Numerical Methods in Engineering Applications

Workshop #04

Elliptic equations (2)

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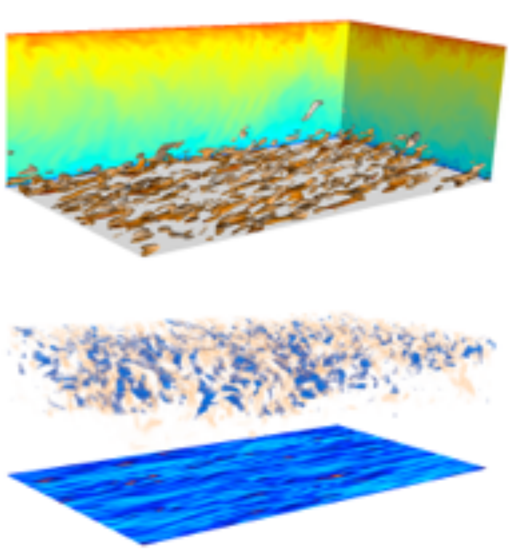
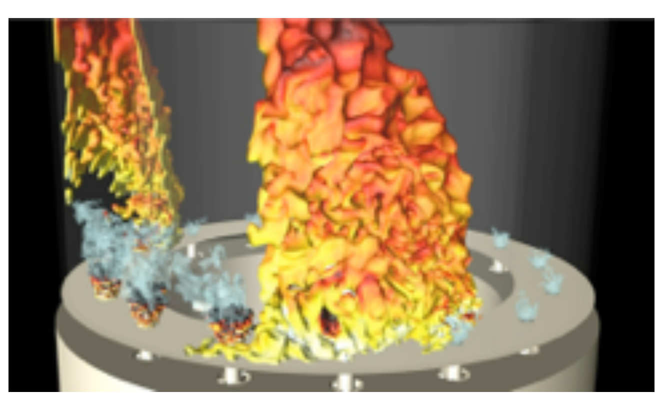
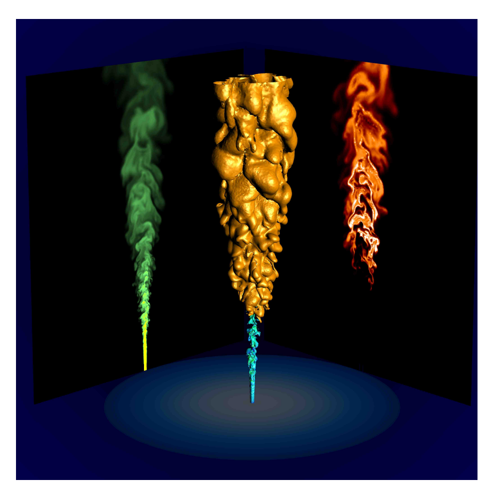
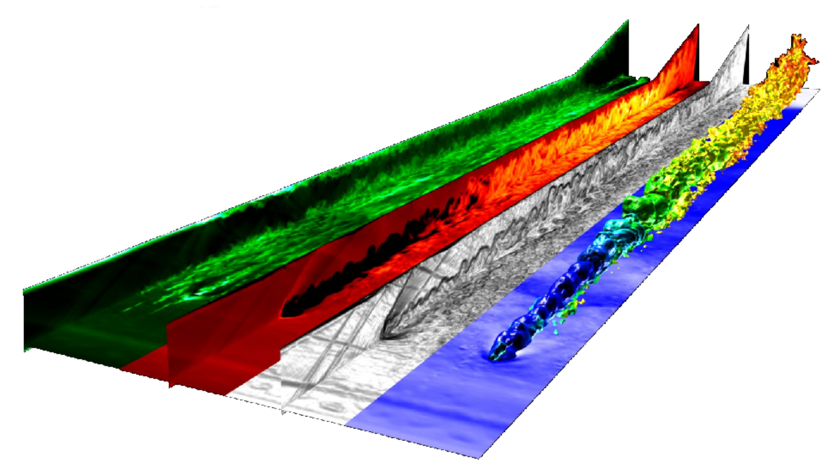




