Numerical Methods in Engineering Applications

Session #2
Ordinary Differential Equations

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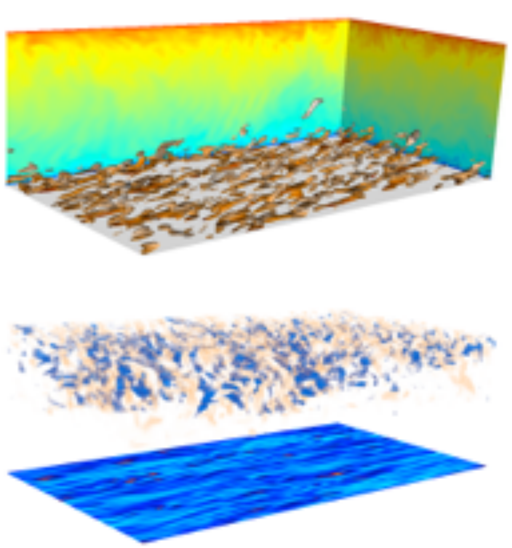
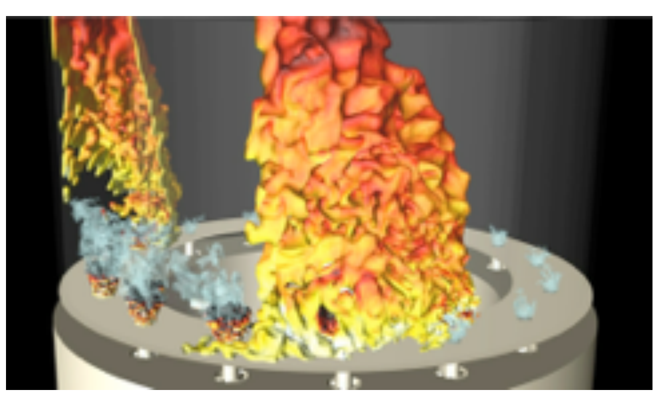
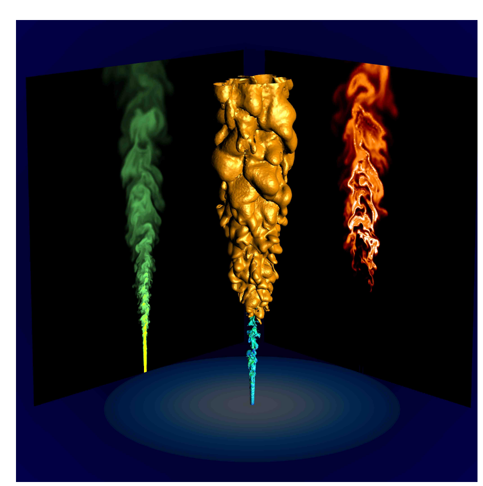
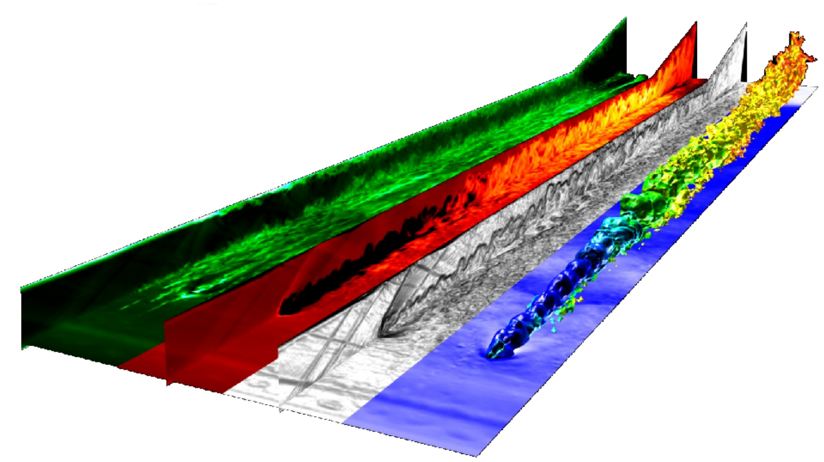





Course contents

● Theoretical lecture
■ Problem-solving workshop

I. Basics on numerical approximations

- Introduction and Finite Differences. ● ■ ■
- Numerical solution of ordinary differential equations. ● ■

II. Solving large linear equations systems: Applications to steady heat equation.

- Elliptic PDE 1. ● ■ ■
- Elliptic PDE 2. ● ■

III. Methods for unsteady advection/diffusion problems

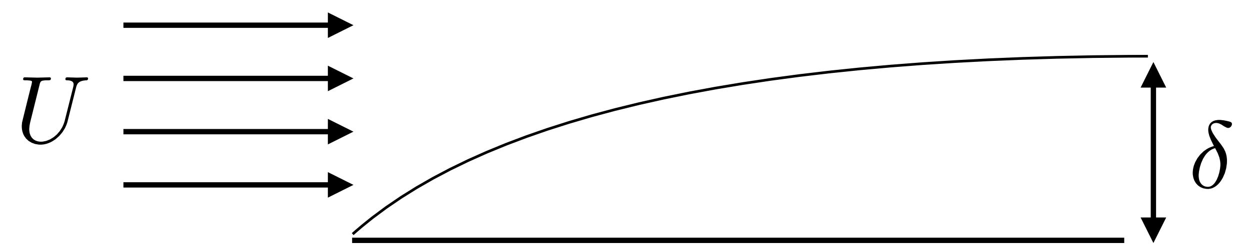
- Hyperbolic and parabolic PDE: Explicit methods. ● ■
- Characterization of numerical errors. ● ■
- Hyperbolic and parabolic PDE: Implicit methods. ● ■

IV. Towards computational fluid dynamics

- Methodology in numerical computations. ■ ■
- Incompressible Flow equations. ● ■
- Semi-Implicit method for incompressible flows. ● ■
- Final project on incompressible flow. ■ ■

Example : Blasius equation

- **Study of a laminar boundary layer on a flat plate**



- **Looking for auto-similar solutions yields the Blasius equation**

$$u(x,y) = U f'(\eta) \text{ où } \eta = \frac{y}{\sqrt{\frac{x\nu}{U}}}$$

Blasius: $$\boxed{f''' + \frac{1}{2} f f'' = 0}$$ **with** $$\left| \begin{array}{l} f(0) = 0 \\ f'(0) = 0 \\ f'(+\infty) = 1 \end{array} \right.$$

- **Resolution => velocity profiles and the skin friction coefficient**

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2f''(0)\left(\frac{Ux}{\nu}\right)^{-1/2} \Longrightarrow \boxed{f''(0) = ???}$$

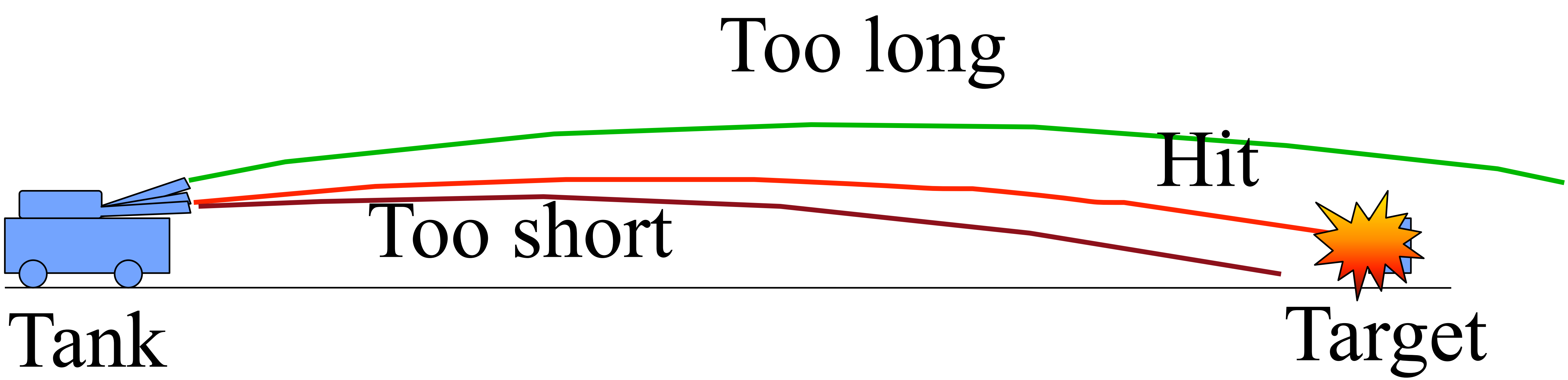
Example : Blasius equation

- Boundary value problem solved by a shooting method

$$f''' + \frac{1}{2} f f'' = 0$$

$$\left| \begin{array}{l} f(0) = 0 \\ f'(0) = 0 \\ f''(0) = s \end{array} \right. \qquad \left| \begin{array}{l} f(0) = 0 \\ f'(0) = 0 \\ f'(+\infty) = 1 \end{array} \right.$$

