Contents

Math Handbook

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If I have seen further, it is by standing on the shoulders of giants. - Isaac Newton (1676)

1 Laws of Mathematics

- Associative Laws

$a + (b + c) = (a + b) + c$: For addition
$a(bc) = (ab)c$: For multiplication
Associativity is the direction of operation processing (right to left, or left to right)

- Commutative Laws

$a + b = b + a$: For addition
$ab = ba$: For multiplication

- Distributive Laws

$a(b + c) = ab + ac$: For multiplication
$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}, a \neq 0$: For division

2 Logic

2.1 Sentential Logic

De Morgan's Laws

- $\neg(P \wedge Q) = \neg P \vee \neg Q$
- $\neg(P \vee Q) = \neg P \wedge \neg Q$

Commutative Laws

- $P \wedge Q = Q \wedge P$
- $P \vee Q = Q \vee P$

Associative Laws

- $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$
- $P \vee (Q \vee R) = (P \vee Q) \vee R$

Idempotent Laws (an element of a set is unchanged in value if operated on by itself)

- $P \wedge P = P$
- $P \vee P = P$

Distributive Laws

- $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

Absorption Laws

- $P \vee (P \wedge Q) = P$
- $P \wedge (P \vee Q) = P$

Double Negation Law

- $\neg\neg P = P$

Tautology Laws

- $P \wedge (\text{tautology}) = P$
- $P \vee (\text{tautology}) = \text{tautology}$
- $\neg(\text{tautology}) = \text{contradiction}$

Contradiction Laws

- $P \wedge (\text{contradiction}) = \text{contradiction}$
- $P \vee (\text{contradiction}) = P$
- $\neg(\text{contradiction}) = \text{tautology}$

Conditional Laws

- $P \rightarrow Q = \neg P \vee Q$

- $P \to Q$, converse statement: $Q \to P$
- $P \to Q = \neg Q \to \neg P$: contrapositive law
- $P \vee S \to Q = (P \to Q) \wedge (S \to Q)$
- $(P \to Q) \wedge (P \to S) = P \to Q \wedge S$
- $(P \to Q) \wedge (R \to \neg Q) = P \to \neg R$
- $P \to Q$ means:
  Q, if P or Q is a necessary condition for P
  P only if Q i.e. P is a sufficient condition for Q
- $P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$: Biconditional law

2.2 Quantificational Logic

- $\exists$: existential quantifier (distributes over disjunction)
- $\forall$: universal quantifier (distributes over conjunction)

and $\forall\, (xP(x)) \equiv \neg\exists x \neg P(x)$
πχ. όλα είναι κόκκινα $\equiv$ δεν υπάρχει κανένα που να μην είναι κόκκινο.

Quantifier Negation Laws
- $\neg\exists x P(x) = \forall x \neg P(x)$
- $\neg\forall x P(x) = \exists x \neg P(x)$

Abbreviation rules
- $\forall x\big(x \in A \to P(x)\big) = \forall x \in A\ P(x)$
- $\exists x\big(x \in A \to P(x)\big) = \exists x \in A\ P(x)$

3 Set Theory

- Zermelo–Fraenkel set theory (ZFC - C stands for axiom of Choice) is the standard form of axiomatic set theory and as such is the most common foundation of mathematics. It is one of several axiomatic systems that were proposed in the early twentieth century to formulate a theory of sets free of paradoxes such as Russell's paradox.
- Axiom of Choice, or AC, is an axiom of set theory equivalent to the statement that the Cartesian product of a collection of non-empty sets is non-empty. It states that for every indexed family $(S_i)_{i \in I}$ of nonempty sets there exists an indexed family $(x_i)_{i \in I}$ of elements such that $x_i \in S_i \; \forall i \in I$. The axiom of choice says that given any collection of bins, each containing at least one object, it is possible to make a selection of exactly one object from each bin.
- Truth set of $P(x) = \{x | P(x)\}$
- $B = \{x \mid \underbrace{x \text{ is a prime number}}_{\text{elementhood test for the set B}} \}$
- $A \cap B = (x \in A) \wedge (x \in B)$: Conjunction
- $A \cup B = (x \in A) \vee (x \in B)$: Disjunction
- $A \backslash B = x \in A \wedge x \notin B$: A without B
- Predicate: A Boolean-valued function $P : X \to \{\text{true, false}\}$: called "The predicate on X".
- $A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$: Symmetric difference of A and B
- $A \cap B \neq \varnothing \to \exists x (x \in A \wedge x \in B)$
- $A \cap B = \varnothing \to \neg \exists x (x \in A \wedge x \in B)$
- $\exists! x \; P(x) = \begin{cases} \exists x P(x) \wedge \forall y \forall z \big[(P(y) \wedge P(z)) \to y = z \big] \\ \exists x \big[P(x) \wedge \forall y (P(y) \to y = x) \big] \\ \exists x \big[P(x) \wedge \neq \exists y (P(y) \wedge y \neq x) \big] \end{cases}$
- Singleton Set: A set that contains only one element.
- Alphabet: A set with finite number of elements.
- $C \backslash A = C \cap (\Omega \backslash A)$, $U = \Omega$ ie. the universal set
- A Convex set is a region such that, for every pair of points within the region, every point on the straight line segment that joins the pair of points is also within the region. For example, a solid cube is a convex set, but anything that is hollow or has an indent, for example, a crescent shape, is not convex.
- A set is called denumerable exactly when it can be put in one-to-one correspondence with the set of natural numbers.
- A set $\mathcal{S}$ is countable if there is a sequence $r_1, r_2, r_3, \ldots$ which consists of all the elements of $\mathcal{S}$.
- A linear order on a set S satisfies two properties:

– For any $a, b \in S$, exactly one of $a < b, a = b$ or $a > b$ is true.
– For all $a, b, c \in S$, if $a < b$ and $b < c$ then $a < c$ (transitivity).
Examples of sets with a natural linear order are integers, floats, characters and strings in C.

- A set of elements (vectors) in a vector space V is called a basis (or a set of basis vectors) if the vectors are linearly independent and every vector in the vector space is a linear combination of this set. In more general terms, a basis is a linearly independent spanning set.
- A basis function is an element of a particular basis for a function space. Every continuous function in the function space can be represented as a linear combination of basis functions, just as every vector in a vector space can be represented as a linear combination of basis vectors.
- Idempotence is a property of certain operations that they can be applied multiple times without changing the result of the initial application.

For example, the absolute value unary operation or function is idempotent, since $||x|| = x$

Cartesian Product

Cartesian Product is the collection of all ordered pairs of two given sets such that the first elements of the pairs are chosen from one set and the second elements from the other set; this procedure generalizes to an infinite number of sets.

- $A \times B = (a, b)|a \in A \wedge b \in B$: Cartesian product of sets A, B

Properties

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
- $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$
- $A \times \varnothing = \varnothing$
- $\big[(A = B) \vee (A = \varnothing \wedge B = \varnothing)\big] \rightarrow A \times B = B \times A$

3.1 Family Sets

- $\cap\mathcal{F} = \{x \mid \forall A \in \mathcal{F}(x \in A)\} = \{x \mid \forall A(A \in \mathcal{F} \rightarrow x \in A)\}$: Family Set $\mathcal{F}$
- $\cup\mathcal{F} = \{x|\exists A \in \mathcal{F}(x \in A) = \{x|\exists A(A \in \mathcal{F} \wedge x \in A)\}$

Alternative notation for family sets:

- $\cup\mathcal{F} = \bigcap\limits_{i \in I} A_i = \{x|\forall i \in I(x \in A_i)\}$
- $\cup\mathcal{F} = \bigcup\limits_{i \in I} A_i = \{x|\exists i \in I(x \in A_i)\}$
- Ordered pair: an ordered pair (a, b) is a pair of objects. The order in which the objects appear in the pair is important, ie. the ordered pair (a, b) is different from the ordered pair (b, a) unless $a = b$. (In contrast, the unordered pair $\{a, b\}$ equals the unordered pair $\{b, a\}$).
- $x \in A \rightarrow x \in \cup\{A\}$
- $A \in \mathcal{F} \rightarrow A \subseteq \cup\mathcal{F}$
- $A(A \in \mathcal{F} \wedge x \in A) \rightarrow x \in \cup\mathcal{F}$
- $(x \in A \wedge x \in \cap\mathcal{F}) \rightarrow A \subseteq \cap\mathcal{F}$
- $A \subseteq B = \forall x(x \in A \rightarrow x \in B)$

- $A \not\subseteq B = \exists x(x \in A \wedge x \notin B)$
- $(A \subseteq B \wedge A \neq B) \rightarrow A \subset B$, ie. A is a proper subset of B
- $\mathcal{P}(A) = \{x | x \subseteq A\}$: Power set of A
- $B = \mathcal{P}(A) \rightarrow \forall x(x \in B \rightarrow x \subseteq A)$
- $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$
- If A has n elements then $\mathcal{P}(A)$ has 2^n elements and $\mathcal{P}_2(A)$ has $\dfrac{n(n-1)}{2}$ elements

Indexed Family notation of a set:

- $A = \{x_i | i \in I\} = \{x | \exists i \in I(x = x_i)\}$, I : index set
- $x = \in \{x_i | i \in I\} = \exists i \in I(x = x_i)$

4 Algebra

4.1 Algebraic Identities

- $a^2 - b^2 = (a+b)(a-b)$
- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
- $(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a)$
- $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$: Euler's Identity
- $a + b + c = 0 \rightarrow a^3 + b^3 + c^3 = 3abc$
- $a^2 + b^2 > 0 \rightarrow a \neq 0 \vee b \neq 0$
- $a^2 + b^2 = 0 \rightarrow a = 0 \wedge b = 0$
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

- $a \propto b \rightarrow \dfrac{a}{b} = \text{σταθ.}$, αν τα ποσά a, b είναι ανάλογα, τότε $\exists k(a = kb)$ ο λόγος τους είναι σταθερός
- a, b ομόσημοι $\leftrightarrow a \cdot b > 0 \leftrightarrow a \backslash b > 0$
- a, b ετερόσημοι $\leftrightarrow a \cdot b < 0 \leftrightarrow a \backslash b < 0$
- $(a > b) \wedge (c > d) \rightarrow a + c > b + d$
- $(a > b) \wedge (c > d) \rightarrow a \cdot c > b \cdot d$
- $\forall a, b \in \mathbb{R}^* \left(\dfrac{a}{b} + \dfrac{b}{a} \geqslant 2 \right)$
- $\forall a, b \in \mathbb{R}^* \forall n \in \mathbb{N}(n \geqslant 2) \left(\dfrac{a_1}{a_2} + \dfrac{a_2}{a_3} + \ldots + \dfrac{a_{n-1}}{a_n} + \dfrac{a_n}{a_1} \geqslant n \right)$

- Numbers that satisfy polynomial equations, are called algebraic numbers. All algebraic numbers are connected with the integers.
- A transcendental number is a real or complex number that is not algebraic. A complex number is algebraic if both its real and imaginary part is algebraic.
- A transcendental function is an analytic function that does not satisfy a polynomial equation.

- Σύστημα 2 εξισώσεων (ε) και (έ) $\rightarrow l \cdot$ (ε) $+ l' \cdot$ έ) : γραμμικός συνδυασμός των (ε) και (ε′). Συστήματα που προκύπτουν με γραμμικό συνδυασμό είναι ισοδύναμα.
- $\forall n \in \mathbb{N} \left[a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}) \right]$
- $\forall n \in \mathbb{N} \left[a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1}) \right]$
- $x \ll 1 \rightarrow (1 \pm x)^n \approx 1 \pm nx$

- $a, b \in \mathbb{R}\Big[\big(a^2 + b^2 \geqslant ab\big) \wedge \big(-(a^2+b^2) \leqslant -ab\big)\Big]$
- $\forall a, b \in \mathbb{R}\Big[a^2 + b^2 \geqslant ab \wedge -(a^2+b^2) \leqslant -ab\Big]$
- $\forall a_1, a_2, ..., a_n \in \mathbb{R}^*, \forall n \in \mathbb{N}(n \geqslant 2)\Big(\dfrac{a_1 + a_2 + ... + a_n}{n} \geqslant \sqrt{a_1 a_2 \cdots a_n}\Big)$: Arithmetic - Geometric mean inequality
- $\forall a_1, a_2, ..., a_n \in \mathbb{R}^*, \forall n \in \mathbb{N}(n \geqslant 2)\Big(\dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \geqslant \sqrt{a_1 a_2 \cdots a_n}\Big)$: Harmonic - Geometric mean inequality
- $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
- $a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) = (a+b+c)\cdot(a-b+c)\cdot(a+b-c)(a-b-c)$: De Moivre Identity

Gauchy Identities

- $(a+b)^3 - a^3 - b^3 = 3ab(a+b)$
- $(a+b)^5 - a^5 - b^5 = 5ab(a+b)(a^2+ab+b^2)$
- $(a+b)^7 - a^7 - b^7 = 7ab(a+b)(a^2+ab+b^2)^2$

Εξισώσεις της μορφής: $ax^4 + bx^3 + cx^2 + bx + a = 0$, με $a \neq 0$, λέγονται αντίστροφες. Τις λύνουμε θέτοντας: $x + \dfrac{1}{x} = y$

- $1\text{grad} = \dfrac{9}{10}\ of\ 1^o \vee 1\text{grad} = \dfrac{\pi}{200}\ of\ 1\text{rad}$
- $45^o 36' 18" = 45^o + \big(\dfrac{36}{60}\big)^o + \big(\dfrac{18}{60\cdot 60}\big)^o = 45.605^o$

4.2 Absolute Values

- $|a| = a \leftrightarrow a \geqslant 0$
- $|a| = -a \leftrightarrow a \leqslant 0$
- $|x| > p \leftrightarrow (x < -p) \vee (x > p)$
- $|x| < p \leftrightarrow -p < x < p$
- $|x| = a \leftrightarrow (x = a) \vee (x = -a)$
- $\forall a \in \mathbb{R}(|a|^2 = a^2)$
- $|a \cdot b| = |a| \cdot |b|$
- $|P(x)| \geqslant x \leftrightarrow P(x) \leqslant -|x| \wedge P(x) \geqslant |x|\ ,\ \forall x \in \mathbb{R}$
- $|P(x)| \leqslant x \leftrightarrow -|x| \leqslant P(x) \leqslant |x|$
- $\big||a| - |b|\big| \leqslant |a \pm b| \leqslant |a| + |b|\ , \forall a, b \in \mathbb{R}$
- $\forall x, x_0 \in \mathbb{R}, p \in \mathbb{R}^+\big(|x - x_0| < p \leftrightarrow x_0 - p < x < x_0 + p\big)$
- $\forall x, x_o \in \mathbb{R}, p \in \mathbb{R}^+\big(|x - x_0| > p \leftrightarrow x < x_0 - p \vee x > x_0 + p\big)$
- $\forall x, x_0 \in \mathbb{R}, p \in \mathbb{R}^+\big[|x - x_0| < p \leftrightarrow d(x, x_0) < p \leftrightarrow x \in (x_0 - p, x_0 + p)\big]$
- $\forall x, x_0 \in \mathbb{R}, p \in \mathbb{R}^+\big[|x - x_0| > p \leftrightarrow d(x, x_0) > p \leftrightarrow x \in (-\infty, x_0 - p) \cup (x_0 + p, +\infty)\big]$
- $|a - b| = |b - a|$

4.3 Powers, Radicals

- $\forall k \in \mathbb{N}^* \left[(\sqrt[n]{a})^k = \sqrt[n]{a^k} \right]$
- $\forall a \geqslant 0 \ (a^{1/n} = \sqrt[n]{a}$
- $a\sqrt[n]{b} = \sqrt[n]{a^n \cdot b}$
- $\sqrt[k]{\sqrt[n]{a}} \ \sqrt[k \cdot n]{a}$
- $a < b \leftrightarrow \sqrt[n]{a} < \sqrt[n]{b} \ , \forall a \geqslant 0$
- $a^n = \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text{ times}}$
- $a^n \cdot a^m = a^{n+m}$
- $a^n \cdot b^n = (a \cdot b)^n$
- $\dfrac{a^n}{a^m} = a^{n-m}$
- $a^{-n} = \dfrac{1}{a^n}$
- $\dfrac{a^n}{b^n} = (\dfrac{a}{b})^n$
- $(a^n)^m = a^{n \cdot m}$
- $a^{n^m} = a^{(n^m)}$
- $a^{n/m} = \sqrt[m]{a^n}$
- $\sqrt[m]{(a^n)} = a^{n/m}$
- $a^0 = 1$
- $0^n = 0 \ , \forall n > 0$
- $1^n = 1 \ , \forall n \in \mathbb{R}$
- $(-1)^n = \begin{cases} 1 & n = 2k \ , n, k \in \mathbb{Z} \\ -1 & n = 2k+1 \end{cases}$
- $\forall a \in \mathbb{R}, n \in \mathbb{Z} (-a)^n = \begin{cases} a^n & n = 2k, k \in \mathbb{Z} \\ -a^n & n = 2k+1 \end{cases}$
- $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$
- $\tilde{X} = \sqrt{\bar{X}^2}$, $\tilde{X}$: ενεργός τιμή του μεγέθους X

  $\bar{X}$: μέση τιμή του μεγέθουςX
- Surds are numbers left in square root form, or cube root form etc. They are therefore irrational numbers.

4.4 Logarithms

Το $\log_b x$ είναι ο εκθέτης στον οποίο πρέπει να υψώσουμε το b, για την εύρεση του x

- $y = log_b x \leftrightarrow x = b^y \ , x > 0, b > 0, b \neq 1$
- $log_b(x \cdot y) = log_b x + log_b y$

- $log_b(x/y) = log_b x - log_b y$
- $log_b(x^n) = n log_b x$
- $log_b x = \frac{log_a x}{log_a b}, log_a x \cdot log_a b = log_b x \big(log_a b = \frac{1}{log_b a}\big)$
- $log_b b = 1$
- $log_b 1 = 0 \leftrightarrow 1 = b^0$
- $log_e a = \ln\ a$: Natural logarithm, $e = 2.71828$: Euler's number
- $b^{log_b x} = x$, αφού antilog $_b(log_b(x)) = x$
- $log_b b^x = x$
- $log_a b \cdot log_b a = 1$
- $log_a x = log_a^2 x^2$
- $log_a \theta + log_{\frac{1}{a}} \theta = 0$
- $a > b \rightarrow log_a b < 1$
- $a < b \rightarrow log_a b > 1$
- $a = b \rightarrow log_a b = 1$
- $x^{log\ x} = a^{log\ x}$
- $a^x = e^{x\ \ln\ \ a}$, αφού: $a = e^{\ln\ \ a}$
- $n^{\log\log n} = e^{\log n \log\log n}$ (natural logarithms)

4.5 Theorems

- Remainder Theorem: Polynomial $P(x)$ division with $x - p$, yields
$P(x) = (x - \rho) \cdot \pi(x) + \upsilon$, $\upsilon = P(\rho)$
- Factor Theorem: A polynomial $F(x)$ has a factor $x - k$ iff $f(k) = 0$(i.e. k is a root of f)
- Rational Root Theorem: Suppose polynomial equation with interger coefficients: $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$. Then the rational solution: $x_0 = p/q$ (expressed in lowest terms) of the equation, satisfies: (a) p is an integer factor of the constant term a_0 and (b) q is an integer factor of the leading coefficient a_n.
- Ταυτότητα Ευκλείδειας διαίρεσης: $\Delta(\chi) \cdot \pi(\chi) + \upsilon(\chi)$ $\Delta(\chi)$: Διαιρετέος, δ(χ): διαιρέτης, π(χ): πηλίκο, υ(χ): υπόλοιπο

- Horner's Method

a_n	a_{n-1}	a_{n-2}	$\ldots$	a_1 (a_{n-n+1})
	$a_n x_0$	$a_n x_0^2 + a_{n-1} x_0$	$\ldots$	$a_n x_0^{n-1} + \ldots + a_2 x_0$
a_n	$a_n x_0 + a_{n-1}$	$a_n x_0^2 + a_{n-1} x_0 + a_{n-2}$	$\ldots$	$a_n x_0^{n-1} + a_{n-1} x_0^{n+2} + \ldots + a_2 x_0 + a_1$

a_0	x_0
$a_n x_0^n + \ldots + a_2 x_0^2 + a_1 x_0$	

- $\Delta(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \delta(x) = x_0$
- $\pi(x) = a_n x^{n-1} + (a_n x_0 + a_{n-1}) x^{n-2} + (a_n x_0^2 + a_{n-1} x_0 + a_{n-2}) x^{n-3} + \ldots + (a_n x_0^{n-1} +$

$a_{n-1}x_0^{n-2} + \ldots + a_2x_0 + a_1)$
- $\upsilon(x) = \Delta(x_0)$

- The simplest form of factorization is the extraction of the HCF from an expression.
- Fundamental Theorem of Algebra: Every non-constant single-variable polynomial with complex coefficients has at least one complex root (Alternatively:) Every polynomial expression $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ can be written as a product of n linear factors in the form: $f(x) = a_n(x - r_1)(x - r_2)(\ldots)(x - r_n), a_i, r_i \in \mathbb{C}$
- If $a_1, a_2, a_3, \ldots, a_n$ are the roots of:

$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$ $(p_0 \neq 0)$, then
- sum of the roots $= -p_1/p_0$
- sum of the roots, two at a time $= p_2/p_0$
- sum of the roots, three at a time $= -p_3/p_0$
- sum of the roots, n at a time $= (-1)^n p_n/p_0$

- Transforming a cubic: $x^3 + ax^2 + bx + c = 0$, to its reduced form: $y^3 + py + q = 0$, by the substitution: $x = y - \dfrac{a}{3}$, only when $a > 0$.
- Tartaglia's solution for a real root of a cubic equation of the form:

$x^3 + ax + b = 0, a > 0$ is:

$$x = \left\{ -\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{\frac{1}{3}} + \left\{ -\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{\frac{1}{3}}$$

- Solution of a Quartic equation. There are three ways:

1. Numerically,
2. Ferrari-Cardano procedure, or one of its kin,
3. By design, it could be one of the relatively few such equations that collapse, because there are some very simple roots.

- Continued fraction (CF): an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on. There are two kinds of continued fractions, 1. Finite (or terminated) continued fractions and 2. Infinite continued fractions. In a finite continued fraction, the iteration/recursion is terminated after finitely many steps by using an integer in lieu of another continued fraction. In contrast, an infinite continued fraction is an infinite expression. In either case, all integers in the sequence, other than the first, must be positive. The integers a_i are called the coefficients, or terms of the continued fraction.
- Finite continued fraction:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Such a continued fraction is sometimes represented as: $[a_0, a_1, \ldots, a_n]$, denoting the coefficients.

- Infinite continued fraction:

$$[a_0, a_1, a_2, \ldots] = \lim_{n\to\infty} x_n$$

5 Trigonometry

Trigonometric Identities

- $\sin(x) = \sin(\theta) \leftrightarrow (x = 2\kappa\pi + \theta) \vee (x = 2\kappa\pi + (\pi - \theta))$
- $\cos(x) = \cos(\theta) \leftrightarrow (x = 2\kappa\pi + \theta) \vee (x = 2\kappa\pi - \theta)$
- $\tan(x) = \tan(\theta) \leftrightarrow x = \kappa\pi + \theta$
- $\cot(x) = \cot(\theta) \leftrightarrow x = \kappa\pi + \theta$
- $\sin^2(x) + \cos^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\operatorname{cosec}^2(x) - \cot^2(x) = 1$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$
- $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$
- $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{1}{\cot(x)}$
- $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$
- $\sec(x) = \dfrac{1}{\cos(x)}$
- $\sin(A + B) = \sin A \ \cos B + \cos A \ \sin B$
- $\sin(A - B) = \sin A \ \cos B - \cos A \ \sin B$
- $\cos(A + B) = \cos A \ \cos B - \sin A \ \sin B$
- $\cos(A - B) = \cos A \ \cos B + \sin A \ \sin B$
- $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \ \tan B}$
- $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \ \tan B}$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\cos^2 x = \dfrac{1 - \cos 2x}{2}$
- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$
- $\sin A + \sin B = 2\sin\left(\dfrac{A + B}{2}\right)\cos\left(\dfrac{A - B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\dfrac{A - B}{2}\right)\cos\left(\dfrac{A + B}{2}\right)$
- $\cos A + \cos B = 2\cos\left(\dfrac{A + B}{2}\right)\cos\left(\dfrac{A - B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\dfrac{A + B}{2}\right)\sin\left(\dfrac{A - B}{2}\right)$

- $a, b \neq 0 \longrightarrow \forall x \in \mathbb{R} : \alpha \ \sin(x) + \beta \ \cos(x) = \rho \ \sin(x+\phi)$,

$\rho = \sqrt{\alpha^2+\beta^2}, \phi \in \mathbb{R} : \tan(\phi) = \dfrac{\beta}{\alpha}$

- $\sin(-\omega) = \sin(\omega)$
- $\cos(-\omega) = \cos(\omega)$

Γωνίες με άθροισμα 180 μοίρες ($\omega + \omega' = 180^o$)
- $\sin(180^o - \omega) = \sin(\omega)$
- $\cos(180^o - \omega) = -\cos(\omega)$
- $\tan(180^o - \omega) = -\tan(\omega)$
- $\cot(180^o - \omega) = -\cot(\omega)$

Γωνίες που διαφέρουν κατά 180 μοίρες ($\omega' = 180^o + \omega$)
- $\sin(180^o + \omega) = -\sin(\omega)$
- $\cos(180^o + \omega) = -\cos(\omega)$
- $\tan(180^o + \omega) = \tan(\omega)$
- $\cot(180^o + \omega) = \cot(\omega)$

Γωνίες με άθροισμα 90 μοίρες $\omega' + \omega = 90^o$
- $\sin(90^o - \omega) = \cos(\omega) = -\sin(\omega - 90^o)$
- $\cos(90^o - \omega) = \sin(\omega) = \cos(\omega - 90^o)$
- $\tan(90^o - \omega) = \cot(\omega)$
- $\cot(90^o - \omega) = \tan(\omega)$

- $\tan(\theta) = \dfrac{a}{b} \rightarrow \sin(\theta) = \dfrac{a}{\sqrt{a^2+b^2}} \vee \cos(\theta) = \dfrac{b}{\sqrt{a^2+b^2}}$
- $\sin(2a) = \dfrac{2\tan\ a}{1+\tan^2 a}$
- $\cos(2a) = \dfrac{1-\tan^2 a}{1+\tan^2 a}$
- $\tan^2 a - \sin^2 a = \tan^2 a \cdot \sin^2 a$
- $\sin(a+b) \cdot \sin(a-b) = \sin^2 a - \sin^2 b$
- $\sin(3x) = 3\sin x - 4\sin^3 x$
- $\cos(3x) = 4\cos^3 x - 3\cos x$
- $\tan(3x) = 3\tan x - \tan^3 x$
- $\cot(nx) - \tan(nx) = 2\cot(2nx)$
- $\cos(\sin^{-1}(x)) = \sin(cos^{-1}(x)) = \sqrt{1-x^2}$
- $\tan(\sec\ ^{-1}(x) = \sqrt{x^2-1}$
- $\cos(\tan^{-1}(x)) = \dfrac{1}{\sqrt{1+x^2}}$

Ταυτότητες για στοιχεία τριγώνου
- $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

- $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$
- $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$
- $\sin 2A+\sin 2B+\sin 2C=4\sin A\ \sin B\ \sin C$
- $\cos 2A+\cos 2B+cos2C=1-4\cos A\ \cos B\ \cos C$
- $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\ \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$
- $\tan\frac{A}{2}\cdot\tan\frac{B}{2}+\tan\frac{B}{2}\cdot\tan\frac{C}{2}+\tan\frac{C}{2}\cdot\tan\frac{A}{2}=1$
- $\cot A\cdot\cot B+\cot B\cdot\cot C+\cot C\cdot\cot A=1$
- $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R:$ Νόμος Ημιτόνων,

όπου R : ακτίνα περιγεγγραμένου κύκλου του τριγώνου.

Νόμος Συνημιτόνων
- $a^2=b^2+c^2-2bc\ \cos A$
- $b^2=a^2+c^2-2ac\ \cos B$
- $c^2=a^2+b^2-2ab\ \cos C$

- $\dfrac{a-b}{a+b}=\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\tan\left(\dfrac{A-B}{2}\right)\cdot\tan\left(\dfrac{C}{2}\right):$ Νόμος Εφαπτομένων

- Με το νόμο των ημιτόνων μπορούμε να επιλύσουμε ένα τρίγωνο, όταν δίνονται 1. Μια πλευρά και δύο γωνίες του, ή 2. Δύο πλευρές και μια από τις μη περιεχόμενες γωνίες του
- Με το νόμο των συνημιτόνων μπορούμε να υπολογίσουμε μια οποιαδήποτε πλευρά ενός τριγώνου, αρκεί να δοθούν οι άλλες δύο και η περιεχόμενη τους γωνία.
- Ο νόμος εφαπτομένων μπορεί να χρησιμοποιηθεί σε ένα τρίγωνο, εάν δίνονται: 1. δύο πλευρές και η περιεχόμενη γωνία τους, ή 2. αν είναι γνωστές δύο γωνίες και μια πλευρά του

- $\sin\left(\frac{\pi}{8}\right)=\dfrac{\sqrt{2-\sqrt{2}}}{2}$
- $\cos\left(\frac{\pi}{8}\right)=\dfrac{\sqrt{2+\sqrt{2}}}{2}$
- $\tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1$
- $\cot\left(\frac{\pi}{8}\right)=\sqrt{2}+1$
- $\sin\left(\frac{\pi}{12}\right)=\dfrac{\sqrt{6}-\sqrt{2}}{4}$
- $\cos\left(\frac{\pi}{12}\right)=\dfrac{\sqrt{6}-\sqrt{2}}{4}$
- $\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$
- $\cot\left(\frac{\pi}{12}\right)=2+\sqrt{3}$

6 Proofs

A proof is a method for ascertaining truth.

A mathematical proof is a verification of a proposition, by a chain of logical deductions from a set of axioms.

Good proofs have seven characteristics (CCC-BE-OO). They are:

- correct
- complete
- clear
- brief
- elegant
- (well) organized
- ordered

6.1 Proof Techniques

- To prove a Goal of the form:

1. $\neg P$: a) Reexpress it as a positive statement
   b) Use proof by contradiction, i.e. assume P and try to reach a contradiction.
2. $P \to Q$: a) Assume P is true and prove Q.
   b) Prove the contrapositive, i.e. assume Q is false and prove that P is false.
3. $P \wedge Q$: Prove P and Q separately. In other words treat P and Q as two separate goals.
4. $P \vee Q$: a) Assume P is false and prove Q, or assume Q is false and prove P.
   b) Use proof by cases. In each case either prove P, or prove Q.
5. $P \leftrightarrow Q$: prove $P \to Q$ and $Q \to P$.
6. $\forall x P(x)$: Let x stand for an arbitrary object and prove $P(x)$. (if the letter x already stands for something in the proof, you will have to use a different letter.)
7. $\exists x P(x)$: Find a value of x that makes $P(x)$ true. Prove $P(x)$ for this value of x.

8. $\exists! x P(x)$:
   a) Prove $\exists x P(x)$ (existence) and $\neg y\,(P(y) \wedge y \neq x)$ (uniqueness).
   b) Prove the equivalent statement $\exists x\,[P(x) \wedge \forall y(P(y) \rightarrow y = x)]$, or some other similar one.

9. $\forall n \in \mathbb{N} P(n)$: a) Mathematical Induction: Prove $P(0)$ (base case) and $\forall n \in \mathbb{N}(P(n) \rightarrow P(n+1))$ (inductive step).
   b) Strong Induction: Prove $\forall n \in \mathbb{N}\,[(\forall i < n P(i)) \rightarrow P(n)]$.
   We can prove a stronger form of an induction, thus proving the more lenient form.

- To use a <u>Given</u> of the form:

1. $\neg P$: a) Reexpress as a positive statement.
   b) In a proof by contradiction you can reach a contradiction by proving P.
2. $P \rightarrow Q$: a) If you are also given P, or you can prove that P is true, then you can conclude that Q is true.
   b) Use the contrapositive.
3. $P \wedge Q$: Treat this as two givens P and Q.
4. $P \vee Q$: a) Use proof by cases. In case 1 assume P and in case 2 assume Q.
   b) If you are also given $\neg P$, or you can prove $\neg P$ then you can conclude Q. Similarly, if you know $\neg Q$, then you can conclude P.
5. $P \leftrightarrow Q$: Treat this as two givens: $P \rightarrow Q$ and $Q \rightarrow P$.
6. $\forall x P(x)$: You can plug in any value, say a, for x, and conclude $P(a)$.
7. $\exists x P(x)$: Introduce a new variable, say x_0, into the proof, to stand for a parrticular object for which $P(x_0)$ is true.

- Techniques that can be used in any proof:

1. Proof by Contradiction: Assume the goal is false and derive a contradiction.
2. Proof by Cases: Consider several cases that are exhaustive, i.e. that include all the posibilities. Prove the goal in each case.

6.2 Problem Solving

<u>Methodology - 4 Phases</u>

1. Understand the problem. See clearly what is required.

2. Figure out how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a plan.

3. Carry out the plan.

4. Look back at the completed solution, review, examine and discuss it. This way we consolidate our knowledge and develop our ability to solve problems.

Key suggestions

Rephrase your problem. This will provide a different perspective, which will stimulate your brain cells and awaken more memories and ideas from slumber.

- Did you examine all the data **(p.t.f.)** / the hypothesis **(p.t.p)** ?
- Did you use the whole condition (what links the data to the unknown) ?
- If at first it doesn't seem possible to satisfy the complete solution, we have two options. Whether we can solve a related (simpler analogous) problem and whether we can solve a part of the original problem.
- Restate the givens in order for them to match some mathematical definition. Having used the definition you eliminate the technical term. Then proceed using the definition. Can you restate the problem still differently ?
- At first visualize the problem as a whole as clearly and as vividly as you can. Do not concern yourself with details.
- Isolate the principal parts of the problem. The hypothesis and the conclusion are the principal parts of a "problem to prove". The unknown, the data and the conditions are the principal parts of a "problem to find". Go through the principal parts of the problem. Consider them one by one, consider them in turn, consider them in various combinations, relating each detail to other details and each to the whole of the problem.
- Start when you feel sure of your grasp of the main connection and you feel confident that you can supply the minor details that may be needed.
- Convince yourself about the correctness of each step by formal reasoning, or by intuitive insight, or both ways if you can.
- If your problem is very complex you may distinguish "great" steps and "small" ones. First check the great steps and get down to the smaller ones afterwards. Can you see clearly that the step is correct ? Yes, i can see it clearly and distinctly. Intuition is ahead, but could formal reasoning overtake it ? Can you also PROVE that it is correct ?

- When you reach the result, scrutinize the method that led you to the solution, try to see its point and try to make use of it for other problems.
- If you cannot solve the proposed problem try to solve first some related problem.
- Solve by Generalization, Specialization, Analogy, Decomposing and Recombining.
  - Generalization: Pick from the set an object that does not comply with it.
  - Specialization: find a special case of the original problem.
- Look for other hints and clues that may have been stated in the problem. Read the problem carefully.

Possible ways for solving problems

- Inference by analogy.
- Inference by induction (induction is naturally based on analogy). Induction tries to find regularity and coherence behind the observations. Its most conspicuous instruments are generalization, specialization, analogy.
- Analysis (or solution backwards, or regressive reasoning). We start from what is required, we take it for granted. We inquire from what antecedent the desired result could be derived. We pass from antecedent to antecedent, until we eventually come upon something already known or admittedly true.
- Synthesis (or progressive reasoning). We start from the point which we reached last of all in the analysis, from the thing already known or admittedly true. We derive from it what preceded it in the analysis and go on making derivations until we finally succeed in arriving to what is required. Synthesis retraces faithfully the steps of the analysis.

Remember

- We cannot hope to solve any worthwhile problem without intense concentration. In order to keep the attention alive, the object on which it is directed must unceasingly change.
- Don't rush, or you will most certainly make mistakes. Be calm and carry on.
- First check for any constrictions that must be set before starting up the problem.

6.2.1 Analytical vs Numerical solutions

1. Analytical solutions can be obtained exactly with pencil and paper,
2. Numerical solutions cannot be obtained exactly in finite time and typically cannot be solved using pencil and paper.

7 Relations

- The set R is a relation from A to B : $R \subset A \times B$
- Domain of R is the set: Dom $(F) = \{a \in A | \exists b \in B((a,b) \in B)\}$
- Range of R is the set: Ran $(R) = \{b \in B | \exists a \in A((a,b) \in R)\}$
- Inverse of R is the relation R^{-1} from B to A, defined as:

$R^{-1} = \{(b,a) \in B \times A | (a,b) \in R\}$
- Composition of sets $R \subset A \times B$ and $S \subset B \times C$ is the relation:

$S \circ R = \{(a,c) \in A \times C | \exists b \in B((a,b) \in R \wedge (b,c) \in S)\}$

Alternative notations for $(a,b) \in R$ are aRb, $R(a,b)$.

- Also $(a,c) \in A \times C \begin{cases} (a,b) \in R \wedge (b,c) \in S \rightarrow (a,c) \in (S \circ R) \\ or,\ (b,c) \in S \wedge (a,bc) \in R \rightarrow (a,c) \in (S \circ R) \\ or,\ (a,b) \in R \wedge (b,c) \notin S \rightarrow (S \circ R) \end{cases}$
- $(R^{-1})^{-1} = R$
- (Dom (R^{-1}) = Ran (R)
- (Ran (R^{-1}) = Dom (R)
- $T \circ (S \circ R) = (R \circ S) \circ R$
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
- If A is a set, then $i_A = \{(x,y) \in A \times A | x = y\}$ is the identity relation on A. Every element of A is related to itself <u>only</u>. e.g. $A = \{1,2,3\} \rightarrow i_A = \{(1,1),(2,2),(3,3)\}$]
- Suppose R is a binary relation on A (i.e. $R \subset A \times A = A^2$). Then R is <u>reflexive</u> (on A), if $\forall x \in A\,[(x,x) \in R]$, i.e. every element of A is related to itself. Alternatively, R is reflexive iff $i_A \subset R$
- R is <u>symmetric</u>, if $\forall x,y \in A\,[(x,y) \in R \rightarrow (y,x) \in R]$. Alternatively, R is symmetric iff $R = R^{-1}$.
- R is <u>transitive</u>, if $\forall x,y,z \in A\,[(xRy \wedge yRz) \rightarrow xRz]$. Alternatively, R is transitive iff $R \circ \subset R$.
- R is <u>antisymmetric</u>, if $\forall a,b \in A\,[(a,b) \in R \wedge (b,a) \in R \rightarrow a = b]$. Alternatively, R is antisymmetric if $\forall a,b \in A\,[R(a,b) \wedge a \neq b \rightarrow \neg R(b,a)]$
- R is <u>asymmetric</u>, if $\forall a,b \in A(aRb \rightarrow \neg bRa)$
- A relation $R \subset A \times A$ is called a <u>partial order</u> (on A), if it is

1. reflexive,
2. transitive and
3. antisymmetric.

In a particular context, it can be stated that R is a partial order (on A), if an object can be at least as large as another.

- A relation $R \subset A \times A$ is called a <u>total order</u>, if it is a partial order and in addition it has the property: $\forall x,y \in A\ (xRy \vee yRx)$.
- Two distinct elements are called "<u>comparable</u>" when one of them is greater than the other. This is the definition of "comparable". When you have a partially ordered set, some

pairs of elements can be not comparable. i.e. you can have two elements x and y such that $x \leqslant y$ is false and $y \leqslant x$ is also false. A total order ensures that all items of this set are comparable.

- A relation $R \subset A \times A$ is called a preorder if it is

  1. reflexive and
  2. transitive.

- A binary relation R is a relation on a set A, i.e. $R \subset A \times A$, or $R \subset A^2$. Thus it is a collection of ordered pairs of elements of A. The terms correspondence, dyadic relation and 2-place relation are synonyms for binary relation.

- R is called an equivalence relation (on A) if it is

  1. reflexive,
  2. symmetric and
  3. transitive.

- Suppose R is a partial order on a set $A, B \subset A,\ b \in B$ and $a \in A$. Then

  → b is called an R-smallest element of B, if $\forall x \in B\,[(b, x) \in R]$.

  → b is called an R-minimal element of B, if $\neg\exists x \in B(xRb \wedge x \neq b)$.

  → b is the largest element of B, if $\forall x \in B(xRb)$. Alternatively for set theory, S is the largest set of the family $\mathcal{F} : \exists S \in \mathcal{F}\ \forall T \in \mathcal{F}(T \subset S)$.

  → b is a maximal element of B, if $\neg\exists x \in B(bRx \wedge x \neq b)$.

  → a is called a lower bound for B, if $\forall x \in B(aRx)$

  → a is called an upper bound for B, if $\forall x \in B(xRa)$

- Let U be the set of all upper bounds for B and let L be the set of all lower bounds. Then

  → if U has a smallest element, then this smallest element is called the least upper bound (l.u.b.) of B.

  → if L has a largest element, then this largest element is called the greatest lower bound (g.l.b) of B.

7.1 Closures

- A relation $S \subset A \times A$ is the reflexive closure of R if it has the following 3 properties:

  1. $R \subset S$,
  2. S is reflexive,
  3. for every relation $T \subset A \times A$, if $R \subset T$ and T is reflexive, then $S \subset T$.

- Auseful reflexive closure: $S = R \cup i_A$
- A relation $S \subset A \times A$ is the symmetric closure of R, if:

  1. $R \subset S$,
  2. S is symmetric,
  3. for every relation $T \subset A \times A$, if $R \subset T$ and T is symmetric, then $S \subset T$.

- Useful symmetric closure: $S = R \cup R^{-1}$
- A relation $S \subset A \times A$ is the transitive closure of R, if:

  1. $R \subset S$,
  2. S is transitive,
  3. For every relation $T \subset A \times A$, if $R \subset T$ and T is transitive, then $S \subset T$.