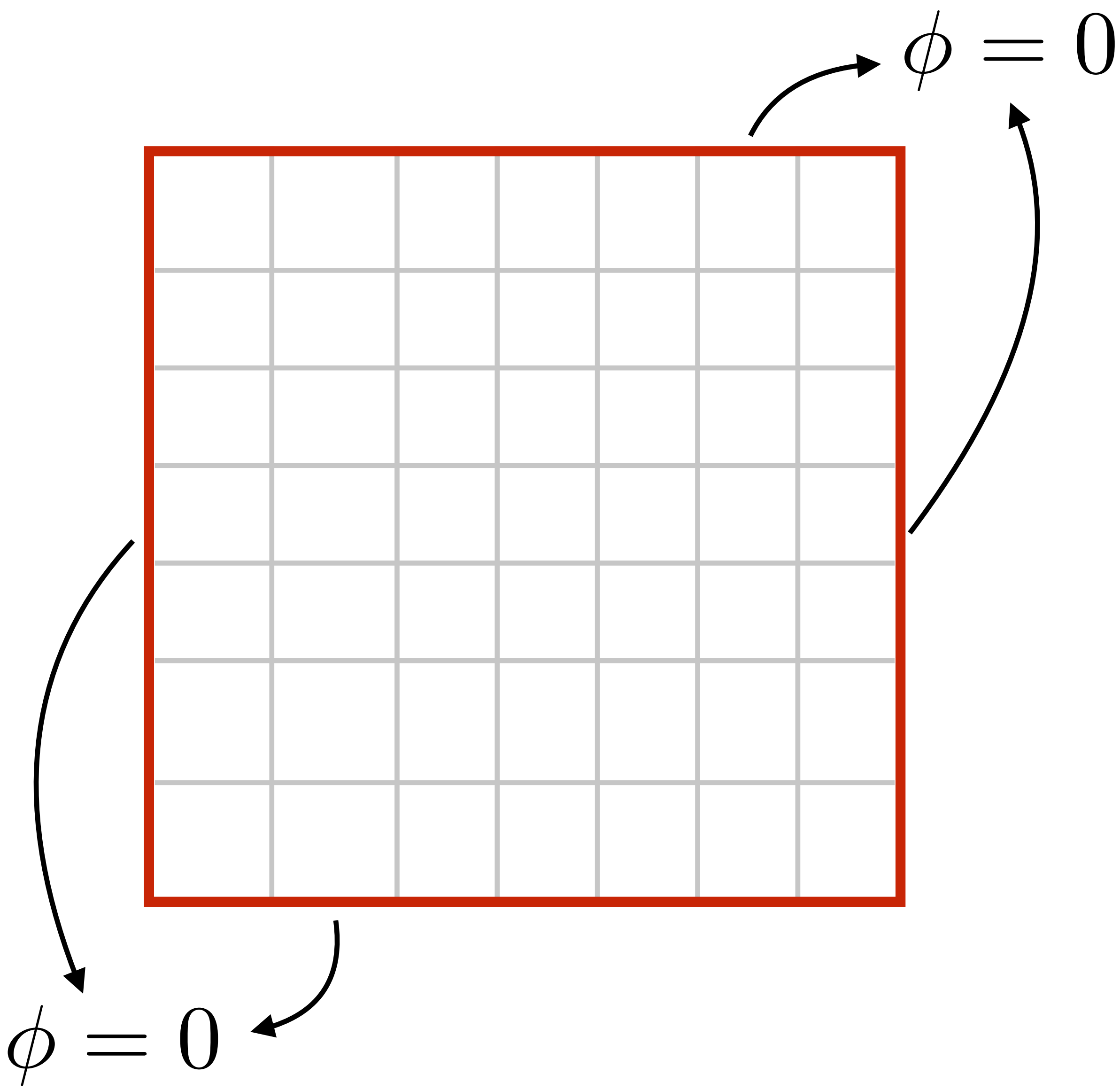




Objectives of Workshop #3

- **SOR method**
- **Conjugate Gradient methods**
- **Multi-block**

Poisson equation with Dirichlet boundary conditions



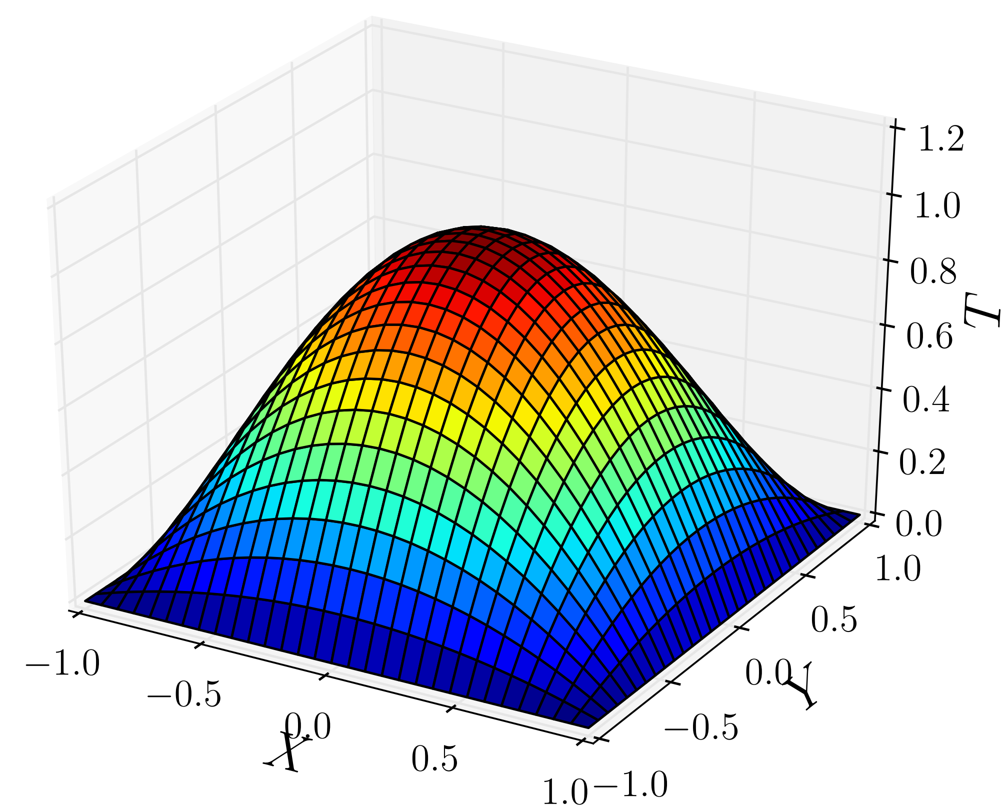
$$\Delta\phi = f(x, y)$$

with

$$f(x, y) = 2(x^2 + y^2 - 2)$$

Solution method:

- **Direct** (given)
- **Jacobi**
- **Gauss-Seidel**



Poisson equation with Dirichlet boundary conditions

SOR method

$$\phi_{i,j}^{(k+1)} = (1-\omega)\phi_{i,j}^{(k)} + \frac{\omega}{\frac{2}{\Delta x^2}+\frac{2}{\Delta y^2}}\left(\frac{\phi_{i+1,j}^{(k)}+\phi_{i-1,j}^{(k+1)}}{\Delta x^2}+\frac{\phi_{i,j+1}^{(k)}+\phi_{i,j-1}^{(k+1)}}{\Delta y^2}-f_{i,j}\right)$$

Conjugate Gradient Method

Compute $r_0 := b - Ax_0, p_0 := r_0.$

For $j = 0, 1, \ldots,$ *until convergence Do:*

$\quad\alpha_j := (r_j, r_j)/(Ap_j, p_j)$

