STABILITY THRESHOLDS ON RATIONAL FANO T-VARIETIES OF COMPLEXITY ONE

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Abstract. We compute $\alpha(X)$ and $\delta(X)$ on rational Fano T-varieties of complexity one.

1. Introduction

2. Preliminaries

We collect some concepts and results of T-varieties that will be used in this note. We refer the reader to [4] for more details. We will restrict ourselves to the case of rational complexity one T-varieties.

Let N be the lattice with dual M and let $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ be the associated $\mathbb{Q}$ vector spaces.

Definition 2.1. *A polehedral divisor $\mathcal{D}$ on $\mathbb{P}^1$ with $\operatorname{tail}(\mathcal{D})=\sigma$ is a formal sum*

$$\mathcal{D}:=\sum_{P\in\mathbb{P}^1}\mathcal{D}_P\otimes P,$$

where each $\mathcal{D}_P$ is a polyhedral in $N_{\mathbb{Q}}$ with tail cone σ. Moreover, only finitely many $\mathcal{D}_P$ differ from σ.

Definition 2.2. *$\mathcal{D}$ is called a p-divisor if*

$$\deg\mathcal{D}:=\sum_{P}\mathcal{D}_P\subsetneq\sigma.$$

Note that, we allow the coefficients of $\mathcal{D}$ to be empty set and we use the convention $\mathcal{D}_P+\emptyset=\emptyset$.

Definition 2.3. *Given a polehedral divisor $\mathcal{D}$ on $\mathbb{P}^1$ with $\operatorname{tail}(\mathcal{D})=\sigma$, the locus of $\mathcal{D}$ is defined by*

$$\operatorname{Loc}\mathcal{D}:=\mathbb{P}^1\backslash\Big(\bigcup_{\mathcal{D}_P=\emptyset}P\Big).$$

For any $u\in\sigma^\vee\cap M$, we put

$$\mathcal{D}(u):=\sum_{P}\min_{v\in\mathcal{D}_P}\langle u,v\rangle P.$$

Note that, this is a $\mathbb{Q}$-divisor on $\operatorname{Loc}\mathcal{D}$.

Given a p-divisor $\mathcal{D}$, we define a M-graded sheaf of rings

$$\mathcal{A}(\mathcal{D}) := \bigoplus_{u\in\sigma^\vee\cap M} \mathcal{O}_{\operatorname{Loc}\mathcal{D}}(\mathcal{D}(u))\chi^u.$$

We then define two T-varieties

$$\widetilde{X}(\mathcal{D}) := \mathbf{Spec}_{\operatorname{Loc}\mathcal{D}}\,\mathcal{A}(\mathcal{D}) \text{ and } X(\mathcal{D}) := \operatorname{Spec}\Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{A}(\mathcal{D})).$$

Note that $T = \operatorname{Spec}\mathbb{C}[M]$ and we have the following diagram

$$\begin{tikzcd} & \widetilde{X}(\mathcal{D}) \arrow[dl, "r"'] \arrow[dr, "\pi"] & \\ X(\mathcal{D}) \arrow[rr, dashed, "\psi=\pi\circ r^{-1}"'] & & \operatorname{Loc}\mathcal{D} \end{tikzcd}$$

Here r is a proper birational T-equivariant contraction morphism and π is a good quotient map.

An important fact for us is that, $\widetilde{X}(\mathcal{D})$ is *toroidal*, meaning that $\widetilde{X}(D)$ is locally biholomorphic to a toric variety. In particular, when we are working locally on $\widetilde{X}(\mathcal{D})$, we can apply standard toric tools. See [11, 15] for related discussions.

In this note, we will deal with T-invariant divisors, so let us recall some results from [13]. There are two types of prime T divisors on $\widetilde{X}(\mathcal{D})$: *vertical divisors* and *horizontal divisors*. Vertical divisors are in bijection with the pair (P, v), where $P \in \mathbb{P}^1$ and v is a vertex of $\mathcal{D}_P$; we denote the corresponding divisor by $D_{P,v}$. Likewise, horizontal divisors are in bijection with rays ρ of σ; we denote the corresponding divisor by D_ρ. Under the contraction morphism $r : \widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$, all the vertical divisors will survive, while some horizontal divisors will be contracted. Those D_ρ that are not contratced can be characterized by *extremal rays*: rays ρ of σ such that $\rho \cap \deg\mathcal{D} = \emptyset$. The prime T divisors on $X(\mathcal{D})$ are exactly those that are not contracted by r.

The T-variety $X(\mathcal{D})$ defined above is affine. To construct complete T-varieties, we need the following.

