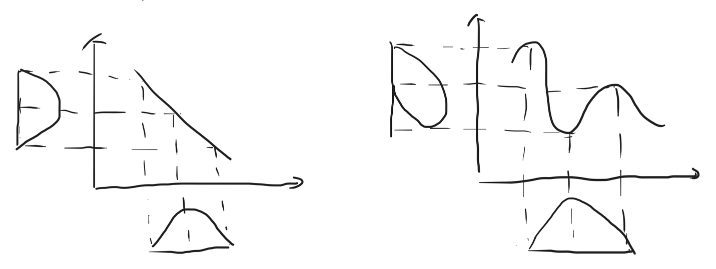
① Nonlinear System

$x_t = g(x_{t-1}, u_t) + \varepsilon_t$ ↳ system noise

$y_t = h(x_t) + \delta_t$ ↳ measurement noise

Linear Non linear



② Non linear System Solution

State Estimaton	Model	Assumed Distribution	Computation Cost
Kalman Filter	Linear	Gaussian	Low
Extended Kalman Filter	Locally Linear	Gausian	Low Medium
Unscented Kalman Filter	Nonlinear	Gaussian	Medium
Particle Filter	Nonlinear	NonGaussia	High

③ Extended Kalman Filter — 1st ordel Taylor Series

Motion Model: $x_k = f(x_{k-1}, u_k) + w_k$ $\quad w_k \sim N(0, Q_k)$

Measurement Model: $z_k = h(x_k) + v_k$ $\quad v_k \sim N(0, R_k)$

locally Linearization – Taylar Expasion

$$f(x_{k-1}, u_k) \approx f(\mu_{k-1}, u_k) + \left.\frac{\partial f(x_{k-1}, u_k)}{\partial x_{k-1}}\right|_{\mu_{k-1}} (x_{k-1} - \mu_{t-1})$$

$$h(x_k) \approx h(\bar{\mu}_k) + \left.\frac{\partial h(x_k)}{\partial x_k}\right|_{\bar{\mu}_k} (x_k - \bar{\mu}_k)$$

$\to F_k$, $\to H_k$ — Jacobians $\Rightarrow$

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ \vdots \end{bmatrix} \qquad G_x = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Step 1. Predict

Predicted State Estimate $\quad \hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k)$

Predicted Covariance Estimate $\quad P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1|k-1}, u_k}$$

Step 2. Update

Innovation / measurement residual $\quad \tilde{y}_k = z_k - h(\hat{x}_{k|k-1})$

Innovation covariance $\quad S_k = H_k P_{k|k-1} H_k^T + R_k$

Near-Optimal Kalman Gain $\quad K_k = P_{k|k-1} H_k^T S_k^{-1}$

Updated State Estimate $\quad \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$

Updated Covariance Estimate $\quad P_{k|k} = (I - K_k H_k) P_{k|k-1}$

$$H = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k|k-1}}$$

Initialize

$$\hat{x}_{0|0} = E[x(t_0)]$$

$$P_{0|0} = E\left[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))^T\right]$$

④ Unscented Kalman Filter – 3rd order Talyor Series

1) Unscented Transformation — using sigma vector $\mathcal{X}_i$

x, mean $\bar{x}$, covariance P_x

$$\mathcal{X}_i = \begin{cases} \bar{x} & i = 0 \\ \bar{x} + \left(\sqrt{(L+\lambda) P_x}\right)_i & i = 1, 2, \cdots L \\ \bar{x} - \left(\sqrt{(L+\lambda) P_x}\right)_{i-L} & i = L+1, \cdots 2L \end{cases}$$

Sigma vector

- $\lambda = \alpha^2 (L + \kappa) - L$
- L : dimension of x
- α : determine the spread of sigma points around $\bar{x}$, small, eg 1e-3
- κ : secondary scaling parameter, usually = 0
- $\left(\sqrt{(L+\lambda) P_x}\right)_i$: the ith row of matrix square root

$$W_i = \begin{cases} W_0^{(m)} = \lambda / (L + \lambda) & i = 0 \\ W_0^{(c)} = \lambda / (L+\lambda) + (1 - \alpha^2 + \beta) & i = 0 \\ W_i^{(m)} = W_i^{(c)} = 1 / \{2(L+\lambda)\} & i = 1, 2, \cdots 2L \end{cases}$$

weights

- β : prior knowledge of distribution of x

$$y_i = g(\mathcal{X}_i)$$

$$\bar{y} \approx \sum_{i=0}^{2L} W_i^{(m)} y_i$$

$$P_y \approx \sum_{i=0}^{2L} W_i^{(c)} \{y_i - \bar{y}\}\{y_i - \bar{y}\}^T$$

2) Step 1 : Initialize

$$\hat{x}_0 = E[x_0]$$

$$P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$$

$$\hat{x}_0^a = E[x^a] = \begin{bmatrix} \hat{x}_0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_0^a = E[(x_0^a - \hat{x}_0^a)(x_0^a - \hat{x}_0^a)^T] = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & P_v & 0 \\ 0 & 0 & P_n \end{bmatrix}$$

Step 2 : Sigma Points

$$\begin{bmatrix} x \\ v \\ n \end{bmatrix} \longrightarrow \begin{aligned} x_k &= F(x_{k-1}, w) + B v_{k-1} \\ y_k &= x_k + n_k \end{aligned}$$

$$\mathcal{X}_{k-1}^a = \left[\hat{x}_{k-1}^a \quad \hat{x}_{k-1}^a \pm \sqrt{(L+\lambda) P_{k-1}^a}\right] \longrightarrow \begin{bmatrix} \mathcal{X}^x \\ \mathcal{X}^v \\ \mathcal{X}^n \end{bmatrix}$$

Time Update

$$\mathcal{X}_{k|k-1}^x = F[\mathcal{X}_{k-1}^x, \mathcal{X}_{k-1}^v]$$

$$\hat{x}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{X}_{i,k|k-1}^x$$

$$P_k^- = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1}^x - \hat{x}_k^-][\mathcal{X}_{i,k|k-1}^x - \hat{x}_k^-]^T$$

$$\mathcal{Y}_{k|k-1} = H[\mathcal{X}_{k|k-1}^x, \mathcal{X}_{k-1}^n]$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{Y}_{i,k|k-1}$$

Measurement

$$P_{\tilde{y}_k \tilde{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{Y}_{i,k|k-1} - \hat{y}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{y}_k^-]^T$$

$$P_{x_k y_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{x}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{y}_k^-]^T$$

$$K = P_{x_k y_k} P_{\tilde{y}_k \tilde{y}_k}^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K(y_k - \hat{y}_k^-)$$

$$P_k = P_k^- - K P_{\tilde{y}_k \tilde{y}_k} K^T$$