THE ASYMPTOTIC BEHAVIOR OF BERGMAN KERNELS ON POLARIZED KÄHLER MANIFOLDS

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Abstract. On polarized Kähler manifolds with Sobolev constant upper bound and Ric lower bound, we give a better estimate for the Bergman kernels.

1. Introduction

In this note, we denote by (X, ω, L, h) a polarized Kähler manifold, where X is an n-dimensional compact Kähler manifold, L is an ample line bundle on X, and h is a smooth Hermitian metric on L such that

$$-\sqrt{-1}\partial\bar{\partial}\log h = \omega.$$

Now for any $k \in \mathbb{N}_+$, using ω and h, we define an L^2 Hermitian inner product $\langle \cdot\ , \cdot \rangle$ on the vector space $H^0(X, L^k)$ by letting

$$\langle s_1, s_2 \rangle = \int_X (s_1, s_2)_{h^k} \frac{(k\omega)^n}{n!},\ \forall s_1, s_2 \in H^0(X, L^k). \tag{1.1}$$

(Note that there is an additional k^n factor in (1.1)! See also Remark 1.6.)

Using this Hermitian inner product, we can pick an orthonormal basis $s_0, s_1, ..., s_{N_k}$ of $H^0(X, L^k)$ and we define a function

$$\rho_{\omega,k}(x) := \sum_{i=0}^{N_k} |s_i|^2_{h^k}(x),\ x \in X. \tag{1.2}$$

Note that $\rho_{\omega,k}$ is called the Bergman kernel of (X, L^k, ω) and it is independent of the choices of h and the orthonormal basis.

Recall that we have the following well-known asymptotic behavior of $\rho_{\omega,k}$ as $k \to \infty$:

$$\rho_{\omega,k} = \frac{1}{(2\pi)^n}\big(1 + \frac{S(\omega)}{2}k^{-1} + O(k^{-2})\big), \tag{1.3}$$

where $S(\omega)$ denotes the scalar curvature of ω (see e.g. [10]). This result can be thought of a pointwise Riemann-Roch theorem on (X, L).

Note that (1.3) is only proved for a fixed polarized Kähler manifold. In practice, we often consider a *family* of polarized Kähler manifolds (like in Cheeger-Colding theory). In Donaldson-Sun[3, Conjecture 5.15], the authors conjectured that, similar asymptotic behavior as in (1.3) should hold *uniformly* whenever the polarized family satisfies certain

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geometric conditions. In this note we make some progress towards this conjecture and prove the following.

Theorem 1.4. *Given $A < \infty$, there exists a large integer $D = D(n, A)$ and two constants $b = b(n, A) > 0$, $B = B(n, A) < \infty$ such that the following holds. Let (X, ω, L, h) be a polarized Kähler manifold that satisfies*
(1) $C_S(X, \omega) \leq A$;
(2) $\mathrm{Ric}(\omega) > -\omega$.
Then for any $k \in \mathbb{N}_+$, we have $b \leq \rho_{\omega,Dk} \leq B$.

We remark that, in the above theorem, the condition $C_S(X, \omega) \leq A$ means that the following Sobolev inequality holds:

$$(\int_X |u|^{\frac{2n}{n-1}}\omega^n)^{\frac{n-1}{n}} \leq A(\int_X u^2\omega^n + \int_X |\nabla u|^2\omega^n),\ \forall u \in W^{1,2}(X).$$

The formulation of Theorem 1.4 is inspired by the work of R. Bamler[1]. The key point is that, the conditions on the Sobolev constant and Ricci curvature are **preserved** if we rescale (X, ω, L, h) by large integers (just as in [1], the conditions on the μ-entropy and scalar curvature are preserved if we blow up the Ricci flow).

To be more precise, given any $k \in \mathbb{N}_+$, we may rescale (X, ω, L, h) as follows:

$$\tilde{\omega} := k\omega,\ \tilde{L} := L^k,\ \tilde{h} := h^k.$$

Then $(X, \tilde{\omega}, \tilde{L}, \tilde{h})$ is again a polarized Kähler manifold (in this note, any rescaling will be of this form). If ω satisfies

$$C_S(X, \omega) \leq A,\ \mathrm{Ric}(\omega) > -\omega.$$

Then it is easy to see that

$$C_S(X, \tilde{\omega}) \leq A,\ \mathrm{Ric}(\tilde{\omega}) > -\frac{1}{k}\tilde{\omega}.$$

In other words, rescaling will only make things better (as in Bamler's work). And it should be emphasized that, in this note, whenever we raise L to L^k, the underlying Kähler form ω will be rescaled to $k\omega$ accordingly, so that everything will work consistently (this explains why we use $(k\omega)^n$ as the volume form in (1.1)). This kind of treatment also appeared in [2, 8, 4].