Definition 2.4. *A finite set $\mathcal{S} = \{\mathcal{D}^i\}$ of p-divisors on $\mathbb{P}^1$ is called a divisorial fan if the intersection of two p-divisors from $\mathcal{S}$ is a common face of both and $\mathcal{S}$ is closed under taking intersections. For any point $P \in \mathbb{P}^1$, the set $\mathcal{S}_P := \{\mathcal{D}^i_P\}$ is called a slice. $\mathcal{S}$ is called complete if each slice $\mathcal{S}_P$ is a complete polyhedral subdivision of $N_\mathbb{Q}$.*

By [3], given a complete divisorial fan $\mathcal{S}$, one can glue $X(\mathcal{D}^i)$ together to obtain a complete T variety $X(\mathcal{S})$. Any Cartier divisor on $X(\mathcal{S})$ is linearly equivalent to a Cartier T-divisor. Note that Cartier T-divisors on $X(\mathcal{S})$ can be described by support functions in $\operatorname{CaSF}(\mathcal{S})$ (see [13] for its definition).

3. Log canonical threshold of T-divisors

Let X be a rational Fano T-variety of complexity one with log terminal singularities. Note that X can be characterized by a complete divisorial fan $\mathcal{S}$. Let us fix a canonical divisor $K_{\mathbb{P}^1} = \sum a_P \cdot P$ on $\mathbb{P}^1$. Then the canonical divisor K_X can be represented by (see [13, theorem 3.21])

$$K_X = \sum_{P,v} (\mu(v)a_P + \mu(v) - 1)D_{P,v} - \sum_{\text{extremal } \rho} D_\rho.$$

Here v is any vertex in $\mathcal{S}_P$ and $\mu(v) \in \mathbb{N}_+$ is the smallest integer such that $\mu(v)v \in N$. Note that $D_{P,v}$ is the vertical T-divisor associated to (P, v) and D_ρ is the horizontal T-divisor associated to the ray $\rho \in \mathrm{tail}(\mathcal{S})$.

Then we can find a support function $h \in \mathrm{CaSF}(\mathcal{S})$ corresponding to $-K_X$ such that

$$-K_X = -\sum_{P,v} \mu(v)h_p(v)D_{P,v} - \sum_{\text{extremal } \rho} h_t(n_\rho)D_\rho \tag{3.1}$$

where h_t be the linear part of h and n_ρ are the primitive vectors of ρ (cf. [13, Corollary 3.19]). We can then construct a polytope

$$\Box_h := \bigcap_\rho \{\langle u, n_\rho\rangle \geq h_t(n_\rho)\},$$

where ρ run through all the rays in $\mathrm{tail}(\mathcal{S})$.

Moreover, we define

$$\Psi_P(u) := h_P^*(u) := \min_{v \in \mathcal{S}_P^{(0)}} \{\langle u, v\rangle - h_P(v)\},\ u \in \Box_h,$$

where $\mathcal{S}_P^{(0)}$ denotes the set of vertices in the slice $\mathcal{S}_P$. We put

$$\Psi := \sum_{P \in \mathbb{P}^1} \Psi_P \otimes P.$$

Then the pair $(\Box_h, \Psi)$ is a divisorial polytope associated to the polarized pair $(X, -K_X)$ (cf. [9, Section 3]). For $k \in \mathbb{N}_+$, we have

$$H^0(X, -kK_X) = \bigoplus_{u \in k\Box_h \cap M} H^0(\mathbb{P}^1, \mathcal{O}(\lfloor k\Psi(u/k)\rfloor)). \tag{3.2}$$

For any effective T-invatiant $\mathbb{Q}$-Cartier divisor $D \sim_{\mathbb{Q}} -K_X$, we can explicitly calculate $\mathrm{lct}(X, D)$ using the combinatoric description of X. More specifically, suppose that for some $k \in \mathbb{N}_+$, kD is Cartier. Then we can find $f \in K(\mathbb{P}^1)$ and $u \in k\Box_h \cap M$ such that

$$kD = \mathrm{div}(f\chi^u) - \sum_{P,v} k\mu(v)h_P(v)D_{P,v} - \sum_{\text{extremal } \rho} kh_t(n_\rho)D_\rho.$$

Lemma 3.3. *We have*

$$\mathrm{lct}(X, D) = \min\left\{\frac{-h_t(n_\rho)}{\langle u/k, n_\rho\rangle - h_t(n_\rho)}\right\}_\rho \cup \left\{\frac{1}{\mu(v)(\mathrm{ord}_P f/k + \langle u/k, v\rangle - h_P(v))}\right\}_{P,v}.$$

