Algebraic invariants and canonical metrics

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Outline

Canonical metrics

- The existence of canonical metrics on Kähler manifolds is an important problem in Geometric analysis.
  e.g. Kähler-Einstein metrics, Kähler-Ricci solitons, constant scalar curvature metrics...
- If we allow certain singularities, then we can also study conical Kähler-Einstein metrics, weak Kähler-Einstein metrics and so on...
- Essentially, these are PDE problems on manifolds.

How do you find canonical metrics

- **Continuity method**
  Calabi conjecture
  Aubin-Yau
- **Geometric flow**
  Ricc flow
  Calabi flow
- **Variational method**
  K-energy
  Ding energy
- **But canonical metrics do not always exists!**

Obstructions

- **Topological obstruction**
  The first Chern class
- **Analytic obstruction**
  The Automorphism group
  Futaki invariant
- **Algebraic obstruction**
  K-stability

Yau-Tian-Donaldson conjecture

The existence of canonical metric is equivalent to K-stability.

But K-stability is very hard to check. So we need effective criterion.

Alpha invariant

The first criterion was found by Prof. Gang Tian in 1987, which is known as the α-invariant.

Delfinition of α-invariant (log canonical thershold)

Let X be a $\mathbb{Q}$-Fano variety over $\mathbb{C}$. Then the α-invaraint of X is defined by

$$\alpha(X) = \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

Tian's criterion

Let X be an n dimensional Fano manifold. Suppose that $\alpha(X) > \frac{n}{n+1}$, then there exists a Kähler form $\omega \in 2\pi c_1(X)$ such that

$$\mathrm{Ric}(\omega) = \omega$$

Some examples

Using α or α_G-invariant, the following manifolds admit Kähler-Einstein metrics.

- m-dimension Fermat hypersurface with degree greater than $m-1$.
- Blow-up of $\mathbf{CP}^2$ at 3,4,5,7 or 8 general points.

However, blow-up of $\mathbf{CP}^2$ at 6 general points (i.e. cubic surface) was not easy to deal with...

Unfortunately

The α-invariant of cubic surface is possibly **equal** to $\frac{2}{3}$. So Tian's criterion cannot be applied in this case!

In 1990, Tian proved the existence of Kähler-Einstein metrics on cubic surfaces using much more involved techniques: compactness of moduli spaces and partial C^0 estimate...

Delta invariant

In 2016, Fujita-Odaka introduced δ-invariant. This invariant is more closely related to K-stability.

Definition of δ-invaraint

Suppose that X is a $\mathbb{Q}$-Fano variety. For a sufficiently large and divisible integer k, consider a basis $s_1, \cdots, s_{d_k}$ of the vector space $H^0(\mathcal{O}_X(-kK_X))$, where $d_k = h^0(\mathcal{O}_X(-kK_X))$. For this basis, consider $\mathbb{Q}$-divisor

$$\frac{1}{kd_k}\sum_{i=1}^{d_k}\{s_i = 0\} \sim_{\mathbb{Q}} -K_X.$$

Any $\mathbb{Q}$-divisor obtained in this way is called a k-basis type (anticanonical) divisor. Let

$$\delta_k(X) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every } k\text{-basis type } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Then let

$$\delta(X) = \limsup_{k\in\mathbb{N}} \delta_k(X).$$

Delta invariant

In 1990, Prof. Tian proposed the following in his paper.

Tian:

The author believes that the existence of Kähler-Einstein metric with positive scalar curvature should be closely related to the geometry of pluri-anti-canonical divisors.

This expectation is made precise by the following result.

Fujita-Odaka (2016), Blum-Jonsson (2017)

A Fano manifold X is uniformly K-stable if and only if the $\delta(X) > 1$.

But this new invariant is much harder to compute!

- The first attempt to compute this new invariant was made by Park-Won in 2016. And they showed that, the delta invariant of a cubic surface is bigger or equal to $\frac{36}{31}$. So it is K-stable and hence admits a Kähler-Einstein metric.

Joint work with Y. Rubinstein and I. Cheltsov

In 2018, we developed a geometric method to compue delta invariant.

I. Cheltsov, K. Zhang

The delta invariant of a smooth cubic surface is actually bigger or equal to $\frac{6}{5}$.

This result improves Park-Won's estimate.
We also considered log Fano surfaces.

I. Cheltsov, Y. Rubinstein, K. Zhang

The *log* delta invariant of $\mathbb{P}^1 \times \mathbb{P}^1$ blowing up at more than 6 general points on a $(1, 2)$-curve is bigger than one.

This result give a family of new examples of surfaces admitting conic Kähler-Einstein metrics.

Greatest Ricci lower bound

There is an analytic invariant defined on Fano manifold, which is known as the greatest Ricci lower bound (first studied by Tian in 1992).

The greatest Ricci lower bound (β-invaraint)

Let X be a Fano manifold. The greatest Ricci lower bound $\beta(X)$ is defined by

$$\beta(X) := \sup\{\lambda > 0 \mid \exists\, \omega \in 2\pi c_1(X) \text{ such that } Ric(\omega) > \lambda\omega\ \}.$$

Roughly speaking, $\beta(X)$ measures how far X is from being a Kähler-Einstein manifold. So it is an interesting problem to find the exact value of $\beta(X)$.

Z (2018)

On a Fano manifold X, one has

$$\beta(X) = \min\{1, \delta(X)\}$$

Analytic delta invariant

Inspired by the previous result, we can introduce an analytic delta invariant.

Analytic δ-invariant

Given a Fano manifold (X, ω) with $\omega \in 2\pi c_1(X)$. Let $\mathcal{M}_\omega(\cdot)$ be Mabuchi's K-energy. Let $I_\omega(\cdot)$ and $J_\omega(\omega)$ be the standard I, J functionals defined on the space of Kähler potentials. The analytic δ-invariant of X defined by

$$\tilde{\delta}(X) := \sup\{\delta > 0 \mid \mathcal{M}_\omega \geq (\delta - 1)(I_\omega - J_\omega)\}$$

We expect the following to hold.

Expectation

For a Fano manifold X, one has

$$\delta(X) = \tilde{\delta}(X).$$

One direction is known: $\delta(X) \geq \tilde{\delta}(X)$, by the work of Boucksom-Jonsson using non-Archimedean approach (2018). When $\tilde{\delta}(X) \leq 1$, this expectation is true since both δ and $\tilde{\delta}$ coincide with the greatest Ricci lower bound.

Thanks for your attention!