PARTIAL C^0 ESTIMATE ALONG THE GENERALIZED KÄHLER RICCI FLOW

KEWEI ZHANG

ABSTRACT. In this note, we derive partial C^0 estimate along the generalized Kähler Ricci flow.

1. INTRODUCTION

In this note, we are interested in the *generalized* Kähler-Ricci flow on Fano manifolds, mainly due to the reason that it can be used to study the existence of generalized Kähler-Einstein metrics (see e.g. [17, 9, 4]). Indeed, as shown in [4], generalized KRF will always converge to a generalized KE metric if there is one, generalizing [14]. The purpose of this note is to study the Bergman kernel along the generalized KRF. In particular, using the recent result in [8] we show that, the partial C^0 estimate of the Bergman kernel holds along the generalized KRF.

To be more precise, let X be an n-dimensional compact Kähler manifold X and L an ample line bundle on X. Assume that there is a nonnegative closed $(1,1)$ form $\alpha \in 2\pi(c_1(X) - c_1(L))$. Note that in this setting, X is automatically Fano. We fix an initial metric $\omega_0 \in 2\pi c_1(L)$ and consider the following generalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t + \alpha \tag{1.1}$$

starting from ω_0.

This flow preserves the cohomology class of ω_t and exists for all time. Along the flow, we choose a family of smooth Hermitian metrics h_t on the line bundle L such that the curvature form of h_t satisfies

$$-\sqrt{-1}\partial\bar{\partial}\log h_t = \omega_t. \tag{1.2}$$

Now for any $k \in \mathbb{N}$, using ω_t and h_t, we can define a Hermitian inner product $\langle \cdot\,,\cdot\rangle_t$ on the vector space $H^0(X, kL)$ by letting

$$\langle s_1, s_2\rangle_t = \int_X (s_1, s_2)_{h_t^k}\frac{\omega_t^n}{n!},\ \forall s_1, s_2 \in H^0(X, kL).$$

Using this Hermitian inner product, we can pick an orthonormal basis $s_0, s_1, ..., s_{N_k}$ of $H^0(X, kL)$ and define

$$\rho_{\omega_t,k}(x) := \sum_{i=0}^{N_k} |s_i|^2_{h_t^k}(x),\ x \in X. \tag{1.3}$$

Note that $\rho_{\omega_t,k}$ is called the Bergman kernel of (X, kL, ω_t), which is independent of the choices of h_t and the orthonormal basis.

The main result of this note is the following

Theorem 1.4. *There exists a positive constant $b = b(n, \omega_0, \alpha)$ and a large integer $k = k(n, \omega_0, \alpha)$ such that*

$$\rho_{\omega_t,k} \geq b > 0$$

along the flow (1.1).

This theorem gives a definite lower bound of the Bergman kernels along the generalized KRF. An estimate of this kind is often referred to as Tian's partial C^0 estimate in the literature, which plays a crucial role in the study of Yau-Tian-Donaldson correspondence. To derive such estimates, one needs to understand the compactness of certain moduli space, which is a highly nontrivial problem. Indeed, for the usual KRF (when $L = -K_X$ and $\alpha = 0$), Theorem 1.4 had remained unknown for some time and was studied by various authors (see e.g. [2, 13, 6]). As shown in [13], the partial C^0 estimate along the KRF will follow from the celebrated Hamilton-Tian conjecture. For instance, relying on the L^4 bound of Ricci curvature, Tian-Zhang [13] proved the Hamilton-Tian conjecture for dimension $n \leq 3$ and hence the partial C^0 estimate follows in this case (for $n = 2$, this was also proved by Chen-Wang [2] using the L^2 bound of the curvature tensor). For general dimensions, the Hamilton-Tian conjecture was proved in [1, 3] so the partial C^0 estimate holds as a consequence. So to prove Theorem 1.4, one possible approach is to establish the Hamilton-Tian compactness along the generalized flow (1.1), which however seems to be out of reach at present. In this note we present a shortcut to prove Theorem 1.4, using the recent compactness result obtained in [8], where a general version of Tian's partial C^0 conjecture has been confirmed. Our method is very likely well known to experts. But since this is not documented in literature, we write it down for reader's convenience.

