CentraleSupelec **Numerical Methods, S8**

Parabolic and hyperbolic PDEs: Explicit methods

Highlights:

- Solving 1D advection and diffusion problems
- Fourier criterion
- CFL condition

The goal of the present workshop is to first solve simple parabolic and hyperbolic problems where analytical solutions are known: 1D unsteady heat equation and 1D advection. The stability criteria are illustrated with these examples using different numerical methods. The last problem considers a parabolic problem of paramount physical importance: the thin boundary layer in the presence of pressure gradient.

1 1D Unsteady diffusion

Let's consider the unsteady heat equation for the scaled temperature profile $\hat{T} = \frac{T-T_w}{T_c^0-T_w}$:

$$\frac{\partial \hat{T}}{\partial t} = a\frac{\partial^2 \hat{T}}{\partial x^2},$$

with the corresponding boundary conditions

$$\begin{cases} \hat{T}(x=0,t) & = 0 \\ \hat{T}(x=L,t) & = 0 \end{cases},$$

and initial profile,

$$\hat{T}(x,0) = \sin\left(\frac{\pi x}{L}\right).$$

The length of the domain is $L = 1$ cm and the thermal diffusivity is $a = 10^{-4}$ m^2/s. The analytical solution of the problem is known and is given by:

$$\hat{T}(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{-\frac{\pi^2}{L^2}at}$$

- Solve the discrete parabolic PDE using forward Euler for time integration and the centered-2$^{\text{nd}}$-order difference formula for space discretization.
- Check the importance of setting the time step by selecting the value of the Fourier number for the sake of numerical stability.
- Compare the numerical solution $\hat{T}(x = L/2, t)$ with the analytical solution.
- Multiply the number of spatial points by 10, what is the critical issue of using explicit methods to tackle such a parabolic PDE?

2 1D advection

Advection of a profile with constant velocity c is described by the following PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

The initial profile is set as

$$u(x, t = 0) = u_0(x) = \exp\left(-\frac{(x - x_0)^2}{\delta^2}\right),$$

with $x_0 = 1$ m and $\delta = 0.2$ m. The domain length is $L = 12$ m and the advection velocity is set to $c = 10$ m/s. Since the solution is conveyed from left to right, a boundary condition is required only on the left side of the domain, a particularity of the hyperbolic nature of the PDE. This boundary condition is set as

$$u(x = 0, t) = 0.$$

The analytical solution is a translation of the initial profile:

$$u(x, t) = u_0(x - ct).$$

- Check the conditional stability of the discretized problem with forward Euler for time integration and 1$^{\text{st}}$-order upwind for space discretization.

- Implement a second-order centered space discretization instead. What do you observe?
- Implement the second-order centered space discretization along with the leap-frog scheme.
- Implement the second-order centered space discretization along with the third-order Runge-Kutta method RK3.
- Choose any combination and test it. Is the stability behavior consistent with the theory?
- For all tested methods, plot on the same figure the different solutions and the analytical one after half a residence time in the computational domain, *i.e.* $t = L/(2c)$. What do you think of the quality of your results? Solutions exhibit strong numerical diffusion and/or dispersion.

3 Thin boundary layer in the presence of pressure gradient

Boundary layers determine many flow characteristics in fluid mechanics such as head losses in pipes, drag on moving objects, stall of a plane, viscous heating around supersonic vehicles ... Boundary layers are typically very thin (if they remain attached) and are characterized by the presence of viscous effects that cannot be neglected. For laminar thin boundary layers, the bidimensional flow over a flat plate is described by the velocity components (u, v), and is governed by (i) the continuity equation (a.k.a. local balance equation of mass) which is written for incompressible flows as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

and (ii) by the balance of momentum along the x-direction:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}, \tag{2}$$

where ρ is the density, ν the kinematic viscosity and $\frac{\mathrm{d}p}{\mathrm{d}x}$ is the pressure gradient set by the outer flow onto the boundary layer. For water, $\rho = 1000$ kg/m^3 and $\nu = 10^{-6}$ m^2/s. The general steady Navier-Stokes equations would have yielded a non-linear elliptic set of PDEs, which is cumbersome to solve. Thanks to the thin layer approximation, the obtained equations 1 and 2 are parabolic and can be solved step by step in the x-direction starting from an incoming profile at $x = 0$.

