STABILITY THRESHOLDS AND CANONICAL METRICS

ABSTRACT. These notes are written for the several talks that the author delivered in the Autumn of 2020 at various places. The goal of these notes is to review some history of stability thresholds in Kähler geometry and present some recent progress in this area.

CONTENTS

1. MOTIVATION: KÄHLER–EINSTEIN PROBLEM

A central problem in Kähler geometry is to find canonical metrics on a given compact Kahler manifold (X, ω). One important class of canonical metrics is the Kähler-Einstein (KE) metric. A Kähler metric is KE if the Ricci form of the Kähler metric is a constant multiple of the Kähler form, namely,

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

As we know, the Ricci form of a Kähler metric must lie in the first Chern class of the manifold:

$$\mathrm{Ric}(\omega) \in c_1(X).$$

Therefore, a necessary condition for the existence of KE metric is that the first Chern class of the manifold has a sign:

$$\begin{cases} c_1(X) < 0, & K_X \text{ ample (canonically polarized)} \\ c_1(X) = 0, & K_X \equiv 0 \text{ (Calabi–Yau)} \\ c_1(X) > 0, & -K_X \text{ ample (Fano)} \end{cases}$$

The study of KE metrics has a long history. In the cases where the first Chern class is zero or negative, the uniqueness of the KE metric was proved by Calabi in the 1950s, and the existence of such a metric was obtained in 1978 by Yau and Aubin. However, when the first Chern class is positive (i.e., for Fano manifolds), the situation is much more complicated. It turns out that there are obstructions to the existence of KE metrics on Fano manifolds. The first obstruction was found by Matsushima in 1957, which says that the automorphism group of a KE Fano manifold must be reductive. In 1983 another obstruction was found by Futaki, who defined an holomorphic invariant (which we now call Futaki invariant) and it was

Date: September 2020.

shown that the Futaki invariant must vanish if the Fano manifold admits a KE metric. Summing up, we have

$$\begin{cases} c_1(X) < 0, & \text{KE always exists} \\ c_1(X) = 0, & \text{KE always exists} \\ c_1(X) > 0, & \text{there are obstructions!} \end{cases}$$

So it is natural to ask, when does a Fano manifold admit a KE metric? Regarding this problem, many significant results were obtained in history. For instance, in 1990, Tian completely solved the existence problem for Fano (del Pezzo) surfaces and showed that the existence of KE metrics is equivalent to the reductivity of the automorphism group; in 2004, Xujia Wang and Xiaohua Zhu showed that there exists a KE metric on toric Fano manifolds if and only if the Futaki invaraint vanishes. For general Fano manifolds, the existence of KE metrics is more difficult to characterize. In 1992, Ding-Tian defined a *generalized Futaki invariant* for a deformation family of Fano manifolds, and based on this, in 1997, Tian introduced an algebro-geometric notion called *K-stability*. This notion was later reformulated by Donaldson using more algebraic language. And the famous Yau–Tian–Donaldson conjecture says that, the existence of KE metrics on Fano manifolds is equivalent to K-stability. This conjecture was solved by Tian and Chen-Donaldson-Sun independently in 2012.

Theorem 1.1 (Tian, Chen–Donaldson–Sun)**.** *A Fano manifold admits a KE metric if and only if it is K-stable.*

Question 1.1. *How do we test K-stability?*

2. Alpha invariant

Given a general Fano manifold, it is very difficult to test its K-stability and hence the existence of KE metric cannot be easily determined. So it is an important problem to find a **computable** criterion that one can use to determine if the manifold admits a KE metric or not. In history, the first effective criterion was found in 1987 by Tian, which is now known as the α-invariant.

2.1. **Analytic definition.** Given a compact Kähler manifold (X,ω), put

$$\mathcal{H}_\omega := \left\{\varphi \in C^\infty(X,\mathbb{R}) \middle| \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0\right\}.$$

This is called the space of Kähler potentials. Locally, each $\varphi \in \mathcal{H}_\omega$ is the sum a smooth function and a *plurisubharmonic* (psh) function.

Lemma 2.1 (Hömander)**.** *If ψ is a plurisubharmonic function on $B_R(0) \subset \mathbb{C}^n$, satisfying $\psi(0) \geq -1$ and $\psi \leq 0$, on $B_R(0)$, then for every $\rho \in [R/2, e^{-1/2}R)$ there exists a constant C depending only on R, ρ, n such that*

$$\int_{B_\rho(0)} e^{-\psi} d\mu \leq C.$$

Globalizing this estimate, we have

Theorem 2.2 (Tian, 1987)**.** *There exist two positive constants α and C such that*

$$\int_X e^{-\alpha(\varphi - \sup\varphi)\omega^n} \leq C \text{ for any } \varphi \in \mathcal{H}_\omega.$$

So the following definition makes sense.

Definition 2.3. *The alpha invariant of* (X, ω) *is defined by*

$$\alpha(X, [\omega]) := \sup \left\{ \alpha > 0 \middle| \sup_{\varphi \in \mathcal{H}_\omega} \int_X e^{-\alpha(\varphi - \sup \varphi)} \omega^n < +\infty \right\}.$$

This invariant only depends on the Kähler class $[\omega]$. When X is Fano and $[\omega] = c_1(X)$, we denote

$$\alpha(X) := \alpha(X, [\omega]).$$

Theorem 2.4 (Tian, 1987)**.** *Let* X *be an* n*-dimensional Fano manifold. Suppose that* $\alpha(X) > \frac{n}{n+1}$*, then* X *admits a KE metric.*

When X admits a compact group G action, one can also define α_G-invariant, which characterizes the existence of G-invariant KE metrics. Note that α-invariant provides the first **computable** criterion for KE metrics on Fano manifolds, which yields many new examples.

Examples. Any Fermat hypersurface

$$X_{m,p} := \left\{ [z_0, ..., z_{n+1}] \in \mathbb{P}^{n+1} \middle| z_0^p + ... + z_{n+1}^p = 0 \right\}$$

with $p = m$ or $m + 1$ admits a KE metric.