Proof. Let us pick $\mathcal{D} \in \mathcal{S}$ and work on $\widetilde{X}(\mathcal{D})$. We have

$$\begin{cases} K_{\widetilde{X}} = \sum_{P,v} \big(\mu(v)a_P + \mu(v) - 1\big) D_{P,v} - \sum_\rho D_\rho \\ r^*K_X = \sum_{P,v} \mu(v)h_P(v)D_{P,v} + \sum_\rho h_t(n_\rho)D_\rho \\ r^*D = \sum_{P,v} \mu(v)\big(\mathrm{ord}_P f/k + \langle u/k, v\rangle - h_p(v)\big) D_{P.v} + \sum_\rho \big(\langle u/k, n_\rho\rangle - h_t(n_\rho)\big) D_\rho \end{cases}$$

By our choice of h, we have

$$\mu(v)h_P(v) = \mu(v)a_P + \mu(v) - 1.$$

Thus we obtain

$$A_X(D_{P,v}) = 1 \text{ and } A_X(D_\rho) = -h_t(n_\rho). \tag{3.4}$$

Here $A_X(\cdot)$ denotes the discrepancy of any prime divisor over X. Note that $-h_t(n_\rho) > 0$ as X is log terminal.

For $\lambda > 0$, we have

$$r^*(K_X+\lambda D)-K_{\widetilde{X}} = \sum_{P,v} \lambda\mu(v)\big(\mathrm{ord}_P f/k+\langle u/k, v\rangle - h_P(v)\big)D_{P,v} + \sum_\rho \big(1+\lambda\langle u, n_\rho\rangle + (1-\lambda)h_t(n_\rho)\big)D_\rho.$$

Now we use [15, Lemma 4.2]. Since $\widetilde{X} \xrightarrow{r} X$ is a toroidal desingularization, the pair $(X, \lambda D)$ is log canonical if and only if

$$\begin{cases} \lambda\mu(v)(\mathrm{ord}_P f/k + \langle u/k, v\rangle - h_P(v)) \le 1 \\ 1 + \lambda\langle u, n_\rho\rangle + (1-\lambda)h_t(n_\rho) \le 1 \end{cases}$$

Or equiavalently,

$$\begin{cases} \lambda \le \frac{A_X(D_{P,v})}{\mathrm{ord}_{D_{P,v}}(D)} = \frac{1}{\mu(v)(\mathrm{ord}_P f/k + \langle u/k, v\rangle - h_p(v))} \\ \lambda \le \frac{A_X(D_\rho)}{\mathrm{ord}_{D_\rho}(D)} = \frac{-h_t(n_\rho)}{\langle u, n_\rho\rangle - h_t(n_\rho)} \end{cases}$$

Thus we obtain the desired formula for $\mathrm{lct}(X, D)$. ∎

As a consequence, we get the following formula for Tian's α-invariant.

Theorem 3.5. *We have*

$$\alpha(X) = \alpha_T(X) = \min_{u \in \Box_h} \left\{ \frac{-h_t(n_\rho)}{\langle u, n_\rho\rangle - h_t(n_\rho)} \right\}_\rho \cup \left\{ \frac{1}{\mu(v)(\sum_{Q\neq P} \Psi_Q(u) + \langle u, v\rangle - h_p(v))} \right\}_{P,v}.$$

Proof. It is well known that (cf. [8, 14])

$$\alpha(X) = \lim_k \alpha_k(X).$$

We will give an formula for $\alpha_k(X)$ and then let $k \to \infty$.

As shown in [15, Propostion 2.6], to compute $\alpha_k(X)$, it is enough to look at T-divisors. So it suffices to consider $f\chi^u \in H^0(X, -kK_X)$, where $f \in K(\mathbb{P}^1)$ and $u \in k\Box_h \cap M$. Note that (cf. (3.2))

$$f \in H^0(\mathbb{P}^1, \mathcal{O}(\lfloor k\Psi(u/k)\rfloor)).$$

This immediately implies that

$$-\lfloor k\Psi_P(u/k)\rfloor \le \mathrm{ord}_P f \le \sum_{Q\neq P}\lfloor k\Psi_Q(u/k)\rfloor. \tag{3.6}$$

Combining this with Lemma 3.3, we obtain

$$\alpha_k(X)=\min_{u\in k\Box_h\cap M}\left\{\frac{-h_t(n_\rho)}{\langle u/k,n_\rho\rangle-h_t(n_\rho)}\right\}_\rho\cup\left\{\frac{1}{\mu(v)(\sum_{Q\neq P}\lfloor k\Psi_Q(u/k)\rfloor/k+\langle u/k,v\rangle-h_p(v))}\right\}_{P,v}.$$

