# A quantization approach to the cscK problem

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# Outline

The cscK problem

# The cscK problem

Let $(X, \omega)$ be an $n$-dimensional compact Kähler manifold. Our goal is to find a Kähler metric $\omega^* \in \{\omega\}$ such that

$$S(\omega^*) = \mathrm{tr}_{\omega^*} \mathrm{Ric}(\omega^*) = \bar{S},$$

where $\bar{S} = 2\pi n \frac{c_1(X) \cdot \{\omega\}^{n-1}}{\{\omega\}^n}$ is the average of the scalar curvature. Such a metric is called a constant scalar curvature Kähler (cscK) metric.

To find the cscK metric, it amounts to solving a coupled system of equations:

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n, \\ \Delta_{\omega_\varphi} F = \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega) - \bar{S}. \end{cases}$$

We assume throughout that $\mathrm{Aut}(X)$ is discrete. Our discussions below hold for general $\mathrm{Aut}(X)$ as well, but are a bit more subtle.

# Some energy functionals

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$, and set

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X, \mathbb{R}) | \omega_\varphi := \omega + dd^c\varphi > 0\}.$$

Put $V := \int_X \omega^n$. For any $u \in \mathcal{H}_\omega$, define

$$E(u) := \frac{1}{(n+1)V} \int_X u \sum_{i=0}^{n} \omega^i \wedge \omega_u^{n-i}.$$

$$J(u) := \frac{1}{V} \int_X u\omega^n - E(u).$$

$$Ent(u) = Ent(\omega^n, \omega_u^n) := \frac{1}{V} \int_X \log \frac{\omega_u^n}{\omega^n} \omega_u^n.$$

Note that by Jensen's inequality, one has $Ent(u) \geq 0$.

# Some energy functionals

Define Ricci energy:

$$\mathcal{J}(u) := \frac{1}{V}\int_X u\,\mathrm{Ric}(\omega)\wedge\sum_{i=0}^{n-1}\omega^i\wedge\omega_u^{n-1-i} - \bar{S}E(u),$$

where $\bar{S}$ is the average of the scalar curvature. Mabuchi's K-energy is defined by

$$K(u) := Ent(u) - \mathcal{J}(u).$$

Direct calculation gives:

$$\frac{d}{dt}\Big|_{t=0}K(u+tf) = \frac{1}{V}\int_X f(\bar{S}-S(\omega_u))\omega_u^n.$$

So cscK metrics are critical points of $K$.

# Another functional: Ding energy

In the study of Kähler Einstein metrics on Fano manifolds, we also use Ding energy, which is defined as

$$D(u) := -\log \int_X e^{f_\omega - u}\omega^n - E(u),$$

where $f_\omega$ is the Ricci potential of $\omega$. Namely, $\mathrm{Ric}(\omega) = \omega + dd^c f_\omega$.

- The critical points of $D$ are Kähler–Einstein metrics, which coincide with cscK metrics in the Fano case.
- Ding energy has much better variational properties than K energy does. So in the Kähler–Einstein problem we mainly use Ding energy.
- But for general cscK case, there is NO Ding energy! This is one of the main stumbling blocks in the cscK problem.

# The metric completion: $\mathcal{E}^1$ space

For any $\varphi \in \mathcal{PSH}_\omega$, note that $\varphi_j := \max\{-j, \varphi\}$ decreases to $\varphi$ pointwise, along which, $E(\varphi_j)$ decreases as well. Put

$$E(\varphi) := \lim_{j \to +\infty} E(\varphi_j).$$

Define

$$\mathcal{E}^1 := \{\varphi \in \mathcal{PSH}_\omega | E(\varphi) > -\infty\}.$$

This is called the finite energy space. For any $u, v \in \mathcal{E}^1$, the $d_1$-distance between them is given by

$$d_1(u, v) := E(u) + E(v) - 2E(P(u, v)),$$

where $P(u, v) = \sup\{\varphi \in \mathcal{PSH}_\omega : \varphi \leq \min\{u, v\}\}$. Note that $(\mathcal{E}^1, d_1)$ is a complete metric length space (by Darvas, 2015).

