano varieties.

1.2. **Valuative characterization of the δ-invariant.** Let $\pi : Y \to X$ be a proper birational morphism and let $F \subset Y$ be a prime divisor F in Y. Such an F will be called a divisor *over* X. Let

$$S_m(L,F) := \frac{1}{md_m}\sum_{j=1}^{\tau_m(L,F)} \dim H^0(Y, m\pi^*L - jF)$$

denote the *expected vanishing order of* L *along* F *at level* m. Here

$$\tau_m(L,F) := \sup_{0\neq s\in H^0(X,mL)} \mathrm{ord}_F(s)$$

denotes the *pseudo-effective threshold of* L *along* F *at level* m. Then a basic but important linear algebra lemma due to Fujita–Odaka [18] says that

$$S_m(L,F) = \sup\left\{\mathrm{ord}_F(D)\,\middle|\, m\text{-basis divisor } D \text{ of } L\right\},$$

and this supremum is attained by any m-basis divisor D arising from a basis $\{s_i\}$ that is *compatible* with the filtration

$$H^0\big(Y, m\pi^*L\big) \supset H^0\big(Y, m\pi^*L - F\big) \supset \cdots \supset H^0\big(Y, m\pi^*L - (\tau_m(L,F)+1)F\big) = \{0\},$$

meaning that each $H^0(Y, m\pi^*L - jF)$ is spanned by a subset of the $\{s_i\}_{i=1}^{d_m}$. Then it is easy to deduce that

$$\delta_m(L) = \inf_F \frac{A_X(F)}{S_m(L,F)}.$$

As $m \to \infty$, one has

$$S(L,F) := \lim_{m\to\infty} S_m(L,F) = \frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)} \mathrm{vol}(\pi^*L - xF)dx,$$

which is called the *expected vanishing order of* L *along* F. Then Blum–Jonsson [2] further show that, the limit of $\delta_m(X,L)$ also exists, and is equal to

$$\delta(L) = \inf_F \frac{A_X(F)}{S(L,F)}.$$

Another closely related valuative invariant is Tian's α-invariant [30], which can be defined as

$$\alpha(L) := \inf_F \frac{A_X(F)}{\tau(L,F)},$$

where $\tau(L,F) := \lim \tau_m(L,F)/m$, the *pseudo-effective threshold of* L *along* F.

An important property of $S(L,F)$ is illustrated by the following result of K. Fujita, who shows that $S(L,F)$ can be viewed as the coordinate of the barycenter of certain Newton–Okounkov body along the "F-axis", and hence the well-known Brunn–Minkovski inequality in convex geometry gives the following estimate.

Proposition 1.2 (Barycenter inequality in [16])**.** *For any F over X, one has*

$$\frac{\tau(L,F)}{n+1} \le S(L,F) \le \frac{n\tau(L,F)}{n+1}.$$

One should think of τ and S as the non-Archimedean analogues of the I and $I-J$ functionals of Aubin, and it is shown by Boucksom–Jonsson that the above result holds for general valuations as well (cf. [6, Theorem 5.13]). An immediate consequence of Proposition 1.2 is the following:

$$\frac{n+1}{n}\alpha(L) \le \delta(L) \le (n+1)\alpha(L). \tag{1.1}$$

1.3. **Delta invariant with higher moment.** $S(L,F)$ can be treated as the *first moment* of the vanishing order of L along F. In general for $p\geq 1$, one can also consider the p-th moment of the vanishing order of L along F. More precisely, given a basis $\{s_i\}$ of $H^0(X,mL)$ that is compatible with the filtration induced by F, put

$$S_m^{(p)}(L,F):=\frac{1}{d_m}\sum_{i=1}^{d_m}\left(\frac{\mathrm{ord}_F(s_i)}{m}\right)^p.$$

In the K-stability literature, related L^p notions (for test configurations) have been introduced and studied by Donaldson [12], Dervan [11], Hismoto [21] et. al. Our formulation of $S^{(p)}$ is inspired by the recent work of Han-Li [19], where a sequence of Monge–Ampère energies with higher moments was considered. We will give a more formal definition of this in Section 2. As $m\to\infty$, one has (see Lemma 2.2)

$$S^{(p)}(L,F):=\frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)}px^{p-1}\,\mathrm{vol}(L-xF)dx.$$

So in particular $S(L,F)=S^{(1)}(L,F)$. One main point of this article is to show that, most properties established for S in the literature hold for $S^{(p)}$ as well; see Section 2 for more details.

Extending Fujita's barycenter inequality to the p-th moment, we have

Theorem 1.3. *Given any divisor F over X, one has*

$$\frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(p+n+1)}\tau(L,F)^p\leq S^{(p)}(L,F)\leq\frac{n}{n+p}\tau(L,F)^p.$$

Here $\Gamma(\cdot)$ is the gamma function.

Letting $p\to\infty$, one obtains the following

Corollary 1.4. *One has*

$$\tau(L,F)=\lim_{p\to\infty}S^{(p)}(L,F)^{1/p}.$$

In the pluri-potential viewpoint, $S^{(p)}(L,F)$ has the following interpretation.

Theorem 1.5. *Let φ_t^F be the weak geodesic ray induced by F (cf. Section 6 for the setup and definition) and assume that φ_t^F has C^1 regularity, then*

$$S^{(p)}(L,F)^{1/p}=\frac{d_p(0,\varphi_t^F)}{t}\text{ for all }t\geq 0,$$

where d_p denotes the Finsler metric introduced by Darvas (see (6.5)).

The set of moments $\{S^{(p)}\}$ can also be used to construct various kinds of valuative thresholds for (X,L). It turns out that α and δ are only two special ones. To be more precise, one can put

$$\delta^{(p)}(L):=\inf_F\frac{A_X(F)}{S^{(p)}(L,F)^{1/p}}.\tag{1.2}$$

As we shall see in Proposition 4.4, here one can also take inf over all valuations, which yields the same invariant.

Now the following question arises naturally.

Question 1.1. *What can we say about the $\delta^{(p)}$-invariant?*

First, it is interesting to note that $\{\delta^{(p)}(L)\}_{p\geq 1}$ is a decreasing family of valuative invariants with

$$\delta(L)=\delta^{(1)}(L)\text{ and }\alpha(L)=\lim_{p\to\infty}\delta^{(p)}(L).$$

Moreover, observe that the above valuative formulation of $\delta^{(p)}(L)$ also makes sense when L is merely a big $\mathbb{R}$-line bundle. We shows that the continuity of δ established in [33] holds for $\delta^{(p)}$ as well.

Theorem 1.6. *$\delta^{(p)}(\cdot)$ is a continuous function on the big cone.*

Furthermore, we have the following result, generalizing the work of Blum-Liu [3].

Theorem 1.7. *Let $\pi:X\to T$ is a projective family of varieties and L is a π-ample Cartier divisor on X. Assume that T is normal, X_t is klt for all $t\in T$ and $K_{X/T}$ is $\mathbb{Q}$-Cartier. Then the function*

$$T\ni t\mapsto\delta^{(p)}(X_t,L_t)$$

is lower semi-continuous.

And also, in the Fano setting, it turns out that $\delta^{(p)}$ can be used to detect K-stability.

Theorem 1.8. *Let X be a $\mathbb{Q}$-Fano variety of dimension n. If*

$$\delta^{(p)}(-K_X) > \frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p},$$

then X is uniformly K-stable and hence admits a Kähler–Einstien metric.

For $p=1$ this is simply the δ-criterion of Fujita-Odaka [18] (which says that $\delta(-K_X)>1$ implies uniform K-stability), while for $p=\infty$ this recovers the α-criterion of Tian [30] and Odaka–Sano [25] (which says that $\alpha(-K_X)>\frac{n}{n+1}$ implies uniform K-stability). Thus Theorem 1.8 provides a bridge between α and δ-invariants and gives rise to a family of valuative criterions for the existence of Kähler–Einstein metrics.

