Lecture 20:

Physically-Based Animation and PDEs

Computer Graphics

CMU 15-462/15-662, Fall 2015

Last time: Optimization

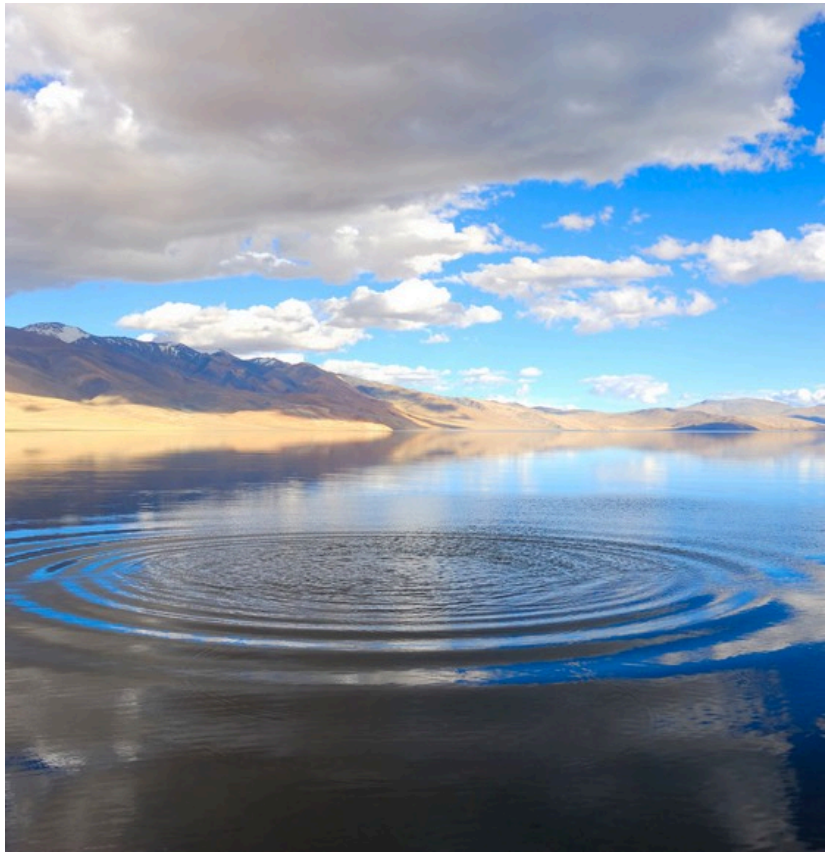
- Graphics as optimization
- Many complex criteria/constraints
- Technique: numerical optimization
  - pick initial guess
  - ski downhill
  - keep fingers crossed!
- Today: return to differential equations
  - saw ODEs—derivatives in time
  - now PDEs—also have derivatives in space
  - describe many natural phenomena (water, smoke, cloth, ...)
  - recent revolution in CG/visual effects

Partial Differential Equations (PDEs)

- ODE: Implicitly describe function in terms of its time derivatives
- Like any implicit description, have to solve for actual function
- PDE: Also include space derivatives in description

ODE—rock flies through air

PDE—rock lands in pond



To make a long story short…

- **Solving ODE looks like *"add a little velocity each time"***

$$q_{k+1} = q_k + \tau f(q)$$

- **Solving a PDE looks like *"take weighted combination of neighbors to get velocity (…and add a little velocity each time)"***

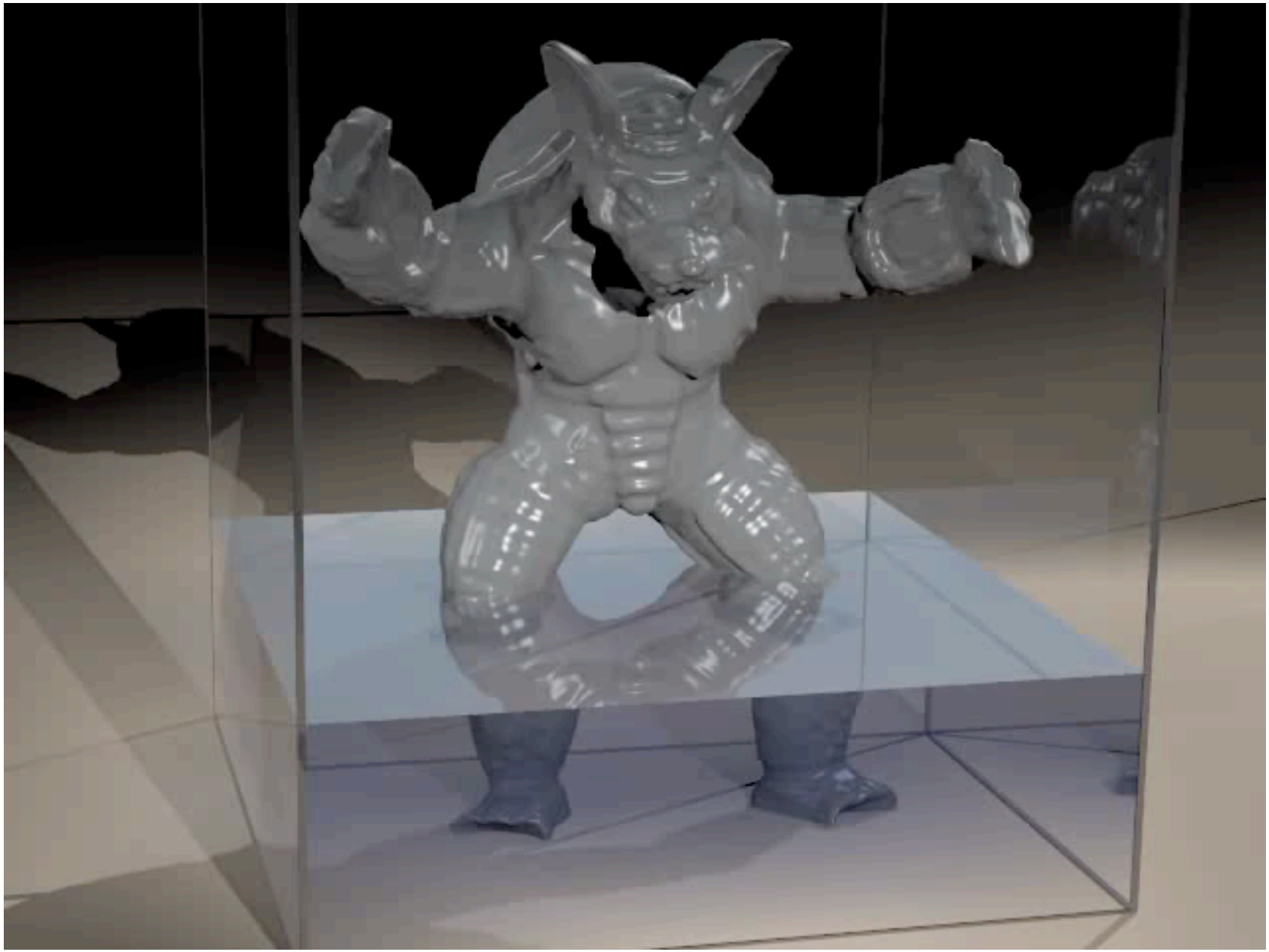
	1	
1	-4	1
	1	

$f(q)$

$$q_{k+1} = q_k + \tau f(q)$$

…obviously there is a *lot* more to say here!

Liquid Simulation in Graphics



Losasso, F., Shinar, T. Selle, A. and Fedkiw, R., *"Multiple Interacting Liquids"*

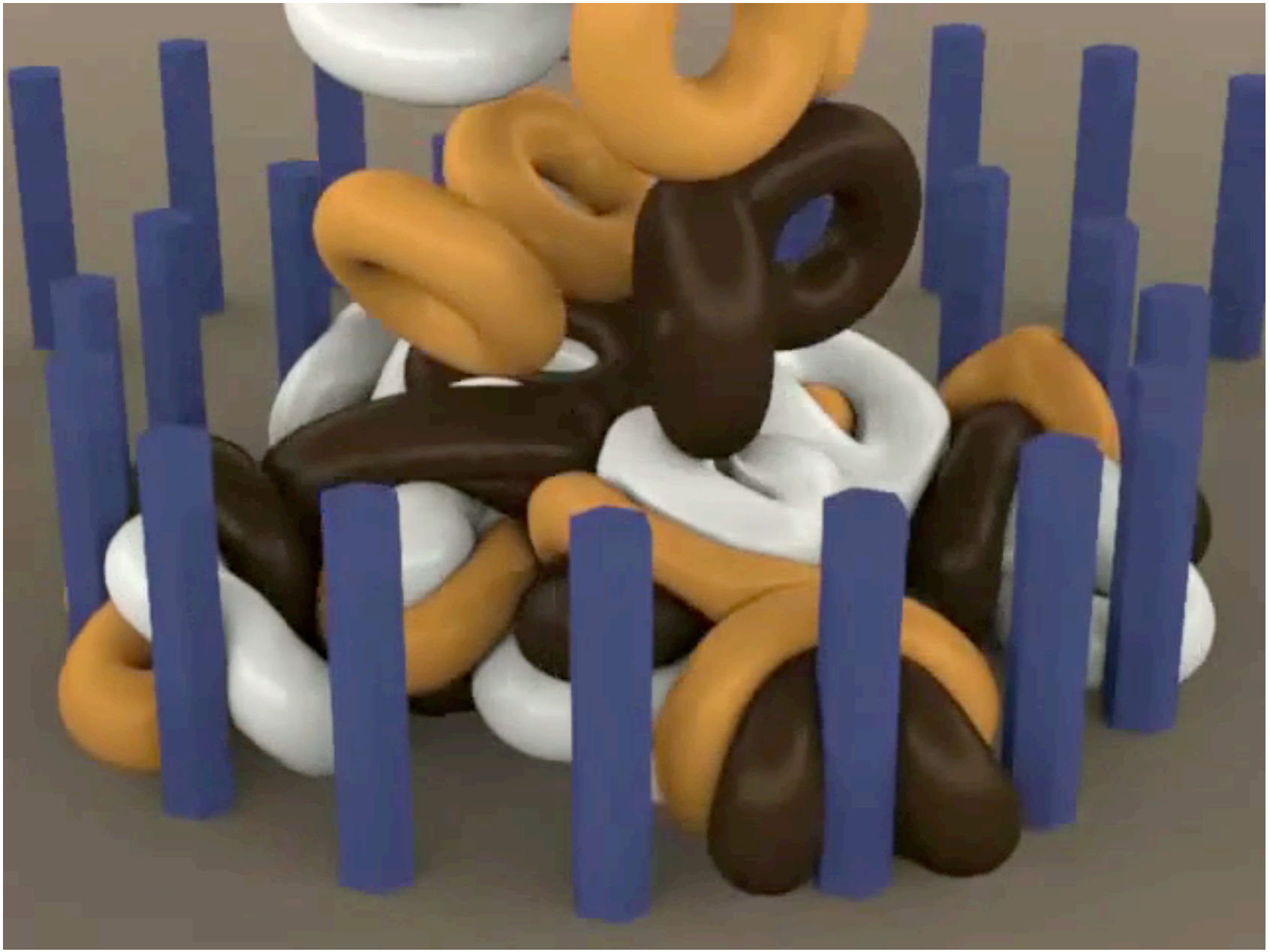
Smoke Simulation in Graphics

S. Weißmann, U. Pinkall. ***"Filament-based smoke with vortex shedding and variational reconnection"***

Cloth Simulation in Graphics

Zhili Chen, Renguo Feng and Huamin Wang, *"Modeling friction and air effects between cloth and deformable bodies"*