# Extending functionals to $\mathcal{E}^1$

All the functionals defined above can be extended to $\mathcal{E}^1$. All of them are continuous with respect to the $d_1$-topology, except the entropy

$$Ent(u) = \frac{1}{V} \int_X \log \frac{\omega_u^n}{\omega^n} \omega^n,$$

which could take $+\infty$ at some points in $\mathcal{E}^1$, and it is only lower semi-continuous with respect to the $d_1$-topology. Thus,

$$K(u) \leq \liminf_{i \to \infty} K(u_i)$$

for a sequence $\{u_i\}$ $d_1$-converging to $u$. While for Ding energy in the Kähler–Einstein problem, we have

$$D(u) = \lim_{i \to \infty} D(u_i)$$

# Properness

**Definition**

We say a functional $F(\cdot)$ is proper if there exists $\varepsilon > 0$ and $C > 0$ such that
$$F(u) \geq \varepsilon J(u) - C \text{ for any } u \in \mathcal{E}^1.$$

**Remark:** one should view $J$ as a distance function on $\mathcal{H}_\omega$.

**Conjecture (Tian's properness conjecture, 90s)**

$(X, \omega)$ admits a cscK metric iff $K(\cdot)$ is proper.

Tian's conjecture is central in Kähler geometry and has attracted much work over the past two decades, leading to developments in geometric analysis, algebraic geometry and pluripotential theory. Recently, this conjecture has been solved, based on the work of Berman–Berndtsson (2017, JAMS), Darvas–Rubinstein (2017, JAMS) and Chen–Cheng (2021, JAMS) et. al.

# Convexity of K-energy

**Theorem (Berman–Berndtsson 2014, Berman–Darvas–Lu 2015)**

*$K$-energy is convex along geodesics in $\mathcal{E}^1$.*

**Theorem (Tian, Berman–Berndtsson, Darvas–Rubinstein, Berman–Darvas–Lu, Chen–Cheng et. al.)**

*The following are equivalent.*

1. *$X$ admits a cscK metric in $\{\omega\}$.*
2. *$(X, \omega)$ is geodesic stable. Namely,*

$$K\{u_t\} := \lim_{t\to\infty} \frac{K(u_t)}{t} > 0$$

*for any non-trivial geodesic ray $\{u_t\}_{t\in[0,\infty)}$.*

**Remark:** The general principle is that: the properness of a convex function is equivalent to geodesic stable.

# The picture for geodesic stable

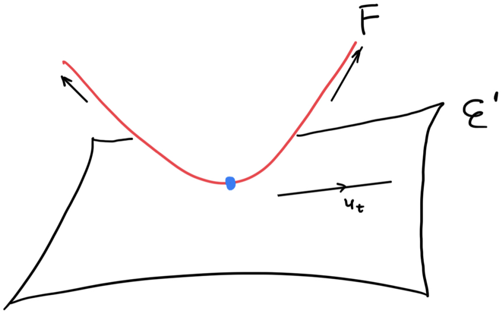



Figure: geodesic stable

Assume that $(X, L)$ is polarized, where $L$ is an ample line bundle on $X$ and $\omega \in c_1(L)$.

**The uniform Yau–Tian–Donaldson conjecture**

$(X, L)$ admits a cscK metric in $c_1(L)$ iff $(X, L)$ is uniformly K-stable.

Here the notion K-stability condition is purely algebraic. It was originally due to Prof. Tian (1997) in the Fano setting, based on Ding–Tian (1992). Later in 2002, it was generalized by Donaldson to projective varieties.
The uniform stability notion was suggested by Székelyhidi, Dervan and Boucksom–Hisamoto–Jonsson.
It can be described as follows:

# Algebraic rays

For $m \gg 1$, choose a one parameter family of basis $\{s_i^{(t)}\}_{t \geq 0}$ of $H^0(X, mL)$, we get a family $\{X_t\}_{t \geq 0}$ of embedded $X$ in $\mathbb{CP}^{N_m}$. The restriction of the Fubini–Study metric of $\mathbb{CP}^{N_m}$ on $X_t$ induces a family of Kähler metrics $\omega_t \in c_1(L)$. Write $\omega_t = \omega + dd^c u_t$. So we obtain a ray $\{u_t\}_{t \geq 0}$ in $\mathcal{H}_\omega$. Such $\{u_t\}$ is called an algebraic ray.