The rest of this note is organized as follows. In Section 2, we recall some standard results in the literature. In Section 3, we apply Calabi-Yau theorem along the flow (1.1) and we derive an interesting diameter bound. In Section 4 we prove Theorem 1.4. In Section 5 we give an application of Theorem 1.4, following [6, 7].

Remark 1.5. After writing this note, the author was informed that Theorem 1.4 had already been known to Prof. Gang Tian and Zhenlei Zhang using a similar argument.

2. Preliminaries

In this part, we recall some standard results of generalized KRF. These results were originally stated for KRF (cf. [11]) and were extended to the generalized setting in [9, 4].

For any Kähler form $\omega \in [\omega_0]$, let $f_\omega \in C^\infty(X, \mathbb{R})$ be the generalized Ricci potential of ω, which is uniquely determined by

$$\mathrm{Ric}(\omega) = \omega + \alpha + \sqrt{-1}\partial\bar{\partial} f_\omega, \ \int_X e^{f_\omega} \omega^n = V, \tag{2.1}$$

where $V = (2\pi c_1(L))^n$ is the volume of the Kähler class.

The following result is essentially due to Perelman.

Theorem 2.2 ([11, 9, 4]). *Along the flow (1.1), there exists a uniform constant C such that*

$$|f_{\omega_t}| + |\nabla f_{\omega_t}| + |\Delta f_{\omega_t}| + diam(X, \omega_t) \leq C.$$

Here the gradient, Laplacian and the norms are all taken with respect to the evolving metric ω_t. The constant C only depends on the dimension n, the volum V, the L^2-Sobolev constant of (X, ω_0), $|\nabla f_{\omega_0}|$ and $|\Delta f_{\omega_0}|$.

We also have a uniform Sobolev inequality along the generalized KRF.

Theorem 2.3 ([16, 15, 9, 4]). *Along the flow (1.1), there exists a uniform constant C_S such that for any $u \in W^{1,2}(X)$, we have*

$$(\int_X |u|^{\frac{2n}{n-1}} \omega_t^n)^{\frac{n-1}{n}} \leq C_S(\int_X u^2 \omega_t^n + \int_X |\nabla u|^2 \omega_t^n).$$

Here C_S only depends on the dimension n, the volum V, the L^2-Sobolev constant of (X, ω_0), $|\nabla f_{\omega_0}|$ and $|\Delta f_{\omega_0}|$.

The following is Futaki's weighted Poincaré inequality in the generalized setting (which still holds since α is nonnegative). Note that this lemma will play a crucial role in our proof of Theorem 1.4.

Lemma 2.4 ([5, 9, 4]). *Let $\omega \in [\omega_0]$ be any Kähler form. Then for any function $u \in W^{1,2}(X)$ with $\int_X u e^{f_\omega} \omega^n = 0$, we have*

$$\int_X u^2 e^{f_\omega} \omega^n \leq \int_X |\nabla u|^2 e^{f_\omega} \omega^n.$$

3. Applying Calabi-Yau along the flow

The flow (1.1) preserves the cohomological class of ω_t, so we have

$$\omega_t + \alpha \in 2\pi c_1(X).$$

Therefore we can apply Calabi-Yau theorem to obtain a family of Kähler forms $\eta_t \in [\omega_0]$ such that

$$Ric(\eta_t) = \omega_t + \alpha, \tag{3.1}$$

which is equivalent to the following Monge-Ampère equation

$$\eta_t^n = e^{f_{\omega_t}} \omega_t^n. \tag{3.2}$$

Since η_t and ω_t are in the same Kähler class, we can write

$$\eta_t = \omega_t - \sqrt{-1}\partial\bar{\partial}\phi_t \tag{3.3}$$

for some $\phi_t \in C^\infty(X, \mathbb{R})$. It is clear that ϕ_t and f_{η_t} only differ by a constant (recall (2.1)). The main result of this section is the following

Theorem 3.4. *There exists a uniform constant $C = C(n, \omega_0, \alpha)$ depending only on the initial data such that*

$$osc_X \phi_t + diam(X, \eta_t) \leq C.$$

The proof will be divided into two parts. We first prove the oscillation estimate following Yau's approach and then derive the diameter bound.

