Kalman Filter in Control Engineering

- Bayers Filter for the Gaussian Linear Case
- Everything is Gaussian
- Optimal Solution for Linear model and Gaussian Distribution

$e = x - \hat{x}$

$P = \mathbb{E}(e \cdot e^T)$

- prediction: $\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

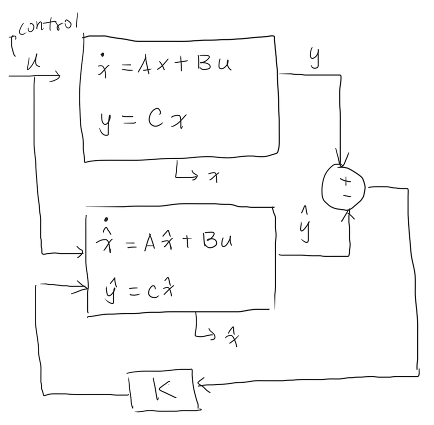
  $P_k^- = A P_{k-1} A^T + Q$ ← measure noise $\varepsilon_t \sim N(0, Q)$ → This ↑ P

  Update: $K_k = \dfrac{P_k^- C^T}{C P_k^- C^T + R}$

  $\hat{x}_k = \hat{x}_k^- + K_k(y_k - C\hat{x}_k^-) = \underbrace{A\hat{x}_{k-1} + Bu_k}_{\text{Model}} + \underbrace{K_k(y_k - C\hat{x}_k^-)}_{\text{Correction}}$

  $P_k = (I - K_k C) P_k^-$ → ↓ P

- Mechanism



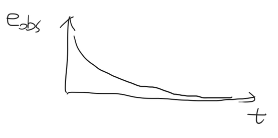
if $\hat{x} \to x \Rightarrow e_{obs} = x - \hat{x} \to 0$

$\because \begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \end{cases}$

$\therefore \dot{e}_{obs} = A e_{obs} + KC e_{obs}$

$\Rightarrow e_{obs} = e^{[(A-KC)t]} e_{obs}(0)$

if $A - KC < 0$



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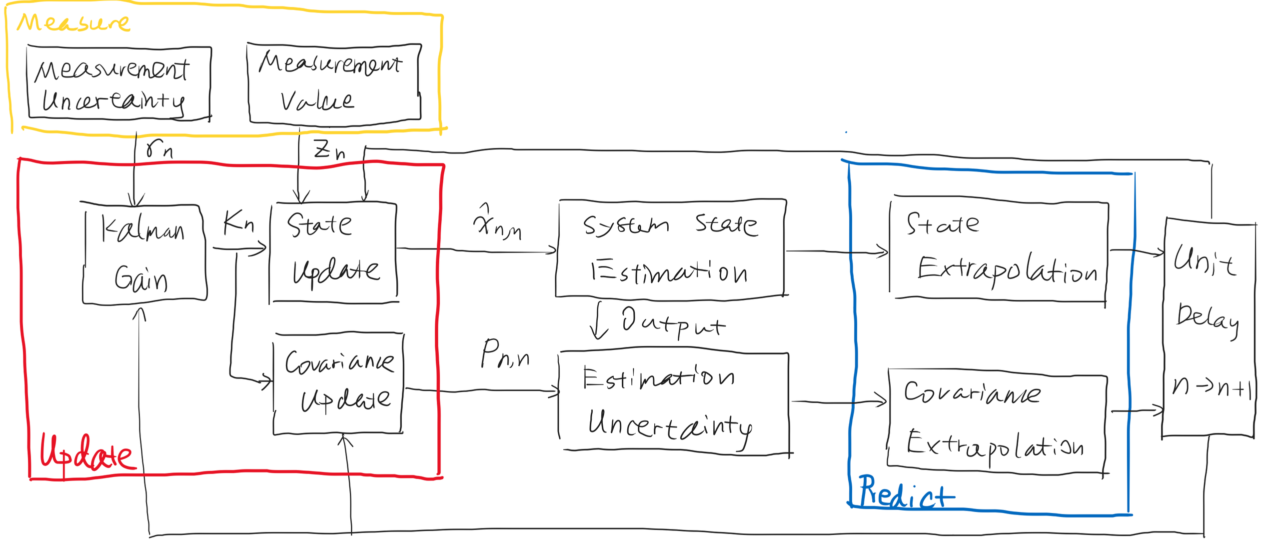

- 1D Estimation

  1) Kalman Gain

  $K_n = \dfrac{P_{n,n-1}}{P_{n,n-1} + r_n}$ → estimate uncertainty / → measurement uncertainty