Under our convention, the Bergman kernel enjoys the following rescaling property:

$$\rho_{l\omega,k} = \rho_{\omega,kl},\ \forall k, l \in \mathbb{N}_+. \tag{1.5}$$

Here $\rho_{l\omega,k}$ denotes the Bergman kernel of $(X, L^{kl}, l\omega)$, so it coincides with the Bergman kernel of (X, L^{kl}, ω). Note that, this property will play an important role in the proof of Theorem 1.4.

Remark 1.6. If one prefers to use the following L^2 inner product:

$$\langle s_1, s_2\rangle = \int_X (s_1, s_2)_{h^k}\frac{\omega^n}{n!},\ \forall s_1, s_2 \in H^0(X, L^k).$$

Then one would get an additional facor k^n for the Bergman kernel $\rho_{\omega,k}$. In particular, in the statement of Theorem 1.4, one would have

$$bk^n \leq \rho_{\omega,Dk} \leq Bk^n, \forall k \in \mathbb{N}_+.$$

The rest of this note is organized as follows. In Section 2, we recall some standard facts that will be useful for us. In Section 3 we prove Theorem 1.4 and some applications will be given in Section 4.

2. PRELIMINARIES

In this part, we recall some standard results in the literature.

Lemma 2.1 ([5]). *For any $A < \infty$, there exists a constant $\kappa = \kappa(n, A) > 0$ such that the following holds. Let (X, g) be a $2n$-dimensional compact Riemannian manifold. Assume that $C_S(X, g) \leq A$, then we have*

$$\mathrm{vol}(B(x, 1)) \geq \kappa, \ \forall x \in X.$$

Lemma 2.2. *For any $A < \infty$, there exists a constant $B = B(n, A) > 0$ such that the following holds. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that $C_S(X, \omega) \leq A$. Then for any section $s \in H^0(X, L)$, we have*

$$||s||^2_{L^\infty} \leq B||s||^2_{L^2}.$$

Proof. We have

$$\Delta|s|^2_h = |\nabla s|^2_h - n|s|^2_h.$$

So the result follows from the standard Moser iteration. □

Lemma 2.3. *For any $A < \infty$, we have the following fact. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that $C_S(X, \omega) \leq A$. Then for any $k, l \in \mathbb{N}_+$, we have*

$$\rho_{\omega,kl} \geq \frac{(\rho_{\omega,k})^l}{l^n B^{l-1}},$$

where B is determined in Lemma 2.2.

Proof. Consider an arbitrary point $x \in X$. We may assume that $\rho_{\omega,k}(x) \neq 0$, then there exists $s \in H^0(X, L^k)$ with

$$||s||^2_{L^2} = \int_X |s|^2_{h^k} \frac{(k\omega)^n}{n!} = 1$$

such that

$$\rho_{\omega,k}(x) = |s|^2_{h^k}(x).$$

Now we look at $s^l \in H^0(X, L^{kl})$. Using Lemma 2.2, it is clear that

$$||s^l||^2_{L^2} = \int_X (|s|^2_{h^k})^l \frac{(kl\omega)^n}{n!} \leq l^n (||s||^2_{L^\infty})^{l-1} ||s||^2_{L^2} \leq l^n B^{l-1}.$$

So we get

$$\rho_{\omega,kl}(x) \geq \frac{|s^l|^2_{h^{kl}}(x)}{||s^l||^2_{L^2}} \geq \frac{(\rho_{\omega,k}(x))^l}{l^n B^{l-1}}.$$

□

3. Proof of the main result

To prove Theorem 1.4, we will make use of the following compactness result obtained in [6].

Proposition 3.1. *([6, Proposition 3.1]) Given $A < \infty$, there is a large integer $K_0 = K_0(n, A)$ and two constants $\epsilon = \epsilon(n, A) > 0$, $c = c(n, A) > 0$ with the following property. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that*

(1) $C_S(X, \omega) \leq A$;
(2) $\mathrm{Ric}(\omega) > -\epsilon\omega$.