Lemma 3.5. *There exists a uniform constant $C = C(n, \omega_0, \alpha)$ depending only on the initial data such that*

$$osc_X \phi_t \leq C.$$

Proof. This is standard. We include a proof for reader's convenience, following the exposition in [12]. For simplicity, we will abbreviate the subscript t. We may assume that

$$\int_X \phi \eta^n = \int_X \phi e^{f_\omega} \omega^n = 0.$$

So it is enough to derive a uniform bound for $||\phi||_{C^0}$.

First, we have

$$\begin{aligned}
\int_X |\phi| \omega^n \geq \int_X -\phi \omega^n = \int_X \phi(\eta^n - \omega^n) \\
&= -\int_X \phi \sqrt{-1}\partial\bar{\partial}\phi \wedge \sum_{i=0}^{n-1} \eta^i \wedge \omega^{n-1-i} \\
&= \int_X \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \sum_{i=0}^{n-1} \eta^i \wedge \omega^{n-1-i} \\
&\geq \int_X \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \omega^{n-1} \\
&= \frac{1}{n} \int_X |\nabla\phi|^2 \omega^n.
\end{aligned}$$

Then we apply Futaki's weighted Poincaré inequality (Lemma 2.4) and Hölder inequality to derive

$$\begin{aligned}
\int_X \phi^2 e^{f_\omega} \omega^n &\leq \int_X |\nabla\phi|^2 e^{f_\omega} \omega^n \\
&\leq e^{\sup_X f_\omega} \int_X |\nabla\phi|^2 \omega^n \leq n e^{\sup_X f_\omega} \int_X |\phi| \omega^n \\
&\leq n e^{\mathrm{osc}_X f_\omega} \int_X |\phi| e^{f_\omega} \omega^n \leq n e^{\mathrm{osc}_X f_\omega} (\int_X |\phi|^2 e^{f_\omega} \omega^n)^{\frac{1}{2}} (\int_X e^{f_\omega} \omega^n)^{\frac{1}{2}}
\end{aligned}$$

Thus we get

$$\int_X \phi^2 e^{f_\omega} \omega^n \leq n^2 e^{2\mathrm{osc}_X f_\omega} V.$$

Now using the fact that $|f_\omega|$ is uniformly bounded (recall Theorem 2.2), we immediately get an L^2 bound:

$$||\phi||_{L^2(\omega)} = (\int_X \phi^2 \omega^n)^{\frac{1}{2}} \le C_1 \tag{3.6}$$

for some constant $C_1 = C(n, \omega_0, \alpha)$.

Now for any $p \ge 1$, using the fact the $x|x|^{p-1}$ is a differentiable function with derivative $p|x|^{p-1}$, we have

$$\begin{aligned}
(e^{\sup_X f_\omega} - 1) \int_X |\phi|^p \omega^n &\ge \int_X |\phi|^p (\eta^n - \omega^n) \\
&\ge \int_X \phi|\phi|^{p-1}(\eta^n - \omega^n) \\
&= -\int_X \phi|\phi|^{p-1}\sqrt{-1}\partial\bar{\partial}\phi \wedge \sum_{i=0}^{n-1} \eta^i \wedge \omega^{n-1-i} \\
&= p\int_X |\phi|^{p-1}\sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge \sum_{i=0}^{n-1} \eta^i \wedge \omega^{n-1-i} \\
&\ge \frac{4p}{(p+1)^2} \int_X \sqrt{-1}\partial(\phi|\phi|^{\frac{p-1}{2}}) \wedge \bar{\partial}(\phi|\phi|^{\frac{p-1}{2}}) \wedge \omega^{n-1} \\
&= \frac{4p}{(p+1)^2 n} \int_X |\nabla(\phi|\phi|^{\frac{p-1}{2}})|^2 \omega^n.
\end{aligned}$$

It then follows that (keeping in mind that $|f_\omega|$ is uniformly bounded)

$$\int_X |\nabla(\phi|\phi|^{\frac{p-1}{2}})|^2 \omega^n \le C_2 p \int_X |\phi|^p \omega^n$$

for some constant $C_2 = C(n, \omega_0, \alpha)$. Then applying Sobolev inequality (Theorem 2.3) to the function $\phi|\phi|^{\frac{p-1}{2}}$, we get

$$(\int_X |\phi|^{\frac{n(p+1)}{n-1}})^{\frac{n-1}{n}} \le C_S(C_2 p \int_X |\phi|^p \omega^n + \int_X |\phi|^{p+1}\omega^n). \tag{3.7}$$