We are looking for s



Example : Blasius equation

- **Vectorial ODE, Non-linear, Boundary value problem**

Equations

$$\frac{dX}{d\eta} = Y$$

$$\frac{dY}{d\eta} = Z$$

$$\frac{dZ}{d\eta} = -\frac{1}{2}XZ$$

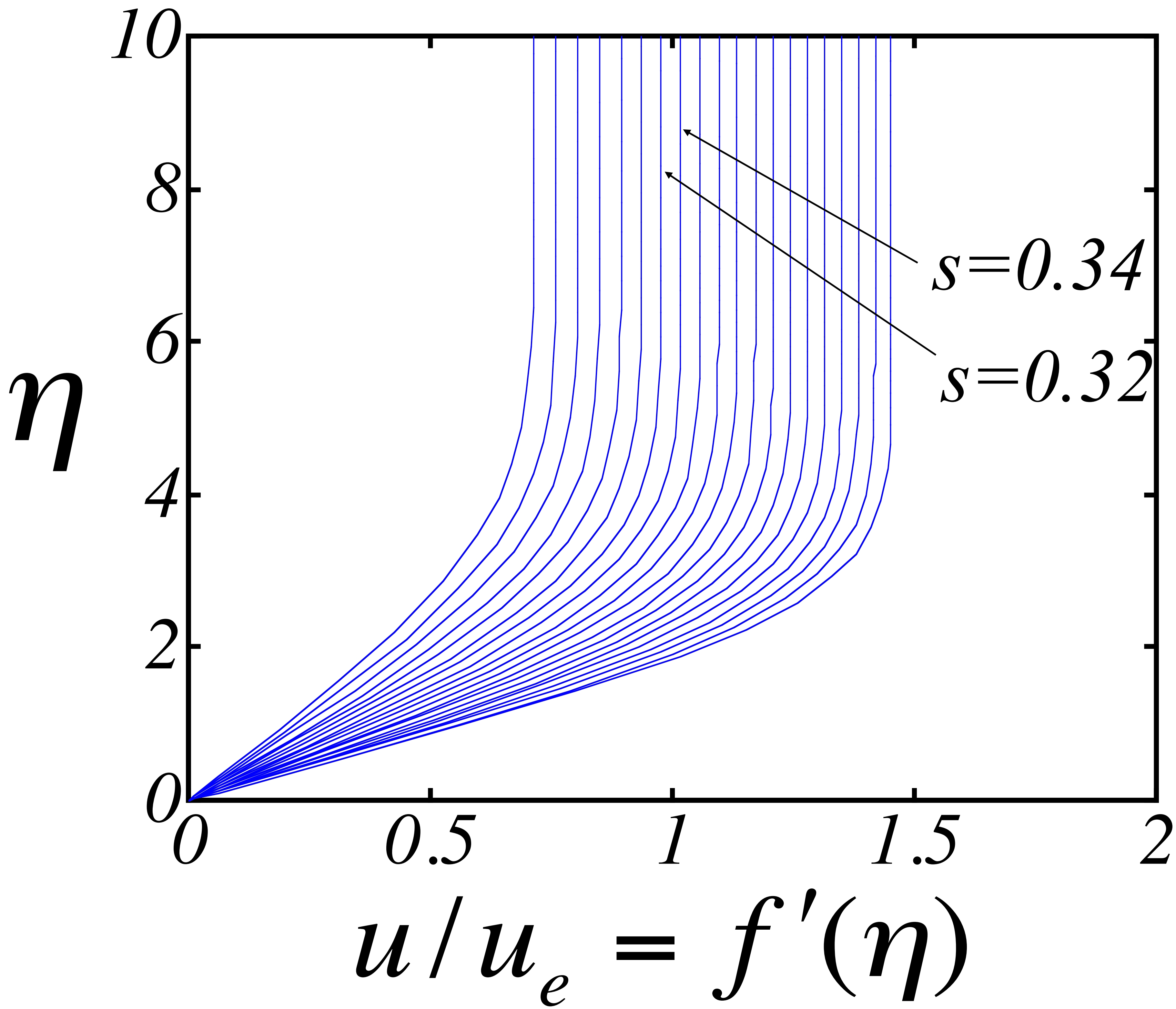
Initial conditions

$$X(0) = 0,$$

$$Y(0) = 0,$$

$$Z(0) = \boxed{s}$$

Example : Blasius equation



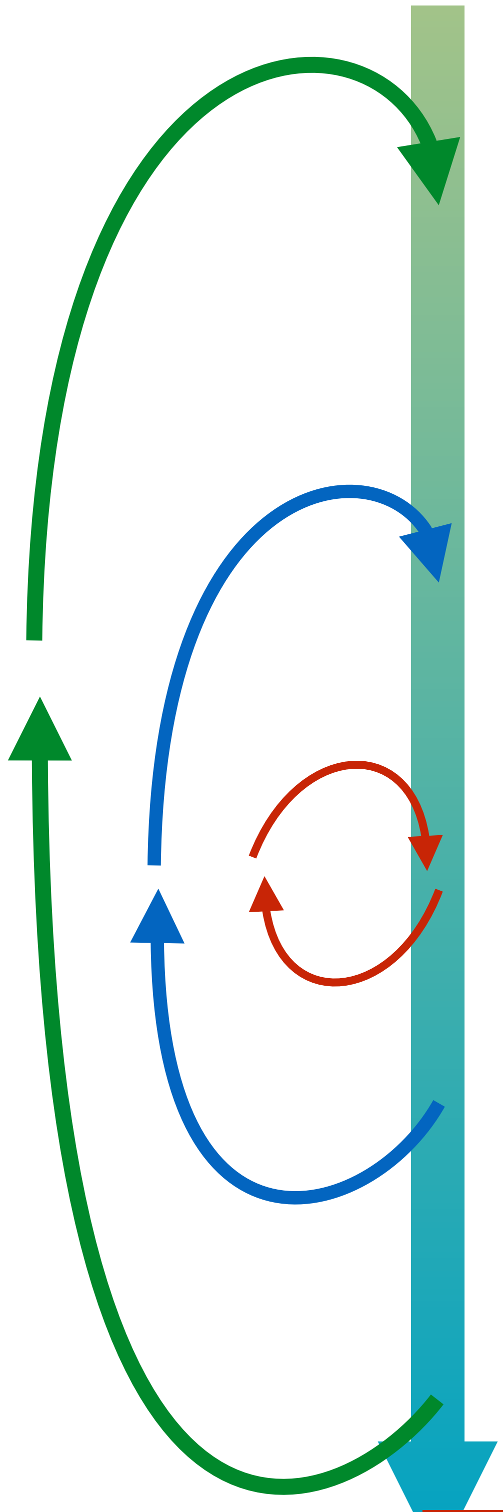
Today's contents

- Finite differences
  - Accuracy analysis: Modified wavenumber

- ODE
  - Forward Euler method
  - Numerical stability, Implicit vs Explicit
  - Backward Euler Method
  - Trapezoidal method
  - Linearization of implicit methods
  - Stiffness