- Suppose R is a relation on A. Then R is said to be irreflexive if $\forall x \in A\,[(x, x) \notin R]$.
- $R \subset A \times A$ is called a strict partial order if it is irreflexive and transitive.
- $R \subset A \times A$ is called a strict total order if it is a strict partial order and in addition it satisfies the requirement of trichotomy: $\forall x, y \in A(xRy \vee yRx \vee x = y)$
- Suppose R is an equivalence relation on a set A and $x \in A$. Then the equivalence class of x with respect to R is the set $[x]_R = \{y \in A | yRx\}$ (or $[x]$ if R is clear from contect).
- The set of all equivalence classes of elements of A is called A modulo R and it is denoted by A R. Thus, $A/R = \{[x]_R \,| x \in A\} = \{X \subset A | \exists x \in A(X = [x]_R)\}$.
- Suppose A is a set and $\mathcal{F} \subset P(A)$. We will say that $\mathcal{F}$ is pairwise disjoint if every pair of distinct elements of $\mathcal{F}$ are disjoint, or in other words: $\forall X, Y \in \mathcal{F}\,(X \neq Y \rightarrow X \cap Y = \varnothing)$.
- $\mathcal{F}$ is called a partition of A if it has the following properties:

  1. $\cup\mathcal{F} = A$,
  2. $\mathcal{F}$ is pairwise disjoint,
  3. $\forall X \in \mathcal{F}\,(X \neq \subset)$

- Suppose R is an equivalence relation on A. Then:

→ $\forall x \in A\ x \in [x]$ i.e. xRx (or $z \in [y] \to zRy$)

→ $\forall x, y \in A$ ($y \in [x]$ iff $[y] = [x]$

- $C_m = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x \equiv y (\text{mod } m)\}$ is an equivalence relation on $\mathbb{Z}$.
- There is an equivalence relation R on A such that $A/R = \mathcal{F}$.

8 Functions

- $f(x) = g(x)$: Equivalence. An equality of functions
- $x^2 = 25$: Equation. The equality holds only for a few values of x!
- Suppose f is a relation from A to B. Then f is called a function from A to B, denoted as $f : A \to B$, if $\forall a \in A \exists! b \in B\,[(a, b) \in f]$. Dom $(f) = A$, Ran $(f) = \{f(a) | a \in A\} \subset B$.
- In contrast with a function, a mapping is a relation which may map an element of its domain to multiple elements of its range.
- $\forall a \in A\ \forall b \in B (b = f(a) \leftrightarrow (a, b) \in f)$, b is the value of f at a, of "f of a".
- Δύο συναρτήσεις f, g λέγονται ίσες όταν

  $\to$ έχουν το ίδιο πεδίο ορισμού A

  $\to$ $\forall x \in A (f(x) = g(x))$

- Αν f, g είναι δύο συναρτήσεις με πεδίο ορισμού A, B αντιστοίχως, τότε ονομάζουμε σύνθεση της f με την g και τη συμβολίζουμε με $g \circ f$, τη συνάρτηση:
$g \circ f = (g \circ f)(x) = g(f(x))$.
Η $g \circ f$ ορίζεται εφόσον $A \neq \varnothing$, όπου $A = \text{Dom}\ (f)$. Αν ορίζεται και η $f \circ g = f(g(x))$, τότε οι $f \circ g$ και $g \circ g$ δεν είναι υποχρεωτικά ίσες.
- Αν f, g, h είναι τρείς συναρτήσεις και ορίζεται η $h \circ (g \circ f)$, τότε ορίζεται και η $(h \circ g) \circ f$ και ισχύει: $h \circ (g \circ f) = (h \circ g) \circ f$. Τη συνάρτηση αυτή τη λέμε σύνθεση των f, g και h και τη συμβολίζουμε με $h \circ g \circ f$.
- $f \uparrow \Delta$, $\Delta \subset \text{Dom}\ (f)$, όταν $\forall x_1, x_2 \in \Delta[x_1 < x_2 \to f(x_1) < f(x_2)]$ f γνησίως αύξουσα συνάρτηση / increasing function.
- $f \downarrow \Delta$, $\Delta \subset \text{Dom}\ (f)$, όταν $\forall x_1, x_2 \in \Delta[x_1 < x_2 \to f(x_1) > f(x_2)]$ f γνησίως φθίνουσα συνάρτηση / decreasing function.
- Έστω $f, A = \text{Dom}\ (f)$:

  $\to$ $\forall x \in A[\exists! x_0 \in A(f(x_0) \leqslant f(x)) \to f(x_0) = \min(f(x))]$

  $\to$ $\forall x \in A[\exists! x_0 \in A(f(x_0) \geqslant f(x)) \to f(x_0) = \max(f(x))]$

- Έστω $f : A \to \mathbb{R}$. Αν υπάρχει η αντίστροφη της $g : f(A) \to \mathbb{R}$ έχουμε:
$f(x) = y \leftrightarrow f^{-1}(y) = x$.
- Οι γραφικές παραστάσεις των f και f^{-1} είναι συμμετρικές ως προς την ευθεία: $y = x$.
- Suppose f and g are functions from A to B. If $\forall a \in A(f(a) = g(a))$, then $f = g$.
- Suppose $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and $\forall a \in A$, the value of $g \circ f$ at a is given by the formula $(g \circ f)(a) = g(f(a))$.
$\big[(a, c) \in g \circ f, \text{so} (g \circ f)(a) = c = g(b) = g(f(a))\big]$
- Suppose $f : A \to B$ and $C \subset A$. The set $f \cap (C \times B)$, which is a relation from C to B is called a restriction of f to C, denoted as $f \restriction C$. In other words $f \restriction C = f \cap (C \times B)$. The restriction is obtained by choosing a smaller domain for the original function.
- Suppose $f : A \to A\,[\exists a \in A\ \forall X \in A(f(x) = a)] \to f$ is called a constant function.
- Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We say that "$f$ is big-oh of g" and write

$f(x) = O(g(x))$, to describe the limiting behavior of a function when the argument tends towards a particular value or infinity, usually in terms of simpler functions. So:

1. $f = O(\phi)$ means that $|f| < A \times \phi$, for some constant A and all values of x.

2. $f = o(\phi)$ means that $f/\phi \to 0$.

$O(x)$ and $o(x)$ are the Landau symbols.
• A description of a function in terms of big O notation usually only provides an upper bound on the growth rate of the function.
• Suppose $f : A \to B$. We will say that f is one-to-one (ένα προς ένα), or $1-1$, or injection, or injective if $\neg\exists a_1, a_2 \in A(f(a_1) = f(a_2) \wedge a_1 \neq a_2)$. The situtation that must not occur is that there are two different elements of the domain of f, a_1 and a_2, such that $f(a_1) = f(a_2)$
• Suppose $f : A \to B$. We say that f is onto, or surjection, or subjective if $\forall b \in B \exists a \in A(f(a) = b)$.
This means that every element of B is the image under f of some element of A (μονοσήμαντη).
The definitions that follow are equivalent to those of one-to-one and onto.

→ f is one-to-one iff $\forall a_1, a_2 \in A(f(a_1) = f(a_2) \to a_1 = a_2)$

→ f is onto iff Ran $(f) = B$

• Suppose $f : A \to B$ and $g : B \to C$. It follows $f \circ f : A \to C$.

→ If f and g are both $1-1$, then so if $g \circ f$.

→ If f and g are both onto, then so is $g \circ f$.

• Functions that are both one-to-one and onto are called bijections, or bijectives, or one-to-one correspondences ($1-1$ και επί). Such a function is invertible.
• Suppose $f : A \to B$. Then the following statements are equivalent:

1. f is one-to-one and onto,

2. $f^{-1} : B \to A$,

3. There is a function $g : B \to A$ such that $g \circ f = i_A(A \to B \to A)$ and $f \circ g = i_b(B \to A \to b)$.

• Suppose $g : B \to A$. Then $g = i_A \circ g = g \circ i_B$
• Suppose $f : A \to B$. If there exists a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$ then f is one-to-one and onto and $g = f^{-1}$.
• If there is a function $g : B \to A$ such that $g \circ f = i_A$ then f is one-to-one.
• If there is a function $g : B \to A$ such that $f \circ g = i_B$ then f is onto.
• Suppose $f : A \to B$ and $X \subset A$. Then the image of X under f is the set $f(X)$ defined

as follows:
$f(X) = \{f(x) | x \in X\} = \{b \in B | \exists x \in X(f(x) = b)\}$
If $Y \subset B$, then the inverse image of Y under f is the set $f^{-1}(Y)$ defined as follows:
$f^{-1}(Y) = \{a \in A | f(a) \in Y\}$
- Suppose $f : A \to B$ and W and X are subsets of A. Then $f(W \cap X) \subset f(W) \cap f(X)$. Furthermore, if f is one-to-one, then $f(W \cap X) = f(W) \cap f(X)$.

Algebraic properties of functions
- Functions shifted Left / Right: Given a function $f(x)$ and a value $c > 0$, the graph of $f(x + c)/f(x - c)$ will be a shift of the graph of $f(x)$ left / right by "c" units.
- Functions shifted Up / Down: Given a function $f(x)$ and a value $c > 0$, the graph of $f(x) + c/f(x) - c$ will be a shift of the graph of $f(x)$ up / down by "c" units.
- Functions vertically scaled: Given function $f(x)$, the function $a \cdot f(x)$ will stretch all y-values of $f(x)$ by multiplying each one by $a \in \mathbb{R}$.
- Functions horizontally scaled: Given function $f(x)$, the function $f(\alpha x)$ will adjust all x-values of $f(x)$, by multiplying each one by a.
- A scale is a non-rigid translation in that it does alter the shape and size of the function graph.
- Not all functions are even, or odd, but most can be written as a sum of an even part f_e and an odd f_0 part. Every function f can be written: $f(x) = f_e(x) + f_0(x)$

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \wedge \quad f_0(x) = \frac{f(x) - f(-x)}{2}$$

- A continuous function is, roughly speaking, a function for which sufficiently small changes in the input results in arbitrarily small changes in the output.
- $ax^2 + bx + c = 0,\ a \neq 0 :$ Trinomial

$S = x_1 + x_2 = -\dfrac{b}{a} \quad \wedge P = x_1 \cdot x_2 = \dfrac{c}{a}$: Τύποι του Vieta
$\therefore a(x - x_1)(x - x_2) = 0$
$x^2 - (x_1 + x_2)x + x_1 x_2 = 0 \leftrightarrow x^2 - Sx + P = 0$
- Λογαριθμική συνάρτηση με βάση a είναι η $f : (0, +\infty) \to \mathbb{R}$, με $f(x) - \log_a x$
- Εκθετική μεταβολή: $Q(t) = Q_0 e^{ct}$ $\quad$ ($c > 0$: εκθετική άυξηση $\vee$ ($c > 0$: εκθετική απόσβεση)

Q_0 : αρχική τιμή $@t = 0$
- Factorization of Quadratic $ax^2 + bx + c$, when $a = 1$

  → if c is positive: a) f_1, f_2 are factors of c and both have the sign of b, b) The sum of f_1 and f_2 is b

  → if c is negative: a) f_1, f_2 are factors of c and have opposite signs, the numerically larger having sign of b, b) the difference between f_1 & f_2 ish We finally denote them as $(x - f_1)(x - f_2)$

- Factorization of Quadratic $ax^2 + bx + c$, when $a \neq 1$.

→ We obtain $|ac|$

→ We write down all the possible factors of $|ac|$.

→ We follow similar procedure as above

→ Once we find f_1, f_2 we write them: $ax^2 + f_1x + f_2x + c$ and then this is factorised by grouping.

- If $D = b^2 - 4ac$ is a perfect square, the quadratic has 2 simple factors.

8.1 Even and Odd functions

- Not every function is even, or odd but many can be written as the sum of an even part and an odd part, like so:

for $f(x)$: $f_e(x) = \dfrac{f(x) + f(-x)}{2} \wedge f_0(x) = \dfrac{f(x) - f(-x)}{2}$

Properties

Properties involving Addition and Subtraction
- Odd functions are symmetric in the 1st and 3rd quadrants.
- If a function is odd, its absolute value is even.
- The sum of two even — odd functions is even — odd and any constant multiple of an even — odd function is even — odd.
- The difference between two even odd functions is even — odd.
- The sum of an even and an odd function is neither even, nor odd.

Properties involving Multiplication and Division:
- The product of two even — odd functions is an even — odd function
- The product of an even function and an odd function is an odd function
- The quotient of two even — odd functions is an even — even function
- The quotient of an even function and an odd function is an odd function

Properties involving Composition:
- The composition of two even — odd functions is even — odd.
- The composition of an even function and an odd function is even.
- The composition of either an odd, or an even function with an even function is even (but not vice versa).

Calculus Properties:
- The derivative of an even — odd function is odd — even.
- The integral of an odd function from -A to A is zero (where A is finite and the function has no vertical asymptotes between -A and A).
- The integral of an even function from -A to A is twice the integral from 0 to +A

(where A is finite and the function has no vertical asymptotes between -A and A. This also holds true, when A is infinite, but only if the integral converges).
(The integral of a function is the set of all its antiderivatives.)

Series Properties:
- The MacLaurin series of an even — odd function includes only even — odd powers.
- The Fourier series of a periodic even — odd function includes only consine — sine terms (If it is even it also includes a_0 which may be regarded as $a_n \cos(nx)$ with $n = 0$.)

Periodicity:
- If $f(x) = f(x + \pi)$, the Fourier series for $f(x)$ contains only even harmonics (cosine & sine).
- If $f(x) = -f(x + \pi)$, the Fourier series for $f(x)$ contains only odd harmonics.

9 Graph Theory

- A graph is a nonempty finite set of vertices, along with a set E of 2-element subsets of V. The elements of V are called vertices, the elements of E are called edges.

Example
The graph G (figure 1) is not a regular graph, because it has loop V_1 around a vertex. Such graphs, with loops, are called multigraphs.

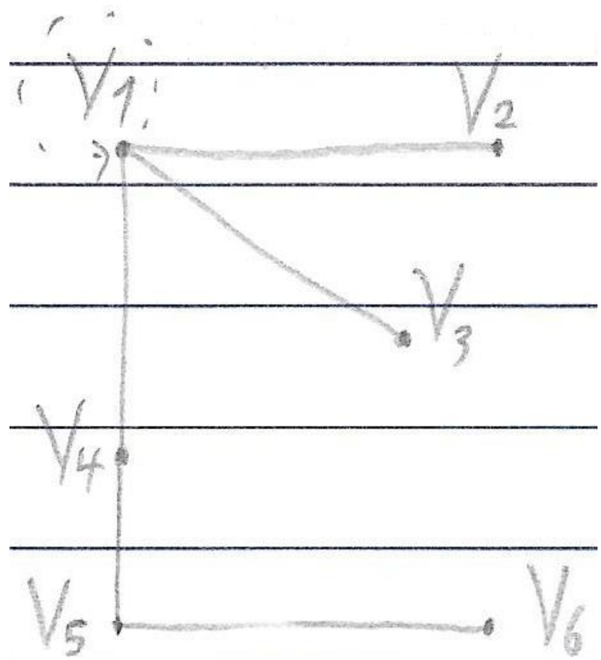


Figure 1: Graph G

Vertex set: $V = \{V_1, V_2, V_3, V_4, V_5, V_6\}$
Edge set: $E = \{\{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_4, V_5\}, \{V_5, V_6\}\}$ defines sets of edges, ie. vertices directly connecting each other.

- Cardinality of a graph is the number of its vertices. eg. $|G| = 6$
- Degree of V_1 = $\deg(V_1) = 3$: The degree of a vertex, say V_1 (Graph G), is the number of vertices it is directly connected with.
- The edges don't need to be straight, as long as the connections are preserved. Such graphs are called isomorphic. For example, graphs G and G' are isomorphic.

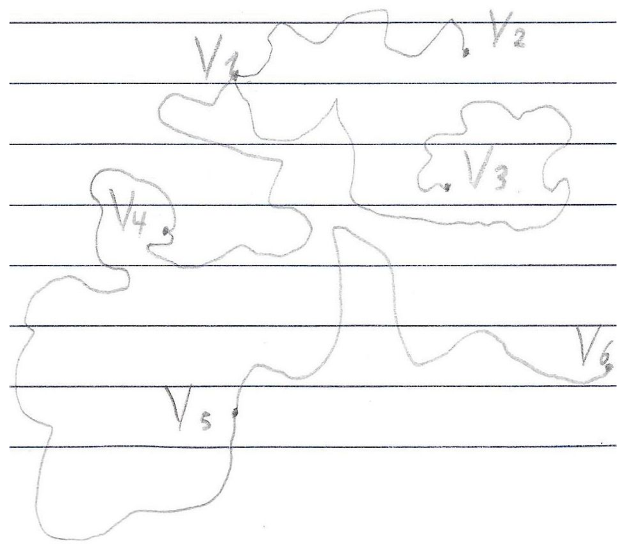


Figure 2: Graph G'

- Adjacency List: We list vertices adjacent to each vertex. eg. For graph G, we have:

$V_1 : V_2, V_3, V_4$
$V_2 : V_1$
$V_3 : V_1$
$V_4 : V_1, V_5$
$V_5 : V_4, V_6$
$V_6 : V_5$

- Adjacency Matrix: In every place of the matrix we insert a 1, if there is a connection between the corresponding vertices, or a 0 if there is not. For graph G, the adjacency matrix is the one pictured below (figure 3)

$$\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 3: Adjacency Matric for the Graph G

- Graph C below (figure 4) is called a circuit, because there is at least one vertex, say D, from which we can start and without ever backtracking or lifting the pen, we can return back to it through a route, specified by the edges of the graph. Possible routes in this case are DABCD, DCBAD.

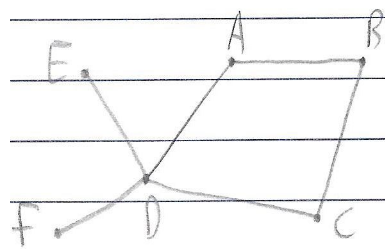


Figure 4: Graph C - A circuit

- A cyclic graph is a graph containing at least one graph cycle. A graph that is not cyclic is said to be acyclic. A cyclic graph possessing exactly one (undirected, simple) cycle is called a unicyclic graph. Cyclic graphs are not trees.
- A tree (figure 5) is an undirected graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree.
- A forest is an undirected graph, all of whose connected components are trees; in other words, the graph consists of a disjoint union of trees. Equivalently, a forest is an undirected acyclic graph.

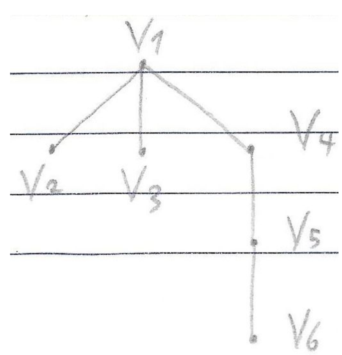


Figure 5: Graph G is in fact a Tree

- A Eulerian trail, or Eulerian path is a trail in a graph which visits every edge exactly once.
- A Eulerian circuit, or Eulerian cycle is an Eulerian trail which starts and ends on the same vertex.

- We have a graph $G = (V, E)$, where V, E the sets of vertices and edges in the graph respectively.

$\sum_{v \in V} d(v) = 2|E|$, where $d(v)$ the grade of vertex v

10 Discrete Math

- Sigma notation definition: $\sum_{i=a}^{b} f(i) \triangleq \begin{cases} 0 & b < a \\ f(a) + \sum_{i=a+1}^{b} f(i) & b \geqslant a \end{cases}$
- Permutation of Indices: $\sum_{n=1}^{N} b_{n+1} = \sum_{n=2}^{N+1} b_n$
- $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n \ \forall n \in \mathcal{Z}$
- Pigeonhole Principle (Αρχή του Περιστερώνα): Με οποιονδήποτε τρόπο να τοποθετήσουμε n περιστέρια σε m φωλιές, με $n > m$ θα υπάρχει τουλάχιστον μια φωλιά με $\lceil n/m \rceil$ περιστέρια.
- Zero-based numbering, or index origin = 0, is a way of numbering in which the initial element of a sequence is assigned the index 0, rather than the index 1 as is typical in everyday non-programming context. Under zero-based numbering, the initial element is sometimes termed the zeroth element, rather than the first element; zeroth is a coined ordinal number corresponding to the number zero.
- $e^x \geqslant 1 + x \ , \ \forall x \in \mathcal{R}$
- $\dfrac{x}{1+x} \leqslant \ln{(1+x)} \leqslant x \ , \ x > -1$

- Golden Ratio (Χρυσή Τομή)

Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum divided by the larger of the two quantities. Some twentieth-century artists and architects, including Le Corbusier and Dalí, have proportioned their works to approximate the golden ratio—especially in the form of the golden rectangle, in which the ratio of the longer side to the shorter is the golden ratio—believing this proportion to be aesthetically pleasing. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and plantlife.

$\phi a = a + bab$

Its value is: $\phi = \dfrac{1+\sqrt{5}}{2} = 1.61803\ 39887 \in \mathcal{Q}$

Properties

- $1 + \dfrac{1}{\phi} = \phi$, which can be arranged into...
- $\phi + 1 = \phi^2$
- $\dfrac{1}{\phi} + \dfrac{1}{\phi^2} = 1$
- Using the quadratic formula for the above, two solutions are obtained:

$\phi = \dfrac{1+\sqrt{5}}{2} = 1.61803\ 39887\ldots$ and $\overline{\phi} = \dfrac{1-\sqrt{5}}{2} = -0.61803\ 39887\ldots$

10.1 Asymptotic Notations and Growth of Functions

- Stirling's approximation: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$, for large n.
- $n! \geqslant \left(\frac{n}{3}\right)^n$
- $\sum_{i=1}^{n} \log(i) = \log(n!) \approx n\log(n)$, ie. $log(n!) = \Theta(n\log(n))$
- $\lim_{x\to\infty} \sqrt[x]{x} = 1$
- If $\lim_{n\to\infty} \frac{T(n)}{g(n)} = 0 \rightarrow T(n) = o\left(g(n)\right)$

ie. $T(n)$ has a much smaller rate of growth to that of $g(n)$ as n grows without measure $\forall n \geqslant n_0$.

- If $\lim_{n\to\infty} \frac{T(n)}{g(n)} = c \; > 0 \rightarrow T(n) = \omega\left(g(n)\right) \;\; \wedge \;\; T(n) = o\left(g(n)\right),$ thus $T(n) = \Theta\left(g(n)\right)$

ie. $T(n)$ and $g(n)$ have the same rates of growth.

- If $\lim_{n\to\infty} \frac{T(n)}{g(n)} = \pm\infty \rightarrow T(n) = \omega\left(g(n)\right)$

ie. $T(n)$ has a much greater rate of growth to that of $g(n)$ as n grows without measure $\forall n \geqslant n_0$.

10.2 Sequences and Series

- Arithmetic Series is a sequence (/series) of numbers in which each differs from the preceding one by a constant quantity.
- $u_n = \alpha + (n-1)d$: General term
- $\sum_{r=0}^{n-1} (a + rd) = \frac{n}{2}\left[2a + (n-1)d\right]$: Sum

or $S_n = \sum_{r=1}^{n} (a + rd) = n\frac{(a_1 + a_n)}{2}$, a_1 is first term & a_n is the last term.

- $b = \frac{a+c}{2}, a < b < c$: Arithmetic mean

- Geometric Series is a series with a constant ratio between successive terms.
- $u_n = a \cdot r^{n-1}$: General term
- $\sum_{k=0}^{n-1} ar^k = a\frac{(1-r^n)}{1-r}$, or more generally ..
- $\sum_{k=m}^{n} r^k = \frac{a(r^m - r^{n+1})}{1-r}$
- $|r| < 1 \longrightarrow \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

- $b = \sqrt{ac} \quad , a < b < c :$ Geometric mean

- Harmonic Series is the series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \log(n) + \mathcal{O}(1)$

- $u_n = \frac{1}{n}, n \in \mathbb{N}^* :$ General term
- $b = \frac{2ac}{a+c} = \frac{2}{\frac{1}{a} + \frac{1}{c}}, a < b < c :$ Harmonic mean
- $H_k = \sum_{i=1}^{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} :$ Harmonic number
- $H_{2^n} \leqslant 1 + n$
- $\frac{x}{x-1} \leqslant 2 \ , \ \forall x \geqslant 2$
- The sum of the first n terms of the harmonic series is given analytically by the nth harmonic number:

$H_n = \sum_{k=1}^{n} \frac{1}{k} = \gamma + \psi_0(n+1)$

where γ is the Euler-Mascheroni constant and $\psi_0(\chi)$ is the digamma function.

- Euler's number: e is a mathematical constant that is the base of the natural logarithm: the unique number whose natural logarithm is equal to one. The number $e \approx 2.71828$ is the limit of $\left(1 + \frac{1}{n}\right)^n$ as n approaches infinity, an expression that arises in the study of compound interest. It can also be calculated as the sum of the infinite series:

$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$

The constant can be characterized in many different ways. For example, e can be defined as the unique positive number a such that the graph of the function $y = a^x$ has unit slope at $x = 0$. The function $f(x) = e^x$ is called the (natural) exponential function. The natural logarithm, or logarithm to base e, is the inverse function to the natural exponential function. The natural logarithm of a positive number k can be defined directly as the area under the curve $y = 1/x$ between $x = 1$ and $x = k$, in which case e is the value of k for which this area equals one (see figure 6).

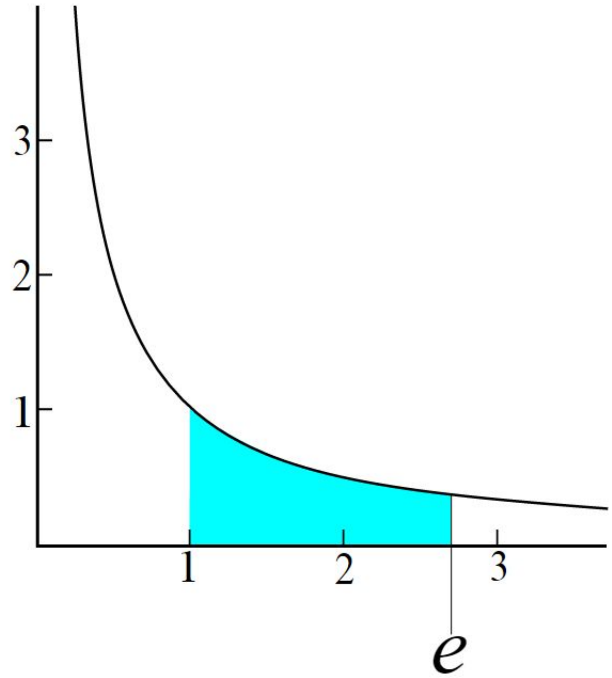


Figure 6: e is the unique number that makes the shaded area under the curve $y = 1/x$ equal to 1

Sum of the Powers of Natural Numbers

- $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{r=k}^{m} r = k \cdot (m-k+1) + \sum_{i=1}^{m-k} i$
- $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$
- $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, |x| < 1$
- $\displaystyle\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, |x| < 1$
- $\displaystyle\sum_{i=m}^{n} c = c \cdot (n-m+1)$
- $\displaystyle\sum_{i=m}^{n} i = \frac{(n-m+1)(n+m)}{2}$

Infinite Series: $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

- $\lim_{n\to\infty} \sum_{k=1}^{\infty} u_k$ is a definite value $\to$ Series is convergent
- $\lim_{n\to\infty} \sum_{k=1}^{\infty} u_k$ is not a definite value $\to$ Series is divergent

Relationship between Summation and Product notations:

$$\prod_{r=\text{sth}}^{n \text{or} \infty} k^r = k^{\sum_{r=\text{sth}}^{n \text{or} \infty} r}$$

$$\sum_{k=1}^{n} \log k = \log\left(\prod_{k=1}^{n} k\right) = \log(n!)$$

Decimal Representation: A decimal representation of a non-negative real number r is an expression in the form of a series, traditionally written as a sum: $r = \sum_{i=0}^{\infty} \frac{a_i}{10^i}$, where a_0 is a nonnegative integer, and $a_1, a_2, \ldots$ are integers satisfying $0 \leqslant a_i \leqslant 9$ called the digits of the decimal representation. The sequence of digits specified may be finite, in which case any further digits a_i are assumed to be 0.

The number defined by a decimal representation is often written more briefly as
$r = a_0.a_1a_2a_3\ldots.$

That is to say, a_0 is the integer part of r, not necessarily between 0 and 9, and $a1, a2, a3, \ldots$ are the digits forming the fractional part of r.

- A sequence is called monotone if it is either increasing, or decreasing.

Tests for Convergence

1. $\lim_{n\to\infty} u_n = 0 \to$ series may be convergent,
   $\lim_{n\to\infty} u_n \neq 0 \to$ series is certainly divergent

2. Comparison test - Useful standard series
   - $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots + \frac{1}{n^p} + \ldots$: P-series

   For $p > 1$ series converges. For $p \leqslant 1$, series diverges
   - $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

3. D'Alembert's ratio test for positive terms
   - $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \longrightarrow$ limit converges
   - $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1 \longrightarrow$ limit diverges

- $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1 \longrightarrow$ inconclusive assessment

4. For General Series
   - $\sigma|u_n|$ converges $\longrightarrow \sigma u_n$ is absolutely convergent
   - $\sigma|u_n|$ diverges $\wedge$ σu_n converges $\longrightarrow \sigma u_n$ is conditionally convergent

5. Limit Comparison Test
   - Suppose b_n : known series, a_n : series under test

   $0 < \lim_{n\to\infty}\frac{a_n}{b_n} < \infty \longrightarrow$ both series behave in like manners

6. Monotone Convergence Theorem
   - Lemma 1: If a sequence of $\mathbb{R}$ is increasing and bounded above, then its supremum is the limit. The supremum is defined as the least upper bound of a sequence / function.
   - Lemma 2: If a sequence of real numers is decreasing and bounded below, then its infimum is the limit.
   - Theorem: If $\{a_n\}$ is a monotone sequence of real numbers (i.e. if $a_n \leqslant a_{n+1}$, for every $n \geqslant 1$, or $a_n \geqslant a_{n+1}$, for every $n \geqslant 1$) then this sequence has a finite limit if and only if the sequence is bounded (a sequence is called "bounded", when it's bounded above and below).

7. <u>Alternating sign test</u>: Based on the alternating harmonic series: $\frac{(-1)^{n+1}}{n}$ If the magnitude of the terms decreases and the signs alternate then the series converges.

- <u>Taylor Series</u>: $f(x_0 + h) = f(x_0) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$

where $f(x)$ is continuous in $Dom(f)$.

- <u>McLaurin Series</u>: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$

where $f(x)$ continuous in $Dom(f)$. McLaurin series describes the function $f(x)$ in terms of its successive derivatives at $x = 0$

Useful / Common Series Expansions

Reminder: In every trigonometric expansion the angle must be in radians

- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$
- $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$
- $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

- $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots$
- $\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \ldots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$
- Binomial Series: $(1 \pm x)^n = 1 \pm nx + \frac{x^2}{2!}n(n-1) \pm \frac{x^3}{3!}n(n-1)(n-2) + \ldots$
- Binomial Expansion (General Case):

$(a+b)^n = ^n C_0 a^n b^0 + ^n C_1 a^{n-1} b^1 + ^n C_2 a^{n-2} b^2 + ^n C_3 a^{n-1} b^3 + \ldots + ^n C_n a^0 b^n$
- $(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp \ldots$: Converges iff $|x| < 1$
- $\frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$
- $\cos^2(x) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \ldots$
- $\sin^{-1}(x) = x + \frac{x^3}{3} + \frac{3x^5}{40} + \ldots$
- $e^{ax} = 1 + ax + \frac{x^2 x^2}{2!} + \frac{a^3 x^3}{3!} + \frac{a^4 x^4}{4!} + \ldots$
- $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$
- $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n\to 0}(1+n)^{\frac{1}{n}}$
- Harmonic numbers are the numbers H_n for $n \geqslant 1$ defined by $H_n = \sum_{i=1}^{n} \frac{1}{i}$

$\forall n, m \in \mathbb{N}\left(n \geqslant m \to H_n - H_m \geqslant \frac{n-m}{n}\right) \quad \sum_{k=1}^{n-1} H_k = nH_n - n$
- $2^n - 1 = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = \sum_{k=0}^{n-1} 2^k$
- $2^n - 2^{n-1} = 2^{n-1}$
- $1 + 3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3^{n+1}-1}{2}$
- The length of $[1 + 2 + 3 + \ldots + 2^n + (2^n+1) + (2^n+2) + (2^n+3) + \ldots + 2^{n+1}], n \geqslant 1$, is a power of 2.
- $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

Fibonacci Sequence

- The definition is given by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$

where $F_0 = 0, F_1 = 1$ or $F_1 = F_2 = 1$

- $F_n = F_{n-1} + F_{n-2} = \dfrac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$
- $\displaystyle\sum_{i=0}^{n} F_{2i} = F_0 + F_2 + F_4 + \ldots + F_{2n} = F_{2n+1} - 1$
- The sequence $a_0, a_1, a_2, \ldots$ is called a generalized Fibonacci sequence, or a Gibonacci sequence, if

$\forall n \geqslant 2(a_n = a_{n-2} + a_{n-1})$.

Also, $\exists s, t \in \mathbb{R} \left[a_n = s\left(\dfrac{1+\sqrt{5}}{2}\right)^2 + t\left(\dfrac{1-\sqrt{5}}{2}\right)^2 \right]$

- The Lucas numbers are the numbers $L_0, L_1, L_2, \ldots$ defines as follows:

$a)L_0 = 2, \quad b)L_1 = 1, \quad \forall n \geqslant 2 L_n = L_{n-2} + L_{n-1} = \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \left(\dfrac{1-\sqrt{5}}{2}\right)^n$

- Triangular numbers: $T_n = \displaystyle\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$: equal to the number of dots composinga triangle with n dots on one side
- $T_n + T_{n-1} = n^2 = (T_n - Tn - 1)^2$

- Dirichlet series: is any series of the form:

$\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

where $s \in \mathcal{C}$ and a is a complex sequence. The Dirichlet series plays a varierty of important roles in analytic number theory (the branch of Mathematics connecting Number Theory / Discrete Math and Calculus).

- $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \ldots$
- Mersenne sequence: $f(x) = 2^n -$, $n \in \mathbb{N}$

10.3 Fourier Series

- Approximates the values of a periodic function: $f(x+T) = f(x)$, where T is the period.
- The Fourier series converges to $f(x)$, if the Dirichlet conditions are satisfied (sufficient conditions).

1. The function $f(t)$ must be defines single valued & periodic.
2. $f(t)$ and $f'(t)$ have at most a finite number of finite discontinuities over a single period - i.e. they are piecewise continuous.

- $f(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$, a_0, a_n, b_n are the Fourier coefficients

$a_0 = \dfrac{2}{T} \displaystyle\int_{-T/2}^{T/2} f(t)dt$

$a_n = \dfrac{2}{T} \displaystyle\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt$, $\qquad b_n = \dfrac{2}{T} \displaystyle\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt$

- Alternative notation for real valued function $f(t)$ with complex coefficients:

$f(t) = \displaystyle\sum_{n=-\infty, n\neq 0}^{\infty} c_n e^{jn\omega t} \quad where c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt = \dfrac{a_n - jb_n}{2} = |c_n|e^{j\phi n}$: Discrete complex spectrum

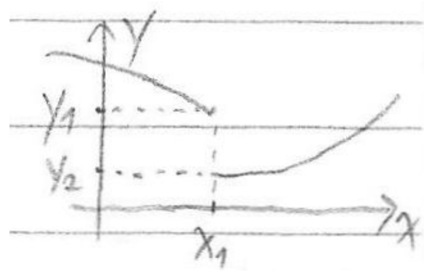
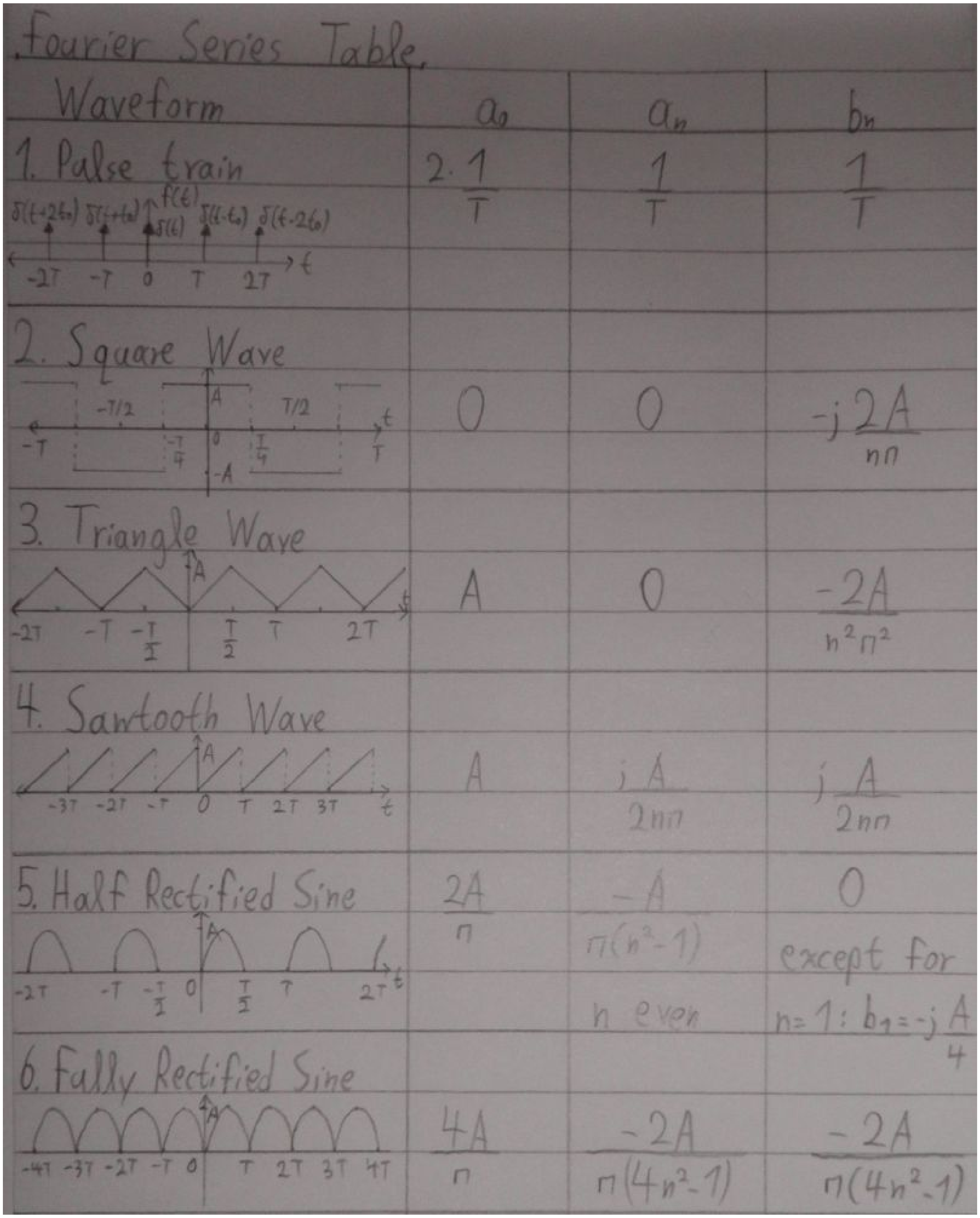


Figure 7: Sum of Fourier series at a finite discontinuity

- Sum of Fourier series at a finite discontinuity: At $x = x_1$ series for $f(x)$ converges to the value: $1/2(f(x_{1-}) + f(x_{1+})) = 1/2(y_1 + y_2)$
- Alternative notation: $f(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin(n\omega t + \phi_n)$

$c_n = \sqrt{a_n^2 + b_n^2}$, $\quad \phi_n = \tan^{-1}(\dfrac{a_n}{b_n})$

- The constant (D.C.) term $a_0/2$ is to raise, or lower the entire waveform on the y-axis.



Fourier Series Table.

Waveform	a_0	a_n	b_n
1. Pulse train	$2 \cdot \frac{1}{T}$	$\frac{1}{T}$	$\frac{1}{T}$
2. Square Wave	0	0	$-j\frac{2A}{n\pi}$
3. Triangle Wave	A	0	$\frac{-2A}{n^2\pi^2}$
4. Sawtooth Wave	A	$j\frac{A}{2n\pi}$	$j\frac{A}{2n\pi}$
5. Half Rectified Sine	$\frac{2A}{\pi}$	$\frac{-A}{\pi(n^2-1)}$ n even	0 except for $n=1$: $b_1=-j\frac{A}{4}$
6. Fully Rectified Sine	$\frac{4A}{\pi}$	$\frac{-2A}{\pi(4n^2-1)}$	$\frac{-2A}{\pi(4n^2-1)}$

Figure 8: Fourier Series Table

11 Probability Theory

- Αν σε ν εκτελέσεις ενός πειράματος ένα ενδεχόμενο A πραγματοποιείται k φορές, τότε ο λόγος $\dfrac{k}{\nu}$ ονομάζεται σχετική συχνότητα του A και συμβολίζεται με f_A.
- Στατιστική ομαλότητα ή Νόμος των Μεγάλων Αριθμών: Οι σχετικές συχνότητες πραγματοποίησης ενός πειράματος, που εκτελείται κάτω από αμετάβλητες συνθήκες, σταθεροποιούνται γύρω από κάποιους αριθμούς (όχι πάντοτε ίδιους), καθώς ο αριθμός των δοκιμών του πειράματος επαναλαμβάνεται απεριόριστα.

