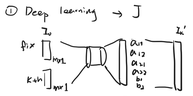
2D US Kidney Motion Tracking

① Deep learning → J



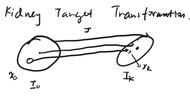
$I_k \hat{\sim} J\, I_0$

$I_k' = J^{-1} I_k$

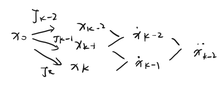
$$J = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix}$$ affine Transformation

Data Requirements:
1. same organ plan
- different B-mode add robustness but not accurate
- different Patients add application range

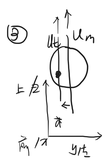
② Kidney Tangent Transformation.



$$x_0 = \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} \qquad x_k = \begin{bmatrix} x_k \\ y_k \\ 1 \end{bmatrix}$$



③



$${}^{E}_{U_1}T = {}^{E}_{U_m}T \cdot {}^{U_m}_{U_n}T$$

$$= \begin{bmatrix} 1 & & & d \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

④ State Space Equation

Assume $\ddot{x}_{k+1} = \ddot{x}_k$, $\ddot{y}_{k+1} = \ddot{y}_k$

$$Y_k = \begin{bmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{bmatrix} \qquad Y_{k+1} = A Y_k + B U_k \leftarrow \text{Inputs}$$

$$= \begin{bmatrix} 1 & & \Delta t & \\ & 1 & & \Delta t \\ & & 1 & \\ & & & 1 \end{bmatrix} Y_k + \begin{bmatrix} \frac{1}{2}\Delta t^2 & \\ & \frac{1}{2}\Delta t^2 \\ \Delta t & \\ & \Delta t \end{bmatrix} \begin{bmatrix} \ddot{x}_k \\ \ddot{y}_k \end{bmatrix}$$

$$Y_k = \begin{bmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \\ \ddot{x}_k \\ \ddot{y}_k \end{bmatrix} \qquad Y_k = A Y_{k-1} = \begin{bmatrix} 1 & & \Delta t & & \frac{1}{2}\Delta t^2 & \\ & 1 & & \Delta t & & \frac{1}{2}\Delta t^2 \\ & & 1 & & \Delta t & \\ & & & 1 & & \Delta t \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \\ \ddot{x}_{k-1} \\ \ddot{y}_{k-1} \end{bmatrix}$$

⑤ Kalman Filter recursive lease squares estimator

Motion Model: $x_k = F_k x_{k-1} + G_k \overset{\text{Input}}{u_k} + \overset{\text{noise}}{w_k}$ ... ①

Measurement Model: $z_k = H_k x_k + v_k$ ... ②

noise

$w_k \sim N(0, Q_k)$ Process/Motion Noise

$v_k \sim N(0, R_k)$ Measurement Noise

Step 1: Prediction

- predicted State: $\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k U_k$ ... ③
- predicted Covariance $P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$ ... ④

Step 2: Update

- Innovation / measurement precite residual: $\tilde{y}_k = \overset{\text{observation/measurement}}{z_k} - H_k \hat{x}_{k|k-1}$ ... ⑤
- Innovation Covariance $S_k = H_k P_{k|k-1} H_k^T + R_k$ ... ⑥
- Optimal Kalman Gain $K_k = P_{k|k-1} H_k^T S_k^{-1}$ ... ⑦
- updated state estimate: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$ ... ⑧
- updated estimate covariance: $P_{k|k} = (I - K_k H_k) P_{k|k-1}$ ... ⑨
- Measurement post fit residual $\tilde{y}_{k|k} = z_k - H_k \hat{x}_{k|k}$ ... ⑩

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1})$$

$$z_k = H_k x_k + v_k \quad ②$$

$$= (I - K_k H_k)\hat{x}_{k|k-1} + K_k (H_k x_k + v_k)$$

$$Y_k = \begin{bmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \\ \ddot{x}_k \\ \ddot{y}_k \end{bmatrix} \qquad Y_k = A Y_{k-1} = \begin{bmatrix} 1 & & \Delta t & & \frac{1}{2}\Delta t^2 & \\ & 1 & & \Delta t & & \frac{1}{2}\Delta t^2 \\ & & 1 & & \Delta t & \\ & & & 1 & & \Delta t \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \\ \ddot{x}_{k-1} \\ \ddot{y}_{k-1} \end{bmatrix}$$

Prelimination

$$E(x_k) = F\bar{x}_{k-1} + G \underset{\text{fixed}}{\bar{u}_k}$$

$$= A^k \bar{Y}_0 + A^k B \bar{U}_k + A^{k-1} B \bar{U} + \cdots B \bar{U}_k$$

$$\Sigma_{x,k} = E\left[(x_k - \bar{x}_k)\underbrace{(x_k - \bar{x}_k)^T}_{= F(x_{k-1} - \bar{x}_{k-1}) + G(u_k - \bar{u}_k)}\right]$$

$$= F \Sigma_{x,k-1} F^T + F \Sigma_{x,u,k} G^T + G \Sigma_{u,x,k} F^T + G \Sigma_{u,k} G^T$$