Next time: Runge-Kutta and Multi-step methods

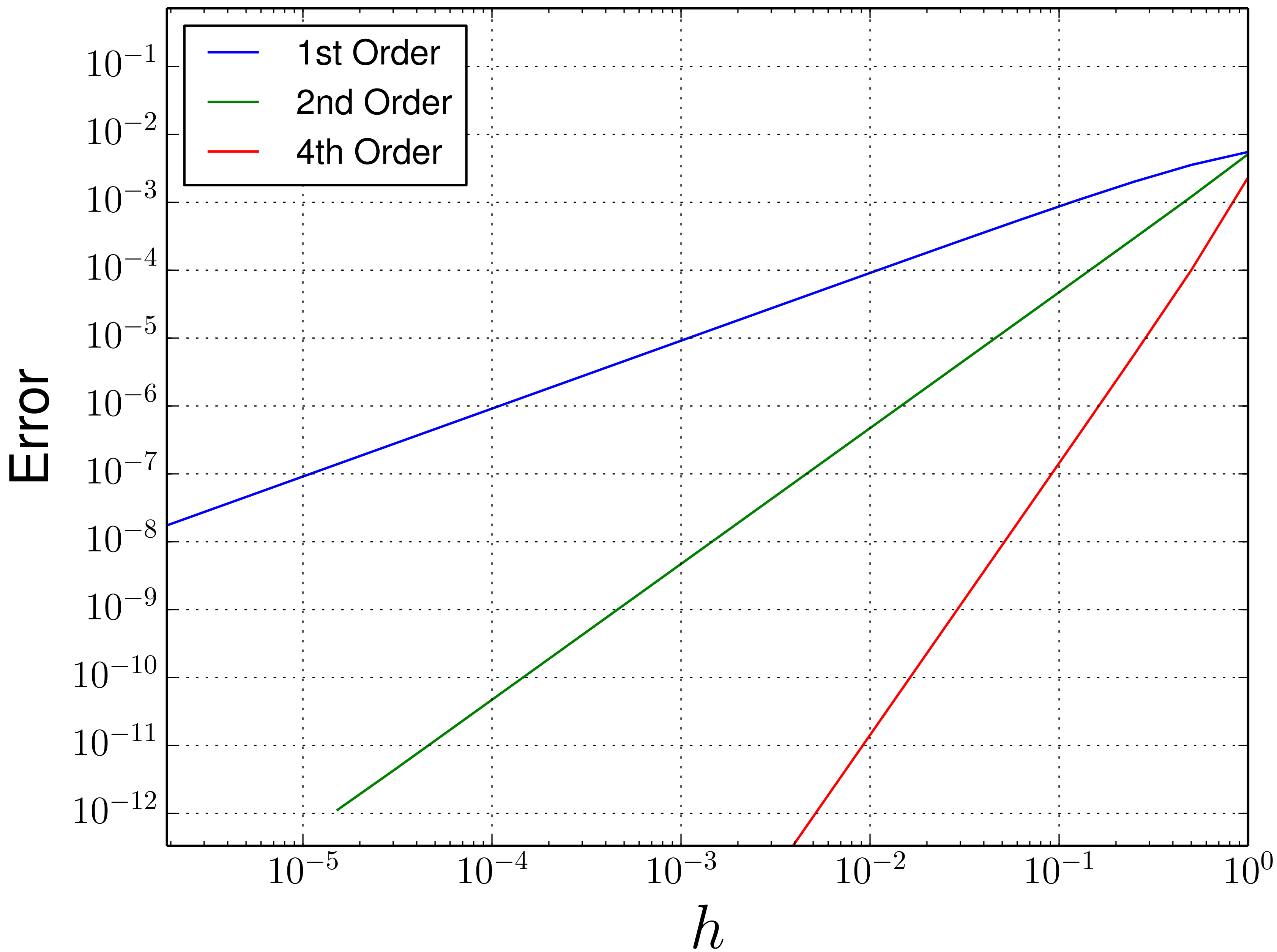
Methodology



- Set the physical problem
- Formalization: Equations and Hypotheses
- Simplifications, symmetries
- Discretization (structures, unstructured mesh)
- Boundary conditions. Initial conditions.
- **Numerical resolution** (finite differences, finite volumes, finite elements, ...)
- Characterization of numerical error (accuracy of the numerical scheme, mesh convergence/adaptation)
- Data post-processing
- Model validation, Verification of hypotheses

