CentraleSupelec **Numerical Methods, S8**

Elliptic Partial Differential Equations (2)

Highlights:

- SOR method
- Conjugate gradient method
- Multi-block

The goal of the present workshop is to solve elliptic partial differential equations of the form:

$$\nabla^2\phi = f(\mathbf{x}) \tag{1}$$

with more advanced iterative methods than the basic Jacobi and Gauss-Seidel methods. The multi-block problem is left as homework.

1 Poisson's equation with SOR and conjugate gradient methods

Let's consider the finite difference formulation of the Poisson equation seen in Workshop 3:

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = f(x_i, y_j) = f_{i,j} \tag{2}$$

where Δx and Δy are the spacing in x and y directions.The objective is to implement two methods: the SOR method that is a overrelaxed modification of the Gauss-Seidel method, and the Conjugate Gradient method (a Krylov projection method for symmetric systems).

1. First, modify the Gauss-Seidel Algorithm from Workshop 3 to implement the SOR method.
2. Evaluate the influence of the parameter ω on the convergence rate of the method. Find the best choice for you problem and compare SOR to Jacobi and Gauss-Siedel methods.
3. Following the Appendix, implement the Conjugate Gradient method. To simplify the implementation, you will have to generate a function that compute the Laplacian of a given 2D scalar field in a matrix-free manner.
4. Compare the Conjugate Gradient method to SOR, Jacobi and Gauss-Siedel, and comment on the convergence rate.

2 Multi-block resolution

Here, a domain decomposition with a multi-block structure mesh is illustrated with the study the cooling of the Vulcain engine, main stage engine of the Arian V space launcher. The hot exhaust gases in the nozzle downstream the engine apply incredible heat loads on the structure which must be cooled (in fact, by using the cryogenic hydrogen fed into the combustion chamber) to sustain the tremendous heat flux. A cross-section of the nozzle geometry and its cooling channels is represented in Figure 1a. Instead of solving this complex problem, making use of the symmetry of the problem helps to restrict to the geometry in Figure 1b, with the represented boundary conditions. To handle such a geometry, we decomposed the problem into structured blocks that will be solve iteratively. Each is connected to each other by a continuity condition. The sole differences between each block are the boundary condition. The objective is to compute the temperature distribution in all blocks, which is done using the Laplace equation with the appropriate boundary conditions.

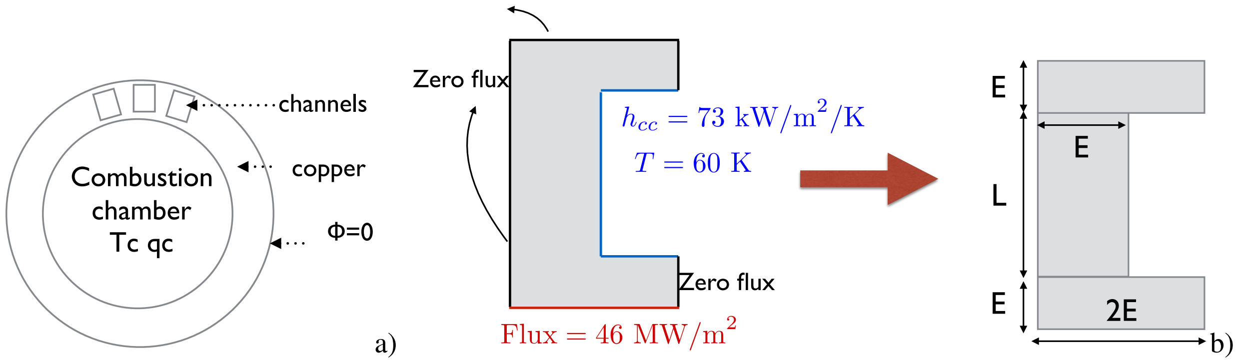

Figure 1: Cross-section of the Vulcain nozzle: The nozzle made of copper is cooled through channels (full (a) and simplified (b) representations). Geometric dimensions: $E = 2$ mm and $L = 10$ mm.

- Start by dealing with two blocks. For each iteration in the global iterative method, the temperature fields in the first block and then the second block are updated sequently with the solver of your choice (Jacobi, Gauss-Seidel, SOR...). The interface shared by the two blocks requires a specific attention: It is yet an undefined boundary condition for each part of the domain. The most simple choice is to set a Dirichlet boundary condition for one block (fixing the temperature values from other block solution) and a Neumann boundary condition for the other block (fixing the temperature derivative from the former block solution).
- Discretization the geometry in Figure 1b, compute the maximum temperature in the domain using a multi-block decomposition. The thermal conductivity of the copper is 366 $W.m^{-1}.K^{-1}$.

A Iterative solvers

Let us consider the following discretized form:

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = f_{i,j} \tag{3}$$

A.1 Succesive OverRelaxation (SOR) method

The SOR method is an overrelaxation of the Gauss-Seidel method:

$$\phi_{i,j}^{(k+1)} = (1-\omega)\phi_{i,j}^{(k)} + \frac{\omega}{\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}} \left(\frac{\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k+1)}}{\Delta x^2} + \frac{\phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k+1)}}{\Delta y^2} - f_{i,j} \right) \tag{4}$$

where $\omega \in]0, 2[$.

A.2 Conjugate gradient method

Let us first define the residual of iteration k:

$$r_{i,j}^{(k)} = f_{i,j} - \frac{\phi_{i+1,j}^{(k)} - 2\phi_{i,j}^{(k)} + \phi_{i-1,j}^{(k)}}{\Delta x^2} + \frac{\phi_{i,j+1}^{(k)} - 2\phi_{i,j}^{(k)} + \phi_{i,j-1}^{(k)}}{\Delta y^2} \tag{5}$$

We also define the application of the RHS to a given vector $p_{i,j}^{(k)}$:

$$(A \cdot p)_{i,j}^{(k)} = \frac{p_{i+1,j}^{(k)} - 2p_{i,j}^{(k)} + p_{i-1,j}^{(k)}}{\Delta x^2} + \frac{p_{i,j+1}^{(k)} - 2p_{i,j}^{(k)} + p_{i,j-1}^{(k)}}{\Delta y^2} \tag{6}$$

Let us consider $p_{i,j}^{(0)} = r_{i,j}^{(0)}$. The iterations of Conjugate Gradient method are then the following:

$$\alpha^{(k)} = \frac{\sum_{i,j} r_{i,j}^{(k)} r_{i,j}^{(k)}}{\sum_{i,j} p_{i,j}^{(k)} \left(A \cdot p\right)_{i,j}^{(k)}} \tag{7}$$

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \alpha^{(k)} p_{i,j}^{(k)} \tag{8}$$

$$r_{i,j}^{(k+1)} = r_{i,j}^{(k)} - \alpha^{(k)} \left(A \cdot p\right)_{i,j}^{(k)} \tag{9}$$

$$\beta^{(k)} = \frac{\sum_{i,j} r_{i,j}^{(k+1)} r_{i,j}^{(k+1)}}{\sum_{i,j} r_{i,j}^{(k)} r_{i,j}^{(k)}} \tag{10}$$

$$p_{i,j}^{(k+1)} = r_{i,j}^{(k+1)} + \beta^{(k)} p_{i,j}^{(k)} \tag{11}$$