

# DELTA INVARIANTS OF PROJECTIVE BUNDLES

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ABSTRACT. We compute the  $\delta$ -invariants of projective bundles of Fano type. This is a baby version of the joint work [19].

## 1. INTRODUCTION

Given an arbitrary Fano manifold  $X$ , it is often the case that  $X$  does not admit any Kähler–Einstein (KE) metric. But still,  $X$  could admit twisted KE or conical KE metrics. To study these metrics and their degenerations, some analytic and algebraic thresholds play important roles. For instance, the greatest Ricci lower bound  $\beta(X)$  of Tian [17] measures how far  $X$  is away from being a KE manifold. As shown in [1, 5],  $\beta(X)$  is equal to the algebraic  $\delta$ -invariant  $\delta(X)$ , which serves as the right threshold for  $X$  to be Ding stability (cf. [?, 1, 4]).

More precisely, suppose that  $X$  does not admit KE metrics. For any  $\mu \in (0, \beta(X))$ , we can find a Kähler form  $\omega \in 2\pi c_1(X)$  such that  $\text{Ric}(\omega) \geq \mu\omega$ . An interesting problem would be to study the Gromov–Hausdorff limit of  $(X, \omega)$  as  $\mu \rightarrow \beta(X)$ . By [13], the limit is homeomorphic to a  $\mathbb{Q}$ -Fano variety, which is supposed to be the optimal degeneration of  $X$  in a suitable sense. To study this problem, it could be enlightening if we have some explicit examples to play with. We refer the reader to [16, 15, 11] for some discussions in this direction. The purpose of this note is to generalize the construction in [16, Section 3.1] to higher dimensions. More precisely, we will use the Calabi symmetry of projective bundles to explicitly construct a family of Kähler metrics with Ricci curvature as positive as possible, with the aid of which we can compute the  $\delta$ -invariants of such manifolds.

To state the main result, let us fix the notation that will be used throughout. Let  $X$  be an  $n$ -dimensional Fano manifold with Fano index  $I(X) \geq 2$ . So we can find an ample line bundle  $L$  such that

$$(1.1) \quad L = -\lambda K_X \text{ for some } \lambda \in (0, 1).$$

We put

$$Y := \mathbb{P}(L^{-1} \oplus \mathcal{O}_X) \xrightarrow{\pi} X.$$

Let  $E_0$  denote the zero section and  $E_\infty$  the infinity section. Then

$$-K_Y = \pi^*(-K_X) + E_0 + E_\infty \sim_{\mathbb{Q}} (1/\lambda + 1)E_\infty - (1/\lambda - 1)E_0$$

is ample and hence  $Y$  is an  $(n+1)$ -dimensional Fano manifold. We put

$$(1.2) \quad \beta_0 := \left( \frac{n+1}{n+2} \cdot \frac{(1/\lambda + 1)^{n+2} - (1/\lambda - 1)^{n+2}}{(1/\lambda + 1)^{n+1} - (1/\lambda - 1)^{n+1}} - (1/\lambda - 1) \right)^{-1}.$$

Using binomial formula, one can easily verify the following elementary fact:

$$(1.3) \quad \beta_0 \in (1/2, 1).$$

The main result is the following

**Theorem 1.1.** *One has*

$$\beta(Y) = \delta(Y) = \min \left\{ \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}, \beta_0 \right\}.$$

In particular,  $Y$  cannot admit KE metrics<sup>1</sup>. But as we shall see in Section 2,  $Y$  does admit a family of twisted conical KE metrics. When  $\delta(X) \geq \lambda + \beta_0(1 - \lambda)$  (this holds for example when  $X$  is K-semistable), we deduce that

$$(1.4) \quad \delta(Y) = \beta_0.$$

As we will show, in this case  $E_0$  computes  $\delta(Y)$ . This generalizes the example  $Y = Bl_1\mathbb{P}^2$  treated in [16]. Indeed, when  $Y = Bl_1\mathbb{P}^2$ , one has  $X = \mathbb{P}^1$ ,  $n = 1$  and  $\lambda = 1/2$ , so that  $\delta(Y) = \beta_0 = 6/7$ , which agrees with the result obtained in [16, 10]. In the case of  $\delta(X) \leq \lambda + \beta_0(1 - \lambda)$ , Theorem 1.1 gives

$$(1.5) \quad \delta(Y) = \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

In this case, there always exists a prime divisor  $F$  over  $X$  computing  $\delta(X)$  (see [4, Theorem 6.7]). This divisor naturally induces a divisor  $\bar{F}$  over  $Y$ , and we will show that  $\delta(Y)$  is computed by  $\bar{F}$  when (1.5) takes place. See Section 5 for an explicit example.

**Remark 1.2.** In [18], Zhuang derived the  $\delta$ -invariants of product spaces. In particular, let  $Y = X \times \mathbb{P}^1$  be the trivial  $\mathbb{P}^1$ -bundle over  $X$ , then

$$\delta(Y) = \min\{\delta(X), 1\}.$$

So to some extent, Theorem 1.1 generalizes this product formula.

The proof of Theorem 1.1 essentially makes use of the natural  $\mathbb{C}^*$ -action on  $Y$ . On the analytic side, this toric action allows us to carry out the momentum construction due to Calabi, from which we will derive a lower bound for  $\beta(Y)$  in Section 2. On the algebraic side, by using this torus action and the lct definition of  $\delta$ -invariant, we show in Section 3 that the obtained lower bound also bounds  $\delta(Y)$  from above, so we conclude the main result. In Section 4 we provide several useful properties of  $Y$ , which will be applied in Section 5 to investigate some concrete examples.

## 2. THE LOWER BOUND

To derive a lower bound for  $\beta(Y)$ , we follow the approach in [16, Section 3.1], using Calabi ansatz to construct a family of Kähler metrics  $\eta \in 2\pi c_1(Y)$  with Ricci curvature as positive as possible. Similar treatment also appears in [12, Section 3.2].