The problem answer

Accuracy order for finite difference formula

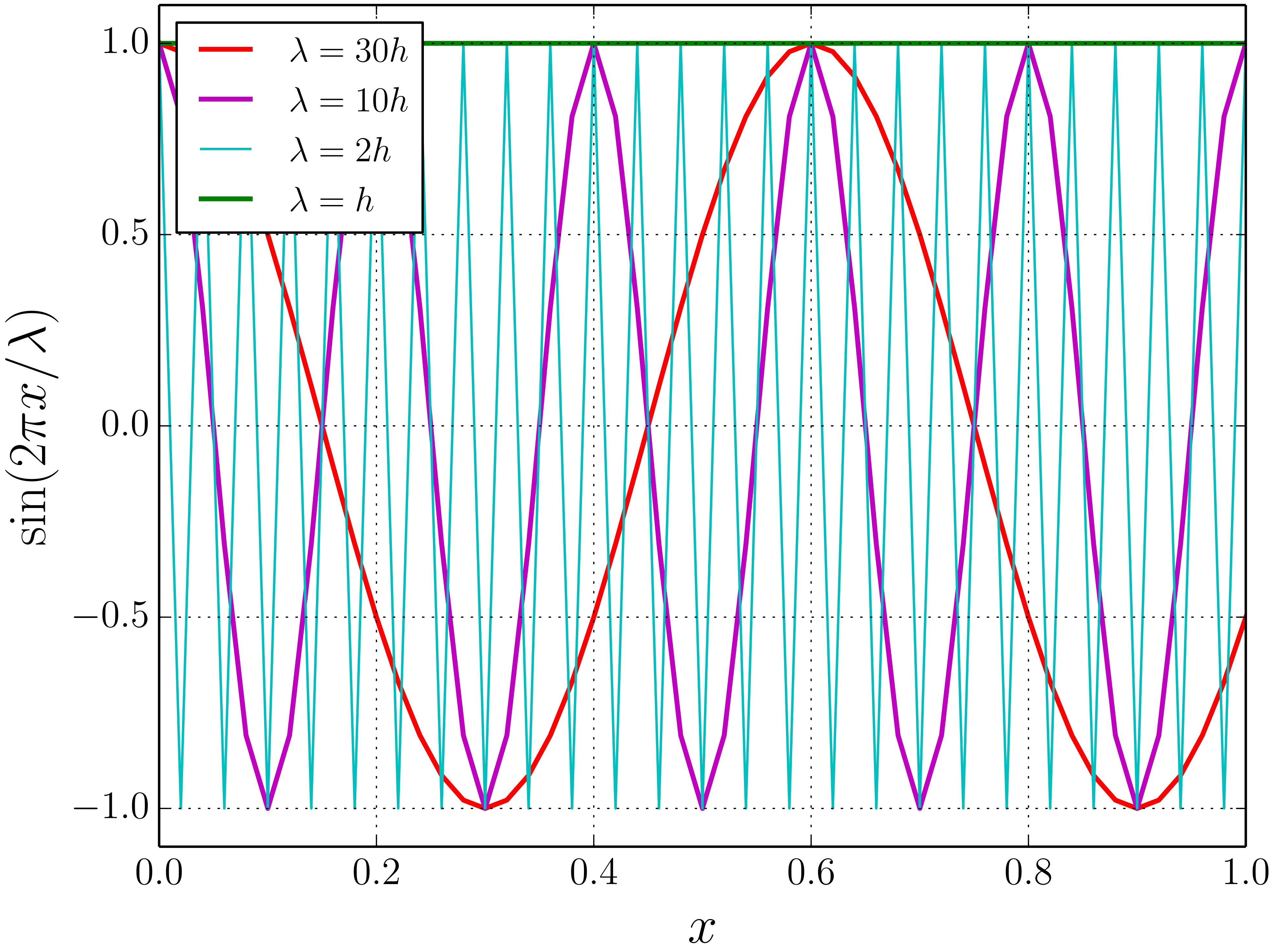


Appoximation of $f'(x)$

$$f(x) = \frac{\sin(x)}{x^3}$$

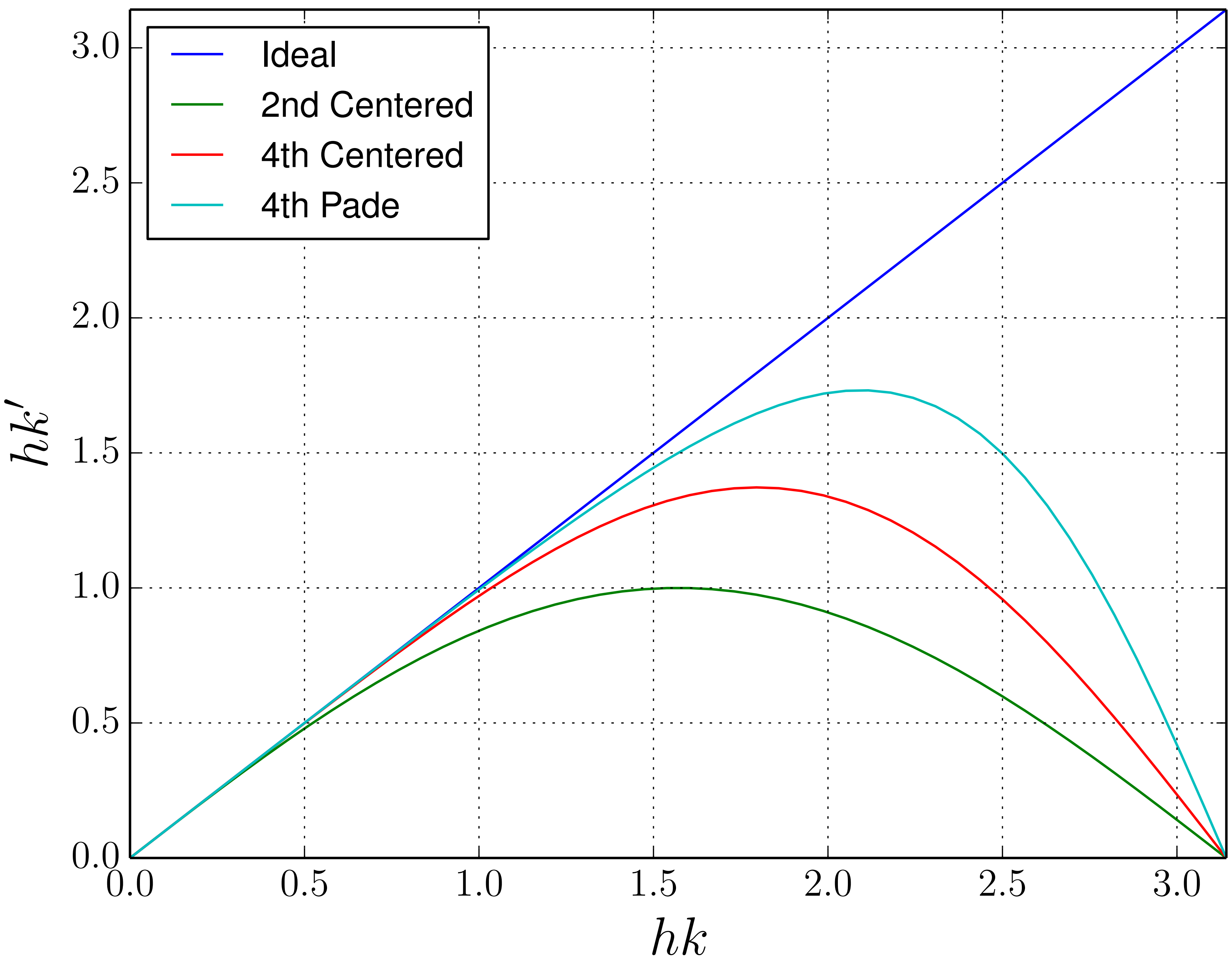
for $x = 4$

Nyqvist-Shannon criterion

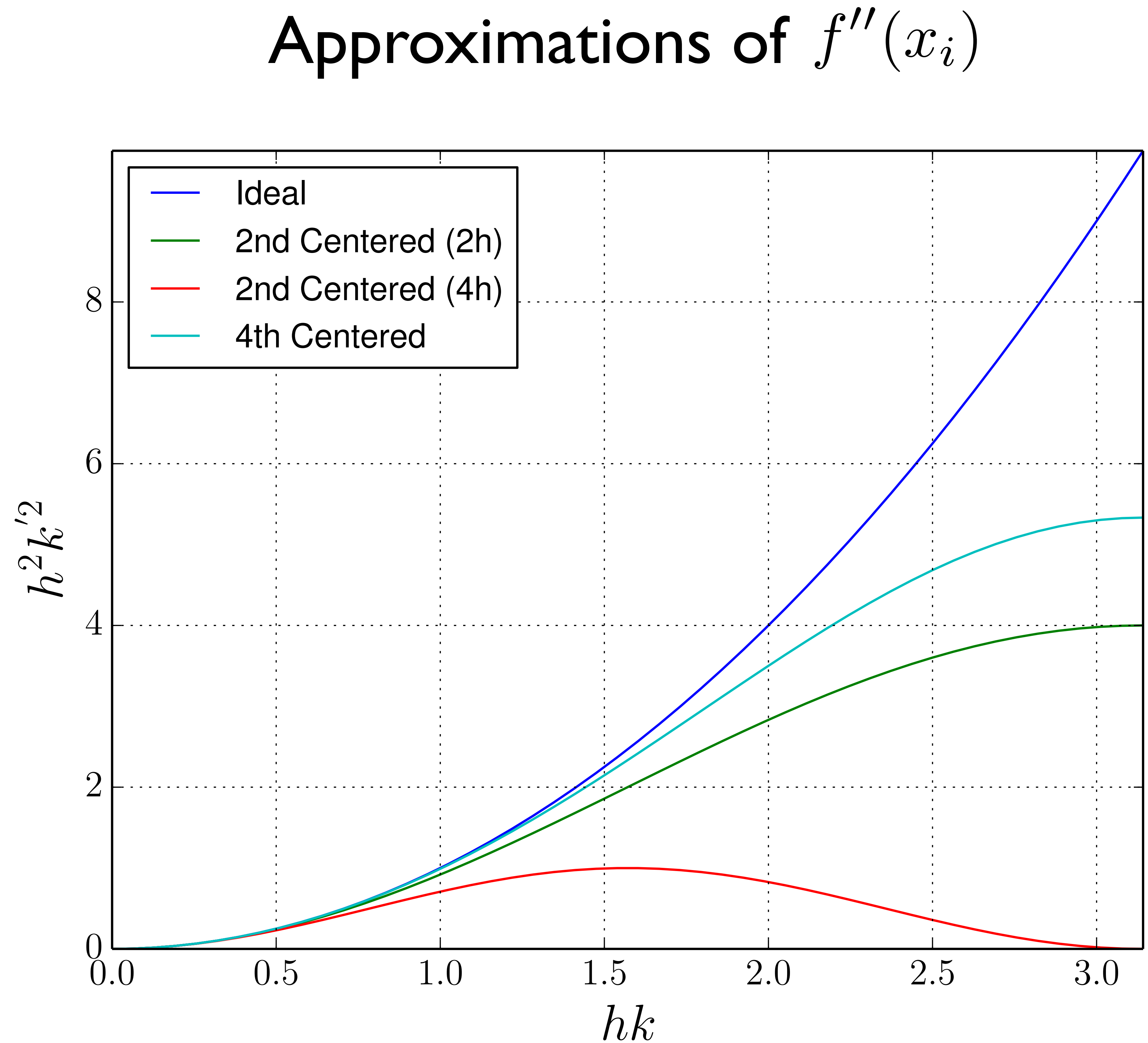


Accuracy analysis: modified wavenumber