Letting $k\to\infty$, we complete the proof. $\square$

4. Delta invariant

We use the same notation as in the previous section. The purpose of this part is to give an explicit formula for the stability threshold $\delta(X)$. Let $d\mu$ denote the Lebesgue measure on $\Box_h$. We put

$$\begin{cases}\mathrm{vol}(\Psi):=\int_{\Box_h}\deg\Psi d\mu,\\ \mathrm{bc}(\Psi):=\frac{1}{\mathrm{vol}(\Psi)}\int_{\Box_h}u\cdot\deg\Psi d\mu,\\ \mathrm{bc}_P:=\frac{1}{\mathrm{vol}(\Psi)}\int_{\Box_h}(\frac{\deg\Psi}{2}-\Psi_P)\deg\Psi d\mu.\end{cases}$$

Here $\deg\Psi=\sum_P\Psi_p$. By straightforward computation, $(\mathrm{bc}(\Psi),\mathrm{bc}_P)$ is exactly the barycenter of the following polytope

$$\Delta_P:=\left\{(u,a)\in M\times\mathbb{Q}\;\middle|\;u\in\Box_h,\ -\Psi_P(u)\le a\le\sum_{Q\neq P}\Psi_Q(u)\right\}. \tag{4.1}$$

Note that, this polytope also appears in [10, 7].

Theorem 4.2. *We have*

$$\delta(X)=\min\left\{\frac{-h_t(n_\rho)}{\langle\mathrm{bc}(\Psi),n_\rho\rangle-h_t(n_\rho)}\right\}_\rho\cup\left\{\frac{1}{\mu(v)(\mathrm{bc}_P+\langle\mathrm{bc}(\Psi),v\rangle-h_p(v))}\right\}_{P,v}.$$

To prove this, we use toroidal desingularization and then apply the argument in [6, §7]. So in the following, we will work locally.

To describe the toroidal desingularization more explicitly, let us pick $P\in\mathbb{P}^1$ and $\mathcal{D}_P\in\mathcal{S}_P$. We put $N'=N\times\mathbb{Z}$, $M'=(N')^\vee$ and consider the Cayley cone

$$\sigma_P:=\mathrm{cone}\big(\mathcal{D}_P\times\{1\}\big)\subset N'_{\mathbb{Q}}.$$

Then $\widetilde{X}_P:=\mathrm{Spec}\,\mathbb{C}[\sigma_P^\vee\cap M']$ gives us a local toric model. The vertex $v\in\mathcal{D}_P$ turns into the ray of σ_P generated by the primitive vector $\big(\mu(v)v,\mu(v)\big)$ and the ray ρ of $\mathrm{tail}(\mathcal{D}_P)$ turns into the ray $(\rho,0)$ of σ_P. Meanwhile, $f\chi^u\in K(\mathbb{P}^1)[M]$ translates into the lattice point $(u,\mathrm{ord}_P f)\in M'$. Note that this construction also appears in the proof of [5, Theorem 2.2].

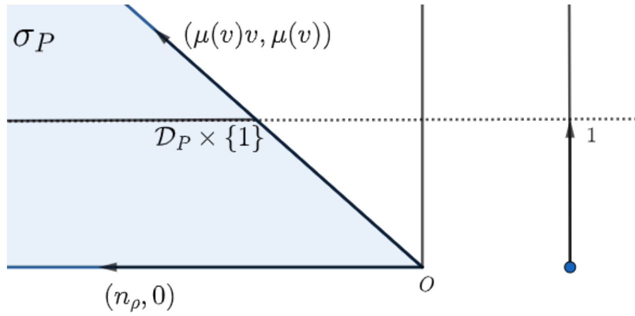

FIGURE 1. Local toric model

On $\widetilde{X}_P$, we can consider the Cartier divisor $r^*(-K_X)$. By standard toric geometry and by our choice of h, we may find $b(\sigma_P)\in M'$ such that

$$\begin{cases}\big\langle b(\sigma_P),(n_\rho,0)\big\rangle = -h_t(n_\rho)\\ \big\langle b(\sigma_P),(\mu(v)v,\mu(v))\big\rangle = -\mu(v)h_P(v)\end{cases}$$

Let us put

$$\psi_P(w) := -\langle b(\sigma_P),w\rangle,\ \ w\in\sigma_P.$$

We will be mainly interested in the semi-invariant elements $f\chi^u\in H^0(X,-kK_X)$. If we put $a=\mathrm{ord}_P f$, then by (3.6), these $f\chi^u$ translate into the lattice points $(u,a)\in k\Delta_P\cap M'$ (recall (4.1)). In other words, the polytope Δ_P captures all the sections in $R(X,-K_X)$ when we are working on $\widetilde{X}_P$ (so Δ_P can be thought of as the Okounkov body of $-K_X$ arising from our localization; see also [12]).