We fix

$$\mu \in (0, \beta(X))$$

and choose Kähler forms  $\omega, \alpha \in 2\pi c_1(X)$  such that

$$(2.1) \quad \text{Ric}(\omega) = \mu\omega + (1 - \mu)\alpha.$$

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<sup>1</sup>By [9],  $Y$  always admits a Kähler Ricci soliton metric.

Then the momentum construction due to Calabi can provide Kähler metrics  $\eta$  on  $Y$  of the form (in special local coordinates)

$$\eta = \lambda \tau \pi^* \omega + \varphi \frac{\sqrt{-1} dw \wedge d\bar{w}}{|w|^2},$$

whose Ricci forms are given by

$$(2.2) \quad \begin{aligned} \text{Ric}(\eta) = & \left( \mu - n\lambda \frac{\varphi}{\tau} - \lambda \varphi' \right) \pi^* \omega + (1 - \mu) \pi^* \alpha \\ & - \varphi \left( n \frac{\varphi}{\tau} + \varphi' \right)' \frac{\sqrt{-1} dw \wedge d\bar{w}}{|w|^2}. \end{aligned}$$

Here  $\varphi = \varphi(\tau)$  with  $\tau \in (1/\lambda - 1, 1/\lambda + 1)$  is a one-variable positive function to be determined and  $w$  denotes the fiberwise coordinate. To cook up  $\eta \in 2\pi c_1(Y)$  with  $\text{Ric}(\eta) \geq \beta \eta$  (possibly in the current sense), we will impose the following conditions for  $\varphi$ :

$$(2.3) \quad \begin{cases} \varphi(1/\lambda - 1) = \varphi(1/\lambda + 1) = 0, \\ \varphi'(1/\lambda - 1) \in (0, 1], \\ \varphi'(1/\lambda + 1) \in [-1, 0), \end{cases}$$

and

$$(2.4) \quad - \left( n \frac{\varphi}{\tau} + \varphi' \right)' = \beta \text{ for } \tau \in (1/\lambda - 1, 1/\lambda + 1),$$

where  $\beta$  is any constant that satisfies

$$(2.5) \quad 0 < \beta \leq \min \left\{ \frac{\mu \beta_0}{\lambda + \beta_0(1 - \lambda)}, \beta_0 \right\}.$$

Let us explain the exact meanings of these conditions. The boundary condition (2.3) makes sure that  $\eta \in 2\pi c_1(Y)$  and  $\eta$  possibly possesses certain amount of edge singularities along  $E_0$  and  $E_\infty$ . Solving the ODE (2.4), we obtain that

$$(2.6) \quad \tau^n \varphi = -\frac{\beta}{n+2} \tau^{n+2} + A \tau^{n+1} + B$$

where

$$\begin{cases} A = \frac{\beta}{n+2} \cdot \frac{(1/\lambda+1)^{n+2} - (1/\lambda-1)^{n+2}}{(1/\lambda+1)^{n+1} - (1/\lambda-1)^{n+1}}, \\ B = \frac{-2\beta}{n+2} \cdot \frac{(1/\lambda^2-1)^{n+1}}{(1/\lambda+1)^{n+1} - (1/\lambda-1)^{n+1}}. \end{cases}$$

From this, we easily derive that

$$(2.7) \quad \begin{cases} \beta_1 := \varphi'(1/\lambda - 1) = \frac{\beta}{\beta_0}, \\ \beta_2 := -\varphi'(1/\lambda + 1) = \frac{\beta(2\beta_0-1)}{\beta_0}. \end{cases}$$

Then (1.3) and (2.5) simply imply that

$$0 < \beta_2 < \beta_1 \leq 1.$$

So  $\eta$  has edge singularities with angles  $\beta_1$  and  $\beta_2$  along  $E_0$  and  $E_\infty$  respectively. Moreover (2.5) also guarantees that

$$(2.8) \quad \begin{aligned} \mu - n\lambda \frac{\varphi}{\tau} - \lambda \varphi' &= \mu - \lambda \beta_1 - \beta(1 - \lambda - \tau) \\ &= (\mu - \lambda \beta / \beta_0 - \beta(1 - \lambda)) + \tau \beta \\ &\geq \tau \beta. \end{aligned}$$

Therefore  $\eta$  satisfies  $\text{Ric}(\eta) \geq \beta\eta$  in the current sense. More precisely,  $\eta$  solves the following twisted Kähler–Einstein edge equation:

$$(2.9) \quad \begin{aligned} \text{Ric}(\eta) = & \beta\eta + (\mu - \lambda\beta/\beta_0 - \beta(1 - \lambda))\pi^*\omega + (1 - \mu)\pi^*\alpha \\ & + 2\pi(1 - \beta/\beta_0)[E_0] + 2\pi(1 - \beta(2\beta_0 - 1)/\beta_0)[E_\infty]. \end{aligned}$$

This implies that (using [5, Theorem 5.7] and [1, Theorem C])

$$(2.10) \quad \beta(Y) = \delta(Y) \geq \delta_\theta(Y) \geq \beta_\theta(Y) \geq \beta,$$

where

$$\theta = \frac{(\mu - \lambda\beta/\beta_0 - \beta(1 - \lambda))}{2\pi}\pi^*\omega + \frac{1 - \mu}{2\pi}\pi^*\alpha + (1 - \beta_1)[E_0] + (1 - \beta_2)[E_\infty]$$

is a semi-positive current in  $(1 - \beta)c_1(Y)$ . Using (2.5) and letting  $\mu \rightarrow \beta(X)$ , we obtain

$$\beta(Y) \geq \min \left\{ \frac{\beta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}, \beta_0 \right\}.$$

Finally, applying [5, Theorem 5.7], we get the following

**Proposition 2.1.** *One has*

$$\delta(Y) \geq \min \left\{ \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}, \beta_0 \right\}.$$

**Remark 2.2.** *There is a direct and purely algebraic proof of this if one uses that fact that  $\delta_T(Y) = \delta(Y)$  (cf. [7]). Here  $T = \mathbb{C}^*$  acts naturally on the fibers. So it suffices to investigate  $T$ -invariant divisor over  $Y$  and the argument in the next section also proceeds to give this lower bound.*

Now let us go back to our motivation mentioned at the very beginning of this paper. We shall study the degeneration of metrics on  $Y$  with positive Ricci curvature as they approach the roof.