2.2. **Quantized alpha invariant.** When $[\omega] = c_1(L)$, there is a natural way to "quantize" $\mathcal{H}_\omega$ using *Fubini–Study* metrics. More precisely, choose a Hermitian metric h on L such that

$$\omega = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h.$$

Consider the vector space $H^0(X, mL)$ for $m \gg 1$. Assume that

$$d_m := \dim H^0(X, mL).$$

Put

$$\mathcal{B}_m := \left\{ \varphi = \frac{1}{m} \log \sum_{i=1}^{d_m} |s_i|^2_{h^m} \middle| \{s_i\} \text{ basis of } H^0(X, mL) \right\}.$$

Then $\mathcal{B}_m \subset \mathcal{H}_\omega$ is called the space of Bergman potentials.

Theorem 2.5 (Tian, 1989)**.** *Any* $\varphi \in \mathcal{H}_\omega$ *can be approximated by a sequence of Bergman potentials* $\varphi_m \in \mathcal{B}_m$ *as* $m \to \infty$*.*

By the work of Catlin–Lu–Zelditch et al. one actually has smooth convergence. Then a natural way to quantize the alpha invariant is to put

$$\alpha_m(X, L) := \sup \left\{ \alpha > 0 \middle| \sup_{\varphi \in \mathcal{B}_m} \int_X e^{-\alpha(\varphi - \sup \varphi)} \omega^n < +\infty \right\}.$$

Combining α and α_m-invariants with other analytic tools (e.g. partial C^0 estimate), we have the following

Theorem 2.6 (Tian, 1990)**.** *A smooth del Pezzo surface admits a KE metric if and only if its automorphism group is reductive. Namely, it is either* $\mathbb{P}^2$*,* $\mathbb{P}^1 \times \mathbb{P}^1$ *or a blowup of* $\mathbb{P}^2$ *at* $3 \le m \le 8$ *(general) points.*

2.3. **Algebraic definition.** Around 2005, algebraic geometers (e.g. I. Cheltsov) began to realize that the α-invairiant is closely related to the *log canonical threshold* (lct) in algebraic geometry. In terms of the complex geometry, lct is also called the *complex singularity exponent*, which can be locally described as follows.

Suppose that f is a holomorphic function defined on a domain in $\mathbb{C}^n$. Then

$$\phi := \log |f|^2$$

is a psh function, which is singular along the zero locus of f. Suppose that p is a vanishing point of f. Then the complex singularity exponent of ϕ at p is defined by

$$c_p(\phi) := \sup \left\{ c > 0 \middle| \, e^{-c\phi} \in L^1_{loc} \textit{ around } p \right\}.$$

Equivalently,

$$c_p(f) := \sup \left\{ c > 0 \middle| \, \frac{1}{|f|^{2c}} \in L^1_{loc} \textit{ around } p \right\},$$

which we also call the complex singularity exponent of f at p.

Now given an effective divisor D on X, locally D is cut out by some holomorphic function f, namely,

$$D = \{f = 0\}.$$

The lct of D is then defined by

$$\operatorname{lct}(X, D) := \inf_{p \in X} c_p(f).$$

Equivalently, choose any smooth Hermitian metric h on the line bundle $\mathcal{O}_X(D)$, one has

$$\operatorname{lct}(X, D) = \sup \left\{ c > 0 \middle| \int_X \frac{1}{|s_D|_h^{2c}} \omega^n < +\infty \right\},$$

where $s_D \in H^0(X, \mathcal{O}_X(D))$ is the defining section of D.

Definition 2.7. *We say the pair* (X, D) *is log canonical if* $\operatorname{lct}(X, D) \geq 1$.

Consider (X, L), where L is an ample line bundle. Define

$$\operatorname{lct}_m(X, L) := \inf \left\{ m \cdot \operatorname{lct}(X, D) \middle| D \in |mL| \right\}.$$

Then by a result of Demailly–Kollár (lower semi-continuity of complex singularity exponent), one has

Proposition 2.8. $\alpha_m(X, L) = \operatorname{lct}_m(X, L)$.

In other words, α_m-invariant measures the "worst" integrability of divisors in the linear system $|mL|$. Equivalently,

$$\alpha_m(X, L) = \sup \left\{ c > 0 \middle| \int_X \frac{1}{\left(|s|^2_{h^m}\right)^{c/m}} \omega^n < +\infty \text{ for all non-zero } s \in H^0(X, mL) \right\}.$$

Using L^2 theory, Demailly (and Y. Shi independently) proved the following result.

Theorem 2.9. *One has*

$$\alpha(X, L) = \lim_m \alpha_m(X, L) = \inf \left\{ \operatorname{lct}(X, D) \middle| \textit{effective } D \sim_{\mathbb{Q}} L \right\}.$$

This algebraic definition also makes sense when X is "singular" (e.g. $\mathbb{Q}$ Fano variety). It is also easier to compute. For instance, the α-invariant of all smooth del Pezzo surfaces S has been calculated by Cheltsov.

$$\alpha(S) = \begin{cases} \frac{1}{3} \text{ if } S = Bl_p\mathbb{P}^2 \text{or } K_S^2 \in \{7,9\}, \\ \frac{1}{2} \text{ if } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5,6\}, \\ \frac{2}{3} \text{ if } K_S^2 = 4, \\ \frac{2}{3} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point}, \\ \frac{3}{4} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points}, \\ \frac{3}{4} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve}, \\ \frac{5}{6} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves}, \\ \frac{5}{6} \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve}, \\ 1 \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves}. \end{cases}$$

2.4. **Valuative characterization of alpha invariant.** There is a more conceptual definition of α-invariant, using the language of "valuations". This formalism is more related to the "non-Archimedean geometry" in the literature.

Let X be a normal projective variety (over $\mathbb{C}$) and let $K := \mathbb{C}(X)$ denote its function field (the field of rational functions on X). Then a valuation is map

$$v : K \to \mathbb{R} \cup \{+\infty\}$$

satisfying

(1) $v(f) = +\infty$ iff $f = 0$;
(2) $v(fg) = v(f) + v(g)$;
(3) $v(f + g) \geq \min\{v(f), v(g)\}$;
(4) $v(\lambda) = 0$ for $\lambda \in \mathbb{C}^*$.

Given a valuation v, there is a unique scheme point $\xi \in X$ such that there is a local inclusion $\mathcal{O}_{X,\xi} \to R_v$ of local rings, where $R_v := \{f \in K \mid v(f) \geq 0\}$ denotes the valuation ring of v. This ξ is called the *center* of v.