To show Theorem 1.8, the key new ingredient is the following monotonicity:

$$\left(\frac{n+p}{n}S^{(p)}(L,F)\right)^{1/p} \text{ is non-decreasing in } p.$$

See Proposition 5.2 and its proof, which involves some measure-theoretic argument.

Finally we can also characterize the borderline case in Theorem 1.8, which generalizes the α-criterion of Fujita [15].

Theorem 1.9. *If an n-dimensional Fano manifold X satisfies that*

$$\delta^{(p)}(-K_X) = \frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p}$$

for some $p\in(1,\infty]$. Then either $X=\mathbb{P}^1$ or X is K-stable. In particular, X admits a Kähler-Einstein metric.

The rest of this article is organized as follows. In Section 2 we introduce $S^{(p)}$ in a more formal way, using filtrations. In Section 3 we prove Theorem 1.6 and in Section 4 we prove Theorem 1.7. Theorem 1.3, Theorem 1.8 and Theorem 1.9 are proved in Section 5. Finally in Section 6 we discuss the relation between $S^{(p)}$ and d_p-geometry and then prove Theorem 1.5.

Acknowledgments. The author would like to thank Mingchen Xia for helpful discussions on test curves and geodesic rays. Special thanks go to Yuchen Liu for valuable comments and for proving Proposition 4.4. He also thanks Yanir Rubinstein for suggesting Theorem 1.9. The author is supported by the China post-doctoral grant BX20190014.

2. Expected vanishing order of higher moments

Let X be a klt projective variety and L an ample line bundle on X. Also fix some $p\geq 1$.

2.1. Divisorial valuations.

The next definition is a natural generalization of the expected vanishing order introduced in [18, 2].

Definition 2.1. *Let $p\in[1,+\infty)$. Given any prime divisor F over X, the p-th moment of the expected vanishing order of L along F at level m is given by*

$$S_m^{(p)}(L,F) := \sup\left\{\frac{1}{d_m}\sum_{i=1}^{d_m}\left(\frac{\mathrm{ord}_F(s_i)}{m}\right)^p \middle| \{s_i\}_{i=1}^{d_m} \text{ is a basis of } H^0(X,mL)\right\}.$$

We also put

$$S^{(p)}(L,F) := \lim_{m\to\infty} S_m^{(p)}(L,F),$$

which is called p-th moment of the expected vanishing order of L along F.

This definition can be reformulated as follows (which in turn justifies the existence of the above limit).

Lemma 2.2. *Given any prime divisor $F\subset Y\xrightarrow{\pi}X$, one has*

$$S_m^{(p)}(L,F) = \frac{1}{d_m}\sum_{j\geq 1}\left(\frac{j}{m}\right)^p\cdot\Big(h^0(m\pi^*L-jF)-h^0(m\pi^*L-(j+1)F)\Big)$$

and

$$S^{(p)}(L,F) = \frac{1}{\mathrm{vol}(L)}\int_0^{+\infty} x^p d(-\mathrm{vol}(L-xF)) = \frac{p}{\mathrm{vol}(L)}\int_0^{+\infty} x^{p-1}\,\mathrm{vol}(L-xF)dx.$$

Proof. For the first statement, we follow the proof of [18, Lemma 2.2]. Given a basis $\{s_i\}$ of $H^0(X, mL)$, for integer $j \geq 0$, let a_j be the number of sections of $\{s_i\}$ that are contained in the subspace $H^0(m\pi^*L - jF)$ when pulled back to Y. Then one has

$$\begin{aligned}\frac{1}{d_m}\sum_{i=1}^{d_m}\left(\frac{\mathrm{ord}_F(s_i)}{m}\right)^p &= \frac{1}{d_m}\sum_{j\geq 1}\left(\frac{j}{m}\right)^p \cdot (a_j - a_{j+1})\\ &= \frac{1}{d_m}\sum_{j\geq 1}\left[\left(\frac{j}{m}\right)^p - \left(\frac{j-1}{m}\right)^p\right]\cdot a_j\\ &\leq \frac{1}{d_m}\sum_{j\geq 1}\left[\left(\frac{j}{m}\right)^p - \left(\frac{j-1}{m}\right)^p\right]\cdot h^0(m\pi^*L - jF)\\ &= \frac{1}{d_m}\sum_{j\geq 1}\left(\frac{j}{m}\right)^p \cdot \Big(h^0(m\pi^*L - jF) - h^0(m\pi^*L - (j+1)F)\Big).\end{aligned}$$

The equality is achieved exactly when $\{s_i\}$ is compatible with the filtration $\{H^0(m\pi^*L - jF)\}_{j\geq 0}$.

The second statement then follows from the theory of filtrated graded linear series and Newton–Okounkov bodies; see e.g. the proof of [7, Lemma 2.7] for an exposition. ☐

Remark 2.3. *Very recently, using $\{S^{(p)}\}$, Han–Li [19] constructed a non-Archimedean analogue of the H-functional of the Kähler–Ricci flow, which were previously studied by Tian–Zhang–Zhang–Zhu [31] and W. He [20]. More precisely, consider*

$$\sum_{k=1}^{\infty}\frac{(-1)^k S^{(k)}(L,F)}{k!} = \frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)} e^{-x}\,\mathrm{vol}(L - xF)dx.$$

Then Han–Li defined

$$H^{\mathrm{NA}}(X,L) := \inf_F\left\{A_X(F) + \log\left(1 - \frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)} e^{-x}\,\mathrm{vol}(L - xF)dx\right)\right\}.$$

As shown by Han–Li [19], this invariant plays significant roles in the study of the Hamilton–Tian conjecture just as the δ-invariant in the Yau–Tian–Donaldson conjecture.

2.2. **Filtrations.** For simplicity we put

$$R := \bigoplus_{m\geq 0} R_m$$

with $R_m := H^0(X, mL)$. We say $\mathcal{F}$ is a filtration of R if for any $\lambda \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{N}$,

(1) $\mathcal{F}^{\lambda'}R_m \supseteq \mathcal{F}^{\lambda}R_m$ for any $\lambda' \leq \lambda$;
(2) $\mathcal{F}^{\lambda}R_m = \bigcap_{\lambda'<\lambda}\mathcal{F}^{\lambda'}R_m$;
(3) $\mathcal{F}^{\lambda_1}R_{m_1}\cdot\mathcal{F}^{\lambda_2}R_{m_2} \subseteq \mathcal{F}^{\lambda_1+\lambda_2}R_{m_1+m_2}$ for any λ_1, λ_2 and $m_1, m_2 \in \mathbb{N}$;
(4) $\mathcal{F}^{\lambda}R_m = R_m$ for $\lambda \leq 0$ and $\mathcal{F}^{\lambda}R_m = \{0\}$ for $\lambda \gg 0$.

We say $\mathcal{F}$ is *linearly bounded* if there exists $C > 0$ such that $\mathcal{F}^{Cm}R_m = \{0\}$ for any $m \in \mathbb{N}$. We say $\mathcal{F}$ is a filtration of R_m if only items (1), (2) and (4) are satisfied.

Following [2], the definition of $S_m^{(p)}$ also extends to filtrations of R_m. More precisely, let $\mathcal{F}$ be a filtration of R_m, the *jumping numbers* of $\mathcal{F}$ are given by

$$0 \leq a_{m,1} \leq a_{m,2} \leq \cdots \leq a_{m,d_m}$$

where

$$a_{m,j} := a_{m,k}(\mathcal{F}) := \inf\{\lambda \in \mathbb{R}_{\geq 0} |\ \mathrm{codim}\mathcal{F}^{\lambda}R_m \geq j\}. \tag{2.1}$$

Then we put (see also [19, (81)])

$$S_m^{(p)}(L,\mathcal{F}) := \frac{1}{d_m}\sum_{j=1}^{d_m}\left(\frac{a_{m,j}}{m}\right)^p \text{ and } T_m(L,\mathcal{F}) := \frac{a_{m,d_m}}{m}.$$

Thus $S_m^{(1)}(L,\mathcal{F}) = S_m(L,\mathcal{F})$ is the rescaled sum of jumping numbers studied in [2].