Suppose that  $X$  admits a KE metric  $\omega_{KE} \in 2\pi c_1(X)$ . (In this case  $\beta(Y) = \beta_0$  by Theorem 1.1). Then as in [16, Section 3.1], for any  $\beta \in (0, \beta_0)$  we can construct a smooth Kähler form  $\omega_\beta \in 2\pi c_1(Y)$  with  $\text{Ric}(\omega_\beta) > \beta\omega_\beta$  such that, as  $\beta \rightarrow \beta_0$ , one has

$$(Y, \omega_\beta) \xrightarrow{G.H.} (Y, \eta),$$

with  $\eta$  solving

$$\text{Ric}(\eta) = \beta_0\eta + (1 - \lambda - \beta_0(1 - \lambda))\pi^*\omega_{KE} + 2\pi(2 - 2\beta_0)[E_\infty].$$

In particular the limit space is still  $Y$ . This generalizes [16], where an  $\eta$  satisfying

$$\text{Ric}(\eta) = \frac{6}{7}\eta + \frac{1}{7}\pi^*\omega_{FS} + 2\pi(1 - \frac{5}{7})[E_\infty]$$

was constructed on  $Bl_1\mathbb{P}^2$ .

Suppose in general that  $X$  does not necessarily admit KE, but  $\beta(X) > \lambda + \beta_0(1 - \lambda)$ . (In this case again  $\beta(Y) = \beta_0$  by Theorem 1.1). We choose  $\mu \in (\lambda + \beta_0(1 - \lambda), \beta(X))$  and hence there are Kähler forms  $\omega, \alpha \in 2\pi c_1(X)$  satisfying (2.1). Then the same construction as in [16, Section 3.1] shows that, for any  $\beta \in (0, \beta_0)$  there is a smooth Kähler form  $\omega_\beta \in 2\pi c_1(Y)$  with  $\text{Ric}(\omega_\beta) > \beta\omega_\beta$  such that, as  $\beta \rightarrow \beta_0$ , one has

$$(Y, \omega_\beta) \xrightarrow{G.H.} (Y, \eta),$$

with  $\eta$  solving

$$\text{Ric}(\eta) = \beta_0 \eta + (\mu - \lambda - \beta_0(1 - \lambda))\pi^* \omega + (1 - \mu)\pi^* \alpha + 2\pi(2 - 2\beta_0)[E_\infty].$$

So as in the previous case, the limit space is still  $Y$  itself. Note that the limit metric  $\eta$  is not unique (as  $\mu$ ,  $\omega$  and  $\alpha$  are allowed to vary).

Finally, suppose that  $\beta(X) \leq \lambda + \beta_0(1 - \lambda)$ . Then by Theorem 1.1,  $\beta(Y) = \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}$ . This case turns out to be more subtle. Firstly, it seems that the Calabi ansatz does not easily provide *smooth* Kähler forms  $\omega_\beta$  such that  $\text{Ric}(\omega) \geq \beta\omega_\beta$  as  $\beta \rightarrow \beta(Y)$ . Secondly, as  $\mu \rightarrow \beta(X)$ , the Kähler form  $\omega$  we chose from the base  $X$  (recall (2.1)) is supposed to develop certain singularities (see [14]), which suggests that  $X$  itself would degenerate in the Gromov-Hausdorff topology to some other  $\mathbb{Q}$ -Fano variety. So at this stage it is unclear how  $Y$  would degenerate. We leave this case to future studies.

**Remark 2.3.** *It is worth mentioning that, Calabi ansatz also applies to projective bundles of higher ranks (see [?] for more general discussions).*

### 3. THE UPPER BOUND

As we have seen, both

$$\beta_0 \text{ and } \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}$$

arise naturally from Calabi's ODE. In this section, by using the definition of  $\delta$ -invariant (cf. [6, 3]), we shall show that they also have purely algebraic interpretations and that they naturally bound  $\delta(Y)$  from *above*, which hence completes the proof of Theorem 1.1.

We begin with the following simple lemma, which justifies the appearance of  $\beta_0$ .

**Lemma 3.1.** *One has*

$$\delta(Y) \leq \frac{A_Y(E_0)}{S_{-K_Y}(E_0)} = \beta_0.$$

*Proof.* This follows from a straightforward calculation. Indeed, one has  $A_Y(E_0) = 1$  and

$$\begin{aligned} S_{-K_Y}(E_0) &= \frac{1}{(-K_Y)^{n+1}} \int_0^\infty \text{Vol}(-K_Y - tE_0) dt \\ &= \frac{1}{(-K_Y)^{n+1}} \int_0^2 \left( (1/\lambda + 1)E_\infty - (t + 1/\lambda - 1)E_0 \right)^{n+1} dt \\ &= \frac{2(1/\lambda + 1)^{n+1} - ((1/\lambda + 1)^{n+2} - (1/\lambda - 1)^{n+2})/(n+1)}{(1/\lambda + 1)^{n+2} - (1/\lambda - 1)^{n+2}} \\ &= \frac{n+1}{n+2} \cdot \frac{(1/\lambda + 1)^{n+2} - (1/\lambda - 1)^{n+2}}{(1/\lambda + 1)^{n+1} - (1/\lambda - 1)^{n+1}} - (1/\lambda - 1). \end{aligned}$$

So the result follows.  $\square$

A combination of Proposition 2.1 and Lemma 3.1 gives the following consequence.

**Corollary 3.2.** *Suppose that*

$$\delta(X) \geq \lambda + \beta_0(1 - \lambda),$$

*then one has*

$$\delta(Y) = \beta_0$$

and  $\delta(Y)$  is computed by the divisor  $E_0 \subseteq Y$ .