Let us denote the point in $M'_{\mathbb{Q}}$ by (u,a), with a being the last component. Now for any $(u,a)\in\Delta_P$, we can define an effective $\mathbb{Q}$-divisor

$$D_{(u,a)} := \sum_v\Big(\big\langle (u,a),(\mu(v)v,\mu(v))\big\rangle\Big)D_{P,v}+\sum_\rho\Big(\big\langle (u,a),(n_\rho,0)\big\rangle\Big)D_\rho + r^*(-K_X).$$

Then $D_{(u,a)}\sim_{\mathbb{Q}} r^*(-K_X)$.

Now we can follow the argument in [6, §7] closely, and we will freely use the notation therein. For any $w\in\sigma_P$, the map

$$\mathbb{C}[\sigma_P^\vee\cap M'] = \bigoplus_{(u,a)\in\sigma_P^\vee\cap M'}\mathbb{C}\cdot\chi^{(u,a)}\to\mathbb{Q}_+$$

defined by

$$\sum_{(u,a)\in\sigma_P^\vee\cap M'} c_{(u,a)}\chi^{(u,a)}\mapsto \min\{\langle (u,a),w\rangle \mid c_{u,a}\neq 0\}$$

gives rise to a valuation on X, which we also denote by w. In [6], such a w is called a toric valuation.

Lemma 4.3. *If* $(u,a) \in \Delta_P$ *and* $w \in \sigma_P$, *then*

$$w(D_{(u,a)}) = \big\langle (u,a), w \big\rangle - \psi_P(w).$$

Proof. This is [6, Lemma 7.1] in our setting. We may pick a sufficiently divisible $k \in \mathbb{N}_+$ such that $(ku, ka) \in M'$ and

$$D_{(u,a)} = k^{-1}\mathrm{div}(\chi^{(ku,ka)+kb(\sigma_P)})$$

So it is clear that

$$w(D_{(u,a)}) = \big\langle (u,a) + b(\sigma_P), w \big\rangle = \big\langle (u,a), w \big\rangle - \psi_P(w).$$

□

Lemma 4.4. *The restriction of the log discrepancy function* A_X *to* $\sigma_P \subset \mathrm{Val}_X$ *is the unique function that is linear on the cone* σ_P *such that*

$$A_X\big((\mu(v)v, \mu(v))\big) = 1 \text{ and } A_X\big((n_\rho, 0)\big) = -h_t(n_\rho).$$

Proof. By [6, Proposition 7.2], we know that $A_{\widetilde{X}_P}$ is linear on σ_P such that

$$A_{\widetilde{X}_P}\big((\mu(v)v, \mu(v))\big) = 1 \text{ and } A_{\widetilde{X}_P}\big((n_\rho, 0)\big) = 1.$$

To compute A_X, we use the relation

$$A_X(w) = A_{\widetilde{X}_P}(w) + w(K_{\widetilde{X}_P/X}).$$

Note that A_X is also linear on σ_P and the desired result follows from (3.4). □

Lemma 4.5. *For* $w \in \sigma_P$, *we have*

$$S(w) = \big\langle (\mathrm{bc}(\Psi), \mathrm{bc}_P), w \big\rangle - \psi_P(w).$$

Proof. This is [6, Corollary 7.7] in our setting. For $k \in \mathbb{N}_+$, we put

$$\Omega_k := \Big\{ (u,a) \in M' \mid u \in k\Box_h \cap M,\ -\lfloor k\Psi_P(u/k) \rfloor \le a \le \sum_{Q \ne P} \lfloor k\Psi_Q(u/k) \rfloor \Big\} \subset k\Delta_P \cap M'.$$

Then we have

$$S_k(w) = \frac{\sum_{(u,a)\in\Omega_k} \Big(\big\langle (u,a), w \big\rangle - k\psi_P(w) \Big)}{k\#\Omega_k}.$$