Αξιωματική Θεμελίωση Πιθανότητας

Έστω $\Omega = \{\omega_1, \omega_2, \ldots, \omega_\nu\}$ δειγματικός χώρος με πεπερασμένο πλήθος στοιχείων. Ισχύουν τα παρακάτω:

- $0 \leqslant P(\omega_i) \leqslant 1 \quad , 1 \leqslant i \leqslant \nu$
- $P(\Omega) = P(\omega_1) + P(\omega_2) + \ldots + P(\omega_\nu) = 1$
- Εάν $\omega_1, \omega_2, \ldots, \omega_\nu$ είναι ανά δύο ασυμβίβαστα, δηλαδή $\omega_i \cap \omega_j = \varnothing$, τότε:

$P(\omega_1 + \omega_2 + \ldots + \omega_\nu) = P(\omega_1) + P(\omega_2) + \ldots + P(\omega_\nu)$

- $P(\varnothing) = 0$
- Αν $P(\omega_i) = \dfrac{1}{\nu}$, τότε έχουμε τον κλασικό ορισμό της πιθανότητας ενός ενδεχομένου (ισοπίθανα ενδεχόμενα):

$$P(A) = \frac{\text{Πλήθος Ευνοϊκών Περιπτώσεων}}{\text{Πλήθος Δυνατών Περιπτώσεων}} = \frac{N(A)}{N(\Omega)}$$

- $P(A \cup B) = P(A + B)$ (alternative notation)
- $P(A \cap B) = P(A \cdot B)$ (alternative notation)
- $P(A') = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (additive law)
- $A \cap B = \varnothing \rightarrow P(A \cup B) = P(A) + P(B)$ (simple additive law)
- $A \subset B \rightarrow P(A) \leqslant P(B)$
- $P(A - B) = P(A) - P(A \cap B) = P(A \cap B')$
- $\forall A, B \in \Omega \ P(B) = P(B \cdot A + B \cdot A')$

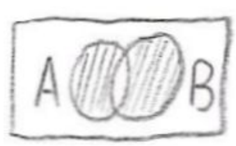


Figure 9: Διάγραμμα Venn της πιθανότητας: $P(A - B) + P(B - A)$

- $P(A - B) + P(B - A) = P(A \cdot B') + P(B \cdot A')$ figure 9
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) > 0$ (δεσμευμένη πιθανότητα)

- $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$ (πολλαπλασιαστικός νόμος των πιθανοτήτων)
- $P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)}$: θεώρημα Bayes, επίσης:

$P(A|B) = \dfrac{P(A)\ P(B|A)}{P(A)\ P(B|A) + P(\overline{A})\ P(B|\overline{A})}$

- Δύο ενδεχόμενα A και B με $P(A) > 0$ και $P(B) > 0$ λέγονται ανεξάρτητα, αν και μόνον αν $P(A \cap B) = P(A) \cdot P(B)$
- A, B ανεξάρτητα ενδεχόμενα $\rightarrow P(A|B) = P(A)$ και $P(B|A) = P(B)$
- $P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_\nu) = \sum_{i=1}^{\nu} P(A|B_i) \cdot P(B_i)$ μόνον αν $B_1, B_2, \ldots, B_\nu$ ασυμβιβαστά ενδεχόμενα και $B_1 + B_2 + \ldots + B_\nu = \Omega$
- $P(A + B + \Gamma) = P(A) + P(B) + P(\Gamma) - P(AB) - P(A\Gamma) - P(B\Gamma) + P(AB\Gamma)$
- $\sum_{i=1}^{\nu} P(x_i|y) = 1$
- An experiment that has a result with more than one possible outcomes is referred to as a random experiment. The only requirement that is made of the outcomes of a random experiment is that they be mutually exclusive. To cater for ranges of possible outcomes we define an event. An event consists of one or more outcomes selected from a list of all possible outcomes.

- Indicator function is a function that returns the value of 1 when something is true and 0 when it is false.
- Indicator Random variable has value 1 if something is going to happen and 0 otherwise.

$$\mathbf{1}[A] = \begin{cases} 1, & \notin \mathcal{A} \\ 0, & x \in \mathcal{A} \end{cases}$$

For N trials the probability of 1's will be $N \cdot \mathbf{1}[A]$ and the long term average value for these N trials will be $P(N)$.

- Expected value of a discrete random variable is R defined as following. Suppose R can take value r_1 with probability p_1, value r_2 with probability p_2, and so on, up to value r_k with probability p_k. Then the expectation of this random variable R is defined as

$$E[R] = r_1 \cdot p_1 + r_2 \cdot p_2 + \ldots + r_k \cdot p_k$$

- Linearity of Expectation: Let R, S be random variables of some probability space and a, b constants. Then the following holds:

$$E\{a \cdot R + b \cdot S\} = a \cdot E\{R\} + b \cdot E\{S\}$$

- Central Limit Theorem: Given certain conditions, the arithmetic mean of a sufficiently large number of iterations of independent random variables, each with a well defined (finite) expected value and finite variance, will be approximately normally distributed,

regardless of the underlying normal distribution.
Αναλυτικότερα: Δίνονται οι Τ.Μ. X_i και $X = X_1 + X_2 + \ldots + X_n$ το άθροισμα τους. Το άθροισμα αυτό αποτελεί μια Τ.Μ. με μέση τιμή: $m = m_1 + m_2 + \ldots + m_n$ και διακύμανση $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$. Το Κ.Ο.Θ. δηλώνει ότι, κάτω από ορισμένες γενικές συνθήκες, η κατανομή $f_X(\chi)$ της X προσεγγίζει την κανονική κατανομή με την ίδια μέση τιμή m και διακύμανση $\sigma^2 : f(\chi) \simeq G\left(\dfrac{\chi - m}{\sigma}\right)$ καθώς το m αυξάνει (θεωρητικά καθώς $m \to \infty$).

- Law of large numbers (LLN): describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value and will tend to become close as more trials are performed.

11.1 Combinatorics

- Βασική Αρχή Απαρίθμησης (κανόνας γινομένου): Έστω ότι μια διαδικασία μπορεί να πραγματοποιηθεί σε ν διαδοχικές φάσεις $\phi_1, \phi_2, \ldots, \phi_\nu$. Αν η φάση ϕ_1 μπορεί να πραγματοποιηθεί με k_1 τρόπους και για καθέναν από αυτούς η φάση ϕ_2 μπορεί να πραγματοποιηθεί με k_2 τρόπους, ... και για καθέναν από όλους αυτούς τους τρόπους η φάση ϕ_ν μπορεί να πραγματοποιηθεί με k_ν τρόπους, τότε η διαδικασία αυτή μπορεί να πραγματοποιηθεί με $k_1 \cdot k_2 \cdot \ldots \cdot k_\nu$ τρόπους.
- Μεταθέσεις (permutations): Στην περίπτωση που πάρουμε και τα ν στοιχεία ενός συνόλου και τα βάλουμε σε μια σειρά, τότε έχουμε μια διάταξη των ν στοιχείων ανά ν, η οποία λέγεται μετάθεση των ν στοιχείων. Το πλήθος των μεταθέσεων θα είναι: $M_\nu^\nu = M_\nu - \nu!$
- Διατάξεις (k-permutations): Διάταξη των ν στοιχείων ενός συνόλου ανά k, με $k \leqslant \nu$, λέγεται καθένας από τους διαφορετικούς τρόπους με τους οποίους μπορούμε να πάρουμε k διαφορετικά στοιχεία του συνόλου και να τα βάλουμε σε μια σειρά. (διατάξεις των ν ανά k :)
$\Delta_k^\nu = \nu(\nu-1)(\nu-2)\ldots(\nu-k+1) = \dfrac{\nu!}{(\nu-k)!}$

Notes:
- The order of elements matters.
- No elements may appear more than once.

- Συνδυασμοί (combinations): Συνδυασμός των ν στοιχείων ενός συνόλου ανά k ονομάζεται κάθε υποσύνολο του συνόλου με k στοιχεία.
(”n choose k”) ${}^nC_k = \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$, $n \geqslant k$

Notes:
- Order doesn’t matter.

- All combinations of sizes of the input sequence (eg. the Power set of the input set): 2^n

- Newton’s Binomial:

$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k \ , \ n \in \mathbb{N} \ \wedge a, b \in \mathbb{R}$

Properties:

- $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, ή $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$ ή $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}$
- $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$

<u>Random Variables - (Τυχαίες Μεταβλητές)</u>

A (discrete) random variable X is a function from a finite, or countably infinite sample space S to the real numbers. It associates a real number with each possible outcome of an experiment, which allows us to work with the probability distribution induced on the resulting set of numbers.

- $P(X \geqslant x) = 1 - P(X \leqslant x)$
- $P(X \overset{\leqslant}{\underset{\geqslant}{=}} x) = P(X \lesseqgtr x) + P(X = x)$
- $F_X(x) = P(X \leqslant x) \ , \ \forall x \in (-\infty, \infty) \ , \ F_X$: <u>Α.Σ.Κ</u> της τ.μ. X.
- Η F_X είναι μη φθίνουσα. Αν $x_1 < x_2$, τότε $F_X(x_1) \leqslant F_X(x_2)$
- $F(+\infty) = 1 \ , \ F(-\infty) = 0$
- $\lim_{x \to \infty} F(X \leqslant x) = 1 \ , \ \lim_{x \to -\infty} F(X \leqslant x) = 0$
- $\forall x \leqslant x_0 (F_X(x_0) = 0 \to F_X(x) = 0)$
- $0 \leqslant F(X \leqslant x) \leqslant 1 \ , \ \forall x$
- $F_X(-x) = F_X(x)$
- F_X : συνεχής από τα δεξιά: $F_X(x) = F_X(x^+)$
- $P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$
- $P(X = x) = F_X(x) - F_X(x^-)$
- $P(x_1 \leqslant X \leqslant x_2) = F_X(x_2) - F_X(x_1^-)$
- <u>Σ.Π.Π.</u>: $f_X(x) \triangleq \dfrac{dF_X(x)}{dx}$ και $F_X(x) \triangleq \int_{-\infty} x f_X(u) du$
- $P(X \in B)$, B : διάστημα $= \int_B f_X(x) dx$
- $P(X = B) = f_X(B)$
- $\int_{-\infty}^{\infty} f_X(x) dx = 1 \ , \ \sum_{x \in A} f_X(x) = 1 \ , \ \int_{-\infty}^{\infty} F_X(x) dx = 1$
- $P(a \leqslant X \leqslant b) = \int_a^b f_X(x) dx$

<u>Expected/Mean Value</u>

- $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$: for P.D.F.
- $E(X) = \sum_i x_i P(x_i)$: for P.M.F.

Properties:
- $E(C) = C$: σταθερά
- $E(C \cdot X) = C \cdot E(X)$
- $E(C \cdot X + b) = C \cdot E(X) + b$

Variance
- $\sigma_X^2 = E(X^2) - \left[E(X)^2\right]$: Variance

Properties:
- Var $(C) = 0$
- Var $(C \cdot X) = C^2 \cdot$ Var (X)
- Var $(C \cdot X + b) = C^2 \cdot$ Var (X)

- Covariance (συνδιασκύμανση) of two r.v.: A measure of how much two r.v. change together.

Cov $(X, Y) = E\big[(X - E[X]) \cdot (Y - E[Y])\big] = E[X \cdot Y] - E[X] \cdot E[Y]$

- Error probability:

$P_e = P(e|X = x_1) \cdot P(X = x_1) + P(e|X = x_0) \cdot P(X \approx x_2) + \ldots + P(e|X = x_n) \cdot P(X = x_n)$
- Q function , X is gaussian r.v.
- $Q(X) = 1 - F_X(x)$, $Q(-x) = 1 - Q(x)$, $Q(x)$: φθίνουσα

11.2 Probability Distributions

Bernoulli trials: A Bernoulli trial is any random experiment (r.e.) whose result has only two outcomes, which we shall call success with probability p and failure with probability q . P(success) $= p$, P(failure) $= q$. Thus $p + q = 1$.

Binomial (Διωνυμική) Distribution: gives the discrete probability distribution $P(n|N)$ of obtaining exactly n successes out of N Bernoulli trials.

$P(n|N) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$, where $\binom{N}{n}$ is the binomial coefficient.

Geometric Distribution: is the D.P.D. of the number X of Bernoulli trials needed to get one success

$P(X = k) = pq^{k-1}$, k trials ($k \in \mathbb{N}$), X : number of attempts until the first success

Properties:

- (i) $E[X] = \frac{1}{p}$,
- (ii) $\text{Var }[X] = \frac{(1-p)}{p^2}$
- (iii) The Expected value / Mean of a D.R.V. following the geometric distribution is the inverse of its parameter. eg. $X_i = \text{Geo}(\frac{6-i}{6}) \to E[X_i] = \frac{6}{6-i}$

It is the discrete analog of the exponential distribution.

<u>Poisson Distribution</u>: is a D.P.D. that expressses the probability of a given number of events k occuring in a fixed interval of values (e.g. time), if these events occur with a known average rate λ (rate parameter) and independently of the time since the last event.
$P(k \text{ events in interval}) = P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$: P.M.F. , $k \in \mathbb{N}$
λ : average number of events per interval.
Properties:

- (i) $E[X] = \lambda$,
- (ii) $\text{Var }[X] = \lambda$

<u>Exponential Distribution</u>: is the P.D. that describes the time between events in a <u>Poisson process</u> (i.e. a process in which events occur continuously and independently at a constant average rate)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \ , & x \geqslant 0 \\ 0 \ , & otherwise \end{cases}$$

λ : rate parameter (slope of curve)
Properties:

- (i) $E[X] = \frac{1}{\lambda}$,
- (ii) $\text{Var }[X] = \frac{1}{\lambda^2}$

<u>Uniform Distribution</u>: is a family of symmetric P.D.'s such that for each member of the family, all intervals of the same length on the distribution's support are equal.

$$\text{C.D.F. : } F_X(x) \begin{cases} 1 \ , & x \geqslant b \\ \frac{x-a}{b-a} \ , & a \leqslant x \leqslant b \\ 0 \ , \ x < a \end{cases}$$

P.D.F.:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} \;, & a \leqslant x \leqslant b \\ 0 \;, & otherwise \end{cases}$$

Properties:

- (i) $E[X] = \dfrac{a+b}{2}$,
- (ii) Var $[X] = \dfrac{(b-a)^2}{12}$

If the random variable X follows the uniform distribution we'll indicate that, by writing: $X \sim \cup(a,b)$

Normal Distribution: A C.R.V. X is a normal, or Gauss R.V. with parameters μ and σ^2 , if its P.D.F. is:

$f_X(\chi) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\chi-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$, σ^2 : variance, μ : mean / expected value

C.D.F.: $F_X(\chi) = \displaystyle\int_{-\infty}^{x} f_X^{(}y)dy \triangleq G(\frac{\chi-\mu}{\sigma})$

The cumulative distribution function is often given tabulated (G).

Standard Normal Distribution: P.D.F.: $y = \phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, where $z = \dfrac{\chi-\mu}{\sigma}$

$\mu = 0$, $\sigma = 1$

Area under the standard normal curve: $P(\alpha \leqslant z \leqslant b)$

P(values within 1s.d. of the mean) = 68%

P(values within 2s.d. of the mean) = 95%

P(values within 3s.d. of the mean) = 99.7%

11.3 Stochastic Processes

- A Markov process, is a stochastic process that satisfies the Markov property (sometimes characterized as ”memorylessness”). A process satisfies the Markov property if one can make predictions for the future of the process based solely on its present state just as well as one could knowing the process's full history, hence independently from such history.
- The Markov property refers to the memoryless property of a stochastic process.
- Memorylessness is a property of certain probability distributions. It usually refers to the cases when the distribution of a ”waiting time” until a certain event, does not depend on how much time has elapsed already. Only two kinds of distributions are memoryless: exponential distributions of non-negative real numbers and the geometric distributions of non-negative integers.

12 Statistics

- $\nu_1 + \nu_2 + \nu_3 + \ldots + \nu_k = \nu$: Το άθροισμα των (απόλυτων) συχνοτήτων είναι ίσο με το μέγεθος του δείγματος ν.
- $f_i = \dfrac{\nu_i}{\nu}$: Σχετική συχνότητα (relative frequency) της τιμής x_i $0 \leqslant f_i \leqslant 1$, αφού $0 \leqslant \nu_i \leqslant \nu$

$f_i(\%) = 100 \cdot f_i$

- Το σύνολο των (x_i, ν_i) αποτελεί την κατανομή συχνοτήτων
- Το σύνολο των (x_i, f_i) ή $(x_i, f_i(\%))$ αποτελεί την κατανομή σχετικών συχνοτήτων
- Για τις ποσοτικές μεταβλητές εκτός από τα ν_i, f_i χρησιμοποιούνται συνήθως και οι λεγόμενες αθροιστικές συχνότητες (cumulative frequencies) N_i και οι αθροιστικές σχετικές συχνότητες (cumulative relative frequencies) F_i, οι οποίες εκφράζουν το πλήθος και το ποσοστό αντίστοιχα των παρατηρήσεων που είναι μικρότερες ή ίσες της τιμής x_i.
- $\nu_k = N_k - N_{k-1} \;\wedge\; f_k = F_k - F_{k-1}$ (υποθέτοντας $x_1 < x_2 < \ldots < x_k$)
- Η γωνία ϕ_i που αντιστοιχεί στο αντίστοιχο κυκλικό διάγραμμα συχνοτήτων (piechart) είναι $\phi_i = \nu_i \dfrac{360^o}{\nu} = 260^o f_i$, για $i = 1, 2, \ldots, k$.
- Σε ιστόγραμμα / διάγραμμα με άνισο πλάτος w_i κλάσεων, το ύψος της κλάσης είναι: $h_i = \dfrac{\nu_i}{w_i}$, ή $h_i^* = \dfrac{f_i(\%)}{w_i}$
- Σε ένα ιστόγραμμα συχνοτήτων το εμβαδόν του ορθογωνίου ισούται με τη συχνότητα της κλάσης αυτής.
- Αν στα ιστογράμματα συχνοτήτων θεωρήσουμε δύο ακόμη υποθετικές κλάσεις, στην αρχή και στο τέλος, με συχνότητα μηδέν και στη συνέχεια ενώσουμε τα μέσα των άνω βάσεων των ορθογωνίων με ευθύγραμμα τμήματα, σχηματίζεται το <u>πολύγωνο συχνοτήτων (frequency polygon)</u>. Το εμβαδό του χωρίου που ορίζεται από αυτό και τον οριζόντιο άξονα είναι ίσο με το άθροισμα των συχνοτήτων, δηλαδή με το μέγεθος του δείγματος ν, ή ίσο με 100 αν πρόκειται για ιστόγραμμα σχετικών συχνοτήτων.

<u>Μέτρα θέσης (Position Metrics)</u>

- $\bar{x} = \dfrac{x_1\nu_1 + x_2\nu_2 + \ldots + x_k\nu_k}{\nu_1 + \nu_2 + \nu_3 + \ldots + \nu_k} = \dfrac{1}{\nu}\sum_{i=1}^{k} x_i\nu_i = \sum_{i=1}^{k} x_i f_i$: Αριθμητικός μέσος όρος (Arithmetic mean)

όπου: $x_1, x_2, \ldots, x_k$: οι τιμές της τ.μ. Χ με συχνότητες $\nu_1, \nu_2, \ldots, \nu_k$ αντίστοιχα και f_i οι αντίστοιχες σχετικές συχνότητες.

- $\bar{x} = \dfrac{x_1w_1 + x_2w_2 + \ldots + x_\nu w_\nu}{w_1 + w_2 + w_3 + \ldots + w_\nu} = \dfrac{\sum\limits_{i=1}^{\nu}}{\sum\limits_{i=1}^{\nu} w_i}$: Σταθμισμένος μέσος όρος (weighted mean)

όπου $x_1, x_2, \ldots, x_\nu$ οι τιμές, με συντελεστές βαρύτητας $w_1, w_2, \ldots, w_\nu$.

- Η διάμεσος (median) δ ενός πληθυσμού ν παρατηρήσεων που έχουν διαταχθεί σε αύξουσα σειρά, είναι η μεσαία παρατήρηση όταν ν περιττός, ή ο μέσος όρος (το ημιάθροισμα) των δύο μεσσαίων παρατηρήσεων όταν ν άρτιος.

- P_k , $k = 1, 2, \dots, 99$: Εκατοστημόρια (percentiles)

Οι τιμές $P_1, P-2, \dots, P_{99}$ χωρίζουν τη συνολική συχνότητα σε 100 ίσα μέρη. Δηλαδή, ορίζουμε ως κ-εκατοστημόριο, ή P_k την τιμή εκείνη για την οποία το πολύ $k\%$ των παρατηρήσεων είναι μικρότερες του P_k και το πολύ $(100-k)\%$ των παρατηρήσεων είναι μεγαλύτερες από την τιμή αυτήν. Ειδική περίπτωση εκατοστημορίων είναι τα $P_{25}, P_{50} = \delta, P_{75}$ που λέγονται τεταρτημόρια. Αναφέρονται και ως Q_1, Q_2, Q_3 αντίστοιχα.

- Η επικρατούσα τιμή M_0 , ή κορυφή (mode) ορίζεται ως η παρατήρηση με τη μεγαλύτερη συχνότητα ν_i. Μπορεί να οριστεί και στην περίπτωση ποιοτικών δεδομένων, ενώ τα προηγούμενα μέτρα θέσης ορίζονται μόνο για ποσοτικά δεδομένα. Είναι δυνατό να υπάρχουν πολλαπλές, ή και καμιά επικρατούσα τιμή.

Μέτρα Διασποράς (Dispersion Metrics)

- Εύρος (Range) $R =$ Μεγαλύτερη παρατήρηση $-$ Μικρότερη παρατήρηση
- $Q = Q_3 - Q_1$: Ενδοτεταρτημοριακό εύρος (interquartile range) $\uparrow Q \Rightarrow \uparrow$ Διασπορά
- $\mathrm{Var}[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - (E\left[X\right])^2 = \sigma^2$: Διακύμανση

$\sigma^2 = \dfrac{1}{\nu}\sum_{i=1}^{\nu}(x_i\nu_i - \bar{x})^2 = \dfrac{1}{\nu}\sum_{i=1}\nu(x_i\nu_i)^2 - \Big(\dfrac{\sum_{i=1}^{\nu} x_i\nu_i}{\nu}\Big)^2$: για μη ομαδοποιημένα δεδομένα

$\sigma^2 = \dfrac{1}{\nu}\sum_{i=1}^{k} x_i^2\nu_i - \Big(\dfrac{\sum_{i=1}^{k} x_i\nu_i}{\nu}\Big)^2$: για ομαδοποιημένα δεδομένα, εδώ x_i είναι οι κεντρικές τιμές της κάθε κλάσης

- $\sigma = \sqrt{\sigma^2}$: Τυπικής απόκλιση (standard deviation)
- $CV = \dfrac{\sigma}{|\bar{X}} \cdot 100(\%)$: Συντελεστής μεταβολής, ή σχετική τυπική απόκλιση (Coefficient of variation, or relative standard deviation)

$\downarrow CV \rightarrow$ Μεγαλύτερη ομοιογένεια στις τιμές

- Mode of grouped data $= L + h\Big(\dfrac{f_m - f_1}{2f_m - f_1 - f_2}\Big)$

L : lower boundary of modal class, h : size of modal class, f_m : frequency of modal class, f_1 : frequency of class preceding the modal class, f_2 : frequency of class proceeding modal class

- Pearson correlation coefficient (r): Gives the strength of a linear relationship between the n values of two variables x_i and y_i.

(Be aware that there may be correlation, but not linear one)

$$r = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]\cdot\left[n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right]}}$$

- Spearman's rank correlation coefficient: It doesn't measure the actual values, but rather the differences (d_i) between the n corresponding values of two categories (/columns of data):

$$r_s = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$$

13 Number Theory

- $\forall n, m \in \mathbb{Z}\left[m \neq 0 \rightarrow \exists! q, r \in \mathbb{Z}\left(n = q \cdot m + r \wedge 0 \leqslant r < |m|\right)\right]$

q and r are called quotient and remainder respectively, when n is divided by m.

- $(n|m \wedge m > 0) \rightarrow r \in \{0, 1, 2, \ldots, m-1\}$
- $r = 0 \rightarrow n = 2q$: άρτιος
- $r = 1 \rightarrow n = 2q + 1$: περιττός
- Well-Ordering principle: Every nonempty set of $\mathbb{N}$ has a smallest element.
- n is even iff n^2 is even.
- $\forall n \in \mathbb{N}\ \forall x \in \mathbb{R}\left[n \text{ is odd} \rightarrow (x+1)(x^n+1)\right]$
- $n \in \mathbb{N} \rightarrow x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$
- $\forall n \in \mathbb{N}^*\left[1 + 3 + 5 + 7 + \ldots + (2n-1)\right] = \nu^2$
- $\forall n \in \mathbb{N}^*\left[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}\right]$
- $\forall n \in \mathbb{N}^*\left(\dfrac{1}{2 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}\right)$
- $\forall n \in \mathbb{N}^* \forall x \in \mathbb{R} - \{1\}\left(1 + x + x^2 + \ldots x^{n-1} = \dfrac{x^n - 1}{x - 1}\right)$
- $\forall n \in \mathbb{N}^* \geqslant 3\left(n^2 > 2n + 1\right)$
- $\forall k, l \in \mathbb{Z}\left(k = 2l + 1 \rightarrow \exists m \in \mathbb{Z}\left(k^2 = 8m + 1\right)\right)$
- $\forall a \in \mathbb{Z}\left[\left(a^2 = 3k \vee a^2 = 3k + 1\right) \wedge k \in \mathbb{Z}\right]$
- $\forall n \in \mathbb{N} \geqslant 2\left[1 + 2 + 2^2 + \ldots + 2^{n-1} : \text{πρώτος} \rightarrow 2^{n-1}(2^n - 1) : \text{τέλειος}\right]$
- A discrete logarithm is an interger k solving the equation: $b^k = g$ where b and g are elements of a group, s.t. $k = \log_b g$

Discrete logarithms are the group theoretic analog of ordinary logarithms, which solve the same equation for b and g in the group of real numbers.

- A number g is a primitive root modulo n if every number a coprime to n is congruent to a power of g modulo n. That is, for every integer a coprime to n, there is an integer k such that $g \cdot k \equiv a \pmod{n}$. Such k is called the index or discrete logarithm of a to the base g mod n.

13.1 Division Properties

- $\forall a \in \mathbb{Z}^*\ (\pm 1|a \wedge \pm a|a)$
- $\forall b \in \mathbb{Z}^*(b|0)$
- $\forall a, b, c \in \mathbb{Z} \wedge b \neq 0$ έχουμε: - $a|b \wedge b|a \rightarrow a = b \vee a = -b$

- $a|b \wedge b|c \rightarrow a|c$
- $a|b \rightarrow \exists \lambda \in \mathbb{Z}(a|\lambda b)$
- $a|b \wedge a|c \rightarrow a|(b+c)$ (Το αντίστροφο φυσικά δεν ισχύει)
- $a|b \wedge b \neq 0 \rightarrow |a| \leqslant |b|$

- $\forall k, \lambda \in \mathbb{Z}\left[(a|b \wedge a|c) \rightarrow a|(kb + \lambda c)\right]$

Το $(kb+\lambda c)\in\mathbb{Z}$ λέγετεαι γραμμικός συνδυασμός των b, c

- $\forall a, m \in \mathbb{Z}\,[(m|a \wedge m > 1) \rightarrow m \nmid (a+1)]$
- $\forall \alpha, \beta \in \mathbb{Z}$ Ο Μ.Κ.Δ. των α, β, όταν ένας τουλάχιστον από τους α, β είναι διάφορος του 0, είναι ο δ και είναι ο μεγαλύτερος από τους θετικούς κοινούς διαιρέτες τους. Δηλαδή, ο δ έχει τις ακόλουθες δύο ιδιότητες:

$\delta|\alpha \wedge \delta|\beta$

$(x|\alpha \wedge x|\beta) \rightarrow x \leqslant \delta$

Λέμε ότι $\delta\,\mathrm{gcd}(a,b)$, ή $\delta=(a,b)$

- <u>Ευκλείδιος Αλγόριθμος</u> Αν $a, b \in \mathbb{N}$ και υ το υπόλοιπο της ευκλείδειας διαίρεσης του a με τον b, τότε: $\mathrm{gcd}(a,b)=\mathrm{gcd}(b,v)$
- $\mathrm{gcd}(a,b)=\mathrm{gcd}(|a|,|b|)$
- $(a,a)=a$
- $(a,0)=a$
- $(a,1)=1$
- $\forall a, b \in \mathbb{N}^*\,(b|a \rightarrow (a,b)=b)$
- $\forall a, b, k \in \mathbb{N}, b \neq 0\,((a,b)=(a-kb,b))$
- $\forall a, b, \in \mathbb{Z}\,((a,b)=1 \rightarrow a, b)$ πρώτοι μεταξύ τους)
- $\forall a, b \in \mathbb{Z}, b \neq 0\,[\delta=\mathrm{gcd}(a,b) \rightarrow \delta = ka+lb]$: Bezout's Identity. Οι k, l δεν είναι μοναδικοί.
- $\forall a, b \in \mathbb{Z}, b \neq 0\,(a, b$πρώτοι μεταξύ τους $\leftrightarrow ka+lb=1)$
- $\forall a, b \in \mathbb{Z}, b \neq 0\left[ka+lb=\delta \rightarrow \left(k(\frac{a}{\delta})+l(\frac{b}{\delta})=1 \leftrightarrow (\frac{a}{\delta},\frac{b}{\delta}=1)\right)\right]$
- $\forall a, b, c \in \mathbb{Z}\,((b,c)=1 \wedge a|bc \rightarrow a|b \vee a|c)$
- $\forall a, b, c \in \mathbb{Z}\,(a|b \cdot c \wedge (a,b)=1 \rightarrow a|c)$
- $\forall k, a, b \in \mathbb{Z}\,[(ka,kb)=k(a,b)]$

Ανάλογες σχέσεις ισχύουν και για περισσότερους από δύο ακεραίους πχ. $[\delta=(a,b,c) \rightarrow \exists k, l, m \in \mathbb{Z}(\delta=ka+lb+mc)]$ και $[\delta=(\alpha,\beta,\gamma) \rightarrow (\alpha/\delta, \beta/\delta, \gamma/\delta)=1]$.

- $\delta=(a,b,c,\ldots)=((a,b),c,d,\ldots)=(a,(b,c),d,\ldots)$
- Ε.Κ.Π των $a, b \in \mathbb{Z}^*$ είναι το μικρότερο από τα θετικά κοινά πολλαπλάσια των a, b.

Συμβολίζεται: $\epsilon=lcm[a,b]$ ή $\epsilon=[a,b]$ και $\epsilon \in \mathbb{N}^*$ έχει τις ακόλουθες ιδιότητες:

$\epsilon=mul(a) \wedge \epsilon=mul(b)$

$[x=mul(a) \wedge x=mul(b)] \rightarrow \epsilon \leqslant x$

- $[a,b]=[|a|,|b|]$
- $b|a \rightarrow [a,b]=a$
- $[a,1]=a$
- $\forall a, b \in \mathbb{N}^*\,[(a,b)\cdot[a,b]=a\cdot b]$
- $\forall a, b \in \mathbb{Z}^*\,[(a,b)\cdot[a,b]=|a|\cdot|b|]$
- Τα κοινά πολλαπλάσια δύο ακεραίων είναι πολλαπλάσια του ΕΚΠ τους
- $\epsilon=[a,b,c,\ldots]=[[a,b],c,\ldots]=[a,[b,c],d,\ldots]$
- <u>Θεώρημα πρώτων αριθμών</u>: # Πρώτων μικρότερων του $x \sim \dfrac{x}{\ln\,(x)}$

Πυκνότητα πρώτων αριθμών μέχρι τον $x : x = \dfrac{1}{\ln(x)}$. Διατύπωση:

$\lim\limits_{x\to\infty} \dfrac{\pi(x)}{x\log(x)} = 1, \quad \pi(x)$: prime counting function (counts # of primes under $x \in \mathbb{Z}$

- $\dfrac{x}{\log(x)}$: approximates $\pi(x)$ as x increases without bound
- Κάθε ακέραιος $p \neq 0, \pm 1$ λέγεται πρώτος, αν οι μόνοι θετικοί διαιρέτες του είναι οι 1 και $|p|$.
- Κάθε θετικός ακέραιος μεγαλύτερος του 1έχει έναν τουλάχιστον πρώτο διαιρέτη
- Αν α είναι ένας σύνθετος ακέραιος με $a > 1$, τότε υπάρχει ένας τουλάχιστον πρώτος αριθμός ρ, τέτοιος ώστε $p|a$ και $p \leqslant \sqrt{a}$.
- Αν ένας πρώτος ρ διαιρεί το γινόμενο οσωνδήποτε ακεραίων, τότε διαιρεί έναν τουλάχιστον, από τους ακεραίους αυτούς.
- Κάθε τεικός ακέραιος α ' 1 μπορεί να γραφεί κατά μοναδικό τρόπο στη μορφή: $\alpha = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ όπου οι $p_1, p_2, \ldots, p_k$ είναι θετικοί πρώτοι με $p_1 < p_2 < \ldots p_k$ και $a_1, a_2, \ldots, a_k \in \mathbb{N}^*$. Τότε λέμε ότι το α είναι γραμμένος στην κανονική του μορφή.
- Αν ο φυσικός αριθμός n δεν είναι τετράγωνο φυσικού τότε ο $\sqrt{n}$ είναι άρρητος
- Ο ευκλείδειος αλγόριθμος βασίζεται στο γεγονός ότι: $\forall a, b, r \in \mathbb{N}, b \neq 0 (a > b \rightarrow a = qb + r \wedge b > r)$
- $\forall a, b, c \in \mathbb{Z}(ax + by = c)$: Μια γραμμική διοφαντική εξίσωση και $\delta = \gcd(a, b)$. Η εξίσωση έχει λύση ανν $\delta|c$. Τότε υπάρχουν άπειρες λύσεις που δίνονται από τους τύπους $x = x_0 + b \cdot n \wedge y = y_0 - a \cdot n), \forall n \in \mathbb{Z}$, όπου (x_0, y_0) μια λύση της γραμμικής διοφαντικής εξίσωσης.
- $\forall a, b, k, l \in \mathbb{Z} \exists m \in \mathbb{N}\, [a = km + \upsilon \wedge b = lm + \upsilon \leftrightarrow a \equiv b (\text{mod } m)]$
- $\forall a, b \in \mathbb{Z}\, [a \equiv b (\text{mod } m) \leftrightarrow m|(a - b)]$

<u>Congruent (ισουπόλοιποι) Numbers</u>

- <u>Ορισμός:</u> Έστω m θετικός ακέραιος. Δύο ακέραιοι a και b λέγονται ισουπόλοιποι με μέτρο m, όταν διαιρούμενοι με m αφήνουν το ίδιο υπόλοιπο.

- $a \equiv b (\text{mod } m) \Leftrightarrow m|(a - b)$
- $a \equiv a (\text{mod } m)$ (ανακλαστική)
- $a \equiv b (\text{mod } m) \rightarrow b \equiv a (\text{mod } m)$ (συμμετρική)
- $\big[a \equiv b (\text{mod } m) \wedge b \equiv c (\text{mod } m)\big] \rightarrow a \equiv c (\text{mod } m)$ (μεταβατική)
- $a \equiv b (\text{mod } m) \wedge c \equiv d (\text{mod } m) \begin{cases} a + c \equiv b + d (\text{mod } m) \\ a - c \equiv b - d (\text{mod } m) \\ a \cdot c \equiv b \cdot d (\text{mod } m) \end{cases}$
- $\forall a, b, c \in \mathbb{Z}\, [\exists m \in \mathbb{N} (a \equiv b (\text{mod } m)) \rightarrow a^n \equiv b^n (\text{mod } m) \wedge n \in \mathbb{N}]$
- Έστω $\delta = \gcd(a, m)$. Τότε $ax \equiv b (\text{mod } m)$ έχει μια λύση, ανν $\delta|b$.
- Αν υ το υπόλοιπο της ευκλείδειας διαίρεσης του $a \in \mathbb{Z}$ με τον $m \in \mathbb{N}^*$, τότε $a \equiv \upsilon (\text{mod } m)$
- $\forall m \in \mathbb{Z}\, [m|b \rightarrow b \equiv 0 (\text{mod } m)]$
- $\forall m \in \mathbb{Z}\, [m \equiv 0 (\text{mod } m)]$

- $\forall a \in \mathbb{Z}\left[a^2 \equiv 0(\text{mod } 8) \vee a^2 \equiv 1(\text{mod } 8) \vee a^2 \equiv 4(\text{mod } 8)\right]$
- $\forall a, b \in \mathbb{Z}(a, b) = 5 \to \left(\frac{a}{\delta}, \frac{b}{\delta}\right) = 1$

Χρήσιμος μετασχηματισμός: $A = \frac{a}{\delta} \to a = A\delta \wedge B = \frac{b}{\delta} \to b = B\delta$

- $\forall k \in \mathbb{Z}\left[n = 2k + 1 \to 9^n + 1 \equiv 0(\text{mod } 10)\right]$

- Euler / Euler-Mascheroni constant:

$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln\,(n)\right) = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx \simeq 0.5772156649\ldots \quad \lfloor x \rfloor$: floor function

- The positive integers i and j are called relatively prime, or coprime if they share no common factors. In other words i and j are coprime if their only common factor is 1.
- Mersenne prime: $M_n = 2^n - 1\ , \quad n \in \mathbb{N}$
- If $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is a perfect number.
- Euler's totient/Phi function $\Phi(n)$: Roughly speaking it measure the "breakability" of a number.

$\Phi(n)$ number of positive integers less than $n \in \mathbb{N}$ (including 1) that do not share a common factor with n, ie. they are coprime with n. eg. $\Phi(8) = 4$ (numbers 1, 3, 5, 7). These integers k are referred to as totatives of n.

$\Phi(n)$ is difficult to compute, except when n is prime. In that case $\Phi(n) = n - 1$ (numbers 1, 2, 3, ... , $n - 1$)

Properties

- $\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b)$
- p is prime $\to \Phi(p) = p - 1$
- m, n coprime $\to m^{\Phi(n)} \equiv 1\ mod\ n$

14 Geometry

14.1 Line Segment

- Αν M εσωτερικό (ή M′ εξωτερικό) σημείο ευθυγράμμου τμήματος AB, λέμε ότι το M διαιρεί εσωτερικά (ή εξωτερικά αντιστοίχως) το ευθύγραμμο τμήμα AB σε λόγο λ, αν και μόνο αν $\lambda = \dfrac{MA}{MB}$. Το σημείο αυτό είναι μοναδικό. Για κάθε περίπτωση ισχύει:

- $MA = \dfrac{\lambda}{\lambda+1} \cdot AB, MB = AB - MA = \dfrac{1}{\lambda+1}AB$ (ή M′ αντί για M)

Figure 10: Συζυγή Αρμονικά σημεία M & M′

- (figure 10) Δύο σημεία M και M′, που διαιρούν εσωτερικά και εξωτερικά το τμήμα AB στον ίδιο λόγο, λέγονται συζυγή αρμονικά των A και B, αν τα τέσσερα σημεία είναι συνευθειακά και επίσης ισχύει ότι: $\dfrac{MA}{MB} = \dfrac{M'A}{M'B}$
- Δύο (ή περισσότερα) ευθύγραμμα σχήματα λέγονται όμοια, όταν οι πλευρές τους είναι ανάλογες (ανάλογο μήκος) και οι γωνίες τους ίσες.
- Το ύψος υ_a ενός τριγώνου $\overset{\triangle}{ABC}$ δίνεται από τον τύπο:

$\upsilon_a = \dfrac{2}{a}\sqrt{\tau(\tau-a)(\tau-b)(\tau-c)}, \tau$: ημιπερίμετρος.

<u>Αναλογίες</u>

- $\dfrac{a}{b} = \dfrac{c}{d} \leftrightarrow ad = bc, \quad \dfrac{a}{b} = \dfrac{b}{c} \leftrightarrow b^2 = ac$
- $\dfrac{a}{b} = \dfrac{c}{d} \leftrightarrow \dfrac{a}{c} = \dfrac{b}{d}$
- $\dfrac{a}{b} = \dfrac{c}{d} \leftrightarrow \dfrac{a \pm b}{b} = \dfrac{c \pm d}{d}, \dfrac{a}{b} = \dfrac{c}{d} \leftrightarrow \dfrac{a}{a \pm b} = \dfrac{c}{c \pm d}$
- $\dfrac{a}{b} = \dfrac{c}{d} = \ldots = \dfrac{k}{l} = \dfrac{a + c + \ldots + k}{b + d + \ldots + l}$
- $a \propto b \leftrightarrow a = \lambda \cdot b$

14.2 Triangle

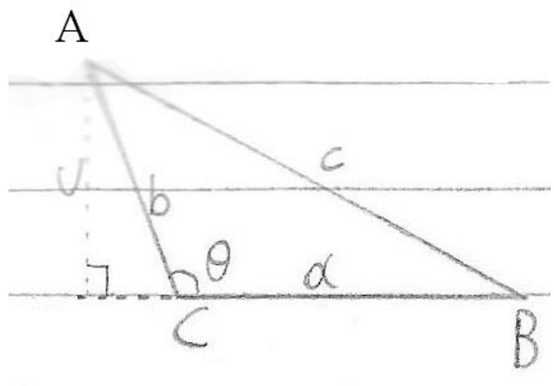


Figure 11: Reference Triangle

- $\Pi = a + b + c$: Περίμετρος (See figure 11)
- $\epsilon = \dfrac{a \cdot v}{2} = \dfrac{ab \cdot sin\theta}{2} = \dfrac{r\Pi}{2}$: Εμβαδό τριγώνου
- $\epsilon = \sqrt{\mathrm{T}(\mathrm{T}-a)(\mathrm{T}-b)(\mathrm{T}-c)}$: Τύπος του Ήρωνα, όπου Τ ημιπερίμετρος τριγώνου

r : η ακτίνα του εγγεγραμένου κύκλου του τριγώνου

- $S = \dfrac{\Pi}{2} = \dfrac{a+b+c}{2}$
- Για ισόπλευρο τρίγωνο έχουμε: $\epsilon = \dfrac{\sqrt{3}}{2}a, \Pi = 3a$, a : πλευρά
- $30^o - 60^o - 90^o$ triangle $\rightarrow 1 : \sqrt{2} : 2$
- $45^o - 45^o - 90^o$ triangle $\rightarrow 1 : 1 : \sqrt{2}$
- $\epsilon = \dfrac{abc}{4R} = \dfrac{1}{2}bc \cdot sinA = \dfrac{1}{2}ac \cdot sinB = \dfrac{1}{2}ab \cdot sinC$ R : ακτίνα περιγεγγραμένου κύκλου του τριγώνου

Ορθογώνιο Τρίγωνο

- $a^2 = b^2 + c^2$: Πυθαγόρειο Θεώρημα
- $a^2 = b^2 + c^2 - 2b \cdot AC$, in $\overset{\triangle}{ABD}$: Γενίκευση Π.Θ. για $\hat{A} < 90^o$, $AC =$ προβ$_b c$
- Εάν $\hat{A} > 90^o \rightarrow \overset{\triangle}{ABC}$ αμβλυγώνιο $\rightarrow a^2 = b^2 + c^2 + 2b \cdot AC$
- The maximum possible altitude of a right-angled triangle is half the hypotenuse. By inscribing the triangle into a circle to see this.