Now let us give an algebraic explanation for the quantity

$$\frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

For any prime divisor  $F$  over  $X$ , we put

$$(3.1) \quad \delta_X(F) := \frac{A_X(F)}{S_{-K_X}(F)}.$$

Let  $\bar{X} \xrightarrow{\phi} X$  be a log resolution of  $X$  such that  $F \subseteq \bar{X}$ . Then we have the following commutative diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{\phi}} & Y \\ \bar{\pi} \downarrow & & \downarrow \pi \\ \bar{X} & \xrightarrow{\phi} & X \end{array}$$

where

$$\bar{Y} := \mathbb{P}(\phi^*(L^{-1} \oplus \mathcal{O}_X)).$$

Set

$$\bar{F} := \bar{\pi}^* F$$

and

$$\delta_Y(\bar{F}) := \frac{A_Y(\bar{F})}{S_{-K_Y}(\bar{F})}.$$

Then it is easy to check that

$$(3.2) \quad A_Y(\bar{F}) = A_X(F).$$

**Proposition 3.3.** *For any prime divisor  $F$  over  $X$ , we have*

$$\delta_Y(\bar{F}) = \frac{\delta_X(F)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

So by taking inf over all  $F$ , we get

**Corollary 3.4.** *We have*

$$(3.3) \quad \delta(Y) \leq \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

Combining this with Proposition 2.1 and Lemma 3.1, Theorem 1.1 follows immediately.

To prove Proposition 3.3, we use the fact that  $Y$  is a  $T$ -variety, where  $T = \mathbb{C}^*$  acts multiplicatively on  $\mathbb{P}^1$ -fibers. So for any  $m \geq 1$ , we have a weight decomposition:

$$(3.4) \quad R_m := H^0(Y, -mK_Y) = \bigoplus_{j \in \mathbb{Z}} R_m^j,$$

where

$$R_m^j := \{s \in H^0(Y, -mK_Y) \mid \tau \cdot s = \tau^j s, \tau \in T\}.$$

More precisely, each  $R_m^j$  consists of those sections that vanish along  $E_0$  with order  $j$ , i.e.,

$$R_m^j := \{s \in H^0(Y, -mK_Y) \mid \text{ord}_{E_0} s = j\}.$$

One can easily compute the dimension of each  $R_m^j$ . Indeed, we write

$$(3.5) \quad -K_X = IH,$$

where  $I := I(X)$  is the Fano index of  $X$  and  $H$  is an ample line bundle on  $X$ . Then for any  $j \in \mathbb{Z}$  we can write

$$(3.6) \quad -mK_Y \sim (jI\lambda + mI(1-\lambda))\pi^*H + jE_0 + (2m-j)E_\infty.$$

Moreover any  $T$ -invariant divisor in  $|-mK_Y|$  can be written in this form. So we deduce that

$$(3.7) \quad \dim_{\mathbb{C}} R_m^j = \begin{cases} h^0(X, (jI\lambda + mI(1-\lambda))H), & 0 \leq j \leq 2m, \\ 0, & \text{otherwise.} \end{cases}$$

Now given a prime divisor  $F$  over  $X$ , let us construct an  $m$ -basis type divisor  $\mathcal{D}_m \sim_{\mathbb{Q}} -K_Y$  that is compatible with the filtration on  $R_m$  induced by  $\text{ord}_{\overline{F}}$ . Note that, for each  $j \in \{1, \dots, 2m\}$ ,  $\text{ord}_F$  induces a filtration of  $R_m^j$ , from which we can choose a compatible basis  $\{s_i^j\}$  with  $i \in \{1, \dots, \dim_{\mathbb{C}} R_m^j\}$ . Let  $D_i^j$  be the divisor cut out by  $s_i^j$  and we put

$$(3.8) \quad \mathcal{D}_m := \frac{1}{m \sum_{k=0}^{2m} \dim R_m^k} \sum_{j=0}^{2m} \sum_{i=1}^{\dim R_m^j} \left( \pi^* D_i^j + jE_0 + (2m-j)E_\infty \right).$$

Then  $\mathcal{D}_m \sim_{\mathbb{Q}} -K_Y$  is an  $m$ -basis type divisor that is compatible with the filtration induced by  $\text{ord}_{\overline{F}}$ . In particular, by the proof of [6, Lemma 2.2],

$$(3.9) \quad \lim_{m \rightarrow \infty} \text{ord}_{\overline{F}}(\mathcal{D}_m) = S_{-K_Y}(\overline{F}).$$

**Lemma 3.5.** *We also have*

$$\lim_{m \rightarrow \infty} \text{ord}_{\overline{F}}(\mathcal{D}_m) = \frac{\lambda + \beta_0(1-\lambda)}{\beta_0} S_{-K_X}(F).$$

Here  $\frac{\lambda + \beta_0(1-\lambda)}{\beta_0} = \frac{n+1}{n+2} \cdot \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}}.$

*Proof.* Note that

$$\text{ord}_{\overline{F}}(\mathcal{D}_m) = \text{ord}_F \left( \frac{\sum_{j=0}^{2m} \sum_{i=1}^{\dim R_m^j} D_i^j}{m \sum_{k=0}^{2m} \dim R_m^k} \right)$$

Moreover we have the following three asymptotic calculations.

- (1) For each  $j$ , the chosen basis  $\{s_i^j\}$  of  $R_m^j$  is adapted to  $\text{ord}_F$ , so by [6, Lemma 2.2], we have

$$\lim_{m \rightarrow \infty} \text{ord}_F \left( \frac{\sum_{i=1}^{\dim R_m^j} D_i^j}{(jI\lambda + mI(1-\lambda)) \dim R_m^j} \right) = S_H(F) = \frac{S_{-K_X}(F)}{I}.$$

This convergence is uniform for all  $j$ .