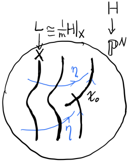



Figure: One parameter family (figure from Chi Li's slides)

**Remark:** An algebraic ray is a sub-geodesic ray. There exists a geodesic ray $\{u'_t\}$ such that $|u_t - u'_t| \leq C$ and Chi Li shows that

$$K\{u_t\} = K\{u'_t\}.$$

# YTD conjecture in terms of rays

**Definition**

$(X, L)$ is called uniformly K-stable if there exists $\delta > 0$ such that

$$K\{u_t\} \geq \delta J\{u_t\}$$

for all algebraic ray $\{u_t\}$. Here $J\{u_t\} := \lim_{t\to\infty} J(u_t)/t$.

Using the language of algebraic rays, the YTD conjecture can be reformulated as follows:

**The uniform YTD conjecture**

$(X, L)$ admits a cscK metrics in $c_1(L)$ iff

$$K\{u_t\} \geq \delta J\{u_t\}$$

for any algebraic ray $\{u_t\}$.

# Main difficulty

The main obstacle in the YTD conjecture is that, $K$ is only lower semi-continuous. So in particular, if $\{u_t^i\}$ is a sequence of algebraic rays converging to $\{u_t\}$, then

$$K\{u_t\} \leq \liminf_{i\to\infty} K\{u_t^i\}.$$

The question is, for any geodesic ray $\{u_t\}$, can we find a suitable sequence of algebraic rays such that (Boucksom–Jonsson conjecture)

$$K\{u_t\} = \lim_{i\to\infty} K\{u_t^i\}?$$

In the Fano case, this problem can be circumvented, with the help of Ding functional, thanks to its continuity.
By solving non-Archimedean Calabi–Yau equations, Chi Li showed that this can be done using model rays.

# Recent progress of Chi Li

**Definition (Boucksom–Jonsson, Chi Li)**

A geodesic ray $\{u_t\}$ is called a model ray, if it can be approximated from below by a sequence of algebraic rays.

**Theorem (Chi Li, 2020)**

*$(X, L)$ admits a cscK metrics in $c_1(L)$ iff*

$$K\{u_t\} \geq \delta J\{u_t\}$$

*for any model ray $\{u_t\}$.*

**Remark:** The model rays constructed by Chi Li are implicit, which are solutions to some non-Archimedean Calabi–Yau equations. Recently, we made progress on this problem by using quantization methods, that we now turn to describe.

Algebraic quantization (established by Tian)

# The Bergman kernel

Let $(X, L)$ be a polarized manifold, i.e., $X$ is a projective manifold and $L$ is an ample line bundle on $X$. Pcik $\omega \in c_1(L)$.
Choose a smooth Hermitian metric $h$ on $L$ such that its Chern curvature form satisfies

$$-dd^c \log h = \omega \in c_1(L),$$

where $dd^c = \sqrt{-1}\partial\bar{\partial}/2\pi$. Note that there is a natural Hermitian inner product on $H^0(X, mL)$ induced by $\omega$ and $h$:

$$(s, t) := \int_X h^m(s, t)\omega^n, \ s, t \in H^0(X, mL).$$

# The Bergman kernel

Let $\{s_i\}$ be an orthonormal basis of $H^0(X, mL)$. Put

$$\rho_m^\omega := \sum_i |s_i|^2_{h^m} \in C^\infty(X, \mathbb{R}).$$

This expression is independent of the choice of the orthonormal basis $\{s_i\}$ and $h$. It only depends on $\omega \in c_1(L)$ and $m$, which is called the Bergman kernel of $\omega$ at level $m$.
Notice that

$$\int_X \rho_m^\omega \omega^n = h^0(X, mL) = \chi(mL) \text{ for } m \gg 1.$$

So the Bergman kernel is a pointwise version of the Hilbert polynomial, which contains geometric information of $\omega$.