Elasticity in Graphics

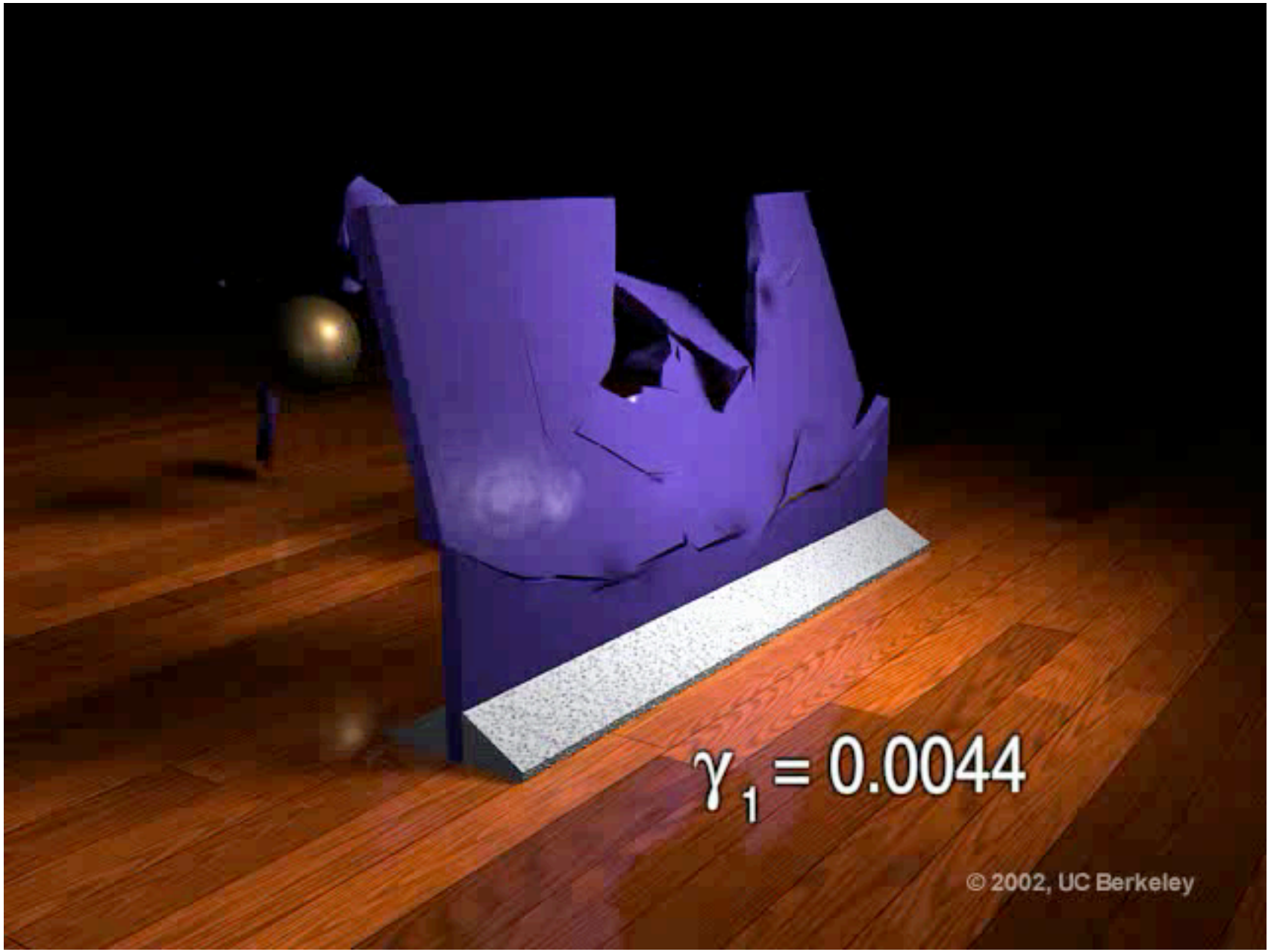


Irving, G., Schroeder, C. and Fedkiw, R., "Volume Conserving Finite Element Simulation of Deformable Models"

Hair Simulation in Graphics

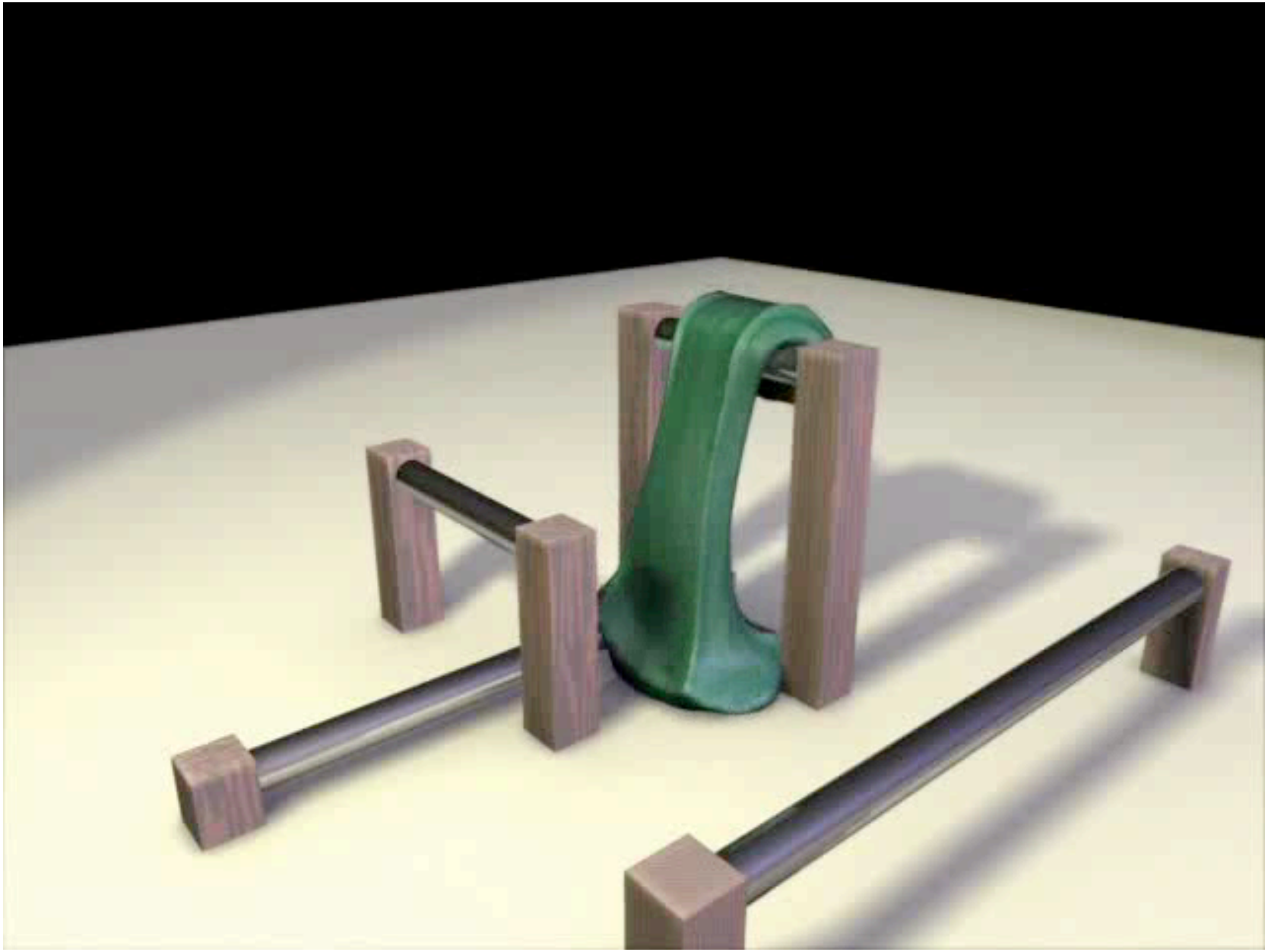
Danny M. Kaufman, Rasmus Tamstorf, Breannan Smith, Jean-Marie Aubry, Eitan Grinspun,
"Adaptive Nonlinearity for Collisions in Complex Rod Assemblies"

