The optimal exponent of certain Moser–Trudinger type inequalities on projective manifolds

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Abstract We show that Fujita–Odaka's δ-invariant coincides with the optimal exponent of certain Moser–Trudinger type inequalities on projective manifolds. As a consequence we give an alternative proof of the uniform Yau–Tian–Donaldson conjecture for the existence of twisted Kähler–Einstein metrics.

1. Motivation

A central problem in Kähler geometry is to find canonical metrics on a given compact Kähler manifold. One important class of canonical metrics is the Kähler-Einstein (KE) metric. A Kähler metric is KE if the Ricci form of the Kähler metric is a constant multiple of the Kähler form. As we know, the Ricci form of a Kähler metric must lie in the first Chern class of the manifold. Therefore, a necessary condition for the existence of KE metric is that the first Chern class of the manifold has a sign.

The study of KE metrics has a long history. In the cases where the first Chern class is zero or negative, the uniqueness of the KE metric was proved by Calabi in the 1950s, and the existence of such a metric was obtained in 1978 by Yau [30] (see also Aubin [1]). However, when the first Chern class is positive (i.e., for Fano manifolds), the situation is much more complicated. It turns out that there are obstructions to the existence of KE metrics on Fano manifolds. The first obstruction was found by Matsushima [22] in 1957, which says that the automorphism group of a KE Fano manifold must be reductive. In 1983 another obstruction was found by Futaki in [19], where he defined an holomorphic invariant (which we now call Futaki invariant) and it was shown that the Futaki invariant must vanish if the Fano manifold admits a KE metric. In 1985, Bando-Mabuchi [2] showed that, if any, the KE metric on a Fano manifold is unique up to biholomorphic automorphisms.

So it is natural to ask, when does a Fano manifold admit a KE metric? Regarding this problem, many significant results were obtained in history. For instance, in 1990, Tian [26] completely solved the existence problem for Fano surfaces and showed that the existence of KE metrics is equivalent to the reductivity of the automorphism groups of Fano surfaces; in 2004, Xujia Wang and Xiaohua Zhu [29] showed that there always exist Kähler-Ricci solitons on toric Fano manifolds and the soliton metric is KE if and only if the Futaki invaraint vanishes.

For general Fano manifolds, the existence of KE metrics is more difficult to characterize. In 1992, Ding-Tian [14] defined a generalized Futaki invariant for a deformation family of Fano manifolds, and based on this, in 1997, Tian [27] introduced an algebro-geometric notion called K-stability. This notion was later reformulated by Donaldson [15] using more algebraic language. And the famous Yau–Tian–Donaldson conjecture says that, the existence of KE metrics on Fano manifolds is equivalent to K-stability. This conjecture was recently solved by Tian [28] and Chen-Donaldson-Sun [9] independently in 2012.

More generally, given a polarized Kähler manifold (X, L), one can try to find *twisted Kähler–Einstein* (tKE) metrics, which will be the main focus of this paper. More precisely, a Kähler form $\omega \in c_1(L)$ is called tKE if it satisfies

$$\mathrm{Ric}(\omega) = \omega + \theta$$

for some smooth form $\theta \in c_1(X) - c_1(L)$. So in particular this equation generalizes the KE equation on Fano manifolds. One would expect that such a equation is probably more difficult to solve than the one in the Fano case and a natural question would be: what is the right notion of stability that governs the solvability of this equation? This paper aims to give an answer to this question.

2. Main Results

Before stating our main results, we need to recall two invariants that will play prominent roles in this paper. One is the algebraic δ-invariant that is related to the notion of *Ding stability*, and another is the analytic δ-invariant, which governs the coercivity of Ding functionals.

We first recall the definition of algebraic δ-invariant, which was first introduced by Fujita–Odaka [17] using basis type divisors, and then reformulated by Blum–Jonsson [5] in a more valuative fashion as follows:

$$\delta(L) := \inf_E \frac{A_X(E)}{S_L(E)}.$$

Here E runs through all the prime divisors *over* X, i.e., E is a divisor contained in some birational model $Y \xrightarrow{\pi} X$ over X. Moreover,

$$A_X(E) := 1 + \mathrm{ord}_E(K_Y - \pi^* K_X)$$

denotes the log discrepancy, and

$$S(E) := S_L(E) := \frac{1}{\mathrm{vol}(L)} \int_0^\infty \mathrm{vol}(\pi^* L - xE)dx$$

denotes the expected vanishing order of L along E.

Historically, the case of the most interest is when $L = -K_X$ and $\theta = 0$, i.e., the Fano case. Regarding the existence of Kähler–Einstein metrics on such manifolds, the notion K-stability was introduced by Tian [27] and later reformulated more algebraically by Donaldson [15]. This stability notion has recently been further polished by Fujita and Li's valuative criterion [18, 21], and we now know (see [5, Theorem B]) that $\delta(-K_X) > 1$ is equivalent to $(X, -K_X)$ being uniformly K-stable, a condition stronger than K-stability (but actually these two are equivalent, at least in the smooth setting). It is also known that uniform K-stability is equivalent to the uniform Ding stability of Berman [3]. More recently Boucksom–Jonsson [5] further extend the definition of uniform Ding stability to general polarizations using δ-invariants, which we will adopt in this article.

Definition 1. We say (X, L) is uniformly Ding stable if $\delta(L) > 1$.

Under the YTD framework, it is expected that such a notion would imply the existence of tKE metrics. In the literature, the most examined case is when $= -K_X$, namely, the Fano setting. By using continuity methods (cf. [9, 28]) or the variational approach (cf. [6]), we now know that one can indeed find a KE metric $\omega \in c_1(X)$ solving

$$\mathrm{Ric}(\omega) = \omega$$

whenever $\delta(-K_X) > 1$.

However, to the author's knowledge, all the known approaches to the above statement does not work well for general polarization L, one main difficulty being that there is no convexity available for twisted K-energy in the non-Fano setting. In what follows we will present a quantization approach to circumvent this difficulty, which allows us to work even without the Fano condition.

