THE δ-INVARIANTS OF PROJECTIVE BUNDLES

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Abstract. We compute the δ-invariants of projective bundles of Fano type with the help of Calabi ansatz.

1. Introduction

Given an arbitrary Fano manifold X, it is often the case that X does not admit any Kähler–Einstein (KE) metric. But still, X could admit twisted KE or conical KE metrics. To study these metrics and their degenerations, some analytic and algebraic thresholds play important roles. For instance, the greatest Ricci lower bound $\beta(X)$ of Tian [19] measures how far X is away from being a KE manifold. As shown in [1, 5], $\beta(X)$ is equal to the algebraic δ-invariant $\delta(X)$, which serves as the right threshold for X to be twisted K-semistable (cf. [7, 4]).

More precisely, suppose that X does not admit KE metrics. For any $\mu \in (0, \beta(X))$, we can find a Kähler form $\omega \in 2\pi c_1(X)$ such that $\mathrm{Ric}(\omega) \geq \mu\omega$. An interesting problem would be to study the Gromov–Hausdorff limit of (X, ω) as $\mu \to \beta(X)$. By [15], the limit is homeomorphic to a $\mathbb{Q}$-Fano variety, which is supposed to be the optimal degeneration of X in a suitable sense. To study this problem, it could be enlightening if we have some explicit examples to play with. We refer the reader to [17, 16, 13] for some discussions in this direction. The purpose of this note is to generalize the construction in [17, Section 3.1] to higher dimensions. More precisely, we will use the Calabi symmetry of projective bundles to explicitly construct a family of Kähler metrics with Ricci curvature as positive as possible, with the aid of which we can compute the δ-invariants of such manifolds.

To state the main result, let us fix the notation that will be used throughout. Let X be an n-dimensional Fano manifold with Fano index $I(X) \geq 2$. So we can find an ample line bundle L such that

$$L = -\lambda K_X \text{ for some } \lambda \in (0,1). \tag{1.1}$$

We put

$$Y := \mathbb{P}(L^{-1} \oplus \mathcal{O}_X) \xrightarrow{\pi} X.$$

Let E_0 denote the zero section and E_∞ the infinity section. Then

$$-K_Y = \pi^*(-K_X) + E_0 + E_\infty \sim_{\mathbb{Q}} (1/\lambda + 1)E_\infty - (1/\lambda - 1)E_0$$

is ample and hence Y is an $(n+1)$-dimensional Fano manifold. We put

$$\beta_0 := \left(\frac{n+1}{n+2} \cdot \frac{(1/\lambda+1)^{n+2} - (1/\lambda-1)^{n+2}}{(1/\lambda+1)^{n+1} - (1/\lambda-1)^{n+1}} - (1/\lambda - 1) \right)^{-1}. \tag{1.2}$$

Using binomial formula, one can easily verify the following elementary fact:

$$\beta_0 \in (1/2, 1). \tag{1.3}$$

The main result is the following

Theorem 1.1. *One has*

$$\beta(Y)=\delta(Y)=\min\left\{\frac{\delta(X)\beta_0}{\lambda+\beta_0(1-\lambda)},\beta_0\right\}.$$

In particular, Y cannot admit KE metrics. But as we shall see in Section 2, Y does admit a family of twisted conical KE metrics provided by the Calabi ansatz. When $\delta(X)\geq\lambda+\beta_0(1-\lambda)$ (this holds for example when X is K-semistable), then we deduce that

$$\delta(Y)=\beta_0. \tag{1.4}$$

As we shall see, in this case E_0 computes $\delta(Y)$. This generalizes the example $Y=Bl_1\mathbb{P}^2$ treated in [17]. Indeed, when $Y=Bl_1\mathbb{P}^2$, one has $X=\mathbb{P}^1$, $n=1$ and $\lambda=1/2$, so that $\delta(Y)=\beta_0=6/7$, which agrees with the result obtained in [17, 12]. In the case of $\delta(X)\leq\lambda+\beta_0(1-\lambda)$, Theorem 1.1 gives

$$\delta(Y)=\frac{\delta(X)\beta_0}{\lambda+\beta_0(1-\lambda)}. \tag{1.5}$$

In this case, there always exists a prime divisor F over X computing $\delta(X)$ (see [4, Theorem 6.7]). This divisor naturally induces a divisor $\overline{F}$ over Y, and we will show that $\delta(Y)$ is computed by $\overline{F}$ when (1.5) takes place. See Section 5 for an explicit example.

Remark 1.2. *In [20], Zhuang derived the δ-invariants of product spaces. In particular, let $Y=X\times\mathbb{P}^1$ be the trivial $\mathbb{P}^1$-bundle over X, then*

$$\delta(Y)=\min\{\delta(X),1\}.$$

So to some extent, Theorem 1.1 generalizes this product formula.

The proof of Theorem 1.1 essentially makes use of the natural $\mathbb{C}^*$-action on Y. On the analytic side, this toruc action allows us to carry out the momentum construction due to Calabi, from which we will derive a lower bound of $\delta(Y)$. On the algebraic side, by using this torus action and the definition of δ-invariant, we show in Section 3 that the obtained lower bound also bounds $\delta(Y)$ from above, which hence finishes the proof. In Section 4 we provide several useful properties of Y, which will be applied in Section 5 to investigate some concrete examples.

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2. The lower bound

To derive the lower bound for $\delta(Y)$, we follow the approach in [17, Section 3.1], using Calabi ansatz to construct a family of Kähler metrics $\eta\in 2\pi c_1(Y)$ with Ricci curvature as positive as possible. Similar treatment also appears in [14, Section 3.2].