A filtration is called an $\mathbb{N}$-filtration if all its jumping numbers are integers. For instance the filtration induced by a divisor over X is $\mathbb{N}$-filtration. Given any filtration $\mathcal{F}$ of R_m, its induced $\mathbb{N}$-filtration $\mathcal{F}_{\mathbb{N}}$ is given by

$$\mathcal{F}_{\mathbb{N}}^{\lambda}R_m := \mathcal{F}^{\lceil\lambda\rceil}R_m \text{ for all } \lambda\in\mathbb{R}_{\geq 0}.$$

Then one has

$$a_{m,j}(\mathcal{F}_{\mathbb{N}}) = \lfloor a_{m,j}(\mathcal{F})\rfloor.$$

The next result is a simple generalization of [2, Proposition 2.11]

Proposition 2.4. *If $\mathcal{F}$ is a filtration on R_m, then*

$$S_m^{(p)}(L,\mathcal{F}) \geq S_m^{(p)}(L,\mathcal{F}_{\mathbb{N}}) \geq \begin{cases} S_m^{(p)}(L,\mathcal{F}) - \frac{1}{m},\ p=1,\\ S_m^{(p)}(L,\mathcal{F}) - \frac{p}{m^{p-1}}S_m^{(1)}(L,\mathcal{F}),\ 1<p<2,\\ S_m^{(p)}(L,\mathcal{F}) - \frac{p}{m}S_m^{(p-1)}(L,\mathcal{F}),\ p\geq 2.\end{cases}$$

Proof. This follows from the elementary inequality:

$$(x-1)^p \geq x^p - px^{p-1} \text{ for } p\geq 1 \text{ and } x\geq 1.$$

□

Now as in [2] let us fix a Newton–Okoukov body $\Delta\subset\mathbb{R}^n$ for L and denote the Lebesgue measure on Δ by ρ. Then any filtration $\mathcal{F}$ of $R = R(X,L) = \oplus_{m\geq 0}R_m$ induces a family of graded linear series $V_\bullet^t$ ($t\in\mathbb{R}_{\geq 0}$) and also a concave function G on Δ. More precisely, $V_\bullet^t = \oplus_m V_m^t$ with

$$V_m^t := \mathcal{F}^{tm}R_m,$$

which induces a Newton–Okounkov body $\Delta^t\subset\Delta$ and

$$G(\alpha) := \sup\{t\in\mathbb{R}_{\geq 0}|\alpha\in\Delta^t\}.$$

Then define

$$S^{(p)}(L,\mathcal{F}) := \frac{1}{\operatorname{vol}(L)}\int_0^\infty pt^{p-1}\operatorname{vol}(V_\bullet^t)dt = \frac{1}{\operatorname{vol}(L)}\int_0^\infty t^p d(-\operatorname{vol}(V_\bullet^t)) = \frac{1}{\operatorname{vol}(\Delta)}\int_\Delta G^p d\rho \tag{2.2}$$

and also put

$$T(L,\mathcal{F}) := \lim_{m\to\infty} T_m(L,\mathcal{F}).$$

One then has

$$\frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(p+n+1)}T(L,\mathcal{F})^p \leq S^{(p)}(L,\mathcal{F}) \leq T(L,\mathcal{F})^p.$$

This generalizes [2, Lemma 2.6] (see also the proof of Theorem 1.3). A simple consequence is that

$$T(L,\mathcal{F}) = \lim_{p\to\infty} S^{(p)}(L,\mathcal{F})^{1/p}. \tag{2.3}$$

The next result naturally generalizes Lemma 2.9, Corollary 2.10 and Proposition 2.11 in [2]. Since its proof is largely verbatim, we omit it.

Proposition 2.5. *The following statements hold.*

(1) One has $S^{(p)}(L,\mathcal{F}) = \lim_m S_m^{(p)}(L,\mathcal{F})$.
(2) For every $\varepsilon>0$ there exists $m_0 = m_0(\varepsilon)>0$ such that

$$S_m^{(p)}(L,\mathcal{F}) \leq (1+\varepsilon)S^{(p)}(L,\mathcal{F})$$

for any $m\geq m_0$ and any linearly bounded filtration $\mathcal{F}$ on R.
(3) If $\mathcal{F}$ is a filtration on R, then $S^{(p)}(L,\mathcal{F}_{\mathbb{N}}) = S^{(p)}(L,\mathcal{F})$.

Let Val_X be the set of real valuations on the function field of X that are trivial on the ground field $\mathbb{C}$. Any $v\in\operatorname{Val}_X$ induces a filtration $\mathcal{F}_v$ on R via

$$\mathcal{F}_v^\lambda R_m := \{s\in R_m \mid v(s)\geq t\}$$

for $m\in\mathbb{N}$ and $\lambda\in\mathbb{R}_{\geq 0}$. Then put

$$S_m^{(p)}(L,v) := S_m^{(p)}(L,\mathcal{F}_v) \text{ and } S^{(p)}(L,v) := S^{(p)}(L,\mathcal{F}_v).$$

Note that $\mathcal{F}_v$ is *saturated* in the sence of [14, Definition 4.4]. To be more precise, let

$$\mathfrak{b}(|\mathcal{F}_v^\lambda R_m|) := \operatorname{Im}\Big(\mathcal{F}_v^\lambda R_m\otimes(-mL)\to\mathcal{O}_X\Big)$$

be the base ideal of $\mathcal{F}_v^\lambda R_m$. Let $\overline{\mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)}$ denote its integral closure (i.e., $\overline{\mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)}$ is the set of elements $f \in \mathcal{O}_X$ satisfying a monic equation $f^d + a_1 f^{d-1} + ... + a_d = 0$ with $a_i \in \mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)^i$).

Lemma 2.6. *For any $\lambda \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{N}$, one has*

$$\mathcal{F}_v^\lambda R_m = H^0(X, mL \otimes \mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)) = H^0(X, mL \otimes \overline{\mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)}).$$

Proof. Put $\mathfrak{a}_\lambda(v) := \{f \in \mathcal{O}_X \mid v(f) \geq \lambda\}$. It is easy to see that $\mathfrak{a}_\lambda(v)$ is integrally closed, i.e., $\overline{\mathfrak{a}_\lambda(v)} = \mathfrak{a}_\lambda(v)$. Moreover by definition,

$$\mathcal{F}_v^\lambda R_m = H^0(X, mL \otimes \mathfrak{a}_\lambda(v)).$$

Thus

$$\mathcal{F}_v^\lambda R_m \subset H^0(X, mL \otimes \mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)) \subset H^0(X, mL \otimes \overline{\mathfrak{b}(|\mathcal{F}_v^\lambda R_m|)}) \subset H^0(X, mL \otimes \mathfrak{a}_\lambda(v)) = \mathcal{F}_v^\lambda R_m.$$

So we conclude. □

For $t \in \mathbb{R}_{\geq 0}$ and $l \in \mathbb{N}$, set

$$V_{m,l}^t := H^0(X, mlL \otimes \overline{\mathfrak{b}(|\mathcal{F}_v^{tm} R_m|)^l}).$$

The previous lemma implies that $V_{m,\bullet}^t := \bigoplus_{l \in \mathbb{N}} V_{m,l}^t$ is a subalgebra of $V_\bullet^t$. Put

$$\tilde{S}_m^{(p)}(L, v) := \frac{1}{m^n \operatorname{vol}(L)} \int_0^\infty p t^{p-1} \operatorname{vol}(V_{m,\bullet}^t) dt.$$

As illustrated in [2, Section 5], the graded linear series $V_{m,\bullet}^t$ can effectively approximate $V_\bullet^t$. The argument therein extends to our L^p setting in a straightforward way. So we record the following result, which generlaizes [2, Theorem 5.3], and leave its proof to the interested reader.