Fracture in Graphics



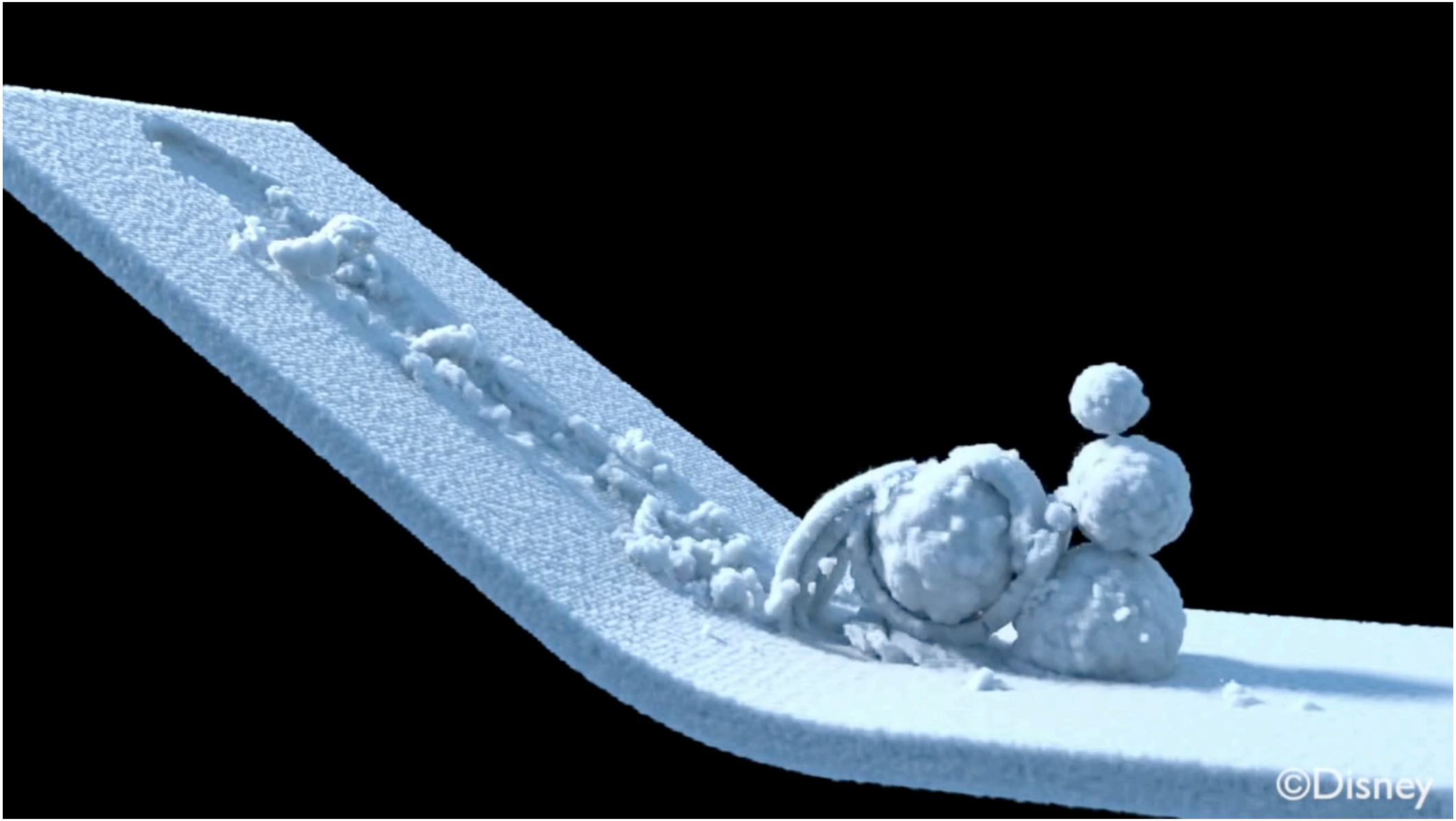
James F. O'Brien, Adam Bargteil, Jessica Hodgins, *"Graphical Modeling and Animation of Ductile Fracture"*

Viscoelasticity in Graphics



Chris Wojtan, Greg Turk, *"Fast Viscoelastic Behavior with Thin Features"*

Snow Simulation in Graphics



Alexey Stomakhin, Craig Schroeder, Lawrence Chai, Joseph Teran, Andrew Selle, *"A Material Point Method For Snow Simulation"*

Definition of a PDE

- **Want to solve for a function of time *and* space**

$$u(t, x)$$

time space

- **Function given implicitly in terms of derivatives:**

$$\dot{u}, \ddot{u}, \frac{d}{dt^3}u, \frac{d}{dt^4}u, \ldots$$

any combination of time derivatives

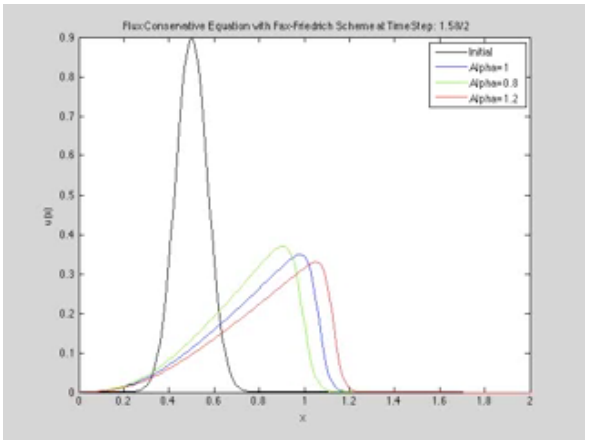
$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^m + nu}{\partial x_i^m \partial x_j^n}, \ldots$$

plus any combination of space derivatives

- **Example:**

$$\dot{u} + uu' = au''$$

(Burgers' equation)



Anatomy of a PDE

- **Linear vs. nonlinear: how are derivatives combined?**

nonlinear!

$$\dot{u} + u u' = a u''$$ (Burgers' equation)

$$\dot{u} = a u''$$ (diffusion equation)

- **Order: how many derivatives in space & time?**

1st order in time

2nd order in space

$$\dot{u} + u u' = a u''$$ (Burgers' equation)

2nd order in time

2nd order in space

$$\ddot{u} = a u''$$ (wave equation)

- **Nonlinear / higher order ⇒ HARDER TO SOLVE!**

Model Equations

- **Fundamental behavior of many important PDEs is well-captured by three model linear equations:**

"Laplacian" (more later!)

LAPLACE EQUATION ("ELLIPTIC") $\Delta u = 0$

"what's the smoothest function interpolating the given boundary data"

HEAT EQUATION ("PARABOLIC") $\dot{u} = \Delta u$

"how does an initial distribution of heat spread out over time?"

WAVE EQUATION ("HYPERBOLIC") $\ddot{u} = \Delta u$

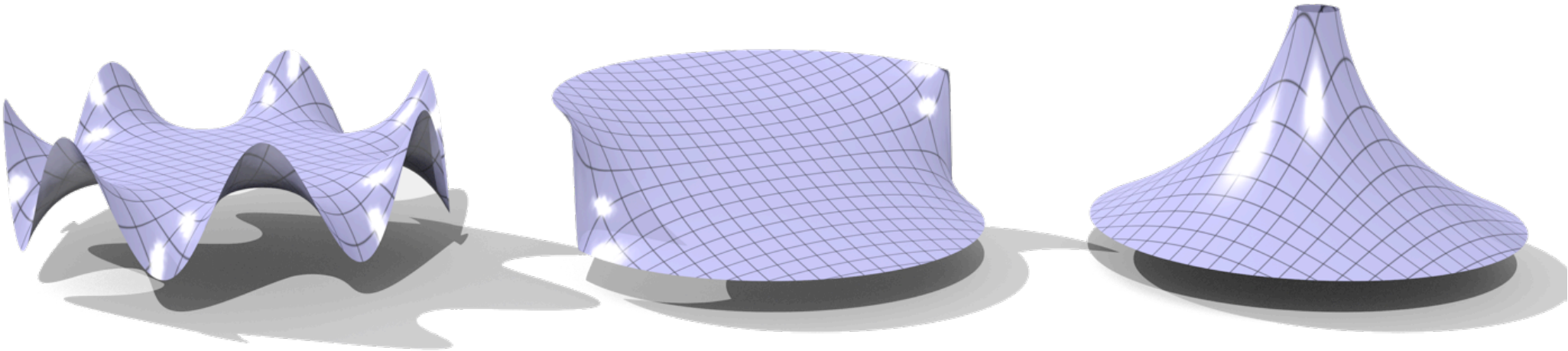
"if you throw a rock into a pond, how does the wavefront evolve over time?"

[NONLINEAR + HYPERBOLIC + HIGH-ORDER]

Solve numerically?

Elliptic PDEs / Laplace Equation

- "What's the smoothest function interpolating the given boundary data?"