We fix

$$\mu\in(0,\beta(X))$$

and choose Kähler forms $\omega,\alpha\in 2\pi c_1(X)$ such that

$$\mathrm{Ric}\,(\omega)=\mu\omega+(1-\mu)\alpha. \tag{2.1}$$

Then the momentum construction due to Calabi can provide Kähler metrics η on Y of the form (in special local coordinates)

$$\eta = \lambda\tau\pi^*\omega + \varphi\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2},$$

whose Ricci forms are given by

$$\begin{aligned}\mathrm{Ric}\,(\eta) &= \left(\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi'\right)\pi^*\omega + (1-\mu)\pi^*\alpha \\ &\quad - \varphi\left(n\frac{\varphi}{\tau} + \varphi'\right)'\frac{\sqrt{-1}dw\wedge d\overline{w}}{|w|^2}.\end{aligned} \tag{2.2}$$

Here $\varphi = \varphi(\tau)$ with $\tau \in (1/\lambda - 1, 1/\lambda + 1)$ is a one-variable positive function to be determined and w denotes the fiberwise coordinate. To cook up $\eta \in 2\pi c_1(Y)$ with $\mathrm{Ric}\,(\eta) \geq \beta\eta$ (possibly in the current sense), we will impose the following conditions for φ:

$$\begin{cases}\varphi(1/\lambda - 1) = \varphi(1/\lambda + 1) = 0, \\ \varphi'(1/\lambda - 1) \in (0, 1], \\ \varphi'(1/\lambda + 1) \in [-1, 0),\end{cases} \tag{2.3}$$

and

$$-\left(n\frac{\varphi}{\tau} + \varphi'\right)' = \beta \text{ for } \tau \in (1/\lambda - 1, 1/\lambda + 1), \tag{2.4}$$

where β is any constant that satisfies

$$0 < \beta \leq \min\left\{\frac{\mu\beta_0}{\lambda + \beta_0(1-\lambda)}, \beta_0\right\}. \tag{2.5}$$

Let us explain the exact meanings of these conditions. The boundary condition (2.3) makes sure that $\eta \in 2\pi c_1(Y)$ and η possibly possesses certain amount of edge singularities along E_0 and E_∞. Solving the ODE (2.4), we obtain that

$$\tau^n\varphi = -\frac{\beta}{n+2}\tau^{n+2} + A\tau^{n+1} + B \tag{2.6}$$

where

$$\begin{cases}A = \frac{\beta}{n+2}\cdot\frac{(1/\lambda+1)^{n+2} - (1/\lambda-1)^{n+2}}{(1/\lambda+1)^{n+1} - (1/\lambda-1)^{n+1}}, \\ B = \frac{-2\beta}{n+2}\cdot\frac{(1/\lambda^2-1)^{n+1}}{(1/\lambda+1)^{n+1} - (1/\lambda-1)^{n+1}}.\end{cases}$$

From this, we easily derive that

$$\begin{cases}\beta_1 := \varphi'(1/\lambda - 1) = \frac{\beta}{\beta_0}, \\ \beta_2 := -\varphi'(1/\lambda + 1) = \frac{\beta(2\beta_0 - 1)}{\beta_0}.\end{cases} \tag{2.7}$$

Then (1.3) and (2.5) simply imply that

$$0 < \beta_2 < \beta_1 \leq 1.$$

So η has edge singularities with angles β_1 and β_2 along E_0 and E_∞ respectively. Moreover (2.5) also guarantees that

$$\begin{aligned}\mu - n\lambda\frac{\varphi}{\tau} - \lambda\varphi' &= \mu - \lambda\beta_1 - \beta(1 - \lambda - \tau) \\ &= (\mu - \lambda\beta/\beta_0 - \beta(1-\lambda)) + \tau\beta \\ &\geq \tau\beta.\end{aligned} \tag{2.8}$$

Therefore η satisfies $\mathrm{Ric}\,(\eta) \geq \beta\eta$ in the current sense. More precisely, η solves the following twisted Kähler–Einstein edge equation:

$$\begin{aligned}\mathrm{Ric}\,(\eta) = \beta\eta &+ \big(\mu - \lambda\beta/\beta_0 - \beta(1-\lambda)\big)\pi^*\omega + (1-\mu)\pi^*\alpha \\ &+ 2\pi(1-\beta/\beta_0)[E_0] + 2\pi\big(1-\beta(2\beta_0-1)/\beta_0\big)[E_\infty].\end{aligned} \tag{2.9}$$

This implies that (using [5, Theorem 5.7] and [1, Theorem C])

$$\beta(Y) = \delta(Y) \geq \delta_\theta(Y) \geq \beta_\theta(Y) \geq \beta, \tag{2.10}$$

where

$$\theta = \frac{(\mu - \lambda\beta/\beta_0 - \beta(1-\lambda))}{2\pi}\pi^*\omega + \frac{1-\mu}{2\pi}\pi^*\alpha + (1-\beta_1)[E_0] + (1-\beta_2)[E_\infty]$$

is a semi-positive current in $(1-\beta)c_1(Y)$. Using (2.5) and letting $\mu \to \beta(X)$, we obtain

$$\beta(Y) \geq \min\left\{\frac{\beta(X)\beta_0}{\lambda + \beta_0(1-\lambda)}, \beta_0\right\}.$$

Finally, applying [5, Theorem 5.7], we get the following

Proposition 2.1. *One has*

$$\delta(Y) \geq \min\left\{\frac{\delta(X)\beta_0}{\lambda + \beta_0(1-\lambda)}, \beta_0\right\}.$$

Remark 2.2. *Suppose that X admits a KE metric $\omega_{KE} \in 2\pi c_1(X)$, then as in* [17]*, for any $\beta \in (0,\beta_0)$ we can construct a smooth Kähler form $\omega_\beta \in 2\pi c_1(Y)$ with* $\mathrm{Ric}\,(\omega_\beta) > \beta\omega_\beta$*, and as $\beta \to \beta_0$, one has*

$$(Y, \omega_\beta) \xrightarrow{G.H.} (Y, \eta),$$

with η solving

$$\mathrm{Ric}\,(\eta) = \beta_0\eta + (1-\lambda-\beta_0(1-\lambda))\pi^*\omega_{KE} + 2\pi(2-2\beta_0)[E_\infty].$$

This generalizes [17, Section 3.1]*, where an η satisfying*

$$\mathrm{Ric}\,(\eta) = \frac{6}{7}\eta + \frac{1}{7}\pi^*\omega_{FS} + 2\pi(1-\frac{5}{7})[E_\infty]$$

was constructed on $Bl_1\mathbb{P}^2$.