# Geometric meaning

For $m \gg 1$, the orthonormal basis $\{s_i\}$ induces an embedding:

$$X \hookrightarrow \mathbb{P}^{N_m - 1},$$

where $N_m := h^0(X, mL)$. Let $\omega_{FS}$ be the Fubini–Study metric on $\mathbb{P}^{N_m-1}$, then

$$\omega_m := \frac{1}{m}\omega_{FS}\Big|_X \in c_1(L),$$

and

$$\omega_m := \omega + dd^c\left(\frac{1}{m}\log\rho_m^{\omega}\right).$$

So the Bergman kernel measures the difference between $\omega$ and $\omega_m$. It connects differential geometric and algebraic geometry.

# Tian's quantization Theorem

The metric $\omega_m$ is called the quantization of $\omega$ at level $m$.
There is a remarkable quantization theorem due to Prof. Tian.

**Theorem (Tian–Catlin–Ruan–Zelditch)**

*As $m \to \infty$, the Bergman kernel has an asymptotic expansion:*

$$\rho_m^\omega = \frac{m^n}{n!} + \frac{S(\omega)}{2n!} m^{n-1} + \ldots \text{ as } m \to \infty.$$

*And one has*

$$\omega_m \xrightarrow{C^\infty} \omega \text{ as } m \to \infty.$$

Philosophy: one can study the geometry of $(X, \omega)$ using $H^0(X, mL)$ as $m \to \infty$.

# Tian's program towards YTD conjecture

In the 90's, Tian proposed the following program to attack the YTD conjecture, in which the partial $C^0$ estimate is an essential step. Roughly speaking, it is a uniform control for the difference between $\omega$ and $\omega_m$.

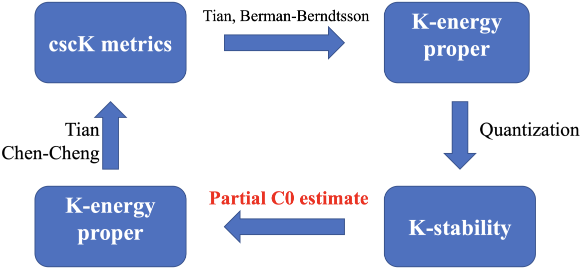



Figure: Tian's program

Transcendental quantization (proposed by Berman)

# Transcendental setting

Let $(X, \omega)$ be a compact Kähler manifold. One can define as before the space of finite energy potentials $\mathcal{E}^1(X, \omega)$.

**Question:** given any $\varphi \in \mathcal{E}^1(X, \omega)$, can we quantize it?

In 2013 R. Berman proposed a way of quantization even when $X$ is not projective. For any $\varphi \in \mathcal{E}^1(X, \omega)$ and $\beta > 0$, let $\varphi^\beta$ be the unique solution of the MA equation

$$(\omega + dd^c \varphi^\beta)^n = e^{\beta(\varphi^\beta - \varphi)} \omega^n.$$

We call $\varphi^\beta$ the quantization of $\varphi$ at level $\beta$. The use of such an equation also appeared in Dai–Wang–Zhou and Chu–Zhou as a way of regularization for $\varphi$. Berman proposed to use

$$e^{\beta \varphi^\beta}$$

as the transcendental Bergman kernel of $\varphi$.

# Why?

When $X$ is polarized with $\omega_\varphi \in c_1(L)$, put

$$\varphi_m := \frac{1}{m} \log \sum_i |s_i|^2_{h^m}.$$

So we have

$$e^{m\varphi_m} = \sum_i |s_i|^2_{h^m}$$

which is indeed the Bergman kernel.
Moreover, the asymptotic expansion of $\varphi_m$ implies that (up to some constant)

$$(\omega + dd^c\varphi_m)^n = e^{m(\varphi_m - \varphi)}\omega^n + O(m^{-1}).$$

# Asymptotic expansion of $\varphi^\beta$

For $\varphi \in \mathcal{E}^1(X,\omega)$ let $\varphi^\beta$ solve $(\omega + dd^c\varphi^\beta)^n = e^{\beta(\varphi^\beta-\varphi)}\omega^n$.