Theorem 2.7. *Let X be a normal projective klt variety and L an ample line bundle on X. Then there exists a constant $C = C(X, L)$ such that*

$$0 \leq S^{(p)}(L, v) - \tilde{S}_m^{(p)}(L, v) \leq \left(\frac{CA(v)}{m}\right)^p$$

for all $m \in \mathbb{N}^*$ and all $v \in \mathrm{Val}_X$ with $A_X(v) < \infty$

3. Continuity in the big cone

In this section we assume that X is a klt projective variety and L a big $\mathbb{R}$-line bundle on X (we refer to [22] for the positivity notions of line bundles). As before, fix some $p \geq 1$. Recall that its $\delta^{(p)}$-invariant is given by

$$\delta^{(p)}(L) := \inf_F \frac{A_X(F)}{S^{(p)}(L, F)^{1/p}},$$

where F runs through all the prime divisors over X. The goal is to show Theorem 1.6. The proof is a slightly modified version of the one in [33, Section 4]. For the reader's convenience we give the details.

Lemma 3.1. *There exists ε_0 only depending n and p such that the following holds. For any big $\mathbb{R}$-line bundle L and any $\varepsilon \in (0, \varepsilon_0)$, let L_ε be any small perturbation of L such that*

$$\text{both } (1+\varepsilon)L - L_\varepsilon \text{ and } L_\varepsilon - (1-\varepsilon)L \text{ are big.}$$

Then we have

$$\delta^{(p)}(L + \varepsilon L_\varepsilon) \leq \delta^{(p)}(L) \leq \delta^{(p)}(L - \varepsilon L_\varepsilon).$$

Proof. We only prove $\delta^{(p)}(L + \varepsilon L_\varepsilon) \leq \delta^{(p)}(L)$, since the other part follows in a similar manner. Let F be any prime divisor over X. It suffices to show

$$S^{(p)}(L + \varepsilon L_\varepsilon, F) \geq S^{(p)}(L, F).$$

To this end, we calculate as follows:

$$\begin{aligned} S^{(p)}(L+\varepsilon L_\varepsilon, F) &= \frac{1}{\operatorname{vol}(L+\varepsilon L_\varepsilon)}\int_0^\infty px^{p-1}\operatorname{vol}(L+\varepsilon L_\varepsilon - xF)dx \\ &\geq \frac{1}{\operatorname{vol}(L+(\varepsilon+\varepsilon^2)L)}\int_0^\infty px^{p-1}\operatorname{vol}(L+(\varepsilon-\varepsilon^2)L-xF)dx \\ &= \left(\frac{1+\varepsilon-\varepsilon^2}{1+\varepsilon+\varepsilon^2}\right)^n \cdot S^{(p)}\Big((1+\varepsilon-\varepsilon^2)L, F\Big) \\ &= \left(\frac{1+\varepsilon-\varepsilon^2}{1+\varepsilon+\varepsilon^2}\right)^n \cdot (1+\varepsilon-\varepsilon^2)^p\cdot S^{(p)}(L,F). \end{aligned}$$

By choosing ε small enough we can arrange that

$$\left(\frac{1+\varepsilon-\varepsilon^2}{1+\varepsilon+\varepsilon^2}\right)^n \cdot (1+\varepsilon-\varepsilon^2)^p \geq 1.$$

This completes the proof. □

Proof of Theorem 1.6. Let L be a big $\mathbb{R}$-line bundle. Fix any auxiliary $\mathbb{R}$-line bundle $S\in N^1(X)_{\mathbb{R}}$. We need to show that, for any small $\varepsilon>0$, there exists $\gamma>0$ such that

$$(1-\varepsilon)\delta^{(p)}(L)\leq \delta^{(p)}(L+\gamma S)\leq (1+\varepsilon)\delta^{(p)}(L).$$

Here $L+\gamma S$ is always assumed to be big (by choosing γ sufficiently small). Notice that for any $\varepsilon>0$, we can wirte

$$L+\gamma S = \frac{1}{1+\varepsilon}\Big(L+\varepsilon\big(L+\frac{(1+\varepsilon)\gamma}{\varepsilon}S\big)\Big).$$

Put

$$L_\varepsilon := L+\frac{(1+\varepsilon)\gamma}{\varepsilon}S.$$

Then by choosing γ small enough, we can assume that

$$\text{both } (1+\varepsilon)L-L_\varepsilon \text{ and } L_\varepsilon-(1-\varepsilon)L \text{ are big.}$$

So from the scaling property (easy to verify from the definition):

$$\delta^{(p)}(\lambda L)=\lambda^{-1}\delta^{(p)}(L) \text{ for } \lambda>0$$

and Lemma 3.1, it follows that

$$\delta^{(p)}(L+\gamma S) = (1+\varepsilon)\delta^{(p)}(L+\varepsilon L_\varepsilon)\leq (1+\varepsilon)\delta^{(p)}(L).$$

We can also write

$$L+\gamma S = \frac{1}{1-\varepsilon}\Big(L-\varepsilon\big(L-\frac{(1-\varepsilon)\gamma}{\varepsilon}S\big)\Big).$$

Then a similar treatment as above yields

$$\delta^{(p)}(L+\gamma S)\geq (1-\varepsilon)\delta^{(p)}(L).$$

In conclusion, for any small $\varepsilon>0$, by choosing γ to be sufficiently small, we have

$$(1-\varepsilon)\delta^{(p)}(L)\leq \delta^{(p)}(L+\gamma S)\leq (1+\varepsilon)\delta^{(p)}(L).$$

This completes the proof. □

4. Semi-continuity in families

In this section we prove Theorem 1.7. Let (X,L) be a polarized klt pair. We set

$$\tilde{\delta}^{(p)}(L) := \inf_v \frac{A_X(v)}{S^{(p)}(L,v)^{1/p}} \tag{4.1}$$

where v runs through all the valuations with $A_X(v)<\infty$.

By [2, Theorem C], $\tilde{\delta}^{(p)}=\delta^{(p)}$ for $p=1$ and ∞, the main reason being that both α and δ-invariants can be defined in terms of the log canonical threshold of certain divisors. However to show that $\tilde{\delta}^{(p)}=\delta^{(p)}$ holds for general p is more tricky; see Proposition 4.4. Leaving this issue aside for the moment, we show that the semi-continuity established by Blum–Liu [3, Theorem B] holds for $\tilde{\delta}^{(p)}$ as well.

Theorem 4.1. *Let $\pi : X \to T$ is a projective family of varieties and L is a π-ample Cartier divisor on X. Assume that T is normal, X_t is klt for all $t \in T$ and $K_{X/T}$ is $\mathbb{Q}$-Cartier. Then the function*

$$T \ni t \mapsto \tilde{\delta}^{(p)}(X_t, L_t)$$

is lower semi-continuous.