Remark 2.3. *It is worth mentioning that, Calabi ansatz also applies to projective bundles of higher ranks (see* [10] *for more general discussions).*

3. The upper bound

As we have seen, both

$$\beta_0 \text{ and } \frac{\delta(X)\beta_0}{\lambda + \beta_0(1-\lambda)}$$

arise naturally from Calabi's ODE. In this section, by using the definition of δ-invariant (cf. [9, 3]), we shall show that they also have purely algebraic interpretations and that they naturally bound $\delta(Y)$ from *above*, which hence completes the proof of Theorem 1.1.

We begin with the following simple lemma, which justifies the appearance of β_0.

Lemma 3.1. *One has*

$$\delta(Y) \leq \frac{A_Y(E_0)}{S_{-K_Y}(E_0)} = \beta_0.$$

Proof. This follows from a straightforward calculation. Indeed, one has $A_Y(E_0)=1$ and

$$
\begin{aligned}
S_{-K_Y}(E_0) &= \frac{1}{(-K_Y)^{n+1}}\int_0^\infty \mathrm{Vol}(-K_Y-tE_0)dt\\
&= \frac{1}{(-K_Y)^{n+1}}\int_0^2 \Big((1/\lambda+1)E_\infty-(t+1/\lambda-1)E_0\Big)^{n+1}dt\\
&= \frac{2(1/\lambda+1)^{n+1}-\big((1/\lambda+1)^{n+2}-(1/\lambda-1)^{n+2}\big)/(n+1)}{(1/\lambda+1)^{n+2}-(1/\lambda-1)^{n+2}}\\
&= \frac{n+1}{n+2}\cdot\frac{(1/\lambda+1)^{n+2}-(1/\lambda-1)^{n+2}}{(1/\lambda+1)^{n+1}-(1/\lambda-1)^{n+1}}-(1/\lambda-1).
\end{aligned}
$$

So the result follows. $\square$

A combination of Proposition 2.1 and Lemma 3.1 gives the following consequence.

Corollary 3.2. *Suppose that*

$$\delta(X)\geq\lambda+\beta_0(1-\lambda),$$

then one has

$$\delta(Y)=\beta_0$$

and $\delta(Y)$ is computed by the divisor $E_0\subseteq Y$.

Now let us give an algebraic explanation for the quantity

$$\frac{\delta(X)\beta_0}{\lambda+\beta_0(1-\lambda)}.$$

For any prime divisor F over X, we put

$$\delta_X(F):=\frac{A_X(F)}{S_{-K_X}(F)}. \tag{3.1}$$

Let $\overline{X}\xrightarrow{\phi}X$ be a log resolution of X such that $F\subseteq\overline{X}$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
\overline{Y} & \xrightarrow{\overline{\phi}} & Y\\
{\scriptstyle\overline{\pi}}\downarrow & & \downarrow{\scriptstyle\pi}\\
\overline{X} & \xrightarrow{\phi} & X
\end{array}
$$

where

$$\overline{Y}:=\mathbb{P}\big(\phi^*(L^{-1}\oplus\mathcal{O}_X)\big).$$

Set

$$\overline{F}:=\overline{\pi}^*F$$

and

$$\delta_Y(\overline{F}):=\frac{A_Y(\overline{F})}{S_{-K_Y}(\overline{F})}.$$

Then it is easy to check that

$$A_Y(\overline{F})=A_X(F). \tag{3.2}$$

Proposition 3.3. *For any prime divisor F over X, we have*

$$\delta_Y(\overline{F})=\frac{\delta_X(F)\beta_0}{\lambda+\beta_0(1-\lambda)}.$$

So by taking inf over all F, we get

Corollary 3.4. *We have*

$$\delta(Y) \leq \frac{\delta(X)\beta_0}{\lambda + \beta_0(1-\lambda)}. \tag{3.3}$$

Combining this with Proposition 2.1 and Lemma 3.1, Theorem 1.1 follows immediately.

To prove Proposition 3.3, we use the fact that Y is a T-variety, where $T = \mathbb{C}^*$ acts multiplicatively on $\mathbb{P}^1$-fibers. So for any $m \geq 1$, we have a weight decomposition:

$$R_m := H^0(Y, -mK_Y) = \bigoplus_{j \in \mathbb{Z}} R_m^j, \tag{3.4}$$

where

$$R_m^j := \{s \in H^0(Y, -mK_Y) \mid \tau \cdot s = \tau^j s,\ \tau \in T\}.$$

More precisely, each R_m^j consists of those sections that vanish along E_0 with order j, i.e.,

$$R_m^j := \{s \in H^0(Y, -mK_Y) \mid \operatorname{ord}_{E_0} s = j\}.$$

One can easily compute the dimension of each R_m^j. Indeed, we write

$$-K_X = IH, \tag{3.5}$$

where $I := I(X)$ is the Fano index of X and H is an ample line bundle on X. Then for any $j \in \mathbb{Z}$ we can write

$$-mK_Y \sim \big(jI\lambda + mI(1-\lambda)\big)\pi^* H + jE_0 + (2m-j)E_\infty. \tag{3.6}$$