There is a special kind of valuation called "divisorial valuations", whose center ξ is, after some modifications (blowups), a prime divisor F, and $v = \lambda \mathrm{ord}_F$ for some $\lambda > 0$. To be more precise, F is contained in some birational model $Y \xrightarrow{\pi} X$ over X and ord_F measure the vanishing order along the generic piont of F after pulling back everything to Y. Such F is called a divisor *over* X. Divisorial valuations are *dense* in the space of all valuations, so in what follows we only consider vauations that are given by some prime divisor over X.

Given $F \subset Y \xrightarrow{\pi} X$ over X, put

$$A_X(F) := 1 + \mathrm{ord}_F(K_Y - \pi^* K_X),$$

which is called the *log discrepancy* of F. Given any effective divisor D on X, one can make sense of $\mathrm{ord}_F(D)$ by putting

$$\mathrm{ord}_F(D) := \mathrm{coeff}_F(\pi^* D).$$

Then one has the following valuative characterization of lct:

$$\mathrm{lct}(X, D) = \inf_F \frac{A_X(F)}{\mathrm{ord}_F(D)}.$$

One can show that this agrees with the previous definition using Hironaka's resolution of singularities.

When $s \in H^0(X, mL)$, $\mathrm{ord}_F(s)$ is defined to be $\mathrm{ord}_F(\{s = 0\})$, where $\{s = 0\}$ denotes the divisor cut out by s. Define

$$\tau_m(L, F) := \max_{0 \neq s \in H^0(X,mL)} \mathrm{ord}_F(s).$$

Proposition 2.10. *One has*

$$\alpha_m(X, L) = \inf_F \frac{A_X(F)}{\tau_m(L, F)/m}.$$

Notice that for $m, k \in \mathbb{Z}_{>0}$,

$$\tau_m(L, F) + \tau_k(L, F) \leq \tau_{m+k}(L, F).$$

So the limit

$$\tau(L, F) := \lim_m \frac{\tau_m(L, F)}{m} = \sup_m \frac{\tau_m(L, F)}{m}$$

exists and is called the *pseudo-effective threshold* of L with respect to F.

Letting $m \to \infty$ and using $\alpha(X, L) = \lim \alpha_m(X, L)$, it is easy to see the following

Proposition 2.11. *One has*

$$\alpha(X, L) = \inf_F \frac{A_X(F)}{\tau(L, F)}.$$

This valuative formula of α-invariant is very useful when studying K-stability from algebrao-geometric viewpoint. See below.

2.5. **Recent results on α-invariants.** We collect some recent results on α-invariant from the literature.

Theorem 2.12 (Odaka–Sano, 2012)**.** *Let X be an n-dimensional $\mathbb{Q}$-Fano variety with $\alpha(X) > \frac{n}{n+1}$. Then X is uniformly K-stable.*

By the recent work of Li–Tian–Wang and Li, this also implies that X admits an admissible "singular" KE metric. So the above result is a generalization of Tian's α-criterion to the singular setting.

When X is smooth, one has the following improvement.

Theorem 2.13 (Fujita, 2016)**.** *Assume that X is a Fano manifold with $\alpha(X) \geq \frac{n}{n+1}$, then X is K-stable and hence X admits a KE.*

This gives an alternative proof of the following results of Tian.

Corollary 2.14. *Any smooth cubic surface in $\mathbb{P}^3$ admits KE.*

However, when X is singular, one has

Theorem 2.15 (Y. Liu–Z. Zhuang, 2019)**.** *There exists singular Fano varieties with $\alpha(X) = \frac{n}{n+1}$ but X is does not admit KE metrics.*

So in particular, the threshold $\frac{n}{n+1}$ is optimal.

One also has a lower bound of $\alpha(X)$.

Theorem 2.16 (Fujita, Y. Liu, 2016)**.** *Assume that X is a K-semistable $\mathbb{Q}$-Fano variety. Then*

$$\alpha(X) \geq \frac{1}{n+1}.$$

Note that $X = \mathbb{P}^n$ achieves this lower bound, and hence $\frac{1}{n+1}$ is optimal as well.

3. Delta invariant

Roughly speaking, $\alpha(X)$ measures the singularities of all the divisors in the pluri-anticanonical system. But as one can see, Theorem 2.4 only gives a sufficient condition for the existence of KE metrics and the condition $\alpha(X) > \frac{n}{n+1}$ turns out to be rather restrictive.

For instance, when X is a smooth cubic surface, we have $\alpha(X) \geq \frac{2}{3}$ (cf. Example of Cheltsov above), so Theorem 2.4 is not directly usable in this case. However Tian still managed to show the existence of KE metrics on cubic surfaces by using $\alpha_{m,2}$-invariant and partial C^0 estimate. In Tian's argument, the singularities of pluri-anticaonical divisors play an essential role. So in his 1990 survey, Tian wrote down the following expectation:

The author believes that the existence of Kähler-Einstein metric with positive scalar curvature should be closely related to the geometry of pluri-anticaonical divisors.

Recently, this expectation has been confirmed with the help of a new invariant introduced by Fujita-Odaka in 2016, which we now describe.

For $m \in \mathbb{Z}_{>0}$, consider a basis $s_1, \cdots, s_{d_m}$ of the vector space $H^0(X, mL)$. For this basis, consider the $\mathbb{Q}$-divisor

$$D := \frac{1}{md_m} \sum_{i=1}^{d_m} \{s_i = 0\} \sim_{\mathbb{Q}} L.$$

Any $\mathbb{Q}$-divisor D obtained in this way is called an m-basis type divisor of L. Let

$$\delta_m\big(X, L\big) := \inf \left\{ \mathrm{lct}(X, D) \middle| D \text{ is } m\text{-basis type of } L \right\}.$$

Then let

$$\delta(X, L) = \limsup_m \delta_m(X, L).$$

This limsup is in fact a limit by the work of Blum–Jonsson. So roughly speaking, $\delta(X, L)$ measures the singularities of basis type divisors of L. When X is Fano and $L = -K_X$, we put

$$\delta(X) := \delta(X, -K_X).$$

Then Tian's expectation can now be stated more rigorously using the following recent result.

Theorem 3.1 (Blum–Jonsson, 2017)**.** *Let X be a $\mathbb{Q}$-Fano variety. The following assertions hold:*

(1) X is K-semistable if and only if $\delta(X) \geqslant 1$;

(2) X is uniformly K-stable if and only if $\delta(X) > 1$.