In what follows we give a sketched proof. To justify that the argument in [3] honestly extends to our setting, we need to spell out the main ingredients used in their proof. First of all, generalizing [3, Proposition 4.10], it is straightforward to obtain that

$$\tilde{\delta}^{(p)}(L) = \inf_{\mathcal{F}} \frac{\mathrm{lct}(X, \mathfrak{b}_\bullet(\mathcal{F}))}{S^{(p)}(L, \mathcal{F})^{1/p}},$$

where $\mathcal{F}$ runs through all non-trivial linearly bounded filtrations of R and $\mathfrak{b}_\bullet(\mathcal{F})$ denote the graded ideal associated to $\mathcal{F}$ (cf. [2, Section 3.6]). Second, we need to introduce a quantized version of $\tilde{\delta}^{(p)}$ by putting

$$\tilde{\delta}^{(p)}_m(L) := \inf_v \frac{A_X(v)}{S^{(p)}_m(L, v)^{1/p}},$$

where v runs through all the valuations with $A_X(v) < \infty$. Then by Proposition 2.5, we have (as in [2, Theorem 4.4])

$$\tilde{\delta}^{(p)}(L) = \lim_{m\to\infty} \tilde{\delta}^{(p)}_m(L). \tag{4.2}$$

Meanwhile, we also need

$$\hat{\tilde{\delta}}^{(p)}_m(L) := \inf_{\mathcal{F}} \frac{\mathrm{lct}(X, \mathfrak{b}_\bullet(\hat{\mathcal{F}}))}{S^{(p)}_m(L, \mathcal{F})^{1/p}}$$

where $\mathcal{F}$ runs through all non-trivial $\mathbb{N}$-filtration of R_m with $T_m(\mathcal{F}) \le 1$ and $\hat{\mathcal{F}}$ denote the filtration of R generated by $\mathcal{F}$ (cf. [3, Definition 3.18]). Then combining Proposition 2.4 with the argument of [3, Proposition 4.17], we derive that

$$\left(\frac{1}{\tilde{\delta}^{(p)}_m(L)}\right)^p - \frac{p}{m}\left(\frac{1}{\alpha(L)}\right)^p \le \left(\frac{1}{\hat{\tilde{\delta}}^{(p)}_m(L)}\right)^p \le \left(\frac{1}{\tilde{\delta}^{(p)}_m(L)}\right)^p. \tag{4.3}$$

Now proceeding as in [3], to conclude Theorem 4.1, it suffices to establish the following two results, which extend Theorem 5.2 and Proposition 6.4 in [3].

Theorem 4.2. *Let $\pi : X \to T$ be a projective $\mathbb{Q}$-Gorenstein family of klt projective varieties over a normal base T and L a π-ample Cartier divisor on X. For each $\varepsilon > 0$ there exits a positive integer $M = M(\varepsilon)$ such that*

$$\hat{\tilde{\delta}}^{(p)}_m(X_t, L_t) - \tilde{\delta}^{(p)}(X_t, L_t) \le \varepsilon$$

for all positive integer m divisible by M and $t \in T$.

Proposition 4.3. *Let $\pi : X \to T$ be a projective $\mathbb{Q}$-Gorenstein family of klt projective varieties over a normal base T and L a π-ample Cartier divisor on X. For $m \gg 0$, the function $T \ni t \mapsto \hat{\tilde{\delta}}^{(p)}_m(X_t, L_t)$ is lower semi-continuous and takes finitely many values.*

To show Theorem 4.2, an intermediate step is to prove (cf. also [3, Proposition 5.16])

$$\tilde{\delta}^{(p)}_m(X_t, L_t) - \tilde{\delta}^{(p)}(X_t, L_t) \le \varepsilon.$$

By the strategy of [3], this can be derived from

$$S^{(p)}(L_t, v) - S^{(p)}_m(L_t, v) \le \varepsilon A_{X_t}(v)^p,$$

which can be proved by generalizing the argument of [3, Theorem 5.13] to our L^p setting (here we need to use Theorem 2.7). Then using (4.3), we conclude Theorem 4.2.

The proof of Proposition 4.3 is a verbatim generalization of [3, Propositon 6.4] so we omit it. Thus by [3, Proposition 6.1] we finish the proof of Theorem 4.1.

Finally, to finish the proof of Theorem 1.7, it suffices to show the following result. The author is grateful to Yuchen Liu for providing the proof.

Proposition 4.4. *One has*

$$\delta^{(p)}(L) = \inf_{F \text{ over } X} \frac{A_X(F)}{S^{(p)}(L, F)^{1/p}} = \inf_{v \in \mathrm{Val}_X} \frac{A_X(v)}{S^{(p)}(L, v)^{1/p}}.$$

Proof. It amounts to proving
$$\tilde{\delta}^{(p)}(L) \geq \delta^{(p)}(L).$$
To show this, it is enough to show that this indeed holds in the following quantized sense:
$$\hat{\tilde{\delta}}_m^{(p)}(L) \geq \inf_F \frac{A_X(F)}{S_m^{(p)}(L,F)^{1/p}}. \tag{4.4}$$
Given this, then one can finish the proof by letting $m \to \infty$ as the left hand side converges to $\tilde{\delta}^{(p)}(L)$ by (4.3) and (4.2) while the right hand converges to $\delta^{(p)}(L)$ by Proposition 2.5 and hence $\tilde{\delta}^{(p)}(L) \geq \delta^{(p)}(L)$ as desired.

To show (4.4), the key point is that, given any $\mathbb{N}$-filtration $\mathcal{F}$ of R_m with $T_m(\mathcal{F}) \leq 1$, the associated graded ideal $\mathfrak{b}_\bullet(\hat{\mathcal{F}})$ is finitely generated (see [3, Lemma 3.20 (2)]). Thus there is a prime divisor F over X computing $\mathrm{lct}(X, \mathfrak{b}_\bullet(\hat{\mathcal{F}}))$. This then yields (as in [3, Lemma 4.16])
$$\frac{\mathrm{lct}(X, \mathfrak{b}_\bullet(\hat{\mathcal{F}}))}{S_m^{(p)}(L,\mathcal{F})^{1/p}} \geq \frac{A_X(F)}{S^{(p)}(L,F)^{1/p}},$$
which finishes the proof. □

Proof of Theorem 1.7. The result follows from Theorem 4.1 and Proposition 4.4. □

Remark 4.5. *Arguing as in* [2, Section 6]*, one can also show that there is always a valuation computing* $\tilde{\delta}^{(p)}(L)$*. Y. Liu proposed a much more challenging question to the author: can one always find a quasi-monomial valuation computing* $\tilde{\delta}^{(p)}(L)$ *(see the work of Blum–Liu–Xu* [4] *proving the case of* δ*-invariant)? After going through* [4]*, it seems to the author that there might be some ingredients missing for the case of* $\delta^{(p)}$*. We leave this problem for future research.*

5. Barycenter inequalities

We prove Theorem 1.3, Theorem 1.8 and Theorem 1.9 in this section. To show Theorem 1.3, the main ingredient is the Brunn–Minkowski inequality for Newton–Okoukov bodies (cf. [23, 13]). While for Theorems 1.8 and 1.9, we also need some measure-theoretic argument from real analysis.