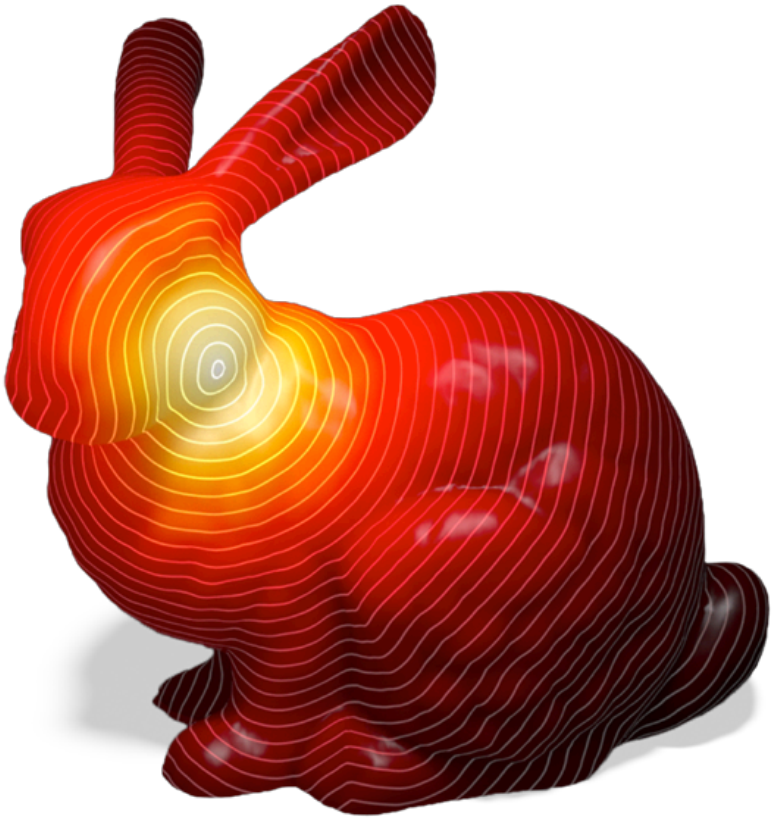


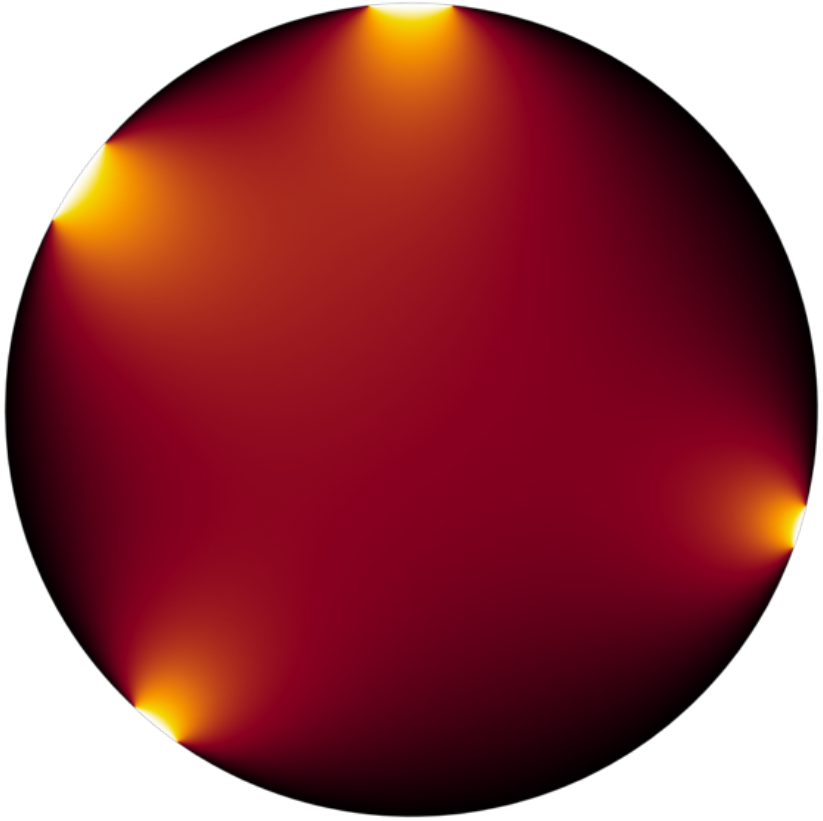
- Conceptually: each value is at the average of its "neighbors"
- Roughly speaking, why is it easier to solve?
- Very robust to errors: just keep averaging with neighbors!

Image from Solomon, Crane, Vouga, "Laplace-Beltrami: The Swiss Army Knife of Geometry Processing"

Parabolic PDEs / Heat Equation

- "How does an initial distribution of heat spread out over time?"





- After a long time, solution is same as Laplace equation!
- Models damping / viscosity in many physical systems

Hyperbolic PDEs / Wave Equation

- "If you throw a rock into a pond, how does the wavefront evolve over time?"

- Errors made at the beginning will persist for a long time! (hard)

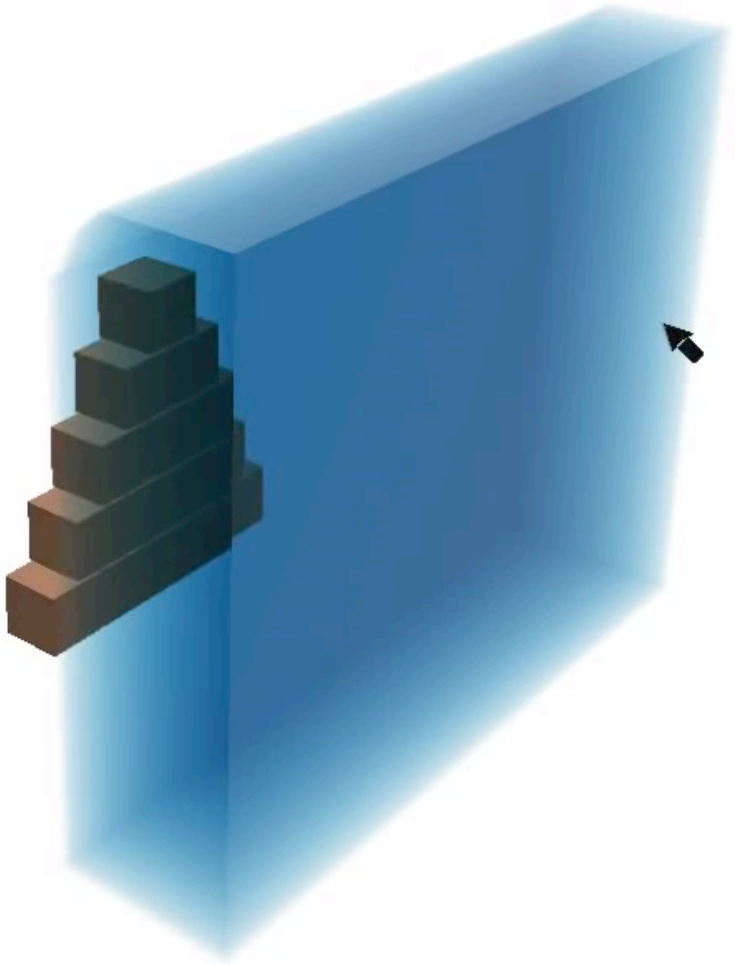
How did we do that?

Numerical Solution of PDEs—Overview

- Like ODEs, many interesting PDEs are difficult/impossible to solve analytically—especially if we want to incorporate data (e.g., user interaction)
- Must instead use numerical integration
- Basic strategy:
  - pick a time discretization (forward Euler, backward Euler...)
  - pick a spatial discretization (TODAY)
  - as with ODEs, run a time-stepping algorithm
- Historically, very expensive—only for "hero shots" in movies
- Computers are ever faster...
- More & more use of PDEs in games, interactive tools, ...

Real Time PDE-Based Simulation (Fire)

Real Time PDE-Based Simulation (Water)

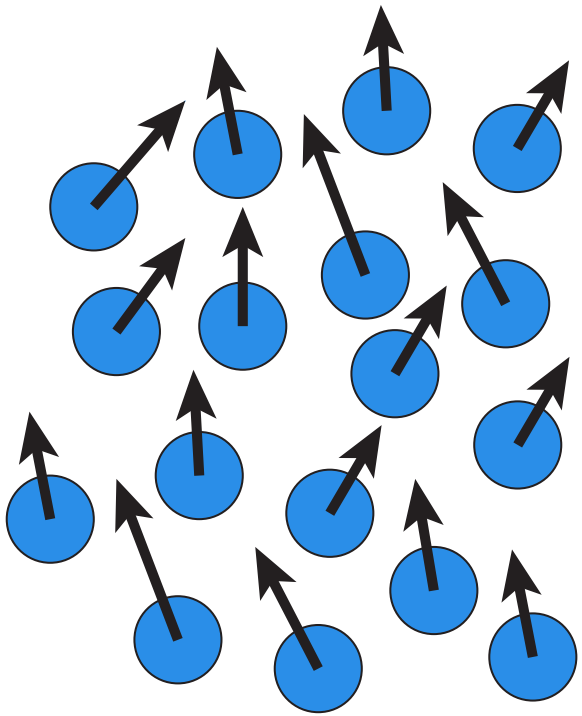


Nuttapong Chentanez, Matthias Müller, *"Real-time Eulerian water simulation using a restricted tall cell grid"*

Lagrangian vs. Eulerian

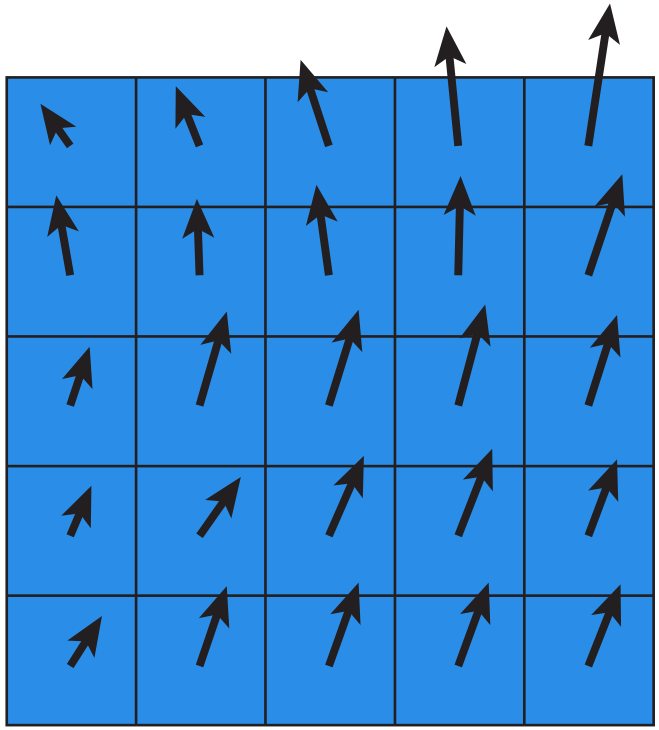
- Two basic ways to discretize space: Lagrangian & Eulerian
- E.g., suppose we want to encode the motion of a fluid

LAGRANGIAN



track position & velocity
of moving particles

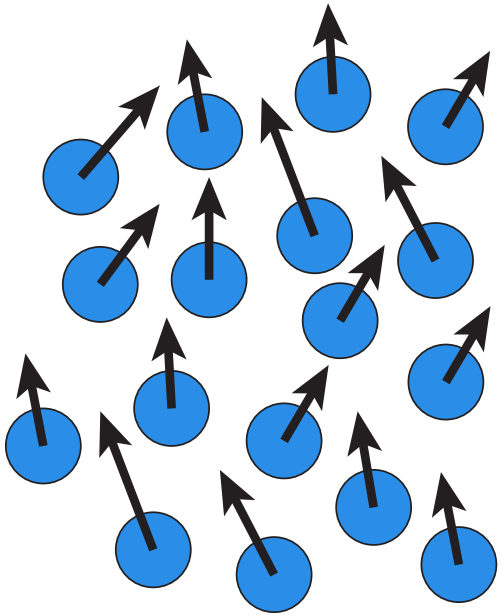
EULERIAN

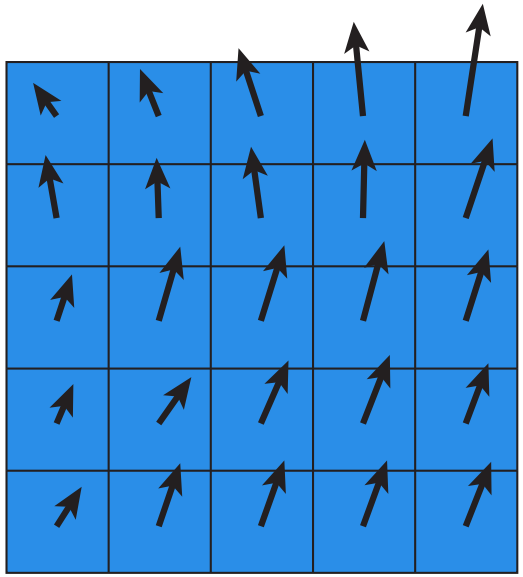


track velocity (or *flux*)
at fixed grid locations

Lagrangian vs. Eulerian—Trade-Offs

- Lagrangian
  - conceptually easy (like polygon soup!)
  - resolution/domain not limited by grid
  - good particle distribution can be tough
  - finding neighbors can be expensive
- Eulerian
  - fast, regular computation
  - easy to represent, e.g., smooth surfaces
  - simulation "trapped" in grid
  - grid causes "numerical diffusion" (blur)
  - need to understand PDEs (but you will!)