(2) One has

$$\begin{aligned}
\frac{\sum_{j=0}^{2m} jI\lambda \dim R_m^j}{m^{n+2}/n!} &= \sum_{j=0}^{2m} \frac{jI\lambda}{m} \cdot \frac{h^0(X, m(jI\lambda/m + I(1-\lambda))H)}{m^n/n!} \cdot \frac{1}{m} \\
&\xrightarrow{m \rightarrow \infty} \frac{1}{I\lambda} \int_0^{2I\lambda} x \text{Vol}((x + I(1-\lambda))H) dx \\
&= \frac{H^n I^{n+1}}{\lambda} \int_0^{2\lambda} t(t + (1-\lambda))^n dt \\
&= \frac{H^n I^{n+1}}{(n+1)\lambda} \left( 2\lambda(1+\lambda)^{n+1} - \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{n+2} \right).
\end{aligned}$$

(3) One has

$$\begin{aligned}
\frac{\sum_{j=0}^{2m} \dim R_m^j}{m^{n+1}/n!} &= \sum_{j=0}^{2m} \frac{h^0(X, m(jI\lambda/m + I(1-\lambda))H)}{m^n/n!} \cdot \frac{1}{m} \\
&\xrightarrow{m \rightarrow \infty} \frac{1}{I\lambda} \int_0^{2I\lambda} \text{Vol}((x + I(1-\lambda))H) dx \\
&= \frac{H^n I^n}{\lambda} \int_0^{2\lambda} (t + (1-\lambda))^n dt \\
&= \frac{H^n I^n}{(n+1)\lambda} \left( (1+\lambda)^{n+1} - (1-\lambda)^{n+1} \right).
\end{aligned}$$

Putting all these together, for  $m \gg 1$ ,

$$\text{ord}_{\overline{F}}(\mathcal{D}_m) = \frac{n+1}{n+2} \cdot \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}} \cdot S_{-K_X}(F) + \epsilon_m,$$

where  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ . So the assertion follows.  $\square$

*Proof of Proposition 3.3.* By (3.2), (3.9) and Lemma 3.5, we have

$$\begin{aligned}
\delta_Y(\overline{F}) &= \frac{A_Y(\overline{F})}{S_{-K_Y}(\overline{F})} \\
&= \lim_{m \rightarrow \infty} \frac{A_Y(\overline{F})}{\text{ord}_{\overline{F}}(\mathcal{D}_m)} \\
&= \frac{A_X(F)\beta_0}{(\lambda + \beta_0(1-\lambda))S_{-K_X}(F)} \\
&= \frac{\delta_X(F)\beta_0}{(\lambda + \beta_0(1-\lambda))}.
\end{aligned}$$

$\square$

So Theorem 1.1 is proved. Proposition 3.3 also implies that, in the case when

$$\delta(Y) = \frac{\delta(X)\beta_0}{\lambda + \beta_0(1-\lambda)},$$

$\delta(Y)$  is computed by some  $\overline{F}$ , where  $F$  is a divisor over  $X$  that computes  $\delta(X)$  (cf. [4, Theorem 6.7]).



## 4. MORE DISCUSSIONS

The purpose of this section is to include some properties of the projective bundle  $Y$ , which can be used to explicitly calculate  $\delta_Y(\bar{F})$  in some special cases. Let  $F \subseteq X$  be a prime divisor. We define the nef threshold of  $F$  to be

$$(4.1) \quad \epsilon_X(F) := \sup\{t > 0 \mid -K_X - tF \text{ is nef}\}.$$

The pseudo-effective threshold of  $F$  is defined as

$$(4.2) \quad \tau_X(F) := \sup\{t > 0 \mid -K_X - tF \text{ is big}\}.$$

Put

$$\bar{F} := \pi^*F.$$

One can define  $\epsilon_Y(\bar{F})$  and  $\tau_Y(\bar{F})$  analogously on  $Y$  as well.

**Lemma 4.1.** *One has*

$$\epsilon_Y(\bar{F}) = (1 - \lambda)\epsilon_X(F).$$

*Proof.* We write

$$-K_Y - t\bar{F} \sim_{\mathbb{R}} \pi^*(-(1 - \lambda)K_X - tF) + 2E_{\infty}.$$

Let  $C \not\subseteq E_0$  be any curve, then for any  $t \in (0, (1 - \lambda)\epsilon_X(F)]$ , one clearly has

$$(-K_Y - t\bar{F}) \cdot C \geq 0.$$

Now consider  $C \subseteq E_0$ . Then by projection formula,

$$(-K_Y - t\bar{F}) \cdot C = (-(1 - \lambda)K_X - tF) \cdot \pi_*C.$$

Thus  $-K_Y - t\bar{F}$  is nef if and only if

$$t \in (0, (1 - \lambda)\epsilon_X(F)].$$

□

**Lemma 4.2.** *For any  $\mathbb{R}$ -divisor  $D \subseteq X$ , we have*

$$\text{Vol}(\pi^*D) = 0.$$

*Proof.* If not, then  $\pi^*D$  is big so there exists an ample  $\mathbb{R}$ -divisor  $A$  and an effective  $\mathbb{R}$ -divisor  $B$  on  $Y$  such that  $\pi^*D \sim_{\mathbb{R}} A + B$ . Then for any generic  $\mathbb{P}^1$ -fiber  $f \subseteq Y$ , one has  $0 = \pi^*D \cdot f = (A + B) \cdot f > 0$ , which is a contradiction. □

**Lemma 4.3.** *Let  $D \subseteq X$  be an  $\mathbb{R}$ -divisor that is not big. Then*

$$\text{Vol}(\pi^*D + aE_0) = 0 \text{ for any } a \geq 0.$$

*Proof.* We make use of the restricted volume. Thinking of  $E_0$  as a copy of  $X$  sitting inside  $Y$ , then for any  $a \geq 0$ , one has

$$(\pi^*D + aE_0)|_{E_0} = D - aL,$$

which is thus not big. Let

$$b := \sup\{a \geq 0 \mid \text{Vol}(\pi^*D + aE_0) = 0\}.$$

So it amounts to showing that  $b = +\infty$ . Assume to the contrary that  $b < +\infty$ . Put

$$f(t) := \text{Vol}(\pi^*D + bE_0 + tE_0), \quad t \in [0, \infty).$$

By the previous lemma,  $f(0) = 0$ . And  $f(t)$  is a non-decreasing positive  $C^1$  function when  $t \in (0, \infty)$  by [2, Theorem A]. Moreover, for any  $t > 0$ , one has

$$\frac{d}{dt}f(t) = n\text{Vol}_{Y|E_0}(\pi^*D + (b+t)E_0) \leq n\text{Vol}(X, D - (b+t)L) = 0.$$