Proof of Theorem 1.3. We follow the argument in [16]. To obtain the lower bound for $S^{(p)}(L,F)$, we use the fact that $\mathrm{vol}(L-xF)^{1/n}$ is a decreasing concave function (cf. [23, Corollary 4.12]), so that
$$\mathrm{vol}(L-xF) \geq \frac{\mathrm{vol}(L)}{\tau^n(L,F)} \cdot \big(\tau(L,F)-x\big)^n$$
and hence
$$\begin{aligned} S^{(p)}(L,F) &= \frac{p}{\mathrm{vol}(L)} \int_0^{\tau(L,F)} x^{p-1}\,\mathrm{vol}(L-xF)dx \\ &\geq \frac{p}{\tau^n(L,F)} \int_0^{\tau(L,F)} x^{p-1}\big(\tau(L,F)-x\big)^n dx \\ &= p\tau(L,F)^p \int_0^1 t^{p-1}(1-t)^n dt \\ &= \frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(p+n+1)}\tau^p(L,F). \end{aligned}$$

To show the upper bound for $S^{(p)}(L,F)$, we assume $n \geq 2$ (the case of $n=1$ is trivial). We use the argument of [16, Proposition 2.1], which shows that there exists a non-negative concave function $f(x)$ for $x \in [0, \tau(L,F)]$ such that
$$f^{n-1}(x)dx = d\bigg(\frac{-\mathrm{vol}(L-xF)}{\mathrm{vol}(L)}\bigg) \text{ for } x \in (0, \tau(L,F)).$$
Thus
$$S^{(p)}(L,F) = \int_0^{\tau(L,F)} x^p f^{n-1}(x)dx.$$

For simplicity set $b := S^{(p)}(L,F)$. Note that $f(x) > 0$ for $x \in (0, \tau(L,F))$. Then $b \in (0, \tau(L,F)^p)$ and by the concavity of $f(x)$, we have

$$\begin{cases} f(x) \geq \frac{f(b)}{b}x, \ x \in [0, b^{1/p}], \\ f(x) \leq \frac{f(b)}{b}x, \ x \in [b^{1/p}, \tau(L,F)]. \end{cases}$$

Thus we have

$$0 = \int_0^{\tau(L,F)} (x^p - b) f^{n-1}(x)dx \geq \int_0^{\tau(L,F)} (x^p - b)\left(\frac{f(b)}{b}x\right)^{n-1} dx = \left(\frac{f(b)}{b}\right)^{n-1} \cdot \left(\frac{\tau(L,F)^{p+n}}{p+n} - b\frac{\tau^n(L,F)}{n}\right),$$

which implies that

$$b \leq \frac{n}{n+p}\tau(L,F)^p,$$

as desired. $\square$

An immediate consequence is the following, which generalizes (1.1).

Corollary 5.1. *One has* $\left(\frac{n+p}{n}\right)^{1/p}\alpha(L) \leq \delta^{(p)}(L) \leq \left(\frac{\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)}\right)^{1/p}\alpha(L)$.

Now we turn to the proof of Theorem 1.8, which can be deduced from the following result.

Proposition 5.2. *Let F be any prime divisor over X. Set*

$$H(p) := \left(\frac{n+p}{n}S^{(p)}(L,F)\right)^{1/p} \ \textit{for } p \geq 1.$$

Then $H(p)$ is non-decreasing in p.

Proof. When $n = 1$, $H(p)$ is a constant, so there is nothing to prove. Then as in the previous proof, we assume $n \geq 2$ and write

$$S^{(p)}(L,F) = \int_0^{\tau(L,F)} x^p f^{n-1}(x)dx$$

for some non-negative concave function $f(x)$ defined on $[0, \tau(L,F)]$. For simplicity set $\tau := \tau(L,F)$. Then it amounts to proving that

$$H(p) = \left(\frac{n+p}{n}\int_0^{\tau} x^p f^{n-1}(x)dx\right)^{1/p}$$

is non-decreasing in p for any non-negative concave function $f(x)$ defined on $[0,\tau]$ with the normalization condition

$$\int_0^{\tau} f^{n-1}(x)dx = 1.$$

To this end, we introduce an auxiliary function:

$$g(x) := \frac{f(x)}{x}, \ x \in (0, \tau].$$

By concavity of $f(x)$, $g(x)$ is differentiable almost everywhere, non-negative and decreasing for $x \in (0,\tau]$. For $s > n-1$, we put

$$K(s) := s\int_0^{\tau} x^{s-1} g^{n-1}(x)dx.$$

Then one has (using $\int_0^{\tau} f^{n-1}(x)dx = 1$)

$$H(p) = \left(\frac{K(n+p)}{K(n)}\right)^{1/p}.$$

To show this is non-decreasing in p, it then suffices to show that

$$K(s) \text{ is log convex for } s > n-1.$$

To see this, for each small $\varepsilon > 0$, using integration by parts for Lebesgue–Stieltjes integration, we have

$$s\int_{\varepsilon}^{\tau} x^{s-1} g^{n-1}(x)dx = \int_{\varepsilon}^{\tau} g^{n-1}(x)dx^s = \int_{\varepsilon}^{\tau} x^s d(-g^{n-1}(x)) + g^{n-1}(\tau) - \varepsilon^s g^{n-1}(\varepsilon).$$

Here we used the fact that, as a monotonic function, g^{n-1} has bounded variation on $[\varepsilon,\tau]$ and hence the measure $d(-g^{n-1}(x))$ is well-defined on (ε,τ). Now observing

$$\lim_{\varepsilon\to 0^-} \varepsilon^s g^{n-1}(\varepsilon) = \lim_{\varepsilon\to 0^-} \varepsilon^{s+1-n} f(\varepsilon) = 0 \text{ for } s > n-1,$$

we derive that

$$K(s)=\int_0^\tau x^s d(-g^{n-1}(x))+g^{n-1}(\tau),$$

where $d(-g^{n-1}(x))$ is understood as a measure on $(0,\tau)$. Set

$$M(s):=\int_0^\tau x^s d(-g^{n-1}(x)) \text{ for } s>n-1,$$

then clearly $M''\geq 0$. Now applying Hölder's inequality to the measure space $\big((0,\tau),d(-g^{n-1}(x))\big)$, we derive that

$$\left(\int_0^\tau \log x\cdot x^s d(-g^{n-1}(x))\right)^2\leq\left(\int_0^\tau (\log x)^2 x^s d(-g^{n-1}(x))\right)\left(\int_0^\tau x^s d(-g^{n-1}(x))\right), \tag{5.1}$$

namely

$$M''M-(M')^2\geq 0.$$

This implies that

$$K''K-(K')^2=M''M-(M')^2+M''g^{n-1}(\tau)\geq 0,$$

so that $K(s)$ is log convex, as desired. □

Remark 5.3. *We also believe that*

$$\left(\frac{\Gamma(n+p+1)}{\Gamma(n+1)\Gamma(p+1)}S^{(p)}(L,F)\right)^{1/p} \text{ is non-increasing in } p.$$

However it seems to the author that the proof of this is much more difficult, which may involve the log concavity of the generalized beta function. If this is indeed true, it then follows that

$$\delta^{(p)}(L)\geq\frac{1}{n+1}\left(\frac{\Gamma(n+p+1)}{\Gamma(n+1)\Gamma(p+1)}\right)^{1/p}\delta(L),$$

which hence generalizes the inequality $\alpha(L)\geq\frac{1}{n+1}\delta(L)$. Meanwhile it will also follow that $\delta^{(p)}(L)$ is continuous in p. We leave this problem to the interested readers.

Proof of Theorem 1.8. Proposition 5.2 implies that, for any F over X and $p\geq 1$,

$$S^{(p)}(L,F)^{1/p}\geq\frac{n+1}{n}\left(\frac{n}{n+p}\right)^{1/p}S(L,F). \tag{5.2}$$

Thus

$$\delta^{(p)}(L)=\inf_F\frac{A_X(F)}{S^{(p)}(L,F)^{1/p}}\leq\frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p}\inf_F\frac{A_X(F)}{S(L,F)}=\frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p}\delta(L).$$

Now in the Fano setting (when $L=-K_X$), we finish the proof by invoking Theorem 1.1 and [1, 24]. □

Finally, we prove Theorem 1.9. This boils down to a carefully analysis on the equality case in the proof Proposition 5.2.