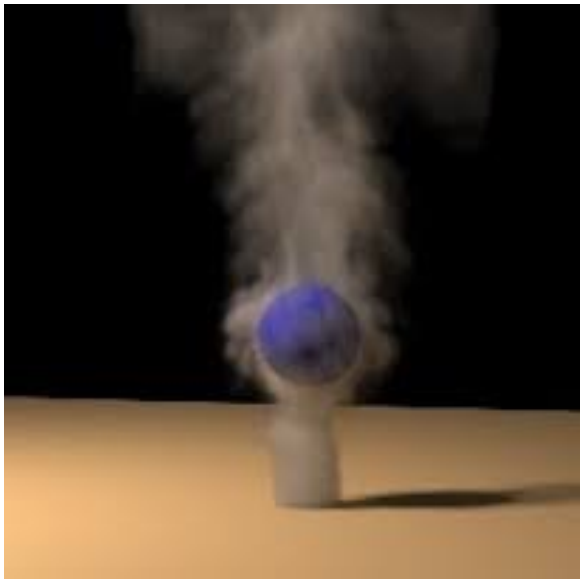
Mixing Lagrangian & Eulerian

- Of course, no reason you have to choose just one!
- Many modern methods mix Lagrangian & Eulerian:
  - PIC/FLIP, particle level sets, mesh-based surface tracking, Voronoi-based, arbitrary Lagrangian-Eulerian (ALE), ...
- *Pick the right tool for the job!*

Maya *Bifrost*

Aside: Which Quantity Do We Solve For?

- Many PDEs have mathematically equivalent formulations in terms of different quantities
- E.g., incompressible fluids:
  - *velocity*—how fast is each particle moving?
  - *vorticity*—how fast is fluid "spinning" at each point?
- Computationally, can make a big difference
- ***Pick the right tool for the job!***



Ok, but we're getting *way* ahead of ourselves.
How do we solve *easy* PDEs?

Numerical PDEs—Basic Strategy

Richard Courant

- Pick PDE formulation
  - Which quantity do we want to solve for?
  - E.g., velocity or vorticity?
- Pick spatial discretization
  - How do we approximate derivatives in space?
- Pick time discretization
  - How do we approximate derivatives in time?
  - When do we evaluate forces?
  - Forward Euler, backward Euler, symplectic Euler, ...
- Finally, we have an *update rule*
- Repeatedly solve to generate an animation

The Laplace Operator

- All of our model equations used the Laplace operator
- Different conventions for symbol:

Δ ← same symbol used for "change"

∇^2 ← same symbol used for Hessian!

- *Unbelievably* important object showing up everywhere across physics, geometry, signal processing, …
- Ok, but what does it mean?
- *Differential operator:* eats a function, spits out its "2nd derivative"
- What does that mean for a function u: $\mathbb{R}^n \to \mathbb{R}$?
  - divergence of gradient
  - sum of second derivatives
  - "average" curvature

div grad

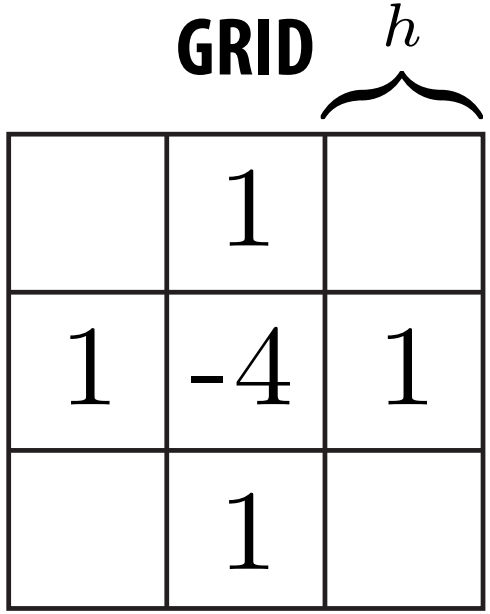
$$\Delta u = \nabla \cdot \nabla u$$

$$\Delta u = \frac{\partial u^2}{\partial x_1^2} + \cdots + \frac{\partial u^2}{\partial x_n^2}$$

Discretizing the Laplacian

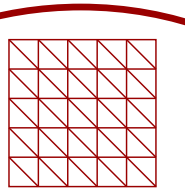
- How do we approximate the Laplacian?
- Depends on discretization (Eulerian, Lagrangian, grid, mesh, ...)
- Two extremely common ways in graphics:

GRID

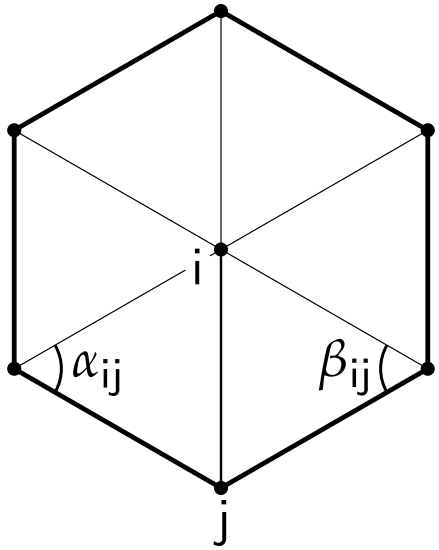


$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$

(actually, this becomes that)



TRIANGLE MESH

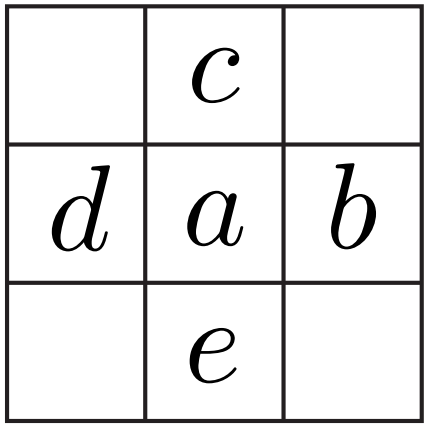


$$\frac{1}{2}\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

- Also not too hard on point clouds, polygon meshes, ...

Numerically Solving the Laplace Equation

- Want to solve $\Delta u = 0$
- Plug in one of our discretizations, e.g.,



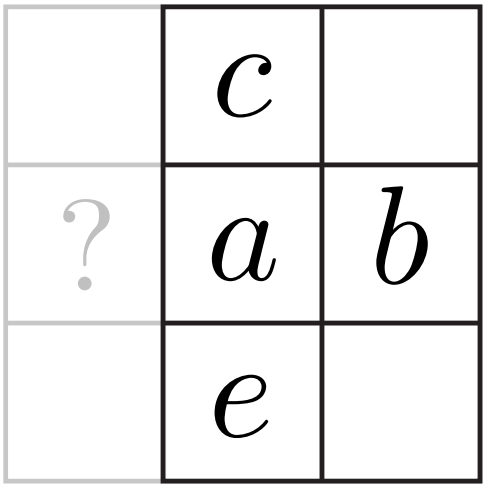
$$\frac{4a - b - c - d - e}{h} = 0$$

$$\iff a = \tfrac{1}{4}(b + c + d + e)$$

- Oh: if we have a solution, then each value must be the average of the neighboring values.
- How do we solve this?
- One idea: keep averaging with neighbors! ("Jacobi method")
- Correct, but *slow*. We'll discuss faster methods next lecture.

Boundary Conditions for Discrete Laplace

- What values do we use to compute averages near the boundary?