More precisely, given any (not necessarily semi-positive) smooth representative $\theta \in c_1(X) - c_1(L)$, we want to investigate the following tKE equation:

$$\mathrm{Ric}(\omega_{tKE}) = \omega_{tKE} + \theta. \tag{1.1}$$

To study this, a crucial input is taken from the work of Ding [13], who essentially showed that the solvability of the above equation is governed by certain Moser–Trudinger type inequality. Inspired by this viewpoint, the author introduced an analytic δ-invariant in [31], which we now turn to describe.

Given any compact Kähler manifold (X, ω), put

$$\mathcal{H}(X,\omega) := \big\{\phi \in C^\infty(X,\mathbb{R}) \big| \omega_\phi := \omega + dd^c\phi > 0\big\}.$$

Let $E : \mathcal{H}(X,\omega) \to \mathbb{R}$ denote the Monge–Ampère energy defined by

$$E(\phi) := \frac{1}{(n+1)V} \sum_{i=0}^{n} \int_X \phi\omega^{n-i} \wedge \omega_\phi^i \text{ for } \phi \in \mathcal{H}(X,\omega).$$

The analytic δ-invariant of (X, ω) is then defined by

$$\delta^A([\omega]) := \sup\left\{\lambda > 0 \middle| \exists C_\lambda > 0 \text{ s.t. } \int_X e^{-\lambda(\phi - E(\phi))}\omega^n < C_\lambda \text{ for any } \phi \in \mathcal{H}(X,\omega)\right\}, \tag{1.2}$$

which does not depend on the choice of ω. As we will see, $\delta^A(L) > 1$ is equivalent the coercivity of certain twisted Ding functional whose critical point gives rise to the desired tKE metric solving (1.1). When $\omega \in c_1(L)$ for some ample line bundle L, we also put

$$\delta(L) := \delta([\omega]).$$

Our first main result can be stated as follows.

Theorem 2 ([32]). *For any ample line bundle L, one has*

$$\delta(L) = \delta^A(L).$$

In particular uniform Ding stability implies the coercivity of twisted Ding functionals and as a consequence, we solve (1.1).

Theorem 3 ([32]). *Assume that (X, L) is uniformly Ding stable. Then for any smooth form $\theta \in c_1(X) - c_1(L)$, there exists a Kähler form $\omega_{tKE} \in c_1(L)$ solving*

$$\mathrm{Ric}(\omega_{tKE}) = \omega_{tKE} + \theta.$$

The proof of Theorem 2 uses the quantization approach initiated in [23], which not only strengthens the work of Berman–Boucksom–Jonsson [6] but also simplifies their argument substantially.

3. Basis divisors and quantization

3.1 Basis divisors

Let (X, L) be a polarized pair, where X is an n-dimensional projective manifold and L is an ample line bundle on X. Recently, Fujita-Odaka [17] introduced an δ-invariant in the study of K-stability of Fano varieties. We begin with a general definition.

Definition 4. For any $m \geqslant 1$, we set

$$d_m := \dim_{\mathbb{C}} H^0(X, mL) > 0.$$

For any basis $s_1, ..., s_{d_m}$ of $H^0(X, mL)$, let D_i be the divisor cut out by s_i and we consider the $\mathbb{Q}$-divisor

$$D = \frac{1}{md_m} \sum_{i=0}^{d_m} D_i,$$

which we call a m-basis divisor of L. We set

$$\delta_m(L) := \sup\{c > 0 |\ (X, cD) \text{ is lc for any } m\text{-basis divisor } D \text{ of } L\}.$$

And we define the delta invariant by

$$\delta(L) := \limsup_{m \to \infty} \delta_m(X, L).$$

We remark that the above limsup is actually a limit (see [5, Theorem A]) and this definition coincides with 4. To be more precise, let $\pi : Y \to X$ be a proper birational morphism and let $F \subset Y$ be a prime divisor F in Y. We say that F *is a divisor over* X. Let

$$S_m(F) := \frac{1}{md_m} \sum_{j=1}^{\infty} \dim H^0(Y, m\pi^\star L - jF)$$

denote the expected vanishing order of L along F at level m. (the sum, of course, only runs up to a certain finite j that will be defined shortly). Also note that one has

$$S_m(F) = \sup \big\{ \mathrm{ord}_F(D) \,:\, m\text{-basis divisor } D \text{ of } L \big\},$$

and this supremum is attained by any m-basis divisor D arising from a basis $\{s_i\}$ that is compatible with the filtration

$$\{H^0(Y, m\pi^\star L - jF)\}_{j=0}^{\tau_m(\pi^\star L, F)}, \text{ where } \tau_m(\pi^\star L, F) := \max\{x \in \mathbb{N} \,:\, H^0(Y, m\pi^\star L - xF) \neq 0\}, \tag{1.3}$$

i.e., each $H^0(Y, m\pi^\star L - jF)$ is spanned by a subset of the $\{s_i\}_{i=1}^{d_m}$ [17, Lemma 2.2] (see [7, Lemma 2.7] for an exposition). Here $\mathrm{ord}_F(D)$ is the vanishing order of $\pi^\star D$ along F. Then by [17],

$$\delta_m(L) = \inf_{F \text{ over } X} \frac{A_X(F)}{S_m(F)}. \tag{1.4}$$

A well-known fact is that this infimum is attained by some F. The above equality actually follows from the following standard fact: for any effective $\mathbb{R}$-divisor $D \subset X$, its *log canonical threshold* is defined by

$$\mathrm{lct}(X, D) := \inf_F \frac{A_X(F)}{\mathrm{ord}_F(D)}. \tag{1.5}$$

Now the key point is that as $m \to \infty$, one has

$$S_m(F) \to S(F)$$

and one actually can find $\varepsilon_m \to 0$ such that (see [5])

$$S_m(F) \leqslant (1 + \varepsilon_m) S(F) \text{ for any } F.$$

These properties easily imply that

$$\delta(L) = \lim_{m \to \infty} \inf_F \frac{A_X(F)}{S_m(F)} = \inf_F \lim_{m \to \infty} \frac{A_X(F)}{S_m(F)} = \inf_F \frac{A_X(F)}{S(F)}.$$

