Task 1 Proof

Friday, 26 April 2019 21:11

## PROPERTIES:

1. $\hat{q} = q_0 + \epsilon q_\epsilon \;,\; \epsilon^2 = 0$
2. $\|\hat{q}\| = \|q_0\| + \epsilon \frac{\langle q_0, q_\epsilon \rangle}{\|q_0\|}$
3. $\hat{q} = \cos(\hat{\theta}/2) + \hat{s}\sin(\hat{\theta}/2) \;,\; \hat{\theta} = \theta_0 + \epsilon\theta_\epsilon$ and $\hat{s} = s_0 + \epsilon s_\epsilon$
4. $\langle s_0, s_\epsilon \rangle = 0$ and $\langle s_0, s_0 \rangle = 1$
5. $\hat{q}^t = e^{t\log(\hat{q})}$
6. $e^{\hat{q}} = \cos(\|\hat{q}\|) + \frac{\hat{q}}{\|\hat{q}\|}\sin(\|\hat{q}\|)$
7. $\log\left(\cos(\hat{\theta}/2) + \hat{s}\sin(\hat{\theta}/2)\right) = \hat{s}\frac{\hat{\theta}}{2}$

## PROOF: (Circled numbers denote the utilized property)

$$\hat{q}^t = e^{t\log(\hat{q})} \quad ⑤$$

$$= e^{t\log\left(\cos(\hat{\theta}/2) + \hat{s}\sin(\hat{\theta}/2)\right)} \quad ③$$

$$= e^{\hat{s}\frac{\hat{\theta}t}{2}} \quad ⑦$$

$$= \frac{t}{2}\left(\theta_0 + \epsilon\theta_\epsilon\right)\cdot\left(s_0 + \epsilon s_\epsilon\right) \quad ③$$

$$= \frac{t}{2}\left(\theta_0 s_0 + \epsilon s_\epsilon + \epsilon\theta_\epsilon s_0 + \cancel{\epsilon^2\theta_\epsilon s_\epsilon}\right) \quad ①$$

$$= \underbrace{\frac{t}{2}\theta_0 s_0}_{x_0} + \epsilon\underbrace{\left(\frac{t}{2}\theta_0 s_\epsilon + \frac{t}{2}\theta_\epsilon s_0\right)}_{x_\epsilon}$$

$$\Rightarrow \hat{s}\frac{\hat{\theta}t}{2} = \hat{x} = x_0 + \epsilon x_\epsilon$$

$$= e^{\hat{x}}$$

$$= \cos\left(\|\hat{x}\|\right) + \frac{\hat{x}}{\|\hat{x}\|}\sin\left(\|\hat{x}\|\right)$$

$$= \|x_0\| + \epsilon\frac{\langle x_0, x_\epsilon\rangle}{\|x_0\|} \quad ②$$

$$= \left\|\frac{t}{2}\theta_0 s_0\right\| + \epsilon\frac{\left\langle \frac{t}{2}\theta_0 s_0, \frac{t}{2}\theta_0 s_\epsilon + \frac{t}{2}\theta_\epsilon s_0\right\rangle}{\left\|\frac{t}{2}\theta_0 s_0\right\|}$$

$$= \sqrt{\left\langle \frac{t}{2}\theta_0 \underline{s_0}, \frac{t}{2}\theta_0 \underline{s_0}\right\rangle} + \epsilon\frac{\left\langle \frac{t}{2}\theta_0 \underline{s_0}, \frac{t}{2}\theta_0 \underline{s_\epsilon}\right\rangle + \left\langle \frac{t}{2}\theta_0 \underline{s_\epsilon}, \frac{t}{2}\theta_\epsilon \underline{s_0}\right\rangle}{\sqrt{\left\langle \frac{t}{2}\theta_0 \underline{s_0}, \frac{t}{2}\theta_0 \underline{s_0}\right\rangle}}$$

$$= \frac{t}{2}\theta_0 + \epsilon\frac{\left(\frac{t}{2}\right)^2\theta_0\theta_\epsilon}{\frac{t}{2}\theta_0} \quad ④$$

$$= \frac{t}{2}\theta_0 + \epsilon\frac{t}{2}\theta_\epsilon = \frac{t}{2}\hat{\theta} \quad ③$$

$$= \cos\left(\frac{\hat{\theta}t}{2}\right) + \frac{\hat{s}\frac{\hat{\theta}t}{2}}{\frac{\hat{\theta}t}{2}}\sin\left(\frac{\hat{\theta}t}{2}\right)$$

$$= \cos\left(\frac{\hat{\theta}t}{2}\right) + \hat{s}\sin\left(\frac{\hat{\theta}t}{2}\right)$$