Notice that

$$\begin{aligned}
\int_X |\phi|^p \omega^n &= \int_{|\phi|<1} |\phi|^p \omega^n + \int_{|\phi|\ge 1} |\phi|^p \omega^n \\
&\le \int_{|\phi|<1} \omega^n + \int_{|\phi|\ge 1} |\phi|^{p+1} \omega^n \le V + \int_X |\phi|^{p+1}\omega^n.
\end{aligned}$$

So it follows from (3.7) that

$$(\int_X |\phi|^{\frac{n(p+1)}{n-1}})^{\frac{n-1}{n}} \le C_3 p(1 + \int_X |\phi|^{p+1}\omega^n) \le 2C_3 p \max\{1, \int_X |\phi|^{p+1}\omega^n\}$$

for some constant $C_3 = C(n, \omega_0, \alpha)$. Thus we obtain

$$\max\{1, ||\phi||_{L^{\frac{n(p+1)}{n-1}}(\omega)}\} \leq (2C_3)^{\frac{1}{p+1}} p^{\frac{1}{p+1}} \max\{1, ||\phi||_{L^{p+1}(\omega)}\}$$

Then standard Moser iteration gives

$$\max\{1, ||\phi||_{L^\infty}\} \leq C_4 \max\{1, ||\phi||_{L^2(\omega)}\}$$

for some constant $C_4 = C(n, \omega_0, \alpha)$. Combining this with (3.6), we finish the proof. □

Since the generalized Ricci potential f_{η_t} and ϕ_t only differ by a constant, the next result follows immediately from Lemma 3.5.

Corollary 3.8. *There exists some constant $C = C(n, \omega_0, \alpha) > 0$ such that*

$$|f_{\eta_t}| \leq C$$

Now we are ready to prove the following diameter bound.

Lemma 3.9. *There exists some constant $C = C(n, \omega_0, \alpha) > 0$ such that*

$$diam(X, \eta_t) \leq C.$$

Proof. For simplicity, we abbreviate the subscript t. By (3.1), it is clear that

$$\mathrm{Ric}(\eta) > 0.$$

If one can prove a uniform C^2 estimate for the Monge-Ampère equation (3.2), then one would get $\mathrm{Ric}(\eta) \geq c > 0$ and the diameter bound follows readily from Myer's theorem. But this approach does not seem to work in our setting since we do not have enough curvature control along the flow. So here we use a different strategy.

We put

$$d := diam(X, \eta),$$

and assume that $d = d_\eta(p, q)$ for two points $p, q \in X$, where d_η is the distance function induced by η. We define

$$d_1(x) := d_\eta(x, p),\ d_2(x) := d_\eta(x, q),\ x \in X.$$

Triangle inequality simply gives

$$d_1(x) + d_2(x) \geq d,\ x \in X.$$

Integrating both sides against the volume form $e^{f_\eta}\eta^n$, we get

$$\frac{1}{V}\int_X d_1 e^{f_\eta}\eta^n + \frac{1}{V}\int_X d_2 e^{f_\eta}\eta^n \geq d.$$

So we may assume that

$$\overline{d_1} := \frac{1}{V}\int_X d_1 e^{f_\eta}\eta^n \geq \frac{d}{2}.$$

Now applying Lemma 2.4 to $d_1(x) - \overline{d_1}$, we get

$$\int_X |d_1 - \overline{d_1}|^2 e^{f_\eta}\eta^n \leq \int_X |\nabla_\eta d_1|^2_\eta e^{f_\eta}\eta^n \leq C(n)\exp(\sup_X f_\eta)\mathrm{vol}(B_\eta(p, d)).$$

On the other hand, since $|d_1 - \overline{d_1}| \geq \frac{d}{4}$ on the ball $B_\eta(p, \frac{d}{4})$, we have

$$\int_X |d_1 - \overline{d_1}|^2 e^{f_\eta} \eta^n \geq \int_{B_\eta(p,\frac{d}{4})} |d_1 - \overline{d_1}|^2 e^{f_\eta} \eta^n \geq \frac{d^2}{C(n)} \exp(\inf_X f_\eta) \mathrm{vol}(B_\eta(p, \frac{d}{4})).$$