3.2 Quantized analytic δ-invariant

Let (X, L) be a polarized Kähler manifold. Let $\mathcal{P}_m$ denote the space of all Hermitian inner products on the complex vector space $H^0(X, mL)$. As observed by Donaldson [16] a fundamental Bott–Chern type functional on $\mathcal{P}_m \times \mathcal{P}_m$ is

$$E_m(H, K) := \frac{1}{md_m} \log\det K^{-1}H. \tag{1.6}$$

In practice it is convenient to fix some H in the first slot and, in the second slot, to pull-back via the isomorphism $\mathrm{FS} : \mathcal{P}_m \to \mathcal{B}_m$,

$$\mathrm{FS}(K) := \frac{1}{m} \log \sum_{i=1}^{d_m} |\sigma_i|^2_{h^m},$$

where $\{\sigma_i\}$ is an(y) orthonormal basis of K, and where $\mathcal{B}_m$ denotes the image of $\mathcal{P}_m$ via FS, also called the m-th Bergman space,

$$\mathcal{B}_m := \left\{ \varphi = \frac{1}{m} \log \sum_{i=1}^{d_m} |\sigma_i|^2_{h^m} \ : \ \{\sigma_i\}_{i=1}^{d_m} \text{ is a basis of } H^0(X, mL) \right\}.$$

This yields a functional $E_m\big(H, \mathrm{FS}^{-1}(\,\cdot\,)\big)$, that we also denote by

$$E_m(H, \varphi) := E_m(H, \mathrm{FS}^{-1}(\varphi)) = E_m(H, K), \quad \text{for } \varphi = \mathrm{FS}(K) \in \mathcal{B}_m.$$

As shown by Donaldson E_m is the natural quantization of E [16, §3]. More precisely, for any $\phi \in \mathcal{H}(X, \omega)$, put

$$H_m^\phi := \int_X (he^{-\phi})^m (\cdot, \cdot)\omega^n.$$

We denote by $\phi^{(m)} \in \mathcal{B}_m$ the *Bergman projection*. It is given by

$$\phi^{(m)} := \frac{1}{m} \log \sum_{i=1}^{d_m} |s_i|^2_{h^m},$$

where $\{s_i\}$ is an H_m^ϕ-orthonormal basis of $H^0(X, mL)$. Then one has

$$\lim_{m\to} E_m(H_m^0, \phi^{(m)}) = E(\phi).$$

Definition 5 ([23]). Let L be an ample line bundle. The m-th analytic δ-invariant is defined by

$$\delta_m^A(L) := \sup \left\{ \delta > 0 \ : \ \sup_{\varphi\in\mathcal{B}_m} \int_X e^{-\delta(\varphi - E_m(H,\varphi))}\omega^n < \infty \right\}.$$

One should think of $\delta_m^A(L)$ as the optimal constant in a *quantized Moser–Trudinger inequality*. Note that this definition depends neither on the choice of H (due to the cocyclic nature of E_m, i.e., $E_m(H,K)+E_m(K,N)=E_m(H,N)$), nor on the choice of h (and hence of ω). In practice we shall often use

$$H_m := H_m^0 = \int_X h^m(\cdot,\cdot)\omega^n$$

as a reference in $\mathcal{P}_m$, in which case we will write

$$E_m(\varphi) := E_m(H_m,\varphi) \text{ for } \varphi \in \mathcal{B}_m.$$

3.3 Comparing E with E_m

Given any $\phi \in \mathcal{H}(X,\omega)$, it has been known since the work of Donaldson that $E(\phi) = \lim_{m\to\infty} E_m(\phi^{(m)})$. But this convergence is not uniform when ϕ varies in $\mathcal{H}(X,\omega)$, which is the main stumbling block in the quantization approach. To overcome this, we recall a quantized maximum principle due to Berndtsson [4].

The setup goes as follows. For any ample line bundle E over X, let g be a smooth positively curved metric on E with $\eta := -dd^c \log g > 0$ being its curvature form. Pick two elements $\phi_0, \phi_1 \in \mathcal{H}(X,\eta)$. It was shown by Chen [8] and more recently by Chu–Tosatti–Weinkove [10] that there always exists a $C^{1,1}$-*geodesic* ϕ_t joining ϕ_0 and ϕ_1. For the reader's convenience, we briefly recall the definition. Let $[0,1] \ni t \mapsto \phi_t$ be a family of functions on $[0,1]\times X$ with $C^{1,1}$ regularity up to the boundary. Let $S := \{0 < \operatorname{Re} s < 1\} \subset \mathbb{C}$ be the unit strip and let $\pi : S \times X \to X$ denote the projection to the second component. Then we say ϕ_t is a $C^{1,1}$-*subgeodesic* if it satisfies $\pi^*\eta + dd^c_{S\times X}\phi_{\operatorname{Re} s} \geqslant 0$. We say it is a $C^{1,1}$-geodesic if it further satisfies the homogenous Monge–Ampère equation: $\left(\pi^*\eta + dd^c_{S\times X}\phi_{\operatorname{Re} s}\right)^{n+1} = 0$.

Now given any $C^{1,1}$ subgeodesic joining ϕ_0 and ϕ_1, one may consider

$$Hilb^{\phi_t} := \int_X g(\cdot,\cdot)e^{-\phi_t},$$

which is a family of Hermitian inner products on $H^0(X, E+K_X)$ joining $Hilb^{\phi_0}$ and $Hilb^{\phi_1}$. Note that we do not need any volume form in the above integral. Then Berndtsson's quantized maximum principle says the following, which in fact holds for subgeodesics with much less regularity; see [11, Proposition 2.12].