**Lemma (Chu–Z, 2021 unpublished)**

*For smooth $\varphi$, there is a complete asymptotic expansion*

$$\varphi^\beta \sim A_0(\varphi,\omega) + \frac{A_1(\varphi,\omega)}{\beta} + \frac{A_2(\varphi,\omega)}{\beta^2} + \ldots, \text{ as } \beta \to \infty.$$

*The leading coefficients of the expansion are given by*

$$\begin{cases} A_0(\varphi,\omega) = \varphi, \\ A_1(\varphi,\omega) = \log \frac{\omega_\varphi^n}{\omega^n}, \\ A_2(\varphi,\omega) = \Delta_{\omega_\varphi} A_1(\varphi,\omega) = \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega) - S(\omega_\varphi). \\ A_3(\varphi,\omega) = \Delta_{\omega_\varphi} A_2(\varphi,\omega) + \frac{1}{2}A_2(\varphi,\omega)^2 - |\sqrt{-1}\partial\bar\partial A_1(\varphi,\omega)|^2_{\omega_\varphi} \end{cases}$$

The next result is a transcendental version of Tian's (and also Darvas–Lu–Rubinstein's) quantization theorem.

**Theorem (Darvas–Z 2025)**

*Let $\varphi \in \mathcal{E}^1(X,\omega)$, then $\varphi^\beta \xrightarrow{d_1} \varphi$ as $\beta \to \infty$.*

Recall that

$$Ent(\varphi) := \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n$$

denotes the entropy of $\varphi$. For general $\varphi \in \mathcal{E}^1(X,\omega)$, it is possible that $Ent(\varphi) = +\infty$. This is the trouble term in the cscK problem.

**Theorem (Darvas–Z 2025)**

*Let $\varphi \in \mathcal{E}^1(X,\omega)$, then $Ent(\varphi^\beta) \leq Ent(\varphi)$, $Ent(\varphi^\beta) < \infty$ for any $\beta > 0$ and $Ent(\varphi^\beta) \nearrow Ent(\varphi)$ as $\beta \nearrow \infty$.*

# Transcendental Partial $C^0$ estimate

We establish a general partial $C^0$ estimate using transcendental quantization.

**Theorem (Darvas-Z 2025)**

*Let $\varphi \in \mathcal{E}^1(X, \omega)$ with $Ent(\varphi) < \infty$. Then there exists $C_0 > 0$ depending only on $X, \omega$ such that for any $\beta > 1$,*

$$d_1(\varphi, \varphi^\beta) \leq \frac{C_0}{\beta}(J(\varphi) + Ent(\varphi)),$$

*where $J(\varphi) = \frac{1}{\int_X \omega^n} \int_X \varphi \omega^n - E_\omega(\varphi)$ is Aubin's J-functional.*

Remark: for the usual algebraic quantization, such an estimate is still missing!

Applications to the cscK problem

# Recall the YTD conjecture

Recall that, using the language of algebraic rays, the YTD conjecture can be reformulated as follows:

**The uniform YTD conjecture**

$(X, L)$ admits a cscK metrics in $c_1(L)$ iff

$$K\{u_t\} \geq \delta J\{u_t\}$$

for any algebraic ray $\{u_t\}$.