1ο Θεώρημα Διαμέσων

- $b^2 + c^2 = 2\mu_a^2 + \dfrac{a^2}{2}, a^2 + c^2 = 2\mu_b^2 + \dfrac{b^2}{2},\ a^2 + b^2 = 2\mu_c^2 + \dfrac{c^2}{2}$

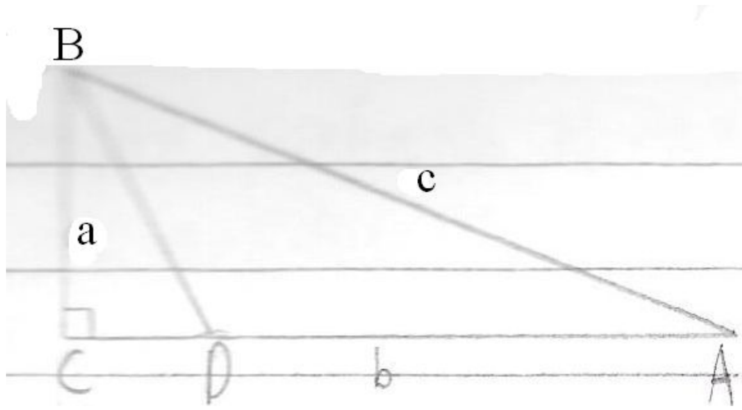


Figure 12: Ορθογώνιο Τρίγωνο

<u>2ο Θεώρημα Διαμέσων</u>

- $b^2 - c^2 = 2a \cdot MD$, $MD =$ προβ$_a \mu_a$

14.3 Quadrilateral

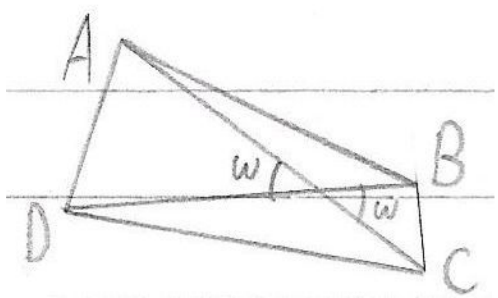


Figure 13: Κυρτό Τετράπλευρο

- (figure 13) Το εμβαδό κυρτού τετραπλεύρου ισούται με το ημιγινόμενο των διαγωνίων του, πολ/μένο με το ημίτονο της γωνίας ω που αυτές σχηματίζουν:
$(ABCD) = 1/2 \cdot AC \cdot BC \cdot sin(\hat{\omega})$

<u>Ορθογώνιο Παραλληλόγραμμο</u>

- $\Pi = 2(\alpha + \beta)$
- $E = \alpha \cdot \beta$

<u>Πλάγιο Παραλληλόγραμμο</u>

- $\Pi = 2(a + b)$
- $E = \alpha \cdot \upsilon = \alpha\beta \sin(\theta)$

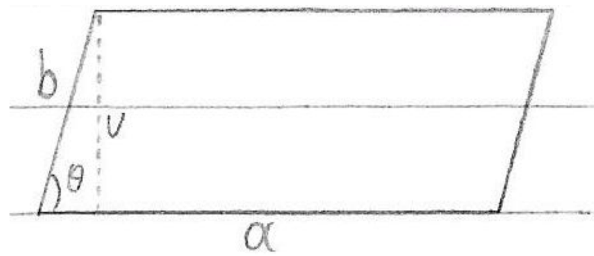


Figure 14: Πλάγιο Παραλληλόγραμμο

<u>Ρόμβος</u>

- $\Pi = 4\alpha$
- $E = \dfrac{\delta_1 \cdot \delta_2}{2} = \alpha \cdot \upsilon$

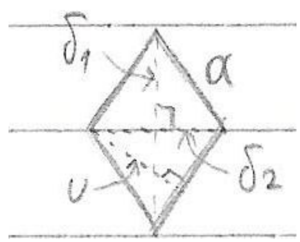


Figure 15: Ρόμβος

<u>Τραπέζιο</u>

- $\Pi = \alpha + \beta + \upsilon \cdot \left(\dfrac{1}{sin\theta} + \dfrac{1}{sin\phi} \right)$
- $E = \dfrac{a+b}{2} \upsilon = \overleftrightarrow{EZ} \cdot \upsilon$
- $\overleftrightarrow{EZ} = \dfrac{a+b}{2}$ (διάμεσος) Η διάμεσος διέρχεται από τα μέσα των διαγωνίων του τραπεζίου.

$BK = KD, A\Lambda = \Lambda C$.

14.4 Regular Polygon

- Ο λόγος των εμβαδών δύο όμοιων πολυγώνων, έστω Ε και Ε′, ισούται με το τετράγωνο του λόγου ομοιότητας τους: $\dfrac{a}{a'} = \dfrac{\upsilon_a}{\upsilon_{a'}} = \lambda \longrightarrow \dfrac{E}{E'} = \lambda^2$

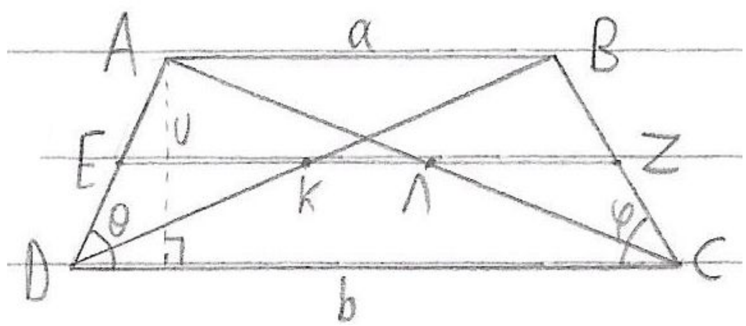


Figure 16: Τραπέζιο

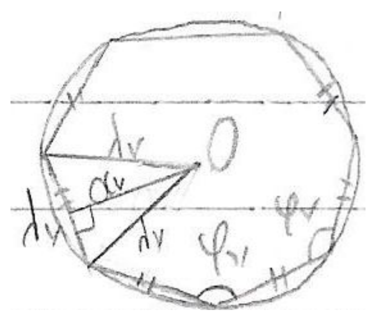


Figure 17: Στοιχεία Κανονικού Πολυγώνου

Σε κάθε κανονικό πολύγωνο ισχύουν οι σχέσεις:

1. $a_v^2 + \dfrac{\lambda_v^2}{4} = R^2$
2. $P_v = v \cdot \lambda_v$: Περίμετρος
3. $\omega_v = \dfrac{360^o}{v}$: Επίκεντρη γωνία
4. $E_v = \dfrac{1}{2} P_v \cdot a_v$: Εμβαδό
5. $\phi_v = 180^o - \omega_v$: Γωνία πολυγώνου

όπου λ_v : πλευρά πολυγώνου, R : ακτίνα κύκλου, a_v : απόστημα, v : πλήθος πλευρών πολυγώνου

- Άθροισμα εσωτερικών γωνιών πολυγώνου: $(2v-4)L$
- Άθροισμα εξωτερικών γωνιών κυρτού πολυγώνου: $4L$

14.5 Stereometry - Polyhedron

- Θεώρημα του Euler (Euler's formula) για τα πολύεδρα: $K + E = A + 2$

Κ: πλήθος κορυφών, Ε: πλήθος εδρών, Α: πλήθος ακμών

- $A = \dfrac{\nu E}{2}$, ν : αριθμός πλευρών κάθε έδρας. Κάθε έδρα έχει και ν κορυφές

- Για μια επιφάνεια A η στερεά γωνία ω που την περιλαμβάνει ορίζεται ως: $\omega = \dfrac{A'}{R^2}$ (sr)

όπου A' η προβολή του Α σε σφαίρα ακτίνας R.
Μονάδα μέτρησης: steradians (sr)

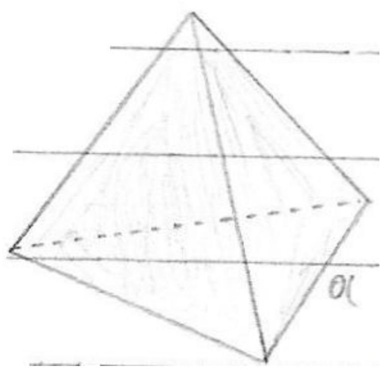


Figure 18: Στοιχεία Τετραέδρου

Τετράεδρο

- Εμβαδό κανονικού τετραέδρου: $E = \sqrt{3}a^2$
- Όγκος κανονικού τετραέδρου: $V = \dfrac{\sqrt{2}a^3}{12}$

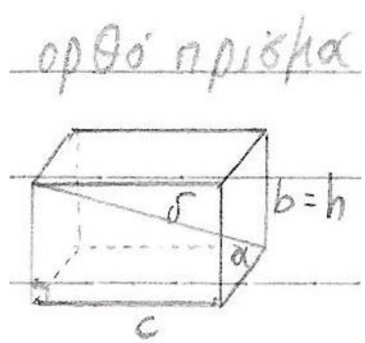


Figure 19: Στοιχεία Ορθογωνίου Παραλληλεπιπέδου

Ορθογώνιο Παραλληλεπίπεδο

- Όγκος: $V = a \cdot b \cdot c = E_B \cdot c$
- Διαγώνιος: $\delta^2 = a^2 + b^2 + c^2$

- Εμβαδό ολικής επιφάνειας: $E_0 = E_\pi + 2E_B$, E_B : εμβαδό βάσης

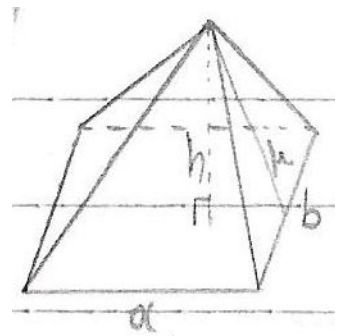


Figure 20: Πυραμίδα

Πυραμίδα

- Εμβαδό: $E = ab + a\sqrt{\left(\frac{b}{2}\right)^2 + h^2} + b\sqrt{\left(\frac{a}{2}\right)^2 + h^2}$
- Όγκος: $V = \frac{E_B h}{3}$, E_B : εμβαδό βάσης

Κανονική Πυραμίδα

(πυραμίδα που η βάση της είναι κανονικό πολύγωνο και τα πλευρικά ύψη είναι όλα ίσα σε μήκος.)

- Εμβαδό παράπλευρης επιφάνειας: $E_\pi = \tau \cdot \mu$
- Εμβαδό ολικής επιφάνειας: $E_0 = \tau(\mu + \alpha)$

τ : ημιπερίμετρος βάσης, μ : παράπλευρο / λοξό ύψος

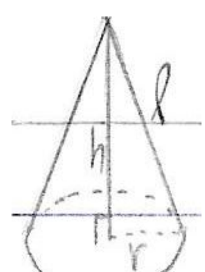


Figure 21: Στοιχεία Κώνου

Κώνος

- Εμβαδό παράπλευρης επιφάνειας: $E_\pi = \pi r l$
- Συνολικό εμβαδό: $E_0 = \pi r l + \pi r^2$, l : λοξό ύψος
- Όγκος κώνου: $V = \frac{\pi r^2 h}{3}$, r : ακτίνα βάσης

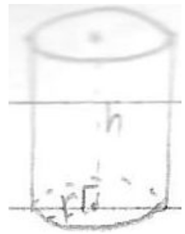


Figure 22: Κύλινδρος

Κύλινδρος

- Παράπλευρο εμβαδόν: $E_\pi = 2\pi rh$
- Ολικό εμβαδό: $E_0 = 2\pi r^2 + 2\pi rh$
- Όγκος: $V = \pi r^2 h$
- Περίμετρος (έπειτα από πλάγια προβολή σε επίπεδο $\Pi = 2(2r + h)$

Σφαίρα

- Εμβαδόν: $E = 4\pi r^2$
- Όγκος: $V = \dfrac{4}{3}\pi r^3$

Μέτρηση κόλουρης (Ισοσκελούς) Πυραμίδας

- Εμβαδό παράπλευρης επιφάνειας: $E_\pi = (\tau + \tau')\mu$

τ, τ' : οι ημιπερίμετροι των βάσεων, μ: παράπλευρο ύψος

- Εμβαδό ολικής επιφάνειας: $E = E_\pi + E_B + E_{B'}$

E_B : Εμβαδό μεγάλης βάσης, $E_{B'}$: Εμβαδό μικρής βάσης

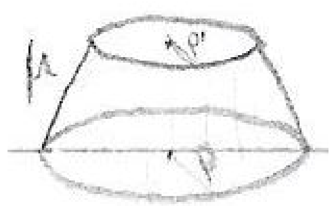


Figure 23: Κόλουρος κώνος

Μέτρηση Κόλουρου Κώνου

- Εμβαδό παράπλευρης επιφάνειας: $E_\pi = \pi \cdot \mu(\rho + \rho')$

μ : παράπλευρο ύψος, ρ, ρ' : μεγάλη και μικρή ακτίνα αντίστοιχα

- Ολικό εμβαδόν: $E_0 = E_\pi + \pi(\rho^2 + \rho'^2)$
- Όγκος κόλουρου κώνου: $V = \dfrac{\pi \upsilon}{3}\left(\rho^2 + \rho'^2 + \rho \cdot \rho'\right)$

14.6 Line

- Γενική μορφή εξίσωσης ευθείας: $Ax + By + \Gamma = 0$, όπου A και B δεν είναι συγχρόνως ίσα με το μηδέν.
- Κανονική μορφή εξίσωσης ευθείας: $y = m \cdot x + b$, b : y-intercept, m : συντελεστής διεύθυνσης ευθείας που διέρχεται από τα σημεία $A(x_1, y_1)$ και $B(x_2, y_2)$,
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y}{x}$
- Εξίσωση ευθείας όταν δίνονται δύο σημεία της:
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$
- Η γενική μορφή εξίσωσης ευθείας σε δύο διαστάσεις είναι:
$\frac{x - x_{\text{αρχ}}}{x_{\text{τελ}} - x_{\text{αρχ}}} = \frac{y - y_{\text{αρχ}}}{y_{\text{τελ}} - y_{\text{αρχ}}}$,
μορφή ισοδύναμη με $m = \frac{y_{\text{τελ}} - y_{\text{αρχ}}}{x_{\text{τελ}} - x_{\text{αρχ}}}$
- Γενική μορφή εξίσωσης ευθείας σε τρείς διαστάσεις:
$\frac{x - x_{\text{αρχ}}}{x_{\text{τελ}} - x_{\text{αρχ}}} = \frac{y - y_{\text{αρχ}}}{y_{\text{τελ}} - y_{\text{αρχ}}} = \frac{z - z_{\text{αρχ}}}{z_{\text{τελ}} - z_{\text{αρχ}}}$
- Πολική μορφή εξίσωσης ευθείας: $r = \frac{mr\cos\theta + b}{\sin\theta}$ όταν $\vartheta = 0$ δεν ορίζεται πολική μορφή εξίσωσης ευθείας
- $m_1 = m_2 \leftrightarrow line1 \parallel line2$
- $m_1 \cdot m_2 = -1 \leftrightarrow line1 \perp line2$
- $\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$: Γωνία ϑ μεταξύ 2 καμπυλών ευθειών m_1, m_2 : κλίσεις των καμπυλών στο σημείο επαφής τους, ή οι κλίσεις των ευθειών
- Απόσταση σημείου $M(x_0, y_0)$ και ευθείας
$\epsilon : Ax + By + \Gamma = 0 : d(M, \epsilon) = \frac{|Ax_0 + By_0 + \Gamma|}{\sqrt{A^2 + B^2}}$

14.7 Miscellaneous

- $n \in \mathbb{N}^*$ chords drawn in a circle in such a way that each chord intersects each other, but no three intersect at one point, cut the circle into $\frac{n^2 + n + 2}{2}$ regions.
- Η γωνία που σχηματίζεται από μια χορδή κύκλου και την εφαπτομένη στο άκρο της χορδής ισούται με την εγγεγραμένη που βαίνει στο τόξο της χορδής. - Γωνία χορδής και εφαπτομένης
- Όταν ένα τετράπλευρο έχει δύο απέναντι γωνίες του παραπληρωματικές και μια πλευρά του φαίνεται από τις απέναντι κορυφές υπό ίσες γωνίες, τότε είναι εγγράψιμο σε κύκλο.
- An open polygon with n sides and k vertexes at infinity will have $(n - k)$ internal angles.
- Orthogonality is the relation of two lines at right angles to one another (perpendicularity), and the generalization of this relation into n dimensions; and to a

variety of mathematical relations thought of as describing non-overlapping, uncorrelated, or independent objects of some kind.

- 1 arcminute is $\frac{1}{60}$th of a degree. 1 arcsecond is $\frac{1}{3600}$th of a degree.

15 Conic Sections

15.1 Circle

- Αν δύο χορδές ΑΒ, ΓΔ, ή οι προεκτάσεις τους τέμνονται σε ένα σημείο Ρ, τότε ισχύει: $PA \cdot PB = P\Gamma \cdot P\Delta$. Στην ειδική περίπτωση της εφαπτομένης, όπου τα σημεία τομής ταυτίζονται (έστω Γ, Δ ≡ Ε), το θεώρημα ισχύει: $PE^2 = PA \cdot PB$. Εάν ΟΡ = δ τότε: $\delta^2 - R^2$ λέγεται δύναμη του Ρ ως προς τον (Ο,Ρ) και συμβολίζεται $\Delta_{(O,R)} = \delta^2 - R^2 = OP^2 - R^2$
- $(x - x_0)^2 + (y - y_0)^2 = r^2$: Εξίσωση κύκλου, κέντρο: $O(x_0, y_0)$, ακτίνα: r Όταν κέντρο είναι: $K(0,0)$,τότε: $x^2 + y^2 = r^2$
- $x^2 + y^2 + Ax + By + C = 0, A^2 + B^2 - 4C > 0$: Γενικευμένη εξίσωση κύκλου κέντρο: $\left(-\frac{A}{2}, -\frac{B}{2}\right)$, ακτίνα: $r = \frac{\sqrt{A^2 + B^2 - 4C}}{2}, A = -2x_0, B = -2y_0$
- Εξίσωση κύκλου σε πολικές συντεταγμένες: $r^2 - 2rr_0cos(\phi - \phi_0) + r_0^2 = R^2$, ακτίνα: R
- $L = 2\pi r$: Περιφέρεια κύκλου
- $E = \pi r^2$: Εμβαδό
- $\pi = \frac{\text{περιφέρεια του κύκλου}}{\text{διάμετρος του κύκλου}} = 3.14\ 159$
- $\frac{\text{Κεντρική γωνία } \vartheta \text{ τόξου (σε } ^o \text{ ή rad)}}{360^o \text{ ή } 2\pi} = \frac{\text{περιφέρεια τόξου } L}{\text{περιφέρεια κύκλου} : 2\pi r}$
- ακτίνιο(rad) = τόξο μήκους r (ίσο με την ακτίνα)
- $\frac{180^o}{\mu} = \frac{\pi}{\alpha}$

μ: μέτρο γωνίας σε μοίρες, α: μέτρο γωνίας σε ακτίνια

- $L = r \cdot a$ ή $L = \frac{\pi \mu r}{180^o}$: Μήκος τόξου
- $E = \frac{1}{2}r^2\alpha$ ή $E = \frac{\mu \pi r^2}{360}$: Εμβαδό κυκλικού τομέα ακτίνας r, τόξου ϑ

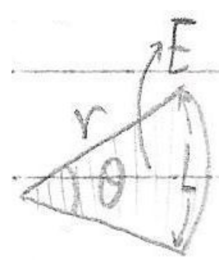


Figure 24: Τόξο κύκλου

Παραμετρικές εξισώσεις κύκλου

- $x = r \cdot \cos(\theta), y = r \cdot \sin(\theta), \theta = [0, 2\pi)$
- $r = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}$: Κύκλος ακτίνας r περιγεγγραμένος σε $A\hat{B}\Gamma$

- Εξίσωση εφαπτομένης κύκλου με κέντρο την αρχή των αξόνων:

$x \cdot x_1 + y \cdot y_1 = r^2 \qquad (x_1, y_1)$: σημείο επαφής κύκλου - ευθείας.

15.2 Parabola

- Παραβολή: ορίζεται ως ο Γ.Τ. των σημείων ενός επιπέδου που ισαπέχουν από δεδομένη ευθεία δ - τη διευθετούσα και σημείο Ε του επιπέδου, εκτός της ευθείας δ.
- $|r|$: απόσταση εστίας από τη διευθετούσα, $r = \dfrac{1}{2a}$: παράμετρος παραβολής ίση με την απόσταση της εστίας από τη διευθετούσα
- Εξίσωση παραβολής με εστία $E(\frac{r}{2}, 0)$ και διευθετούσα $x = -\frac{r}{2}$ σε καρτεσιανές συντεταγμένες είναι: $y^2 = 2rx$.
- Εξίσωση παραβολής με εστία $E(0, \frac{r}{2})$ και $\delta : y = -\frac{r}{2}$ σε καρτεσ. συντ. είναι: $x^2 = 2ry$.
- Γενικής μορφή εξίσωσης παραβολής: $y = ax^2 + bx + c$. ή στη vertex μορφή της:

$y = a(x-h)^2 + k$ Η καμπύλη αυτή είναι παραβολή αν $4ac = b^2$ και τουλάχιστον ένα των a, c διάφορο του μηδενός.

έχει κορυφή το σημείο $K\left(\dfrac{-\beta}{2\alpha}, \dfrac{-\Delta}{4\alpha}\right)$, όπου $\Delta = \beta^2 - 4\alpha\gamma$

εστία $E\left(\dfrac{1-\Delta}{4a}\right)$

και διευθετούσα $y = c - \dfrac{b^2+1}{4a}$

Η γραφική παράσταση της παραβολής $y = \alpha x^2 + \beta x + \gamma, \alpha \neq 0$, άξονα συμμετρίας την κατακόρυφη γραμμή που διέρχεται από την κορυφή Κ και έχει εξίσωση $x = \dfrac{-\beta}{2\alpha}$.

Εάν $\alpha > 0$ *η* y παίρνει ελάχιστη τιμή το y_k, ενώ αν $\alpha < 0$ η y παίρνει ελάχιστη τιμή το y_k.

- Εφαπτομένη παραβολής στο $A(x_0, y_0) \quad y^2 = 2rx$, είναι: $yy_0 = r(x + x_0)$
- Εφαπτομένη παραβολής στο $A(x_0, y_0) \quad x^2 = 2ry$, είναι: $xx_0 = r(y + y_0)$

15.3 Ellipse

- Έλλειψη είναι ο Γ.Τ. των σημείων του επιπέδου των οποίων το άθροισμα των αποστάσεων από δύο σταθερά σημεία - τις εστίες της έλλειψης - Ε και Ε′, είναι σταθερό, ίσο με 2α και μεγαλύτερο της εστιακής απόστασης ΕΕ′
- Σε κάθε έλλειψη ισχύει $(ME') + (ME) = 2\alpha$, Μ: σημείο της έλλειψης
- Εξίσωση έλλειψης με εστίες $E'(-\gamma, 0)$ και $E(\gamma, 0)$ είναι $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$, όπου $\beta = \sqrt{\alpha^2 - \gamma^2}$
- Εξίσωση έλλειψης με εστίες $E'(0, -\gamma)$ και $E(0, \gamma)$ είναι $\dfrac{x^2}{\beta^2} + \dfrac{y^2}{\alpha^2} = 1$, όπου $\beta = \sqrt{\alpha^2 - \gamma^2}$
- Μεγάλος άξονας = 2α
- Μικρός άξονας = 2β

- $\epsilon = \frac{\gamma}{\alpha}$ εκκεντρότητα έλλειψης < 1
- $\frac{\beta}{\alpha} = \sqrt{1-\epsilon^2}$
- $2\beta \leqslant$ (διάμετρος έλλειψης $\leqslant 2\alpha$
- $x = \alpha\cos(\phi)$, $\quad y = \beta\sin(\phi)$: Παραμετρικές εξισώσεις έλλειψης
- 2γ: εστιακή απόσταση
- Γενική μορφή εξίσωσης έλλειψης: $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, κέντρο

$K = (x_0, y_0), E'(x_0-\gamma, 0), E(x_0+\gamma, 0)$
- Εφαπτομένη έλλειψης $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, είναι: $\frac{xx_0}{\alpha^2} + \frac{yy_0}{\beta^2} = 1$ όπου $A(x_0, y_0)$ σημείο επαφής.
- Εφαπτομένη έλλειψης $\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = 1$, είναι: $\frac{xx_0}{\beta^2} + \frac{yy_0}{\alpha^2} = 1$
- Εμβαδό έλλειψης: $A = \pi\alpha\beta$
- Σημείο (x_0, y_0) εντός έλλειψης εάν: $\frac{x_0^2}{\beta^2} + \frac{y_0^2}{\alpha^2} < 1$

15.4 Hyperbola

- Υπερβολή είναι ο Γ.Τ. των σημείων του επιπέδου των οποίων η απόλυτη τιμή της διαφοράς των αποστάσεων από τις εστίες Ε και Ε΄ είναι σταθερή, ίση με 2α και μικρότερη της εστιακής απόστασης (ΕΕ΄).
- Εστιακή απόσταση $= 2\gamma$
- Ένα σημείο Μ είναι σημείο της υπερβολής ανν $|(ME') - (ME)| = 2\alpha$.
- Ισχύει $|(ME') - (ME)| < (EE')$, δλδ. $2\alpha < 2\gamma \leftrightarrow \alpha < \gamma$
- Εξίσωση υπερβολής με $E(\gamma, 0)$ και $E'(-\gamma, 0)$ είναι: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1, \quad \beta = \sqrt{\gamma^2 - \alpha^2}$. Αν

έχει εστίες Ε(0,-γ) και Ε΄(0,γ) τότε η εξίσωση της είναι: $\frac{y^2}{\alpha^2} - \frac{x^2}{\beta^2} = 1$. Κέντρο κανονικής μορφής υπερβολής (0,0)
- Αν α = β έχουμε την ισοσκελής υπερβολής: $x^2 - y^2 = \alpha^2$
- Υπάρχουν δύο ασύμπτωτες της υπερβολής, όπου για την κανονικής μορφής υπερβολή

είναι: $y = -\frac{\beta}{\alpha}x$ και $y = \frac{\beta}{\alpha}x$ ανν $|\lambda| < \beta/\alpha$ όπου λ: συντελεστής διεύθυνσης ασύμπτωτης.
Για την υπερβολή με εξίσωση $\frac{y^2}{\alpha^2} - \frac{x^2}{\beta^2} = 1$ οι ασύμπτωτες της είναι: $y = \frac{\alpha}{\beta}x$ και $y = -\frac{\alpha}{\beta}x$
- $\epsilon = \frac{\gamma}{\alpha}(> 1)$: Εκκεντρότητα υπερβολής
- $\frac{\beta}{\alpha} = \sqrt{\epsilon^2 - 1}$
- Εφαπτομένη υπερβολής $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2}$ στο $M(x_0, y_0)$ είναι: $\frac{xx_0}{\alpha^2} - \frac{yy_0}{\beta^2} = 1$ ενώ η εφαπτομένη

της υπερβολής $\frac{y^2}{\alpha^2} - \frac{x^2}{\beta^2} = 1$ είναι: $\frac{yy_0}{\alpha^2} - \frac{xx_0}{\beta^2} = 1$

- $d(M, \epsilon_1) \cdot d(M, \epsilon_2) = \frac{\alpha^2\beta^2}{\alpha^2\beta^2}$: Το γινόμενο των αποστάσεων ενός σημείου της υπερβολής από τις ασύμπτωτες της είναι σταθερό.

16 Calculus / Mathematical Analysis

16.1 Limits

- Existence of a limit of a function $f(x)$ when:

$\lim_{x \to c} f(x) = L \Leftrightarrow \left(\lim_{x \to c^-} f(x) = L\right) \wedge \left(\lim_{x \to c^+} f(x) = L\right)$
A function may have a limit at a point of its domain, but a different, or no value at that point. The limit describes the behavior of the function, as the latter tends towards a certain point.

- Suppose $f : \mathbb{R} \to \mathbb{R} \wedge x_0, L \in \mathbb{R}$. The limit of f, as x approaches x_0 is L and is written: $\lim_{x \to x_0} f(x) = L$.
- If the following property holds $\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}(0 < |x - x_0| < \delta$ implies $|f(x) - L| < \epsilon)$, then the value of the limit does not depend on the value of $f(x_0)$, nor even that x_0 be in the domain of f, i.e. the limit does not depend on $f(x_0)$ being well - defined. (ϵ = "error", δ = "distance")

Όταν η παραπάνω ιδιότητα ισχύει, τότε η f έχει στο x_0 όριο το $L \in \mathbb{R}$.

- $\lim_{x \to x_0} f(x) = L \leftrightarrow \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$
- $\lim_{x \to x_0} f(x) = L \leftrightarrow \lim_{x \to x_0} (f(x) - L) = 0$
- $\lim_{x \to x_0} f(x) = L \leftrightarrow \lim_{h \to 0} f(x_0 + h) = L$
- $\lim_{x \to x_0} x = x_0$
- $\lim_{x \to x_0} c = c$
- $\lim_{x \to x_0} f(x) > 0 \to f(x) > 0$ κοντά στο x_0
- $\lim_{x \to x_0} f(x) < 0 \to f(x) < 0$ κοντά στο x_0
- Αν οι f, g έχουν όριο στο x_0 και $f(x) \leqslant g(x)$, κοντά στο x_0, τότε: $\lim_{x \to x_0} f(x) \leqslant \lim_{x \to x_0} g(x)$
- Αν υπάρχουν τα πραγματικά όρια των f και g στο x_0, τότε:

$$\to \lim_{x \to x_0} f(x) = L \leftrightarrow \frac{|L|}{2} < |f(x)| < \frac{3}{2}|L|$$

$$\to \lim_{x \to x_0} \sqrt[k]{f(x)} = \sqrt[k]{\lim_{x \to x_0} f(x)}, \text{ εφόσον } f(x) \geqslant 0 \text{ κοντά στο } x_0 \text{ (Well defined radical)}$$

- $\lim_{x \to x_0} (kf(x)) = k \lim_{x \to x_0} f(x), \ \forall k \in \mathbb{R}$
- $\lim_{x \to x_0} (f(x) \pm g(x)) = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x)$
- $\lim_{x \to x_0} (f(x) \cdot g(x)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$
- $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$, εφόσον $\lim_{x \to x_0} g(x) \neq 0$
- $\lim_{x \to x_0} |f(x)| = |\lim_{x \to x_0} f(x)|$
- $\forall \nu \in \mathbb{N}^* \left(\lim_{x \to x_0} (f(x))^\nu = \left[\lim_{x \to x_0} f(x)\right]^\nu \right)$

- $\lim_{n\to\infty} k^n = \begin{cases} \text{undef.}\ , & k \leqslant -1 \\ 0\ , & -1 < k < 1 \\ 1\ , & k = 1 \\ \infty\ , k > 1 & \end{cases}$

16.2 Calculations with 0 and inf

- $-\infty(-\theta) = +\infty$
- $+\infty(-\theta) = -\infty$
- $\theta^{+\infty} = +\infty\ ,\ \theta > 0$
- $\theta^{-\infty} = 0^+\ ,\ \theta > 0$
- $(+\infty)^\theta = +\infty\ ,\ \theta > 0$
- $(+\infty)^{+\infty} = +\infty$
- $\sqrt[\nu]{+\infty} = +\infty\ ,\ \nu \in \mathbb{N}$
- $\sqrt[\infty]{\theta} = 1\ ,\ \theta > 0$
- $\log_\theta(+\infty) = +\infty\ ,\ \theta > 1$
- $\log_\theta(+\infty) = -\infty\ ,\ 0 < \theta < 1$
- $+\infty \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$
- $\dfrac{0}{\pm\infty} = 0$
- $\log_{+\infty} \theta = 0\ ,\ 0^+ αν \theta > 1 ή 0^- αν 0 < \theta < 1$
- $\dfrac{\theta}{0^+} = +\infty\ (\theta > 0)$
- $\dfrac{\theta}{0^-} = -\infty\ (\theta > 0)$
- $\dfrac{\theta}{+\infty} = 0^+$
- $\dfrac{\theta}{-\infty} = 0^-$

16.3 Differential Calculus

- A differentiable function must be continuous at every point in its domain. The converse does not hold: a continuous function need not be differentiable. For example, a function with a bend, cusp, or vertical tangent may be continuous, but fails to be differentiable at the location of the anomaly.

- Squeeze / Sandwitch Theorem (Κριτήριο Παρεμβολής):

Suppose f, g, h are functions. If

→ $h(x) \leqslant f(x) \leqslant g(x)$, close to x_0 $\wedge$

→ $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) = L$,

then $\lim_{x \to x_0} f(x) = L$.
Functions g, h are said to be upper and lower bounds respectively.

- $|\sin(x)| \leqslant |x|, \forall x \in \mathbb{R}$
- $\lim_{x \to x_0} \sin(x) = \sin(x_0),\ \lim_{x \to x_0} \cos(x) = \cos(x_0)$
- $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$
- $\lim_{x \to 0} \dfrac{\cos(x) - 1}{x} = 0$
- $\lim_{x \to x_0} [f(x)]^{g(x)} = \left[\lim_{x \to x_0} f(x)\right]^{\lim_{x \to x_0} g(x)}$, $f(x) \geqslant 0$
- Αν $\lim_{x \to x_0} f(x) = 0$ και η g φραγμένη σε μια περιοχή U του x_0, τότε: $\lim_{x \to x_0} [f(x) \cdot g(x)] = 0$

- Μη πεπερασμένο όριο στο $x_0 \in \mathbb{R}$

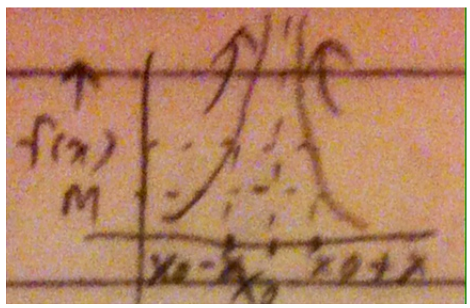


Figure 25: Μη πεπερασμένο όριο στο $x_0 \in \mathbb{R}$

Έστω f ορισμένη σε σύνολο της μορφής $(\alpha, x_0) \cup (x_0, \beta)$. Ορίζουμε:

→ $\lim_{x \to x_0} f(x) = +\infty$, όταν $\forall M \in \mathbb{R}^+$ υπάρχει $\delta > 0$ τέτοιο, ώστε $\forall x \in (\alpha, x_0) \cup (x_0, \beta)$ με $0 < |x - x_0| < \delta$ να ισχύει: $f(x) > M$.

→ $\lim_{x \to x_0} f(x) = -\infty$, όταν $\forall M \in \mathbb{R}^+$ υπάρχει $\delta > 0$ τέτοιο, ώστε $\forall x \in (\alpha, x_0) \cup (x_0, \beta)$ με $0 < |x - x_0| < \delta$ να ισχύει: $f(x) < -M$.

- If $P(x) = a_\nu x^\nu + a_{\nu-1} x^{\nu-1} + \ldots + a_0$: polynomial equation, με $a_\nu \neq 0$ ισχύει:

$\lim_{x \to +\infty} P(x) = \lim_{x \to +\infty} (a_\nu x^\nu) \wedge \lim_{x \to -\infty} P(x) = \lim_{x_{x \to -\infty}} (a_{nu} x^\nu)$

- If $f(x) = \dfrac{a_\nu x^\nu + a_{\nu-1} x^{\nu-1} + \ldots + a_1 x + a_0}{\beta_\kappa x^\kappa + \beta_{\kappa-1} x^{\kappa-1} + \ldots + \beta_1 x + \beta_0}$, $a_\nu \neq 0,\ \beta_\kappa \neq 0$, then:

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(\dfrac{a_\nu x^\nu}{\beta_\kappa x^\kappa}\right) \wedge \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(\dfrac{a_\nu x^\nu}{\beta_\kappa x^\kappa}\right)$

- $\alpha > 1 \to \left(\lim_{x \to -\infty} a^x = 0 \wedge \lim_{x \to \infty} a^x = +\infty\right)$
- $\alpha > 1 \to \left(\lim_{x \to 0} \log_\alpha x = -\infty \wedge \lim_{x \to +\infty} log_\alpha x = -\infty\right)$
- A sequence is every real function $\alpha : \mathbb{N}^* \to \mathbb{R}$. Θα λέμε ότι η ακολουθία α_ν έχει όριο το

$L \in \mathbb{R}$ και θα γράφουμε $\lim_{\nu \to +\infty} \alpha_\nu = L$, όταν $\forall \epsilon > 0, \exists \nu_0 \in \mathbb{N}^*$ τέτοιο, ώστε $\forall \nu > \nu_0$ να ισχύει: $|\alpha_\nu - L| < \epsilon$

- Discontinuity classification:
  1. Removable discontinuity: has a "hole" in its graph, a term in the denominator that cancels out,
  2. Non-Removable discontinuity: Has a "jump" at a point.
- f is continuous at $x_0 \in \text{Dom}\ (f)$, iff $\lim_{x \to x_0} f(x) = f(x_0)$
- Αν f, g συνεχείς στο x_0, τότε είναι συνεχείς στο x_0 και οι συναρτήσεις

$f + g, c \cdot f, c \in \mathbb{R}, f \cdot g, f/g, |f|, \sqrt[\nu]{f}$, με την προυπόθεση ότι ορίζονται σ'ένα διάστημα που περιέχει το x_0.

- If f is continuous at x_0 and g continuous at $f(x_0)$, then their composition $g \circ f$ is continuous at x_0.
- f είναι συνεχής σένα ανοικτό διάστημα (a, b), όταν είναι συνεχής σε κάθε σημείο του (a, b).
- f είναι συνεχής σ'ένα κλειστό διάστημα $[a, b]$, όταν είναι συνεχής σε κάθε σημείο του (a, b) και επιπλέον $\lim_{x \to \alpha^+} f(x) = f(\alpha) \quad \wedge \lim_{x \to \beta^-} f(x) = f(b)$.

- Θεώρημα Bolzano: Έστω μια συνάρτηση f , ορισμένη σε ένα κλειστό διάστημα $[a, b]$. Αν:

  → η f είναι συνεχής στο $[a, b]$ και επιπλέον, ισχύει

  → $f(a) \cdot f(b) < 0$

τότε υπάρχει ένα, τουλάχιστον, $x_0 \in (a, b)$ τέτοιο, ώστε $f(x_0) = 0$, στο ανοικτό διάστημα (a, b).

- Θεώρημα Μέγιστης και Ελάχιστης τιμής: Αν f συνεχής στο $[a, b]$, τότε η f παίρνει στο $[a, b]$ μια μέγιστη τιμή M και μια ελάχιστη τιμή m.
- Αν f γνησίως αύξουσα και συνεχής στο (a, b), τότε το σύνολο τιμών της στο διάστημα αυτό είναι το (A, B), όπου: $A = \lim_{x \to a^+} f(x)$ και $B = \lim_{x \to b^-} f(x)$.

Αν, όμως η f είναι γνησίως φθίνουσα και συνεχής στο (a, b), τότε το σύνολο τιμών της στο διάστημα αυτό είναι το (B, A).

- Έστω f και $A(x_0, f(x_0))$ ένα σημείο της. $\exists \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R} \to$ εφαπτομένη (κλίση)

της f στο A είναι η ευθεία ε που διέρχεται από το A και έχει συντελεστή διέυθυνσης λ.

- f is differentiable at point $x_0 \in \text{Dom}\ (f)$, iff $\exists \lim_{x \to x_0} \dfrac{f(x) - f(x_0))}{x - x_0} \in \mathbb{R}$. This limit is the

derivative of $f @ x_0$ and we denote it as $f'(x_0)$. Thus, $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

- Line $\epsilon : y - f(x_0) = f'(x_0)(x - x_0)$
- If function f is differentiable $@a_0$, then it is also continuous $@x_0$.
- Θεώρημα Rolle: Αν μια συνάρτηση f είναι:

1. συνεχής στο $[a, b]$,

2. παραγωγίσιμη στο (a, b) και

3. ισχύει ότι: $f(a) = f(b)$,

τότε υπάρχει ένα, τουλάχιστον $x_0 \in (a, b)$ τ.ω.: $f'(\xi) = 0$.

- Θεώρημα Μέσης Τιμής (Θ.Μ.Τ.): Έστω συνάρτηση f. Αν

1. f συνεχής στο $[a, b]$,

2. f παραγωγίσιμη στο (a, b),

τότε υπάρχει ένα, τουλάχιστον $x_0 \in (a, b)$ τ.ω.: $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$: που είναι ο μέσος ρυθμός μεταβολής της f στο (a, b).

Suppose function f, continuous among space $\Delta = (a, b)$.

- $\forall x \in \Delta\,(f'(x) > 0) \rightarrow f \uparrow \Delta$
- $\forall x \in \Delta\,(f'(x) < 0) \rightarrow f \downarrow \Delta$
- Θεώρημα Fermat: If f has a stationary point at $x_0 \in \Delta \subset \text{Dom}\,(f)$ and f is differentiable at that point, then $f'(x_0) = 0$.