Moreover any T-invariant divisor in $|-mK_Y|$ can be written in this form. So we deduce that

$$\dim_{\mathbb{C}} R_m^j = \begin{cases} h^0\big(X, \big(jI\lambda + mI(1-\lambda)\big)H\big), & 0 \leq j \leq 2m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Now given a prime divisor F over X, let us construct an m-basis type divisor $\mathcal{D}_m$ that is compatible with the filtration on R_m induced by $\operatorname{ord}_{\overline{F}}$. Note that, for each $j \in \{1, ..., 2m\}$, ord_F induces a filtration of R_m^j, from which we can choose a compatible basis $\{s_i^j\}$ with $i \in \{1, ..., \dim_{\mathbb{C}} R_m^j\}$. Let D_i^j be the divisor cut out by s_i^j and we put

$$\mathcal{D}_m := \frac{1}{m \sum_{k=0}^{2m} \dim R_m^k} \sum_{j=0}^{2m} \sum_{i=1}^{\dim R_m^j} \Big(\pi^* D_i^j + jE_0 + (2m-j)E_\infty\Big). \tag{3.8}$$

Then $\mathcal{D}_m \sim_{\mathbb{Q}} -K_Y$ is an m-basis type divisor that is compatible with the filtration induced by $\operatorname{ord}_{\overline{F}}$. In particular, by the proof of [9, Lemma 2.2],

$$\lim_{m \to \infty} \operatorname{ord}_{\overline{F}}(\mathcal{D}_m) = S_{-K_Y}(\overline{F}). \tag{3.9}$$

Lemma 3.5. *We also have*

$$\lim_{m \to \infty} \operatorname{ord}_{\overline{F}}(\mathcal{D}_m) = \frac{\lambda + \beta_0(1-\lambda)}{\beta_0} S_{-K_X}(F).$$

Here $\frac{\lambda+\beta_0(1-\lambda)}{\beta_0} = \frac{n+1}{n+2} \cdot \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}}$.

Proof. Note that

$$\mathrm{ord}_{\overline{F}}(\mathcal{D}_m) = \mathrm{ord}_F\left(\frac{\sum_{j=0}^{2m}\sum_{i=1}^{\dim R_m^j} D_i^j}{m\sum_{k=0}^{2m}\dim R_m^k}\right)$$

Moreover we have the following three asymptotic calculations.

(1) For each j, the chosen basis $\{s_i^j\}$ of R_m^j is adapted to ord_F, so by [9, Lemma 2.2], we have

$$\lim_{m\to\infty}\mathrm{ord}_F\left(\frac{\sum_{i=1}^{\dim R_m^j} D_i^j}{\big(jI\lambda + mI(1-\lambda)\big)\dim R_m^j}\right) = S_H(F) = \frac{S_{-K_X}(F)}{I}.$$

This convergence is uniform for all j.

(2) One has

$$\begin{aligned}
\frac{\sum_{j=0}^{2m} jI\lambda\dim R_m^j}{m^{n+2}/n!} &= \sum_{j=0}^{2m}\frac{jI\lambda}{m}\cdot\frac{h^0\big(X, m\big(jI\lambda/m + I(1-\lambda)H\big)\big)}{m^n/n!}\cdot\frac{1}{m}\\
&\xrightarrow{m\to\infty}\int_0^{2I\lambda} x\mathrm{Vol}\big((x+I(1-\lambda))H\big)dx\\
&= H^nI^{n+2}\int_0^{2\lambda} t\big(t+(1-\lambda)\big)^n dt\\
&= \frac{H^nI^{n+2}}{n+1}\left(2\lambda(1+\lambda)^{n+1} - \frac{(1+\lambda)^{n+2}-(1-\lambda)^{n+2}}{n+2}\right).
\end{aligned}$$

(3) One has

$$\begin{aligned}
\frac{\sum_{j=0}^{2m}\dim R_m^j}{m^{n+1}/n!} &= \sum_{j=0}^{2m}\frac{h^0\big(X, m\big(jI\lambda/m + I(1-\lambda)H\big)\big)}{m^n/n!}\cdot\frac{1}{m}\\
&\xrightarrow{m\to\infty}\int_0^{2I\lambda}\mathrm{Vol}\big((x+I(1-\lambda))H\big)dx\\
&= H^nI^{n+1}\int_0^{2\lambda}\big(t+(1-\lambda)\big)^n dt\\
&= \frac{H^nI^{n+1}}{n+1}\Big((1+\lambda)^{n+1}-(1-\lambda)^{n+1}\Big).
\end{aligned}$$

Putting all these together, for $m \gg 1$,

$$\mathrm{ord}_{\overline{F}}(\mathcal{D}_m) = \frac{n+1}{n+2}\cdot\frac{(1+\lambda)^{n+2}-(1-\lambda)^{n+2}}{(1+\lambda)^{n+1}-(1-\lambda)^{n+1}}\cdot S_{-K_X}(F) + \epsilon_m,$$

where $\epsilon_m \to 0$ as $m \to \infty$. So the assertion follows. $\square$

Proof of Proposition 3.3. By (3.2), (3.9) and Lemma 3.5, we have

$$\begin{aligned}\delta_Y(\overline{F}) &= \frac{A_Y(\overline{F})}{S_{-K_Y}(\overline{F})}\\ &= \lim_{m\to\infty}\frac{A_Y(\overline{F})}{\mathrm{ord}_{\overline{F}}(\mathcal{D}_m)}\\ &= \frac{A_X(F)\beta_0}{\big(\lambda+\beta_0(1-\lambda)\big)S_{-K_X}(F)}\\ &= \frac{\delta_X(F)\beta_0}{\big(\lambda+\beta_0(1-\lambda)\big)}.\end{aligned}$$

□

So Theorem 1.1 is proved. Proposition 3.3 also implies that, in the case when

$$\delta(Y)=\frac{\delta(X)\beta_0}{\lambda+\beta_0(1-\lambda)},$$

$\delta(Y)$ is computed by some $\overline{F}$, where F is a divisor over X that computes $\delta(X)$ (cf. [4, Theorem 6.7]).