We finish the proof by letting $k \to \infty$. □

Corollary 4.6. *We have*

$$\delta(X) \le \min \left\{ \frac{-h_t(n_\rho)}{\langle \mathrm{bc}(\Psi), n_\rho \rangle - h_t(n_\rho)} \right\}_\rho \cup \left\{ \frac{1}{\mu(v)(\mathrm{bc}_P + \langle \mathrm{bc}(\Psi), v \rangle - h_p(v))} \right\}_{P,v}.$$

Proof. As we know,

$$\delta(X)=\inf_{w\in\mathrm{Val}_X}\frac{A_X(w)}{S(w)}.$$

So we have

$$\delta(X)\leq\inf_{w\in\sigma_P}\frac{A_X(w)}{S(w)},\ P\in\mathbb{P}^1.$$

The result follows from Lemma 4.4 and 4.5 by linearity. □

Now we follow [6, §7.4] closely to show that the inequality in the above corollary is actually an equality.

Set $\widetilde{N}:=N'\times\mathbb{Z}$ and $\widetilde{M}:=M'\times\mathbb{Z}$. We define a cone

$$\Sigma_P:=\mathrm{cone}(\Delta_P\times\{1\})\subset\widetilde{M}_{\mathbb{Q}}$$

and choose $y_1,...,y_{n+1}\in\Sigma_P^\vee\cap\widetilde{N}$ that are linear independent in $\widetilde{N}_{\mathbb{Q}}$ (here $n=\dim X$). Let $\nu:\Sigma_P\cap\widetilde{M}\to\mathbb{Z}_+^{n+1}$ denote the injective map given by

$$\nu(\tilde{u})=\big(\langle\tilde{u},y_1\rangle,\cdots,\langle\tilde{u},y_{n+1}\rangle\big),\ \tilde{u}\in\Sigma_P\cap\widetilde{M}.$$

We endow $\mathbb{Z}_+^{n+1}$ with the lexicorgraphic order. In this way we put a $\mathbb{Z}_+^{n+1}$ order on $\mathbb{C}[\Sigma_P\cap\widetilde{M}]$.

Moreover, we define

$$S:=\bigcup_k\Omega_k\times\{k\}\subset\Sigma_P\cap\widetilde{M},$$

where Ω_k is defined in the proof of Lemma 4.5. Note that each monomial $\chi^{(u,a,k)}\in\mathbb{C}[S]$ can be thought of as a section in $H^0(X,-kK_X)$, and all these monomials form a basis. So as in [6], for $s\in H^0(X,-kK_X)$, we can define its initial term $\mathrm{in}_>(s)$ to be the greatest monomial $\chi^{(u,a,k)}$ in s with respect to the $\mathbb{Z}_+^{n+1}$ order chosen above.

Given a subspace W of $H^0(X,-kK_X)$, we set

$$\mathrm{in}_>(W)=\mathrm{Span}\{\mathrm{in}_>(s)\mid s\in W\}$$

By [6, Proposition 7.10], we have

$$\dim W=\dim\mathrm{in}_>(W).$$

Thus for any filtration $\mathcal{F}$ of $R(X,-K_X)$, if we write $\mathcal{F}_{\mathrm{in}}$ for the filtration defined by

$$\mathcal{F}^\lambda_{\mathrm{in}}H^0(X,-kK_X):=\mathrm{in}_>\left(\mathcal{F}^\lambda H^0(X,-kK_X)\right),$$

for all $\lambda\in\mathbb{R}_+$ and $k\in\mathbb{N}_+$, we get (cf. [6, Proposition 7.13])

$$S(\mathcal{F}_{\mathrm{in}})=S(\mathcal{F}).$$

By [6, Theorem E], there always exists a valuation $w\in\mathrm{Val}_X$ with $A_X(w)<\infty$ such that

$$\delta(X)=\frac{A_X(w)}{S(w)}.$$

And we know that, w must satisfy (cf. Proposition 4.8 therein)

$$\mathrm{lct}\big(\mathfrak{a}_\bullet(w)\big) = A_X(w).$$

Namely, w computes $\mathrm{lct}\big(\mathfrak{a}_\bullet(w)\big)$. Let $\xi := c_X(w)$ denote the center of w. Since we are working locally, we may assume that ξ is contained in our local picture. Let $\mathcal{F}_w$ be the filtration of $R(X, -K_X)$ induced by w. Then by previous discussion, we have

$$S(\mathcal{F}_{w,\mathrm{in}}) = S(\mathcal{F}_w).$$

Meanwhile, we can find a nontrivial toric valuation $w_0 \in \sigma_P$ such that

$$\frac{A_X(w_0)}{w_0\big(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})\big)} \leq \mathrm{lct}\big(\mathfrak{b}_\bullet(\mathcal{F}_w)\big) = \mathrm{lct}\big(\mathfrak{a}_\bullet(w)\big) = A_X(w).$$

By rescaling w_0, we may assume $w_0\big(\mathfrak{b}_\bullet(\mathcal{F}_{w,\mathrm{in}})\big) = 1$, then as in [6, Proposition 7.14], we get

$$A_X(w_0) \leq A_X(w), \text{ and } S(w_0) \geq S(\mathcal{F}_{w,\mathrm{in}}) = S(\mathcal{F}_w).$$

So we see that $\delta(X)$ is indeed computed by some toric valuation $w_0 \in \sigma_P$. Thus Theorem 4.2 is proved.