Suppose that $d_{GH}(B(p, \epsilon^{-1}), B(o, \epsilon^{-1})) < \epsilon$ for a metric cone (V, o). Then there exists an integer $m \leq K_0$, such that

$$\rho_{\omega,m}(p) \geq c > 0.$$

An estimate of this kind originates from the work of Tian[12], and was further developed in [3, 13], which is proved using the so called *peak section method*. The basic philosophy is that, whenever the manifold is close to a metric cone, we can construct suitable holomorphic sections. Note that in [6, Proposition 3.1], (X, ω) is assumed to be locally non-collapsed. This is guaranteed in our setting by Lemma 2.1.

Now notice that, Proposition 3.1 gives the following compactness result.

Proposition 3.2. *Given $A < \infty$, there is a large integer $K_1 = K_1(n, A)$ and a constant $\eta = \eta(n, A) > 0$ with the following property. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that*

(1) $C_S(X, \omega) \leq A$;
(2) $\mathrm{Ric}(\omega) > -\omega$.

Then for any point $p \in X$, there exists an integer $m \leq K_1$ such that

$$\rho_{\omega,m}(p) \geq \eta > 0.$$

Proof. We argue by contradiction. Suppose that the statement is wrong, then there exists $A < \infty$ such that, for any $K_i \to \infty$ and $\eta_i \to 0$, there exists a polarized sequence $(X_i, \omega_i, L_i, h_i)$ satisfying

(1) $C_S(X_i, \omega_i) \leq A$,
(2) $\mathrm{Ric}(\omega_i) > -\omega_i$,

and there exists $p_i \in X_i$ such that for any $m \leq K_i$, we have

$$\rho_{\omega_i,m}(p_i) \leq \eta_i. \tag{3.3}$$

Note that by Lemma 2.1, the sequence (X_i, ω_i, p_i) satisfies local non-collapsing condition. So the standard Cheeger-Colding theory works perfectly. After passing to a subsequence, we may assume

$$(X_i, \omega_i, p_i) \xrightarrow{\text{pointed } GH} (Z, d, p_\infty).$$

Note that Z does not have to be a metric cone. But the blow-up will always be. So we take a sequence of integers $l_j \to \infty$ and by passing to a subsequence we can assume that

$$(Z, \sqrt{l_j} d, p_\infty) \xrightarrow{\text{pointed } GH} (V, o)$$

for some metric cone (V, o). Now we take $\epsilon = \epsilon(n, A)$ from Proposition 3.1. Then for any j sufficiently large, we have

$$d_{GH}\Big(B_{d_j}(p_\infty, \epsilon^{-1}), B(o, \epsilon^{-1})\Big) < \frac{\epsilon}{2}.$$

Here $B_{d_j}(p_\infty, \epsilon^{-1})$ denotes the ball centered at p_∞ measured with respect to the rescaled metric $d_j = \sqrt{l_j} d$. We now fix such an j and put $l = l_j$. Then we clearly have

$$(X_i, l\omega_i, p_i) \xrightarrow{\text{pointed } GH} (Z, d_j, p_\infty).$$

Thus for any i large enough, we have

$$d_{GH}\Big(B_{l\omega_i}(p_i, \epsilon^{-1}), B_{d_j}(p_\infty, \epsilon^{-1})\Big) < \frac{\epsilon}{2}.$$

Here $B_{l\omega}(p_i, \epsilon^{-1})$ denotes the ball centered at p_i measured with respect to the rescaled Kähler form $l\omega_i$. Thus we see that

$$d_{GH}\Big(B_{l\omega_i}(p_i, \epsilon^{-1}), B(o, \epsilon^{-1})\Big) < \epsilon$$

for any sufficiently large i. By adjusting l if necessary, we may further assume that $1/l < \epsilon$. Then Proposition 3.1 can be applied to the polarized manifold $(X_i, l\omega_i, L^l, h_i^l)$ with i sufficiently large. So we can find $m_i \le K_0 = K_0(n, A)$ such that

$$\rho_{l\omega_i, m_i}(p_i) \ge c = c(n, A) > 0,$$

with K_0 and c determined by Proposition 3.1. Now thanks to the rescaling property (1.5), we arrive at

$$\rho_{\omega_i, lm_i}(p_i) \ge c > 0,$$

contradicting our assumption (3.3) whenever i is large enough. □

Now we can apply Lemma 2.3 to refine the statement of Proposition 3.2.

Proposition 3.4. *Given $A < \infty$, there is a large integer $D = D(n, A)$ and a constant $b = b(n, A) > 0$ with the following property. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that*

(1) $C_S(X, \omega) \le A$;
(2) $\mathrm{Ric}(\omega) > -\omega$.