Stationary points

- Let $x_0 \in \Delta$, $(f'(x_0) = 0 \wedge f''(x_0) < 0) \rightarrow f(x_0)$ local maximum.
- Let $x_0 \in \Delta$. $(f'(x_0) = 0 \wedge f''(x_0) > 0) \rightarrow f(x_0)$ local minimum.
- Determining Points of Inflexion (POI): Let $x_0 \in \Delta \subset \text{Dom}\,(f)$

1. We differentiate $y = f(x)$ twice to get $\dfrac{d^2y}{dx^2}$.

2. We solve the equation $\dfrac{d^2y}{dx^2} = 0$.

3. We test to see whether, or not a change of sign occurs in $\dfrac{d^2y}{dx^2}$, at $x = x_0 + a$ and at$x = x_0 - a$. If $f(x_0 + a) \cdot f''(x_0 - a) < 0 \rightarrow x_0$: P.O.I.

- $\forall x \in \Delta(\dfrac{d^2y}{dx^2} > 0) \rightarrow f$ convex / concave-upwards @$\Delta \quad (\cup shape)$
- $\forall x \in \Delta(\dfrac{d^2y}{dx^2} < 0) \rightarrow f$ convex / concave-downwards @$\Delta \quad (\cap shape)$
- $\dfrac{df(x)}{dx}$ = undefined for some $x = x_0$, then $(x_0, f(x_0))$ is a critical point.
- $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- $(\dfrac{f}{g})'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$

- Chain rule: $f'(g(x_0)) = (f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$ ή $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$

- Differerentiation of parametric equations: $x = f(t), y = g(t)$ we find: $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ and

finally we evaluate: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

- Curvature of a curve at a point P. It tells us how quickly the curve is bending in the immediate neighborhood of that point P.

- Radius of curvature: $R = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ (R is large $\rightarrow$ curvature is small)

- Centre of curvature (h, k), of the circle at point $P(x_1, y_1)$:

$h = x_1 - R\sin\theta \ \wedge \ k = y_1 + R\cos\theta$

- Indeterminate Forms: are algebraic expressions obtained in the context of limits. Limits involving algebraic operations are often performed by replacing subexpressions by their limits; if the expression obtained after this substitution does not give enough information to determine the original limit, it is known as an indeterminate form.

Indeterminate forms:

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty - \infty, 0 \cdot \infty, 1^{\infty}$, 0^0 , ∞^0 , $e^{\pm j\infty}$

- L'Hospital's rule for circumventing indeterminate forms: We take the derivatives of both the numerator and the denominator.

- Asymptotes: first express the equation ”on one line”.
- Asymptotes parallel to the x-axis: We equate the coefficient of the highest power of x to zero.
- Asymptotes paralle to the y-axis: We equate the coefficient of the highest power of y to zero.
- Other asymptotes: To find a (general type) asymptote to $y = f(x)$:

1. Substitute $y = mx + c$ in the given equation and simplify.

2. Equate to zero the coefficients of the two highest powers of x and so determine the values of $m \wedge c$.

Symmetry:

- If only even powers of y (x) occur the curve is symmetrical about the x (y) axis.

Limitations

- First, always check for restrictions on the possible range of values that x, or y may have.
- Symmetry about origin: replace both x with $-x$ and y with $-y$. If it's the same equation then it is symmetric.
- Symmetric about x-axis: replace x with $-x$. Check.

- Symmetric about y-axis: replace y with $-y$. Check.

- Differentials dy, dx are finite quantities -not necessarily zero- and can therefore exist alone. $dy = f'(x_0)\ dx$

- Root Mean Square (R.M.S.) value of a function: $y = f(x)$, $y = \sqrt{\frac{1}{b-a}\int_a^b y^2 dx}$

- The General Leibniz rule generalizes the product rule. It states that if f and g are n-times differentiable functions, then the product fg is also n-times differentiable and its nth derivative is given by:

$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$

This can be proved by using the product rule and mathematical induction.

16.3.1 Derivatives

- $f(x) = C \to f'(x) = 0$
- $f(x) = x^k, k \in \mathbb{R} \to f'(x) = kx^{k-1}$
- $f(x) = \sqrt{x}, x \geqslant 0 \to f'(x) = (1/2) \cdot x^{\frac{-1}{2}},\ x > 0$
- $f(x) = \sin(x) \to f'(x) = \cos(x)$
- $f(x) = \cos(x) \to f'(x) = -\sin(x)$
- $f(x) = e^x \to f'(x) = e^x$
- $f(x) = \ln\ (x),\ x > 0 \to f'(x) = \frac{1}{x}$
- $f(x) = \log_a(x) \to f'(x) = \frac{1}{xln(a)}$
- $f(x) = \tan(x), x \neq \pi/2 \to f'(x) = \text{sec}\ ^2(x)$
- $f(x) = \cot(x), x \neq 0 \to f'(x) = -\text{cosec}\ ^2(x)$
- $f(x) = x^{-k}, k \in \mathbb{R}, x \neq 0 \to f'(x) = -kx^{-k-1}$
- $f(x) = a^x, a > 0 \to f'(x) = a^x lna$
- $f(x) = \arcsin(x) = \sin^{-1}(x) \to f'(x) = \frac{1}{\sqrt{1-x^2}}$
- $f(x) = \cos^{-1}(x) \to f'(x) = \frac{-1}{sqrt1 - x^2}$
- $f(x) = \tan^{-1}(x) \to f'(x) = \frac{1}{1+x^2}$
- $f(x) = \cot^{-1}(x) \to f'(x) = \frac{-1}{1+x^2}$
- $f(x) = \sinh(x) \to f'(x) = \cosh(x)$
- $f(x) = \cosh(x) \to f'(x) = \sinh(x)$
- $f(x) = \tanh(x) \to f'(x) = 1 - \tanh^2(x)$
- $f(x) = \coth(x) \to f'(x) = 1 - \coth^2(x)$
- $f(x) = \sinh^{-1}(x) \to f'(x) = \frac{1}{\sqrt{1+x^2}}$

- $f(x) = \cosh^{-1}(x) \to f'(x) = \dfrac{1}{\sqrt{x^2-1}}$
- $f(x) = \tanh^{-1}(x) \to f'(x) = \dfrac{1}{1-x^2}$
- $f(x) = \coth^{-1}(x) \to f'(x) = \dfrac{1}{1+x^2}$
- $f(x) = g^k(x) \to f'(x) = kg^{k-1}(x)g'(x)$
- $f(x) = \sqrt{g(x)}, g(x) \geqslant 0 \to f'(x) = \dfrac{1}{2\sqrt{g(x)}}g'(x),\ g(x) > 0$
- $f(x) = \sin(g(x)) \to f'(x) = g'(x)\cos(g(x))$
- $f(x) = e^{g(x)} \to f'(x) = e^{g(x)} \cdot g'(x)$
- $f(x) = x^x \to f'(x) = x^x \cdot (1 + \ln\, x)$
- $f(x) = \ln\,(g(x)) \to f'(x) = (\dfrac{1}{g(x)} \cdot g'(x)$
- $f(x) = \tan(g(x)) \to f'(x) = \dfrac{1}{\cos^2(g(x))}g'(x)$
- $f(x) = u(x)^{g(x)} \to f'(x) = \left(e^{g(x)\cdot\ln\,(u(x))}\right)'$
- $f(x) = \dfrac{1}{g(x)}, g(x) \neq 0 \to f'(x) = \dfrac{-1}{g^2(x)}g'(x)$
- $f(x) = \log_a(g(x)) \to f'(x) = \dfrac{1}{\ln\,(a)g(x)}g'(x)$
- $f(x) = \sec\,(x) \to f'(x) = \tan(x) \cdot \sec\,(x)$
- $f(x) = \sin(ax) \to f^{(n)}(x) = a^n \cdot \sin(ax + \dfrac{n\pi}{2}$
- $f(x) = \cos(ax) \to f^{(n)}(x) = a^n \cdot \cos(ax + \dfrac{n\pi}{2}$
- $f(x) = e^{ax} \to f^{(n)}(x) = a^n e^{ax}$
- $f(x) = \ln\,(x) \to f^{(n)}(x) = \dfrac{[(-1)^{n-1}(n-1)!]}{x^n}$
- $f(x) = \sinh(ax) \to f^{(n)}(x) = \dfrac{a^n}{2} \cdot \{[1 + (-1)^n]\sinh(ax) + [1 - (-1)^n] \cdot \cosh(ax)\}$
- $f(x) = \cosh(ax) \to f^{(n)}(x) = \dfrac{a^n}{2} \cdot \{[1 + (-1)^n]\cosh(ax) + [1 - (-1)^n] \cdot \sinh(ax)\}$
- $f(x) = x^a, a > 0 \to f^{(n)}(x) = \dfrac{a!}{(a-n)!}x^{a-n}$

16.3.2 Numerical solutions to Derivatives

<u>Numerical Solutions to Derivatives of $f(x)$</u>

- Forward difference formula: $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$

(neglecting terms of order 2 and above) • Backward difference formula:
$f'(x) \approx \dfrac{f(x) - f(x-h)}{h}$
(neglecting terms of order 2 and above) • Central difference formulas: (more accurate)

(neglecting terms of order 3 and above)
$f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$
$f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

16.3.3 Partial Differentiation

- Notation: $f_x(x,y) = \dfrac{\partial}{\partial x} f(x,y)$, $f_{xx}(x,y) = \dfrac{\partial^2}{\partial x^2} f(x,y)$, $f_{yx}(x,y) = \dfrac{\partial^2}{\partial y \cdot \partial x} f(x,y)$
- Partial differentiation with respect to a given variable, say x : We are certain that all other variables, besides x, are held constant for the time being.
- Taylor's theorem for two independent variables:

$f(x+h, y+k) = f(x,y) + \{h f_x(x,y) + k f_y(x,y)\} + \dfrac{1}{2!}\{h^2 f_{xx}(x,y) + 2hk f_{xy}(x,y) + k^2 f_{yy}(x,y)\} + \dfrac{1}{2!}\{k h^2 f_{yxx}(x,y) + k^2 h f_{yyx}(x,y)\} + \ldots$

- A function $f(x)$ is an infinitesimal at x_0, if its limit at x_0 is zero, e.g.

$\lim_{x\to 0} \sin(x) = 0 \quad \therefore x_0 = 0$

- Function's Differential: Suppose $z = f(x, y, w, \ldots, n)$, where $x, y, w, \ldots, n$ are the independent variables of f. A function's differential shows its functions behaviour for small changes in all of its independent variables. With satisfying accuracy it's approximated to be:

$\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y + \dfrac{\partial z}{\partial x}\delta w + \ldots + \dfrac{\partial z}{\partial n}\delta n$ (+ higher order derivatives of $x, y, w, \ldots, n$ of smaller magnitude / value)

- Rates of change: If $z = f(x,y)$, then if z changes with respect to δt and $\delta t \to 0$ then

$\dfrac{\delta z}{\delta t} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\delta y}{\delta t}$

- Parametric functions: If $z = f(x,y)$ and $x = g(u,v)$, $y = h(u,v)$ then $z = k(u,v)$. To find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial y}$ we do:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \ \wedge \ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

- Implicit functions: If $f(x,y) = 0$ is an implicit function, we let $z = f(x,y)$. Then:

$\dfrac{dy}{dx} = -(\dfrac{\partial z}{\partial x} / \dfrac{\partial z}{\partial y})$

- $z = f(x,y) \to \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial x}(\dfrac{\partial z}{\partial y}) = \dfrac{\partial}{\partial y}(\dfrac{\partial z}{\partial x})$
- Inverse functions and Jacobian usage:

If $z = f(x,y)$ and $u = g(x,y)$ and $v = h(x,y)$ then:

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}/J \ , \ \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}/J \ , \ \frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}/J \ , \ \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}/J$$

where the Jacobian of u, v with respect to x, y is:

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

- Inverse Jacobian: $\mathbb{J}(x,y)^{-1} = \mathbb{J}(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = 1\text{div} \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix}$

- Stationary points of functions in 3 dimensions

i.e. $z = f(x,y)$. Four steps in the routine:

1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and solve $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$. If found any, then (x_0, y_0) is a stationary point.
2. If $\left(\dfrac{\partial^2 z}{\partial x^2}\right) \cdot \left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^z}{\partial x \partial y}\right) > 0$ @ (x_0, y_0), then the point (x_0, y_0) is either a maximum, or a minimum (over $(x_0 + h, y_0 + k)$ in any direction from the point).
3. If $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ are both negative then (x_0, y_0) is a maximum. If $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ are both positive then (x_0, y_0) is a minimum.
4. We evaluate the actual minimum, or maximum values, i.e. we find $f(x_0, y_0)$.

- If $\left(\dfrac{\partial^2 z}{\partial x^2}\right) \cdot \left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$, then further, detailed study (most likely) is necessary

to determine the stationary points.

- Lagrange multipliers: Used to find extremal values of a function $u = f(x, y, z, \ldots)$, with

constraint $\phi(x, y, z, \ldots) = 0$. Then we need to find these $x, y, z, \ldots$ points in order to calculate the extremal value $u = f(x, y, z, \ldots)$. For this purpose we solve a system of equations whose number is the one of the unknown variables $+1$ (for the constraint equation). If $u = f(x, y, z)$, with constraint $\phi(x, y, z) = 0$, then we solve the following system of 4 equations:

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0_{(\mathrm{I})}\ ,\ \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0_{(\mathrm{II})}\ ,\ \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0_{(\mathrm{III})}\ ,\ \phi(x,y,z) = 0_{(\mathrm{IV})}$$

- Any expression $dz = Pdx + Qdy$, $P = f(x)$, $Q = g(y)$ is an exact differential if it can

be integrated to determine z. Therefore if $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$, then dz is an exact differential. The concept can be extended to functions of more than two variables.

- Exact differential in 3 independent variables.

$dw = Pdx + Qdy + Rdz$ is an exact differential of $w = f(x, y, z)$, if

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \wedge \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} \wedge \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$

16.4 Integral Calculus

- Αρχική συνάρτηση, ή παράγουσα της f στο Δ ονομάζεται κάθε συνάρτηση F που είναι παραγωγίσιμη στο Δ και ισχύει: $F'(x) = f(x), \ \forall x \in \Delta$.
- Αν F παράγουσα της f στο Δ, τότε όλες οι συναρτήσεις της μορφής $G(x) = F(x) + c, \ c \in \mathbb{R}$ είναι παράγουσες της f στο Δ.
- Το σύνολο όλων των παραγουσών της f στο Δ ονομάζεται αόριστο ολοκλήρωμα της f στο Δ.
- Fundamental Theorem of Calculus: If f is continuous in $[a, b]$, then it has an antiderivative: $F(x) = \displaystyle\int_a^x f(t)dt, \ x \in [a, b]$.

$\therefore \dfrac{dF}{dx} = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$. Corrolary: $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$, i.e. f is Riemann integrable on $[a, b]$.

- $\displaystyle\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$: Integration by parts
- $\displaystyle\int f(g(x))g'(x)dx = \int f(u)du$: Integration by substitution

$u = g(x) \ landdu = g'(x)dx$

- $\displaystyle\int \frac{f'(x)}{f(x)}dx = \ln\left[f(x)\right] + C$
- $\displaystyle\int f(x)f'(x)dx = \frac{f^2(x)}{2} + C \qquad (= \int zdz)$

$z = f(x), \ dz = f'(x)dx$

- To integrate a "function of a linear function of x", simply replace x in the corresponding standard result by the linear expression and divide by the coefficient of x in the linear expression.

- The total area A between a curve $f(x) = y$ and the x axis, from $x = a$ to $x = b$, is given by $A = \displaystyle\int_a^b y \ dx$
- Area between a curve and an intersecting line from $x = a$, to $x = b$ is:

$A = \displaystyle\int_a^b (y_1 - y_2)dx$

- The equation that is "more positive" (from a to b) is y_1 (either the line of the curve)

- Partial Fractions

  1. It is necessary that the degree of the numerator is less than the degree of the denominator.
  2. In such an expression that the denominator can be expressed as a product of simple prime factors, each one of the $ax + b$:
     (a) Write the rational expression with the denominator given as product of its prime

factors,

(b) Each factor then gives rise to a partial fraction of the form: $\dfrac{A}{ax+b}$, where$A$ is a constant whose value is to be determined.

(c) Add the partial fractions together to form a single algebraic fraction whose numerator contains the unknown constants and whose denominator is identical to that of the original expression,

(d) Equate the numerator so obtained with the numerator of the original algebraic fraction,

(e) By substituting appropriate values of x in this equationdetermine the values of the unknown constants. You'll have to solve a system of linear equations whose number is that of the unknown constants.

3. In such a case that the denominator of the original rational expressionn of the form (ax^2+bx+c) is irreducible, it gives rise to a partial fraction of the form: $\dfrac{Ax+b}{ax^2+bx+c}$

4. In such a case that the denominator of the original rational expression contains algebraic repeated facors of the form $ax+b)^n$, these give rise to partial fractions of the form: $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\ldots+\dfrac{A_n}{(ax+b)^n}, \quad n \in \mathbb{N}^*$.

<u>Heavyside's Cover up Method</u>

- $y(x) = \dfrac{v(x)}{B(x-p_1)(x-p_2)\cdots(x-p_\nu)} = \dfrac{k_1}{x-p_1}+\dfrac{k_2}{x-p_2}+\ldots+\dfrac{k_\nu}{x-p_\nu}$
- Αν p_i : απλός πόλος, τότε: $k_i = \left[(x-p_i)y(x)\right]_{x=p_i}$ • Αν p_i : πολλαπλός πόλος,

πολλαπλότητας m, δηλαδή $\dfrac{\nu(x)}{(x-p_i)^m} = \dfrac{k_1}{x-p_i}+\dfrac{k_2}{(x-p_i)^2}+\ldots+\dfrac{k_m}{(x-p_i)^m}$, τότε:

$$\rightarrow k_i = \frac{1}{(i-1)!}\cdot\frac{d^{i-1}}{dx^{i-1}}\left[(x-p_i)^m y(x)\right]_{x=p_i}\ ,\ i=2,3,\ldots,m$$

$$\rightarrow k_i = \left[(x-p_i)^m y(x)\right]_{x=p_i}\ ,\ i=m, \text{ i.e. highest order repeated root.}$$

- Αν p_i : μιγαδικό ζεύγος πόλων, δλδ.:

$\dfrac{\nu(x)}{x-p_i} = \dfrac{k_1}{x-a-jb}+\dfrac{k_2}{x-a+jb}$, τότε $k_1=\bar{k}_2, \quad k_1,k_2\in\mathbb{C}$

$k_1 = \left[(x-a-jb)y(x)\right]_{x=a+jb} = \lim\limits_{x\to a+jb}\left[(x-a-jb)y(x)\right]$

- Mean value of a function f between two limits a and b :

$M = \dfrac{\text{Area}}{b-a} = \dfrac{1}{b-a}\displaystyle\int_a^b y\,dx \qquad , \ b>a\ ,\ y=f(x)$

- <u>Θ.Μ.Τ. ολοκληρωτικού λογισμού</u>: Για f που ορίζεται στο $(a,b)\subset \text{Dom }(f)$ και έστω

$c\in(a,b)$. Τότε: $F'(c) = \dfrac{F(b)-F(a)}{b-a}$

- We get that: $\int_a^b f(x)dx = f(c)(b-a)$ and $f(c) = \frac{\int_a^b f(x)dx}{b-a}$ is the mean value of f in the interval (a, b).
- Integration as a summing process:

when $\delta x \to 0$ $\sum_{x=a}^{x=b} y\delta x = \int_a^b ydx$

- Moment of mass of element about axis $= r \cdot \delta m, \;\; r:$ distance from the axis.
- Volume of a solid of revolution rotating about the x-axis: $V = \int_a^b \pi y^2 dx$, when the function is given in parametric form then: $V = \int_a^b \pi y^2 \frac{dx}{dt} dt$
- Volume of a solid of revolution rotating about the y-axis: $V = 2\int_a^b \pi xydx$
- Surface Area of plane figure: $A = \int_a^b ydx$
- Centroid $\left(\bar{x}, \bar{y}\right)$ of plane figure: $\bar{x} = \frac{1}{A}\int_A x_e dA = \frac{1}{A}\int_a^b xydx$ and

$\bar{y} = \frac{1}{A}\int_A y_e dA = \frac{1}{A}\int_a^b \frac{y}{2} ydx = \frac{1}{2A}\int_a^b y^2 dx$

- Centre of gravity (or mass) of a solid of revolution, revolving about the x-axis:

$\bar{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$ and $\bar{y} = 0$

- Length of a curve: $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$, when the function is given in parametric form (i.e.$x = f(\partial), y = g(\partial)$: $A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{d\partial}\right)^2 + \left(\frac{dy}{d\partial}\right)^2} d\partial$
- Centre of gravity of a solid of revolution that rotates about the y-axis: $\bar{x} = 0$ and

$\bar{y} = \frac{\int_a^b yx^2 dy}{\int_a^b x^2 dy}$

- Surface of revolution generated by the arc of a curve rotating about the y-axis:

$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

- <u>1st rule of Pappus</u>: If an arc of a plane curve rotates about an axis in its plane, the area of the surface generated is equal to the length of the line multiplied by the distance travelled by its centroid.
- <u>2nd rule of Pappus</u>: If a plane figure rotates about an axis in its plane, the volume generated is equal to the area of the figure multiplied by the distance travelled by its centroid.

→ One proviso / caveat in using the rules of Pappus: the axis of rotation must not cut the rotating arc, or plane figure.

- Whenever we have a problem not covered by our standard results, we build up the integral from first principles.
- $KE = \frac{1}{2}mU^2 = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}\omega^2 \sum mr^2 (J)$: Kinetic Energy
- Moment of Inertia (ροπή αδράνειας): $I = \sum mr^2 (\frac{kg}{m^2})$: Physical property of the object.

It's a sum for all of its particles, r : distance from axis of rotation.
The M.O.I. is also called rotational mass, because it does the same thing in rotational dynamics as mass does in linear dynamics.

- $I = Mk^2$, $M = \sum m$, k : radius of gyration or gyradius
- $I = \frac{Md^2}{12}$: M.O.I. for rectangular plate
- $I_{AB} = I_G + Ml^2$: for axis AB $\parallel$ G axis, l : distance$(AB - G)$
- $I = \frac{\pi r^4 \rho}{2} = \frac{Mr^2}{2}$: for normal axis of a circular plane / disc ($p = \frac{m}{v}$: density) and

$I = \frac{Mr^2}{4}$: for axes-diameters of the disc(-oid)

- Perpendicular axes theorem (applies only for thin plates and plane figures): $I_z = I_x + I_y$

where I_x is an axis perpendicular to I_y and I_z is perpendicular to both I_y and I_z.

- Liquid Pressure: $P = W \cdot d$, W : weight, d : depth from sea level.
- Atmospheric or barometric pressure is the pressure exerted by the weight of air in the atmosphere of Earth (or that of another planet). In most circumstances atmospheric pressure is closely approximated by the hydrostatic pressure caused by the weight of air above the measurement point. Low-pressure areas have less atmospheric mass above their location, whereas high-pressure areas have more atmospheric mass above their location. Likewise, as elevation increases, there is less overlying atmospheric mass, so that atmospheric pressure decreases with increasing elevation.
- Total thrust = Area of object · Pressure at its center of gravity: Applies for an object immersed in liquid.
- For second moment of area, or moment of inertia of plane area, we replace mass with the area on every relevant formula. It is a geometrical property of an area that reflects the manner its points are distributes with respect to an arbitrary axis.
- Total thrust of a submerged surface = total area of a face · pressure at its centroid (depth)
- The resultant thrust acts at the centre of pressure, the depth of which: $\bar{\bar{z}} = \frac{k^2}{\bar{z}}$, k^2 is the gyradius on the surface of the liquid.
- Area of polar sector: $A = \int_{\partial_1}^{\partial_2} \frac{1}{2} r^2 d\partial$
- Volume generated by polar plane figure rotating about initial line

$(Ox) : V = \frac{2\pi}{3} \int_{\partial_1}^{\partial_2} r^3 \sin(\partial) d\partial$

- Length of arc of polar curve: $S = \int_{\partial_1}^{\partial_2} \sqrt{r^2 + \left(\frac{dr}{d\partial}\right)^2} d\partial$
- Surface area of revolution of polar curve, generated when rotating about line

$\partial = \pi/2 : S = 2\pi \int_{\partial_1}^{\partial_2} r \cos\partial \sqrt{r^2 + \left(\frac{dr}{d\partial}\right)^2} d\partial$

- If a system of masses S consists of parts, each of which has its center of gravity on the same plane, then this plane also contains the center of gravity of the entire system S.

Using multiple integrals

- Finding area and volume enclosed between a curve $y = f(x)$ and the x-axis: $\sum_{y=y_1}^{y_2} \delta y \cdot \delta x$

: Area of vertical strip, $\delta y \cdot \delta x$: Area of element.

Area $= \sum_{x=x_1}^{x_2} \sum_{y=y_1}^{y_2} \delta y \delta x$

As $\delta y \to 0$, $\delta x \to 0$ $\therefore \int_{x_1}^{x_2} \int_{y_1}^{y_2} dydx = \int_{x_1}^{x_2} y\Big|_{y_1}^{y_2} dx$.

- For curves in polar coordinates:

$A \simeq \sum_{\partial=0}^{\partial_1} \sum_{r=0}^{r_1} r\delta r\delta\partial$, $\delta \to 0$, $\delta\partial \to 0$ $\therefore A = \int_0^{\partial_1} \int_0^{r_1} rdrd\partial$

- Approximate values: We use them whenever we quote a result correct to a stated number of decimal places or significant figures.

Approximate Integration

- Method 1: By Series
- Method 2: By Simpson's rule

The area of a figure from a to b : $A = \int_a^b ydx = \int_a^b f(x)dx$, that cannot be calculated in closed form, can be approximated using Simpson's rule: $A \simeq \frac{S}{3}\left[(F + L) + 4E + 2R\right]$, S : width of each strip, $F + L$: Sum of the first and last ordinates, $2R = 2\cdot$ Sum of the remaining odd numbered ordinates.

Note: each ordinate is used only once.

- Method 3: By the trapezoid rule

- Integration of exact differentials:

$z = \int Pdx = f(x) + g(y)$, where $g(y)$ is a function of y only, and is akin to the constant in a normal integral.

$z = \int Qdy = f(y) + g(x)$. By inspection of the two results we can find $g(x)$ and $g(y)$.

- Exact differential in 3 independent variables:

$dw = Pdx + Qdy + Rdz$ is an exact differential of $w = f(x, y, z)$, if

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \vee \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \wedge \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

- $A = -\oint ydx$: Closed curve integral. The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner.
- Line Integrals, Definition: $I = \int_c f(x,y)ds = \int_c P(x)dx + Q(y)dy$ where c is the prescribed curve and f, or P and Q are functions of x and y.

Properties

1. $\int_c Fds = \int_c \{Pdx + Qdy\}$
2. $\int_{AB} Fds = -\int_{BA} Fds$, i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.
3. Paths of integration:

   → For a path of integration parallel to the y-axis, i.e.
   $x = k, dx = 0 \therefore \int_c Pdx = 0 \therefore I_c = \int_c Qdy$

   → For a path of integration paralle to the x-axis, i.e.
   $y = k \ , \ dy = 0 \therefore \int_c Qdy = 0 \therefore I_c = \int_c Pdx$
4. If the path of integration c joining A to B is divided into two parts AK and KB, then $I_c = I_{AB} = I_{AK} + I_{KB}$.
5. In all cases, the function $y = f(x)$ that describes the path of integration involved must be continuous and single-valued - or dealt with in this way (item 6 below).
6. If the function $y = f(x)$ that describes the path of integration c isn't single valued for part of its extent (figure 26), the path is divided into two sections:
   $(y = f_1(x) \text{from} A \text{to} K) \wedge (y = f_2(x) \text{from} K \text{to} B)$

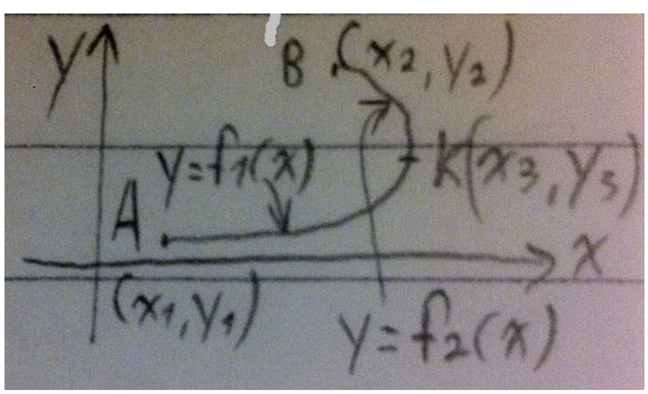


Figure 26: Function divided to multiple sections to be integrated

- Line integral relation with arc length:

$$I = \int_{AB} f ds = \int_{AB} (Pdx + Qdy) = \int_{c^{x_1}}^{x_2} f(x,y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- When x and y are expressed in parametric form, i.e. $x = f(t)$, $y = g(t)$, then:

$$I = \int_c f(x,y) ds = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- If $Pdx + Qdy$ is an exact differential where P, Q and their first derivatives are finite and continuous inside the simply connected region R, then

1. $I = \int_c (Pdx + Qdy)$ is independent of the path of integration where c lies entirely within R
2. $I = \oint_c (Pdx + Qdy)$ is zero when c is a closed curve lying entirely within R.

Same reasoning applies for functions of more than two variables.

- Integration under the Integral Sign is the use of the identity:

$\int_a^b \int_{a_0}^a f(x,a)\ da\ dx = \int_{a_o}^a \int_a^b f(x,a)\ dx\ da$ to compute an integral of the form $\int_{a_0}^a f(x,a)\ da$. It is also referred to as the Leibniz integral rule.

- Green's Theorem: Let P and Q be two functions of x and y that are, along with their first partial derivatives, finite and continuous inside and on the boundary c of a region R in the $x - y$ plane. Then, by definition of G.T.: $\int_R \int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dxdy = -\oint_c (Pdx + Qdy)$

and for a simple closed curve:

$\oint_c (xdy - ydx) = 2\int_R \int dxdy = 2A$, where A is the area of the enclosed figure.

- $\int_R \int\ dxdy =$ Area of region R
- Centre of gravity $(\bar{x}, \bar{y}) M\bar{x} =$ Sum of Moments about x-axis, $M\bar{y} =$ Sum of moments about y-axis.

Sum of moments about x-axis $= \int_R \int xdm = \int_R \int xdydx$,

Sum of moments about y-axis $= \int_R \int ydm$

- Surface integrals: $I = \int_R f(x,y) da = \int_R \int f(x,y) dydx$
- Surface in space: $I = \int_{S-} \phi(x,y,z) ds = \int_R \int \phi(x,y,z) \sec\gamma\ dxdy =$

$\int_R \int \phi(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dxdy\ ,\ \gamma < \frac{\pi}{2}\ ,\ z = f(x,y)$
$\phi(x,y,z)$: function of position on S
Area of the surface $s = \int_s ds = \int_R \int \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dxdy$
R : projection of surface s on $x-y$ plane.

- We project the surface s onto the $x-y$ plane and then we use the variables x,y,z such as: $ds = Adxdy$. A : scaling factor $\in \mathbb{R}$
- The equation of $\phi(x,y,z) =$ constant , represents a surface in space. [just like the equation $f(x,y) =$ constant(e.g.$x+y=1$) represents a line on a surface.]

- Volume integrals

We find: 1) Volume of column, 2) Volume of slice, 3) Total volume

- $V = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\ dzdydx$

- Change of variables in multiple integrals: From Cartesian coordinates (x,y) to Curvilinear coordinates (u,v)

a) Double Integrals, where $x = f(u,v)\ , y = g(u,v)$. Then:
$I = \int_R \int f(x,y)dxdy = \int_R \int F\left(f(u,v), g(u,v)\right) |\frac{\partial(x,y)}{\partial(u,v)}|\ dudv$
and $|\frac{\partial(x,y)}{\partial(u,v)}|\ dudv = dA$: the transformed element of area.
b) Triple Integrals, where $x = f(u,v,w), y = g(u,v,w), z = h(u,v,w)$, i.e. transformation in 3 dimensions.
$I = \int\int\int F(x,y,z)\ dxdydz = \int\int\int G(u,v,w)|\frac{\partial(x,y,z)}{\partial(u,v,w)}|\ dudvdw$
and $|\frac{\partial(x,y,z)}{\partial(u,v,w)}|\ dudvdw = dV$: the transformed element of volume

- $J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix}$

16.4.1 Properties of Integrals

- An integral is the limit of a(n) (infinite) sum.
- $\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt$
- $\int_a^a f(t)dt = 0$

- $\int_a^b f(t)dt = -\int_b^a f(t)dt$
- $\left|\int_a^b f(t)dt \leqslant \int_a^b |f(t)|dt\right|$
- Distance interpretation of integral: Suppose $u(t)$ is the velocity at time t of a particle moving along the x-axis. Then: $\int_a^b |u(t)|\, dt = S^+ + S^- = S$, where:

  → S^+ : total distance traveled in the forward direction

  → S^- : total distance traveled in the backward direction

  → S : total distance traveled. whereas: $\int_a^b u(t)dt = S^+ - S^- = D$

  → S^+ : area bounded by t axis, lines $t = a$ and $t = b$ and the part of the graph, where $f(t) \geqslant 0$.

  → S^- : area bounded by t axis, lines $t = a$ and $t = b$ and the part of the graph, where $f(t) \leqslant 0$.

  → D : net displacement from the original position.

16.4.2 Integrals

- $f(x) = 0 \rightarrow \int f(x)dx = C$
- $f(x) = C \rightarrow \int f(x)dx = Cx = D$
- $f(x) = x^a,\ a \in \mathbb{R} - \{-1\} \rightarrow \int f(x) = \frac{x^{a+1}}{a+1} + C$
- $f(x) = \frac{1}{2\sqrt{x}}, x > 0 \rightarrow \int f(x)dx = \sqrt{x} + C$
- $f(x) = \sin(x) \rightarrow \int f(x)dx = -\cos(x) + C$
- $f(x) = \cos(x) \rightarrow \int f(x)dx = \sin(x) + C$
- $f(x) = \frac{1}{\cos^2(x)} \rightarrow \int f(x)dx = \tan(x) + C$
- $f(x) = \frac{1}{\cos^2(x)} \rightarrow \int f(x)dx = -\cot(x) + C$
- $f(x) = e^x \rightarrow \int f(x)dx = e^x + C$
- $f(x) = a^x\ , a \in \mathbb{R} - \{1\} \rightarrow \int f(x)dx = \frac{a^x}{\ln\, (a)} + C$

- $f(x) = \tan(x) \to \int f(x)dx = -\ln\ (|\cos(x)|) + C$
- $f(x0 = \cot(x) \to \int f(x)dx = \ln\ (|\sin(x)|) + C$
- $f(x) = \cos^2(x) \to \int f(x)dx = \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C$
- $f(x) = \cos^2(x) \to \int f(x)dx = \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + C$
- $f(x) = \ln\ (x) \to \int f(x)dx = xln(x) - x + C$ (by substitution)
- $f(x) = \log_a(a) \to \int f(x)dx = x\log_a x - \frac{x}{\ln\ (a)} + C$
- $f(x) = \frac{1}{x} \to \int f(x)dx = \begin{cases} \ln\ (x) + C\ , & x > 0 \\ \ln\ (-x) + C & x < 0 \end{cases} = \ln\ |x| + C$
- $f(x) = \sin^{-1}(x) \to \int f(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C,\ |x| \leqslant 1$
- $f(x) = \cos^{-1}(x) \to \int f(x)dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C,\ |x| \leqslant 1$
- $f(x) = \tan^{-1}(x) \to \int f(x)dx = x\tan^{-1}(x) - \frac{1}{2}\ln\ |1+x^2| + C$
- $f(x) = \cot^{-1}(x) \to \int f(x)dx = x\cot^{-1}(x) + \frac{1}{2}\ln\ |1+x^2| + C$
- $f(x) = \sinh(x) \to \int f(x)dx = \cosh(x) + C$
- $f(x) = \cosh(x) \to \int f(x)dx = \sinh(x) + C$
- $f(x) = \tanh(x) \to \int f(x)dx = \ln\ |\cosh(x)| + C$
- $f(x) = \coth(x) \to \int f(x)dx = \ln\ |\sinh(x)| + C$
- $f(x) = \sinh^{-1}(x) \to \int f(x)dx = x\sinh^{-1}(x) - \sqrt{x^2+1} + C$
- $f(x) = \cosh^{-1}(x) \to \int f(x)dx = x\cosh^{-1}(x) - \sqrt{x^2-1} + C\ ,\ x > 1$
- $f(x) = \tanh^{-1}(x) \to \int f(x)dx = x\tanh^{-1}(x) + \frac{\ln\ (1-x^2)}{x} + C\ ,\ |x| < 1$
- $f(x) = \coth^{-1}(x) \to \int f(x)dx = c\coth^{-1}(x) + \frac{\ln\ (x^2-1)}{2} + C\ ,\ |x| > 1$
- $f(x) = e^{-x^2} \to \int_{-\infty}^{\infty} f(x)dx = \sqrt{\pi}$
- $f(x) = \tan^2(x) \to \int f(x)dx = \tan x - x + C$

- $f(x) = \dfrac{1}{x^2 + a^2} \rightarrow \displaystyle\int f(x)dx = \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)$
- $f(x) = \dfrac{1}{x^2 - a^2} \rightarrow \displaystyle\int f(x)dx = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right),\ x^2 > a^2$
- $f(x) = \sin(ax) \rightarrow \displaystyle\int f(x)dx = -\frac{1}{a}\cos(ax)$
- $\displaystyle\int f'(x)e^{f(x)}dx = e^{f(x)} + C$
- $\displaystyle\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln\,(a)} + C$
- $\displaystyle\int f'(x)\ \ (f(x))^a\, dx = \frac{[f(x)]^{a+1}}{a+1} + C$
- $f(t) = \delta(t) \rightarrow \displaystyle\int_{-\infty}^{\infty} f(t)dt_i = 1 \wedge \int_{-\infty}^{t} f(t_i)dt_i = u(t)$
- $f(t) = u(t) \rightarrow \displaystyle\int_{-\infty}^{t} f(t_i)\ dt_i = r(t)\ ,\ r(t) = \begin{cases} t\ , & t \geqslant 0 \\ 0\ , & t < 0 \end{cases}$
- $f(x) = \sinh(ax)dx \rightarrow \displaystyle\int f(x)dx = \frac{1}{a}\cosh(ax) + C$
- $f(x) = \sinh^2(ax) \rightarrow \displaystyle\int f(x)dx = \frac{1}{4a}\sinh(2ax) - \frac{x}{2} + C$
- $f(x) = \cosh^2(ax) \rightarrow \displaystyle\int f(x)dx = \frac{1}{4a}\sinh(2ax) + \frac{x}{2} + C$
- $f(x) = \tanh^2(ax) \rightarrow \displaystyle\int f(x)dx = x - \frac{\tanh(ax)}{a} + C$
- $f(x) = \sec\,(x) \rightarrow \displaystyle\int f(x)dx = \ln\,\left(\frac{1+\sin(x)}{\cos(x)}\right) + C$
- $f(Z) = \dfrac{1}{Z^2 - A^2} \rightarrow \displaystyle\int f(Z)dZ = \frac{1}{2A}\ln\,\left(\frac{Z-A}{Z+A}\right) + C$
- $f(Z) = \dfrac{1}{A^2 - Z^2} \rightarrow \displaystyle\int f(Z)dZ = \frac{1}{2A}\ln\,\left(\frac{A+Z}{A-Z}\right) + C$
- $f(Z) = \dfrac{1}{A^2 + Z^2} \rightarrow \dfrac{1}{A}\tan^{-1}\left(\dfrac{Z}{A}\right) + C$
- $f(Z) = \dfrac{1}{\sqrt{A^2 - Z^2}} \rightarrow \displaystyle\int f(Z)dZ = \sin^{-1}\left(\frac{Z}{A}\right) + C$ (the root keeps us from using partial fraction decomposition)
- $f(Z) = \dfrac{1}{\sqrt{A^2 + Z^2}} \rightarrow \displaystyle\int f(Z)dZ = \sinh^{-1}\left(\frac{Z}{A}\right) + C$
- $f(Z) = \dfrac{1}{\sqrt{Z^2 - A^2}} \rightarrow \displaystyle\int f(Z)dZ = \cosh^{-1}\left(\frac{Z}{A}\right) + C$
- $f(Z) = \sqrt{A^2 - Z^2} \rightarrow \displaystyle\int f(Z)dZ = \frac{A^2}{2}\Big[\sin^{-1}\left(\frac{Z}{A}\right) + \frac{Z\sqrt{A^2 - Z^2}}{A^2}\Big] + C$

- $f(Z) = \sqrt{Z^2 + A^2} \rightarrow \int f(Z)dZ = \frac{A^2}{2}\left[\sinh^{-1}\left(\frac{Z}{A}\right) + \frac{Z\sqrt{A^2+Z^2}}{A^2}\right] + C$
- $f(Z) = \sqrt{Z^2 - A^2} \rightarrow \int f(Z)dZ = \frac{A^2}{2}\left[-\cosh^{-1}\left(\frac{Z}{A}\right) + \frac{Z\sqrt{Z^2-A^2}}{A^2}\right] + C$
- $f(x) \int \frac{1}{a + b\sin^2(x) + c\cos^2(x)}dx$, $t = \tan(x)$, $\sin(x) = \frac{t}{\sqrt{1+t^2}}$, $dx = \frac{2dt}{1+t^2}$, $\cos(x) = \frac{1}{\sqrt{1+t^2}}$
- $f(x) \int \frac{1}{a + b\sin(x) + c\cos(x)}dx$, $t = \tan(\frac{x}{2})$, $\sin(x) = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$, $\cos(x) = \frac{1-t^2}{1+t^2}$
- $f(x) = \frac{1}{g(x)} \rightarrow \int f(x)dx = \ln\left(g(x)\right)$, if the highest power of x is 1. Else: resort to partial fraction decomposition method.
- $f(x) = \text{cosec }(x) \rightarrow \int f(x)dx = -\ln\left(|\text{cosec }(x) + \cot(x)|\right) + C$
- $\pi = 2e \int_0^{+\infty} \frac{\cos(x)}{x^2+1}dx$
- $f(x) = \sec(x) \rightarrow \int f(x)dx = \ln\left(|\sec(x) + \tan(x)|\right) + C$
- $f(x) = e^{-x^2} \rightarrow \int_0^{\infty} f(x)dx = \frac{\sqrt{\pi}}{2}$, $\int_{-\infty}^{\infty} f(x)dx = \sqrt{\pi}$
- $f(x) = e^{\frac{-x^2}{2}} \rightarrow \int_{-\infty}^{\infty} f(x)dx = \sqrt{2\pi}$
- $f(x) = e^{-k^2x^2} \rightarrow \int_0^{\infty} f(x)dx = \frac{\sqrt{\pi}}{2k}$, $k > 0$

Integrals of periodic functions (primarily to assist with Fourier series expansions) - applied for any interval of length 2π -

- $\int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = 2\pi$
- $\int_{-\pi}^{\pi} \cos(nx)dx = 0$
- $\int_{-\pi}^{\pi} \sin(nx)dx = 0$
- $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn}$
- $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn}$
- $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$

16.4.3 Reduction Formulae

$n \in \mathbb{N}^*$

- $I_n = \int e^{nx}\sin(x)dx \rightarrow I_n = \dfrac{e^{nx}}{n^2+1}\ (n\sin(x) - \cos(x))$
- $I_n = \int \sin^n(x)dx \rightarrow I_n = -\dfrac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \dfrac{n-1}{n}I_{n-2}$
- $I_n = \int \cos^n(x)dx \rightarrow I_n = \dfrac{1}{n}\cos^{n-1}(x)\cdot\sin(x) + \dfrac{n-1}{n}I_{n-1}$
- $I_n = \int x^n e^x dx \rightarrow I_n = x^n e^x - nI_{n-1}$
- $I_n = \int x^n \sin(x)dx \rightarrow I_n = -x^n\cos(x) + nx^{n-1}\sin(x) - n(n-1)\cdot I_{n-2}$
- $I_n = \int x^n \cos(x)dx \rightarrow I_n = x^n\sin(x) + nx^{n-1}\cos(x) - n(n-1)I_{n-2}$
- $I_n = \int \tan^n(x)dx \rightarrow I_n = \dfrac{\tan^{n-1}(x)}{n-1} - I_{n-2}(+C)$
- <u>Walli's Formula</u>: If $I_n = \int_0^{\pi/2}\sin^n(x)dx$, or $I_n = \int_0^{\pi/2}\cos^n(x)dx$ then:

$I_n = \dfrac{n-1}{n}I_{n-2}$, $I_0 = \dfrac{\pi}{2}$, $I_1 = 1$

or $I_n = \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdot\ldots\cdot\begin{cases}\cdot 1 & (n \text{ odd}) \\ \cdot\dfrac{\pi}{2} & (n \text{ even}\end{cases}$

- $I_n = \int (\ln\,(x))^n\,dx \rightarrow I_n = x\,(\ln\,(x))^n - nI_{n-1}$
- $I_n = \int \cot^n(x)dx$, $n > 1 \rightarrow I_n = -\dfrac{\cot^{n-1}(x)}{n-1} - I_{n-2} =$

$-\dfrac{\cot^{n-1}(x)}{n-1} + \dfrac{\cot^{n-3}(x)}{n-3} - \dfrac{\cot^{n-5}(x)}{n-5} \pm\cdots\pm x + C$

- $I_n = \int \left(x^2+a^2\right)^n dx \rightarrow I_n = \dfrac{1}{2n+1}\left[x(x^2+a^2)^n + 2na^2 I_{n-1}\right]$
- $I_n = \int \sec^n(x)dx \rightarrow I_n = \dfrac{1}{n-1}\tan(x)\sec^{n-2}(x) + \dfrac{n-2}{n-1}I_{n-1}$, $n \geqslant 2$
- $I_n = \int_0^{\pi} e^{-x}\sin^n(x)dx \rightarrow I_n = \dfrac{n(n-1)}{n^2+1}\cdot I_{n-1}$
- $I_n = \int_0^{\frac{\pi}{2}} x\cos^n(x)dx$, $n > 1 \rightarrow I_n = \dfrac{n(n-1)I_{n-2} - 1}{n^2}$
- $I_n = \int_0^{\infty} x^n e^{-ax}dx \rightarrow I_n = \dfrac{n}{a}I_{n-1}\ \therefore I_n = \dfrac{n!}{a^{n+1}}$
- $I_{m,n} = \int_0^{\frac{\pi}{2}}\sin^m(x)\cdot\cos^n(x)dx \rightarrow I_{m,n} = \dfrac{m-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^{m-2}(x)\cos^n(x)\ dx$ or

$I_{m,n} = \dfrac{n-1}{m+n}I_{m-2,n}$

Improper integral: The limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number, or ∞, or $-\infty$ or in some cases, as both endpoints approach limits. Symbolically:

$\lim\limits_{b\to\infty}\displaystyle\int_a^b f(x)dx\Big[=\int_a^\infty f(x)dx\Big]$, or e.g.