Proposition 6. *[4, Proposition 3.1] Let $[0,1] \ni t \mapsto H_t$ be the Bergman geodesic connecting $Hilb^{\phi_0}$ and $Hilb^{\phi_1}$. Then one has*

$$H_t \leqslant Hilb^{\phi_t} \text{ for } t \in [0,1].$$

We will now apply this result to the setting where $E := mL - K_X$ and $g := h^m \otimes \omega^n$. As a consequence, we obtain the following key estimate, which can be viewed as a weak version of the "partial C^0 estimate".

Proposition 7 ([32]). *For any $\varepsilon \in (0,1)$, there exist $m_0 = m_0(X, L, \omega, \varepsilon) \in \mathbb{N}$ such that*

$$E(\phi) \leqslant E_m\big(((1-\varepsilon)\phi)^{(m)}\big) + \varepsilon \sup \phi$$

for any $m \geqslant m_0$ and any $\phi \in \mathcal{H}(X, \omega)$.

Proof. Since the statement is translation invariant, we assume that $\sup \phi = 0$. Let $[0,1] \ni t \mapsto \phi_t$ be a $C^{1,1}$ geodesic connecting 0 and ϕ, with $\phi_0 = 0$ and $\phi_1 = \phi$. The geodesic condition implies that ϕ_t is convex in t so we have

$$\dot{\phi}_0 := \frac{d}{dt}\Big|_{t=0} \phi_t \leqslant 0$$

as $\phi \leqslant 0$. Put $\tilde{\phi}_t := (1-\varepsilon)\phi_t$. Observe that $(he^{-\tilde{\phi}_t})^m \otimes \omega^n$ gives rise to a family of Hermitian metrics on $mL - K_X$, which is in fact a $C^{1,1}$ subgeodesic whenever m satisfies $m\varepsilon\omega \geqslant -\operatorname{Ric}(\omega)$. Indeed, let $S := \{0 < \operatorname{Re} s < 1\} \subset \mathbb{C}$ be the unit strip and let $\pi : S \times X \to X$ denote the projection to the second component. Then $(he^{-\tilde{\phi}_{\operatorname{Re} s}})^m \otimes \omega^n$ induces a Hermitian metric on $\pi^*(mL - K_X)$ over $S \times X$ whose curvature form satisfies

$$\pi^*(m\omega + \operatorname{Ric}(\omega)) + m(1-\varepsilon) dd^c_{S\times X} \phi_{\operatorname{Re} s} \geqslant 0$$

whenever $m\varepsilon\omega \geqslant -\operatorname{Ric}(\omega)$. It then follows from Proposition 6 that

$$H_{m,t} \leqslant H_m^{\tilde{\phi}_t} \text{ for } t \in [0,1],$$

where $[0,1] \ni t \mapsto H_{m,t}$ is the Bergman geodesic in $\mathcal{P}_m(X, L)$ joining H_m^0 and $H_m^{(1-\varepsilon)\phi}$ with $H_{m,0} = H_m^0$ and $H_{m,1} = H_m^{(1-\varepsilon)\phi}$. So we obtain that

$$E_m(FS(H_{m,t})) \geqslant E_m(FS(H_m^{\tilde{\phi}_t})) \text{ for } t \in [0,1],$$

with equality at $t = 0, 1$. Fixing an H_m^0-orthonormal basis $\{s_i\}$ of R_m, then we obtain that

$$E_m\big(((1-\varepsilon)\phi)^{(m)}\big) = \frac{d}{dt}\Big|_{t=0} E_m(FS(H_{m,t})) \geqslant \frac{d}{dt}\Big|_{t=0} E_m(FS(H_m^{\tilde{\phi}_t})) = \frac{1-\varepsilon}{d_m} \int_X \dot{\phi}_0 \Big(\sum_{i=1}^{d_m} |s_i|^2_{h^m} \Big) \omega^n,$$

where the last equality is from a direct calculation using the definition of E_m. Now by the first order expansion of Bergman kernels going back to Tian [25] (with respect to the background metric ω), one has

$$\frac{\sum_{i=1}^{d_m} |s_i|^2_{h^m}}{d_m} \leqslant \frac{1}{(1-\varepsilon)V}$$

for all $m \gg 1$. So we arrive at (recall $\dot{\phi}_0 \leqslant 0$)

$$E_m\big(((1-\varepsilon)\phi)^{(m)}\big) \geqslant \frac{1}{V}\int_X \dot{\phi}_0 \omega^n = E(\phi),$$

where the last equality follows from the well-known fact that E is linear along the geodesic ϕ_t. This completes proof. □

One can also bound E from below in terms of E_m on the Bergman space $\mathcal{B}_m(X,\omega)$.

Proposition 8. *For any $\varepsilon > 0$, there exists $m_0 = m_0(X, L, \omega, \varepsilon) \in \mathbb{N}$ such that*

$$E_m(\phi) \leqslant (1-\varepsilon)E(\phi) + \varepsilon \sup \phi + \varepsilon.$$

for any $m \geqslant m_0$ and $\phi \in \mathcal{B}_m(X,\omega)$.