This implies that  $f(t) = f(0) = 0$  for any  $t > 0$ , a contradiction.  $\square$

**Lemma 4.4.** *One has*

$$\tau_Y(\overline{F}) = (1 + \lambda)\tau_X(F).$$

*Proof.* We write

$$-K_Y - t\overline{F} \sim_{\mathbb{R}} \pi^*(-(1 + \lambda)K_X - tF) + 2E_0.$$

Thus  $-K_Y - t\overline{F}$  is linearly equivalent to a pseudo-effective  $\mathbb{R}$ -divisors for  $t \in [0, (1 + \lambda)\tau_X(F)]$ . Moreover, for any  $t \geq (1 + \lambda)\tau_X(F)$ ,  $-(1 + \lambda)K_X - tF$  is not big, so  $\text{Vol}(-K_Y - t\overline{F}) = 0$  by the previous lemma. The assertion follows.  $\square$

By slightly modifying the argument of Lemma 4.3, the following is clear.

**Lemma 4.5.** *Assume that  $B \subseteq Y$  is an  $\mathbb{R}$ -divisor that is not big when restricted to  $E_0$ . Then*

$$\text{Vol}(B + aE_0) = \text{Vol}(B) \text{ for any } a \geq 0.$$

The next result is of course covered by Proposition 3.3, but we shall give an alternative computational proof, which will be helpful in Section 5.

**Proposition 4.6.** *Assume that  $F \subseteq X$  is a prime divisor with  $\epsilon_X(F) = \tau_X(F)$ , then one has*

$$\delta_Y(\overline{F}) = \frac{\delta_X(F)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

*Proof.* We write

$$\epsilon := \epsilon_X(F)$$

to ease notation. For  $t \in [0, (1 - \lambda)\epsilon]$ , we have

$$\begin{aligned} \text{Vol}(-K_Y - \tau\overline{F}) &= \left( (1/\lambda + 1)E_\infty - (1/\lambda - 1)E_0 - t\overline{F} \right)^{n+1} \\ &= \sum_{i=0}^n C_{n+1}^i (-t)^i \left( (1/\lambda + 1)^{n+1-i} - (1/\lambda - 1)^{n+1-i} \right) L^{n-i} \cdot F^i. \end{aligned}$$

For  $t \in [(1 - \lambda)\epsilon, (1 + \lambda)\epsilon]$ , applying Lemma 4.5, we have

$$\text{Vol}(-K_Y - t\overline{F}) = \text{Vol}\left(-K_Y - t\overline{F} - \left(\frac{t}{\epsilon\lambda} - (1/\lambda - 1)\right)E_0\right).$$

Note that

$$-K_Y - t\overline{F} - \left(\frac{t}{\epsilon\lambda} - (1/\lambda - 1)\right)E_0 \sim_{\mathbb{R}} \pi^*(-(1 + \lambda)K_X - tF) + \left(1/\lambda + 1 - \frac{t}{\epsilon\lambda}\right)E_0$$

is clearly nef for  $t \in [(1-\lambda)\epsilon, (1+\lambda)\epsilon]$  (it suffices to check curves contained in  $E_0$ ), so we get that

$$\begin{aligned} \text{Vol}(-K_Y - t\bar{F}) &= \left( -K_Y - t\bar{F} - \left( \frac{t}{\epsilon\lambda} - (1/\lambda - 1) \right) E_0 \right)^{n+1} \\ &= \left( (1/\lambda + 1)E_\infty - \frac{t}{\epsilon\lambda}E_0 - t\bar{F} \right)^{n+1} \\ &= \sum_{i=0}^n C_{n+1}^i (-t)^i \left( (1/\lambda + 1)^{n+1-i} - \left( \frac{t}{\epsilon\lambda} \right)^{n+1-i} \right) L^{n-i} \cdot F^i. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty \text{Vol}(-K_Y - t\bar{F}) &= \sum_{i=0}^n \int_0^{(1-\lambda)\epsilon} C_{n+1}^i (-t)^i \left( (1/\lambda + 1)^{n+1-i} - (1/\lambda - 1)^{n+1-i} \right) L^{n-i} \cdot F^i dt \\ &\quad + \sum_{i=0}^n \int_{(1-\lambda)\epsilon}^{(1+\lambda)\epsilon} C_{n+1}^i (-t)^i \left( (1/\lambda + 1)^{n+1-i} - \left( \frac{t}{\epsilon\lambda} \right)^{n+1-i} \right) L^{n-i} \cdot F^i dt \\ &= \sum_{i=0}^n C_{n+1}^i \frac{(-1)^i \epsilon^{i+1} ((1+\lambda)^{n+2} - (1-\lambda)^{n+2})}{(i+1)\lambda^{n+1-i}} L^{n-i} \cdot F^i \\ &\quad - \sum_{i=0}^n C_{n+1}^i \frac{(-1)^i \epsilon^{i+1} ((1+\lambda)^{n+2} - (1-\lambda)^{n+2})}{(n+2)\lambda^{n+1-i}} L^{n-i} \cdot F^i. \end{aligned}$$