Thus δ-invariant serves as a criterion for the existence of KE metrics on $\mathbb{Q}$-Fano varieties.

3.1. **Valuative characterization of δ-invariant.** Let $\pi : Y \to X$ be a proper birational morphism and let $F \subset Y$ be a prime divisor F in Y. Let

$$S_m(L,F) := \frac{1}{md_m} \sum_{j=1}^{\tau_m(L,F)} \dim H^0(Y, m\pi^*L - jF)$$

denote the *m-th expected vanishing order of L along F*. Then a basic but important linear algebra lemma due to Fujita–Odaka says that

$$S_m(L,F) = \sup\big\{\mathrm{ord}_F(D) \,:\, m\text{-basis divisor } D \text{ of } L\big\},$$

and this supremum is attained by any m-basis divisor D arising from a basis $\{s_i\}$ that is *compatible* with the filtration

$$H^0\big(Y, m\pi^*L\big) \supset H^0\big(Y, m\pi^*L - F\big) \supset \cdots \supset H^0\big(Y, m\pi^*L - (\tau_m(L,F)+1)F\big) = \{0\},$$

meaning that each $H^0(Y, m\pi^*L - jF)$ is spanned by a subset of the $\{s_i\}_{i=1}^{d_m}$. Then it is easy to deduce that

$$\delta_m(X,L) = \inf_F \frac{A_X(F)}{S_m(L,F)}.$$

As $m \to \infty$, one has

$$S(L,F) := \lim_{m\to\infty} S_m(L,F) = \frac{1}{\mathrm{vol}(L)} \int_0^{\tau(L,F)} \mathrm{vol}(\pi^*L - xF)dx,$$

which is called the *expected vanishing order of L along F*. It is not a trivial fact that the above limit exists, which requires the theory of Newton–Okounkov bodies. Then Blum–Jonsson further show that, the limit of $\delta_m(X,L)$ also exists, and is equal to

$$\delta(X,L) = \inf_F \frac{A_X(F)}{S(L,F)}.$$

An important property of $S(L,F)$ is illustrated by the following result of Fujita, who shows that $S(L,F)$ can be viewed as the coordinate of the barycenter of certain Newton–Okounkov body along the "F-axis", and hence the well-known Brunn–Minkovski inequality in convex geometry gives the following estimate.

Proposition 3.2 (Barycenter inequality)**.** *For any F over X, one has*

$$\frac{\tau(L,F)}{n+1} \le S(L,F) \le \frac{n\tau(L,F)}{n+1}.$$

One should think of τ and S as the non-Archimedean analogues of the I and $I - J$ functional of Aubin. An immediate consequence is the following

Corollary 3.3. *One always has*

$$\frac{n+1}{n}\alpha(X,L) \le \delta(X,L) \le (n+1)\alpha(X,L).$$

3.2. **Delta invariant with higher momentum.** $S(L,F)$ can be treated as the *first momentum* of the vanishing order of L along F. It is also related to the so called "total weight" of the *test configuration* induced by F (when F is *dreamy*). In general one can also consider the k-th momentum of the vanishing order of L along F. More precisely, given a basis $\{s_i\}$ of $H^0(X,mL)$ that is compatible with the filtration induced by F, put

$$S^{(k)}_m(L,F) := \frac{1}{m^k d_m}\sum_{i=1}^{d_m}\mathrm{ord}_F(s_i)^k.$$

Then as $m\to\infty$, one has

$$S^{(k)}(L,F) := \frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)} kx^{k-1}\,\mathrm{vol}(L-xF)dx.$$

So in particular $S(L,F)=S^{(1)}(L,F)$.

Extending Fujita's barycenter inequality to k-th momentum, we have

Proposition 3.4. *Given any divisor F over X, one has*

$$\frac{n!k!}{(n+k)!}\tau(L,F)^k \le S^{(k)}(L,F) \le \frac{n}{n+k}\tau(L,F)^k.$$

Corollary 3.5. *One has*

$$\tau(L,F)=\lim_{k\to\infty}S^{(k)}(L,F)^{1/k}.$$

The set of momentums $\{S^{(k)}\}$ can be used to construct various kinds of valuative thresholds for (X,L). α and δ are only two special ones. For instance one can put

$$\delta^{(k)}(X,L) := \inf_F\frac{A_X(F)}{S^{(k)}(L,F)^{1/k}}.$$

Then

$$\delta(X,L)=\delta^{(1)}(X,L) \text{ and } \alpha(X,L)=\delta^{(\infty)}(X,L).$$

Question 3.1. *What can we say about $\delta^{(k)}$-invariant?*

Very recently, using $\{S^{(k)}\}$, Chi Li constructed a non-Archimedean analogue of the H-functional of the Kähler–Ricci flow, which were previously studied by W. He and Tian–Zhang–Zhang–Zhu. More precisely, consider

$$\sum_{k=1}^\infty\frac{(-1)^kS^{(k)}(L,F)}{k!}=\frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)}e^{-x}\,\mathrm{vol}(L-xF)dx.$$

Then Chi Li defined

$$H^{\mathrm{NA}}(X,L) := \inf_F\left\{A_X(F)+\log\left(1-\frac{1}{\mathrm{vol}(L)}\int_0^{\tau(L,F)}e^{-x}\,\mathrm{vol}(L-xF)dx\right)\right\}.$$

This invariant plays significant roles in the study of Hamilton–Tian conjecture just as δ-invariant in the Yau–Tian–Donaldson conjecture.

3.3. **Bounding the volume of Fano manifolds using delta invariant.** Bounding the volume of Fano manifolds is a classical problem in algebraic geometry, which is related to the boundedness and rationally connectedness of Fano manifolds. It was once a folklore conjectur that

$$(-K_X)^n \leq (n+1)^n.$$

However counterexamples were found by investigating the volume of Fano type projective bundles. Then the above conjecture was modified as follows.

Conjecture 3.6. *Let X be a Fano manifold with Picard number 1. Then*

$$(-K_X)^n \leq (n+1)^n.$$

By the work of Hwang, it is known that this holds for $n \leq 4$. But for higher dimensions this is still open.

In general, one needs additional assumtptions to get the effective volume control of Fano manifolds. In this direction, a remarkable result of K. Fujita says the following (see also the thesis of Y. Liu for some generalizations).