Proof of Theorem 1.9. The case of $p=\infty$ is exactly [15, Theorem 1.2]. So we assume $p\in(1,\infty)$. We follow the strategy of Fujita [15]. Note that $X=\mathbb{P}^1$ clearly satisfies the claimed statement, so assume that $n\geq 2$ and that X is not K-stable with $\delta(-K_X)=1$. Then there exists a dreamy divisor F over X such that

$$A_X(F)=S(-K_X,F).$$

Moreover this F has to achieve the equality in (5.2), namely

$$S^{(p)}(L,F)^{1/p}=\frac{n+1}{n}\left(\frac{n}{n+p}\right)^{1/p}S(L,F) \text{ for some } p>1.$$

Using the notion in the proof of Proposition 5.2, this reads

$$\frac{n+1}{n}\int_0^\tau xf^{n-1}(x)dx=\left(\frac{n+p}{n}\int_0^\tau x^pf^{n-1}(x)dx\right)^{1/p}.$$

Namely,

$$\frac{K(n+1)}{K(n)}=\left(\frac{K(n+p)}{K(n)}\right)^{1/p}.$$

By the log convexity of $K(s)$, this forces that

$$\log K(s) \text{ is affine for } s \in [n, n+p].$$

In particular

$$K''K - (K')^2 = M''M - (M')^2 + M''g^{n-1}(\tau) = 0 \text{ for } s \in [n, n+p].$$

This further forces that

$$M''M - (M')^2 = M''g^{n-1}(\tau) = 0 \text{ for } s \in [n, n+p].$$

In particular, the Hölder inequality (5.1) is an equality, which implies that there exist real numbers $\alpha, \beta \geq 0$, not both of them zero, such that

$$\alpha|\log x| = \beta x^{s/2} \text{ holds for } x \in (0,\tau)\backslash U$$

for some set $U \subset (0,\tau)$ with

$$\int_U d(-g^{n-1}(x)) = 0.$$

This implies that $d(-g^{n-1}(x))$ is a zero measure away from finitely many points in $(0,\tau)$. Therefore

$$g(x) \text{ is a step function.}$$

Now recall that $f(x) = xg(x)$ is concave and hence continuous on $(0,\tau)$. Thus

$$f(x) = Cx \text{ for some constant } C > 0.$$

The normalization condition $\int_0^\tau f^{n-1}(x)dx$ further implies that

$$C^{n-1} = \frac{n}{\tau^n}.$$

Thus

$$A_X(F) = S(-K_X, F) = \frac{n}{\tau^n}\int_0^\tau x^n dx = \frac{n}{n+1}\tau(-K_X, F).$$

Then by [15, Theorem 4.1], $X \cong \mathbb{P}^n$. Now let H be a hyperplane in $\mathbb{P}^n$, straightforward calculation then yields

$$\frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p} = \delta^{(p)}(-K_X) \leq \frac{A_X(H)}{S^{(p)}(-K_X,H)^{1/p}} = \frac{1}{n+1}\cdot\left(\frac{\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)}\right)^{1/p}.$$

This will give us a contradiction. Indeed, consider the function

$$h(x) := x\log n - \sum_{i=1}^{n-1}\log(\frac{x+i}{i}) \text{ for } x \geq 1.$$

Observe that

$$h(1) = \log n - \log n = 0.$$

Moreover, for $x \geq 1$,

$$\begin{aligned} h'(x) &= \log n - \sum_{i=1}^{n-1}\frac{1}{x+i} \\ &\geq \log n - \sum_{i=2}^{n}\frac{1}{i} > \log n - \int_1^n \frac{1}{x}dx = 0. \end{aligned}$$

Here we used $n \geq 2$. Thus

$$h(p) = p\log n - \sum_{i=1}^{n-1}\log\frac{p+i}{i} > 0$$

since we assumed $p > 1$. From this we derive that (recall $x\Gamma(x) = \Gamma(x+1)$)

$$\begin{aligned} \frac{1}{n+1}\cdot\left(\frac{\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)}\right)^{1/p} &= \frac{1}{n+1}\cdot\left(\prod_{i=1}^{n}\frac{p+i}{i}\right)^{1/p} \\ &< \frac{n}{n+1}\left(\frac{n+p}{n}\right)^{1/p}, \end{aligned}$$

which is a contradiction. So we conclude. $\square$

6. Relating to the d_p-geometry of weak geodesic rays

We prove Theorem 1.5 in this section. In fact we will carry out the discussion in a more general fashion using filtrations instead of divisorial valuations. To some extent this section is expository, as most results appearing below should be well-known to experts.

Our setup is as follows. Let (X, L) be a polarized Kähler manifold. We fix a smooth Hermitian metric h on L such that $\omega := -dd^c \log h \in c_1(L)$ defines a Kähler form (here $dd^c := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$). As before put $R := \bigoplus_{m\geq 0} R_m$ with $R_m := H^0(X, mL)$. Let $\mathcal{F}$ be a linearly bounded filtration of R.

Definition 6.1. [28, Section 7] *For any $\tau \in \mathbb{R}$ and $x \in X$, put*

$$\psi^{\mathcal{F}}_{\tau,m}(x) := \sup\left\{\frac{1}{m}\log|s|^2_{h^m}(x)\middle|s \in \mathcal{F}^{\tau m}R_m,\ \sup|s|^2_{h^m} \leq 1\right\}$$

and

$$\psi^{\mathcal{F}}_{\tau} := \left(\lim_{m\to+\infty}\psi_{\tau,m}\right)^*,$$

where $*$ denotes the upper semi-continuous regularization. We call $\psi^{\mathcal{F}}_{\tau}$ the test curve *induced by $\mathcal{F}$.*

Note that $\psi^{\mathcal{F}}_{\tau}$ is non-increasing and concave in τ (since $\mathcal{F}$ is decreasing and multiplicative).

Theorem 6.2. [28, Corollary 7.12] *Consider the Legendre transform*

$$\varphi^{\mathcal{F}}_t := \left(\sup_{\tau\in\mathbb{R}}\{\psi^{\mathcal{F}}_{\tau} + t\tau\}\right)^* \text{ for } t \geq 0. \tag{6.1}$$

Then $\varphi^{\mathcal{F}}_t \in \mathrm{PSH}(X, \omega)$ is a weak geodesic ray emanating from 0.

We call $\varphi^{\mathcal{F}}_t$ the weak geodesic ray induced by $\mathcal{F}$. A priori, the regularity of $\varphi^{\mathcal{F}}_t$ is rather weak. But when $\mathcal{F}$ is a filtration induced by some test configuration (in the sense of [32]), $\varphi^{\mathcal{F}}_t$ has $C^{1,1}$ regularity in t and x variables by [28, Theorem 9.2] and [8, Theorem 1.2] (see also [27, Theorem 1.3]).

Remark 6.3. *When $\mathcal{F} = \mathcal{F}_v$ for some $v \in \mathrm{Val}_X$, we put $\varphi^v_t := \varphi^{\mathcal{F}_v}_t$. When $v = \mathrm{ord}_F$ for some prime divisor F over X, we also write $\varphi^F_t := \varphi^{\mathrm{ord}_F}_t$. This explains the notation in Theorem 1.5.*

Note that $\varphi^{\mathcal{F}}_t(x)$ is convex in t. Dually, one further has

$$\psi^{\mathcal{F}}_{\tau} = \inf_{t\geq 0}\{\varphi^{\mathcal{F}}_t - t\tau\}. \tag{6.2}$$

See [28] for the proof.

An equivalent way of producing the geodesic ray $\varphi^{\mathcal{F}}_t$ is by quantization approach. More precisely, for $m \geq 1$, let $\{a_{m,i}\}_{1\leq i\leq d_m}$ be the set of jumping numbers of $\mathcal{F}$ (recall (2.1)). Now consider the Hermitian inner product

$$H_m := \int_X h^m(\cdot,\cdot)\omega^n$$

on R_m. By elementary linear algebra one can find an H_m-orthonormal basis $\{s_i\}_{1\leq i\leq d_m}$ of R_m such that

$$s_i \in \mathcal{F}^{a_{m,i}}R_m \text{ for each } 1 \leq i \leq d_m.$$

Now set

$$\varphi^{\mathcal{F}}_{t,m} := \frac{1}{m}\sum_{i=1}^{d_m} e^{a_{m,i}t}|s_i|^2_{h^m},\ t \geq 0. \tag{6.3}$$

One can easily verify that $\varphi^{\mathcal{F}}_{t,m}$ does not depend on the choice of $\{s_i\}$. We call $\varphi^{\mathcal{F}}_{t,m}$ the *Bergman geodesic ray induced by $\mathcal{F}$*. Such geodesics goes back to the work of Phone–Sturm [26] and is used to construct geodesic rays in the space of Kähler potentials by approximation. The above Bergman geodesic ray has also been utilized in the recent work [29] to study quantized δ-invariants.