3.1 No pressure gradient

Many theoretical results are available when the pressure gradient is null or is negligible. Indeed, the velocity profiles correspond to the solution of the Blasius solution, which is obtained by looking for auto-similar solutions of equations 1 and 2. The method is here different: Equations 1 and 2 are solved directly without further modifications.

The computational domain is discretized with $N_y + 1$ points in the y-direction between $y = 0$ (the wall location) and $y = H$ the outer boundary condition where the velocity can be assumed uniform. The corresponding y-grid spacing is denoted by Δy. Adopting a step-by-step approach, $N_x + 1$ iterations are carried out in the x-direction starting from the inlet $x = 0$ up to the last computed profile corresponding to the domain x-extent L. The corresponding x-grid spacing is denoted by Δx.

The discrete velocity fields $u_{i,j}$ and $v_{i,j}$ require "initial" profiles and boundary conditions as for any parabolic PDE. The "initial"profile is the inlet velocity profile and is set as a uniform flow:

$$u(x = 0, y) = u_{0,j} = U_{in}$$

$$v(x = 0) = v_{0,j} = 0.$$

The boundary conditions are

$$u(x, y = 0) = u_{i,0} = 0$$

$$v(x, y = 0) = v_{i,0} = 0$$

$$u(x, y = H) = u_{i,N} = U_{ext}(x),$$

where $U_{ext}(x)$ is the external flow profile which can change due to the pressure gradient. With $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$, $U_{ext}(x) = U_{in}$.

- Consider equation 2 to determine a formula to evaluate $u_{i+1,j}$ from known data at $x = x_i$ (examples: $u_{i,j}$, $u_{i,j+1}$, $u_{i,j-1}$, $v_{i,j}$, $v_{i,j-1}$...) using a Forward Euler integration. Define the dimensionless numbers equivalent to the Courant and Fourier numbers.

- Use then equation 1 to obtain a formula to determine $v_{i+1,j}$ from the computed values $u_{i+1,j}$ and the data at $x = x_i$.

- Implement the derived numerical resolution and plot different computed profiles $u(x, y)$ at different axial positions. Be careful with the stability by controlling Δx and monitoring the effective Courant and Fourier numbers properly.

- The profiles show that the boundary layer thickens. Estimate the boundary layer thickness $\delta_{99}(x)$ defined as the point where $u(x, y = \delta_{99}(x)) = 0.99 U_{ext}(x)$. The theoretical behavior $\delta_{99}(x) \approx \sqrt{x}$ is expected away from the leading edge (x=0).

3.2 Effects of pressure gradient

The velocity profiles in the boundary layer are greatly impacted by the presence of a pressure gradient in the external flow. The boundary layer can then thicken faster or can get thinner. The flow can even separate from the wall, yielding a large recirculation bubble with negative values of the streamwise velocity u. In this last case, equation 2 breaks down and cannot be solved beyond the separation point.

Accounting for the pressure gradient modifies the outer boundary condition at $y = H$,

$$u_{i,N} = U_{ext}(x),$$

since $U_{ext}(x)$ changes accounting to the Bernoulli theorem:

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{1}{\rho U(x)}\frac{\mathrm{d}p}{\mathrm{d}x}.$$

- Define a function that returns $\frac{\mathrm{d}p}{\mathrm{d}x}$ which is null in the first half of the computed domain (between $x = 0$ and $x = L/2$) and non-null yet constant in the second half (between $x = L/2$ and $x = L$).
- Add the effect of $\frac{\mathrm{d}p}{\mathrm{d}x}$ and solve the case of a negative pressure gradient. Why is this called a favorable pressure gradient ?
- Solve the case of a positive pressure gradient. Why is this called a unfavorable pressure gradient ?