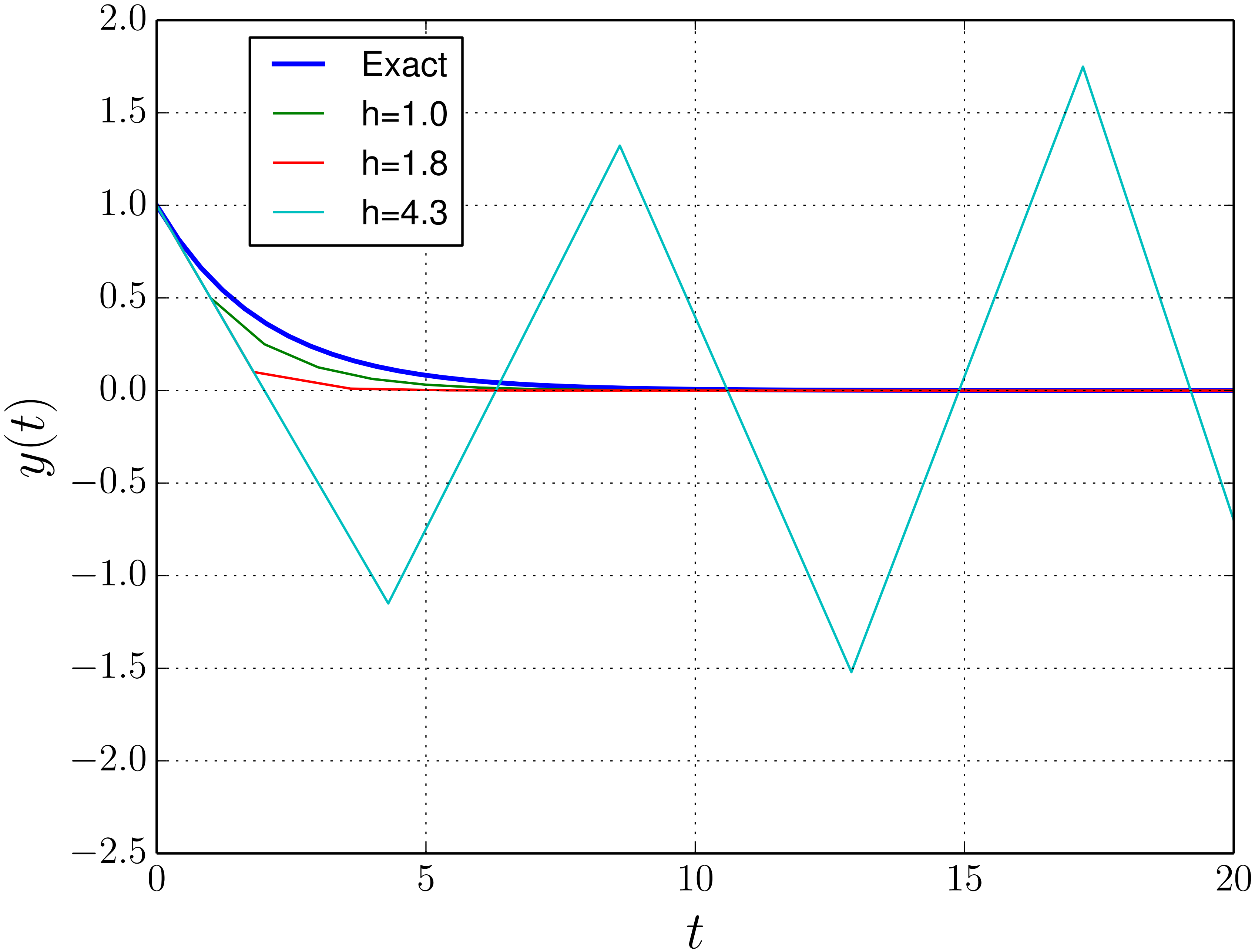
Approximations of $f'(x_i)$



Analyse du schéma: nombre d'onde modifié

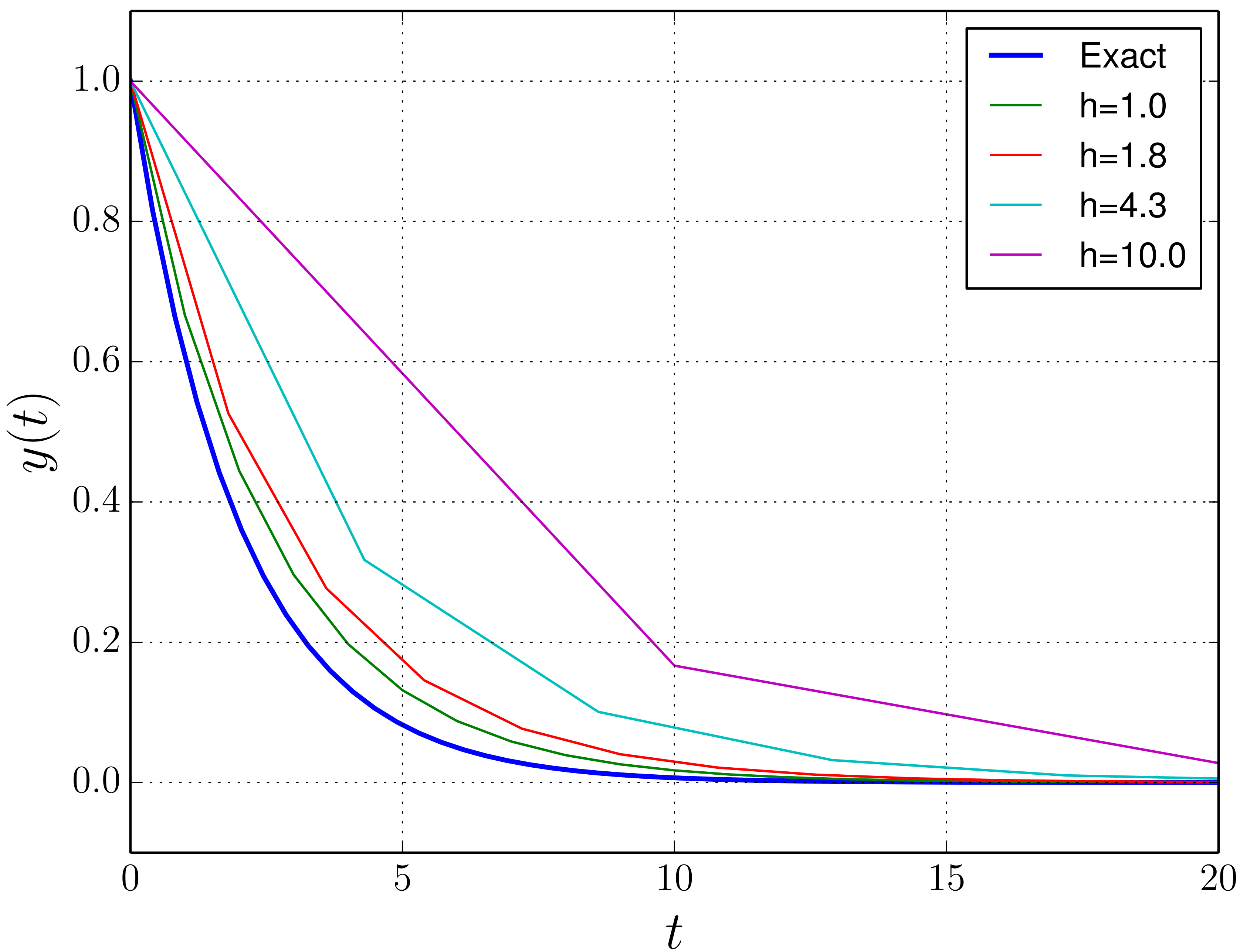


Application of the Forward Euler method



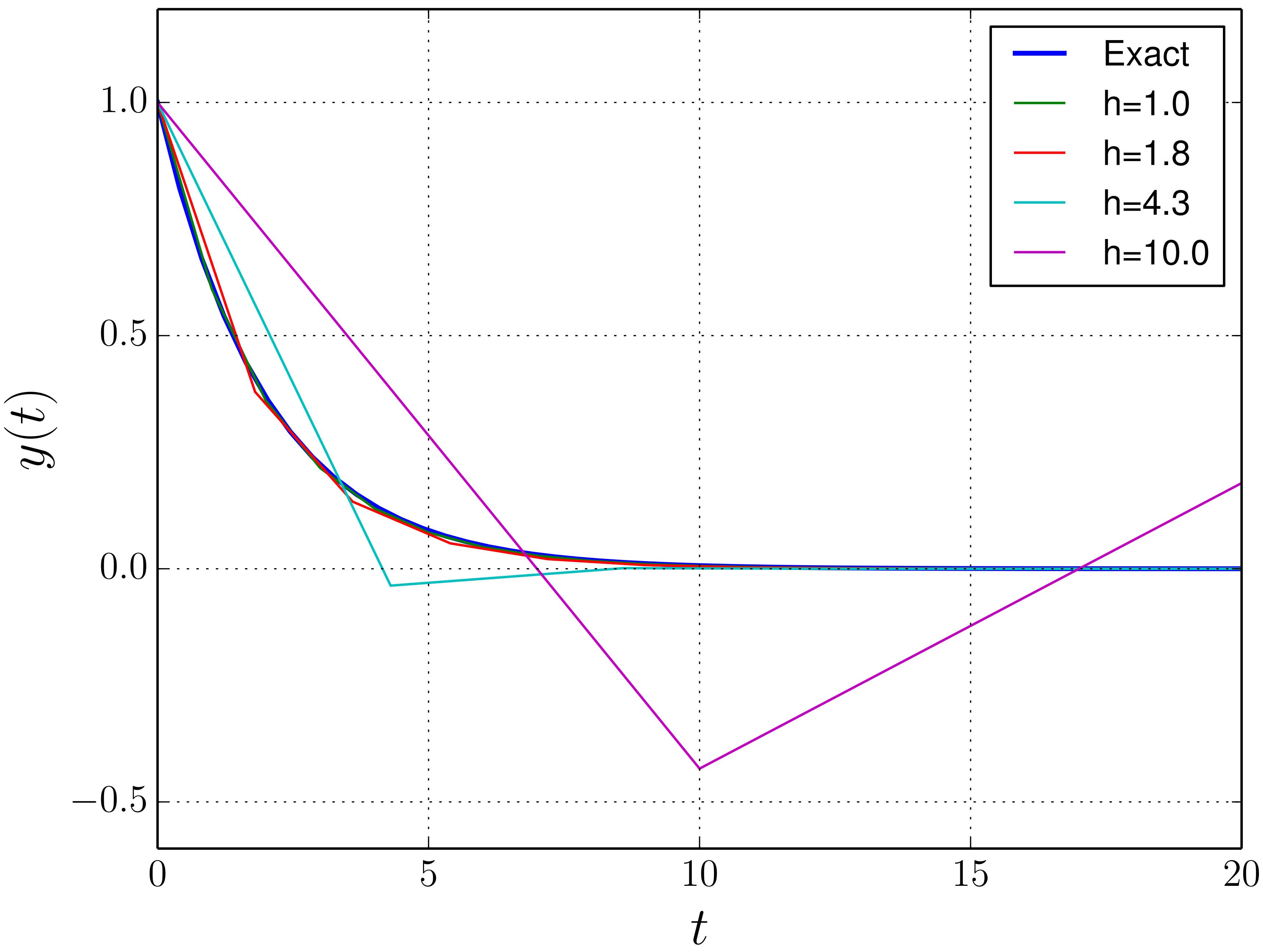
$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = -0.5y \\ y(0) = 1 \end{cases}$$