4. More discussions

The purpose of this section is to include some properties of the projective bundle Y, which can be used to explicitly calculate $\delta_Y(\overline{F})$ in some special cases. Let $F\subseteq X$ be a prime divisor. We define the nef threshold of F to be

$$\epsilon_X(F):=\sup\{t>0\mid -K_X-tF \text{ is nef}\}. \tag{4.1}$$

The pseudo-effective threshold of F is defined as

$$\tau_X(F):=\sup\{t>0\mid -K_X-tF \text{ is big}\}. \tag{4.2}$$

Put

$$\overline{F}:=\pi^*F.$$

One can define $\epsilon_Y(\overline{F})$ and $\tau_Y(\overline{F})$ analogously on Y as well.

Lemma 4.1. *One has*

$$\epsilon_Y(\overline{F})=(1-\lambda)\epsilon_X(F).$$

Proof. We write

$$-K_Y-t\overline{F}\sim_{\mathbb{R}}\pi^*(-K_X-tF)+E_0+E_\infty.$$

Let $C\nsubseteq E_0$ be any curve, then for any $t\in(0,\epsilon_X(F)]$, one clearly has

$$(-K_Y-t\overline{F})\cdot C\geq 0.$$

Now consider $C\subseteq E_0$. Then by projection formula,

$$(-K_Y-t\overline{F})\cdot C=(-(1-\lambda)K_X-tF)\cdot\pi_*C.$$

Thus $-K_Y-t\overline{F}$ is nef if and only if

$$t\in(0,(1-\lambda)\epsilon_X(F)].$$

□

Lemma 4.2. *For any $\mathbb{R}$-divisor $D\subseteq X$, we have*

$$\mathrm{Vol}(\pi^*D)=0.$$

Proof. If not, then π^*D is big so there exists an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor B on Y such that $\pi^*D \sim_{\mathbb{R}} A+B$. Then for any generic $\mathbb{P}^1$-fiber $f\subseteq Y$, one has $0=\pi^*D\cdot f=(A+B)\cdot f>0$, which is a contradiction. $\square$

Lemma 4.3. *Let $D\subseteq X$ be an $\mathbb{R}$-divisor that is not big. Then*

$$\mathrm{Vol}(\pi^*D+aE_0)=0 \text{ for any } a\geq 0.$$

Proof. We make use of the restricted volume. Thinking of E_0 as a copy of X sitting inside Y, then for any $a\geq 0$, one has

$$(\pi^*D+aE_0)|_{E_0}=D-aL,$$

which is thus not big. Let

$$b:=\sup\{a\geq 0 \mid \mathrm{Vol}(\pi^*D+aE_0)=0\}.$$

So it amounts to showing that $b=+\infty$. Assume to the contrary that $b<+\infty$. Put

$$f(t):=\mathrm{Vol}(\pi^*D+bE_0+tE_0),\ t\in[0,\infty).$$

By the previous lemma, $f(0)=0$. And $f(t)$ is a non-decreasing positive C^1 function when $t\in(0,\infty)$ by [2, Theorem A]. Moreover, for any $t>0$, one has

$$\frac{d}{dt}f(t)=n\mathrm{Vol}_{Y|E_0}(\pi^*D+(b+t)E_0)\leq n\mathrm{Vol}(X,D-(b+t)L)=0.$$

This implies that $f(t)=f(0)=0$ for any $t>0$, a contradiction. $\square$

Lemma 4.4. *One has*

$$\tau_Y(\overline{F})=(1+\lambda)\tau_X(F).$$

Proof. We write

$$-K_Y-t\overline{F}\sim_{\mathbb{R}}\pi^*\big(-(1+\lambda)K_X-tF\big)+2E_0.$$

Thus $-K_Y-t\overline{F}$ is linearly equivalent to a pseudo-effective $\mathbb{R}$-divisors for $t\in\big[0,(1+\lambda)\tau_X(F)\big]$. Moreover, for any $t\geq(1+\lambda)\tau_X(F)$, $-(1+\lambda)K_X-tF$ is not big, so $\mathrm{Vol}(-K_Y-t\overline{F})=0$ by the previous lemma. The assertion follows. $\square$

By slightly modifying the argument of Lemma 4.3, the following is clear.

Lemma 4.5. *Assume that $B\subseteq Y$ is an $\mathbb{R}$-divisor that is not big when restricted to E_0. Then*

$$\mathrm{Vol}(B+aE_0)=\mathrm{Vol}(B) \text{ for any } a\geq 0.$$

The next result if of course covered by Proposition 3.3, but we shall give an alternative computational proof, which will be useful in Section 5.