$\lim\limits_{c\to b^-}\displaystyle\int_a^c f(x)dx\ ,\ \lim\limits_{c\to a^+}\int_c^b f(x)dx$

$\displaystyle\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} f(x)dx+\int_0^\infty f(x)dx$

17 Differential Equations

- Solutions to differential equations are functions, as opposed to algebraic equations where the solutions are numbers.
- An nth order differential equation is derived from a function having n arbitrary constants.
- The general, or primitive, solution of the equation contains the arbitrary constant.
- The particular solution of the equation can be found if we are told the value of y for a given value of x. Then we can find a value of C.

17.1 Ordinary Differential Equations (ODEs)

- An <u>ordinary differential equation (ODE)</u> is a differential equation containing one or more functions of one independent variable and its derivatives. The term ordinary is used in contrast with the term partial differential equation which may be with respect to more than one independent variable.

17.1.1 Analytical Solutions to ODEs

<u>Solution of 1st order Differential Equations</u>

- a) By direct integration: $\dfrac{dy}{dx} = f(x)$, gives $y = \displaystyle\int f(x)dx$
- b) By separating the variables: $F(y)\dfrac{dy}{dx} = f(x)$, gives $\displaystyle\int F(y)dy = \int f(x)dx$
- c) Homogeneous equations: Substituting $y = vx$, gives $v + x\dfrac{dv}{dx} = F(v)$, which can be solved by separating the variables.

Homogeneous equations are of the form:
$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$, where all the terms are proportional to either y, or a derivative of x. There's no x <u>alone</u>.

- d) Linear equations - Use of Integrating Factor (IF)

They have the form: $\dfrac{dy}{dx} + Py = Q$, where $P\&Q$ are constants, or functions of x.
Multiplying both sides by the IF $= e^{\int P(x)dx}$, gives
$\dfrac{dy}{dx}\mathrm{IF} + y\dfrac{d\mathrm{IF}}{dx} = Q \cdot \mathrm{IF} \rightarrow \dfrac{d}{dx}\{y\mathrm{IF}\} = Q \cdot \mathrm{IF} \rightarrow y\mathrm{IF} = \displaystyle\int Q \cdot \mathrm{IF}dx$

- e) Bernoulli's equation: $\dfrac{dy}{dx} + Py = Qy^n$

First, divide by y^n. Then put $z = y^{1-n}$. Afterwards it is reduced to type d.

- f) In various cases there may need to happen a specific transformation in order for the differential equation to fall to one of the previous categories. We transform a non-linear O.D.E. to a linear one this way.

Second Order Differential Equations

• Solution of equations of the form: $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)$

Auxiliary equation: $am^2+bm+c=0$

Types of solutions:

a) Real and different roots: $m=m_1 \ \wedge m=m_2$

C.F.: $y=Ae^{m_1x}+Be^{m_2x}$

b) Real and equal roots: $m=m_1(\text{twice})$

C.F.: $y=e^{m_1x}(A+Bx)$

c) Complex roots: $m=a\pm\beta x$

C.F.: $y=e^{ax}(A\cos\beta x+B\sin\beta x)$

• Equations of the form: $\dfrac{d^2y}{dx^2}+n^2y=0$

Auxiliary equation: $m^2=-n^2$

C.F.: $y=A\cos nx+B\sin nx$

• Equations of the form: $\dfrac{d^y}{dx^2}-n^2y=0$

Auxiliary equation: $m^2=n^2$

C.F.: $y=A\cosh(nx)+B\sinh(nx)=e^{nx}\left(\dfrac{A+B}{2}\right)+e^{-nx}\left(\dfrac{A-B}{2}\right)$

• General Solution = Complementary Function + Particular Integral [G.S. = C.F. + P.I.]

• To find C.F. solve: $a\dfrac{d^y}{dx^2}+b\dfrac{dy}{dx}+cy=0$

• Particular Integral exists only when $f(x)\neq 0$, i.e. in inhomogeneous differential equations.

• To find P.I. assume the general form of the R.H.S. It can be of the form:

$f(x)=k\ldots$ Assume: $y_{PI}=C$

$f(x)=kx\ldots$ Assume: $y_{PI}=Cx+D$

$f(x)=kx^2$ Assume: $y_{PI}=Cx^2+Dx+E$

$f(x)=k\sin(x)$, or$k\cos(x)$ Assume: $y_{PI}=C\cos(x)+D\sin(x)$

$f(x)=k\sinh(x)$,or$k\cosh(x)$ Assume: $y_{PI}=C\cosh(x)+D\sinh(x)$

$f(x)=e^{kx}$ Assume: $y_{PI}=Ce^{kx}$

Combine forms where applicable. Afterwards differentiate y_{PI} to get $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, substitute to the L.H.S. to find the constants $C,D,\ldots$ by equating coefficients of the L.H.S. to those in the R.H.S.

• Note: If the general form of the R.H.S. is already included in the C.F., multiply y_{PI} by x as many times as necessary and proceed as before.

• Particular solution: Finding the values of the arbitrary constants A and B, when given the necessary boundary / initial conditions.

• $f(t)=a\cos\sqrt{\dfrac{k}{l}}t+b\sqrt{\dfrac{l}{k}}\sin\sqrt{\dfrac{k}{l}}t$: Behaviour of a system that executes simple harmonic, oscillatory motion with natural frequency $\omega = \sqrt{\dfrac{k}{l}}$. Such a system is called a harmonic oscillator.

- $Af''(t) + Bf'(t) + Cf(t) = 0$: Differential equation describing the position of the harmonic oscillator
- $f'(t)$: damping term, B : damping parameter

With 0 on the R.H.S. only transient terms exist in $f(t)$ (i.e. damping terms that decay over time t). With terms on the R.H.S. steady-state terms emerge.
The R.H.S. is the forcing function. If this function's frequency is the same as the ω of the H.O. then the system will resonate, which means that it will oscillate with increasing amplitude.
Leibnitz n th derivative theorem: $[u = f(x) \wedge v = g(x)] \to (uv)^{(n)} = u^{(n)}v +^n C_1 u^{(n-1)}v^{(1)} +^n C_2 u^{(n-2)}v^{(2)} + \ldots +^n C_{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}$, $u^{(0)} \equiv u$

- Leibnitz - MacLaurin (power series) method for solving O.D.E.'s of the form:

$$\frac{d^y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Steps:
a) Differentiate given equation n times
b) Rearrange the result to obtain the recurrence relation between the derivatives, at $x = 0$ (initial condition).
c) Determine the values of the derivatives at $x = 0$, usually in terms of $y(0)$ and $y'(0)$.
d) Substitute the findings in the MacLaurin expansion for $y = f(x)$.
e) Simplify the result where possible and apply boundary conditions if provided.

- Cauchy-Euler equi-dimensional equations have the structure:

$a_n x^n y^{(n)}(x) + a_{n-1}x^{n-1}y^{(n-1)}(x) + \ldots + a_1 x y'(x) + a_0 y(x) = g(x)$, where the coefficient of the nth derivative contains x^n term.
General solution $= y_H(x) + y_p(x)$, normally. We assume $y_n(x) = kx^n$, find its derivatives and substitute to find as many n's as the degree of the equation. The form of the particular solution depends upon the form of the R.H.S. of the equation.

- Sturm - Liouville systems: $(p(x) \cdot y')' + (q(x) + \lambda r(x))y = 0$, for $a \leqslant x \leqslant b$ and $r(x) > 0$, with boundary conditions $a_1 \cdot y(a) + a_2 \cdot y'(a) = 0$ and $\beta_1 \cdot y(b) + \beta_2 \cdot y'(b) = 0$

Solutions y_n to a Sturm - Liouville system are called eigenvectors, each corresponding to an eigenvalue λ_n for $n = 0, 1, 2, \ldots$

17.1.2 Systems of Differential Equations

- Solving systems of 1st order D.E.s, of the form:

$$\mathbf{F}'(x) = \mathbf{A} \cdot \mathbf{F}(x) \text{ , where } \mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} \text{ \& } \mathbf{F}'(x) = \begin{pmatrix} f_1'(x) \\ f_2'(x) \\ \vdots \\ f_n'(x) \end{pmatrix}$$

1. Find the eigenvalues and eigenvectors of $\mathbf{A}$ and construct the modal matrix $\mathbf{M}$ and spectral matrix $\mathbf{S}$, where eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$.

2. Write the solution of the equation as $\mathbf{F}(x) = \sum_{r=1}^{n} a_r e^{\lambda_r x}\mathbf{C}_r$ and use the boundary conditions to find the values of a_r, for $r = 1, 2, \ldots, n$.

- Solving systems of 2nd order D.E.s, of the form: $\mathbf{F}''(x) = \mathbf{A}\mathbf{F}(x)$

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$.

2. Assuming the eigenvalues are all distinct, find the associated eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$.

3. Write the solution of the equation as $\mathbf{F}(x) = \sum_{r=1}^{n} \left(a_r e^{\sqrt{\lambda_r}\cdot x} + b_r e^{-\sqrt{\lambda_r} x}\right)\mathbf{C}_r$ and use the boundary conditions to determine the values of the constants a_r and b_r , $r = 1, 2, \ldots, n$.

17.2 Partial Differential Equations (PDEs)

- A partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. (ODEs are a special case which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.
- Solution to $u = f(x, y, w, t, \ldots)$
- Linear equations: If $u = u_1, u = u_2, u = u_3, \ldots$ are solutions, then the following is also a solution: $u = u_1 + u_2 + u_3 + \ldots + u_r + \ldots = \sum_{r=1}^{\infty} u_r$
- Wave equation: Simulates transverse vibrations of an elastic string:
$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$, $u = f(x, t)$
- Heat Conduction equation: Heat flows in uniform finite bar: $\dfrac{\partial^2 u}{\partial x} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$, $c^2 = \dfrac{k}{\sigma\rho}$
$k :$ thermal conductivity of material, $\sigma :$ specific heat of the material, $\rho :$ mass per unit length of bar
- Laplace's equation: Distribution of a field over a plane area:
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} + ... = 0$, $u = f(x, y, z, ...)$(u is a scalar function)
It is also written as $\nabla^2 u = 0$, or $\Delta u = 0$
- Solution steps

1. Assume a solution of the form: $u = X(x) \cdot T(t)$

2. Transpose the equation by separation of the variables. Then
   $\dfrac{\partial u}{\partial x} = X'T \ , \ \dfrac{\partial^2 u}{\partial x^2} = X''T \ , \ \dfrac{\partial u}{\partial t} = T'X \ , \ \dfrac{\partial^2 u}{\partial t^2} = T''X$

3. The two solutions are in the form: $X = (A\cos\rho x + B\sin\rho x)\cdot(C\cos c\rho t + D\sin c\rho t)$
   Then $u(x,t) = (A\cos\rho x + B\sin\rho x)\cdot(C\cos c\rho t + D\sin c\rho t)$

4. Putting $c\rho = \lambda$ we apply boundary conditions to determine A and B.

5. List the eigenvalues and eigenfunctions for $n = 1, 2, 3, \ldots$ and determine general solution as an infinite sum.

6. Apply the remaining initial, or boundary conditions and finally determine C_r and D_r by Fourier series techniques.

18 Difference Equations

- Any function f, where its input $n \in \mathbb{Z}$ is restricted to integer values has as output $f(n)$ in the form of a discrete sequence of numbers. Such a function is called a sequence.
- A recurrence relation (Αναδρομική Σχέση) is an equation that recursively defines a sequence, or a multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.
- The term "difference equation" is frequently used to refer to any recurrence relation, but rigorously speaking it is a type of a recurrence relation.
- The prescription of a sequence, say $f(n) = 5n - 2$, written as a recursive equation, with all the unknowns on one side: $f(n+1) - f(n) = 5$, is called a difference equation.
- The order of a difference equation is taken from the maximum number of terms between any pair of terms. In order to solve a difference equation it is necessary to have as many initial terms as the order of the difference equation.
- The solution of a constant coefficient, linear recursive difference equation is of the form: $f(n) = f_h(n) + f_p(n)$
- $f_h(n)$ is the solution to the homogeneous equation:

$a_n f(n) + a_{n-1} f(n-1) + \ldots + a_{n-k} f(n-k) = 0 \ {\scriptstyle (I)}$ (k order)
We assume that the above has a solution in the form of: $f(n) = cw^n$, $c, w \in \mathbb{R}^*$, $n \in \mathbb{Z}$ and we substitute $f(n)$ into the dif. eq. (I).
Factorizing we find the characteristic equation of the difference equation:
$Kw^N \{a_k w^k + a_{k-1} w^{k-1} + \ldots + a_2 w + a_1\} = 0$, where a_k are constants and find the roots $w_k, w_{k-1}, \ldots, w$.
Then $f_h(n) = A \cdot w_k^n + B \cdot w_{k-1}^n + \ldots$.
If we have 2 equal roots, we assume $Bn \cdot w^n$ is also a root of the difference equation (I).

- $f_p(n)$ is the particular solution and to find it we assume a solution to the (now) inhomogeneous equation: $a_n f(n) + a_{n-1} f(n-1) + \ldots + a_{n-k} f(n-k) = g(n) \ {\scriptstyle (II)}$, according to:

$$\begin{bmatrix} g(n) & \text{Particular solution} \\ \text{polynomial term} n^m & C_m n^m + C_{m-1} n^{m-1} + \ldots + C_1 n + C_0 \\ \text{exponential} a^n & C a^n \\ a^n \cos(bn) \ , \ a^n \sin(bn) & a^n \left(C_1 \cos(bn) + C_2 \sin(bn) \right) \end{bmatrix}$$

We substitute the particular solution into (II) and equate coefficients between the L.H.S. and R.H.S.

- The general solution is $f(n) = f_h(n) + f_p(n)$
- Finally, given the initial conditions, we find the constants $A, B, C, \ldots$ of the $f_h(n)$.

19 Complex Numbers

• A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, satisfying the equation $i^2 = -1$. In this expression, a is the real part and a is the imaginary part of the complex number. If $z = a + bi$, then Re $(z) = a \quad$ Im $(z) = b$.

- $j = \sqrt{-1}$, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$
- $z = a + jb$, Re $(z) = a \ \wedge\ $ Im $(z) = b$
- $\bar{z} = a - jb$: Conjugate
- $z = r\,(\cos\theta + j\sin\theta)$: Polar form (figure 27)

$a = r\cos\theta\ ,\ b = r\sin\theta\ ,\ \theta = \angle z = \tan^{-1}\left(\dfrac{b}{a}\right)\ ,\ |z| = r$

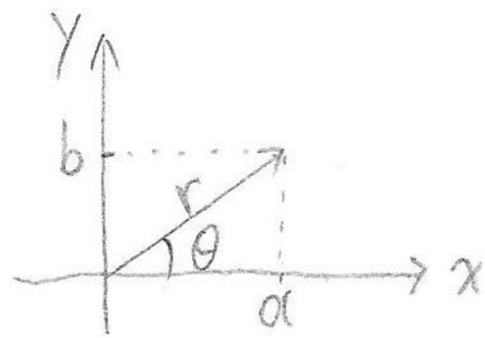


Figure 27: Argand Diagram

• A phasor (portmanteau for phase vector) is a term primarily used in circuit analysis, used to represent the amplitude and phase angle of a complex number (usually a voltage, or current).
• To find the correct angle θ, we should beware that there are two angles between $0°$ and $360°$, the tangent of which has the value b/a. Always draw a sketch of the complex number to ensure we have the right quadrant.
• Those complex numbers that evaluate to 1 when raised to some power p are called the complex roots of unity, ie. $z^p = 1 \Leftrightarrow (a + i \cdot b)^p = 1$
For each N, there are exactly N complex numbers z such that $z^N = 1$.
The numbers $\cos\left(\dfrac{2\pi k}{N}\right) + i\sin\left(\dfrac{2\pi k}{N}\right)$ for $k = 0, 1, \ldots, N-1$ can be easily shown to have this property.

- $z = |z|\underline{\lfloor \mathrm{z}}\ (\angle z > 0)$, ή $z = |z|\overline{\lceil \mathrm{z}}\ (\angle z < 0)$
- $r = \sqrt{a^2 + b^2}$
- $\tan(\theta) = \dfrac{b}{a}$

- $|z|^2 = z \cdot \bar{z} = r^2$
- $|z| = |\bar{z} = | - z|$
- $|z^\nu| = |z|^\nu$
- $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$: Oyler's identity
- $z = re^{j\theta}$: Exponential form. Here θ be in radians
- $\ln z = \ln r \pm j\theta$: Logarithm of complex number
- $z^n = [r(\cos\theta + j\sin\theta]^n = r^n(\cos(n\theta) + j\sin(n\theta))$ De'Moivre's Therorem, we can use it to find the roots of complex numbers
- $z + \dfrac{1}{z} = 2\cos\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$
- $z - \dfrac{1}{z} = j2\sin\theta$
- $z^n - \dfrac{1}{z^n} = j2\sin(n\theta)$
- $\text{Από} e^{jx} = \cos(x) + j\sin(x) :) \cos(x) = \dfrac{e^{jx} + e^{-jx}}{2}$, $\sin(x) = \dfrac{e^{jx} - e^{-jx}}{2j}$

Hyperbolic Functions

- $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$
- $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
- $e^{\pm x} = \cosh(x) \pm \sinh(x)$
- $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- $|f(e^{j\omega})|^2 = f(e^{j\omega}) \cdot f(e^{-j\omega})$: Ταυτότητα του μέτρου
- $\sinh^{-1}(x) = \ln\left[x + \sqrt{x^2 + 1}\right]$
- $\cosh^{-1}(x) = \pm\ln\left[x + \sqrt{x^2 - 1}\right]$
- $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, output $\in [-1, 1]$
- $\sin^{-1}(x) = -j\ln\left[\sqrt{1 - x^2} + jx\right]$, $-1 \leqslant x \leqslant 1$
- $\cos^{-1}(x) = -j\ln\left[j\sqrt{1 - x^2} + x\right]$, $-1 \leqslant x \leqslant 1$
- $\tan^{-1}(x) = \dfrac{1}{2}j\ln\left(\dfrac{1 - jx}{1 + jx}\right)$, $\forall x \in \mathbb{R}$
- $\coth(x) = \dfrac{1}{\tanh(x)}$
- $\text{sech}\ (x) = \dfrac{1}{\cosh(x)}$
- $\text{cosech}\ (x) = \dfrac{1}{\sinh(x)}$
- $\cosh^2(x) - \sinh^2(x) = 1$
- $\text{sech}\ ^2(x) = 1 - \tanh^2(x)$

- $\operatorname{cosech}{}^2(x) = \coth^2(x) - 1$
- $\sinh(2x) = 2\sinh(x)\cosh(x)$
- $\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 1 + 2\sinh^2(x) = 2\cosh^2(x) - 1$

19.1 Complex Trigonometric & Hyperbolic Identities

- $\sin(jx) = j\sinh(x)$
- $\sinh(jx) = j\sin(x)$
- $\cos(jx) = \cosh(x)$
- $\cosh(jx) = \cos(x)$
- $\tan(jx) = j\tanh(x)$
- $\tanh(jx) = j\tan(x)$
- $\mathrm{Re}\left[\sinh(z)\right] = \sinh(a)\cos(b)$
- $\mathrm{Im}\left[\sin(z)\right] = \cosh(a)\sin(b)$
- $\mathrm{Re}\left[\cosh(z)\right] = \cosh(a)\cos(b)$
- $\mathrm{Im}\left[\cos(z)\right] = \sinh(a)\sin(b)$
- $|\cosh(z)|^2 = \sinh^2(a) + \cos^2(b)$
- $|\sinh(z)|^2 = \sinh^2(a) + \cos^2(b)$
- $|x - y| = R \rightarrow x - y = Re^{j\theta}\ ,\ x, y \in \mathbb{C}$
- $\sin(z) = \dfrac{e^{jz} - e^{-jz}}{2j}$
- $\cos(z) = \dfrac{e^{jz} + e^{-jz}}{2}$
- $\tan(z) = \dfrac{\sin(z)}{\cos(z)} = \dfrac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}}$
- $\cot(z) = \dfrac{e^{jz} + e^{-jz}}{e^{jz} - e^{-jz}}$
- $\cos(-z) = \cos(z)$
- $\sin(-z) = -\sin(z)$
- $\cos^2(z) + \cos^2(z) = 1$
- $\cosh(z) = 0 \leftrightarrow z_0 = \pm j\left(n + \dfrac{1}{2}\right)\pi\ ,\ n = 1, 2, \ldots$
- $\sinh(z) = 0 \leftrightarrow z_0 = \pm jn\pi\ ,\ n = 1, 2, \ldots$
- $\cosh\left(\sinh^{-1}(x)\right) = \sqrt{x^2 + 1}$
- $\sinh\left(\cosh^{-1}(x)\right) = \sqrt{x^2 - 1}$
- $\arctan\left(\dfrac{a}{S}\right) + \arctan\left(\dfrac{S}{a}\right) = \dfrac{\pi}{2}$

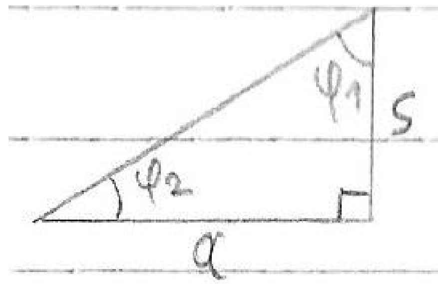


Figure 28: Visual representation of the identity

- $C \cdot f(t) \cdot \text{Re}\,\{g(t)\} = C\text{Re}\,\{f(t) \cdot g(t)\} = \text{Re}\,\{C \cdot f(t) \cdot g(t)\}$

Re , orIm , $\mathbb{Z}$, etc . . . , where$C \in \mathbb{C}$

- Hermitian function: A complex function where its complex conjugate is equal to the original function with the variable changed in sign, i.e. $f^*(x) = f(-x)$
- In linear algebra for an Hermitian matrix: $A^H = A^{-1} = \left(A^T\right)^*$ Properties
- $\text{Re}\,\{f(x)\} = \text{Re}\,\{f(-x)\}$
- $\text{Im}\,\{f(-x)\} = -\text{Im}\,\{f(x)\}$,
- $|f(-x)| = |f(x)|$
- $\angle f(-x) = -\angle f(x)$
- Also: $f \bigstar g = f \bigstar g$, if f is Hermitian and
- $f \bigstar g = g \bigstar f$, if both f and g are Hermitian

- Transformation equation: $z = x + jy \ \wedge w = u + jv$

Mappings from the z plane, $z = f(x, y)$, onto the w-plane
$w = f(z)$ $z = x + jy$, $w = u + jv$, $u = g(x, y)$, $v = h(x, y)$

- Linear Transformation: $w = az + b$, where a and b are real or complex. A straight line in the z-plane maps onto a corresponding straight line the w-plane.
- Types of Linear transformations: $w = az + b$

a) Translation: given by b
b) Magnification: given by $|a|$
c) Rotation: given by arg$a$

- Non-linear transformations

(a) $w = z^2$: A straight line through the origin maps onto a corresponding straight line through the origin in the w-plane. A straight line not passing through the origin maps onto a parabola.
(b) $w = \dfrac{1}{z}$ (inversion): A straight line, or a circle maps onto a straight line, or a circle in the w-plane. A straight line may be regarded as a circle of infinite radius. ($z = 1/w$: to find the equation)
(c) $w = \dfrac{az + b}{cz + d}$ (bilinear transformation) - with a, b, c, d real, or complex.

- Mapping of a region depends on the direction of development.

Right (Left) hand regions map onto right (left) hand regions.

19.2 Complex Analysis

- Complex analysis studies the properties of functions of complex numbers.

- Derivative of a function of a single real variable: $y = f(x)$

$\delta y = f(x + \delta x) - f(x)$

$$\frac{dy}{dx} = \lim_{\delta x \to} \left\{ \frac{f(x+\delta x) - f(x)}{\delta x} \right\} = \frac{(y + \delta y) - y}{\delta x}$$

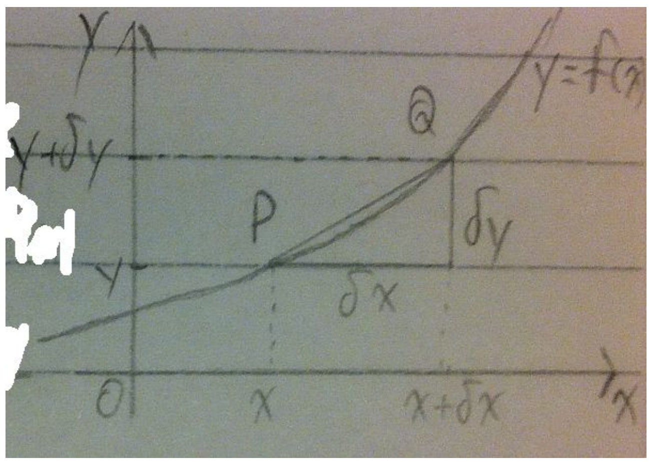


Figure 29: Derivative of a function

- Derivative of a function of a complex variable: $w = f(z)$

$$\left[\frac{dw}{dz}\right]_{z_0} = \lim_{\delta z \to 0} \left\{ \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} \right\} = \frac{(w + \delta w) - w}{\delta z} = \lim_{Q \to P} \frac{P'Q'}{PQ}$$

If this limiting value exists, the function is said to be differentiable at P.

- $dz = dx + jdy$, $dw = du + jdv$
- A function $w = f(z)$ is said to be regular, or holomorphic, or analytic, at a point $z = z_0$, if is defined and single valued and has a derivative at every point at and around z_0.

Points in a region where $f(z)$ ceases to be regular (eg. its derivative does not exist) are called singular points, or singularities. A function of a complex variable that is analytic over the entire finite complex plane is called an entire function. Examples of entire functions are polynomials, e^z, $\sin(z)$ and $\cos(z)$.

In other words, a holomorphic function is a complex-valued function, of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain.

- The real and imaginary parts of an analytic function are both harmonic and form a conjugate pair of functions.

- A meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all of D except for a set of isolated points, which are poles of the function. Every meromorphic function on D can be expressed as the ratio between two holomorphic functions (with the denominator not constant 0) defined on D : any pole must coincide with a zero of the denominator.

- Cauchy - Riemann Equations: A necessary condition for $w = f(z) = u + jv$ to be regular at $z = z_0$ is that $u = g(x, y)$ and $v = h(x, y)$ and their partial derivatives are continuous and that in the neighborhood of $z = z_0$:
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

- Complex Integration: $\int w dz = \int f(z) dz = \int (u dx - v dy) + j \int (v dx + u dy)$
- Contour Integration: Evaluation of line integrals in the z-plane.
- Cauchy's Theorem: If $f(z)$ is regular at every point within and on a closed curve c, then
$\oint_c f(z) dz = 0$

- Deformation of Contours

(**a**) Singularity at A (Multiple singularities may occur.
(**b**) Restored to a closed curve.
(**c**) $\oint_c f(z) dz = \oint_{c_1} f(z) dz$
For $\oint_c f(z) dz$, where $f(z) = \frac{1}{(z-a)^n}$, $n = 1, 2, 3, \ldots$
$\oint_c f(z) dz = \begin{cases} 0 & n \neq 1 \\ 0 & n = 1 \text{and} c \text{does not enclose} a \\ j2\pi & n = 1 \text{and} c \text{does enclose} a \end{cases}$ In general we have to consider if the singularity is enclosed within the contour c (if that's not explicitly stated, or visible).

- If $f'(z_0) = 0$, at $z = z_0$, then z_0 is a critical point.
- Conformal transformation: mapping in which angles are preserved in size and sense of rotation.

Conditions:

1. $w = f(z)$ must be a regular function of z.
2. $f'(z)$, ie. $\frac{dw}{dz} \neq 0$ at a point of intersection.

- Schwarz - Christoffel transformation: maps any polygon in the z-plane onto the entire upper half of the w-plane and the boundary of the polygon onto the real axis of the w-plane. $\frac{dz}{dw} = A\left(w - u_1\right)^{\frac{a_1}{\pi}-1} \cdot \left(w - u_2\right)^{\frac{a_2}{\pi}-1} \ldots \left(w - u_n\right)^{\frac{a_n}{\pi}-1}$
We find z from $\frac{dz}{dw}$ and finally the transformation function $w - f(z)$.

  1. Any three points $u_1, u_2, \ldots, u_n$ can be selected on the u-axis. $a_1, a_2, \ldots, a_n$ are the corresponding angles formed at the points $u_1, u_2, \ldots, u_n$ respectively.
  2. One such point can be chosen at infinity.
  3. Infinite open polygons are regarded as limiting cases of closed polygons.

- If $f(z)$ is a function in the complex variable z, analytic at $z = 0$, then its McLaurin series can be found by
$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \ldots$

- Radius and circle of convergence: The radius of the circle within which a series expansion is valid is called the raius of convergence of the series and the circle is called the circle of convergence. The radius of convergence can be found using the ratio test for convergence. When an expression is expanded in a McLaurin series, the circle of convergence is always centered on the origin and the radius of convergence is determined by the location of the first singular point met as $|z|$ moves out from the origin.
- If $f(z)$ has a singular point at z_0 and for some natural number n the $\lim_{z \to z_0} \left\{(z - z_0)^n f(z)\right\} = L \neq 0$, then the singularity is called a pole of order n (the pole's multiplicity).
- If $f(z)$ has a singular point at z_0, but $\lim_{z \to z_0} \left\{f(z)\right\}$ exists then the singular point is called a removable singularity.

- Complex Taylor Series: Provided $f(z)$ is analytic inside and on a simple closed curve c, the Taylor series expansion of $f(z)$ about a point z_0 which is interior to c is given as:
$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2 f''(z_0)}{2!} + \ldots + \frac{(z - z_0)^n f(z_0)^{(n)}}{n!} + \ldots$
where, here the expansion is about the point z_0 which is the centre of the circle of convergence. The circle of convergence is given as $|z - z_0| = R$, where R is the radius of convergence. McLaurin's series is a special case of Taylor's series where $z_0 = 0$. It is easiest to derive the series with the new origin centered at z_0. For this purpose we can apply the transformation $u = z - z_0$.
- A derivable function of a complex variable is equal to the sum of its Taylor series.
- Laurent's series: The Laurent series expansion provides a series expansion valid within an annular region centered on the singular point. Let $f(z)$ be singular at $z = z_0$ and let c_1 and c_2 be two concentric circles centered on z_0. Then if $f(z)$ is analytic in the annular region, between c_1 and c_2 and c is any concentric circle lying within the annular region

between c_1 and c_2, we can expand $f(x)$ as a Laurent series in the form:

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots = \sum_{n \to -\infty}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{1}{2\pi j} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$: Cauchy integral formula

for $r < |z-z_0| < R$

- The part of the Laurent series that contains negative powers of the variable is called the principal part of the series and the remaining terms constitute what is called the analytic part of the series. If in the principal part the highest power of $1/z$ is n, then the function possesses a pole of order n and if the principal part contains an infinite number of terms, the function possesses an essential singularity. An essential singularity is a singularity that is neither a pole, nor a removable singularity.
- Removable singularities, poles and essential singularities are the three types that constitude the isolated singularities.

- Residues: In the Laurent series:

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

the coefficient a_{-1} is referred to as the residue of $f(z)$. In other words, the residue of a function $f(z)$ is the coefficient of its $\frac{1}{z-a}$ term $(n-1)$, when the function is expanded into its series representation.

- Calculating residues: $a_{-1} = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \right]$

where n is the order of the pole.

- Cauchy's Residue Theorem: Provided $f(z)$ is analytic at all points inside and on the simple closed contour c, apart from the single isolated singularity at z_0 which is interior to c, then $\oint_c f(z)dz = 2\pi j a_{-1}$

The residue theorem extends to the case where the contour contains a finite number of singularities. If $f(z)$ is analytic inside an on the simple closed contour c except at the finite number of points $z_0, z_1, z_2, \ldots$ each with a Laurent series expansion and each with corresponding residues $a_{-1}, a_{-1}, a_{-1}, \ldots$ then $\oint_c f(z)dz = 2\pi j \left\{ a_{-1} + a_{-1} + a_{-1} + \ldots \right\}$

- Evaluation of certain real integrals

The Residue theorem can be very usefully employed to evaluate integrals of real functions.

- Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$

use $z = e^{j\theta}$ and $\cos\theta = \frac{(e^{j\theta} + e^{-j\theta}}{2} = \frac{z + z^{-1}}{2}$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} = z - z^{-1}2j$
and $dz = je^{j\theta}d\theta = jzd\theta$
so that $d\theta = \dfrac{dz}{jz}$. Convert the integral into a contour around the unit circle centered on the origin and use the Residue theorem.

- Integrals of the form: $\int_{-\infty}^{\infty} F(x)dx$ and $\int_{-\infty}^{\infty} F(x)\begin{cases}\sin x\\ \cos x\end{cases} dx$

Consider integrals of the form $\oint F(z)dz$ and $\oint F(z)e^{jz}dz$ respectively, where the contour c is a semicircle with the diameter lying along the real axis. The principle is that the integral can be evaluated by the Residue theorem and then the contour can be expanded to cover the required extent of the real axis, the integration along the semicircle giving a zero contribution.

- An analytic function is a function that is locally given by a convergent power series. There exist both real analytic functions and complex analytic functions, categories that are similar in some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. A function is analytic if and only if its Taylor series about x_0 converges to the function in some neighborhood for every x_0 in its domain.

19.3 Quaternions

Quaternions is a number system that extends the complex numbers. They are generally represented in the form:
$a + bi + cj + dk$
where a, b, c, and d are real numbers, and i, j, and k are the fundamental quaternion units.

- The quaternions are defined by the following equation:

$$i^2 = j^2 = k^2 = ijk = -1$$

20 Numerical Analysis

• Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis. The field of numerical analysis developped as a necessity, because already known things from pure maths were unable to be applied in various branches of problems. So approximations had to be devised.

Newton-Raphson Iterative method

→ First we try to find an approximate root of our real function in question, its x value. Call it x_1.

→ Then we try to find a better approximation of that root, by using: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

→ The algorithm may not always work, especially when $f'(x_0)$ is very small.

→ It is used to find numerical solutions

• Special case for finding $\sqrt{a}$, a : number: $x_{i+1} = 0.5(x_i + a/x_i)$, $i \in \mathbb{N}$ (x_0 is found by bisection)
Modified / Enchanced Newton-Raphson method
When $f'(x_0)$ is very small then the Newton-Raphson method won't work, since x_1 might diverge from the real root. In such a case we calculate instead:
$x_1 = x_0 \pm h$, where $h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$. We choose $+h$ when $f(x_0)$ and $f''(x_0)$ have opposite signs and $-h$ when they are of equal sign.
Afterwards we continue normally with $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Interpolation: A method of constructing new data points within the range of a discrete set of data points.
• Linear, Graphical (or approximate)
From 2 arbitrary points $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ in a figure we want to find out the value y somewhere in between x_1 and x_2, say @ x. We assume a linear dependence between y_1 and y_2. Then:
$y = y_1 + (x - x_1)\dfrac{y_2 - y_1}{x_2 - x_1}$
• Gregory-Newton formula using forward finite differences
$f_p = f_0 + p\Delta f_0 + \dfrac{p(p-1)}{2!}\Delta^2 f_0 + \dfrac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \dots$
• Gregory-Newton formula using backward finite differences
$f_p = f_0 + p\Delta f_{-1} + \dfrac{p(p+1)}{2!}\Delta^2 f_{-2} + \dfrac{p(p+1)(p+2)}{3!}\Delta^3 f_{-3} + \dots$
Gauss interpolation using central finite differences

- Gauss forward formula

$f_p = f_0 + p\delta f_{o+\frac{1}{2}} + \dfrac{p(p-1)}{2!}\delta^2 f_0 + \dfrac{(p+1)p(p-1)}{3!}\delta^3 f_{o+\frac{1}{2}} + \dfrac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0$

- Gauss backward formula

$f_p = f_0 + p\delta f_{o-\frac{1}{2}} + \dfrac{p(p+1)}{2!}\delta^2 f_0 + \dfrac{(p+1)p(p-1)}{3!}\delta^3 f_{o-\frac{1}{2}} + \dfrac{(p+2)(p+1)p(p-1)}{4!}\delta^4 f_0 + \ldots$

in all cases: $h = x_1 - x_0$, $x_1 > x_0$: points that their $f(x_1), f(x_0)$ values are given and are closest to x_p. Also $x_0 < x_p < x_1$ $p = \dfrac{x_p - x_0}{h}$

- Lagrangian Interpolation

The interpolation polynomial of degree n passes through $n+1$ points. To find the value x, not given in the data points $(x_j, f(x_j))$ we follow the formula: $p(x) = \sum_{j=0}^{n} l_j(x) f(x_j)$, where:

$l_j(x) = \prod_{0 \leqslant i \leqslant n-1 \ \ i \neq j} \dfrac{x - x_i}{x_j - x_i}$

Lagrangian interpolation can also be used when the domain points are not equally spaced, sontrary to previous methods.