Derivations

$$P_{k|k} = Cov(x_k - \hat{x}_{k|k})$$

$$= Cov\{x_k - [\hat{x}_{k|k-1} + K_k(z_k - H_k \hat{x}_{k|k-1})]\}$$

$$= Cov\{x_k - [\hat{x}_{k|k-1} + K_k(H_k x_k + v_k - H_k \hat{x}_{k|k-1})]\}$$

$$= Cov[(I - K_k H_k)(x_k - \hat{x}_{k|k-1}) - K_k v_k]$$

$$= (I - K_k H_k) Cov(x_k - \hat{x}_{k|k-1})(I - K_k H_k)^T + K_k Cov(v_k) K_k^T$$

$$= (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k^T$$

$$\Downarrow$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k^T K_k^T + \underbrace{K_k H_k P_{k|k-1} H_k^T K_k^T + K_k R_k K_k^T}_{K_k S_k K_k^T \ ⑥}$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} = (I - K_k H_k) P_{k|k-1} \quad ⑨$$

$$K_k S_k K_k^T = P_{k|k-1} H_k^T K_k^T \quad ⑪$$

$$tr(A+B) = tr(A) + tr(B)$$

$$tr(A^T) = tr(A)$$

$$\frac{\partial tr(Ax)}{\partial x} = \frac{\partial tr(xA)}{\partial x} = A$$

$$\frac{\partial tr(Ax^T)}{\partial x} = \frac{\partial tr(x^T A)}{\partial x} = A^T$$

$$\frac{\partial tr(x^T A x)}{\partial x} = x^T(A + A^T)$$

$$\frac{\partial\, tr(P_{k|k})}{\partial K_k} = -2(H_k P_{k|k-1})^T + 2K_k S_k = 0$$

$$K_k = P_{k|k-1} H_k^T S_k^{-1} \quad ⑪$$

Extended Kalman Filter (nonlinear)

Motion Model: $x_k = f(x_{k-1}, u_m) + w_k$ $\quad w_k \sim N(0, Q_k)$

Measurement Model: $z_k = h(x_k) + v_k$ $\quad v_k \sim N(0, R_k)$

Step 1. Predict

Predicted State Estimate $\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, U_k)$

Predicted Covariance Estimate $P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1|k-1}, U_k}$$

Step 2. Update

Innovation / measurement residual $\tilde{y}_k = z_k - h(\hat{x}_{k|k-1})$

Innovation covariance $S_k = H_k P_{k|k-1} H_k^T + R_k$

Near-Optimal Kalman Gain $K_k = P_{k|k-1} H_k^T S_k^{-1}$

Updated State Estimate $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$

Updated Covariance Estimate $P_{k|k} = (I - K_k H_k) P_{k|k-1}$

$$H = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k|k-1}}$$

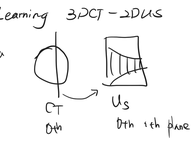
Initialize $\hat{x}_{1|0} = E[x(t_0)]$

$$P_{0|0} = E\left[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))^T\right]$$

3DCT – 2DUS Kidney Motion Tracking

① Deep Learning 3DCT – 2DUS

Data



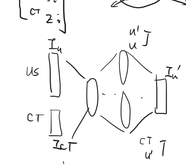
Application:
1. slight deformation 1) more of the feature structure can be recognized
2. multiple patient 2) as rigid
3. different CT during one respiration.

$${}^{u}P_j = \begin{bmatrix} {}^{u}x_j \\ {}^{u}y_j \\ 0 \end{bmatrix}$$

$${}^{CT}P_i = \begin{bmatrix} {}^{CT}x_i \\ {}^{CT}y_i \\ {}^{CT}z_i \end{bmatrix}$$

centroid

$$\begin{bmatrix} {}^{CT}x_i \\ {}^{CT}y_i \\ {}^{CT}z_i \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{u}x_i \\ {}^{u}y_i \\ 0 \\ 1 \end{bmatrix}$$



$$\Rightarrow {}^{E}_{CT}T = {}^{E}_{U}T\, {}^{U}_{CT}T\, {}^{CT}_{CT'}T$$

Real Move

② CT: known shape

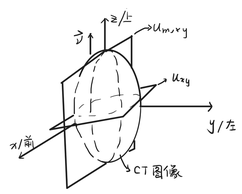
US: known current Ep

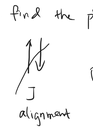
CT → US: where is the longitudinal plane

since respiration, one direction has the maximal displacement

connected with organ shape/structure.

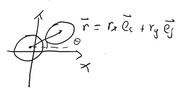
Goal: find the plane in organ that exists maximal respiratory motion





③ Follow the same 2D motion Tracking ②③④⑤

Kidney Motion Sine



$$\vec{r} = r_x \vec{e}_x + r_y \vec{e}_y$$

$$\|\vec{r}\| = A \sin(\omega t + \varphi)$$

$$\begin{cases} r_x = r\cos\theta = A\cos\theta \sin(\omega t + \varphi) \\ r_y = r\sin\theta = A\sin\theta \sin(\omega t + \varphi) \end{cases}$$

$$x_k = \begin{bmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \\ \ddot{x}_k \\ \ddot{y}_k \end{bmatrix} = f(x_{k-1})$$

$r_x = A\cos\theta \sin(\omega t + \varphi)$

$r_x(t + \Delta t) = A\cos\theta \sin(\omega(t + \Delta t) + \varphi)$

$\sin(\omega t + \varphi + \omega\Delta t)$

$\sin(\omega t + \varphi)\cos\omega\Delta t + \cos(\omega t + \varphi)\sin\omega\Delta t$

?