① Theory

1) Discrete Fourier Transform (1D)

$$X(k) = \underbrace{\sum_{n=0}^{N-1} x(n)\, e^{-2\pi i \frac{nk}{N}}}_{}$$

computing consuming

2) Symmetries in DFT

$$X(N+k) = \sum_{n=0}^{N-1} x(n)\cdot e^{-2\pi i \frac{n(k+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\cdot \underset{\substack{\| \\ 1}}{e^{-2\pi i n}} \cdot e^{-2\pi i \frac{nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\cdot e^{-2\pi i \frac{nk}{N}}$$

$$= X(k)$$

3) FFT - Cooley and Tukey Algorithm

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-2\pi i \frac{nk}{N}}$$

even + odd

$$= \sum_{m=0}^{N/2-1} \left(x(2m)\, e^{-2\pi i \frac{2mk}{N}} + x(2m+1)\, e^{-2\pi i \frac{(2m+1)k}{N}} \right)$$

if $N(x) = 2m$ is factor

$$= \underbrace{\sum_{m=0}^{N/2-1} x(2m)\cdot e^{-2\pi i \frac{mk}{N/2}}}_{\text{new iteration}} + \underbrace{\sum_{m=0}^{N/2-1} x(2m+1)\cdot e^{-2\pi i \frac{mk}{N/2}}}_{\text{new iteration}} \cdot e^{-2\pi i \frac{k}{N}}$$

factor is similar

factor compute $O(N^2) \longrightarrow O((N/2)^2)$

if continue split $\rightarrow O(N \log N)$

$k \in [0, N]$ half → half → half

② Code Ooura-fft

1) cdft (int n, int isgn, double *a, int *ip, double *w)

$$X[k] = \sum_{n=0}^{N-1} x(n)\, e^{-2\pi i \frac{nk}{N}} \qquad 0 \le k < N$$

- $2n$ = length(a), mode(n,2) = 0
- Input ($0 \le j < N$) → Output ($0 \le k < N$)
  - $a[2j] = Re(x[j])$ $\xrightarrow{\text{cdft}}$ $a[2k] = Re(X[k])$
  - $a[2j+1] = Im(x[j])$ → $a[2k+1] = Im(X[k])$
- ip : work area for bit reversal, len(ip) $\ge 2+\sqrt{n}$
- $W[0 \cdots \frac{n}{2}-1]$: if ip[0] = 0, W[] and ip will be initialized
- isgn → $a = a \times \frac{1}{n}$

2) rdft (int n, int isgn, double *a, int *ip, double *w)

$$R[k] = \sum_{n=0}^{N-1} a[n] \cos\left(2\pi \frac{nk}{N}\right) \qquad 0 \le k \le \frac{N}{2}$$

$$I[k] = \sum_{n=0}^{N-1} a[n] \sin\left(2\pi \frac{nk}{N}\right) \qquad 0 < k < \frac{N}{2}$$

- n = length(a), mode(n,2) = 0
- Output
  - $a[2k] = R[k]$
  - $a[2k+1] = I[k]$
  - $a[1] = R[n/2]$

$$A[k] = \sum_{n=0}^{N-1} a[n]\, e^{2\pi i \frac{nk}{N}} \qquad 0 \le k \le \frac{N}{2}$$

- $x[n] = a[2n] + i\, a[2n+1]$
- cdft

$$X[k] = \sum_{n=0}^{N/2-1} x[i]\cdot e^{2\pi i \frac{nk}{N/2}} \qquad 0 \le k < N$$