Proof. For any $\varphi \in \mathcal{B}_m$, we may write $\varphi = \frac{1}{m}\log\sum_{i=1}^{d_m} e^{\lambda_i}|s_i|^2_{h^m}$ for some H_m-orthonormal basis $\{s_i\}$ and $\lambda_i \in \mathbb{R}$. Set $\lambda_{max} := \max_i\{\lambda_i\}$ and $\varphi(t) := \frac{1}{m}\log\sum_{i=1}^{d_m} e^{\lambda_i t}|s_i|^2_{h^m}$, $t \geqslant 0$. Note that E is convex along Bergman geodesics (cf. [16, Proposition 1]). Thus

$$\begin{aligned}
E(\varphi) &= E(\varphi_1) - E(\varphi_0) + E(\varphi_0)\\
&\geqslant \frac{d}{dt}\Big|_{t=0} E(\varphi_t) + E(\varphi_0)\\
&= \frac{1}{V}\int_X \dot{\varphi}_0 \omega^n_{\varphi_0} + E(\varphi_0)\\
&= \frac{1}{mV}\int_X \frac{\sum_{i=1}^{d_m}\lambda_i|s_i|^2_{h^m}}{\sum_{i=1}^{d_m}|s_i|^2_{h^m}}\omega^n_{\varphi_0} + E(\varphi_0)\\
&= \frac{1}{mV}\int_X \frac{\sum_{i=1}^{d_m}(\lambda_i - \lambda_{max})|s_i|^2_{h_m}}{\rho_m}\omega^n_{\frac{1}{m}\log\rho_m} + \frac{\lambda_{max}}{m} + E(\frac{1}{m}\log\rho_m),
\end{aligned}$$

where $\rho_m := \sum_{i=1}^{d_m}|s_i|^2_{h^m}$. For each $m \gg 1$ let $\{\sigma_i\}_{i=1}^{d_m}$ be an H_m-orthonormal basis of $H^0(X, mL)$. By the classical Bergman kernel asymptotics,

$$\frac{\rho_m}{d_m} \xrightarrow{C^\infty} \frac{1}{V}, \tag{1.7}$$

implying that

$$\mathcal{B}_m \ni \frac{1}{m}\log\rho_m \xrightarrow{C^\infty} 0. \tag{1.8}$$

Using (1.7) and (1.8), one finds $\varepsilon_m \to 0$ (independent of $\varphi \in \mathcal{B}_m$) such that

$$\begin{aligned} E(\varphi) &\geqslant \frac{1+\varepsilon_m}{md_m}\int_X \sum_{i=1}^{d_m}(\lambda_i - \lambda_{max})|s_i|^2_{h^m}\omega^n + \frac{\lambda_{max}}{m} - \varepsilon_m \\ &= \frac{1+\varepsilon_m}{md_m}\sum_{i=1}^{d_m}(\lambda_i - \lambda_{max}) + \frac{\lambda_{max}}{m} - \varepsilon_m \\ &= (1+\varepsilon_m)\left(E_m(\varphi) - \frac{\lambda_{max}}{m}\right) + \frac{\lambda_{max}}{m} - \varepsilon_m. \end{aligned}$$

Then the desired inequality follows. □

4. Proving the main result

Let (X, L) be a polarized Kähler manifold. The goal of this chapter is to prove Theorem 2, namely,

$$\delta(L) = \delta^A(L).$$

The first step is to show that $\delta_m(L) = \delta^A_m(L)$ whenever m is sufficiently large. The second step is then to push everything to limit as $m \to \infty$.

1.0.1 4.1 At level m

The goal of this part is to show the following result.

Theorem 9. *For all $m \gg 1$, one has*

$$\delta_m(L) = \delta^A_m(L).$$

In Proposition 10 we show that to compute δ_m-invariant, it is enough to consider all the orthonormal basis of $H^0(X, mL)$ with respect to a fixed Hermitian inner product $H \in \mathcal{P}_m$. We apply this to conclude that $\delta_m(L) \leqslant \delta^A_m(L)$ (Corollary 11).

Proposition 10. *For any $H \in \mathcal{P}_m$,*

$$\delta_m(L) = \sup\left\{\delta > 0 \,:\, \sup_{\{s_i\}H\text{-}o.n.b.}\int_X \frac{\omega^n}{\prod_{i=1}^{d_m}|s_i|_{h^m}^{\frac{2\delta}{md_m}}} < \infty\right\}.$$

Proof. We claim that

$$\delta_m(L;H) = \sup\left\{\delta > 0 \,:\, \int_X \frac{\omega^n}{\prod_{i=1}^{d_m} |s_i|_{h^m}^{\frac{2\delta}{md_m}}} < \infty, \quad \text{for all } H\text{-orthonormal bases } \{s_i\}_{i=1}^{d_m}\right\}. \tag{1.9}$$

Indeed, denote the RHS of (1.9) by $\tilde{\delta}_m(L;H)$. Then clearly $\delta_m(L;H) \leqslant \tilde{\delta}_m(L;H)$. If $\delta_m(L;H) < \tilde{\delta}_m(L;H)$, then we can find $\delta \in (\delta_m(L;H), \tilde{\delta}_m(L;H))$ and a sequence of H-orthonormal bases $\{s_i^{(j)}\}_{i=1}^{d_m}$ such that

$$\lim_{j\to\infty} \int_X \frac{\omega^n}{\prod_{i=1}^{d_m} |s_i^{(j)}|_{h^m}^{\frac{2\delta}{md_m}}} = \infty.$$

Up to a subsequence, $\{s_i^{(j)}\}$ converges smoothly to an H-orthonormal basis $\{s_i^{(\infty)}\}$. Then by the lower semi-continuity of complex singularity exponents [12, Theorem 0.2(3)],

$$\lim_{j\to\infty} \int_X \frac{\omega^n}{\prod_{i=1}^{d_m} |s_i^{(j)}|_{h^m}^{\frac{2\delta}{md_m}}} = \int_X \frac{\omega^n}{\prod_{i=1}^{d_m} |s_i^{(\infty)}|_{h^m}^{\frac{2\delta}{md_m}}} < \infty,$$

a contradiction. This proves (1.9).