Proposition 4.6. *Assume that $F\subseteq X$ is a prime divisor with $\epsilon_X(F)=\tau_X(F)$, then one has*

$$\delta_Y(\overline{F})=\frac{\delta_X(F)\beta_0)}{\lambda+\beta_0(1-\lambda)}.$$

Proof. We write

$$\epsilon:=\epsilon_X(F)$$

to ease notation. For $t \in [0, (1-\lambda)\epsilon]$, we have

$$\begin{aligned}\mathrm{Vol}(-K_Y - \tau\overline{F}) &= \Big((1/\lambda+1)E_\infty - (1/\lambda-1)E_0 - t\overline{F}\Big)^{n+1} \\ &= \sum_{i=0}^{n} C_{n+1}^i(-t)^i\Big((1/\lambda+1)^{n+1-i} - (1/\lambda-1)^{n+1-i}\Big)L^{n-i}\cdot F^i.\end{aligned}$$

For $t \in [(1-\lambda)\epsilon, (1+\lambda)\epsilon]$, applying Lemma 4.5, we have

$$\mathrm{Vol}(-K_Y - t\overline{F}) = \mathrm{Vol}\Big(-K_Y - t\overline{F} - \big(\frac{t}{\epsilon\lambda} - (1/\lambda-1)\big)E_0\Big).$$

Note that

$$-K_Y - t\overline{F} - \big(\frac{t}{\epsilon\lambda} - (1/\lambda-1)\big)E_0 \sim_{\mathbb{R}} \pi^*\big(-(1+\lambda)K_X - tF\big) + \big(1/\lambda + 1 - \frac{t}{\epsilon\lambda}\big)E_0$$

is clearly nef for $t \in [(1-\lambda)\epsilon, (1+\lambda)\epsilon]$ (it suffices to check curves contained in E_0), so we get that

$$\begin{aligned}\mathrm{Vol}(-K_Y - t\overline{F}) &= \Big(-K_Y - t\overline{F} - \big(\frac{t}{\epsilon\lambda} - (1/\lambda-1)\big)E_0\Big)^{n+1} \\ &= \Big((1/\lambda+1)E_\infty - \frac{t}{\epsilon\lambda}E_0 - t\overline{F}\Big)^{n+1} \\ &= \sum_{i=0}^{n} C_{n+1}^i(-t)^i\Big((1/\lambda+1)^{n+1-i} - \big(\frac{t}{\epsilon\lambda}\big)^{n+1-i}\Big)L^{n-i}\cdot F^i.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\infty \mathrm{Vol}(-K_Y - t\overline{F}) &= \sum_{i=0}^{n}\int_0^{(1-\lambda)\epsilon} C_{n+1}^i(-t)^i\Big((1/\lambda+1)^{n+1-i} - (1/\lambda-1)^{n+1-i}\Big)L^{n-i}\cdot F^i dt \\ &\quad + \sum_{i=0}^{n}\int_{(1-\lambda)\epsilon}^{(1+\lambda)\epsilon} C_{n+1}^i(-t)^i\Big((1/\lambda+1)^{n+1-i} - \big(\frac{t}{\epsilon\lambda}\big)^{n+1-i}\Big)L^{n-i}\cdot F^i dt \\ &= \sum_{i=0}^{n} C_{n+1}^i\frac{(-1)^i\epsilon^{i+1}\big((1+\lambda)^{n+2} - (1-\lambda)^{n+2}\big)}{(i+1)\lambda^{n+1-i}}L^{n-i}\cdot F^i \\ &\quad - \sum_{i=0}^{n} C_{n+1}^i\frac{(-1)^i\epsilon^{i+1}\big((1+\lambda)^{n+2} - (1-\lambda)^{n+2}\big)}{(n+2)\lambda^{n+1-i}}L^{n-i}\cdot F^i.\end{aligned}$$

Thus

$$\begin{aligned}S_{-K_Y}(\overline{F}) &= \frac{1}{(-K_Y)^{n+1}}\int_0^\infty \mathrm{Vol}(-K_Y - t\overline{F})dt \\ &= \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}}\sum_{i=0}^{n} C_{n+1}^i(-\lambda)^i\epsilon^{i+1}\frac{L^{n-i}\cdot L^i}{(i+1)L^n}\frac{n+1-i}{n+2} \\ &= \frac{n+1}{n+2}\cdot\frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}}\sum_{i=0}^{n} C_n^i\frac{(-\lambda)^i\epsilon^{i+1}L^{n-i}\cdot F^i}{(i+1)L^n}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned} S_{-K_X}(F) &= \frac{1}{(-K_X)^n}\int_0^{\epsilon}\mathrm{Vol}(-K_X-tF) \\ &= \frac{1}{L^n}\sum_{i=0}^{n}\int_0^{\epsilon} C_n^i(-\lambda t)^i L^{n-i}\cdot F^i dt \\ &= \sum_{i=0}^{n} C_n^i \frac{(-\lambda)^i\epsilon^{i+1}L^{n-i}\cdot F^i}{(i+1)L^n}. \end{aligned}$$

Thus we arrive at

$$\begin{aligned} S_{-K_Y}(\overline{F}) &= \frac{n+1}{n+2}\cdot\frac{(1+\lambda)^{n+2}-(1-\lambda)^{n+2}}{(1+\lambda)^{n+1}-(1-\lambda)^{n+1}}\cdot S_{-K_X}(F) \\ &= \lambda\big(1/\beta_0+(1/\lambda-1)\big)S_{-K_X}(F), \end{aligned}$$

so that

$$\delta_Y(\overline{F}) = \frac{\delta_X(F)\beta_0}{\lambda+\beta_0(1-\lambda)}.$$

□

5. Example

In this section we give an example such that

$$\delta(Y) = \frac{\delta(X)\beta_0}{\lambda+\beta_0(1-\lambda)}.$$

To search for such examples, we need to work in high dimensions. In the literature, explicit calculations for $\int_0^{\infty}\mathrm{Vol}(L-tF)dt$ have been carried out many times in dimension 2 and 3 (see e.g., [5, 6, 8]). Note that in these cases, the computation is relatively simple, mainly due to the fact that there is no small contraction in dimension 2 or 3, and one only needs to get rid of those divisors that is contained in the non-nef locus of $L-tF$. However, in higher dimensions, the non-nef locus could have large codimension, which makes the computation more subtle. In fact, as shown in [8, Section 8], one needs to run certain MMP to do the computation. In this section we take the opportunity to illustrate how this can be done in dimension 4.