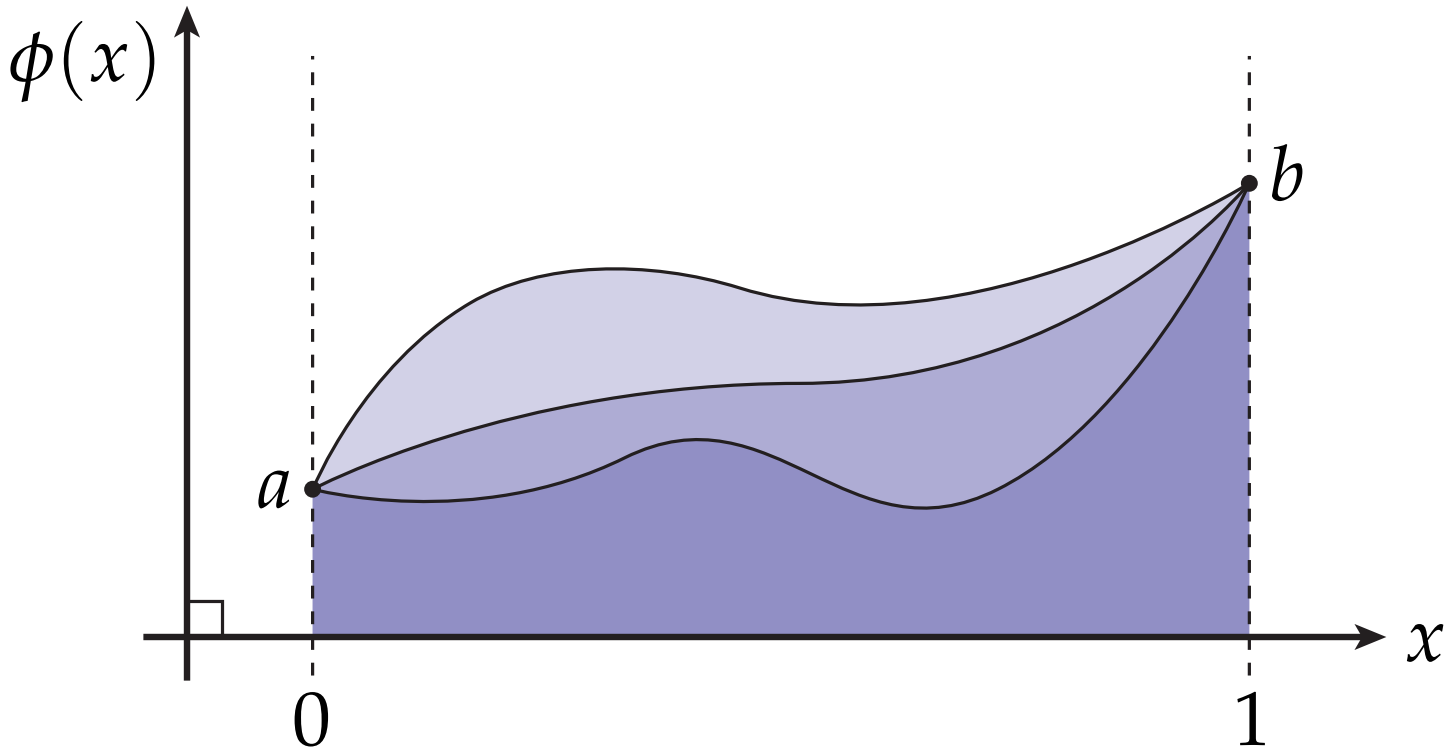


$$a = \frac{1}{4}(b + c + ? + e)$$

- A: We get to choose—this is the data we want to interpolate!
- Two basic boundary conditions:
  1. *Dirichlet*—boundary data always set to fixed values
  2. *Neumann*—specify derivative (difference) across boundary
- Also mixed (*Robin*) boundary conditions (and more, in general)

Dirichlet Boundary Conditions

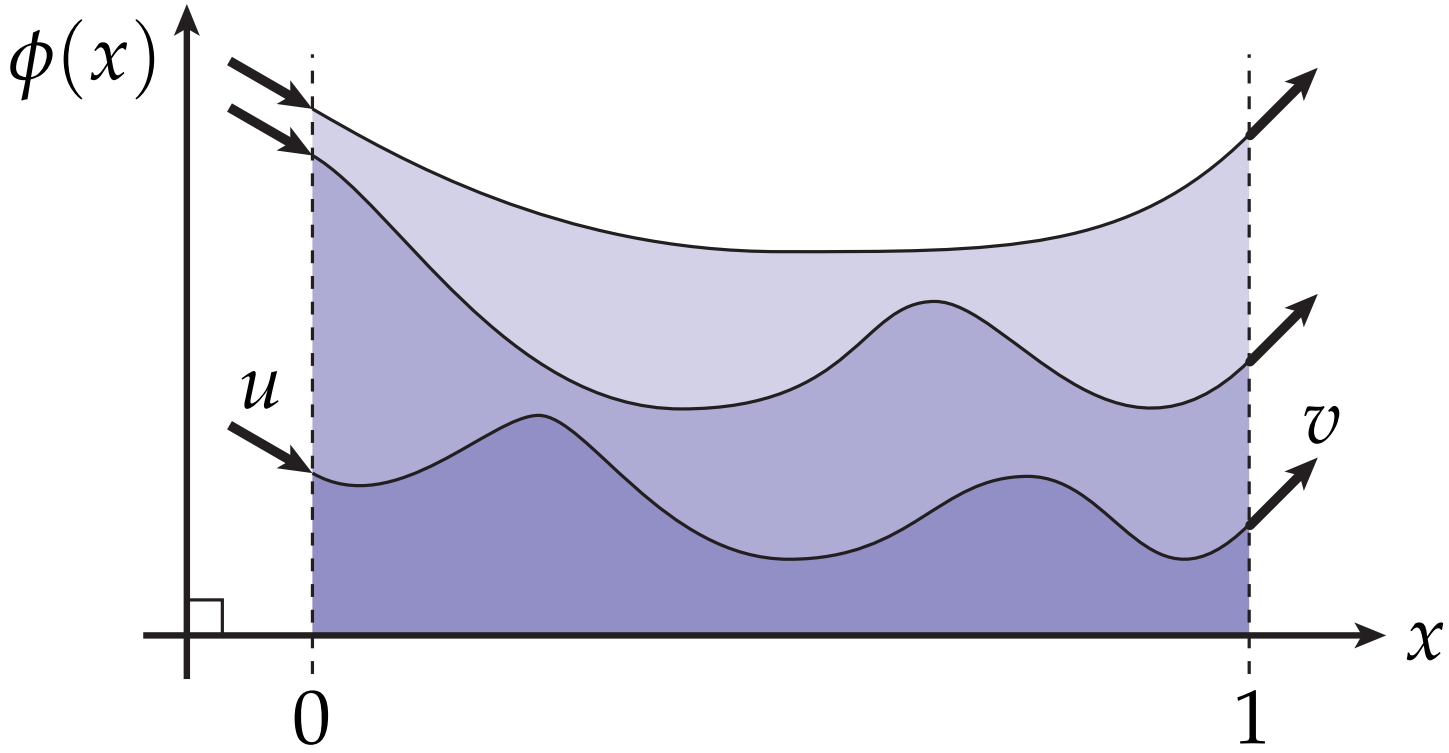
- Let's go back to smooth setting, function on real line
- *Dirichlet* means "prescribe values"
- E.g., $\Phi(0)=a$, $\Phi(1)=b$



- Many possible functions "in between"!

Neumann Boundary Conditions

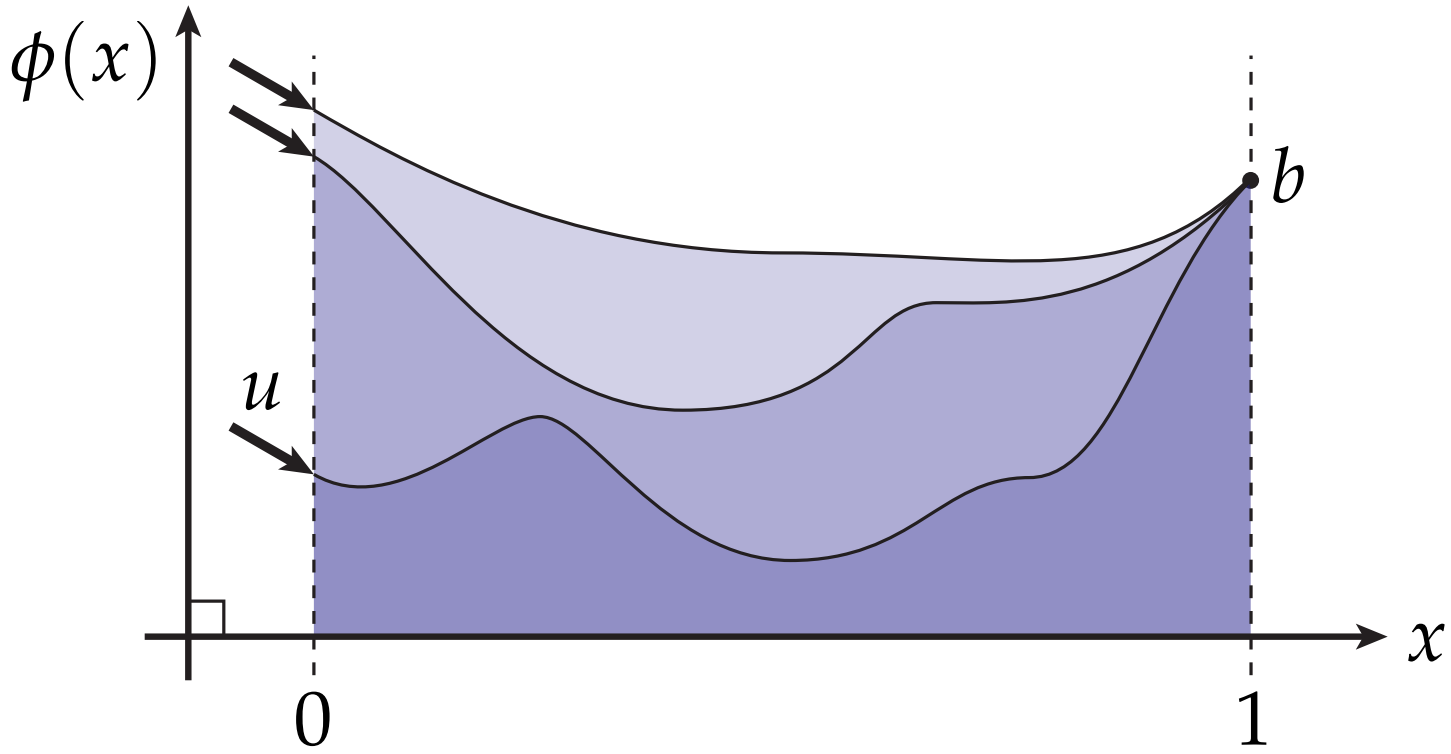
- *Neumann* means "prescribe derivatives"
- E.g., $\Phi'(0)=u$, $\Phi'(1)=v$



- Again, many possible functions!