Now for any F over X, we consider the filtration (1.3). Given $H \in \mathcal{P}_m$, observe that one can choose a compatible H-orthonormal basis $\{s_i\}$ so $S_m(F) = \mathrm{ord}_F(\pi^\star D)$ with D the basis divisor associated to $\{s_i\}_{i=1}^{d_m}$. Namely,

$$S_m(F) = \sup\big\{\mathrm{ord}_F(D) \,:\, m\text{-basis divisor } D \text{ arising from } H\text{-orthonormal basis}\big\}. \tag{1.10}$$

Combining (1.5) and (1.10),

$$\delta_m(L) = \inf\big\{\mathrm{lct}(X,D) \,:\, m\text{-basis divisor } D \text{ arising from } H\text{-orthonormal basis}\big\}.$$

Thus, by the analytic interpretation of lct [20, §8],

$$\delta_m(L) = \sup\left\{\delta > 0 \,:\, \int_X \frac{\omega^n}{\prod_{i=1}^{d_m} |s_i|_{h^m}^{\frac{2\delta}{md_m}}} < \infty, \quad \text{for all } H\text{-orthonormal bases } \{s_i\}_{i=1}^{d_m}\right\}.$$

Combining this with (1.9) concludes the proof. □

Corollary 11. $\delta_m(L) \leqslant \delta_m^A(L)$.

Proof. We first reformulate Definition 5. Fix a reference Hermitian inner product $H \in \mathcal{P}_m$. Then E_m (1.6) can be expressed by

$$E_m(H,\varphi) = \frac{1}{md_m} \log\det\big[H(\sigma_i,\sigma_j)\big]_{i,j=1}^{d_m}, \tag{1.11}$$

for any $\varphi = \mathrm{FS}(K) = \frac{1}{m}\log\sum_{i=1}^{d_m}|\sigma_i|^2_{h^m} \in \mathcal{B}_m$, where $\{\sigma_i\}$ is K-orthonormal. By linear algebra, for any basis $\{\sigma_i\}$ of $H^0(X, mL)$, after a unitary transformation, one may diagonalize it so that

$$\sigma_i = \mu_i^{1/2} s_i$$

for some H-orthonormal basis $\{s_i\}$, with $\mu_i > 0$. Using such convention, one can also write

$$E_m(H,\varphi) = \frac{1}{md_m}\log\prod_{i=1}^{d_m}\mu_i. \tag{1.12}$$

Note that a different choice of H will only shift E_m by a constant. Thus, Definition 5 becomes

$$\delta_m^A(L) = \sup\left\{\delta > 0 \ : \ \sup_{\substack{\{s_i\} H\text{-o.n.b.}\\ \mu_i > 0}} \int_X \frac{\prod_{i=1}^{d_m}\mu_i^{\frac{\delta}{md_m}}}{\left(\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m}\right)^{\frac{\delta}{m}}}\omega^n < \infty\right\}. \tag{1.13}$$

By the arithmetic mean–geometric mean inequality,

$$\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m} \geqslant d_m\Big(\prod_{i=1}^{d_m}\mu_i\Big)^{\frac{1}{d_m}}\Big(\prod_{i=1}^{d_m}|s_i|^2_{h^m}\Big)^{\frac{1}{d_m}},$$

thus for any H-orthonormal basis $\{s_i\}$ and parameters $\mu_i > 0$,

$$\int_X \frac{\prod_{i=1}^{d_m}\mu_i^{\frac{\delta}{md_m}}}{(\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m})^{\frac{\delta}{m}}}\omega^n \leqslant \left(\frac{1}{d_m}\right)^{\frac{\delta}{m}} \cdot \int_X \frac{\omega^n}{\prod_{i=1}^{d_m}|s_i|^{\frac{2\delta}{md_m}}_{h^m}},$$

and the statement follows from Proposition 10. □

We now turn to proving the harder inequality,

$$\delta_m(L) \geqslant \delta_m^A(L). \tag{1.14}$$

Our strategy is as follows. Fix a prime divisor F over X. We find an H-orthogonal basis $\{s_i\}$ of $H^0(X, mL)$ such that the integral

$$\int_X \frac{\prod_{i=1}^{d_m}\mu_i^{\frac{\delta}{md_m}}}{\left(\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m}\right)^{\frac{\delta}{m}}}\omega^n \tag{1.15}$$

has no uniform upper bound for an appropriate choice of positive numbers $\{\mu_i\}$, whenever δ satisfies $\delta > \frac{A_X(F)}{S_m(F)}$. This implies $\delta_m^A(L) \leqslant \inf_F \frac{A_X(F)}{S_m(F)} = \delta_m(L)$, i.e., (1.14), which, when combined with Corollary 11 will conclude the proof of Theorem 9.

Definition 12. Let F be a prime divisor over X and let $\{s_i\}$ be an H-orthonormal basis of $H^0(X, mL)$ compatible with the filtration (1.3). The Bergman geodesic ray associated to $(F, \{s_i\})$ is

$$\varphi_F(t) := \frac{1}{m} \log \sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)} |s_i|^2_{h^m} \in \mathcal{B}_m, \quad t \in \mathbb{R}. \tag{1.16}$$

A simple, but key, observation is that the m^{th} expected vanishing can be viewed as the slope of the Monge–Ampère energy.

Lemma 13. *Let $\varphi(t)$ be defined by (1.16). Then, $E_m(H, \varphi_F(t)) = tS_m(F)$.*

Proof. By (1.12), $E_m(H, \varphi_F(t)) = \frac{t}{md_m} \sum_{i=1}^{d_m} \mathrm{ord}_F(s_i)$. For any basis $\{s_i\}$ compatible with the filtration,

$$\begin{aligned}
\sum_{i=1}^{d_m} \mathrm{ord}_F(s_i) &= \sum_{j=0}^{\infty} j \Big[\dim H^0(Y, m\pi^\star L - jF) - \dim H^0(Y, m\pi^\star L - (j+1)F) \Big] \\
&= \sum_{j=1}^{\infty} \dim H^0(Y, m\pi^\star L - jF),
\end{aligned}$$

where in the last line we used that $H^0(Y, m\pi^\star L - jF)$ vanishes for large enough j (recall (1.3)). Thus, by the definition of S_m the proof is complete. □

Fix $F \subset Y$ over X and let $\{s_i\}$ be a basis as in the proof of Lemma 13. We evaluate (1.15) along the Bergman geodesic $\varphi_F(t)$ of Definition 12, i.e., put $\mu_i(t) = e^{t\,\mathrm{ord}_F(s_i)}$, and use Lemma 13,

$$\begin{aligned}
(1.15)(t) &= \int_X \frac{e^{t\delta S_m(F)}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)} |s_i|^2_{h^m} \right)^{\frac{\delta}{m}}} \omega^n \\
&= e^{t(\delta S_m(F) - A_X(F))} \int_X \frac{e^{tA_X(F)}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)} |s_i|^2_{h^m} \right)^{\frac{\delta}{m}}} \omega^n.
\end{aligned} \tag{1.17}$$

Now the key estimate is the following.

