Numerical Methods in Engineering Applications

Workshop #01-2:

Python & Finite differences

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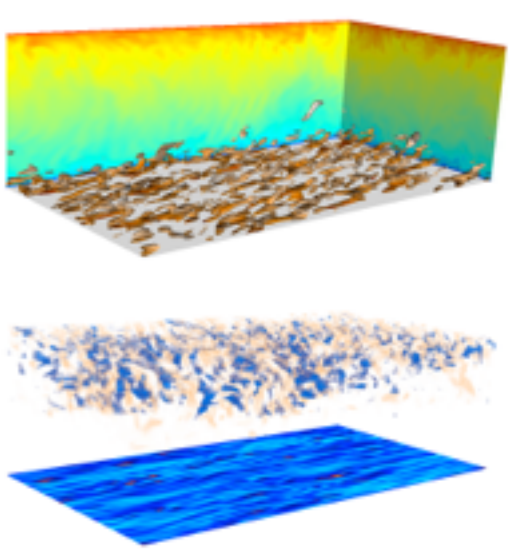
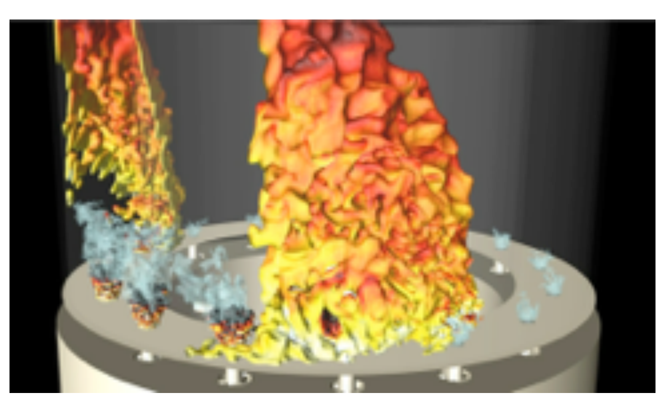
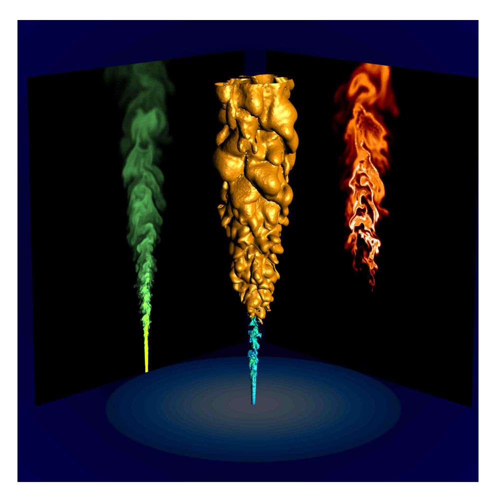
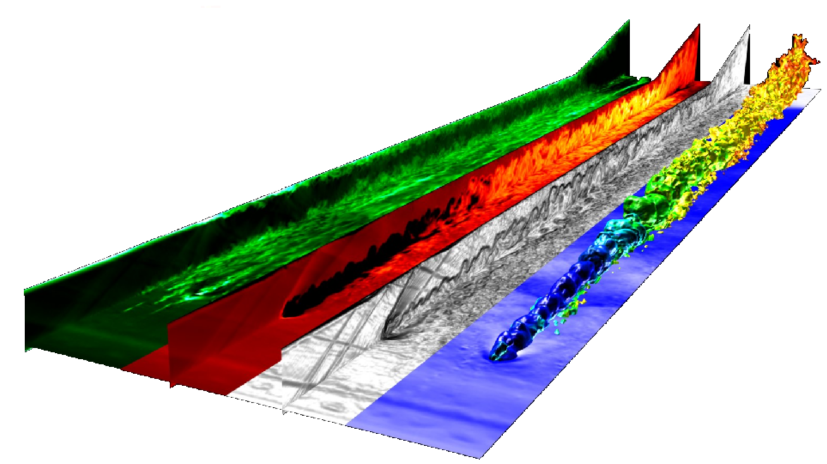
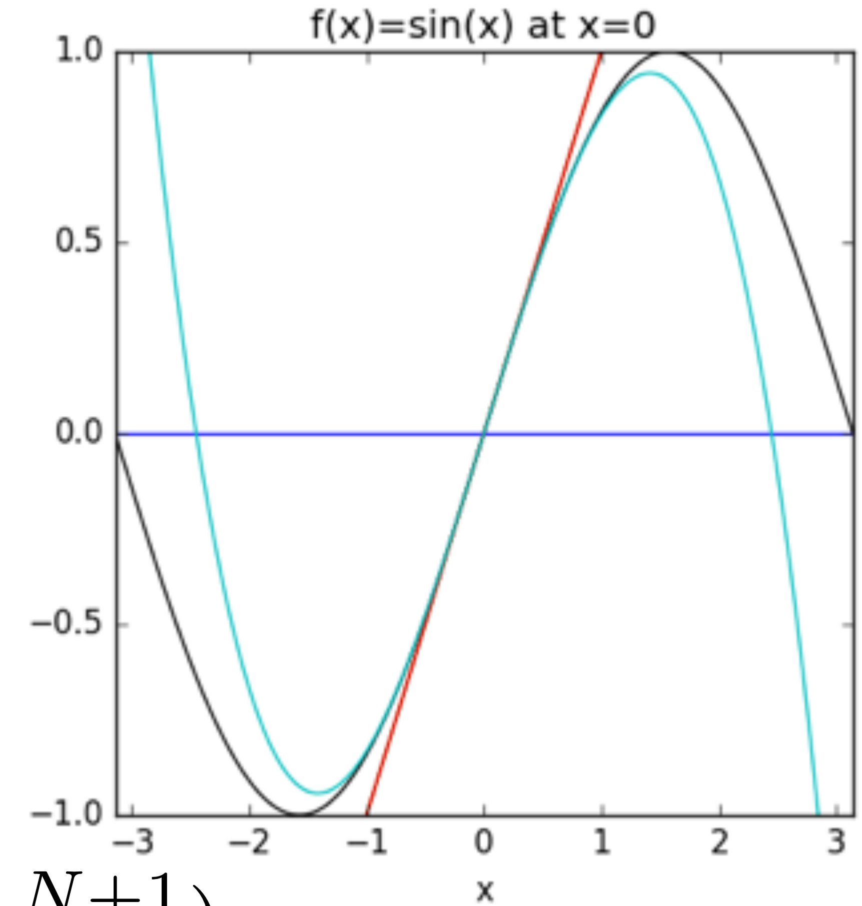