Application of the Backward Euler method



$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = -0.5y \\ y(0) = 1 \end{cases}$$

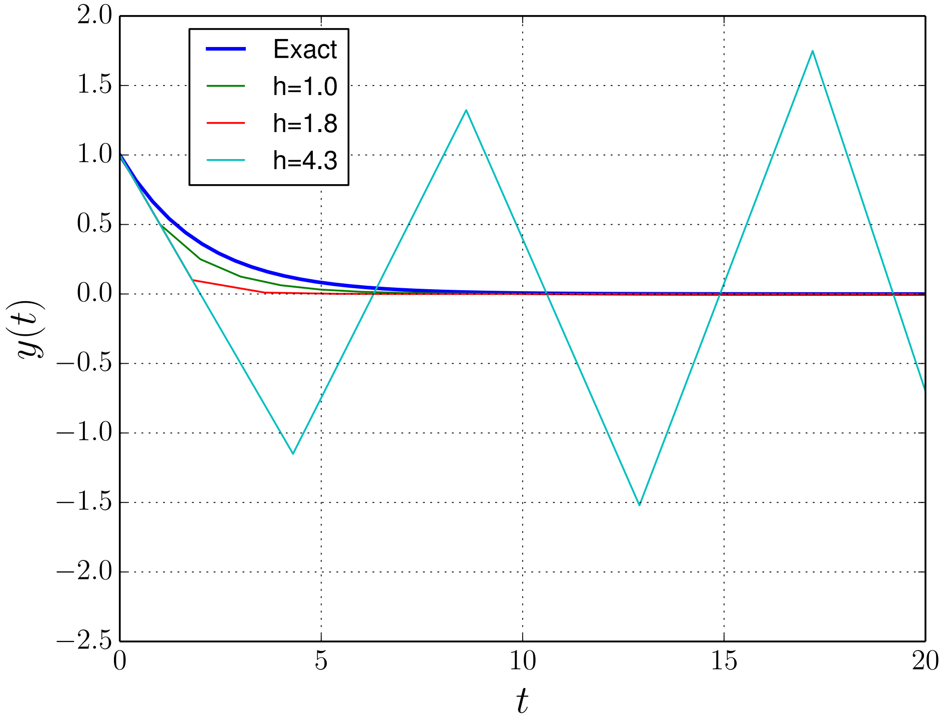
Application of the trapezoidal method



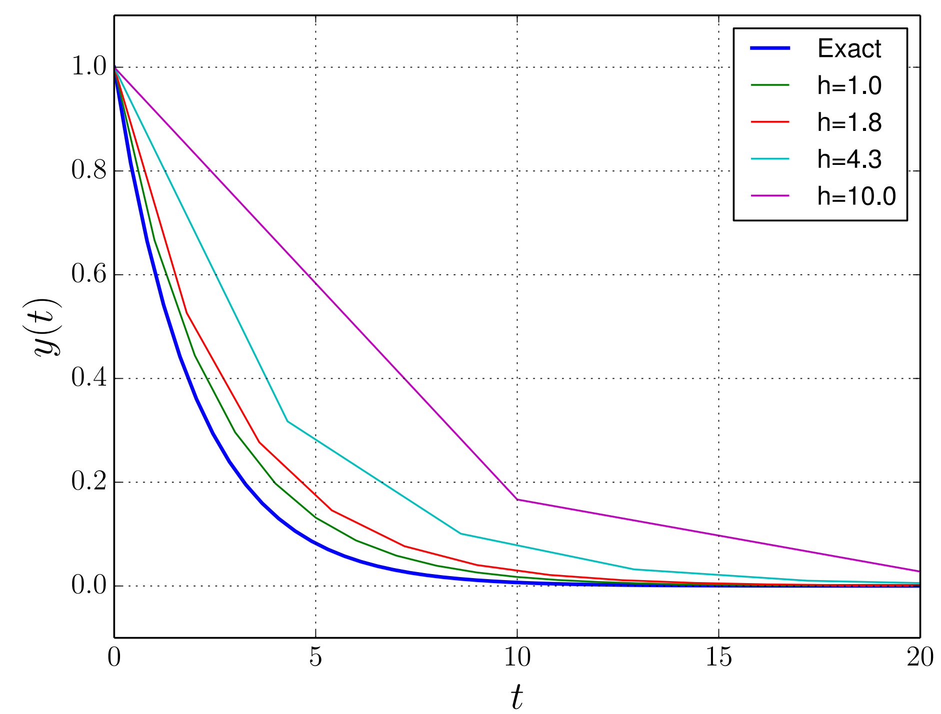
$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = -0.5y \\ y(0) = 1 \end{cases}$$