Then we have

$$\rho_{\omega, D}(p) \ge b > 0, \ \forall p \in X.$$

Proof. We choose $D = (K_1)!$, where $K_1 = K_1(n, A)$ is the integer determined in the previous proposition. So for any $m \leq K_1$, D is divisible by m. Now for any $p \in X$, Proposition 3.2 guarantees that there exists $m_p \leq K_1$ and $\eta = \eta(n, A) > 0$ such that

$$\rho_{\omega, m_p}(p) \geq \eta > 0.$$

Now applying Lemma 2.3, we get

$$\rho_{\omega, D}(p) \geq \frac{(\rho_{\omega, m_p}(p))^{D/m_p}}{(D/m_p)^n B^{D/m_p - 1}} \geq \frac{\min\{1, \eta^D\}}{D^n B^{D-1}} > 0.$$

So we choose $b = \frac{\min\{1, \eta^D\}}{D^n B^{D-1}}$ and finish the proof. □

Finally, we are able to prove Theorem 1.4.

Proof of the main result. Let (X, ω, L, h) be a polarized Kähler manifold that satisfies

(1) $C_S(X, \omega) \leq A$;
(2) $\mathrm{Ric}(\omega) > -\omega$.

For any $k \in \mathbb{N}_+$, if we put

$$\tilde{\omega} := k\omega, \ \tilde{L} := L^k, \ \tilde{h} := h^k.$$

Then we would get

$$C_S(X, \tilde{\omega}) \leq A \text{ and } \mathrm{Ric}(\tilde{\omega}) > -\frac{1}{k}\tilde{\omega}.$$

So the upper bound of $\rho_{\omega,k}$ for each $k \in \mathbb{N}_+$ follows directly if we apply Lemma 2.2 to $(X, \tilde{\omega}, \tilde{L}, \tilde{h})$. Also note that Proposition 3.4 can be applied to the polarized pair $(X, \tilde{\omega}, \tilde{L}, \tilde{h})$. So we find $D = D(n, A)$ and $b = b(n, A) > 0$ such that

$$\rho_{\tilde{\omega}, D}(p) \geq b > 0, \ \forall p \in X.$$

Finally, the rescaling property (1.5) gives

$$\rho_{\omega, Dk} \geq b > 0,$$

as desired. □

4. Applications

The first application is a refinement of the partial C^0 estimate obtained in [6]. This also verifies a stronger version of Tian's conjecture [11].

Theorem 4.1. *For any $A < \infty$, there is a large integer $D = D(n, A)$ and a constant $b = b(n, A) > 0$ with the following property. Let (X, ω, L, h) be an n-dimensional polarized Kähler manifold such that*

(1) $\mathrm{diam}(X, \omega) \leq A$;
(2) $\mathrm{Ric}(\omega) > -\omega$.

Then we have

$$\rho_{\omega, Dk} \geq b > 0, \ \forall k \in \mathbb{N}_+.$$

Proof. This follows immediately from Theorem 1.4 since we have a uniform upper bound for the Sobolev constant $C_S(X,\omega)$. Indeed, since $\omega \in 2\pi c_1(L)$ is in a integral cohomology class, we get

$$\int_X \omega^n = (2\pi c_1(L))^n \geq (2\pi)^n.$$

In other words, the volume of (X,ω) is uniformly bounded from below. Combining this with diameter upper bound and Ricci lower bound, we get a uniform upper bound for the Sobolev constant (cf. [7]). $\square$

We can also extend the argument in this note to Kähler-Ricci solitons. More specifically, we have the following result, improving [8, Theorem 1.1].

Theorem 4.2. *There exists $D = D(n) < \infty$ and $b = b(n) > 0$ with the following property. Let X be an n-dimensional Fano manifold and suppose that $\omega \in 2\pi c_1(X)$ satisfies*

$$\mathrm{Ric}(\omega) = \omega + \mathcal{L}_\xi \omega$$

for some holomorphic vector field ξ on X. Namely (X,ω,ξ) is a Kähler-Ricci soliton. Then we have

$$\rho_{\omega,Dk} \geq b > 0, \ \forall k \in \mathbb{N}_+.$$

Here $\rho_{\omega,Dk}$ denotes the Bergman kernel of $(X, -K_X^{Dk}, \omega)$.

The techniques are more or less standard (following [16, 14, 8]). We outline a proof for reader's convenience.