Theorem 3.7 (K. Fujita, 2015)**.** *Let X be a Fano manifold admitting a KE metric. Then $(-K_X)^n \leq (n+1)^n$ and the equality holds iff $X \cong \mathbb{P}^n$.*

This result is proved using δ-invariant as the assumption implies that $\delta(X) \geq 1$, which in turn can give us an effective bound for $(-K_X)^n$.

Recently, by using the greatest Ricci lower bound and Newton–Okounkov bodies, K. Zhang extends Fujita's result to transcendental Kähler forms with positive Ricci curvature.

Theorem 3.8 (K. Zhang, 2020)**.** *Let (X, ω) be a Kähler manifold with*

$$\mathrm{Ric}(\omega) \geq (n+1)\omega.$$

Then one has

$$\mathrm{vol}(X, \omega) \leq \mathrm{vol}(\mathbb{P}^n, \omega_{FS}),$$

and the equality holds iff (X, ω) is biholomorphically isometric to $(\mathbb{P}^n, \omega_{FS})$.

This result can be thought of as a Kähler analogue of the classical volume comparison of Bishop.

3.4. **Cumputation of δ-invariant.** Given its definition, δ-invariant is not easy to compute! The first attempt was made by Park and Won in 2016. They estimated the δ-invariant of all the smooth del Pezzo surfaces using "Newton polygons" and gave a purely algebraic proof of Theorem 2.6. However for higher dimensional Fano manifolds, we need other tools to compute δ-invariant.

In 2018, a more geometric approach (multiplicity estimates and inversion of adjunction) to compute δ-invariant were established by Cheltsov–Rubinstein–Zhang. They applied their methods to a family of asymptotically log Fano surfaces and showed that there exist Kähler–Einstein edge metrics with sufficiently small cone angles. Meanwhile, Cheltsov–Zhang also applied this geometric approach to smooth del Pezzo surfaces and gave another proof of the K-stability of cubic surfaces. This method was also applied to singular del Pezzo surfaces by Cheltsov–Park–Shramov.

Very recently, H. Ahmadinezhad–Z. Zhuang further modifed this geometric approach using "admissible flags", which yields more new examples of K-stable Fano varieties.

When the variety enjoys certain symmetry, e.g., when there a torus T acting on X (i.e. T-variety), it is shown by Blum–Jonsson (for toric varieties) and Golota (for general T-varieties) that it is enough to investigate all the T-invariant divisors over X. So in particular, the δ-invariant of toric varieties can be easily computed using its combinatoric data. For instance, suppose that a toric polarized variety (X, L) is determined by a polytope

$$P := \bigcap_i \left\{ u \in \mathbb{R}^n \middle| \langle u, v_i \rangle + a_i \geq 0 \right\}.$$

Then its δ-invariant is given by

$$\delta(X, L) = \min_i \left\{ \frac{1}{\langle b_P, v_i \rangle + a_i} \right\},$$

where

$$b_P := \frac{\int_P x dx}{\int_P dx}$$

denotes the barycenter of P.

In the toric Fano case, $a_i = 1$ for all i. So one reads

$$\delta(X) = \min_i \left\{ \frac{1}{\langle b_P, v_i \rangle + 1} \right\}.$$

Observe! This formula coincides with Chi Li's formula for *the greatest Ricci lower bound* of toric Fano manifolds.

3.5. **The greatest Ricci lower bound.** We recall the following definitions.

Definition 3.9 (The greatest Ricci lower bound)**.** *Let X be a Fano manifold. Put*

$$\beta(X) := \sup\{\lambda > 0 \mid \exists\ \omega \in c_1(X) \text{ such that } Ric(\omega) > \lambda\omega\ \}.$$

This invariant was first studied (implicitly) by Tian in 1992. Roughly speaking, $R(X)$ measures how far X is from being a KE manifold. So it is an interesting problem to calculate the value of $\beta(X)$.

It turns out that $\beta(X)$ can also be used to test K-(semi)stability of X.

Theorem 3.10 (C. Li)**.** *Let X be a Fano manifold. Then the following are equivalent.*

(1) X is K-semistable;
(2) $R(X) = 1$.

For toric Fano manifold, Chi Li obtained the formula for $\beta(X)$, which coincides with Blum–Jonsson's formula for $\delta(X)$. So it is natural to expect that $\delta(X)$ and $\beta(X)$ should be closely related for general Fano manifolds as well. Inspired by this, K. Zhang proved the following general fact in his thesis.

Theorem 3.11 (K. Zhang, 2018)**.** *Let X be a Fano manifold. Then we have*

$$\beta(X) = \min\{\delta(X), 1\}.$$

See also Berman–Boucksom–Jonsson for an independent proof using non-Archimedean geometry.

3.6. **Analytic delta invariant.** The study of $\beta(X)$ has a long history going back to Tian's work in 1987. $\beta(X)$ measures how far the following continuity path can go:

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

It is well known that the solvability of this twisted KE equation is controlled by the coercivity of Mabuchi's K-energy (will be recalled below). In other words, $\beta(X)$ can be thought of as a coercivity threshold for K-energy. So Theorem 3.11 suggests that $\delta(X)$ can also be interpreted as a coercivity threshold for some energy functional.

More precisely, let (X, ω) be a compact Kähler manifolds. Put

$$V := \int_X \omega^n.$$

The I-functional $I_\omega(\cdot)$ is defined to be

$$I_\omega(\varphi) := \frac{1}{V}\int_X \varphi(\omega^n - \omega_\varphi^n) = \frac{\sqrt{-1}}{V}\int_X \sum_{i=0}^{n-1} \partial\varphi \wedge \bar\partial\varphi \wedge \omega^{n-i-1} \wedge \omega_\varphi^i,\ \varphi \in \mathcal{H}_\omega.$$

The J-functional $J_\omega(\cdot)$ is defined to be

$$J_\omega(\varphi) := \int_0^1 \frac{I_\omega(s\varphi)}{s} ds = \frac{\sqrt{-1}}{V}\int_X \sum_{i=0}^{n-1} \frac{n-i}{n+1} \partial\varphi \wedge \bar\partial\varphi \wedge \omega^{n-i-1} \wedge \omega_\varphi^i,\ \varphi \in \mathcal{H}_\omega.$$

Now assume that X is Fano and $[\omega] = c_1(X)$. Then by $\partial\bar\partial$-lemma, there exists a unique normalized Ricci potential $f_\omega \in C^\infty(X, \mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial f_\omega \text{ and } \int_X e^{f_\omega}\omega^n = V.$$

Then Ding functional is defined by

$$D(\varphi) := J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \log\left(\frac{1}{V}\int_X e^{f_\omega - \varphi}\omega^n\right),\ \varphi \in \mathcal{H}_\omega.$$

And the Mabuchi K-energy is defined by

$$M(\varphi) := \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n - \frac{1}{V}\int_X f_\omega\omega_\varphi^n - (I_\omega - J_\omega)(\varphi),\ \varphi \in \mathcal{H}_\omega.$$

Here

$$H_\omega(\varphi) := \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n$$

is called the *entropy* of φ.