Frobenius method for solving differential equations of the form:
$y'' + P(x) \cdot y' + Q(x) \cdot y = 0$, under the conditions:
(a) If functions $P(x) \wedge Q(x)$ are such that are both finite when x is put equal to zero, $x = 0$ is called an ordinary point of the equation.
(b) If $x \cdot P(x)$ and $x^2 \cdot Q(x)$ remain finite at $x = 0$, then $x = 0$ is called a regular singular point of the d.eq.
- If P and Q do not satisfy either of the conditions stated in (a), or (b), then $x = 0$ is called an irregular singular point of the d.eq. and the method can't be applied.

→ Solution: We assume a series solution of the form:
$y = x^c(a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots)$, $a_0 \neq 0$.

1. Differentiate to find y' and y''. They will be:
$y' = a_0 c x^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots + a_r(c+r)x^{c+r-1} + \ldots$
$y'' = a_0 c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \ldots + a_r(c+r)(c+r-1)x^{c+r-2} + \ldots$

2. Substitute in the equation.

3. Equate coefficients of corresponding powers of x on each side of the equation - usually written with zero on the R.H.S. and find the recurrence relation whose termin values all of the of the expansions

4. Coefficient of the lowest power of x gives the indicial equation from which the values of c are obtained, $c = c_1$ and $c = c_2$.

   - Case 1: c_1 and c_2 differ by a quantity not an integer.
   Substitute $c = c_1$ and $c = c_2$ in the series for y.

- Case 2: c_1 and c_2 differ by an integer and make a coefficient indeterminate when $c = c_1$. Substitute $c = c_1$ to obtain the complete solution.
- Case 3: c_1 and c_2 ($c_1 < c_2$) differ by an integer and make a coefficient infinite when $c = c_1$. Replace a_0 by $k(c - c_1)$. Two independent solutions are then obtained by putting $c = c_1$ in the new series for y and for $\dfrac{\partial y}{\partial c}$. In general if $c_1 - c_2 = n$ where n is a non zero integer, the solution is of the form:
$y = (1 + k\ln(x))x^{c_1}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c_2}\{b_0 + b_1x + b_2x^2 + \ldots\}$
- Case 4: c_1 and c_2 are equal. Substitute $c = c_1$ in the series for y and for $\dfrac{\partial y}{\partial c}$. Make the substitution after differentiating. In general if $c_1 = c_2 = c$, the solution is of the form: $y = (1 + k\ln(x))x^{c}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c}\{b_1x + b_2x^2 + \ldots\}$

5. For each value of c, consider values of $r = 1, 2, 3, \ldots$ as needed and from the recurrence relation find the constants $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0 or a_1.

6. Final solution is the sum of the series solutions found for the different values of c.

20.0.1 Numerical Solutions to ODEs

Numerical Solutions to ODEs of order 1
[the subindex number is the # of iteration. With 0 we denote the initial conditions]
- Suppose we have the equation: $y' = f(x, y)$, with $y = y_0$ at $x = x_0$, for $x_0 : h = x_n$

1. Euler's Method
$y_1 = y_0 + h \cdot y_0'$ (h is the step value, i.e. $x_{n+1} = x_n + h$)

2. Euler-Cauchy method
$x_1 = x_0 + h$
$\bar{\bar{y}}_1 = y_0 + h \cdot y_0'$: Auxilliary value of y
$y_1 = y_0 + 1/2 \cdot h \left[y_0' + f(x_1, \bar{\bar{y}}_1)\right]$
$y_1' = f(x_1, y_1)$

3. Runge - Kutta method
$x_1 = x_0 + h$
$k_1 = h \cdot f(x_0, y_0) = h \cdot y_0'$
$k_2 = h \cdot f(x_0 + 1/2 \cdot h, y_0 + 1/2 \cdot k_1)$
$k_3 = h \cdot f(x_0 + 1/2 \cdot h, y_0 + 1/2 \cdot k_2)$
$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$
$\Delta y_0 = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$y_1 = y_0 + \Delta y_0$
$y_1' = f(x_1, y_1)$

Numerical Solutions to ODEs of order 2

- Suppose we have the equation: $y'' = f(x, y, y')$, with $y = y_0$ and $y' = y'_0$ at $x = x_0$, for $x = x_0 : h : x_n$.

1. Euler's 2nd order method
   $y_1 = y_0 + h \cdot y'_0 + \dfrac{h^2}{2!} \cdot y''_0$
   $y'_1 = y'_0 + h \cdot y''_0$

2. Ruge-Kutta method
   $x_1 = x_0 + h$
   $k_1 = 1/2 \cdot h^2 \cdot f(x_0, y_0, y'_0) = 1/2 \cdot h^2 \cdot y''_0$
   $k_2 = 1/2 \cdot h^2 \cdot f(x_0 + 1/2h, y_0 + 1/2 \cdot h \cdot y'_0 + 1/4k_1, y'_0 + k_1/h)$
   $k_3 = 1/2 \cdot h^2 \cdot f(x_0 + 1/2h, y_0 + 1/2 \cdot h \cdot y'_0 + 1/4k_1, y'_0 + k_2/h)$
   $k_4 = 1/2 \cdot h^2 \cdot f(x_0 + h, y_0 + h \cdot y'_0 + k_3, y'_0 + 2k_3/h)$
   $P = 1/3 \cdot (k_1 + k_2 + k_3)$
   $Q = 1/3 \cdot (k_1 + 2k_2 + 2k_3 + k_4)$
   $y_1 = y_0 + h \cdot y'_0 + P$
   $y'_1 = y'_0 + Q/h$
   $y''_1 = f(x_1, y_1, y'_1)$

- Predictor-Corrector numerical method for solving D.E.s

Employs a more accurate technique, where, instead of starting with just a single set of initial values, we use a collection of previously calculated values.
Suppose we have the equation: $y' = f(x, y)$, with $y = y_0$ and $y' = y'_0$ at $x = x_0$, for $x : h : x_n$. Then

Predictor: $\begin{cases} \bar{y}_{i+1} = y_i + 1/2 \cdot h(3f(x_i, y_i) - f(x_{i-1}, y_{i-3})) & i = 1, 2, \ldots \\ \bar{y}_1 = y_0 + h \cdot f(x_0, y_0) & i = 0 \end{cases}$

Corrector: $y_{i+1} + 1/2 \cdot h \cdot (f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})) \quad i = 0, 1, 2, 3, \ldots$

20.0.2 Numerical Solutions to PDEs

- Central difference formulas for partial direvatives.

[(i column, j row) h, k : distance between grid points on x, y plane respectively.]

$$\frac{\partial f(x, y)}{\partial x}\Big|_{ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$
$$\frac{\partial f(x, y)}{\partial y}\Big|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

- If $f(x, y)$ is signle valued, then to every domain point (x, y) there corresponds a single range point $f(x, y)$.
- Grid alues: The value of $f(x, y)$ at the ijth grid point is denoted by:

$f_{i,j} \equiv f(x_0 + ih, y_0 + jk)$

- Computational molecules: The P.D.E.: $a\dfrac{\partial f(x,y)}{\partial x} + b\dfrac{\partial f(x,y)}{\partial y} = c$, evaluated at the ijth grid point, is $a\dfrac{\partial f(x,y)}{\partial x}\Big|_{ij} + b\dfrac{\partial f(x,y)}{\partial y}\Big|_{ij} = c$ and is by the central difference formula: $\dfrac{a}{2h}(f_{i+1,j} - f_{i-1,j}) + \dfrac{b}{2k}(f_{i,j+1} - f_{i,j-1}) = c$ which is in turn represented by the composite computational molecule below (figure 30).

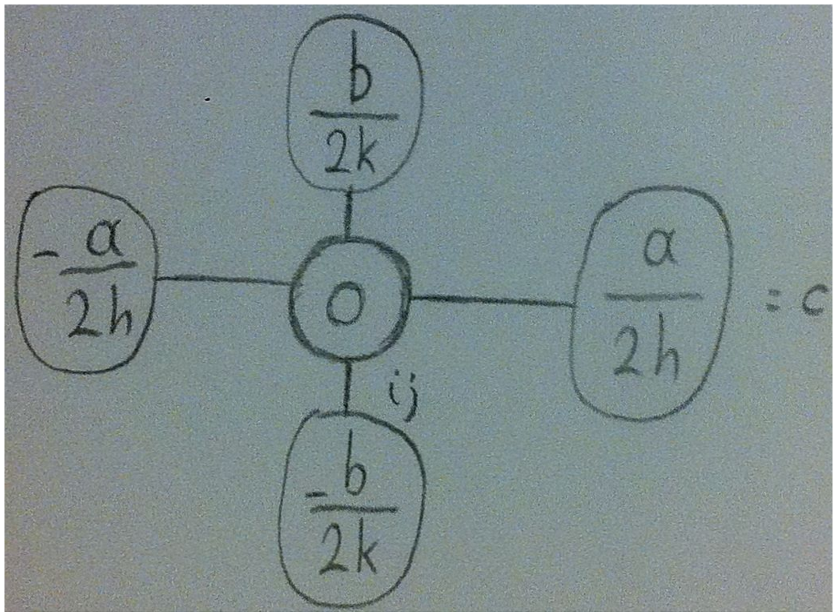


Figure 30: Computational Molecule example

- The procedure to solve a 1st order P.D.E. is:

  1. Draw the function's domain with the grid overlaid.
  2. On the drawing enter the values of $f(x,y)$ that can be obtained from the boundary conditions.
  3. Label the grid points at which $f(x,y)$ is to be evaluated, with capital letters.
  4. Construct the central difference equation that represents the numerical approximation to the P.D.E.
  5. Construct the computational molecule for the P.D.E.
  6. Lay the centre of the molecule on each of the lettered grid points in turn and derive a set of simultaneous linear equations - the unknowns being represented by the letters at the grid points.

7. Write the simultaneous equations in matrix form: $\mathbf{Ax} = \mathbf{b}$
8. Find $\mathbf{A}^{-1}$ and computer the solution: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, finding the $f(x, y)$ values in the lettered grid points.

- If derivative boundary conditions exist, the grid is extended over the boundary of the function domain by adding additional points outside the domain appropriately, with values the value of the aforementioned derivative at the specified boundary point, i.e. @ $\frac{\partial f(x,y)}{\partial x}\Big|_{x=x_0=C}$ (x_0 is b.point and C =value)

- Second Order P.D.E.s: The most general form is:

$$a(x,y)\frac{\partial^2 f}{\partial x^2} + b(x,y)\frac{\partial^2 f}{\partial x \partial y} + c(x,y)\frac{\partial^f}{\partial y^2} + d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y} + g(x,y) = 0$$

- If $b^2 - 4ac < 0$, then the P.D.E. is called an elliptic equation.
- If $b^2 - 4ac > 0$, then the P.D.E. is called a hyperbolic equation.
- If $b^2 - 4ac = 0$, then the P.D.E. is called a parabolic equation.
- Central difference formulas for 2nd partial derivatives:

$\frac{\partial^2 f(x,y)}{\partial x^2}\Big|_{ij} \approx \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}$

$\frac{\partial^2 f(x,y)}{\partial y^2}\Big|_{ij} \approx \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2}$

- Time Dependent Equations: To use a central difference formula for the derivative with respect to t, would require knowledge of $f(x,t)$ for values $t < 0$. Consequently, for a derivative with respect to t we use the forward difference formula: $\frac{\partial f(x,t)}{\partial t}\Big|_{ij} \approx \frac{f_{i,j+1} - f_{i,j}}{k}$

So the P.D.E.: $\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$, becomes $f_{i,j+1} = f_{i,j} + \frac{k}{h^2}\big(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}\big)$, where it can be shown that there will be no growth of rounding errors when evaluating this equation if: $\frac{k}{h^2} \leqslant \frac{1}{2}$

- The Crank - Nicolson procedure makes the assumption that the P.D.E. can be satisfied at points in time halfway between two grid points. That is:

$\frac{\partial^2 f(x,t)}{\partial x^2}\Big|_{i,j+1/2} = \frac{\partial f(x,t)}{\partial t}\Big|_{i,j+1/2}$ This gives:

$\frac{\partial f(x,t)}{\partial t}\Big|_{i,j+1/2} = \frac{f_{i,j+1} - f_{i,j}}{2(k/2)} = \frac{f_{i,j+1} - f_{i,j}}{k}$

$\frac{\partial^2 f(x,t)}{\partial x^2}\Big|_{i,j+1/2} = \frac{1}{2h^2}\big(f_{i-1,j} - 2f_{i,j} + f_{i+1,j} + f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}\big)$

Thus: $-f_{i,j+1} + \frac{k}{2h^2}\big(f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}\big) = -f_{i,j} - \frac{k}{2h^2}\big(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}\big)$

, with no restriction on the value of $\frac{k}{2h^2}$ (usually we make it equal to 1).

21 Vectors

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$: Αντιμεταθετική ιδιότητα
- $(\vec{a}+\vec{b}+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$: Προσεταιριστική ιδιότητα
- $\vec{AB}\leftleftarrows\vec{\Gamma\Delta}\rightarrow\vec{AB},\vec{\Gamma\Delta}$: Ομόρροπα διανύσματα
- $\vec{AB}\leftrightarrows\vec{\Gamma\Delta}\rightarrow\vec{AB},\vec{\Gamma\Delta}$: Αντίρροπα διανύσματα
- $\vec{AB}=-\vec{BA}$
- $\vec{AB}=\vec{OB}-\vec{OA}$ O: σημείο αναφοράς

$\vec{OA},\vec{OB}$: διανυσματικές ακτίνες, ή διανύσματα θέσεως του A και B

- $\left||\vec{a}|-|\vec{b}|\right|\leqslant|\vec{a}+\vec{b}|\leqslant|\vec{a}|+|\vec{b}|$
- $\vec{a}+\vec{0}=\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}$
- $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\mu\vec{a},\ \lambda\in\mathbb{R}$
- $(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a},\ \lambda,\mu,\in\mathbb{R}$
- $\lambda(\mu\vec{a})=(\lambda\mu)\vec{a},\ \lambda,\mu\in\mathbb{R}$
- $\forall\lambda,\mu\in\mathbb{R}\left[(\lambda\vec{a}=\lambda\vec{b}\wedge\lambda\neq 0)\rightarrow\vec{a}=\vec{b}\right]$
- $\forall\lambda,\mu\in\mathbb{R}\left[(\lambda\vec{a}=\mu\vec{a}\wedge\vec{a}\neq 0)\rightarrow\lambda=\mu\right]$
- $\vec{a},\vec{b}$ δύο διανύσματα $,\vec{b}\neq 0\rightarrow\left[\vec{a}\parallel\vec{b}\leftrightarrow\exists\lambda\in\mathbb{R}(\vec{a}=\lambda\vec{b})\right]$
- Διανυσματική ακτίνα μέσου τμήματος: $\vec{AB}:\ \vec{OM}=\dfrac{\vec{OA}+\vec{OB}}{2}$
- Οι συντεταγμένες (x,y) του διανύσματος με άκρα τα σημεία αρχής $A(x_1,y_1)$ και τέλους $B(x_2,y_2)$ δίνονται από τις σχέσεις: $x=x_2-x_1$ & $y=y_2+y_1$
- $\vec{a}=(x,y)\rightarrow|\vec{a}|=\sqrt{x^2+y^2},|\vec{a}|$: μέτρο διανύσματος $\vec{a}$
- Η απόσταση των συμείων $A(x_1,y_1)$ & $B(x_2,y_2)$ είναι ίση με $(AB)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
- Συντεταγμένες κέντρου βάρους τριγώνου $\overset{\triangle}{AB\Delta}$, με $A(x_1,y_1),B(x_2,y_2)$ & $\Gamma(x_3,y_3)$ είναι: $x=\dfrac{x_1+x_2+x_3}{3}$ & $y=\dfrac{y_1+y_2+y_3}{3}$.
- $\vec{a}\parallel\vec{b}\leftrightarrow\det(\vec{a},\vec{b})=0,\ \det(\vec{a},\vec{b})=\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\end{bmatrix},\vec{a}=(x_1,y_1),\vec{b}=(x_2,y_2)$
- $\lambda=\tan(\phi)=\dfrac{y}{x}$: συντελεστής διεύθυνσης του διανύσματος $\vec{a}=(x,y),\phi$: γωνία που σχηματίζει το διάνυσμα $\vec{a}$ με τον άξονα $x'x$.
- $\vec{a}\parallel\vec{b}\leftrightarrow\lambda_1=\lambda_2$: Συνθήκη παραλληλίας δύο διανυσμάτων $\vec{a},\vec{b}$ με συντελεστές διεύθυνσης λ_1,λ_2 αντίστοιχα.
- $\vec{a}=\lambda\vec{b}$. Αν $\vec{a},\vec{b}$ ομόρροπα, τότε $\lambda\geqslant 0$. Αν $\vec{a},\vec{b}$ αντίρροπα τότε $\lambda<0$.
- $\vec{a},\vec{b}\neq 0\rightarrow\left[\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos(\phi)\right]$: Εσωτερικό γινόμενο διανυσμάτων $\vec{a}$ και $\vec{b}$. (Scalar Product)

$\hat{\phi}=$ γωνία που σχηματίζουν τα διανύσματα $\vec{a},\vec{b}$ μεταξύ τους.

- $\vec{a} \perp \vec{b} \leftrightarrow \vec{a} \cdot \vec{b} = 0$
- $\vec{a} \parallel \vec{b} \leftrightarrow \vec{a}\vec{b} = |\vec{a}||\vec{b}|$
- $\vec{a} \leftrightarrows \vec{b} \leftrightarrow \vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$
- $\vec{a} \cdot \vec{a} = \vec{a}^2 = |\vec{a}|^2$
- $\vec{a} = (x_1, y_1) \wedge \vec{b}(x_2, y_2) \to \vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2$: Αναλυτική έκφραση εσωτερικού γινομένου διανυσμάτων $\vec{a}, \vec{b}$.
- $(\lambda\vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda\vec{b}) = \lambda(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot (\vec{b} + \vec{\gamma}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{\gamma}$ Επιμεριστική ιδιότητα
- $(\vec{a}, \vec{b} \nparallel y'y \wedge \vec{a} \perp \vec{b}) \leftrightarrow \lambda_{\vec{a}} \cdot \lambda_{\vec{b}} = -1$
- $\vec{\nu}, \vec{a} \neq \vec{0}$, με κοινή αρχή $\to \vec{a} \cdot \vec{\nu} = \vec{a}\,\text{προβ}_{\vec{a}}\vec{\nu}$ $\qquad$ $\text{προβ}_{\vec{a}}\vec{\nu} = \vec{OM_1}$

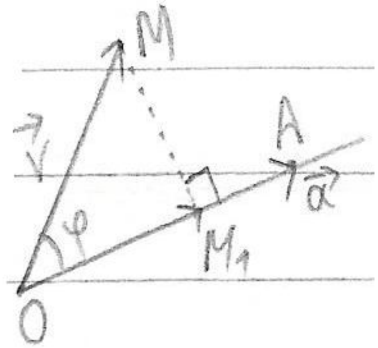


Figure 31: Προβολή διανύσματος

- Εμβαδό τριγώνου: $(A\overset{\triangle}{B}\Gamma) = \dfrac{1}{2}\left|\det\left(\vec{AB}, \vec{A\Gamma}\right)\right|$
- G centroid of $A \overset{triangle}{B} C \to \vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$

- <u>Direction cosines</u>: The direction of a vector in 3 dimensions is determined by the angles which the vector makes with the three axes of reference.

Let $\vec{OP} = a\vec{i} + b\vec{j} + c\vec{k}$ be a vector.

Then: $\dfrac{a}{r} = \cos(a) = l, \qquad \dfrac{b}{r} = \cos(b) = m, \qquad \dfrac{c}{r} = \cos(\gamma) = n$

$a = \angle(i, r), \qquad b = \angle(j, r), \qquad \gamma = \angle(k, r)$

$r = \sqrt{a^2 + b^2 + c^2}, \qquad a, b, c$: 3 direction consines of vector $r = \vec{OP}$.

- <u>Vector/Cross Product</u> of two vectors: It acts in a direction perpendicular to both $\vec{a}$ and $\vec{b}$

$$\therefore \left| \underbrace{\vec{a}}_{\text{(thumb)}} \times \underbrace{\vec{b}}_{\text{(index finger)}} \right| = \underbrace{|a| \cdot |b| \sin\theta}_{\text{(middle finger) (can represent the area of a parallelogram)}}$$

$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$ ($\vec{a}, \vec{b}$ and $\left|\vec{a} \times \vec{b}\right|$ form a right handed set)

$\vec{a} \times \vec{b} = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = -\vec{b} \times \vec{a}$

Cross product represents the signed area of the parallelogram formed by the points, or vectors considered.

- Angle between two vectors. Let $\vec{a}, \vec{b}$ vectors with direction cosines $[l, m, n]$ and $[l', m', n']$ respectively. Then $\cos\theta = \cos\angle(\vec{a}, \vec{b}) = ll' + mm' + nn'$

It can also be found by the scalar product of $\vec{a}, \vec{b}$. • If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ then the area of the formed triangle is: $(A\overset{\triangle}{B}C) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$

- For perpendicular vectors: $ll' + mm' + nn' = 0$.
- For parallel vectors: $ll' + mm' + nn' = 1$.
- $\vec{A} \times (\vec{B} + \vec{C} = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$: Distributive property
- Scalar Triple product of three vectors $\vec{A}(a_x, a_y, a_z), \vec{B}(b_x, b_y, b_z)and\vec{C}(c_x, c_y, c_z)$:

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \left|\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}\right|$

- $\vec{A} \cdot (\vec{B} \times \vec{C} = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$: Unchanged by cyclic change of vectors. Sign reversed by non-cyclic change.
- $\left|\vec{A} \cdot (\vec{B} \times \vec{C})\right| = |\vec{A}| \cdot |\vec{B}| \cdot |\vec{C}| \cdot |\sin(\theta) \cdot \cos(\theta)| \equiv$ volume of the parallelepiped with 3 adjacent sides defined by $\vec{A}, \vec{B}, \vec{C}$.

$\hat{\theta} = \angle(\vec{B}, \vec{C}), \hat{\phi} = \angle(\vec{A}, \vec{n}), |\vec{n}| = 1$

- $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \rightarrow \vec{A}, \vec{B}, \vec{C}$ vectors are coplanar
- Vector Triple Product $\vec{A} \times (\vec{B} \times \vec{C}) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ \begin{bmatrix} b_y & b_z \\ c_y & c_z \end{bmatrix} & \begin{bmatrix} b_z & b_x \\ c_z & c_x \end{bmatrix} & \begin{bmatrix} b_x & b_y \\ c_x & c_y \end{bmatrix} \end{bmatrix}$
- $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$
- $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{C} \cdot \vec{A})\vec{B} - (\vec{C} \cdot \vec{B})\vec{A}$
- Vector Function: $\vec{A}(u) = a_x(u) \cdot \vec{i} + a_y(u)\vec{j} + a_z(u) \cdot \vec{k}$
- $\frac{d\vec{A}}{du} = \frac{da_x}{du}\vec{i} + \frac{da_y}{du}\vec{j} + \frac{da_z}{du}\vec{k}$: Differentiation of vectors
- $T = \frac{dA/du}{|dA/du|}$: Unit (tangent) vector parallel to the tangent to the curve at P figure 32

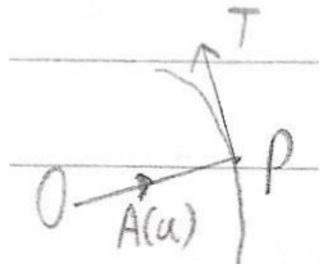


Figure 32: Unit Tangent Vector

- $\int_a^b \vec{A}(u)du = \vec{i}\int_a^b a_x du + \vec{j}\int_a^b a_y du + \vec{k}\int_a^b a_z du$: Integration of vectors
- Most (if not all) standard calculus rules apply in vector functions.
- Transforming vector $\vec{A}$ into unit vector $\hat{a}$:

$\hat{a} = \dfrac{a_x\vec{i} + a_y\vec{j} + a_z\vec{k}}{\left|\vec{A}\right|} = \dfrac{\vec{A}}{\left|\vec{A}\right|}$, with the direction of the original vector $\vec{A}$

- $d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$
- $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$
- $\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$
- $\int_c \vec{F} \cdot d\vec{r} = \int_c F_x dx + \int_c F_y dy + \int_c F_z dz$

21.1 Vector Analysis

- If every point $P(x, y, z)$ of a region R of space has associated with it, a scalar quantity $\phi(x, y, z)$, then $\phi(x, y, z)$ is a scalar function and a scalar field is said to exist in the region R.
- Similarly, if every point $P(x, y, z)$ of a region R has associated with it a vector quantity $\vec{F}(x, y, z)$, then $\vec{F}(x, y, z)$ is a vector function and a vector field is said to exist in the region R.
- Gradient of a scalar function $\phi(x, y, z)$

grad $(\phi) = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = \left\{\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right\}\phi = \nabla\phi$

where $\nabla \equiv \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)$: vector differential operator, or del, or nabla operator and ϕ is continuously differentiable with respect to its variables x, y, z, throughout the region R.

The gradient is a generalization of the usual concept of derivative to functions of several variables. If $f(x_1, ..., x_n)$ is a differentiable, real-valued function of several variables, its gradient is the vector whose components are the n partial derivatives of f. It is thus a vector-valued function.

Similarly to the usual derivative, the gradient represents the slope of the tangent of the graph of the function. More precisely, the gradient points in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the graph in that direction, ie. the direction of grad (ϕ) gives the direction in which the maximum rate of change of ϕ occurs.

- Total differential of $\phi(x, y, z)$:

grad $(\phi) \cdot d\vec{r} = \left(\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}\right) \cdot \left(dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k}\right) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$

Gradient Properties

- $\nabla(A + B) = \nabla A + \nabla B$: Grad of Sums
- $\nabla(A \cdot B) = A(\nabla B) + B(\nabla A)$: Grad of Products

- Directional Derivative $\frac{d\phi}{ds} = \hat{a}$ grad $(\phi) = \hat{a}\nabla\phi$

where $\hat{a}$ is a unit vector in a stated direction. It gives the rate of change of ϕ with distance measured in the direction of $\hat{a}$.

- Unit normal vector N to surface: $\phi(x, y, z) =$ constant, $\vec{N} = \frac{\nabla\phi}{|\nabla\phi|}$ (at a point $P(x, y, z)$)

- Divergence (div) of a vector function $\vec{A}$:

div $(\vec{A}) = \nabla \cdot \vec{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$ (notice the dot $\cdot$ in ∇)

Divergence is a vector operator that produces a signed scalar field giving the quantity of a vector field's source at each point. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

As an example, consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air is cooled and thus contracting, the divergence of the velocity has a negative value.

- If $\nabla \cdot \vec{A} = 0$ for all points, then A is called a solenoidal vector.

- Curl of a vector function $\vec{A}$:

$$\text{curl } (A) = \nabla \times \vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{bmatrix}$$

The curl is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl of that point is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point. The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.

- If $\nabla \times \vec{A} = 0$ then the vector field $\vec{A}$ is said to be irrotational.

- Multiple Vector Operations

  - curl grad $\phi = \nabla \times (\nabla\phi) = 0$
  - div curl $\vec{A} = \nabla \cdot (\nabla \times \vec{A}) = 0$
  - div grad $\phi = \nabla \cdot (\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = \partial^2\phi = \Delta$: Laplacian of ϕ
  - $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$

- Surface Integrals

We note that a surface is defined by $\phi(x, y, z,) =$ constant

a) Scalar field $V(x, y, z)$:

$\displaystyle\int_s V d\vec{s} = \int_s V\hat{n}ds$, where $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

b) Vector field $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$

$\displaystyle\int_s \vec{F} \cdot d\vec{s} = \int_s \vec{F} \cdot \hat{n}ds$,

where $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$ unit normal vector

- A vector field $\vec{F}$ is conservative if

  1. $\displaystyle\oint_c \vec{F} \cdot d\vec{r} = 0$, for all closed curves
  2. curl $\vec{F} = 0$
  3. $\vec{F} =$ grad V

The line integral of a conservative vector field is independent of the path of integration between the two end points.

- Harmonic function: is a twice continuously differentiable function $f(x, y, z, ...)$ that satisfies Laplace's equation.
- A smooth function is a function that has derivatives of all orders everywhere in its domain. The smoothness of a function is a property measured by the number of derivatives it has, which are continuous.

- Divergence (Gauss's) Theorem

For a closed surface S enclosing a region V in a vector field $\vec{F}$: $\displaystyle\int_v \text{div } \vec{F}dV = \int_S \vec{F} \cdot d\vec{S}$

for a closed surface the normal vectors at all points are drawn in an outward direction.

- Stoke's Theorem

Let an open surface S bounded by a simple closed curve c. Then $\int_S \text{curl } \vec{F} \cdot d\vec{S} = \oint_c \vec{F} \cdot d\vec{r}$

- The unit normal $\hat{n}$ is drawn in a right - handed screw sense (common right hand rule).

(convention)

- Sign convention for the surfaces is represented by the image below (figure 33)

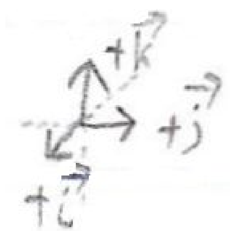


Figure 33: Sign convention for surfaces

22 Determinants

- Considering the equations:

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \\ a_3x + b_3y + c_3z + d_3 = 0 \end{cases}$$,we can write them in determinant form:

$$\frac{x}{\begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}} = \frac{-y}{\begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}} = \frac{z}{\begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}} = \frac{-1}{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}$$

and thus find any unknown. We can extend this method for any number of equations and unknowns, noting the alternating "+", "-" signs. The above can also be written as:

$$\frac{x}{\Delta_1} = -\frac{y}{\Delta_2} = \frac{z}{\Delta_3} = -\frac{1}{\Delta_0},$$

where:

Δ_1 : the determinant of coefficients omitting the x-terms
Δ_2 : the determinant of coefficients omitting the y-terms
Δ_3 : the determinant of coefficients omitting the z-terms
Δ_0 : the determinant of coefficients omitting the constant terms

- $\Delta = 0 \leftrightarrow$ System of equations is consistent.

for example:

$$a_1\Delta_x + b_1\Delta_y + c_1\Delta_0 = 0 \rightarrow a_1 \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} + b_1 \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + c_1 \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = 0 \rightarrow \Delta = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

- <u>Properties of Determinants</u>

They can be applied to determinants of any order.

1. The value of a determinant remains unchanged if rows are changed to columns and columns to rows
$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

2. If two rows (or two columns) are interchanged, the sign of the determinant is changed.
$$\begin{bmatrix} a_2 & b_2 \\ a_1 & b_1 \end{bmatrix} = - \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

3. If two rows (or two columns) are identical, the value of the determinant is zero.
$$\begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix} = 0$$

4. If the determinants of any one row (or column) are all multiplied by a common factor, the determinant is multiplied by that factor

$$\begin{bmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{bmatrix} = k \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

5. If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.
   $\begin{bmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $\begin{bmatrix} a_1 & b_1 \\ a_1 + ka_1 & b_2 + kb_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

23 Matrices

- A matrix is a set of elements arranged in rows and columns to form a rectangular array. A matrix is simply an array of numbers. There is no arithmetical connection between the elements.
- To add or subtract two matrices, they must be of the same order $i \times j$.
- $k(a_{ij}) = (ka_{ij})$, k : scalar, a_{ij} = matrix $i \times j$
- Two matrices can be multiplied together only when the number of columns of the first matrix is equal to the number of rows of the second matrix.
- A^T : Transpose of amtrix A : The rows and columns are interchanged.
- Square matrix is a matrix of order $m \times m$.
- A square matrix is symmetrix iff $a_{ij} = a_{ji}$.
- A square matrix is skew-symmetric iff $a_{ij} = -a_{ji}$ αντισυμμετρικός.
- Diagonal matrix is a square matrix with all elements zero except those on the leading diagonal.
- Unit matrix is a diagonal matrix with all its elements equal to unity on the leading diagonal. It is denoted with **I**. The unit matrix behaves much like the unit factor in ordinary algebra and arithmetic.
- If $\mathbf{A} = (a_{ij})$ is a square matrix, we can form a determinant of its elements. Each element a_{ij} of a square matrix $\mathbf{A}$ has a corresponding cofactor A_{ij} that is $(-1)^{i \pm j}$ times the determinant of the matrix formed by deleting the i-th row and j-th column from $\mathbf{A}$.

e.g. $|A| = \det(A) = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix} = 45$, when $\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$

The minor of 2 is $\begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix} = -24$. Place sign is $+$. Therefore the cofactor of the element 2 is $+1 \cdot (-24) = -24$.

- To form the inverse of a square matrix $\mathbf{A}$:

1. Evaluate the determinant of $\mathbf{A}$, i.e. $\mathbf{A} = \det(A)$
2. Form the matrix $\mathbf{C}$ of the cofactors of $|\mathbf{A}|$.
3. Write the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^T$, to obtain the adjoint matrix of $\mathbf{A}$.
4. Divide each element of $\mathbf{C}^T$ by $\det(A)$. The resulting matrix is the inverse $\mathbf{A}^{-1}$ of the original matrix $\mathbf{A}$.

- $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = 1$
- Validation: $\mathbf{A} \times \text{adj }(A) = |\mathbf{A}| \times \mathbf{I}$ must be true.
- Solution of a set of n linear equations with n unknowns.

$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \rightarrow \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} \rightarrow \mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$

$\mathbf{b}$: the matrix of the constant terms.

- Gaussian elimination method for solving systems of equations

Suppose we have the following equations:

$$x_1 + 2x_2 - 3x_3 = 3$$

$$2x_1 - x_2 - x_3 = 11$$

$$3x_1 + 2x_2 + x_3 = -5$$

Our goal is to modify the matrix of coefficients $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ in order to look like an upper triangular matrix.

1. Step 1: We form the augmented matrix: $\begin{bmatrix} 1 & 2 & -3 & | & 3 \\ 2 & -1 & -1 & | & 11 \\ 3 & 2 & 1 & | & -5 \end{bmatrix}$

2. Step 2: We use any elementary matrix operations at our disposal, in order to form an upper triangular matrix.

3. Step 3:Starting from the bottom equation we find one unknown, as it is immediately given to us, x_n. We proceed to the exact upper row to find the other unknown x_{n-1} etc.

Notes
- These operations are permissible since we are dealing with the coefficients of both sides of the equations.

- Eigenvalues

In equations of the form $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\big(\mathbf{A} = (a_{ij}) :$ square matrix, $\mathbf{x} :$ column matrix $\big)$ we do: $\big(\mathbf{A} - \lambda\mathbf{I}\big)\mathbf{x} = 0 :$ For this set of homogeneous linear equations (i.e. right hand constants are all zero) to have a non-trivial solution, $|\mathbf{A} - \lambda\mathbf{I}|$ must be 0. $|\mathbf{A} - \lambda\mathbf{I}|$ is the characteristic determinant of $\mathbf{A}$ and $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the characteristic equation. On expanding the determinant, this gives a polynomial of degree n and the solution of the characteristic equation gives the values of λ, i.e. the eigenvalues of $\mathbf{A}$.

- Eigenvectors: Substitution of each eigenvalue λ to $|\mathbf{A} - \lambda\mathbf{I}|\,\mathbf{x} = 0$ gives rise to a corresponding eigenvector, in the form of $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \beta \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \beta :$ constant. In matrices the term "vector" indicates a row matrix, or a column matrix. We usually pick $B = 1$ to obtain the most simple eigenvector. • A square matrix is called singular if and only if its determinant is zero. Such a matrix is not invertible. Otherwise, the matrix is non-singular

- The rank of an $n \times m$ matrix $\mathbf{A}$ is the order of the largest square, non-singular sub-matrix. That is, the largest square sub-matrix whose determinant is non-zero. If $n = m$, so making $\mathbf{A}$ itself square, then this sub-matrix could be the matrix $\mathbf{A}$ itself.

- A set of n simultaneous equations in n unknowns is consistent if the rank of the coefficient matrix $\mathbf{A}$ is equal to the rank of the augmented matrix $\mathbf{A}_b$. In particular if the rank of both $\mathbf{A}$ and $\mathbf{A}_b$ is equal to n then a unique solution exists. Else, if the rank of $\mathbf{A}$ and of $\mathbf{A}_b$ is equal to m, where $m < n$, then there will be an infinite number of solutions for the equations. If their rank $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}_b)$ then no solution exists.
- A matrix $\mathbf{A}$ is invertible, iff $|\mathbf{A}| \neq 0$.
- Two matrices, $\mathbf{A}$ and $\mathbf{B}$, are said to be <u>equivalent</u> if $\mathbf{B}$ can be obtained from $\mathbf{A}$ by a sequence of elementary transformations.
- A <u>combines coefficient matrix</u> contains the coefficients of the corresponding unknowns from both sides of an equation.
- <u>Elementary matrix operations</u>:

  1. Intercharging two rows,
  2. Multiplying each element of a row by the same non-zero scalar quantity.
  3. Adding, or subtracting corresponding elements from those of another row.

Eigenvectors make understanding linear transformations easy. They are the "axes" (directions) along which a linear transformation acts simply by "stretching/compressing" and/or "flipping"; eigenvalues give you the factors by which this compression occurs.

<u>Matrix Transformations</u>

- $\mathbf{U} = \mathbf{T} \cdot \mathbf{X}$, where $\mathbf{T}$ is a transformation matrix, which transforms a vector in the $x - y$ plane to a corresponding vector in the $u - v$ plane. Similarly, $\mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$ performs the inverse transformation.

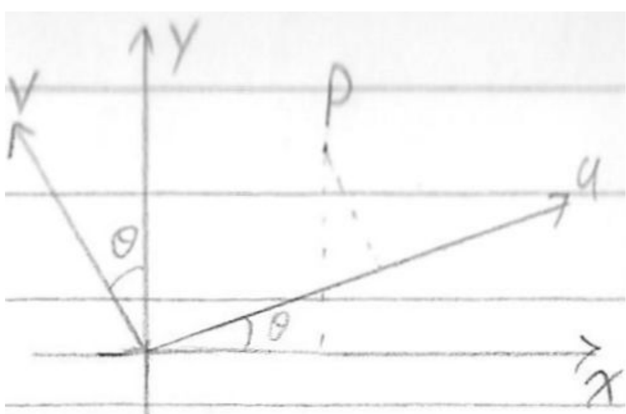


Figure 34: Rotation of axes

- Rotation of axes

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

- Cayley-Hamilton theorem: Every square matrix satisfies its own characteristic equation.
- Modal Matrix: If the n eigenvectors $\mathbf{x}_i$ of a square matrix $\mathbf{A}$ are arranged as columns, the modal matrix of $\mathbf{A}$, denoted by $\mathbf{M}$, is formed. i.e. $\mathbf{M} = \big(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\big)$.
- Spectral Matrix: A diagonal matrix with the eigenvalues only on the main diagonal, denoted by $\mathbf{S}$. This process is called diagonalisation.
- The relationship between $\mathbf{M}$ and $\mathbf{S}$ is: $\mathbf{M}^{-1} \cdot \mathbf{A} \cdot \mathbf{M} = \mathbf{S}$.
- Considering two non-equal matrices $\mathbf{A}, \mathbf{B}$. If $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ we say that the two matrices commute.

- A square matrix that is not invertible is called singular, or degenerate. A square matrix is singular if and only if its determinant is 0. Singular matrices are rare in the sense that a square matrix randomly selected from a continuous uniform distribution on its entries will almost never be singular.
- An orthogonal matrix is a square matrix with real entries whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors), i.e.: $Q^T Q = Q Q^T = I$, where I is the identity matrix. This means that a matrix is orthogonal if its transpose is equal to its inverse.

24 Coordinate Systems

24.1 2-D Coordinate Systems

1. Cartesian Coordinate System: The standard orthogonal 2-dimentional coordinate system, with (x, y) coordinates.

2. Polar Coordinate System: A 2-dimentional coordinate system (r, θ) figure 35 where r = radius, θ = azimuth
$x = r\cos\theta$ $y = r\sin\theta$ $r = \sqrt{x^2 + y^2}$ $\tan\theta = \dfrac{y}{x}$

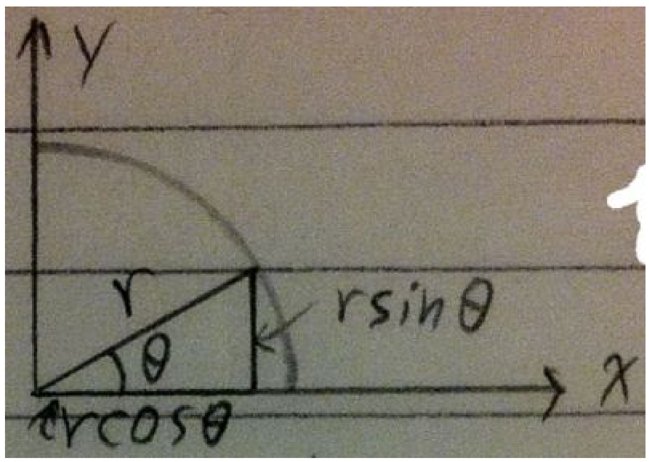


Figure 35: Representation of the Polar (planar) Coordinate System

24.2 3-D Coordinate Systems

1. Cartesian Coordinates (x, y, z) (figure 36)
First octant: $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$

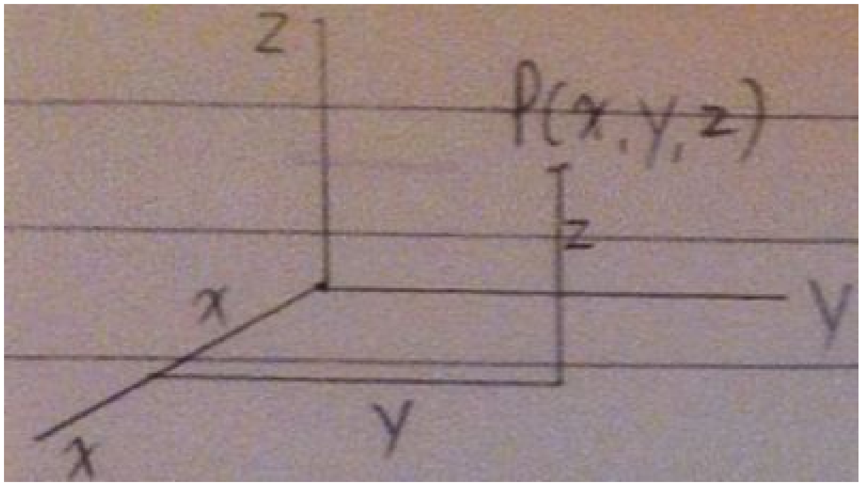


Figure 36: 3-Dimensional Coordinate System

2. Cylindrical Coordinates (r, θ, z) , $r \geqslant 0$ (figure 37)
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$
$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(\dfrac{y}{x})$ $z = z$
Cylindrical coordinates are useful when an axis of symmetry occurs.

