Task 2.3 Derivation

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$$E(I) = \int_\Omega \underbrace{\left[(I(x) - I_0(x))^2 + \lambda \| \nabla I(x) \|^2 \right]}\, dx$$

$$E(I) = \int_\Omega L(I, \nabla I, x)\, dx$$

The Gâteaux derivative:

$$\delta E(I;h) = \lim_{\alpha \to \infty} \frac{1}{\alpha} \left[E(I + \alpha h) - E(I) \right] \quad \text{and} \quad \delta E(\hat{I}; h) = 0 \,,\ \forall h$$

$$= \lim_{\alpha \to \infty} \frac{1}{\alpha} \int_\Omega \left[L(I + \alpha h, \nabla I + \alpha \nabla h, x) - L(I, \nabla I, x) \right] dx$$

Taylor expansion....

$$= \lim_{\alpha \to \infty} \frac{1}{\alpha} \int_\Omega \left[\cancel{L(I, \nabla I, x)} + \frac{\partial L}{\partial I} \alpha h + \frac{\partial L}{\partial (\nabla I)} \alpha \nabla h + O(\alpha^2) \cancel{- L(I, \nabla I, x)} \right] dx$$

L'Hopital rule...

$$= \int_\Omega \left[\frac{\partial L}{\partial I} h + \frac{\partial L}{\partial (\nabla I)} \nabla h \right] dx = \int_\Omega \frac{\partial L}{\partial I} h \, dx + \int_\Omega \frac{\partial L}{\partial (\nabla I)} \nabla h \, dx$$

Integral by parts....

$$= \int_\Omega \frac{\partial L}{\partial I} h \, dx + \cancel{\frac{\partial L}{\partial \nabla I} \cdot h \Big|_\Omega} - \int h \nabla \cdot \left(\frac{\partial L}{\partial (\nabla I)} \right) dx$$

$$= \int_\Omega \left[\frac{\partial L}{\partial I} - \nabla \cdot \left(\frac{\partial L}{\partial (\nabla I)} \right) \right] h(x) \, dx$$

(Using highlighted knowledge about extremum conditions.)

$$\frac{\partial L}{\partial I} - \nabla \cdot \left(\frac{\partial L}{\partial (\nabla I)} \right) = 0$$

Plug $L(I, \nabla I, x)$...

$$2(I - I_0) - \nabla \cdot (2 \lambda \nabla I) = 0$$

$$\cancel{2}(I - I_0) = \cancel{2} \lambda \operatorname{div}(\nabla I)$$

$$\boxed{I_0 = I - \lambda \operatorname{div}(\nabla I)}$$