**Recall:** the main difficulty in the YTD conjecture is that, $K$ is only lower semi-continuous. So in particular, if $\{u_t^i\}$ is a sequence of algebraic rays converging to $\{u_t\}$, then

$$K\{u_t\} \leq \liminf_{i\to\infty} K\{u_t^i\}.$$

# Recent progress

Using transcendental quantization, we construct a family of quantized $K^\beta$ energies, with $\beta \in (0, \infty)$ be a parameter, and they satisfy the following nice properties:

1. $K \geq K^\beta$ and $K^\beta \nearrow K$ as $\beta \nearrow \infty$.
2. $K$ energy proper $\xLeftrightarrow{\text{partial } C^0 \text{ estimate}}$ $K^\beta$ proper for some $\beta \gg 1$.
3. $K^\beta$ is continuous on $\mathcal{E}^1$.
4. For any geodesic ray $\{u_t\}$, we can find an approximating sequence of algebraic rays such that

$$K^\beta\{u_t\} = \lim_{i\to\infty} K^\beta\{u_t^i\}.$$

# A YTD theorem

**Definition (Darvas–Z, 2025)**

We say $(X,\omega)$ is uniformly $K^\beta$-stable if there exists $\delta > 0$ such that

$$K^\beta\{u_t\} \geq \delta J\{u_t\}$$

for any algebraic ray $\{u_t\}$.

**A YTD theorem (Darvas–Z, 2025)**

The following are equivalent.

1. $X$ admits a cscK metric in $c_1(L)$.
2. $(X,L)$ is uniformly $K^\beta$-stable for some $\beta > 0$.

By $K \geq K^\beta$ we easily have $K^\beta$-stability $\Longrightarrow$ $K$-stability.
But the reverse direction is still open!
Yes in the Fano case, by using MMP (BCHM, Li–Xu).

# The construction of $K^\beta$

For any $\beta > 0$ and $u \in \mathcal{E}^1$, let $u^\beta \in \mathcal{E}^1$ be the unique solution of the following Aubin–Yau type equation:

$$(\omega + dd^c u^\beta)^n = e^{\beta(u^\beta - u)}\omega^n.$$

Then let

$$Ent^\beta(u) := \beta(E(u^\beta) - E(u))$$

and

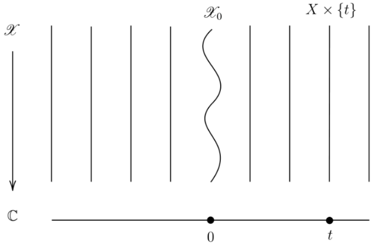
$$K^\beta(u) := Ent^\beta(u) - \mathcal{J}(u).$$

From the construction, $K^\beta(u)$ is only implicitly determined! Given an algebraic ray $\{u_t\}$, it is highly non-trivial to show that $K^\beta\{u_t\} := \liminf_{t\to\infty} K^\beta(u_t)/t$ has algebraic meaning.

# Test configuration

Let $(X, L)$ be polarized. Pick $\omega \in c_1(L)$. A test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ is a $\mathbb{C}^*$-equivariant partial compactification over $\mathbb{P}^1$ of $(X, L) \times \mathbb{C}^*$ which consists of a normal variety $\mathcal{X}$, a flat projective morphism $\pi : \mathcal{X} \to \mathbb{P}^1$, an ample $\mathbb{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$, a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ lifting the standard one on $\mathbb{C} \subset \mathbb{P}^1$, and an identification of the fiber $(\mathcal{X}_1, \mathcal{L}_1)$ with $(X, L)$. This notion is due to Ding, Tian and Donaldson.

Algebraic rays are in 1-1 correspondence to test configurations.



# Slope formulas

Given a geodesic ray $\{u_t\}$ of a test configuration $(\mathcal{X}, \mathcal{L})$.

- By Tian, Paul, Phong, Sturm, Boucksom, Hisamoto, Jonsson and Li, one has $J\{u_t\} = L^n \cdot \mathcal{L} - \frac{\mathcal{L}^{n+1}}{n+1}$ and

$$K\{u_t\} = \mathcal{L}^n \cdot K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1} + \frac{\bar{S}\mathcal{L}^{n+1}}{(n+1)} + K_X \cdot \mathcal{L}^n.$$

  Here $K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1} := K_{\mathcal{X}} - \pi^* K_{X\times\mathbb{P}^1} - \mathcal{X}_{0,red} + \mathcal{X}_0$.