Thus

$$\begin{aligned} S_{-K_Y}(\bar{F}) &= \frac{1}{(-K_Y)^{n+1}} \int_0^\infty \text{Vol}(-K_Y - t\bar{F}) dt \\ &= \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}} \sum_{i=0}^n C_{n+1}^i (-\lambda)^i \epsilon^{i+1} \frac{L^{n-i} \cdot L^i}{(i+1)L^n} \frac{n+1-i}{n+2} \\ &= \frac{n+1}{n+2} \cdot \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}} \sum_{i=0}^n C_n^i \frac{(-\lambda)^i \epsilon^{i+1} L^{n-i} \cdot F^i}{(i+1)L^n}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} S_{-K_X}(F) &= \frac{1}{(-K_X)^n} \int_0^\epsilon \text{Vol}(-K_X - tF) \\ &= \frac{1}{L^n} \sum_{i=0}^n \int_0^\epsilon C_n^i (-\lambda t)^i L^{n-i} \cdot F^i dt \\ &= \sum_{i=0}^n C_n^i \frac{(-\lambda)^i \epsilon^{i+1} L^{n-i} \cdot F^i}{(i+1)L^n}. \end{aligned}$$

Thus we arrive at

$$\begin{aligned} S_{-K_Y}(\bar{F}) &= \frac{n+1}{n+2} \cdot \frac{(1+\lambda)^{n+2} - (1-\lambda)^{n+2}}{(1+\lambda)^{n+1} - (1-\lambda)^{n+1}} \cdot S_{-K_X}(F) \\ &= \lambda(1/\beta_0 + (1/\lambda - 1)) S_{-K_X}(F), \end{aligned}$$

so that

$$\delta_Y(\bar{F}) = \frac{\delta_X(F)\beta_0}{\lambda + \beta_0(1-\lambda)}.$$

□

## 5. EXAMPLE

In this section we give an example such that

$$\delta(Y) = \frac{\delta(X)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

To search for such examples, we need to work in high dimensions. In the literature, explicit calculations for  $\int_0^\infty \text{Vol}(L - tF)dt$  have been carried out many times in dimension 2 and 3 (see e.g., [5, ?, ?]). Note that in these cases, the computation is relatively simple, mainly due to the fact that there is no small contraction in dimension 2 or 3, and one only needs to get rid of those divisors that is contained in the non-nef locus of  $L - tF$ . However, in higher dimensions, the non-nef locus could have large codimension, which makes the computation more subtle. In fact, as shown in [?, Section 8], one needs to run certain MMP to do the computation. In this section we take the opportunity to illustrate how this can be done in dimension 4.

Let  $X = Bl_1\mathbb{P}^3$ . Note that  $X$  itself is a  $\mathbb{P}^1$ -bundle. Let  $F_0$  be the exceptional divisor and  $F_\infty$  be the pull back of a general hyperplane in  $\mathbb{P}^3$ . Then  $-K_X = 4F_\infty - 2F_0$ . Simple calculation shows that  $\epsilon_X(F_0) = \tau_X(F_0) = 2$ , and by Corollary 3.2, we have

$$\delta(X) = \delta_X(F_0) = \frac{14}{17}.$$

We take  $L = 2F_\infty - F_0$  and  $Y = \mathbb{P}(L^{-1} \oplus \mathcal{O}_X)$ , with  $E_0$  and  $E_\infty$  being the zero and infinity sections respectively. Then we have  $n = 3$ ,  $\lambda = 1/2$ , so that  $\beta_0 = 50/71$ . Therefore

$$\frac{\delta_X(F_0)\beta_0}{\lambda + \beta_0(1 - \lambda)} = \frac{1400}{2057}.$$

So by Theorem 1.1,

$$\delta(Y) = \min \left\{ \frac{1400}{2057}, \frac{50}{71} \right\} = \frac{1400}{2057}.$$

Put  $\overline{F_0} := \pi^*F_0$ . Let us explicitly verify that  $\overline{F_0}$  computes  $\delta(Y)$ . Indeed,  $\epsilon_Y(\overline{F_0}) = 1$  and  $\tau_Y(\overline{F_0}) = 3$ . And we have (by the proof of Proposition 4.6)

$$\text{Vol}(-K_Y - t\overline{F_0}) = \begin{cases} (3E_\infty - E_0 - t\overline{F_0})^4 = 560 - 104t - 48t^2 - 8t^3, & t \in [0, 1], \\ (3E_\infty - tE_0 - t\overline{F_0})^4 = 567 - 108t - 54t^2 - 12t^3 + 7t^4, & t \in [1, 3]. \end{cases}$$

From this we obtain that

$$\begin{aligned} S_{-K_Y}(\overline{F_0}) &= \frac{1}{560} \int_0^1 (560 - 104t - 48t^2 - 8t^3)dt \\ &\quad + \frac{1}{560} \int_1^3 (567 - 108t - 54t^2 - 12t^3 + 7t^4)dt \\ &= \frac{2057}{1400}. \end{aligned}$$

Therefore

$$\delta_Y(\overline{F_0}) = \frac{1400}{2057}.$$

So we do have the equality:

$$\delta(Y) = \delta_Y(\overline{F_0}) = \frac{\delta_X(F)\beta_0}{\lambda + \beta_0(1 - \lambda)} = \frac{1400}{2057}.$$

Now choose a prime divisor  $H \in |F_\infty - F_0|$ . Then we have  $\epsilon_X(H) = 2$  and  $\tau_X(H) = 4$ . Moreover  $\delta_X(H) = 14/15$  and

$$\frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1 - \lambda)} = \frac{280}{363}.$$

Consider  $\overline{H} := \pi^*H$ . Then  $\epsilon_Y(\overline{H}) = 1$  and  $\tau_Y(\overline{H}) = 6$ . In the following we verify that

$$\delta_Y(\overline{H}) = \frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1 - \lambda)}.$$

Of course this holds true by Proposition 3.3, but we would like to prove this by directly computing the integrand  $\text{Vol}(-K_Y - t\overline{H})$  for  $t \in [0, 6]$ , which requires some interesting tools that might be useful in other context.