Thus we get

$$d^2 \leq C(n) \exp(\mathrm{osc}_X f_\eta) \frac{\mathrm{vol}(B_\eta(p, d))}{\mathrm{vol}(B_\eta(p, \frac{d}{4}))}.$$

Using $Ric(\eta) > 0$ and relative volume comparison, we have

$$d^2 \leq C(n) \exp(\mathrm{osc}_X f_\eta) 4^{2n}.$$

Then the desired diameter bound follows from Corollary 3.8. □

4. Proof of the main result

As shown in the previous section, along the flow (1.1), if we consider the Kähler form $\eta_t \in [\omega_0]$ such that $\mathrm{Ric}(\eta_t) = \omega_t + \alpha$, then we have

$$Ric(\eta_t) > 0, \ \mathrm{vol}(X, \eta_t) = C(n, [\omega_0]), \ diam(X, \eta_t) \leq C(\omega_0, \alpha).$$

In particular, Cheeger-Colding theory works perfectly for the family $\{(X, \eta_t)\}_{t \geq 0}$. And thanks to the recent compactness result obtained in [8], we have the following partial C^0 estimate.

Theorem 4.1. [8, Theorem 1.2] *There exists a positive constant $b = b(n, \omega_0, \alpha)$ and a large integer $k = k(n, \omega_0, \alpha)$ such that*

$$\rho_{\eta_t, k} \geq b > 0$$

along the flow (1.1).

To finish the proof of Theorem 1.4, it is enough to notice the following

Lemma 4.2. *For each $k \in \mathbb{N}$, there exists a constant $C = C(n, k, \omega_0, \alpha) > 0$ such that*

$$C^{-1} \rho_{\eta_t, k} \leq \rho_{\omega_t, k} \leq C \rho_{\eta_t, k}$$

along the flow (1.1).

Proof. Given a family of smooth Hermitian metrics $\{h_t\}$ on L with

$$-\sqrt{-1}\partial\bar{\partial} \log h_t = \omega_t,$$

we consider

$$\widetilde{h_t} := e^{f_{\eta_t}} h_t.$$

Then it is clear that

$$-\sqrt{-1}\partial\bar{\partial} \log \widetilde{h_t} = \eta_t.$$

Now recall that we have the following uniform control (cf. Theorem 2.2 and Corollary 3.8)

$$|f_{\omega_t}| + |f_{\eta_t}| \leq C(n, \omega_0, \alpha).$$

So the Hermitian metrics h_t and $\widetilde{h}_t$ are fiber-wise comparable. Meanwhile, as volume forms, ω_t^n and η_t^n are comparable as well (recall (3.2)). So it follows easily from the definition of Bergman kernel that $\rho_{\omega_t,k}$ and $\rho_{\eta_t,k}$ are comparable as desired. □

5. An application

In this part we give an application of Theorem 1.4, following Jiang's work [6] closely (see also [7]). Our setup is as follows. Let X be an n-dimensional compact Kähler manifold with an ample line bundle L. Let $\omega \in 2\pi c_1(L)$ be an Kähler form with $V = \int_X \omega^n$ being the volume. Suppose that (X,ω) satisfies the following two conditions:

- there exists a closed nonnegative $(1,1)$ form $\alpha \in 2\pi c_1(-K_X - L)$ such that the scalar curvature $R(\omega)$ satisfies $R(\omega) - \mathrm{tr}_\omega \alpha \geq -\Lambda$ for some constant $\Lambda \geq 0$;
- (X,ω) satisfies the following L^2-Sobolev inequality:

$$\left(\int_X |f|^{\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq C_S \int_X |\nabla f|^2 \omega^n \tag{5.1}$$

  for any $f \in W^{1,2}(X)$ with $\int_X f\omega^n = 0$.

Then we have the following partial C^0 estimate:

Theorem 5.2. *One has*

$$\rho_{\omega,k} \geq b > 0$$

for some k, b only depending on n, V, Λ and C_S.

The proof of this result is essentially contained in [6] and there are two main ingredients— the regularization property of Ricci flows and the fact that the Bergman kernels at different time slices are comparable. Note that similar argument was also exploited in [7]. A simple observation is that, all the estimates in [6] hold analogously for the generalized KRF if we replace the scalar curvature R by the twisted scalar curvature $R - \mathrm{tr}_\omega \alpha$ in the argument. And the proof of Theorem 5.2 is morally the same as the one for [6, Theorem 1.5]. Note that in the statement of [6, Theorem 1.5], one can replace Ricci lower bound and diameter upper bound by other geometric conditions, since these two bounds are essentially used to get the lower bound of scalar curvature, the Sobolev inequality and the lower bound of Green's function. In our setting we use the inequality (5.1) to replace the bounds on Ricci and diameter and the argument in [6] works identically for our purpose. So in the following we only outline the proof and omit some details.