Let $X = Bl_1\mathbb{P}^3$. Note that X itself is a $\mathbb{P}^1$-bundle. Let F_0 be the exceptional divisor and F_∞ be the pull back of a general hyperplane in $\mathbb{P}^3$. Then $-K_X = 4F_\infty - 2F_0$. Simple calculation shows that $\epsilon_X(F_0)=\tau_X(F_0)=2$, and by Corollary 3.2, we have

$$\delta(X) = \delta_X(F_0) = \frac{14}{17}.$$

We take $L = 2F_\infty - F_0$ and $Y = \mathbb{P}(L^{-1}\oplus\mathcal{O}_X)$, with E_0 and E_∞ being the zero and infinity sections respectively. Then we have $n=3$, $\lambda = 1/2$, so that $\beta_0 = 50/71$. Therefore

$$\frac{\delta_X(F_0)\beta_0}{\lambda+\beta_0(1-\lambda)} = \frac{1400}{2057}.$$

So by Theorem 1.1,

$$\delta(Y) = \min\left\{\frac{1400}{2057},\frac{50}{71}\right\} = \frac{1400}{2057}.$$

Put $\overline{F_0} := \pi^* F_0$. Let us explicitly verify that $\overline{F_0}$ computes $\delta(Y)$. Indeed, $\epsilon_Y(\overline{F_0}) = 1$ and $\tau_Y(\overline{F_0}) = 3$. And we have (by the proof of Proposition 4.6)

$$\mathrm{Vol}(-K_Y - t\overline{F_0}) = \begin{cases} (3E_\infty - E_0 - t\overline{F_0})^4 = 560 - 104t - 48t^2 - 8t^3, & t \in [0,1], \\ (3E_\infty - tE_0 - t\overline{F_0})^4 = 567 - 108t - 54t^2 - 12t^3 + 7t^4, & t \in [1,3]. \end{cases}$$

From this we obtain that

$$\begin{aligned} S_{-K_Y}(\overline{F_0}) &= \frac{1}{560}\int_0^1 (560 - 104t - 48t^2 - 8t^3)dt \\ &\quad + \frac{1}{560}\int_1^3 (567 - 108t - 54t^2 - 12t^3 + 7t^4)dt \\ &= \frac{2057}{1400}. \end{aligned}$$

Therefore

$$\delta_Y(\overline{F_0}) = \frac{1400}{2057}.$$

So we do have the equality:

$$\delta(Y) = \delta_Y(\overline{F_0}) = \frac{\delta_X(F)\beta_0}{\lambda + \beta_0(1-\lambda)} = \frac{1400}{2057}.$$

Now choose a prime divisor $H \in |F_\infty - F_0|$. Then we have $\epsilon_X(H) = 2$ and $\tau_X(H) = 4$. Moreover $\delta_X(H) = 14/15$ and

$$\frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1-\lambda)} = \frac{280}{363}.$$

Consider $\overline{H} := \pi^* H$. Then $\epsilon_Y(\overline{H}) = 1$ and $\tau_Y(\overline{H}) = 6$. In the following we verify that

$$\delta_Y(\overline{H}) = \frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1-\lambda)}.$$

Of course this holds true by Proposition 3.3, but we would like to prove this by directly computing the integrand $\mathrm{Vol}(-K_Y - t\overline{H})$ for $t \in [0,6]$, which requires some interesting tools that might be useful in other context.

- For $t \in [0,1]$, as $-K_Y - t\overline{H}$ is nef, we have
$$\begin{aligned} \mathrm{Vol}(-K_Y - t\overline{H}) &= (3E_\infty - E_0 - t\overline{H})^4 \\ &= 560 - 312t + 48t^2. \end{aligned}$$
- For $t \in [1,2]$, we write
$$-K_Y - t\overline{H} \sim_{\mathbb{R}} (6-t)\overline{F_\infty} - (3-t)\overline{F_0} + 2E_0,$$
Its non-nef locus is $S := E_0 \cap \overline{F_0}$, which is a copy of $\mathbb{P}^2$ sitting inside Y and whose normal bundle is ismorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. The numerical class of curves in S generates an extremal ray in $\overline{NE}(Y)$. Let $Y \xrightarrow{\alpha} Z$ be the contraction of this ray and let $Y^+ \xrightarrow{\alpha^+} Z$ be the flip of α. Then by [8, Section 8], Y^+ is the ample model of $-K_Y - t\overline{H}$ for $t \in (1,2)$.

$$\begin{tikzcd} Y \arrow[rr, dashed, "\phi"] \arrow[dr, "\alpha"'] & & Y^+ \arrow[dl, "\alpha^+"] \\ & Z & \end{tikzcd}$$

Note that Y^+ can be explicitly constructed as follows (cf. [11]): blow up the non-nef locus S, then we will get an exceptional divisor that is isomorphic to $\mathbb{P}^2\times\mathbb{P}^1$, whose normal bundle is ismorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(-1)$; contracting this divisor in the other direction, we get Y^+, which is a smooth projective 4-fold. For any effective divisor D on Y, let D^+ denote its strict transform on Y^+. Then for $t\in[1,2]$, straightforward computation gives

$$\begin{aligned}\mathrm{Vol}(-K_Y-t\overline{H})&=(-K_{Y^+}-t\overline{H}^+)^4\\&=559-308t+42t^2+4t^3-t^4.\end{aligned}$$

- Let $t\in[2,3]$. Thinking of E_0 as a copy of $Bl_1\mathbb{P}^3$, then for any point $p\in E_0$, there exists a curve $C\subseteq E_0$ pass through p such that

$$(-K_Y-t\overline{H})\cdot C=\big(-(t-2)F_\infty+(t-1)F_0\big)\cdot\pi_*C<0.$$

So E_0 is contained in the non-nef locus of $-K_Y-t\overline{H}$. Subtracting certain amount of E_0, we derive that (one can also directly apply Lemma 4.5 here), for $t\geq 2$,

$$\mathrm{Vol}(-K_Y-t\overline{H})=\mathrm{Vol}\big(-K_Y-t\overline{H}-(t/2-1)E_0\big).$$