  2) State Update Equation

  $\hat{x}_{n,n} = \hat{x}_{n,n-1}(1 - K_n) + K_n z_n$ → measurement

  3) State Extrapolation Equation

  $\hat{x}_{n+1,n} = \hat{x}_{n,n} + \Delta t \, \dot{\hat{x}}_{n,n}$

  4) Covariance Update

  $P_{n,n} = (1 - K_n) P_{n,n-1}$

  5) Covariance Extrapolation

  $P_{n+1,n} = P_{n,n}$

- Algorithm

  1. $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ $\left(\begin{matrix} \mu_{t-1} \Leftarrow E[x_{t-1}] \\ \Sigma_{t-1} \Leftarrow P \end{matrix}\right)$

  2. Prediction

     $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

     $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T$

  3. Correction & Update

     $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

     $\mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t)$

     $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

  4. Return μ_t, Σ_t

- Math

1) Kalman Filter recursive lease squares estimator

Motion Model: $x_k = F_k x_{k-1} + G_k \underset{\text{Input}}{u_k} + \underset{\text{noise}}{w_k}$ ... (1)

Measurement Model: $z_k = H_k x_k + \underset{\text{noise}}{v_k}$ ... (2)

$w_k \sim N(0, Q_k)$ Process/Motion Noise

$v_k \sim N(0, R_k)$ Measurement Noise

Step 1: Prediction

- predicted State: $\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k$ -- (3)
- predicted Covariance: $P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$ -- (4)

Step 2: Update

- Innovation / measurement prefit residual: $\tilde{y}_k = z_k - H_k \hat{x}_{k|k-1}$ -- (5) (→ observation/measurement)
- Innovation Covariance: $S_k = H_k P_{k|k-1} H_k^T + R_k$ --- (6)
- Optimal Kalman Gain: $K_k = P_{k|k-1} H_k^T S_k^{-1}$ --- (7)
- updated state estimate: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$ -- (8)
- updated estimate covariance: $P_{k|k} = (I - K_k H_k) P_{k|k-1}$ -- (9)
- measurement post fit residual: $\tilde{y}_{k|k} = z_k - H_k \hat{x}_{k|k}$ -- (10)

$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(\underset{\substack{\| \\ H_k x_k + v_k \ (2)}}{z_k} - H_k \hat{x}_{k|k-1})$

$= (I - K_k H_k)\hat{x}_{k|k-1} + K_k(H_k x_k + v_k)$

2) Prelimination

$E(x_k) = F\bar{x}_{k-1} + G\underset{\substack{\| \\ \text{fixed}}}{\bar{u}_k}$

$= A^k \bar{y}_0 + A^k B\bar{u}_k + A^{k-1} B\bar{u} + \cdots B\bar{u}_k$

$\Sigma_{x,k} = E[(x_k - \bar{x}_k)\underbrace{(x_k - \bar{x}_k)}_{= F(x_{k-1} - \bar{x}_{k-1}) + G(u_k - \bar{u}_k)}{}^T]$

$= F\Sigma_{x,k-1}F^T + F\Sigma_{x,u,k}G^T + G\Sigma_{u,x,k}F^T + G\Sigma_{u,k}G^T$

3) Derivations

$P_{k|k} = Cov(x_k - \hat{x}_{k|k})$ ↓ (8)

$= Cov\{x_k - [\hat{x}_{k|k-1} + K_k(z_k - H_k \hat{x}_{k|k-1})]\}$ ↙ (2)

$= Cov\{x_k - [\hat{x}_{k|k-1} + K_k(H_k x_k + v_k - H_k \hat{x}_{k|k-1})]\}$

$= Cov[(I - K_k H_k)(x_k - \hat{x}_{k|k-1}) - K_k v_k]$

$= (I - K_k H_k) Cov(x_k - \hat{x}_{k|k-1})(I - K_k H_k)^T + K_k Cov(v_k) K_k^T$

$= (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k^T$

$\Downarrow$

$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k^T K_k^T + \underbrace{K_k H_k P_{k|k-1} H_k^T K_k^T + K R_k K_k^T}_{K_k S_k K_k^T \ (6)}$

$tr(A+B) = tr(A) + tr(B)$

$tr(A^T) = tr(A)$

$\dfrac{\partial tr(AX)}{\partial X} = \dfrac{\partial tr(XA)}{\partial X} = A$

$\dfrac{\partial tr(AX^T)}{\partial X} = \dfrac{\partial tr(X^T A)}{\partial X} = A^T$

$\dfrac{\partial tr(X^T A X)}{\partial X} = X^T(A + A^T)$

$\dfrac{\partial tr(P_{k|k})}{\partial K_k} = -2(H_k P_{k|k-1})^T + 2K_k S_k = 0$

$K_k = P_{k|k-1} H_k^T S_k^{-1}$ -- (11)

$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} = (I - K_k H_k) P_{k|k-1}$ (9)

$\underset{(11)}{K_k S_k K_k^T} = P_{k|k-1} H_k^T K_k^T$