We consider the generalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t + \alpha$$

starting from ω. Then the lower bound of $R(\omega) - \mathrm{tr}_\omega \alpha$ and the Sobolev inequality (5.1) will give us the following Sobolev inequality along the flow:

$$\left(\int_X |f|^{\frac{2n}{n-1}}\omega_t^n\right)^{\frac{n-1}{n}} \leq A\left(\int_X |\nabla f|^2 \omega_t^n + (R(\omega_t) - \mathrm{tr}_{\omega_t}\alpha + B)\int_X f^2 \omega_t^2\right) \tag{5.3}$$

for any $f \in W^{1,2}(X)$, where A, B are positive constants only depending on n, V, Λ and C_S (cf. [15, 16, 4]). Then we can follow the argument in [6, Section 2,3] (see also [7]) to deduce that

$$|\Delta f_{\omega_t}| + |\nabla f_{\omega_t}|^2 \leq \frac{C}{t^{n+1}},\ t \in (0,1],$$

where f_{ω_t} is the generalized Ricci potential along the flow and the constant C only depends on n, V, Λ and C_S. Note that the lower bound of Green's function for (X, ω) is also involved in this estimate (see [6, (3.6)]), which can be controlled in our setting by n, V and C_S (cf. Lemma A.1). Then applying Theorem 2.2 to the flow $\{\omega_t\}$ for $t \in [\frac{1}{2}, 1]$, we obtain

$$|f_{\omega_t}| + |R(\omega_t - \mathrm{tr}_{\omega_t}\alpha)|^2 \leq C,\ t \in [\frac{1}{2}, 1],$$

where C only depends on n, V, Λ and C_S. Now we can go through the proof of Theorem 1.4 to find that

$$\rho_{\omega_t,k} \geq b > 0,\ t \in [\frac{1}{2}, 1]$$

for some k, b only depending on n, V, Λ and C_S. Finally, following the proof of [6, Theorem 5.8], we get

$$\rho_{\omega,k} \geq C^{-1}\rho_{\omega_t,k} \geq C^{-1}b > 0, t \in [\frac{1}{2}, 1]$$

for some constant $C = C(n, V, \Lambda, C_S) > 0$. So we finish the proof of Theorem 5.2.

Remark 5.4. Form the proof of Theorem 5.2, one can see that the nonnegative form α itself only appears as an auxiliary term and does not play any essential role in the argument. So it is likely that the condition on the lower bound of $R(\omega) - \mathrm{tr}_\omega\alpha$ can be replaced by other geometric (or topological) conditions.

Appendix A. Lower bound of Green's function

Lemma A.1 ([10]). *Given (X, ω) as in Section 5, the Green's function $G(x, y)$ of (X, ω) is bounded from below by a constant only depending on n, V and C_S.*

Proof. We sketch a proof for reader's convenience. Let

$$H(x, y, t) := \frac{1}{V} + \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

be the heat kernel of (X, ω). Here $\lambda_1 \leq \lambda_2 \leq ...$ are the eigenvalues of Δ_ω and ϕ_i's are the corresponding eigenfunctions (i.e. $\Delta_\omega\phi_i = -\lambda_i\phi_i$) such that

$$\int_X \phi_i\phi_j\omega^n = \delta_{ij}.$$

Then the Green's function $G(x, y)$ is given by

$$G(x, y) := \int_0^\infty (H(x, y, t) - \frac{1}{V})dt.$$

To get a lower bound of $G(x, y)$, it suffices to prove the following standard fact:

$$\left|H(x,y,t)-\frac{1}{V}\right| \leq \frac{C(n,C_S)}{t^n},\ t>0.$$

To this end, we put

$$H_1(x,y,t) := H(x,y,t)-\frac{1}{V} = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y).$$