- The most non-trivial result in Darvas–Z (2025) is

$$K^\beta\{u_t\} = \frac{\beta(\langle\mathcal{L}_\beta^{n+1}\rangle - \mathcal{L}^{n+1})}{n+1} + \frac{\bar{S}\mathcal{L}^{n+1}}{(n+1)} + K_X \cdot \mathcal{L}^n.$$

Here $\mathcal{L}_\beta := \mathcal{L} + \frac{1}{\beta} K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1}$ and $\langle\mathcal{L}_\beta^{n+1}\rangle$ denotes movable intersection product. We have $\langle\mathcal{L}_\beta^{n+1}\rangle = \mathrm{vol}(\mathcal{L}_\beta)$.

Note: $\frac{d}{dt}\big|_{t=0} \mathrm{vol}(\mathcal{L} + tK^{\log}_{\mathcal{X}/X\times\mathbb{P}^1})/(n+1) = \mathcal{L}^n \cdot K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1}$.

# How to prove the slope formula?

There are several technical ingredients:

- We observe (by variational principle) that for any $u \in \mathcal{E}^1$,

$$Ent^{\beta}(u) = \sup_{v \in \mathcal{H}_\omega} \left( -\log \int_X e^{\beta(v-u)} \frac{\omega^n}{V} + \beta(E(v) - E(u)) \right).$$

- For any geodesic ray $\{u_t\}$, we prove (very non-trivial)

$$Ent^{\beta}\{u_t\} = \sup_{\{v_t\}} \left( L^{\beta}(\{v_t\}, \{u_t\}) + \beta(E\{v_t\} - E\{u_t\}) \right),$$

where the sup is over algebraic rays, and $L^{\beta}(\{v_t\}, \{u_t\})$ denotes the following generalized radial Ding functional:

$$L^{\beta}(\{v_t\}, \{u_t\}) := \liminf_{t \to \infty} \left( -\frac{1}{t} \log \int_X e^{\beta(v_t - u_t)} \frac{\omega^n}{V} \right).$$

# How to prove the slope formula?

- Assume that $\{u_t\}$ is an algebraic ray associated to $(\mathcal{X}, \mathcal{L})$, we need to find a maximizer $\{v_t\}$ for the sup above. We show that, the geodesic ray $\{v_t\}$ of the modified test configuration

$$\left(\mathcal{X}, \mathcal{L} + \frac{1}{\beta} K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1}\right)$$

attains the sup! This step requires careful analysis of the singularities of $v_t$ as $t \to \infty$.

Here $K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1} := K_{\mathcal{X}} - \pi^* K_{X\times\mathbb{P}^1} - \mathcal{X}_{0,red} + \mathcal{X}_0$.

**Remark:** The line bundle $\mathcal{L}_\beta := \mathcal{L} + \frac{1}{\beta} K^{\log}_{\mathcal{X}/X\times\mathbb{P}^1}$ is only big on $\mathcal{X}$, meaning that it ample on a Zariski open set but is degenerate along some subvariety. So $(\mathcal{X}, \mathcal{L}_\beta)$ is a model in the sense of Chi Li. But note that our models are very explicit.

# A new YTD theorem

Our analysis also yields a new YTD theorem, improving the previous one of Chi Li.

**Theorem (Darvas–Z, 2025)**

*The following are equivalent.*

1. $X$ *admits a cscK metric in* $c_1(L)$.
2. $(X, L)$ *is uniformly* $K^\beta$*-stable.*
3. $(X, L)$ *is uniformly K-stable for models of the form* $(\mathcal{X}, \mathcal{L}_\beta)$. *Namely, there exists* $\delta > 0$ *such that*

$$K(\mathcal{X}, \mathcal{L}_\beta) \geq \delta J(\mathcal{X}, \mathcal{L}_\beta)$$

*for any test configuration* $(\mathcal{X}, \mathcal{L})$ *and any* $\beta \geq 1$.

**Remark:** See also Boucksom–Jonsson and M. Piccione–Witt Nyström for related recent advances on the YTD conjecture.

Thanks for your attention!