5. Examples

5.1. Let us consider $X = Bl_p\mathbb{P}^2$. Of course, X is a smooth toric Fano variety. But we would like to interpret it as a T-variety of complexity one. To do this, we can use toric downgrade (cf. [13, Example 2.9]; see also [2, Section 11]).

Let $N = \mathbb{Z} \oplus \mathbb{Z}$ with P and P' being the projections from N to the first and the second component respectively. Via P', the fan Σ corresponding to X on $N_{\mathbb{Q}}$ projects down to the fan Σ' corresponding to $\mathbb{P}^1$ on $\mathbb{Q}$ (see Figure 2).

Note that $\Sigma' = \{\{0\}, \mathbb{Q}_+, \mathbb{Q}_-\}$. Suppose that the rays $\mathbb{Q}_+$ and $\mathbb{Q}_-$ correspond to the toric divisors 0, ∞ on $\mathbb{P}^1$ respectively. For each cone $\sigma_i \in \Sigma$, we set

$$\Delta_0(\sigma_i) := P(P'^{-1}(1) \cap \sigma_i),\ \Delta_\infty(\sigma_i) := P(P'^{-1}(-1) \cap \sigma_i).$$

And we put

$$\mathcal{D}_i := \Delta_0(\sigma_i) \otimes 0 + \Delta_\infty(\sigma_i) \otimes \infty.$$

Then we get the following 4 polyhedral divisors on $\mathbb{P}^1$:

$$\begin{cases} \mathcal{D}_0 = [0, \infty) \otimes 0 + \emptyset \otimes \infty \\ \mathcal{D}_1 = [-1, 0] \otimes 0 + \emptyset \otimes \infty \\ \mathcal{D}_2 = (-\infty, -1] \otimes 0 + (-\infty, 0] \otimes \infty \\ \mathcal{D}_3 = \emptyset \otimes 0 + [0, \infty) \otimes \infty \end{cases}$$

Note that $\deg \mathcal{D}_0 = \deg \mathcal{D}_1 = \deg \mathcal{D}_3 = \emptyset$ and $\deg \mathcal{D}_2 = (-\infty, -1]$. These polyhedral divisors give rise to a divisorial fan $\mathcal{S}$ corresponding to X. The slices $\mathcal{S}_0$ and $\mathcal{S}_\infty$ are illustrated in figure 2. Moreover, we have

$$\mathrm{tail}(\mathcal{S}) = \{\{0\}, \rho_+, \rho_-\},$$

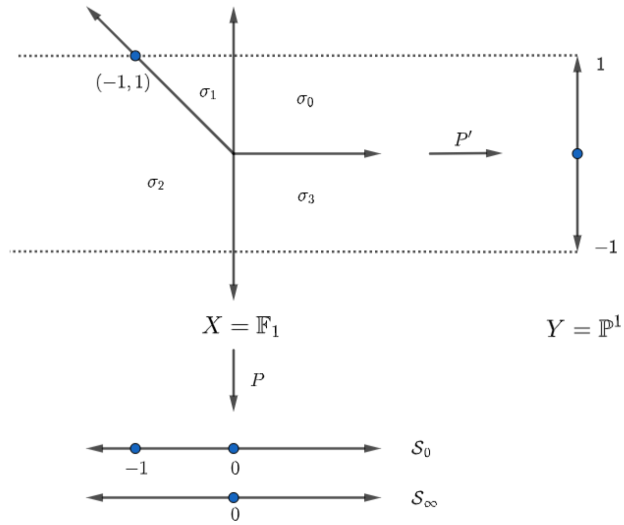

FIGURE 2. Toric downgrade

where $\rho_+ = \mathbb{Q}_+$ and $\rho_- = \mathbb{Q}_-$ are the rays in tail($\mathcal{S}$). Note that

$$\rho_+ \cap \deg \mathcal{D}_0 = \rho_+ \cap \deg \mathcal{D}_3 = \emptyset,\ \rho_- \cap \deg \mathcal{D}_2 \neq \emptyset,$$

so that ρ_+ is extremal while ρ_- is not (cf. [13, Definition 3.18]). Meanwhile, in $\mathcal{S}_0$ there are two vertices and in $\mathcal{S}_\infty$ there is only one vertex. Thus the Picard number of X is given by (see [13, Corollary 3.20])

$$\rho_X = 1 + 1 + (2-1) + (1-1) - 1 = 2,$$

which is of course a standard fact since $X = Bl_p\mathbb{P}^2$.