Note that

$$-K_Y-t\overline{H}-(t/2-1)E_0\sim_{\mathbb{R}}(6-t)\overline{F_\infty}-(3-t)\overline{F_0}+(3-t/2)E_0,$$

whose non-nef locus is again $S=E_0\cap\overline{F_0}$. Thus for $t\in[2,3]$ we have

$$\begin{aligned}\mathrm{Vol}(-K_Y-t\overline{H})&=\mathrm{Vol}(-K_Y-t\overline{H}-(t/2-1)E_0)\\&=\big(-K_{Y^+}-t\overline{H}^+-(t/2-1)E_0^+\big)^4\\&=567-324t+54t^2-t^4/2.\end{aligned}$$

- For $t\in[3,6]$, write

$$-K_Y-t\overline{H}-(t/2-1)E_0\sim_{\mathbb{R}}(6-t)\overline{F_\infty}+(t-3)\overline{F_0}+(3-t/2)E_0.$$

Thinking of $\overline{F_0}$ as a copy of $Bl_1\mathbb{P}^3$, for any point $p\in\overline{F_0}$, we can find a curve $C\subseteq\overline{F}_0$ passing through p with

$$\big(-K_Y-t\overline{H}-(t/2-1)E_0\big)\cdot C<0.$$

Thus $\overline{F_0}$ is contained in the non-nef locus. Subtracting it, we obtain, for $t\geq 3$, that

$$\begin{aligned}\mathrm{Vol}(-K_Y-t\overline{H})&=\mathrm{Vol}(-K_Y-t\overline{H}-(t/2-1)E_0-(t-3)\overline{F_0})\\&=\mathrm{Vol}((6-t)\overline{F_\infty}+(3-t/2)E_0)\\&=\frac{(6-t)^4}{2^4}\mathrm{Vol}(2\overline{F_\infty}-E_0)\\&=\frac{(6-t)^4}{81}\mathrm{Vol}(3\overline{F_\infty}-1.5E_0)\\&=\frac{(6-t)^4}{81}\mathrm{Vol}(-K_Y-3\overline{H})\\&=\frac{(6-t)^4}{2}.\end{aligned}$$

In conclusion, we have [1]

$$\operatorname{Vol}(-K_Y - t\overline{H}) = \begin{cases} 560 - 312t + 48t^2, & t \in [0,1]; \\ 559 - 308t + 42t^2 + 4t^3 - t^4, & t \in [1,2]; \\ 567 - 324t + 54t^2 - t^4/2, & t \in [2,3]; \\ (6-t)^4/2, & t \in [3,6]. \end{cases}$$

Integrating over $[0,6]$, we obtain that

$$S_{-K_Y}(\overline{H}) = \frac{1}{(-K_Y)^4}\int_0^6 (\operatorname{Vol}(-K_Y) - t\overline{H})dt = \frac{363}{280}.$$

So we have verified that

$$\delta_Y(\overline{H}) = \frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1-\lambda)}$$

even when $\epsilon_X(H) \neq \tau_X(H)$.

This computation suggests that, it is impractical to prove Proposition 3.3 by a direct computation using MMP.

We end this note by giving one more exmaple.

Example 5.1. *Let $X = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}}^2(1)$ and $Y = \mathbb{P}(L^{-1} \oplus \mathcal{O}_X)$. Then $n = 2$ and $\lambda = 1/3$ so that $\beta_0 = 14/17$. Pick a point $p \in X$ and blow it up. Let G denote the exceptional divisor over p. Then we have $\epsilon_X(G) = \tau_X(G) = 3$. Moreover,*

$$A_X(G) = 2 \text{ and } S_{-K_X}(G) = \frac{1}{9}\int_0^3 (9 - t^2)dt = 2.$$

Thus $\delta_X(G) = 1$ and

$$\frac{\delta_X(G)\beta_0}{\lambda + \beta_0(1-\lambda)} = \frac{14}{15}.$$

Now let $f := \pi^{-1}(p)$ be the $\mathbb{P}^1$-fiber over p. Let $\overline{Y} \xrightarrow{\sigma} Y$ be the blowup along f. Let $\overline{G} \subseteq \overline{Y}$ be the exceptional divisor of σ. As a divisor over Y, $\overline{G}$ satisfies $A_Y(\overline{G}) = 2$, $\epsilon_Y(\overline{G}) = 2$ and $\tau_Y(\overline{G}) = 4$. And also,

$$\begin{aligned} S_{-K_Y}(\overline{G}) &= \frac{1}{(-K_Y)^3}\int_0^4 \operatorname{Vol}(\sigma^*(-K_Y) - t\overline{G})dt \\ &= \frac{1}{56}\int_0^2 \operatorname{Vol}(\sigma^*(-K_Y) - t\overline{G})dt + \frac{1}{56}\int_2^4 \operatorname{Vol}(\sigma^*(-K_Y) - t\overline{G} - (t-2)\sigma^* F_0)dt \\ &= \frac{1}{56}\int_0^2 (56 - 6t^2)dt + \frac{1}{56}\int_2^4 (64 - 12t^2 + 2t^3)dt \\ &= \frac{15}{7}. \end{aligned}$$

Therefore $\delta_Y(\overline{G}) = 14/15$, so that

$$\delta_Y(\overline{G}) = \frac{\delta_X(G)\beta_0}{\lambda + \beta_0(1-\lambda)}.$$

[1] It is interesting to notice that $\operatorname{Vol}(-K_Y - t\overline{H})$ is C^3-differentiable (but not C^4) for $t \in (0,6)$.

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