Lemma 14. *There exists $C > 0$ such that*

$$\int_X \frac{e^{tA_X(F)}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)} |s_i|^2_{h^m} \right)^{\frac{\delta}{m}}} \omega^n > C > 0, \ \textit{for all } t \geqslant 0.$$

Proof. Let $Z := \pi(F)$ denote the center of the divisorial valuation ord_F on X. We will show that the desired estimate follows from a local calculation around Z. More percisely, Let Ω be a tubular neighborhood around Z. It suffices to find some $\eta > 0$ such that

$$\int_\Omega \frac{e^{tA_X(F)}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)}|s_i|^2_{h^m}\right)^{\frac{\delta}{m}}}\omega^n \geqslant \eta > 0$$

for any $t \geqslant 0$. We will achieve this estimate by pulling back everything to Y. Note,

$$K_Y = \pi^\star K_X + (A_X(F) - 1)F + D, \tag{1.18}$$

where D is some divisor whose support does not contain F. Then we choose a small enough coordinate chart

$$\Big(U, (z_1, \cdots, z_n)\Big) \subseteq Y,$$

centered around some smooth point of F with the following properties:
1. U is away from all the other exceptional divisors of π (i.e., $U \cap \mathrm{Supp}(D) = \emptyset$);
2. Over U, one has $F = \{z_1 = 0\}$;

U contains the polydisk $\mathbb{D} := \Big\{(z_1, ..., z_n) \,:\, |z_i| \leqslant 1,\ \forall i\Big\}$;
3. $\pi^\star(mL)$ is trivialized over $\mathbb{D}$, so that each $\pi^\star s_i$ can be represented as $\pi^\star s_i = z_1^{\mathrm{ord}_F(s_i)} g_i(z)$, where $g_i(z)$ is some holomorphic function on $\mathbb{D}$;
4. In the above trivialization, there exists some constant $C > 0$ such that $h^m < C,\ |g_i|^2 < C,\ \forall i$.
Using (1.18) and (2), the volume form $\pi^\star\omega^n$ can be replaced (up to some bounded factor) by

$$|z_1|^{2A_X(F)-2}(\sqrt{-1})^n dz_1 \wedge \overline{dz_1} \wedge \cdots \wedge dz_n \wedge \overline{dz_n},$$

since we are working away from D.

Therefore, to finish the proof of Lemma 14, it suffices to find some constant $c > 0$ such that for any $t \geqslant 0$,

$$\int_{\mathbb{D}} \frac{e^{tA_X(F)}|z_1|^{2A_X(F)-2}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)}|z_1|^{2\mathrm{ord}_F(s_i)}|g_i|^2 h^m\right)^{\frac{\delta}{m}}}(\sqrt{-1})^n dz_1\wedge\overline{dz_1}\wedge\cdots\wedge dz_n\wedge\overline{dz_n} \geqslant c > 0.$$

Using condition (5) above, it suffices to bound

$$
\begin{aligned}
J(t) &:= \sqrt{-1}\int_{|z_1|\leqslant 1} \frac{e^{tA_X(F)}|z_1|^{2A_X(F)-2}}{\left(\sum_{i=1}^{d_m} |e^{t/2}z_1|^{2\mathrm{ord}_F(s_i)}\right)^{\frac{\delta}{m}}} dz_1\wedge d\bar{z_1}\\
&= \sqrt{-1}\int_{|w|\leqslant e^{t/2}} \frac{|w|^{2(A_X(F)-1)}}{\left(\sum_{i=1}^{d_m} |w|^{2\mathrm{ord}_F(s_i)}\right)^{\frac{\delta}{m}}} dw\wedge d\bar{w}\\
&\geqslant \sqrt{-1}\int_{|w|\leqslant 1} \frac{|w|^{2(A_X(F)-1)}}{\left(\sum_{i=1}^{d_m} |w|^{2\mathrm{ord}_F(s_i)}\right)^{\frac{\delta}{m}}} dw\wedge d\bar{w},
\end{aligned}
$$

where in the last inequality we used $t \geqslant 0$. This last integral is some positive quantity depending only on δ, m, $A_X(F)$ and $\{\mathrm{ord}_F(s_i)\}_{1\leqslant i\leqslant d_m}$. This completes the proof of Lemma 14. □

Corollary 15. $\delta_m(L) \geqslant \delta_m^A(L)$.

Proof. By (1.17) and Lemma 14, $\lim_{t\to\infty}(1.15)(t) = \infty$ if $\delta > \frac{A_X(F)}{S_m(F)}$. Thus, by (1.13), $\delta_m^A(L) \leqslant \frac{A_X(F)}{S_m(F)}$. Taking the infimum over all F and using (1.4) we conclude. □

Proof of Theorem 9. This follows from Corollaries 11 and 15. □

4.2 Taking the limit

We begin by recalling the α-invariant of Tian [24]. Set

$$
\alpha(L) := \sup\left\{\alpha > 0 \,\middle|\, \exists C_\alpha > 0 \text{ s.t. } \int_X e^{-\alpha(\phi-\sup\phi)}\omega^n < C_\alpha \text{ for all } \phi\in\mathcal{H}(X,\omega)\right\}. \tag{1.19}
$$

Note that $\alpha(L)$ will be used several times in what follows, as it can effectively control bad terms when doing integration.

Now we are ready to prove the following main result.

Theorem 16. *Let L be an ample line bundle, then one has $\delta^A(L) = \delta(L)$*

Proof. The proof splits into two steps.