Then it is easy to verify

$$H_1(x,x,2t) = \int_X H_1(x,y,t)^2\omega^n(y).$$

Taking time derivative, we get

$$\partial_t H_1(x,x,2t) = -2\int_X |\nabla_y H_1(x,y,t)|^2\omega^n(y). \tag{A.2}$$

On the other hand we have

$$\begin{aligned} H_1(x,x,2t) &= \int_X H_1^2(x,y,t)\omega^n(y) \\ &\leq \left(\int_X |H_1(x,y,t)|^{\frac{2n}{n-1}}\omega^n(y)\right)^{\frac{n-1}{n+1}} \left(\int_X |H_1(x,y,t)|\omega^n(y)\right)^{\frac{2}{n+1}} \\ &\leq 2^{\frac{2}{n+1}}\left(\int_X |H_1(x,y,t)|^{\frac{2n}{n-1}}\omega^n(y)\right)^{\frac{n-1}{n+1}}. \end{aligned}$$

Combining this with (5.1) and (A.2), we get

$$\partial_t H_1(x,x,2t) \leq -C(n,C_S)H_1(x,x,2t)^{\frac{n+1}{n}},$$

so that

$$\partial_t(H_1(x,x,t)^{-\frac{1}{n}}) \geq C(n,C_S).$$

Integrating this from ϵ to t and using the asymptotic behavior $H_1(x,x,\epsilon)^{-\frac{1}{n}} \to 0$ as $\epsilon \to 0$, we arrive at

$$H_1(x,x,t) \leq \frac{C(n,C_S)}{t^n},\ t>0.$$

Now using the fact $\left|H_1(x,y,t)\right| \leq H_1(x,x,t)^{\frac{1}{2}}H_1(y,y,t)^{\frac{1}{2}}$, we obtain

$$\left|H(x,y,t)-\frac{1}{V}\right| \leq \frac{C(n,C_S)}{t^n},\ t>0.$$

$\square$

References

[1] R. Bamler, Convergence of Ricci flows with bounded scalar curvature, arXiv:1603.05235, to appear in Annals of Math.

[2] X.X. Chen, B, Wang, Space of Ricci flows I, Comm. Pure Appl. Math. **65** (2012), 1399-1457.

[3] X.X. Chen, B. Wang, Space of Ricci flows II, preprint, arXiv:1405.6797.

[4] T. Collins, G. Székelyhidi, The twisted Kähler-Ricci flow, J. Reine Angew. Math. 716 (2016), 179-205.

[5] A. Futaki, Kähler-Einstein metrics and integral invariants, Lecture notes in mathematics, Springer-Verlag, 1988

[6] W. Jiang, Bergman kernel along the Kähler-Ricci flow and Tian's conjecture, J. Reine Angew. Math. **717** (2016), 195-226.

[7] W. Jiang, F. Wang, X. Zhu, Bergman kernels for a sequence of almost Khler-Ricci solitons. Ann. Inst. Fourier. **67** (2017), 1279-1320.

[8] G. Liu, G. Székelyhidi, Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below, preprint, arXiv:1804.08567 (2018).

[9] J. Liu, The generalized Kähler Ricci flow, J. Math. Anal. Appl. **408** (2013), 751-761

[10] P. Li, Geometric Analysis, Cambridge University Press, 2012

[11] N. Sesum, G. Tian, Bounding scalar curvature and diameter along the Kähler Ricci Flow (after Perelman), J. Inst. of Math. Jussieu, **7** (2008), 575-587.

[12] J. Song, B. Weinkove, An introduction to the Kähler-Ricci flow, 89-188, Lecture Notes in Math., 2086, Springer, Cham, 2013.

[13] G. Tian, Z. Zhang, Regularity of Kähler-Ricci flows on Fano manifolds, Acta Math. **216** (2016), 127-176.

[14] G. Tian, X. Zhu, Convergence of Kähler-Ricci flow, J. Amer. Math. Soc., **20** (2007), 675-699.

[15] R. Ye, The logarithmic Sobolev and Sobolev inequalities along the Ricci flow, Commun. Math. Stat. **3** (2015), 1-36.

[16] Q.S. Zhang, A uniform Sobolev Inequality under the Ricci flow, Inter. Math. Res. Not. (2007), 17p.

[17] X. Zhang, X.W. Zhang, Generalized Kähler-Einstein metrics and energy functionals, Canad. J. Math. 66 (2014) 1413-1435.