Proof. Given a Kähler-Ricci soliton (X,ω,ξ), we can find a potential function $u \in C^\infty(X,\mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \omega - \sqrt{-1}\partial\bar{\partial}u, \text{ with } u_{ij} = u_{\bar{i}\bar{j}} = 0.$$

As pointed out in [9], we can assume that

$$|u| + |\nabla u|^2 + |\Delta u| \leq C_1 \text{ for some } C_1 = C_1(n).$$

And also, we have

$$C_S(X,\omega) \leq A \text{ for some } A = A(n).$$

Meanwhile, if we use Z. Zhang's trick (see [16]) and put

$$\eta := e^{-\frac{u}{n-1}}\omega,$$

then we have

$$-C_2 \leq \mathrm{Ric}(\eta) \leq C_2, \text{ for some } C_2 = C_2(n).$$

Note that, these estimates are the key ingredients to prove the partial C^0 estimate in [8, Theorem 1.1]. Now the important observation is that, these estimates are **preserved** if we rescale ω by some intergers greater than 1.

To be more precise, for any $k \in \mathbb{N}_+$, if we put

$$\tilde{\omega} := k\omega,$$

then we would get

$$|u| + k|\nabla_{\tilde{\omega}} u|^2 + k|\Delta_{\tilde{\omega}} u| \le C_1 \text{ and } C_S(X, \tilde{\omega}) \le A.$$

Meanwhile, if we put

$$\tilde{\eta} := e^{-\frac{u}{n-1}} \tilde{\omega},$$

then we have

$$-\frac{C_2}{k} \le \mathrm{Ric}(\tilde{\eta}) \le \frac{C_2}{k}.$$

So as one can see, rescaling makes things better.

Now the proof can be carried out in the same manner as we did in Section 3. The key result is Proposition 4.3 below (compare Proposition 3.1). With this in hand, we can then follow the argument of Proposition 3.2 (using the Cheeger-Colding-Tian theory developed in [16, 14]) to obtain a large integer $K = K(n)$ and $\eta = \eta(n) > 0$ such that, for any rescaled Kähler-Ricci soliton $(X, \tilde{\omega}, \xi)$ and any point $p \in X$, there exists $m_p \le K$ such that

$$\rho_{\tilde{\omega}, m_p}(p) \ge \eta > 0.$$

Then the same argument as in the proof of Proposition 3.4 gives a large integer $D = D(n)$ and $b = b(n) > 0$ such that

$$\rho_{\tilde{\omega}, D}(p) \ge b > 0, \ \forall p \in X.$$

Finally the rescaling property (1.5) completes the proof. □

Proposition 4.3. *There is a large integer $K_0 = K_0(n)$ and two constants $\epsilon = \epsilon(n) > 0$, $c = c(n) > 0$ with the following property. Let $(X, \tilde{\omega})$ be an n-dimensional Fano manifold with $\tilde{\omega} \in 2\pi k c_1(X)$ for some $k \in \mathbb{N}_+$ such that $1/k < \epsilon$. Assume that there exists a potential function $u \in C^\infty(X, \mathbb{R})$ such that*

$$\mathrm{Ric}(\tilde{\omega}) = \frac{1}{k}\tilde{\omega} - \sqrt{-1}\partial\bar{\partial}u, \ \text{with } u_{ij} = u_{\bar{i}\bar{j}} = 0.$$

Namely $\tilde{\omega}$ is a rescaled Kähler-Ricci soliton metric (with sufficiently large scaling factor k). Also assume that

$$d_{GH}\big(B_{\tilde{\omega}}(p, \epsilon^{-1}), B(o, \epsilon^{-1})\big) < \epsilon$$

for a metric cone (V, o). Then there exists an integer $m \le K_0$, such that

$$\rho_{\tilde{\omega}, m}(p) \ge c > 0.$$

Proof. This is essentially contained in [8, Section 5]. We argue by contradiction. Suppose that for $k_i \to \infty$, we have a sequence of blowing-up Kähler-Ricci solitons $(X_i, \tilde{\omega}_i, \xi_i)$ with

$$\mathrm{Ric}(\tilde{\omega}_i) = \frac{1}{k_i}\tilde{\omega}_i - \sqrt{-1}\partial\bar{\partial}u_i$$

and

$$(X, \tilde{\omega}_i, p_i) \xrightarrow{\text{pointed } GH} (V, o)$$

for some metric cone (V, o). Then by [14], we know that V is Ricci flat away from a closed singular set with codimension at least 4 and the convergence takes place in C^∞ topology on the regular part. Then the argument in [3, 13] can be applied in this setting (see also

[15]) to make sure that, there exists $K < \infty$ and $c > 0$ such that, for any sufficiently large i, there exists some $m_i \leq K$ such that $\rho_{\tilde{\omega}_i, m_i}(p_i) \geq c > 0$. □

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