- For  $t \in [0, 1]$ , as  $-K_Y - t\overline{H}$  is nef, we have

$$\begin{aligned} \text{Vol}(-K_Y - t\overline{H}) &= (3E_\infty - E_0 - t\overline{H})^4 \\ &= 560 - 312t + 48t^2. \end{aligned}$$

- For  $t \in [1, 2]$ , we write

$$-K_Y - t\overline{H} \sim_{\mathbb{R}} (6 - t)\overline{F_\infty} - (3 - t)\overline{F_0} + 2E_0,$$

Its non-nef locus is  $S := E_0 \cap \overline{F_0}$ , which is a copy of  $\mathbb{P}^2$  sitting inside  $Y$  and whose normal bundle is isomorphic to  $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ . The numerical class of curves in  $S$  generates an extremal ray in  $\overline{NE}(Y)$ . Let  $Y \xrightarrow{\alpha} Z$  be the contraction of this ray and let  $Y^+ \xrightarrow{\alpha^+} Z$  be the flip of  $\alpha$ . Then by [?, Section 8],  $Y^+$  is the ample model of  $-K_Y - t\overline{H}$  for  $t \in (1, 2)$ .

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y^+ \\ & \searrow \alpha & \swarrow \alpha^+ \\ & Z & \end{array}$$

Note that  $Y^+$  can be explicitly constructed as follows (cf. [8]): blow up the non-nef locus  $S$ , then we will get an exceptional divisor that is isomorphic to  $\mathbb{P}^2 \times \mathbb{P}^1$ , whose normal bundle is isomorphic to  $\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$ ; contracting this divisor in the other direction, we get  $Y^+$ , which is a smooth projective 4-fold. For any effective divisor  $D$  on  $Y$ , let  $D^+$  denote its strict transform on  $Y^+$ . Then for  $t \in [1, 2]$ , straightforward computation gives

$$\begin{aligned} \text{Vol}(-K_Y - t\overline{H}) &= (-K_{Y^+} - t\overline{H}^+)^4 \\ &= 559 - 308t + 42t^2 + 4t^3 - t^4. \end{aligned}$$

- Let  $t \in [2, 3]$ . Thinking of  $E_0$  as a copy of  $Bl_1\mathbb{P}^3$ , then for any point  $p \in E_0$ , there exists a curve  $C \subseteq E_0$  pass through  $p$  such that

$$(-K_Y - t\overline{H}) \cdot C = (- (t - 2)F_\infty + (t - 1)F_0) \cdot \pi_*C < 0.$$

So  $E_0$  is contained in the non-nef locus of  $-K_Y - t\overline{H}$ . Subtracting certain amount of  $E_0$ , we derive that (one can also directly apply Lemma 4.5 here), for  $t \geq 2$ ,

$$\text{Vol}(-K_Y - t\overline{H}) = \text{Vol}(-K_Y - t\overline{H} - (t/2 - 1)E_0).$$

Note that

$$-K_Y - t\overline{H} - (t/2 - 1)E_0 \sim_{\mathbb{R}} (6 - t)\overline{F_\infty} - (3 - t)\overline{F_0} + (3 - t/2)E_0,$$

whose non-nef locus is again  $S = E_0 \cap \overline{F_0}$ . Thus for  $t \in [2, 3]$  we have

$$\begin{aligned} \text{Vol}(-K_Y - t\overline{H}) &= \text{Vol}(-K_Y - t\overline{H} - (t/2 - 1)E_0) \\ &= (-K_{Y^+} - t\overline{H}^+ - (t/2 - 1)E_0^+)^4 \\ &= 567 - 324t + 54t^2 - t^4/2. \end{aligned}$$

- For  $t \in [3, 6]$ , write

$$-K_Y - t\overline{H} - (t/2 - 1)E_0 \sim_{\mathbb{R}} (6 - t)\overline{F_\infty} + (t - 3)\overline{F_0} + (3 - t/2)E_0.$$

Thinking of  $\overline{F_0}$  as a copy of  $Bl_1\mathbb{P}^3$ , for any point  $p \in \overline{F_0}$ , we can find a curve  $C \subseteq \overline{F_0}$  passing through  $p$  with

$$(-K_Y - t\overline{H} - (t/2 - 1)E_0) \cdot C < 0.$$

Thus  $\overline{F_0}$  is contained in the non-nef locus. Subtracting it, we obtain, for  $t \geq 3$ , that

$$\begin{aligned} \text{Vol}(-K_Y - t\overline{H}) &= \text{Vol}(-K_Y - t\overline{H} - (t/2 - 1)E_0 - (t - 3)\overline{F_0}) \\ &= \text{Vol}((6 - t)\overline{F_\infty} + (3 - t/2)E_0) \\ &= \frac{(6 - t)^4}{2^4} \text{Vol}(2\overline{F_\infty} - E_0) \\ &= \frac{(6 - t)^4}{81} \text{Vol}(3\overline{F_\infty} - 1.5E_0) \\ &= \frac{(6 - t)^4}{81} \text{Vol}(-K_Y - 3\overline{H}) \\ &= \frac{(6 - t)^4}{2}. \end{aligned}$$

In conclusion, we have <sup>2</sup>

$$\text{Vol}(-K_Y - t\overline{H}) = \begin{cases} 560 - 312t + 48t^2, & t \in [0, 1]; \\ 559 - 308t + 42t^2 + 4t^3 - t^4, & t \in [1, 2]; \\ 567 - 324t + 54t^2 - t^4/2, & t \in [2, 3]; \\ (6 - t)^4/2, & t \in [3, 6]. \end{cases}$$

Integrating over  $[0, 6]$ , we obtain that

$$S_{-K_Y}(\overline{H}) = \frac{1}{(-K_Y)^4} \int_0^6 (\text{Vol}(-K_Y - t\overline{H})) dt = \frac{363}{280}.$$

So we have verified that

$$\delta_Y(\overline{H}) = \frac{\delta_X(H)\beta_0}{\lambda + \beta_0(1 - \lambda)}$$

<sup>2</sup>It is interesting to notice that  $\text{Vol}(-K_Y - t\overline{H})$  is  $C^3$ -differentiable (but not  $C^4$ ) for  $t \in (0, 6)$ .

even when  $\epsilon_X(H) \neq \tau_X(H)$ .

The above calculation suggests that, it is impractical to prove Proposition 3.3 by a direct computation using MMP.

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