Definition 3.12. *The Ding functional (resp. Mabuchi functional) is called coercive if there exist $\varepsilon, C > 0$ such that $D \geq \varepsilon J_\omega - C$ (resp. $M \geq \varepsilon J_\omega - C$) on $\mathcal{H}_\omega$.*

Theorem 3.13 (Tian, 1997)**.** *Let X be a Fano manifold without non-trivial holomorphic vector field. Then X admits KE iff Ding functional (equivalently, Mabuchi functional) is coercive.*

However, for general Fano manifold, Ding or Mabuchi functional might not be coercive. Instead, one can consider the following coercivity threshold (this is called the analytic delta invariant in the thesis of K. Zhang).

$$\delta^A(X) := \sup\left\{\lambda > 0 \,\middle|\, \exists\, C_\lambda > 0 \text{ s.t. } H_\omega(\varphi) \geq \lambda(I_\omega - J_\omega)(\varphi) - C_\lambda, \forall \varphi \in \mathcal{H}(X, \omega)\right\}.$$

Recall that, the work of Szèkelyhidi implies that

$$\beta(X) = \min\{1, \delta^A(X)\}.$$

Combining this with Theorem 3.11, one is naturally led to the following expectation.

Conjecture 3.14. *One has*

$$\delta(X) = \delta^A(X).$$

So far, this conjecture is still open! But one direction is known.

Proposition 3.15. *We have $\delta^A(X) \leq \delta(X)$.*

This can be shown either using the non-Archimedean approach of Boucksom–Jonsson or by a quantization argument (see the discussion in Section 4).

To study $\delta^A(X)$, it has been observed recently by K. Zhang that an alternative definition for δ^A using Moser–Trudinger inequality could be more helpful. In fact, one has the following

Proposition 3.16 (K. Zhang, 2020)**.** *We have*

$$\delta^A(X) = \sup\left\{\delta > 0 \,\middle|\, \sup_{\varphi\in\mathcal{H}_\omega}\int_X e^{-\delta(\varphi - E_\omega(\varphi))}\omega^n < +\infty\right\}.$$

Here E_ω is called the Monge–Ampère energy, which is defined by

$$E_\omega(\varphi) := \frac{1}{V}\int_X \varphi\omega^n - J_\omega(\varphi) = \frac{1}{(n+1)V}\int_X \sum_{i=0}^n \varphi\omega^i \wedge \omega_\varphi^{n-i},\ \varphi \in \mathcal{H}_\omega.$$

Observe that the above definition for $\delta^A(X)$ is very similar to Tian's analytic definition of $\alpha(X)$ (recall Definition 2.3), the only difference being that $\sup\varphi$ is replaced by $E_\omega(\varphi)$.

Remark 3.17. Using the language of test configurations, $\sup\varphi$ corresponds to the maximal weight (i.e., the pseudo-effective threshold $\tau(L, F)$) of the $\mathbb{C}^*$-action on the central fiber, while $E_\omega(\varphi)$ corresponds to the total weight (i.e., the expected vanishing order $S(L, F)$). This will be clear when we take the quantization approach; see Lemma 4.2 below.

4. Quantized delta invariant and balanced metrics

Now we present some recent progress of Rubinstein–Tian–Zhang. To study δ and δ^A-invariants, we follow Tian's quantization approach for α-invariant.

As before, we consider a polarized pair (X, L) and look at the Bergman space

$$\mathcal{B}_m \text{ for } m \gg 1.$$

Our main contribution is to provide an analytic counterpart of the δ_m-invariant, denoted δ_m^A. Moreover, as it turns out, the analytic approach in the δ-setting yields a number of useful applications to canonical metrics that are new, and completely absent from the α-setting. While each α_m-invariant does not have a clear geometric application, the analytic m-th stability threshold δ_m^A that we introduce here turn out to characterize "balanced metrics". Moreover, they serve as coercivity thresholds for certain *quantized Ding functionals*. They are very much computable, in some instances more so than their algebraic counterparts that up until now were unknown even for $\mathbb{P}^n$. Since we show that the two actually coincide,

$$\delta_m = \delta_m^A,$$

this proves quite useful in a number of situations. Moreover, via the work of Blum–Jonsson, this shows that our analytic invariants converge to the Fujita–Odaka δ, and so this gives via the Yau–Tian–Donaldson framework a new approach to existence and stability. Once the connection of our invariants to balanced metrics is proven, one realizes that this framework is quite flexible, and indeed we show it extends general polarized manifolds X and characterizes twisted KE metrics, Kähler–Ricci solitons, coupled KE metrics, among other canonical metrics. This ties quite neatly with work of Donaldson, Berman–Boucksom–Guedj–Zeriahi, Berman–Witt Nyström, Berman–Bouckom–Jonsson and others on relations between balanced metrics, stability, and existence of canonical metrics.

4.1. **Quantized Monge–Ampère energy.** To describe our invariant $\delta_m^A(L)$ we introduce some additional notation. Let $\mathcal{P}_m$ denote the space of all Hermitian inner products on the complex vector space $H^0(X, mL)$. As observed by Donaldson a fundamental Bott–Chern type functional on $\mathcal{P}_m \times \mathcal{P}_m$ is

$$E_m(H, K) := \frac{1}{md_m} \log\det K^{-1}H.$$

In practice it is convenient to fix some H in the first slot and, in the second slot, to pull-back via the isomorphism $\mathrm{FS} : \mathcal{P}_m \to \mathcal{B}_m$,

$$\mathrm{FS}(K) := \frac{1}{m} \log \sum_{i=1}^{d_m} |\sigma_i|^2_{h^m},$$

where $\{\sigma_i\}$ is an(y) orthonormal basis of K. This yields a functional $E_m\big(H, \mathrm{FS}^{-1}(\,\cdot\,)\big)$, that we also denote by

$$E_m(H, \varphi) := E_m(H, \mathrm{FS}^{-1}(\varphi)) = E_m(H, K), \quad \text{for } \varphi = \mathrm{FS}(K) \in \mathcal{B}_m.$$

As shown by Donaldson E_m is the natural quantization of E.