Objectives of Workshop #1

- **Finite differences**
  - Derivation of finite difference formulae
  - Implementation
  - Validation and error evaluation

- **Introduction to Ordinary Differential Equation**
  - Derivation of Euler's explicit method
  - Implementation

Finite differences



- **Taylor series expansion on uniform mesh**

$$f(x+h) = f(x) + \sum_{k=1}^{\infty} \frac{h^k}{k!} \left.\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right|_x$$

- **Approximation of derivatives**
  - truncation at a given order

$$f(x+h) = f(x) + \sum_{k=1}^{N} \frac{h^k}{k!} \left.\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right|_x + \mathcal{O}(h^{N+1})$$

  - Low order approximation using single Taylor series

$$N=1, \quad h\frac{\mathrm{d}f}{\mathrm{d}x} + \mathcal{O}(h^2) = f(x+h) - f(x)$$

$$\Rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{f(x+h)-f(x)}{h} + \mathcal{O}(h)$$

  - Higher order using a composition of Taylor series at different h

Taylor tables

- **Determine the highest order approximation for a given stencil**

$$\frac{\mathrm{d}f}{\mathrm{d}x} + \sum_{k=-l}^{Q} a_k f(x+kh) = \mathcal{O}(h^?)$$

- **Taylor tables**

	$f(x)$	$f'(x)$	f''	$\dots$	$\mathrm{d}^l f/\mathrm{d}x^l$
f'	0	1	0	$\dots$	0
$f(x)$	a_0	0	0	$\dots$	0
$f(x+h)$	a_1	$a_1 h$	$a_1 h^2/2$	$\dots$	$a_1 h^l/l!$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$f(x+kh)$	a_k	$a_1 kh$	$a_1 (kh)^2/2$	$\dots$	$a_1 (kh)^l/l!$