Comparaison of methods

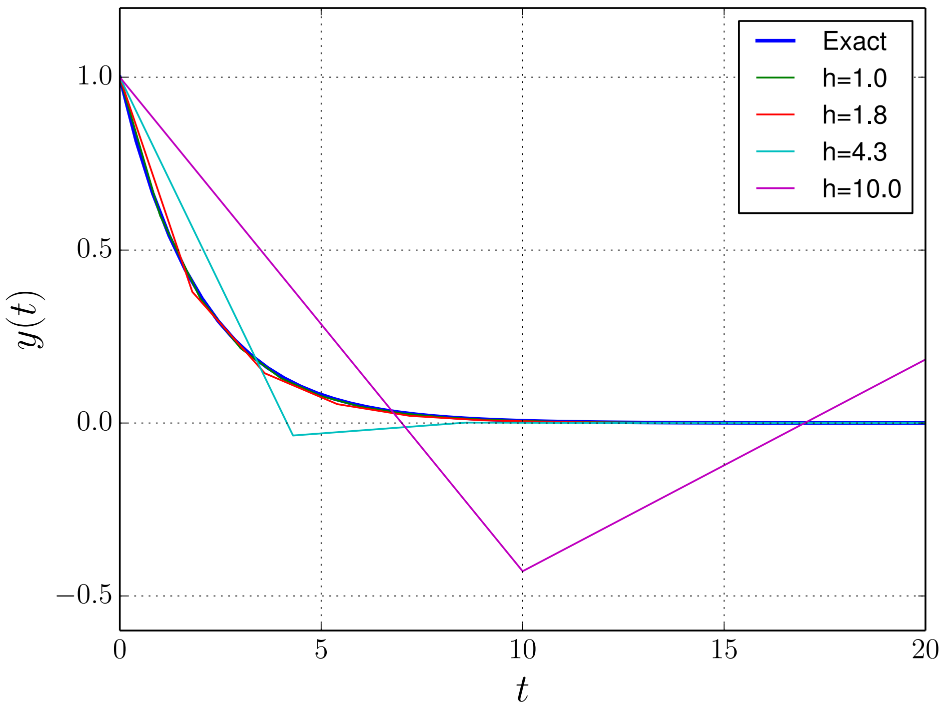
Forward Euler



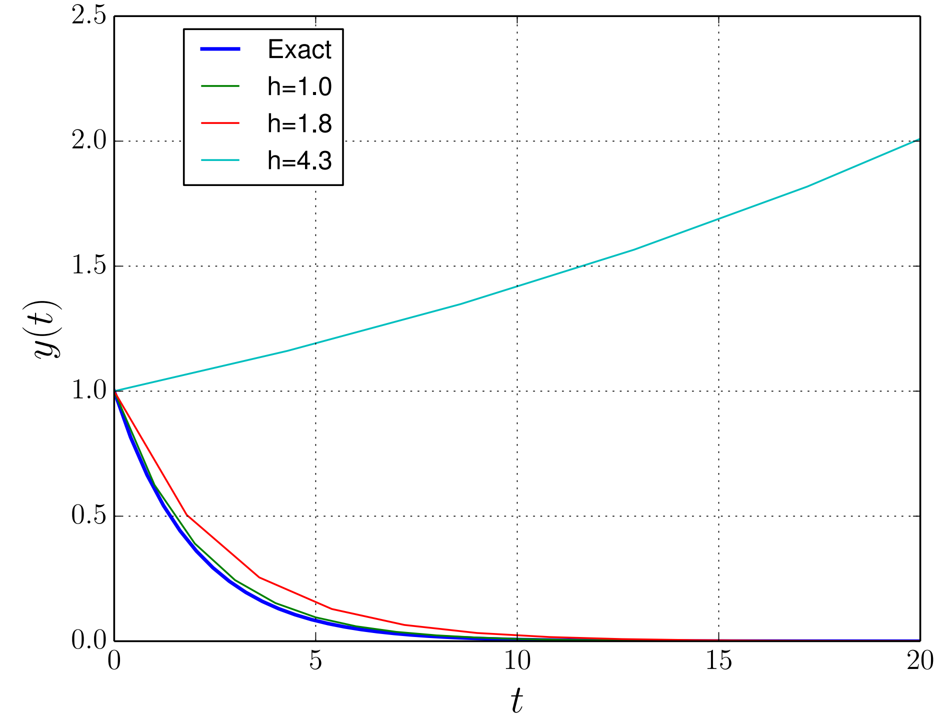
Backward Euler



Trapezoidal

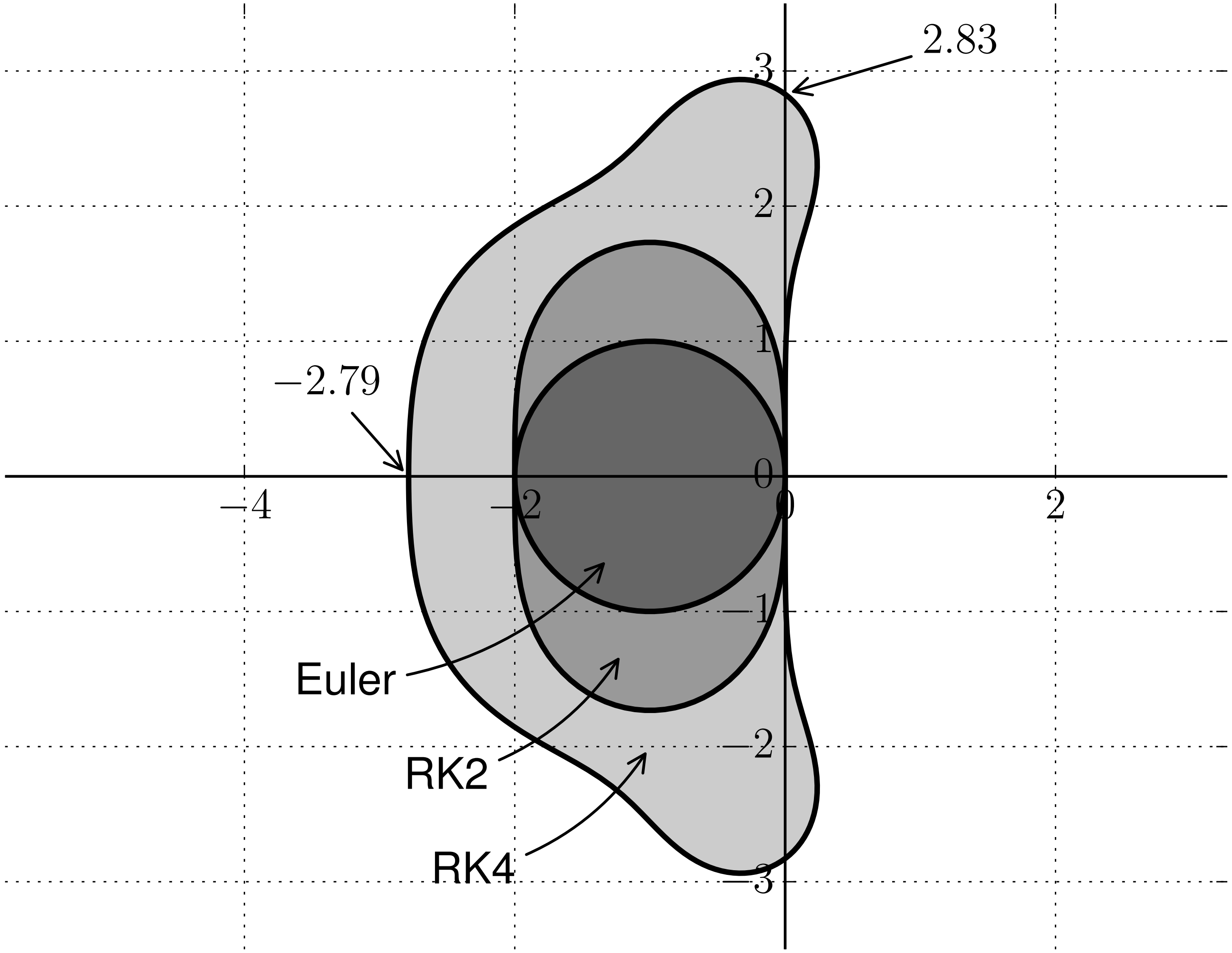


RK2



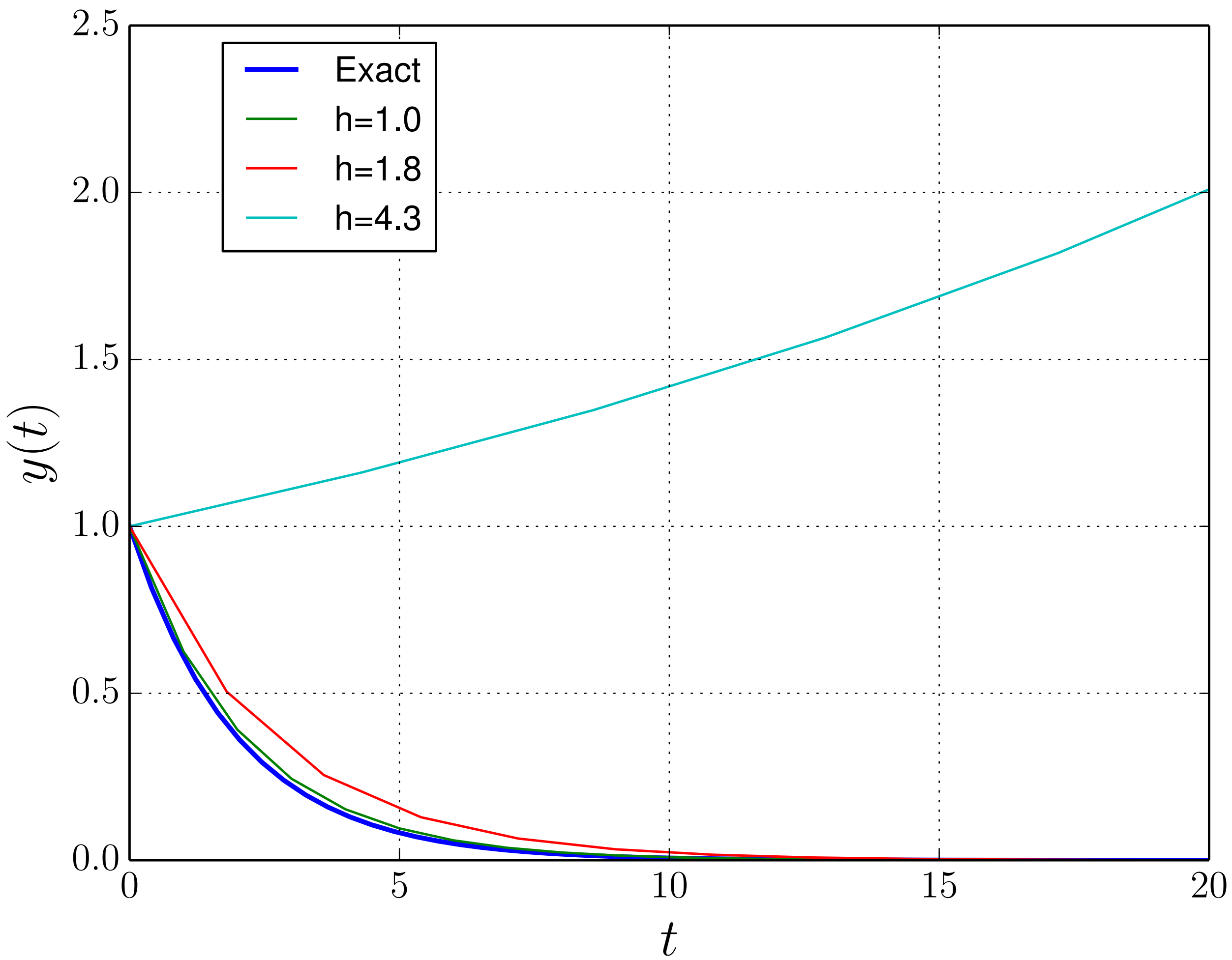
Comparaison of stability regions

For the conditionally stable methods :
Euler, RK2, RK4

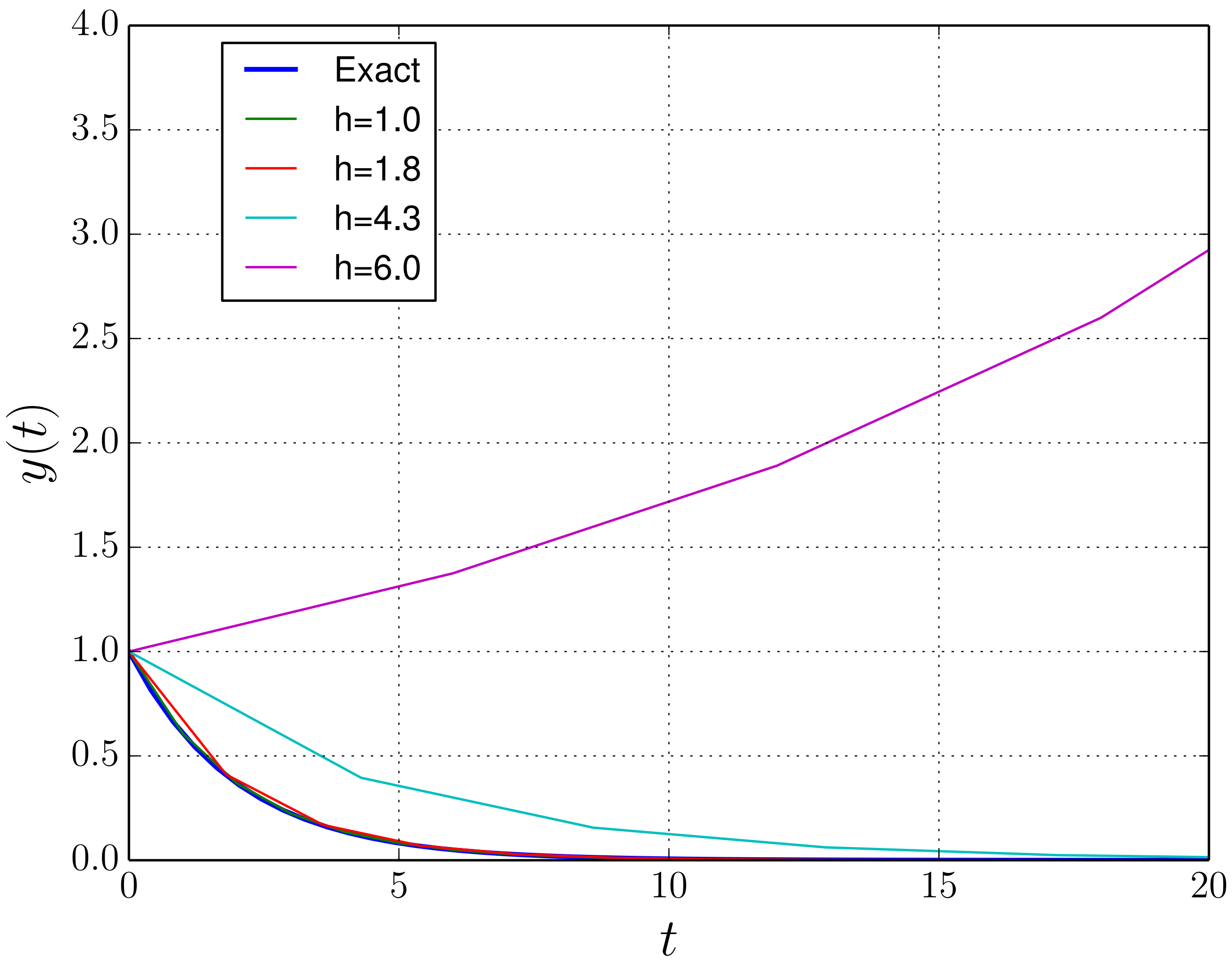


Comparaison of RK2/RK4

RK2



RK4



Projects

Project #01 to hand out (slides in PDF) for April, 13th

Send PDF slides to ronan.vicquelin@centralesupelec.fr and aymeric.vie@centralesupelec.fr

- First slide : names (2 people)+ problem title
- Slide #2 : sum up the problem to solve
- Self-sufficient slides => clear, detailed enough, synthetic
- Explain the approach, discuss your choices
- Describe numerical method, very briefly if seen in class, specify details related to the study
- Show and analyse results
- How sure are you that your results are correct ?

- Plots :
  Readable, clear
  axis names
  units
  legend

- Last slide : highlight results and conclusions