To be more precise, we set

$$H_m := \int_X h^m(\cdot, \cdot)\omega^n$$

and put

$$E_m(\varphi) := E_m(H_m, \varphi) \text{ for } \varphi \in \mathcal{B}_m.$$

Now given any $\varphi \in \mathcal{H}_\omega$, let

$$\varphi_m := \frac{1}{m} \log \sum_{i=1}^{d_m} |\sigma_i|^2_{h^m},$$

where $\{\sigma_i\}$ is any orthonormal basis of the following L^2-inner product

$$\int_X (he^{-\varphi})^m(\,\cdot\,,\,\cdot\,)\omega_\varphi^n.$$

Then,

$$E_m(\varphi_m) \to E(\varphi) \text{ as } m \to \infty.$$

This can be proved using the asymptotics of Bergman kernels (recall Theorem 2.5)

4.2. δ^A_m**-invariant.** Given these notions, we can naturally quantize δ^A-invariant (cf. Propositon 3.16) as follows.

Definition 4.1. *The m-th analytic δ^A-invariant is defined by*
$$\delta^A_m(X,L) := \sup\left\{\delta>0 \,\middle|\, \sup_{\varphi\in\mathcal{B}_m}\int_X e^{-\delta(\varphi-E_m(\varphi))}\omega^n<\infty\right\}.$$

Again this definition is very similar to Tian's analytic formulation of $\alpha_m(X,L)$, the only difference being that $\sup\varphi$ is replaced by $E_m(\varphi)$.

Using linear algebra, one can reformulate the above definition. First note that E_m can be expressed by
$$E_m(\varphi)=\frac{1}{md_m}\log\det\big[H_m(\sigma_i,\sigma_j)\big]=\frac{1}{md_m}\log\det\left[\int_X h^m(\sigma_i,\sigma_j)\omega^n\right],$$
for any $\varphi=\mathrm{FS}(K)=\frac{1}{m}\log\sum_{i=1}^{d_m}|\sigma_i|^2_{h^m}\in\mathcal{B}_m$, where $\{\sigma_i\}$ is K-orthonormal. By linear algebra, after a unitary transformation, one may diagonalize the basis so that
$$\sigma_i=\mu_i^{1/2}s_i$$
for some H_m-orthonormal basis $\{s_i\}$, with $\mu_i>0$. Using such convention, one can also write
$$E_m(\varphi)=\frac{1}{md_m}\log\prod_{i=1}^{d_m}\mu_i.$$
Thus we have
$$\delta^A_m(X,L)=\sup\left\{\delta>0\,\middle|\,\sup_{\substack{\{s_i\}\,H_m\text{-o.n.b.}\\ \mu_i>0}}\int_X\frac{\prod_{i=1}^{d_m}\mu_i^{\frac{\delta}{md_m}}}{\left(\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m}\right)^{\frac{\delta}{m}}}\omega^n<+\infty\right\}.$$
Using such convention and the fact that $|\sup\varphi-\frac{1}{m}\log\max\{\mu_i\}|=o(1)$, one can also reformulate $\alpha_m(X,L)$ as follows:
$$\alpha_m(X,L)=\sup\left\{\alpha>0\,\middle|\,\sup_{\substack{\{s_i\}\,H_m\text{-o.n.b.}\\ \mu_i>0}}\int_X\frac{\max\{\mu_i\}^{\frac{\alpha}{m}}}{\left(\sum_{i=1}^{d_m}\mu_i|s_i|^2_{h^m}\right)^{\frac{\alpha}{m}}}\omega^n<+\infty\right\}.$$

Observe that,
$$\left(\prod_{i=1}^{d_m}\mu_i\right)^{\frac{1}{d_m}}$$
is the geometric mean of the eigenvalues μ_i while $\max\{\mu_i\}$ is the maximun of μ_i. As we will explain below, they can be related to the expected vanishing order S_m and the pseudo-effective threshold τ_m, respectively. (Taking logarithm and using the language of test configuration, $\sum_{i=1}^{d_m}\log\mu_i$ corresponds to the "total weight" while $\max\{\log\mu_i\}$ is the "maximal weight".)

More precisely, given a divisor $F\subset Y\xrightarrow{\pi}X$ over X, as before, we consider the filtration
$$H^0\big(Y,m\pi^*L\big)\supset H^0\big(Y,m\pi^*L-F\big)\supset\cdots\supset H^0\big(Y,m\pi^*L-(\tau_m(L,F)+1)F\big)=\{0\}$$

induced by ord_F. One may choose an H_m-orthonormal basis $\{s_i\}$ that is compatible with this filtration and put

$$\varphi_F := \frac{1}{m} \log \sum_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i)} |s_i|^2_{h^m}.$$

Now a simple but important observation is

Lemma 4.2. *One has*

(1) $\tau_m(L,F) = \max\{\mathrm{ord}_F(s_i)\} = \sup \varphi_F + o(1)$*;*
(2) $S_m(L,F) = \frac{1}{md_m}\sum_{i=1}^{d_m} \mathrm{ord}_F(s_i) = E_m(\varphi_F)$.

The basis $\{s_i\}$ and together with the eigenvalues $\{e^{\mathrm{ord}_F(s_i)}\}$ will induce a one-parameter degeneration $\mathcal{X}_t$ inside $\mathbb{P}^{d_m-1}$ with $\mathcal{X}_1 \cong X$. Then as $t \to 0$, the limit cycle $\mathcal{X}_0$ will admit a $\mathbb{C}^*$-action, which induces a $\mathbb{C}^*$ action on the vector space $H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(1))$. Then $S_m(L,F)$ is exactly the normalized total weight of this $\mathbb{C}^*$-action. This explains why non-Archimedean objects (e.g. a divisor F over X) is related to the generalized Futaki invatiant and K-stability.