- **Nullify the sum in each column**

- **Order is given by the last column constraint that is not verified**

The explicit Euler's method for ODEs

- **Ordinary Differential Equations**

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = F(t, Y(t))$$

- **Time-marching method**

$$Y(t_{n+1}) = Y_n + \int_{t_n}^{t_{n+1}} F(\tau, Y(\tau))\mathrm{d}\tau$$

- **First order approximation of the RHS**

$$F(\tau, Y(\tau)) = F(t_n, Y(t_n))$$

- **First order Explicit Euler method**

$$Y(t_{n+1}) = Y_n + \Delta t F(t_n, Y(t_n))$$

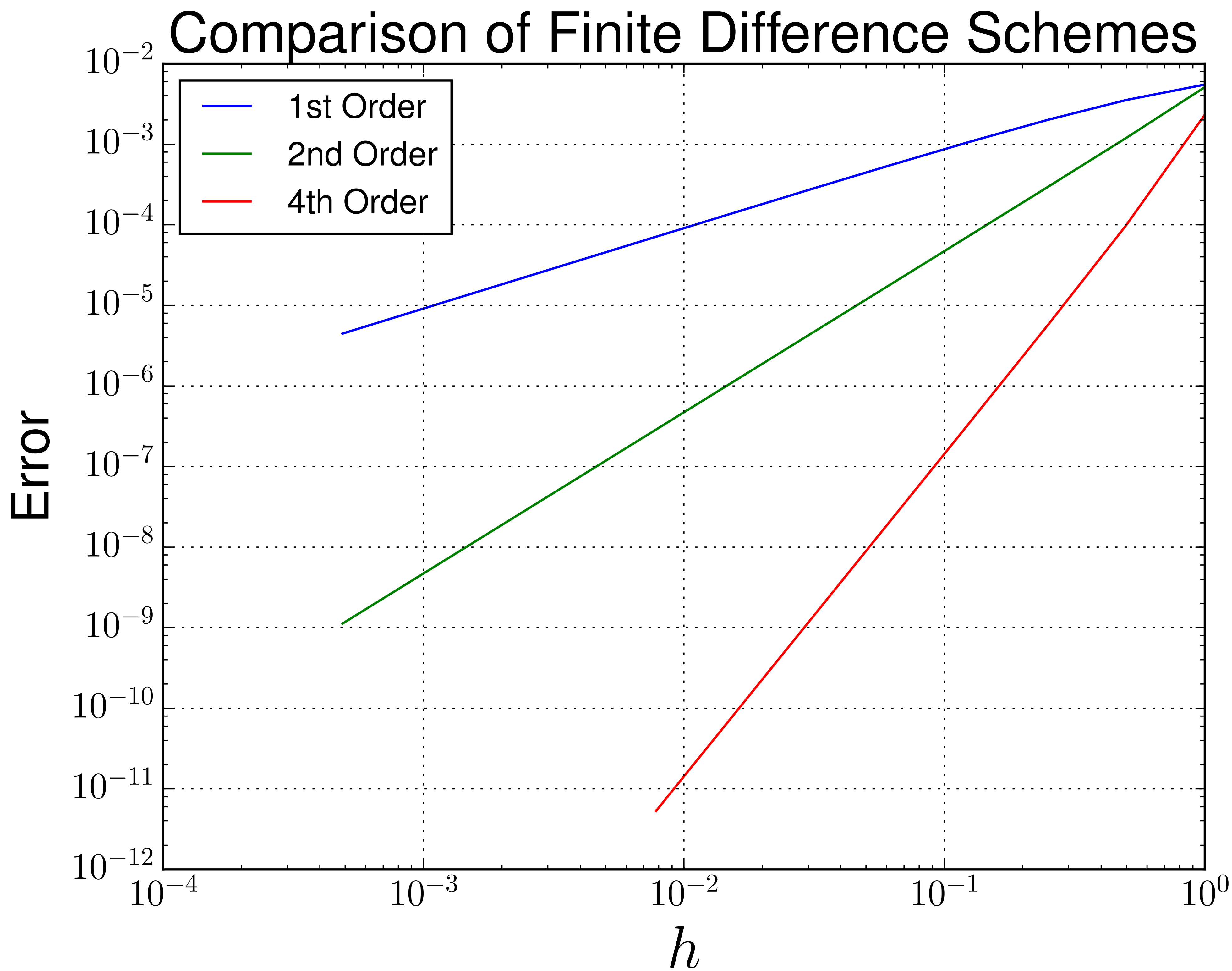
Workshop 1-2

- **Derivation of finite difference approximations for first and second order derivatives**