Both Neumann & Dirichlet

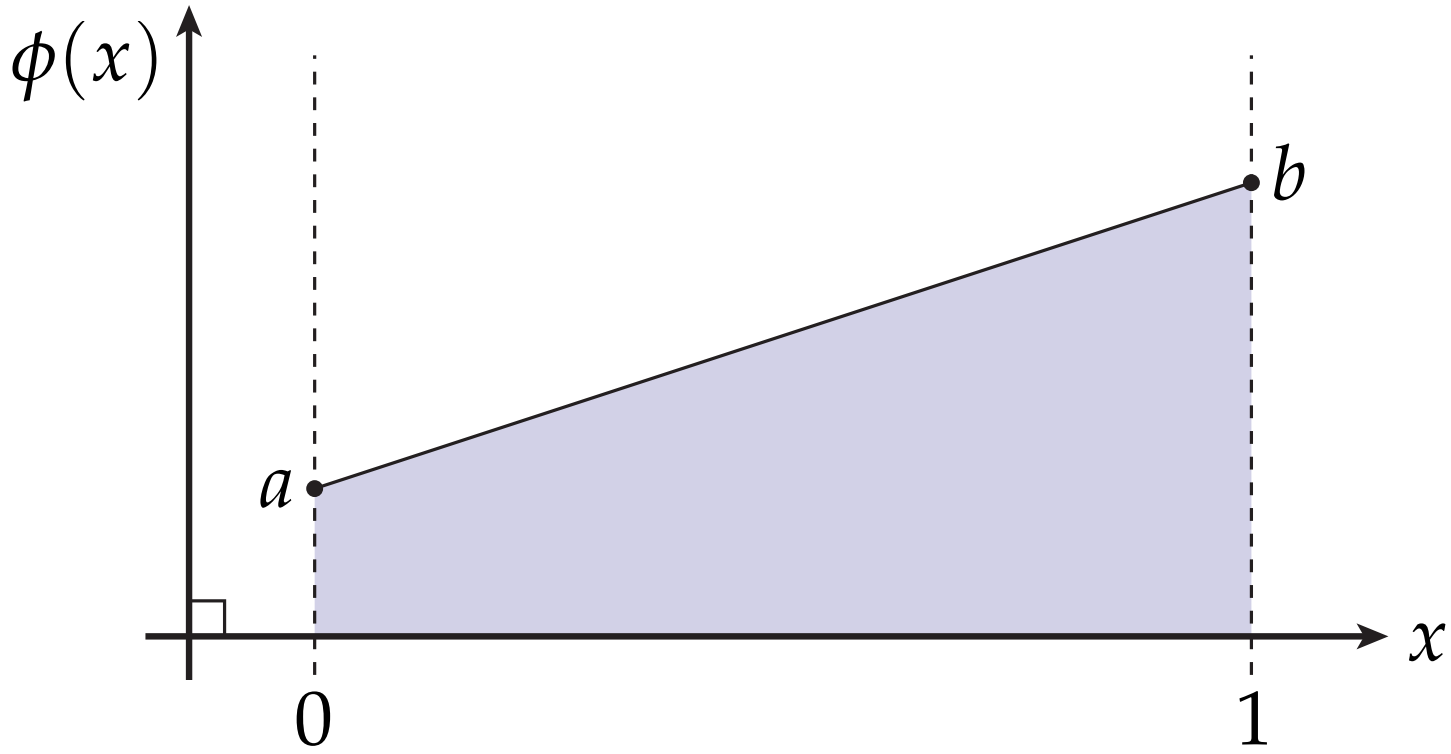
- Or: prescribe some values, some derivatives
- E.g., $\Phi'(0)=u$, $\Phi(1) = b$



- Q: What about $\Phi'(1)=v$, $\Phi(1) = b$? Does that work?
- Q: What about $\Phi'(0) + \Phi(0) = p$, $\Phi'(1) + \Phi(1) = q$? (*Robin*)

1D Laplace w/ Dirichlet BCs

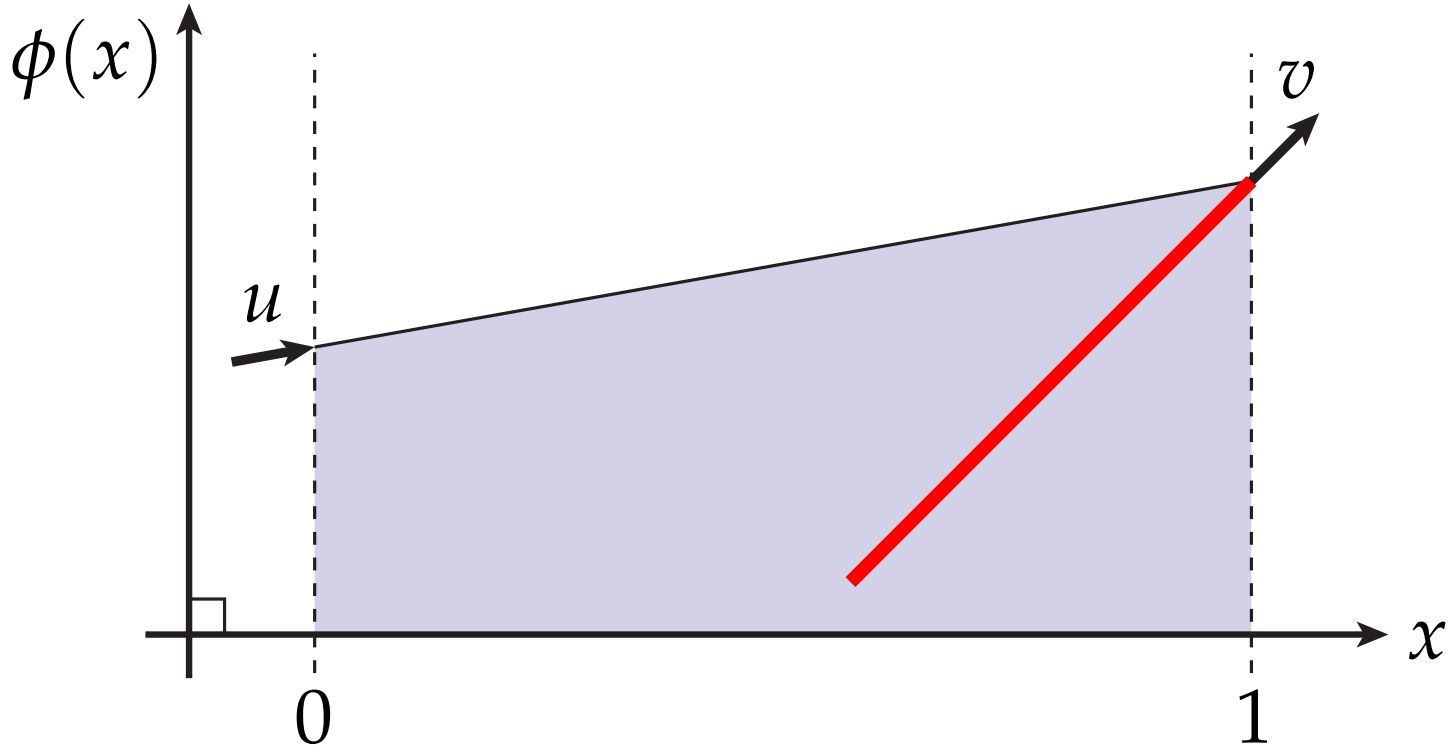
- 1D Laplace: $\partial^2\Phi/\partial x^2 = 0$
- Solutions: $\Phi(x) = cx + d$
- Q: Can we *always* satisfy given Dirichlet boundary conditions?



- Yes: a line can interpolate any two points.

1D Laplace w/ Neumann BCs

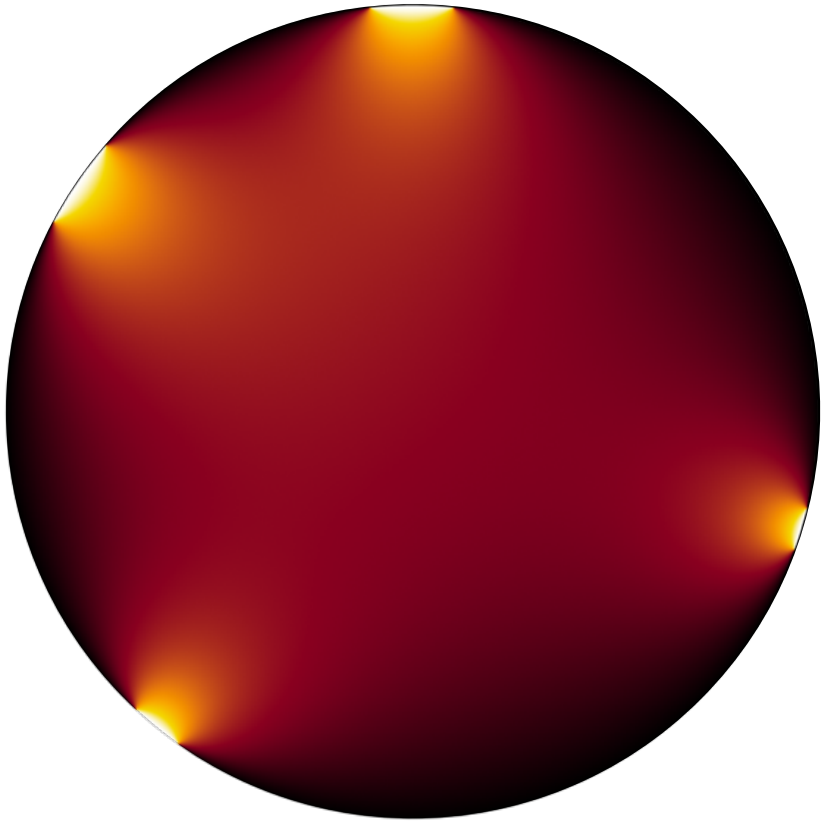
- What about Neumann BCs?
- Q: Can we prescribe the derivative at both ends?



- No! A line has only one slope.
- In general, solution to a PDE may not exist for given BCs.