4.3. **The identity $\delta^A_m = \delta_m$.** Regarding the δ^A_m-invariant, we have the following

Theorem 4.3 (Rubinstein–Tian–Zhang, 2020)**.** *One has*

$$\delta^A_m(X,L) = \delta_m(X,L).$$

The proof of this has two steps. First, applying the lower semi-continuity of complex singularity exponent of Demailly–Kollár, we show that it suffices to consider basis divisors associated to H_m-orthonormal basis of $H^0(X,mL)$. Then geometric mean inequality

$$\sum_{i=1}^{d_m} \mu_i |s_i|^2_{h^m} \geq d_m \Big(\prod_{i=1}^{d_m} \mu_i\Big)^{\frac{1}{d_m}} \Big(\prod_{i=1}^{d_m} |s_i|^2_{h^m}\Big)^{\frac{1}{d_m}},$$

will give us

$$\delta^A_m(X,L) \geq \delta_m(X,L).$$

Second, using Lemma 4.2 we can draw connections between Bergman geodesics and divisorial valuations. Then a local computation around the center of a divisorial valuation gives us a uniform integral control along Bergman geodesics:

$$\int_X \frac{e^{tA_X(F)}}{\Big(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F s_i} |s_i|^2_{h^m}\Big)^{\frac{\delta}{m}}} \omega^n > C > 0, \text{ for all } t \geq 0.$$

We combine this with the valuative description of $\delta_m(L)$ to conclude

$$\delta^A_m(X,L) \leq \delta_m(X,L).$$

For the reader's convenience, we now sum up as follows.

- Algebraic definition of α_m:

$$\alpha_m(L) = \sup\left\{\lambda > 0 \,\middle|\, \big(X, \frac{\lambda}{m} D\big) \text{ is log canonical for any effective divisor } D \in |mL|\right\}.$$

- Analytic definition of α_m:

$$\alpha_m(L) = \sup\left\{\lambda > 0 \,\middle|\, \exists C_\lambda > 0 \text{ s.t. } \int_X e^{-\lambda(\varphi - \sup\varphi)}\omega^n < C_\lambda,\ \forall\varphi\in\mathcal{B}_m\right\}$$

$$= \sup\left\{\lambda > 0 \,\middle|\, \exists C_\lambda > 0 \text{ s.t. } \int_X \frac{\max_i\{\mu_i^{\frac{2\lambda}{m}}\}}{\left(\sum_{i=1}^{d_m}\mu_i^2|s_i|^2_{h^m}\right)^{\frac{\lambda}{m}}}\omega^n < C_\lambda, \forall \text{orthonormal } \{s_i\} \text{ and } \{\mu_i\}\right\}.$$

- Algebraic definition of δ_m:

$$\delta_m(L) = \sup\left\{\lambda > 0 \,\middle|\, (X, \lambda D) \text{is log canonical for any } m\text{-basis divisor } D \text{ of } L\right\}$$

- Analytic definition of δ_m:

$$\delta_m(L) = \sup\left\{\lambda > 0 \,\middle|\, \exists C_\lambda > 0 \text{ s.t. } \int_X e^{-\lambda(\varphi - E_m(\varphi))}\omega^n < C_\lambda,\ \forall\varphi\in\mathcal{B}_m\right\}$$

$$= \sup\left\{\lambda > 0 \,\middle|\, \exists C_\lambda > 0 \text{ s.t. } \int_X \frac{\prod_{i=1}^{d_m}\mu_i^{\frac{2\lambda}{m d_m}}}{\left(\sum_{i=1}^{d_m}\mu_i^2|s_i|^2_{h^m}\right)^{\frac{\lambda}{m}}}\omega^n < C_\lambda, \forall \text{orthonormal } \{s_i\} \text{ and } \{\mu_i\}\right\}.$$

4.4. **Quantized Ding energy and balanced metrics.** Assume that (X, ω) is a Fano manifold with $\omega \in c_1(X)$. Recall that the Ding functional is given by

$$D(\varphi) := -\log\frac{1}{V}\int_X e^{f_\omega - \varphi}\omega^n - E(\varphi),\ \text{for } \varphi\in\mathcal{H}_\omega.$$

Here f_ω is the normalized Ricci potential of ω. It is well-known that the critical point of Ding functional is KE metric.

Nowing using the quantization approach, we may consider the following quantized Ding funtional

$$D_m(\varphi) := -\log\frac{1}{V}\int_X e^{f_\omega - \varphi}\omega^n - E_m(\varphi),\ \text{for } \varphi\in\mathcal{B}_m.$$

The critical point of this funtional is called *anti-canonically balanced metric.*

Definition 4.4. *D_m is said to be coercive if there exist $\lambda > 0$ and $C > 0$ such that*

$$D_m(\varphi) \geq \lambda(\sup\varphi - E_m(\varphi)) - C,\ \textit{for all } \varphi\in\mathcal{B}_m.$$

One should think of $\sup\varphi - E_m(\varphi)$ as a "distance funciton" on $\mathcal{B}_m$.

The following result is standard, which follows from a variational argument and the Berndtsson convexity.

Proposition 4.5. *Assume that X has no non-trivial holomorphic vector field. Then D_m is coercive iff there is an anti-canonically balanced metric in $\mathcal{B}_m$.*

With the help of δ_m-invariant, we now have the following

Proposition 4.6 (Rubinstein–Tian–Zhang, 2020)**.** *D_m is coercive iff $\delta_m(X) > 1$.*

In particular, δ_m-invariant is the coercivity threshold of the quantized Ding functional and characterizes the existence of balanced metrics. This can be thought of as a "finite dimensional Yau–Tian–Donaldson theorem", the proof of which follows from the definition of $\delta_m^A(X)$ and a refinement of the argument for Theorem 4.3

Another consequence of our quantization approach is the following estimate.

Proposition 4.7. *One has $\delta^A(X) \leq \delta(X)$.*

Proof. We sketch the proof. For any $0 < \lambda < \delta^A(X)$, we want to show $\delta(X) \geq \lambda$. For simplicity we assume that $\lambda = 1$ (the general case follows by rescaling the Käherl class). Then Ding funtional is coercive. Using the asymptotics of Bergman kernels, the quantized Ding fucntional is also coercive for $m \gg 1$. Then we have $\delta_m(X) > 1$, and hence $\delta(X) \geq 1$, as desired. □

Question 4.1. *How to establish the other direction*

$$\delta^A(X) \geq \delta(X)?$$