- **Implementation and evaluation of the order of accuracy**

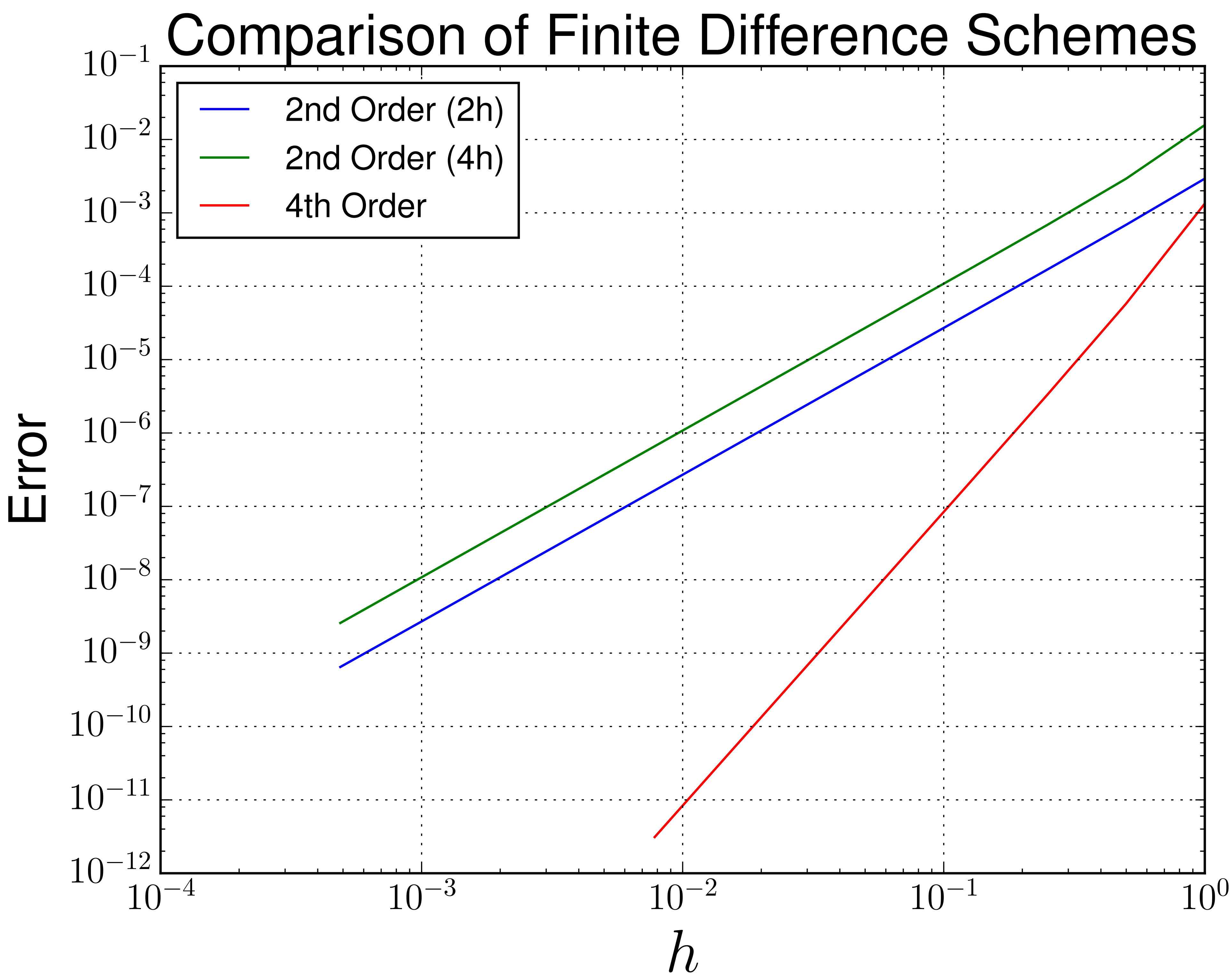
- **Introduction to Ordinary Differential Equations**

f'(x)