$\quad x_{j+1} := x_j + \alpha_j p_j$

$\quad r_{j+1} := r_j - \alpha_j A p_j$

$\quad\beta_j := (r_{j+1}, r_{j+1})/(r_j, r_j)$

$\quad p_{j+1} := r_{j+1} + \beta_j p_j$

EndDo

Poisson equation with Dirichlet boundary conditions

Conjugate Gradient Method applied to heat equation

residual: $$r_{i,j}^{(k)} = f_{i,j} - \frac{\phi_{i+1,j}^{(k)} - 2\phi_{i,j}^{(k)} + \phi_{i-1,j}^{(k)}}{\Delta x^2} + \frac{\phi_{i,j+1}^{(k)} - 2\phi_{i,j}^{(k)} + \phi_{i,j-1}^{(k)}}{\Delta y^2}$$

laplacian of p: $$(A \cdot p)_{i,j}^{(k)} = \frac{p_{i+1,j}^{(k)} - 2p_{i,j}^{(k)} + p_{i-1,j}^{(k)}}{\Delta x^2} + \frac{p_{i,j+1}^{(k)} - 2p_{i,j}^{(k)} + p_{i,j-1}^{(k)}}{\Delta y^2}$$

iterations with

$$p_{i,j}^{(0)} = r_{i,j}^{(0)}$$

$$\alpha^{(k)} = \frac{\sum_{i,j} r_{i,j}^{(k)} r_{i,j}^{(k)}}{\sum_{i,j} p_{i,j}^{(k)} \left(A \cdot p\right)_{i,j}^{(k)}}$$