Theorem 6.4. [28, Theorem 9.2] *One has*

$$\varphi^{\mathcal{F}}_t = \lim_{m\to+\infty}\left[\sup_{k\geq m}\varphi^{\mathcal{F}}_{t,k}\right]^*.$$

We remark that although [28, Theorem 9.2] is only stated for filtrations induced from test configurations, one can easily verify that the argument therein carries over to our setting.

The main contribution of this section is the following result, which generalizes Theorem 1.5. Note that when $p=1$, this should be well-known to experts and as we shall see, for general $p \geq 1$, this follows from the work of Hisamoto [21].

Theorem 6.5. *Assume that $\varphi_t^{\mathcal{F}}$ has C^1 regularity in t and x variables. Then one has*

$$S^{(p)}(L,\mathcal{F})^{1/p} = \frac{d_p(0,\varphi_t^{\mathcal{F}})}{t} \text{ for all } t \geq 0,$$

where d_p denotes the Finsler metric introduced by Darvas (see (6.5)).

As we have mentioned, the regularity assumption in the above result automatically holds when $\mathcal{F}$ is induced by a test configuration. In what follows we give the proof. Note that by the regularity of $\varphi_t^{\mathcal{F}}$, the time derivative $\dot{\varphi}_t^F(x)$ is well-defined for any point $x \in X$ and $t \geq 0$.

Lemma 6.6. *For each point $x \in X$ and $t \geq 0$, $\dot{\varphi}_t^F(x) \geq 0$.*

Proof. Note that $\dot{\varphi}_t^F(x)$ is non-decreasing in t, so it suffices to show that

$$\dot{\varphi}_0^{\mathcal{F}}(x) \geq 0.$$

This follows readily from the construction (see also [21, Lemma 4.2]). Indeed, by $\mathcal{F}^0 R_m = R_m$ and the standard Tian–Yau–Zelditch expansion, one clearly has

$$\psi_0^{\mathcal{F}}(x) = \varphi_0^{\mathcal{F}}(x) = 0.$$

Thus by (6.2),

$$\dot{\varphi}_0^{\mathcal{F}}(x) = \lim_{t\to 0^+} \frac{\varphi_t^{\mathcal{F}}(x)}{t} \geq \lim_{t\to 0^+} \frac{\psi_0^{\mathcal{F}}(x)}{t} = 0,$$

as desired. □

The non-negativity of $\dot{\varphi}_t^{\mathcal{F}}$ makes it convenient to consider the following L^p Finsler speed of the geodesic; see the survey of Darvas [9] for more information on this subject. Set for $p \geq 1$

$$||\dot{\varphi}_t^{\mathcal{F}}||_p := \left(\frac{1}{V}\int_X (\dot{\varphi}_t^{\mathcal{F}})^p(\omega + dd^c\varphi_t^{\mathcal{F}})^n\right)^{1/p}. \tag{6.4}$$

Here $V := \mathrm{vol}(L) = \int_X \omega^n$ and $(\omega + dd^c\varphi_t^{\mathcal{F}})^n$ is understood as the *non-pluripolar* Monge–Ampère measure defined in [5]. As shown in [9], (6.4) is independent of t. The d_p-distance from 0 to $\varphi_t^{\mathcal{F}}$ is then given by

$$d_p(0,\varphi_t^{\mathcal{F}}) := \int_0^t ||\dot{\varphi}_s^{\mathcal{F}}||_p ds. \tag{6.5}$$

Proposition 6.7. *One has*

$$S^{(p)}(L,\mathcal{F})^{1/p} = ||\dot{\varphi}_t^{\mathcal{F}}||_p \text{ for any } t \geq 0.$$

Thus $S^{(p)}(L,\mathcal{F})^{1/p}$ is exactly equal to the L^p speed of the geodesic ray induced by $\mathcal{F}$ itself.

Proof. We recall the main result of Hisamoto [21], who dealt with filtrations that are induced from test configurations. After extending his argument to our setting (for general filtrations but with additional regularity assumption on $\varphi_t^{\mathcal{F}}$), we derive that for each $t \geq 0$

$$\frac{n!}{m^n}\sum_{i=1}^{d_m} \delta_{\frac{a_{m,i}}{m}} \xrightarrow{m\to+\infty} (\dot{\varphi}_t^{\mathcal{F}})_*(\omega + dd^c\varphi_t^{\mathcal{F}})^n$$

as measures on $\mathbb{R}$. Thus we obtain that

$$\begin{aligned} S^{(p)}(L,\mathcal{F}) &= \lim_{m\to+\infty} S_m^{(p)}(L,\mathcal{F}) \\ &= \lim_{m\to+\infty} \frac{1}{d_m}\sum_{i=1}^{d_m}\left(\frac{a_{m,i}}{m}\right)^p \\ &= \frac{1}{\mathrm{vol}(L)}\int_{\mathbb{R}} x^p (\dot{\varphi}_t^{\mathcal{F}})_*(\omega + dd^c\varphi_t^{\mathcal{F}})^n \\ &= \frac{1}{V}\int_X (\dot{\varphi}_t^{\mathcal{F}})^p(\omega + dd^c\varphi_t^{\mathcal{F}})^n. \end{aligned}$$

This completes the proof. □

Proof of Theorem 6.5. A direct consequence of the previous proposition and (6.5). □

Under the assumtion of Theorem 6.5, letting $p \to +\infty$ and using (2.3), one deduces that

$$T(L, \mathcal{F}) = \sup_X \dot{\varphi}_t^{\mathcal{F}}.$$

One can also take $p = 1$. Then Theorem 6.5 immediately implies that

$$S(L, \mathcal{F}) = \frac{E(\varphi_t^{\mathcal{F}})}{t} \text{ for all } t \geq 0, \tag{6.6}$$

where $E(\cdot)$ denotes the Monge–Ampère energy defined by

$$E(\varphi) := \frac{1}{(n+1)V} \int_X \sum_{i=0}^n \omega^{n-i} \wedge (\omega + dd^c\varphi)^i, \ \varphi \in \mathcal{H}_\omega.$$

This energy is known to be linear along $\varphi_t^{\mathcal{F}}$ and hence (6.6) follows from the variation formula of $E(\cdot)$.

Now assume that F is a prime divisor over X such that φ_t^F has C^1 regularity (this holds for instance when F is *dreamy* in the sense of [17], which then induces a *special* test configuration). Then Proposition 5.2 implies that

$$\left(\frac{n+p}{n}\right)^{1/p} \frac{d_p(0, \varphi_t^F)}{t} \text{ is non-decreasing in } p,$$

Taking $p = 1$ and $p = +\infty$, we obtain that

$$\frac{n+1}{n} \cdot \frac{E(\varphi_t^F)}{t} \leq \sup_X \dot{\varphi}_t^F \text{ for all } t \geq 0.$$

On the other hand, by Proposition 1.2 we have

$$(n+1) \cdot \frac{E(\varphi_t^F)}{t} \geq \sup_X \dot{\varphi}_t^F \text{for all } t \geq 0.$$

These inequalities are in general not valid for arbitrary weak geodesic rays. Thus geodesic rays constructed from algebraic data may enjoy additional properties. See also [10] for related discussions in this direction.

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