Now let us fix a canonical divisor $K_{\mathbb{P}^1} = -2 \cdot \infty$. Then by [13, theorem 3.21] we have

$$-K_X = 2D_{\infty,0} + D_{\rho_+}.$$

This is a T-Cartier divisor on X so it corresponds to a support function h on $\mathcal{S}$ (see [13, Proposion 3.10]). We can explicitly write down h (see Figure 3).

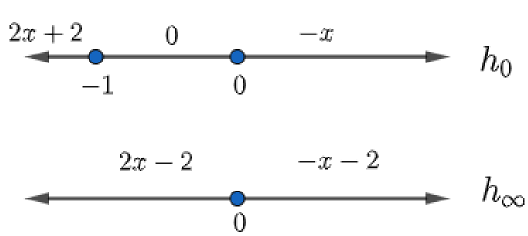

FIGURE 3. Support function h

The linear part h_t of h is given by

$$h_t = \begin{cases} -x, & x \geq 0, \\ 2x, & x \leq 0. \end{cases}$$

So we get a polytope (cf. [13, Definition 3.22])

$$\square_h := [-1, 2].$$

We can also compute the Legendre transform of h on $\square_h$:

$$\Psi_0(u) := h_0^*(u) = \min\{0, -u\}, \ \Psi_\infty(u) := h_\infty^*(u) = 2, \text{ for } u \in [-1, 2].$$

We put

$$\Psi(u) = \Psi_0(u) \otimes 0 + \Psi_\infty(u) \otimes \infty.$$

Then we get a divisorial polytope $(\square_h, \Psi)$ associated to the polarized pair $(X, -K_X)$. Moreover, we have

$$\deg \Psi(u) = \Psi_0(u) + \Psi_\infty(u) = \begin{cases} 2, & -1 \leq u \leq 0, \\ 2 - u, & 0 \leq u \leq 2. \end{cases}$$

We can easily compute

$$\mathrm{vol}(\Psi) = \int_{-1}^{2} \deg \Psi(u) du = 4.$$

So we get (cf. [13, Proposition 3.31])

$$\mathrm{vol}(-K_X) = 2! \cdot \mathrm{vol}(\Psi) = 8,$$

which is again a standard fact since $X = Bl_p\mathbb{P}^2$. We can also compute (recall (3.2))

$$h^0(X, -K_X) = (\deg \Psi(-1) + 1) + (\deg \Psi(0) + 1) + (\deg \Psi(1) + 1) + (\deg \Psi(2) + 1) = 9.$$

The Futaki character of $(\square_h, \Psi)$ is given by (cf. [16, Theorem 3.15])

$$\mathrm{bc}(\Psi) = \frac{1}{\mathrm{vol}(\Psi)} \int_{-1}^{2} u \deg \Psi(u) du = \frac{1}{12}.$$

We can calculate $\alpha(X)$:

$$\alpha(X) = \min_{u \in \square_h} \left\{ \frac{-h_t(1)}{u - h_t(1)}, \frac{-h_t(-1)}{-u - h_t(-1)}, \frac{1}{\Psi_\infty(u)}, \frac{1}{\Psi_\infty(u) - u}, \frac{1}{\Psi_0(u) + 2} \right\} = \frac{1}{3}.$$

We can also calculate $\delta(X)$. We have

$$\mathrm{bc}_0 = \frac{1}{4} \int_{-1}^{2} (\frac{\deg \Psi}{2} - \Psi_0) \deg \Psi = \frac{7}{6}, \ \mathrm{bc}_\infty = \frac{1}{4} \int_{-1}^{2} (\frac{\deg \Psi}{2} - \Psi_\infty) \deg \Psi = -\frac{7}{6}.$$

So $\delta(X)$ is simply given by

$$\delta(X) = \min \left\{ \frac{1}{1 + 1/12}, \frac{2}{2 - 1/12}, \frac{1}{7/6}, \frac{1}{7/6 - 1/12}, \frac{1}{2 - 7/6} \right\} = \frac{6}{7}.$$

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