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \alpha^{(k)} p_{i,j}^{(k)}$$

$$r_{i,j}^{(k+1)} = r_{i,j}^{(k)} - \alpha^{(k)} \left(A \cdot p\right)_{i,j}^{(k)}$$

$$\beta^{(k)} = \frac{\sum_{i,j} r_{i,j}^{(k+1)} r_{i,j}^{(k+1)}}{\sum_{i,j} r_{i,j}^{(k)} r_{i,j}^{(k)}}$$

$$p_{i,j}^{(k+1)} = r_{i,j}^{(k+1)} + \beta^{(k)} p_{i,j}^{(k)}$$

Poisson equation with Dirichlet boundary conditions

Run time (ms)

$$\boxed{\eta = 10^{-3}}$$

9.9Go

0.263Go

	10x10	20x20	40x40	80x80	160x160
Direct	0,7	5	187	13000	1060000
Jacobi	25	392	5510	127000	1362000
Gauss-Seidel	11	200	2888	78000	1034000
SOR	20	86	392	2260	30000
Conjugate Gradient	2	14	103	945	6950

Poisson equation with Dirichlet boundary conditions

$$\boxed{\eta = 10^{-3}}$$

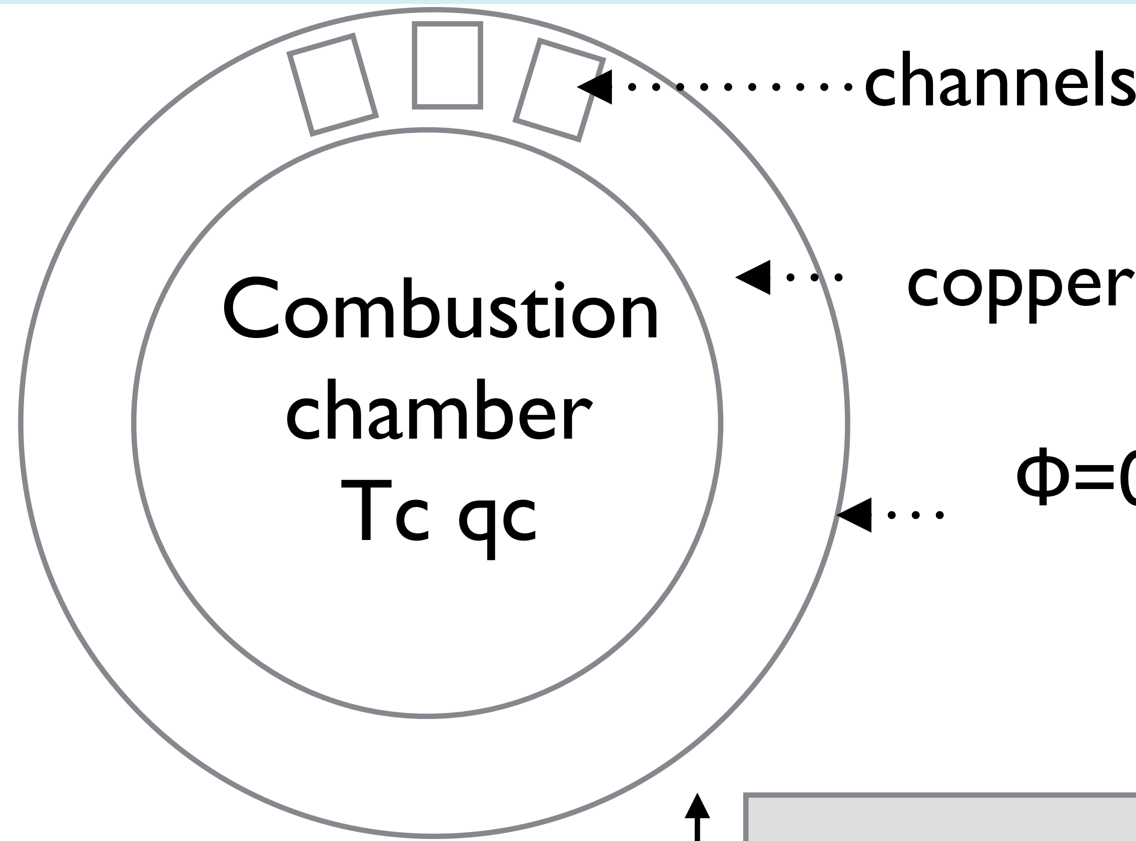
Iterations

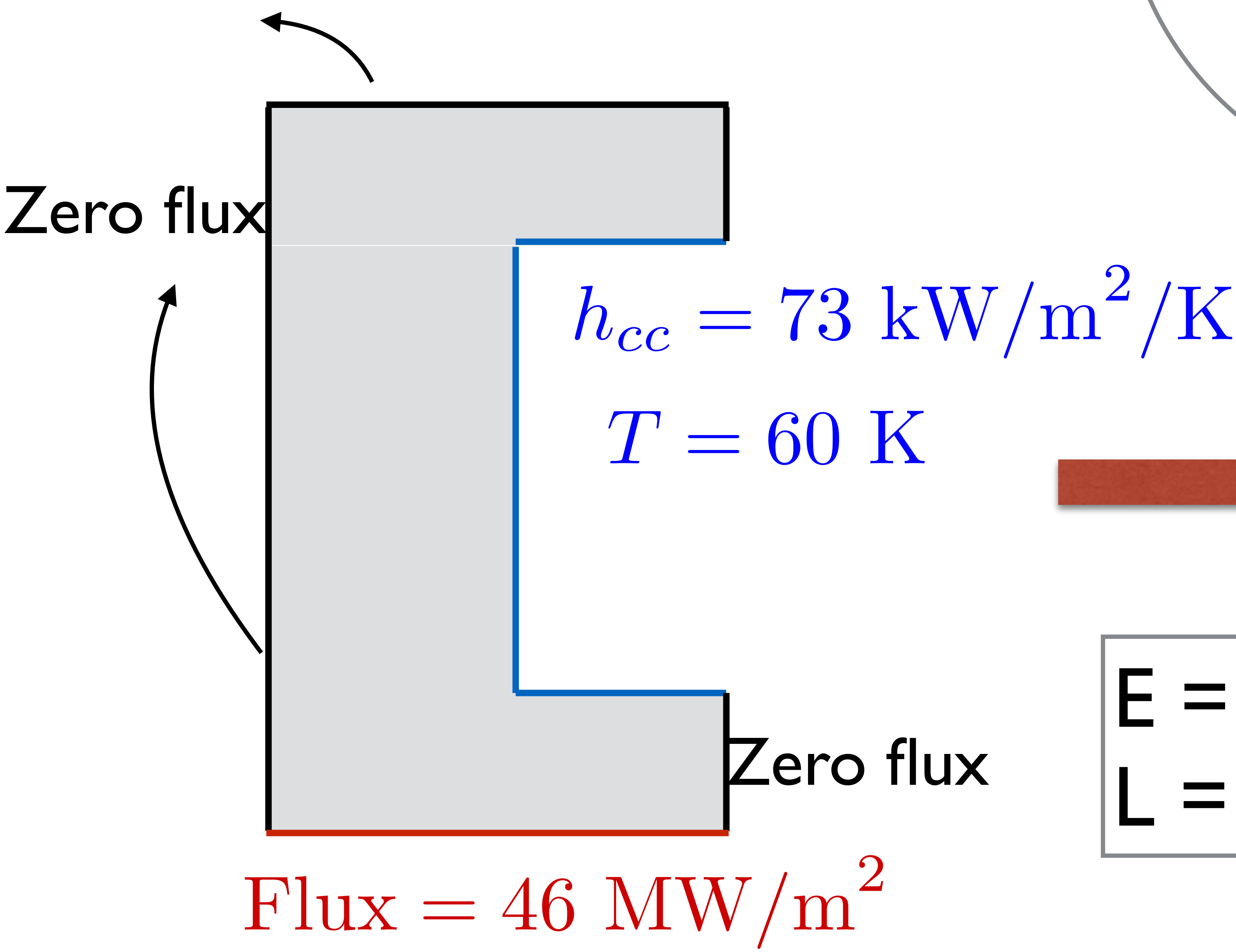
	10x10	20x20	40x40	80x80	160x160
Jacobi	88	352	1281	4400	14306
Gauss-Seidel	45	177	642	2200	7154
SOR	63	65	75	118	382
Conjugate Gradient	6	13	26	53	107

Multi-block resolution

<u>Cooling of the Vulcain engine</u>

What is Tmax?





E = 2 mm
L = 10 mm