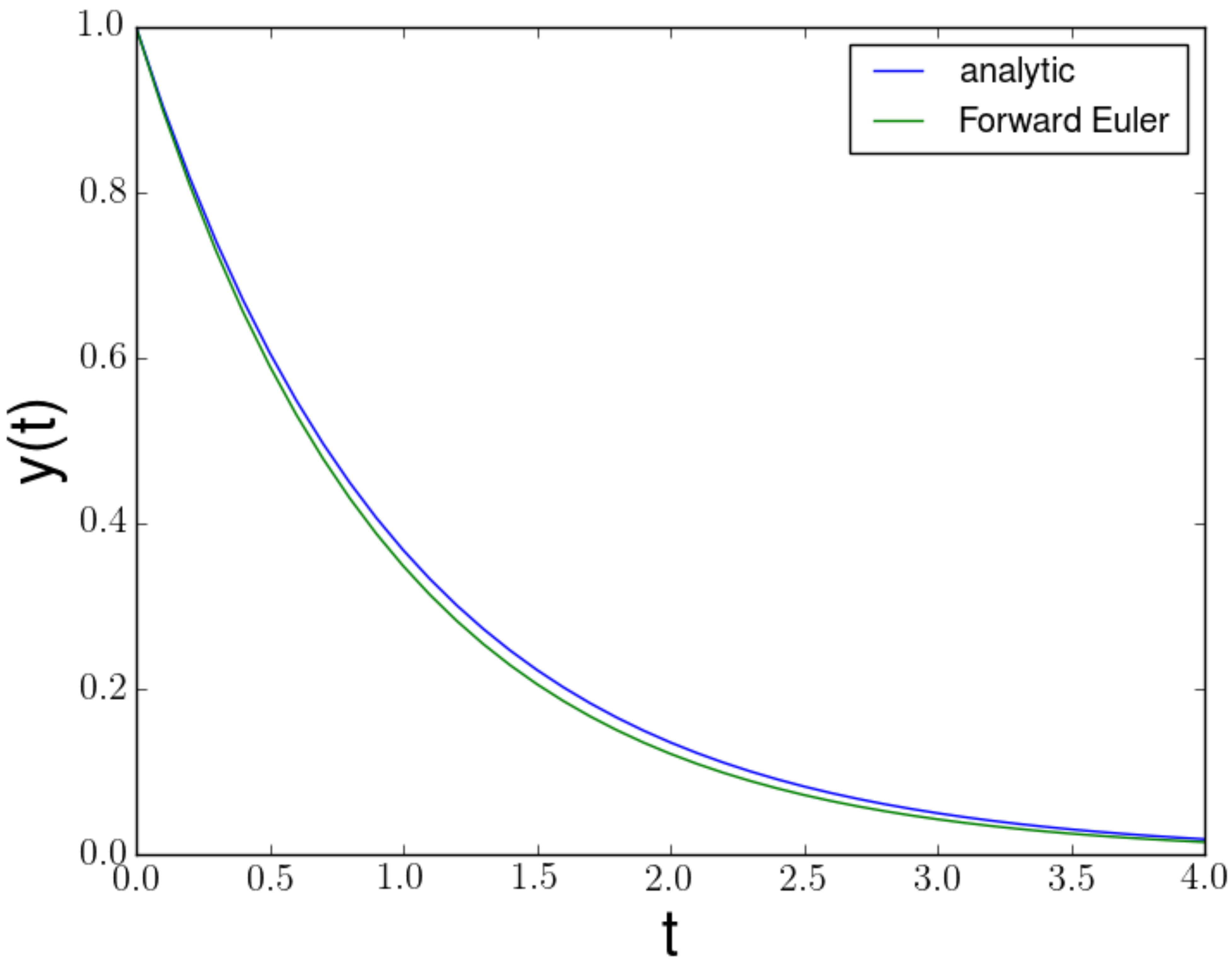


Comparison of Finite Difference Schemes
1st Order
2nd Order
4th Order
Error
h
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

f''(x)


Comparison of Finite Difference Schemes
2nd Order (2h)
2nd Order (4h)
4th Order
Error
h
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

ODE



analytic
Forward Euler
y(t)
t
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

ODE