Step 1: $\delta^A(L) \leqslant \delta(L)$.

By Theorem 9, it suffices to show that, for any $\lambda\in(0,\delta^A(L))$ one has $\delta_m(L) > \lambda$ for all $m \gg 1$. In other words, for any $m \gg 1$, we need to find some constant

$C_{m,\lambda} > 0$ such that

$$\int_X e^{-\lambda(\phi - E_m(\phi))}\omega^n < C_{m,\lambda} \text{ for all } \phi \in \mathcal{B}_m(X,\omega).$$

Assume that $\sup \phi = 0$. For any small $\varepsilon > 0$, by Proposition 8 and Hölder's inequality,

$$\begin{aligned}\int_X e^{-\lambda(\phi - E_m(\phi))}\omega^n &\leqslant \int_X e^{-\lambda(\phi-(1-\varepsilon)E(\phi))+\lambda\varepsilon} d\mu_\theta = e^{\lambda\varepsilon} \cdot \int_X e^{-\lambda(1-\varepsilon)(\phi - E(\phi))} \cdot e^{-\lambda\varepsilon\phi}\omega^n \\ &\leqslant e^{\lambda\varepsilon}\left(\int_X e^{\frac{-\lambda(1-\varepsilon)}{1-\frac{\lambda\varepsilon}{\alpha}}(\phi - E(\phi))}\omega^n\right)^{1-\frac{\lambda\varepsilon}{\alpha}}\left(\int_X e^{-\alpha\phi}\omega^n\right)^{\frac{\lambda\varepsilon}{\alpha}}\end{aligned}$$

holds for all $m \geqslant m_0(X, L, \omega, \varepsilon)$, where $\alpha \in (0, \alpha(L))$ is some fixed number. We may fix $\varepsilon \ll 1$ such that

$$\frac{\lambda(1-\varepsilon)}{1-\frac{\lambda\varepsilon}{\alpha}} < \delta^A(L).$$

Then by (1.2) and (1.19), there exist $C_\lambda > 0$ and $C_\alpha > 0$ such that

$$\int_X e^{-\lambda(\phi - E_m(\phi))}\omega^n < e^{\lambda\varepsilon}(C_\lambda)^{1-\frac{\lambda\varepsilon}{\alpha}}(C_\alpha)^{\frac{\lambda\varepsilon}{\alpha}}$$

for all $\phi \in \mathcal{B}_m(X,\omega)$ whenever m is large enough. This proves the assertion.

Step 2: $\delta^A(L) \geqslant \delta(L)$.

It suffices to show that, for any $\lambda \in (0, \delta(L))$, there exists $C_\lambda > 0$ such that

$$\int_X e^{-\lambda(\phi - E(\phi))} d\mu_\theta < C_\lambda \text{ for any } \phi \in \mathcal{H}(X,\omega).$$

Again assume that $\sup \phi = 0$. Fix any number $\alpha \in (0, \alpha(L))$. Let also $\varepsilon > 0$ be a sufficiently small number, to be fixed later. Set $\tilde{\phi} := (1-\varepsilon)\phi$. Then by Proposition 7 and the generalized Hölder inequality, for any $m \geqslant m_0(X, L, \omega, \varepsilon)$, we can write

$$\begin{aligned}\int_X e^{-\lambda\left(\phi - E(\phi)\right)} d\mu_\theta &\leqslant \int_X e^{-\lambda\left(\phi - E_m(\tilde{\phi}^{(m)})\right)} d\mu_\theta = \int_X e^{\lambda\left(\tilde{\phi}^{(m)} - \tilde{\phi}\right)} \cdot e^{-\lambda\left(\tilde{\phi}^{(m)} - E_m(\tilde{\phi}^{(m)})\right)} \cdot e^{-\lambda\varepsilon\phi} d\mu_\theta \\ &\leqslant \left(\int_X e^{m(\tilde{\phi}^{(m)} - \tilde{\phi})}\omega^n\right)^{\frac{\lambda}{m}}\left(\int_X e^{\frac{-\lambda\left(\tilde{\phi}^{(m)} - E_m(\tilde{\phi}^{(m)})\right)}{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}}}\omega^n\right)^{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}}\left(\int_X e^{-\alpha\phi}\omega^n\right)^{\frac{\lambda\varepsilon}{\alpha}} \\ &= (d_m)^{\frac{\lambda}{m}}\left(\int_X e^{\frac{-\lambda\left(\tilde{\phi}^{(m)} - E_m(\tilde{\phi}^{(m)})\right)}{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}}}\omega^n\right)^{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}}\left(\int_X e^{-\alpha\phi}\omega^n\right)^{\frac{\lambda\varepsilon}{\alpha}},\end{aligned}$$

where we used the fact

$$\int_X e^{m(\tilde{\phi}^{(m)} - \tilde{\phi})}\omega^n = d_m$$

in the last equality. We now fix $\varepsilon \ll 1$ and $m \gg m_0(X, L, \omega, \varepsilon)$ such that

$$\frac{\lambda}{1 - \frac{\lambda}{m} - \frac{\lambda\varepsilon}{\alpha}} < \delta_m(L).$$

Then by Definition (5) and (1.19) there exist $C_{m,\lambda} > 0$ and $C_\alpha > 0$ (recall $\sup \phi = 0$) such that

$$\int_X e^{-\lambda(\phi - E(\phi))}\omega^n < (d_m)^{\frac{\lambda}{m}} \cdot (C_{m,\lambda})^{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}} \cdot (C_\alpha)^{\frac{\lambda\varepsilon}{\alpha}}.$$

Note that all the constants are uniform, independent of ϕ. So we finally arrive at $\int_X e^{-\lambda(\phi - E(\phi))}\omega^n < C_\lambda$ for some uniform $C_\lambda > 0$, as desired.

□

Proof of Theorem 3. The result follows from Theorem 2 and the property of Ding functionals. □

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