2D Laplace w/ Dirichlet BCs

- 2D Laplace: $\Delta \Phi = 0$
- Q: Can satisfy any Dirichlet BCs? (given data along boundary)

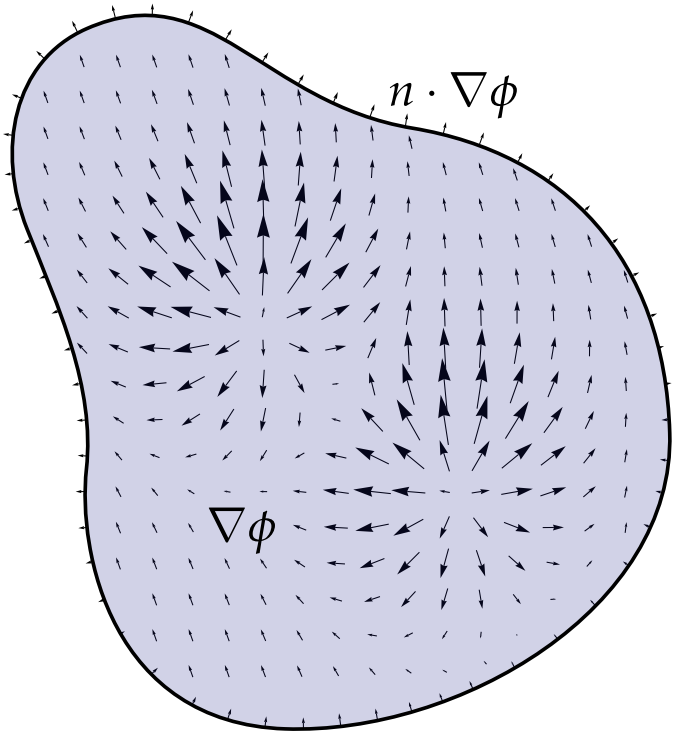


- Yes: Laplace is long-time solution to heat flow
- Data is "heat" at boundary. Then just let it flow…

2D Laplace w/ Neumann BCs

- What about Neumann BCs for $\Delta\Phi=0$?
- Neumann BCs prescribe derivative in normal direction: $n\cdot\nabla\Phi$
- Q: Can it always be done? (Wasn't possible in 1D...)
- In 2D, we have the *divergence theorem:*

$$\int_{\partial\Omega} n\cdot\nabla\phi = \int_{\Omega} \nabla\cdot\nabla\phi = \int_{\Omega} \Delta\phi \stackrel{!}{=} 0$$



- Should be called, "what goes in must come out theorem!"
- Can't have a solution unless the net flux through the boundary is zero.
- Numerical software will *not* always tell you if there's a problem! (Especially if you wrote it yourself...)

Solving the Heat Equation

- Back to our three model equations, want to solve *heat eqn.*

$$\dot{u} = \Delta u$$

- Just saw how to discretize Laplacian
- Also know how to do time (forward Euler, backward Euler, ...)
- E.g., forward Euler:

$$u^{k+1} = u^k + \Delta u^k$$

- Q: On a grid, what's our overall update now at $u_{i,j}$?

$$u_{i,j}^{k+1} = u^k + \tfrac{\tau}{h^2}\left(4u_{i,j}^k - u_{i+1,j}^k - u_{i-1,j}^k - u_{i,j+1}^k - u_{i,j-1}^k\right)$$

- *Not* hard to implement! Loop over grid, add up some neighbors.

Solving the Wave Equation

- Finally, wave equation:

$$\ddot{u} = \Delta u$$

- Not much different; now have 2nd derivative in time
- By now we've learned two different techniques:
  - Convert to two 1st order (in time) equations:

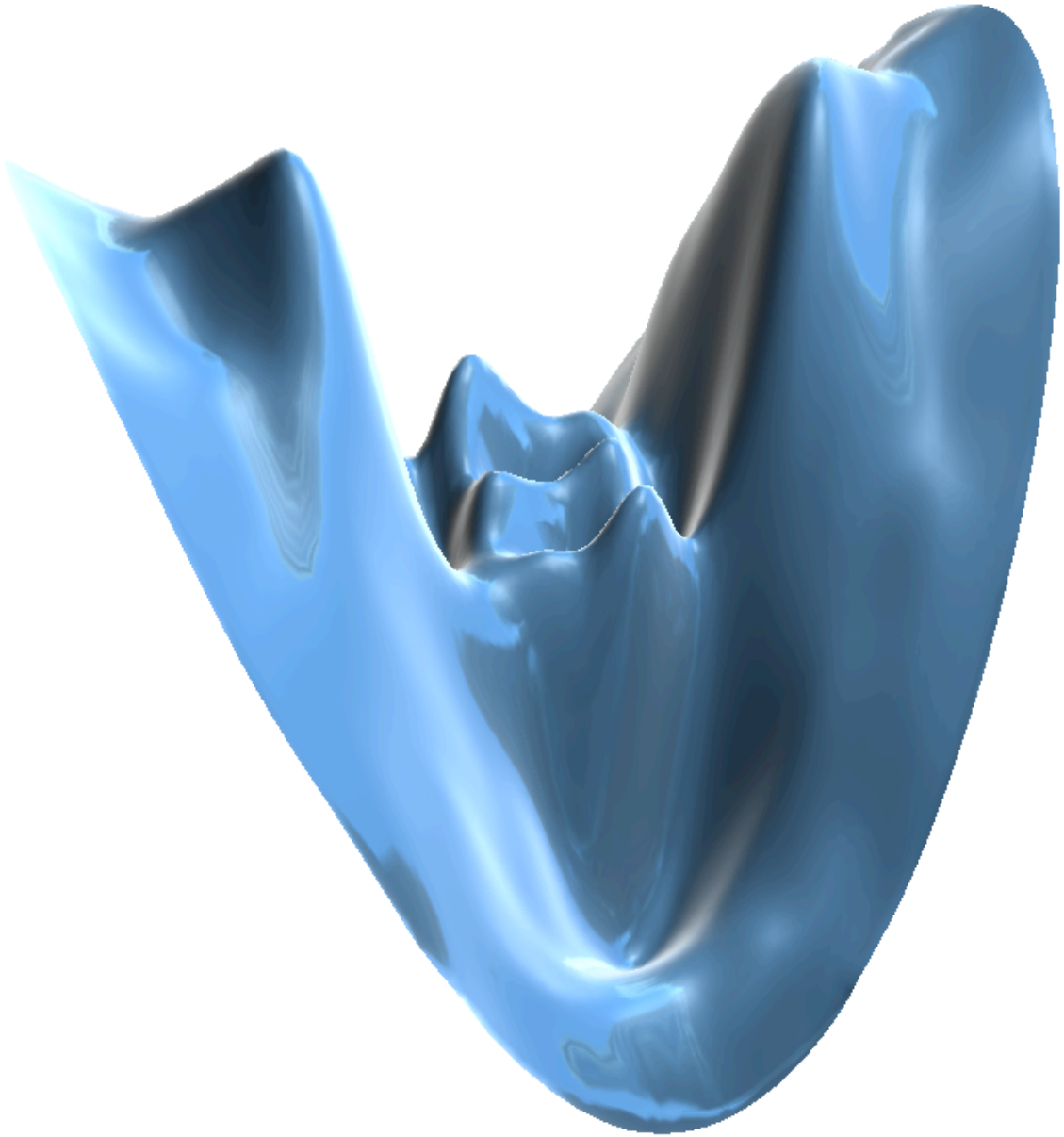
$$\dot{u} = v, \quad \dot{v} = \Delta u$$

  - Or, use centered difference (like Laplace) in time:

$$\frac{u^{k+1} - 2u^k + u^{k-1}}{\tau^2} = \Delta u^k$$

- Plus all our choices about how to discretize Laplacian.
- *So many choices!* And many, many *(many)* more we didn't discuss.

DEMO: Model PDEs on a Triangle Mesh

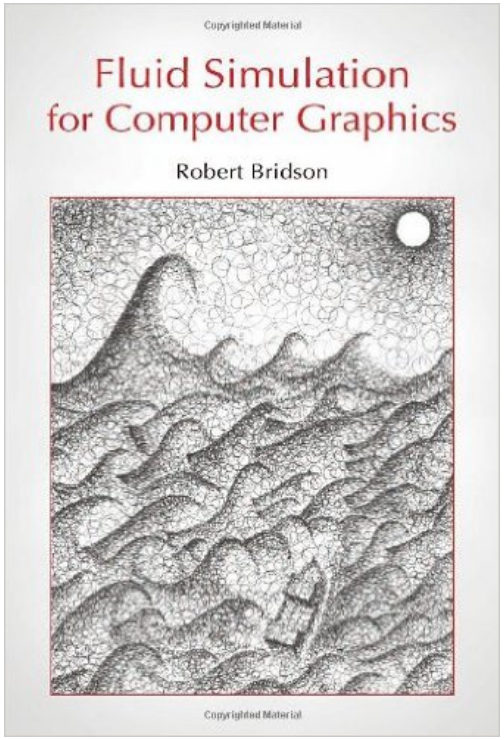


Wait, what about all those cool fluids and stuff?

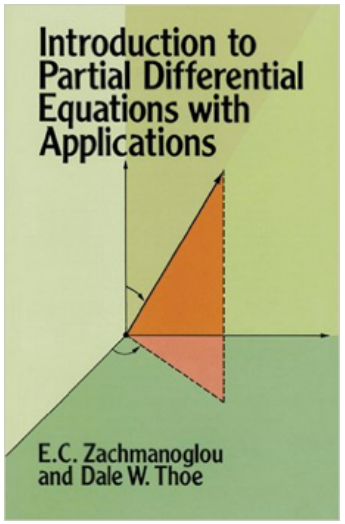
Want to Know More?

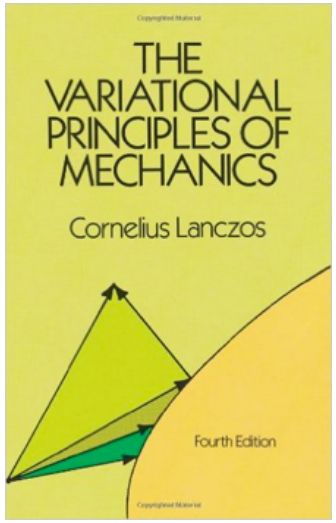
- There are some good books:
- And papers:

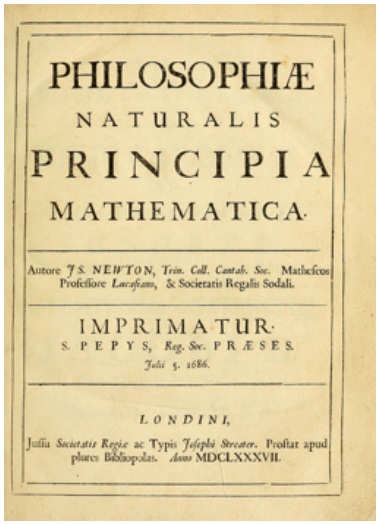
http://www.physicsbasedanimation.com/



- Also, what did the folks who *wrote* these books & papers read?







Next up: solving linear equations