ECE 1508 Multiuser Information Theory
Prof. Wei Yu

Homework Set #2

1. Cover and Thomas, Problem 9.8. *Parallel Gaussian channels.*

2. Cover and Thomas, Problem 9.9. *Vector Gaussian channel.*

3. Cover and Thomas, Problem 9.12. *Time-varying channel.*

4. Cover and Thomas, Problem 9.17. *Impulse power.*

5. *Pulse Position Modulation (PPM):* In the previous problem, you find that the impulse scheme is suboptimal. In this problem, we show that if you are allowed to choose where to place the impulse (among the time indices $i = 1, 2, \cdots, n$), this PPM scheme is capacity achieving on n uses of the Gaussian channel $Y_i = X_i + Z_i$ at low SNR.

   (a) Let $Z_1, \cdots, Z_n$ be i.i.d. Gaussian random variables $\mathcal{N}(0, \sigma^2)$. Let $U = \max_i\{Z_i\}$. Let $t > 0$. Justify each of the following steps:

   $$\exp(t\mathbb{E}[U]) \leq \mathbb{E}[\exp(tU)] = \mathbb{E}\max_i\{\exp(tZ_i)\} \leq \sum_{i=1}^{n} \mathbb{E}[\exp(tZ_i)] = n\exp(t^2\sigma^2/2).$$

   You may recognize the moment generating function of Gaussian random variable in the last step.

   (b) Rewrite $\mathbb{E}[U] \leq \frac{\ln n}{t} + \frac{t\sigma^2}{2}$. Minimize over t to arrive at the inequality

   $$\mathbb{E}[U] \leq \sigma\sqrt{2\ln(n)}.$$

   (c) Use the Chernoff bound to show that for all $a > 0$

   $$\text{Prob}\{U \geq \sigma(\sqrt{2\ln n} + a)\} \leq \exp(-a^2/2).$$

   (d) In PPM, information is encoded in the position of the pulse. Over n uses of the Gaussian channel, we set $X_i = \sqrt{nP}$ for *one* of $i = 1, \cdots, n$, and zero elsewhere. Please express the information rate R of PPM per channel use in nats, assuming that there is no detection error.

   (e) Please argue that it is possible to choose a constant L that depends on some (arbitrarily small) P_e target, and then by setting

   $$\sqrt{nP} = \sigma(\sqrt{2\ln n} + L)$$

   we can achieve a bounded target probability of detection error (e.g. $P_e = 10^{-6}$).

(f) Consider the low SNR regime, i.e., $\frac{P}{\sigma^2} \ll 1$. Please write down an approximation of the Gaussian channel capacity in the low SNR limit, call it C. Show that for the achievable rate R of PPM under P_e, we have

$$C < R(1 + \delta)$$

for some δ that goes to zero as SNR goes to zero. Hence, PPM is capacity approaching at low SNR.

6. *Achievable Rate of Pulse Amplitude Modulation (PAM):* Please generate the following plots in a graph with SNR (in dB) on the x-axis and achievable rates on the y-axis.

   - Gaussian channel capacity $C = \frac{1}{2}\log(1 + \text{SNR})$.
   - The capacity of a 2-PAM input additive Gaussian noise channel as a function of SNR.
   - Repeat for 4-PAM, 8-PAM and 16-PAM, assuming uniform distribution on the input.

   Which constellation(s) are optimum at low SNR? At high SNR, please numerically find the value of shaping loss (in dB), defined as the amount of extra power needed to achieve the same rate as the Shannon capacity limit.

   Please provide the derivations of the mutual information expressions for numerical evaluation and the code for generating the plot.

7. *Shaping Loss*: In this problem we will find the theoretical limit of shaping loss of PAM at high SNR. Consider the following two input distributions over n uses of the additive white Gaussian noise channel $Y_i = X_i + Z_i$, where X_i is subject to power constraint P:

   - $X_i^{(1)} \sim \mathcal{N}(0, P)$ i.i.d.
   - $X_i^{(2)} \sim \text{Unif}[-L, L]$ i.i.d.

   We write the mutual information $I(X; Y) = h(Y) - h(Y|X)$, where $h(Y|X) = h(Z)$ is the same in both cases, and approximate $h(Y) \approx h(X)$ in high SNR.

   (a) What is the relationship between P and L for which we have $h(X^{(1)}) = h(X^{(2)})$?

   (b) Find the amount of extra power needed for $X^{(2)}$ in order to achieve the same mutual information as $X^{(1)}$. The amount of extra power is a multiplicative factor which can be found analytically. Please find this analytic expression. Please also find its numerical value in dB.

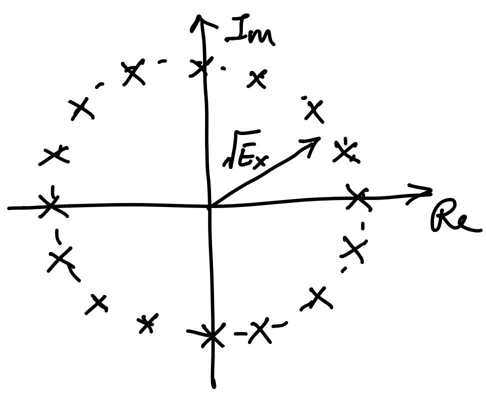
   (c) Compare the theoretical shaping loss with the numerical value found in the previous problem.

8. *Phase Shift Keying on Complex AWGN Channel:* Suppose that we use a phase-shift keying (PSK) modulation on a complex additive white Gaussian noise channel where the i.i.d. noises in the real and imaginary components both have a variance of $\frac{\sigma^2}{2}$. An M-PSK

modulation has M constellation points equally spaced on a circle of radius $\sqrt{P}$ from the origin, where P is the power of the constellation. Suppose that we wish to achieve a symbol probability of error of 10^{-6}.

Please derive an expression of the achievable rate of PSK signalling as a function of the SNR, defined as P/σ^2. (You may approximate the distance between two neighbouring constellation points on a circle of radius r as $2\pi r/M$.)

At high SNR, does the achievable rate of the PSK constellation have the same SNR scaling as the capacity of the complex AWGN channel?



e.g. 16 - PSK