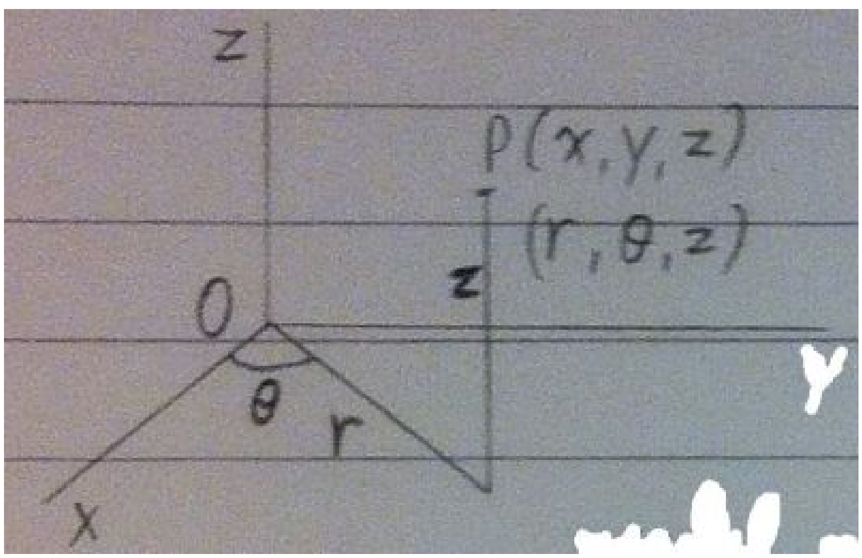


Figure 37: Cylidrical Coordinate System

3. Spherical Coordinates (r, θ, ϕ) , $r \geqslant 0$ (figure 38)
$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \cos^{-1}\left(\dfrac{z}{r}\right)$ $\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$
Spherical coordinates are useful where a centre of symmetry occurs.

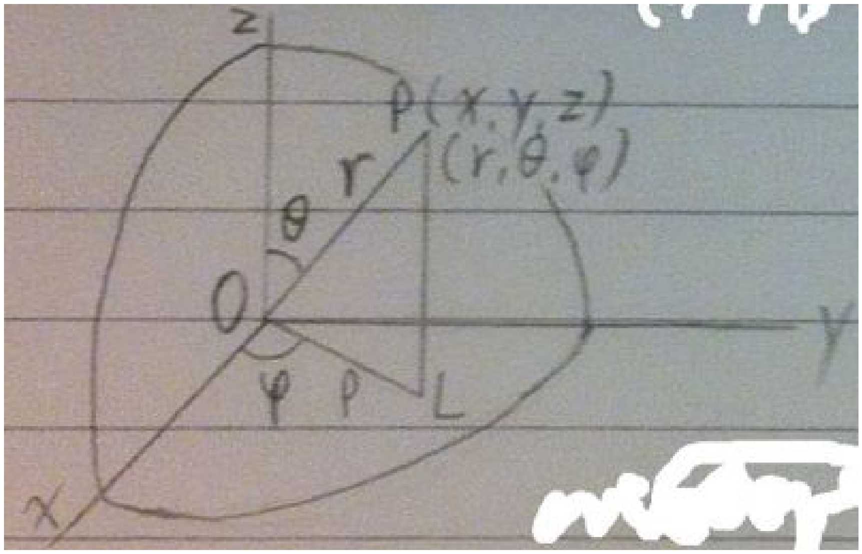


Figure 38: Spherical Coordinate System

24.3 Moment of Area

Element of Area in Polar Coordinates: $\delta a = r\delta r\delta\theta$ (figure 39)

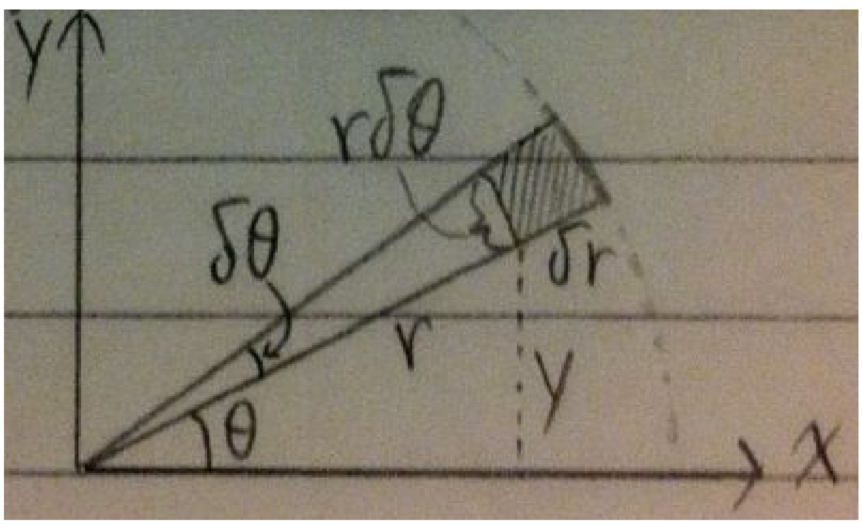


Figure 39: Element of Area in Polar Coordinates

24.4 Element of Volume

1. Cartesian coordinates: $\delta v = \delta x\delta y\delta z$
(figure 40)

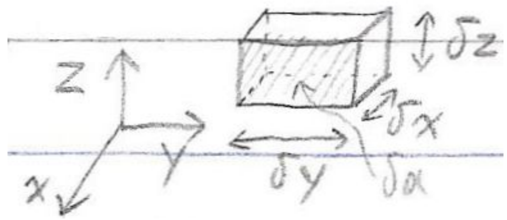


Figure 40: Element of Volume in Cartesian Coordinates

2. Cylindrical coordinates: $\delta v = r\delta\theta\delta r\delta z = r\delta r\delta\theta\delta z$
(figure 41)

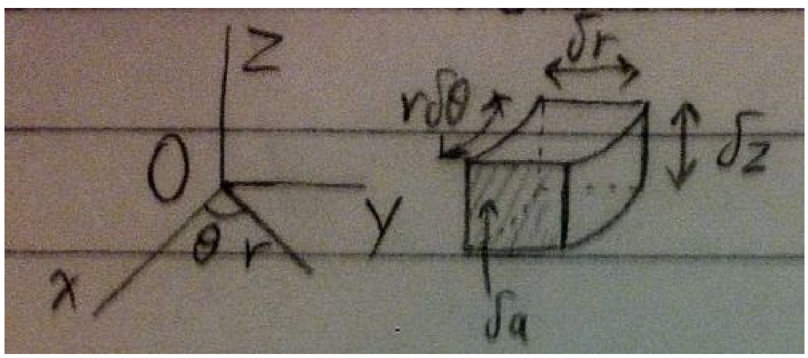


Figure 41: Element of Volume in Cylidrical Coordinates

3. Spherical coordinates: $\delta v = \delta v r\delta\theta \; r\sin\phi\delta\phi = r^2\sin\theta\delta r\delta\theta\delta\phi$ (figure 42)

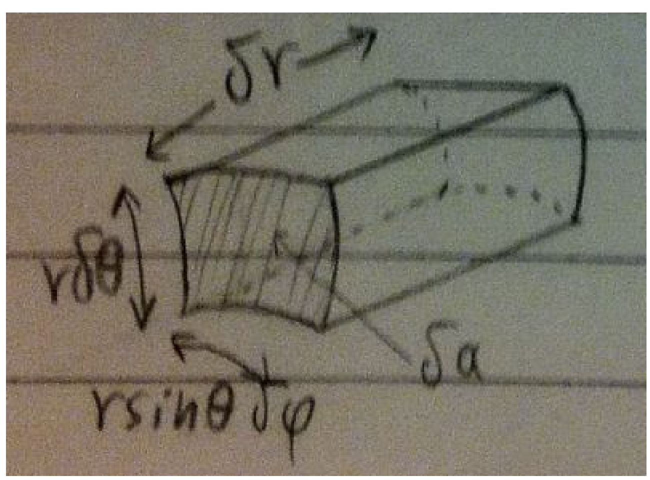


Figure 42: Element of Volume in Spherical Coordinates

[δa = hedra]

24.5 Element of Area in Space

1. Cartesian coordinates: $\delta a = \delta x \delta y$
2. Cylindrical coordinates: $\delta a = r\delta\theta\delta z$
3. Spherical coordinates: $\delta a = r^2 \sin\theta\delta\theta\delta\phi$

24.6 Curvilinear Coordinates

- Curvilinear coordinates is a coordinate system for Euclidean space in which the coordinate lines may be curves, as well as the coordinate surfaces. These coordinates may be derived from a set of Cartesian coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. As such, in the general case (3D):
$u = f(x, y, z)$, $v = g(x, y, z,)$, $w = h(x, y, z)$
- If the coordinate curves for u and v forming the network cross at right angles, the system of coordinates is said to be orthogonal. That is, if $\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$ then u and v are orthogonal.

- Orthogonal Coordinate system in space (Curvilinear)

a) Cartesian Rectangular Coordinates (x, y, z)
$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$, Scale factors: $h_x = h_y = h_z = 1$
b) Cylindrical Polar Coordinates (r, θ, z)
$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j} + z\vec{k}$

Base unit vectors

$$\vec{I} = \frac{\partial \vec{r}}{\partial r} / \left|\frac{\partial \vec{r}}{\partial r}\right|$$

$$\vec{J} = \frac{\partial \vec{r}}{\partial \theta} / \left|\frac{\partial \vec{r}}{\partial \theta}\right|$$

$$\vec{K} = \frac{\partial \vec{r}}{\partial z} / \left|\frac{\partial \vec{r}}{\partial z}\right|$$

Scale factors

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = 1$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r$$

$$h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$$

$$\vec{F} = F_r\vec{I} + F_\theta\vec{J} + F_z\vec{K}$$

c) Spherical Polar Coordinates (r, θ, ϕ)
$\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\sin\phi\vec{j} + r\cos\theta\vec{k}$

Base unit vectors
$\vec{I} = \dfrac{\partial\vec{r}}{\partial r} \Big/ \left|\dfrac{\partial\vec{r}}{\partial r}\right|$
$\vec{J} = \dfrac{\partial\vec{r}}{\partial\theta} \Big/ \left|\dfrac{\partial\vec{r}}{\partial\theta}\right|$
$\vec{K} = \dfrac{\partial\vec{r}}{\partial\phi} \Big/ \left|\dfrac{\partial\vec{r}}{\partial\phi}\right|$

Scale factors
$h_r = \left|\dfrac{\partial\vec{r}}{\partial r}\right| = 1$
$h_\theta = \left|\dfrac{\partial\vec{r}}{\partial\theta}\right| = r$
$h_\phi = \left|\dfrac{\partial\vec{r}}{\partial\phi}\right| = r\sin\theta$

$$\vec{F} = F_r\vec{I} + F_\theta\vec{J} + F_\phi\vec{K}$$

- <u>General Orthogonal Curvilinear coordinates (u, v, w)</u>

$x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$
$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
$\dfrac{\partial\vec{r}}{\partial u} = h_u\vec{I}$, where $h_u = \left|\dfrac{\partial\vec{r}}{\partial u}\right|$
$\left|\dfrac{\partial\vec{r}}{\partial v}\right| = h_v\vec{J}$, where $h_v = \left|\dfrac{\partial\vec{r}}{\partial v}\right|$
$\left|\dfrac{\partial\vec{r}}{\partial w}\right| = h_w\vec{K}$, where $h_w = \left|\dfrac{\partial\vec{r}}{\partial w}\right|$

- Element of arc: $ds = \left(h_u^2du^2 + h_v^2dv^2 + h_w^2dw^2\right)^{\frac{1}{2}}$
- Element of volume: $dV = h_uh_vh_wdudvdw = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}dudvdw$

25 Transforms

25.1 Laplace Transforms

- $\mathcal{L}\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)dt = F(s)$: Definition (single sided)

where $e^{-st}f(t)$ must converge as $t \to \infty$, $s = \sigma + j\omega$complex frequency , $\sigma > 0$ and $f(t)$ is a continuous, or piecewise continuous, function.
- $f(t) = \mathcal{L}^{-1}\{F(s)\}$: Inverse Laplace transform
- $s = \sigma + j\omega$: Complex frequency. σ : represents the transient component, ω : represents the steady state component

Properties
- Both the LaPlace transform and its inverse are linear transforms. This means that:

1. $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$
   $\mathcal{L}^{-1}\{F(s) \pm G(s)\} = \mathcal{L}^{-1}\{F(s)\} \pm \mathcal{L}^{-1}G(s)\}$
2. $\mathcal{L}\{kf(t)\} = k\mathcal{L}\{f(t)\}$
   $\mathcal{L}^{-1}\{kF(s)\} = k\mathcal{L}^{-1}\{F(s)\}$
3. Laplacian of a derivative: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
4. Laplacian of higher derivatives:
   $\mathcal{L}\{f^{(\nu)}(t)\} = s^{\nu}F(s) - s^{\nu-1}f(0) - s^{\nu-2}f'(0) - s^{\nu-3}f''(0) - \ldots - f^{(\nu-1)}(0)$
5. ${}^{(\nu)}(s) = (-1)^n\mathcal{L}\{t^n f(t)\}$
6. $F(s-k) = \mathcal{L}\{e^{kt}f(t)\}$
7. $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$ (1st Shift Theorem)
8. $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$ (Multiplication by t^n)
9. $\mathcal{L}\{\frac{f(t)}{t}\} = \int_{\sigma=s}^{\infty} F(\sigma)d\sigma$, provided that $\exists \lim_{t\to 0}\left(\frac{f(t)}{t}\right)$
10. $\mathcal{L}\{u(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$, $c \in \mathbb{R}$: 2nd Shift Theorem, where $F(s) = \mathcal{L}\{f(t)\}$
11. $\mathcal{L}\{f(t) * g(t)\} = F(s)\cdot G(s) = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$: Convolution theorem
12. $\mathcal{L}\{\bar{f}(t)\} = \frac{1}{1-e^{-s\mathrm{T}}}F(s)$
    where $F(s) = \int_{0\ (\mathrm{or} -\mathrm{T}/2)}^{\mathrm{T}(\mathrm{or}\mathrm{T}/2)} e^{-st}f(t)dt$
    $\bar{f}(t)$ is a periodic function, with period T.

13. $\mathcal{L}\{f(t) \cdot \delta(t-c)\} = f(c)e^{-cs}$

14. To find inverse transforms involving periodic functions, expand the $(1 - e^{-cs})$ term in the denominator as the binomial series: $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

15. $\displaystyle\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(s)}{s}$

16. $f(t) \cdot g(t) \leftrightarrow F(s) * G(s)$

17. $f(\dfrac{t}{a})u(t) \leftrightarrow aF(as)\ ,\ \forall a \in \mathbb{R}^+$

18. $f(\dfrac{t}{a} - b)u(t) \leftrightarrow ae^{-sab}F(as)\ ,\ \forall a \in \mathbb{R}^+$

19. $f(0_+) = \lim\limits_{s\to\infty}[sF(s)]$: Initial Value Theorem, where $f(t)$ is a one-sided function

20. $\lim\limits_{t\to\infty}[f(t)] = \lim\limits_{s\to 0}[sF(s)]$: Final Value theorem,
provided that the $\lim\limits_{t\to\infty}[f(t)]$ exists, i.e. $f(t)$ has a final value.

- Using the LaPlace transform we can solve equations of the form:

$a_n f^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \ldots a_2 f''(t) + a_1 f'(t) + a_0 f(t) = g(t)$, where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are known constants, $g(t)$ is a known expression of t and the values of $f(t)$ and its derivatives are known at $t = 0$.
This type of equation is called a linear, constant-coefficient, inhomogeneous differential equation and the values of $f(t)$ and its derivatives are called boundary conditions. The method to find the solution is:
a) Take the LaPlace Transform of both sides.
b) Find the expression $F(s) = \mathcal{L}\{f(t)\}$ in the form of an algebraic fraction.
c) Separate $F(s)$ into its partial fractions.
d) Find $\mathcal{L}^{-1}\{F(s)\}$ to find the solution $f(t)$.

Table of LaPlace Transforms

$$
\begin{bmatrix}
f(t) & \mathcal{L}\{f(t)\} = F(s) \\
a & \dfrac{a}{s} \ , \ s > 0 \\
e^{-at}u(t) & \dfrac{1}{s+a} \ , \ s > -a \\
t^n u(t) & \dfrac{n!}{s^{n+1}} \ , \ n \in \mathbb{N}^* \\
\sin(at) \cdot u(t) & \dfrac{a}{s^2+a^2} \ , \ s > 0 \\
\cos(at) \cdot u(t) & \dfrac{s}{s^2+a^2} \ , \ s > 0 \\
\sinh(at) \cdot u(t) & \dfrac{a}{s^2-a^2} \ , \ s > |a| \\
\cosh(at) \cdot u(t) & \dfrac{s}{s^2-a^2} \ , \ s > |a| \\
u(t-c) & \dfrac{e^{-cs}}{s} \\
\delta(t-c) & e^{-cs} \\
\ln\left(\dfrac{t}{\mathrm{T}}\right) u(t) & -\dfrac{\mathrm{T}}{s}\left|\ln\left(\mathrm{T}s\right) + \gamma\right| \\
f(t)\sin(ct)u(t) & \dfrac{1}{2j}\left[F(s-jc) - F(s+jc)\right] \\
f(t)\cos(ct)u(t) & \dfrac{1}{2}\left[F(s-jc) + F(s+jc)\right] \\
a^{ct} \cdot u(t) & \dfrac{1}{s - c\ln\left(a\right)} \ , \ s > c \\
ln(a) & \\
r(t-T) = (t-T)u(t-T) & \dfrac{1}{s^2} \cdot e^{-Ts}
\end{bmatrix}
$$

25.1.1 Convolution

- $c(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$ (flip and slide)

Properties

1. $f(t) * h(t) = h(t) * f(t)$
2. $[f(t) * h(t)] * c(t) = f(t) * [h(t) * c(t)]$
3. $f(t) * [h(t) + c(t)] = f(t) * h(t) + f(t) * c(t)$
4. $f(t) * \delta(t) = f(t)$
5. $f(t) * \delta(t - t_0)) = f(t - t_0)$

- The convolution of two functions $x(t)$, $h(t)$ is obtained by:

1. Changing variable t to the dummy variable T.
2. Reversing one of them, say $h(T)$ to form $h(-T$ [if it isn't already in the form $h(-T)$].
3. Shifting $h(-T)$ by t units to the left, $h(t-T)$.
4. Taking the product of $x(t)$ and $h(t-T)$ and integrating with respect to T.
   $t \Rightarrow T \Rightarrow t \Rightarrow T$

25.2 Z Transform

- $Z\{f[n]\} = F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$, $n \in \mathbb{Z}$: Definition (bilateral)

$f[n]\,(= f(n))$ is a discrete function.

Properties

1. $Z\{af[n] + bg[n]\} = aZ\{f[n]\} + bZ\{g[n]\}$: Linearity
2. $F(z) = Z\{f[n]\} \to Z\{f[n+m]\} = z^m F(z) - \left[z^m f[0] + z^{m-1} f[1] + \ldots + z f[m-1]\right]$: Shifting Left
3. $F(z) = Z\{f[n]\} \to Z\{f[n-m]\} = z^{-m} F(z)$: Shifting Right
4. $F(z) = Z\{f[n]\} \to Z\{a^n f[n]\} = F\left(\frac{z}{a}\right)$: Translation
5. $\lim_{n\to\infty} f[n] = \lim_{z\to 1} \left\{ \left(\frac{z-1}{z}\right) F(z) \right\}$: Final Value Theorem, provided that $\lim_{n\to\infty} f[n]$ exists.
6. $f(0) = \lim_{z\to\infty} \{F(z)\}$: Initial Value Theorem
7. $F(z) = Z\{f[n]\} \to Z\{nf[n]\} = -zF'(z)$: Derivative of the transform

Table of Z Transforms

$$\begin{bmatrix} f[n] & F(z) \quad R.O.C. \\ \delta[n] & 1 \\ u[n] & \dfrac{z}{z-1} \ , \ |z|>1 \\ nu[n] & \dfrac{z}{(z-1)^2} \ , \ |z|>1 \\ n^2u[n] & \dfrac{z(z+1)}{(z-1)^3} \ , \ |z|>1 \\ n^3u[n] & \dfrac{z(z^2+4z+1)}{(z-1)^4} \ , \ |z|>1 \\ a^nu[n] & \dfrac{z}{z-a} \ , \ |z|>|a| \\ na^nu[n] & \dfrac{az}{(z-a)^2} \ , \ |z|>|a| \\ \delta[n-c] & z^{-cn} \\ e^{-an}\cdot u[n] & \dfrac{z}{z-e^{-a}} \ , \ |z|>e^{-a} \\ \sin[an]\cdot u[n] & \dfrac{\sin(a)\cdot z}{z^2-2\cos(a)\cdot z+1} \ , \ |z|>1 \\ b^n\sin[an]\cdot u[n] & \dfrac{b\sin(a)\cdot z}{z^2-2b\cos(a)\cdot z+b^2} \ , \ |z|>b \\ \cos[an]\cdot u[n] & \dfrac{z(z-\cos(a))}{z^2-2\cos(a)\cdot z+1} \ , \ |z|>1 \\ b^n\cdot\cos[an]\cdot u[n] & \dfrac{z(z-b\cos(a))}{z^2-2b\cos(a)\cdot z+b^2} \ , \ |z|>b \\ Cu[n] \ , \ C\in\mathbb{C} & \dfrac{Cz}{z-1} \ , \ |z|>1 \end{bmatrix}$$

25.2.1 Sampling

If a continuous function $f(t)$ is sampled at equal intervals, the resulting sequence has a Z transform that is related to the Laplace transform of the piecewise function created $f^*(t)$ from the sequence of sampled values.

$\mathcal{L}\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = Z\{f(kT)\}$

where $\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\}$

$f^*(t) = \begin{cases} f(kT) \ , & \text{if } t=k \ , \ \text{and } z=e^{sT} \\ 0 \ , & \text{otherwise} \end{cases}$

25.3 Fourier Transform

- Definition: If

$(a) f(t)$ and $f'(t)$ are piecewise continuous in every finite interval, and $(b) f(t)$ is absolutely integrable in $(-\infty,\infty)$, that is $\int_{-\infty}^{\infty} |f(t)|dt$ is finite, then

$(\rightarrow)F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \mathcal{F}\{f(t)\}$ (or $f(t)$'s spectrum) &

$(\leftarrow)\ f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = \mathcal{F}^{-1}\{F(\omega)\}$

- The Fourier transform $F(\omega)$ is a complex function, so $F(\omega) = |F(\omega)|\, e^{j\phi(\omega)}$, where $|F(\omega)|$ is the continuous amplitude spectrum and $\phi(\omega)$ is the continuous phase spectrum.
- The Fourier transform describes a waveform $f(t)$ into the frequency domain, just like the complex c_n coefficients of f.s. for periodic signals.

Properties

1. Fourier cosine transformation: $f(t)$ even function $\leftrightarrow F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \ \in \mathbb{R}\mathrm{e}$
2. Fourier sine transformation: $f(t)$ odd function
$\leftrightarrow F(\omega) = j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt \in \mathbb{I}\mathrm{m} = 2j\int_{0}^{\infty} f(t)\sin(\omega t)dt \in \mathbb{I}\mathrm{m}$
3. Linearity: $\mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega)$
4. Time shifting: $\{(\sqcup) = F(\omega) \rightarrow \mathcal{F}\{f(t - t_0)\} = e^{j\omega t_0}F(\omega)$
5. Frequency shifting: $\mathcal{F}\{f(t)\} = F(\omega) \rightarrow \mathcal{F}\{f(kt)\} = \frac{1}{|k|}F\left(\frac{\omega}{k}\right)$
6. Symmetry: $\mathcal{F}\{f(t)\} = F(\omega) \rightarrow \mathcal{F}\{F(\omega)\} = f(-\omega)$
7. Differentiation: $\mathcal{F}\{f(t)\} = F(\omega) \rightarrow \mathcal{F}\left\{\frac{d^n}{dt^n}f(t)\right\} = (n\omega)^n F(\omega)$
8. Convolution property:
$\big[\mathcal{F}\{f(t)\} = F(\omega) \wedge \mathcal{F}\{g(t)\} = G(\omega)\big] \rightarrow \big[\mathcal{F}\{f(t) * g(t)\} = F(\omega)\cdot G(\omega)\big]$
9. Integration: $\mathcal{F}\big[\int_{-\infty}^{t} \chi(\tau)d\tau\big] = \frac{\mathrm{X}(\omega}{j\omega} + \frac{1}{2}\mathrm{X}(0)\delta(\omega)$
10. The Fourier transform of a real signal is a Hermitian function.
11. Rayleigh Energy Theorem: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Fourier Transform Table

$$
\begin{bmatrix}
f(t) & F(\omega)\ (\omega = 2\pi f) \\
\delta(t) & 1 \\
1 & \delta(\omega) \\
\delta(t - t_0) & e^{-j\omega t_0} \\
e^{j\omega_0 t} & \delta(\omega - \omega_0) \\
\cos(\omega_0 t) & \dfrac{1}{2}\delta(\omega - \omega_0) + \dfrac{1}{2}\delta(\omega + \omega_0) \\
\sin(\omega_0 t) & \dfrac{-1}{2j}\delta(\omega + \omega_0) + \dfrac{1}{2j}\delta(\omega + \omega_0) \\
\Pi(t) & \text{sinc }(\omega) \\
A\text{sinc }(2\omega_0 t) & (\dfrac{A}{2\omega_0} \cdot \Pi(\dfrac{\omega}{2\omega_0} \\
\dfrac{1}{t} & -j\pi \cdot \text{sgn }(\omega) \\
\text{sinc }^2(t) & \Lambda(\omega) \\
u(t) & \dfrac{1}{2}\delta(\omega) + \dfrac{1}{j\omega} \\
e^{-at}u(t)\ ,\ a > 0 & \dfrac{1}{j\omega + a} \\
te^{-at}u(t)\ ,\ a > 0 & \dfrac{1}{(j\omega + a)^2} \\
e^{-a|t|} & \dfrac{2a}{\omega^2 + a} \\
e^{-\pi t^2} & e^{-\pi\omega^2} \\
\Pi_\alpha(t) = \begin{cases} 1/\alpha\ , & \dfrac{-a}{2} < t < \dfrac{a}{2} \\ 0\ , & otherwise \end{cases} & \text{sinc }\left(\dfrac{\omega a}{2}\right) \\
\Lambda(t) = \begin{cases} t + 1\ , & -1 \leqslant t < 0 \\ 1 - t\ , & 0 \leqslant t < 1 \\ 0\ , \text{otherwise} & \end{cases} & \text{sinc }^2(\omega) \\
R_X(t_1, t_2) & S_X(\omega) \\
R_XY(t_1, t_2) & S_XY(\omega) \\
e^{-\alpha t}\cos(\omega_0 t)u(t)\ ,\ a > 0 & \dfrac{a + j\omega}{\omega_0^2 + (a + j\omega^2)^2} \\
e^{-\alpha t}\sin(\omega_0 t)u(t)\ ,\ a > 0 & \dfrac{\omega_0}{\omega_0^2 + (\alpha + j\omega)^2} \\
\cos(\omega_0 t)u(t) & \dfrac{j\omega}{\omega_0^2 - \omega^2} + \dfrac{\pi}{2}\big[\delta(\omega + \omega_0 + \delta(\omega - \omega_0)\big] \\
\sin(\omega_0 t)u(t) & \dfrac{\omega_0}{\omega_0^2 - \omega^2} + j\dfrac{\pi}{2}\big[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\big]
\end{bmatrix}
$$

26 Special Functions

Heaviside unit step function: $f(t) = u(t-c) \begin{cases} 0 & t < c \\ 1 & t > c \end{cases}$

Properties

- $u(t) + u(-t) = 1$

Ramp function: $r(t-c) = \begin{cases} t & t > c \\ 0 & t < c \end{cases}$

Properties

- $r(t) = \int_{-\infty}^{t} u(\tau) d\tau$
- $r(t-c) = tu(t-c)$
- $r(t-c) = (t-c)u(t-c)$
- $\frac{d}{dt} r(t) = u(t)$
- $\frac{d}{dt} (r(t-c)) = (t-c)u(t-c)$

Signum Function: $\text{sgn }(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases}$

Properties

- $\forall x \in \mathbb{R}(x = \text{sgn }(x) \cdot |x|)$
- $\text{sgn }(x) = \frac{x}{|x|} = \frac{|x|}{x}$
- $\frac{d|x|}{dx} = \text{sgn }(x)$
- $\text{sgn }(t) + 1 = 2u(t)$
- $\text{sgn }(t) = u(t) - u(-t)$

Unit Impulse / Dirac Delta: $\delta(t)$

$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$, where $f(t)$ is a continuous function @$t = a$, $a \in \mathbb{R}$

- In function terms: $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$

Properties

- $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$
- $\int_{p}^{q} \delta(t-a) = 1$, $p < a < q$

- $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$
- $\frac{d}{dt} u(t) = \delta(t)$
- $\delta'(t) = \frac{d}{dt}\delta(t)$: Unit doublet
- $\delta(t) = \delta(-t)$, $\delta(t-\tau) = \delta(\tau - t)$ (Even)
- $f(t)\delta(t-c) = f(c)\delta(t-c)$, $f(t)$: continuous function at $t = c$: Sampling property
- $f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$: Construction of $f(t)$ by the sum of all its samples T, or,

for discrete functions: $x[n] = \sum k = -\infty^{\infty} x[k]\delta[n-k]$
- $\delta(-t) = \delta(t)$

Sinc function: $\text{sinc }(x) = \frac{\sin(\pi x)}{\pi x}$, $\text{sinc }(0) = 1$
Properties
- $\text{sinc }(x) = \text{sinc }(-x)$: Even function
- $\text{sinc }(n) = 0$, $n \in \mathbb{Z}^*$
- $\int_{-\infty}^{\infty} \text{sinc }(x) dx = 1$

$\therefore \int_{-\infty}^{0} \text{sinc }(x) dx = \int_{0}^{\infty} \text{sinc }(x) dx = \frac{1}{2}$

Sine Integral: $Si(t) = \int_{0}^{t} \frac{\sin(x)}{x} dx$
Properties
- $Si(0) = 0$
- $\lim_{t \to \infty} Si(t) = Si(\infty) = \frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin(x)}{x} dx$
- $\int_{0}^{t} \text{sinc }(x) dx = \frac{Si(\pi t)}{\pi}$
- $\frac{d}{dx}[\frac{Si(\pi x)}{\pi}] = \text{sinc }(x)$

Periodic Functions
$\bar{f}(t) = f(t) + f(t-T) + f(t-2T) + f(t-3T) + \ldots + f(t-nT)$, $n \in \mathbb{Z}$, $T \in \mathbb{N}$, with
$f(t) = f(t-T) = f(t-2T) = \ldots = f(t-nT)$

Orthogonal Functions: If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \leqslant x \leqslant b$ and $\int_{a}^{b} f(x) \cdot g(x) dx = 0$, then the two functions are orthogonal to each other on the aforementioned interval.

Kronecker delta function: $\delta(i,j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Boxcar function: $\Pi_w(t-c) = \text{boxcar } \dfrac{t-c}{w} = \begin{cases} 1 & c-w < t < c+w \\ 1/2 & (t = c+w) \vee (t = c-w) \\ 0 & (t < c-w) \vee (t > c+w) \end{cases}$

Rectangular / Top-hat function: Boxcar function centered at origin ($c = 0$):

$\text{rect } (\dfrac{t}{w}) = \Pi(\dfrac{t}{w}) = \begin{cases} 1 & |t| < w/2 \\ 1/2 & |t| = w/2 \\ 0 & otherwise \end{cases} = w\Pi(t) = \Pi_w(t)$

Properties

- Unit area
- $\displaystyle\int_{-\infty}^{\infty} \lim_{w \to 0} \{\Pi_w(t)\}\, dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$
- Triangle function: $\Lambda_w(t) = \text{tri } (t) = \Lambda(t) = \begin{cases} \dfrac{w+t}{w^2} & -w < t < 0 \\ \dfrac{w-t}{w^2} & 0 < t < w \\ 0 & |t| < w \end{cases}$

Properties

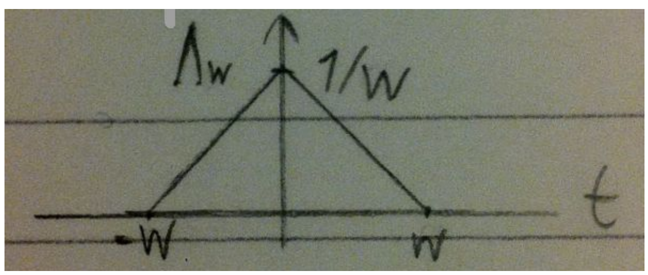


Figure 43: Triangle Function

Bessel's equation & Bessel functions

- Equation: $x^2 y'' + xy' + (x^2 - v^2) = 0$, v is a real constant.

If we express its two solutions in terms of gamma functions, we obtain the Bessel function of the 1st kind of order v - provided v is not a negative integer:

$J_v(x) = \left(\dfrac{x}{2}\right)^2 \left\{ \dfrac{1}{\Gamma(v+1)} - \dfrac{x^2}{2^2 \cdot 1!\Gamma(v+2)} + \dfrac{x^4}{2^4 \cdot 2! \cdot \Gamma(v+3) - \dots} \right\}$ • Also,

$J_{-v}(x) = \left(\dfrac{x}{2}\right)^{-v} \left\{ \dfrac{1}{\Gamma(1-v)} - \dfrac{x^2}{2 \cdot 1! \cdot \Gamma(2-v)} + \dfrac{x^4}{2^2 \cdot 2! \cdot \Gamma(3-v)} - \dots \right\}$

provided that v is not a positive integer.

Therefore, complete solution is: $y = A \cdot J_v(x) + B \cdot J_{-v}(x)$

When $v = n \in \mathbb{Z}$, then:

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{n!} - \frac{1}{(n+1)!}\cdot\left(\frac{x}{2}\right)^2 + \frac{1}{2!\cdot(n+2)!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\cdot(n+3)!}\left(\frac{x}{2}\right)^6 + \dots\right\}$$

Legendre's Equation: $(1-x^2)y'' - 2xy' + k(k+1)y = 0$,
where $k \in \mathbb{R}$ a constant. Solution by Frobenius gives:

$c = 0: \; y = a_0\{1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k-2)(k+1)(k+3)}{4!}x^4 - \dots\}$

$c = 1: \; y = a_1\{x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \dots\}$

When $k \in \mathbb{Z}$, one series terminates. The resulting polynomial $P_n(x)$, is a Legendre polynomial, with a_0 or a_1 being chosen so that the polynomial has unit value @$x-1$, ie. $P_n(1) = 1$.

- Legendre polynomials can be derived by Rodrigues formula:

$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$,

or by the generating function: $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$, $|t| < 1$.

- Legendre polynomials are mutually orthogonal ie. if $m \neq n$, then

$\int_{-1}^{1} P_m(x)\cdot P_n(x)dx = 0$. The orthogonality of the Legendre polynomials permits any polynomial to be writeen as a finite series of Legendre polynomials.

Gamma Function

The Gamma function is an extension of the factorial function, with its argument shifted down by 1, to the realm of real and complex numbers. That is:
$n \in \mathbb{N}^* \to \Gamma(n) = (n-1)!$

- Definition: $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$, converges for $x > 0$

$\Gamma(x+1) = x\Gamma(x) \leftrightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x}$: Recurrence relation

Properties

- If $x = n \in \mathbb{N}^* \to \Gamma(n+1) = n!\Gamma(1) = n!$
- $\Gamma(1) = 1$
- $\Gamma(0) = \infty$
- $\Gamma(-n) = \pm\infty$, ($-\infty$ if n odd) $\vee$ (∞ if n even)
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$, $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$

$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, $\Gamma(-\frac{3}{2}) = \frac{4\sqrt{\pi}}{3}$

- Duplication formula: $\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}$
- For large n : $\Gamma(n+1) \approx \sqrt{2\pi n}n^n e^{-n}$

Digamma Function

- Definition: Two different definitions are given. The first:

$\Psi(z) \triangleq \dfrac{d}{dz}\ln\ (\Gamma(z)) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ defined as the logarithmic derivative of $\Gamma(z)$ and

$F(z) = \dfrac{d}{dz}\ln\ (z!)$ defined as the logarithmic derivative of the factorial function.

- The two are connected by the relationship:

$F(z) = \Psi(z+1)$

- The nth derivative of $\Psi(z)$ is called the polygamma function , denoted $\psi_n(z)$. Thus the notation $\psi_0(z) = \Psi(z)$ is frequently used for the digamma function itself.

<u>Beta Function</u>

- Definition: $\mathrm{B}(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx$,converges for$m > 0 \wedge n > 0$
- Alternative Definition: $\mathrm{B}(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}(\theta)\cdot\cos^{2n-1}(\theta)$

Properties

- $\mathrm{B}(m,n) = \mathrm{B}(n,m)$
- $\mathrm{B}(m,n) = \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathrm{B}(m-1,n-1)$
- $\mathrm{B}(k,1) = \mathrm{B}(1,k) = \dfrac{1}{k}$
- $\mathrm{B}(1,1) = 1$
- $\mathrm{B}(\frac{1}{2},\frac{1}{2}) = \pi$
- $\mathrm{B}(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$, $m,n \in \mathbb{N}^*$
- Relationship between Beta and Gamma Functions:

$\mathrm{B}(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$, $\forall m,n$

<u>Error Function</u>:

- Definition: $\mathrm{erf}\ (x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$

Properties

- $\mathrm{erf}\ (x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$
- Complementary error function: $\mathrm{erfc}\ (x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_x^{\infty} e^{-t^2}dt = 1 - \mathrm{erf}\ (x)$
- $\mathrm{erf}\ (-x) = -\mathrm{erf}\ (x)$: Odd function
- $\mathrm{erf}\ (\infty) = 1$, $\mathrm{erfc}\ (\infty) = 0$
- $\mathrm{erf}\ (0) = 0$, $\mathrm{erfc}\ (0) = 1$
- Area beneath the Gaussian P.D.: $\Phi(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt = 1$

<u>Elliptic Functions</u>

a) Standard Forms: (valid for $0 \leqslant \phi \leqslant \dfrac{\pi}{2}$ $0 < k < 1$

- Of the 1st kind: $F(k, \phi) = \displaystyle\int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$
- Of the 2nd kind: $E(k, \phi) = \displaystyle\int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} d\theta$
- In general, if an integrand is a rational expression of x and of $\sqrt{P(x)}$ where $P(x)$ is a polynomial in x of degree 3 or 4, then the integral is said to be elliptic.
- In each case if $\phi = \dfrac{\pi}{2}$ then the integral is said to be complete and it is denoted by $F\left(k, \dfrac{\pi}{2}\right) = K(k)$ and $E\left(k, \dfrac{\pi}{2}\right) = E(k)$.

b) Alternative forms of elliptic functions: (valid for $0 \leqslant x \leqslant 1 \ \wedge \ 0 < k < 1$)

- Of the 1st kind: $F(k, x) = \displaystyle\int_0^{x} \frac{du}{\sqrt{(1 - u^2)(1 - k^2 u^2)}}$
- Of the 2nd kind: $E(k, x) = \displaystyle\int_0^{x} \sqrt{\frac{1 - k^2 u^2}{1 - u^2}} du$

In some tables k, x are quoted as $\sin(\theta), \sin(\phi)$ respectively, thus
$\theta = \sin^{-1}(k)$, $\phi = \sin^{-1}(\phi)$

- Riemann Zeta function: $\zeta(s)$ is a function of a complex variable $s = \sigma + it$ that analytically continues the sum of the Dirichlet series for when the real part of s is greater than 1. It is equal to the generalization of the harmonic series. It is equal to the following, with the caveat that $\sigma > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \frac{1}{(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})(1 - \frac{1}{7^s})(1 - \frac{1}{11^s})}$$

We notice that the Riemann function can be written in a product form over the Prime numbers. This discovery, attributed to Euler, means that they Riemann Zeta function encodes information about the prime numbers.

- Lambert-**W** function:

Also known as the Omega function, is a set of functions namely the branches of the inverse relation of the function $f(z) = ze^z$, where e^z is the exponential function and z is any complex number:

$$z = f^{-1}(ze^z) = \mathbf{W}(ze^z)$$

By substituting $z_0 = ze^z$ we get the defining equation for the W function (and for the W relation in general):

$$z_0 = \mathbf{W}(z_0)e^{\mathbf{W}(z_0)}$$

for any complex number z_0.

We can approximate the function as follows:

$$\phi(x,r) = 1 + \sum_{k=1}^{[r]} \frac{x^k\big[r-(k-1)\big]^k}{k!}$$

Now consider the following series of approximations, where r is assumed to be sufficiently large. The first one is:

$$\tilde{\mathbf{W}}^1(x,r) = \frac{1}{r}\ln\,\phi(x,r)$$

Subsequent approximations are defined recursively by:

$$\tilde{\mathbf{W}}^{n+1}(x,r) = \frac{1}{r}\ln\left[\frac{\tilde{\mathbf{W}}^n\big(1+\tilde{\mathbf{W}}^n\big)}{x}\phi(x,r)\right]$$

Example: For $x = 2000$, even r as low as 80 gives quite accurate results:
$\tilde{\mathbf{W}}^5(2000,80) \approx 5.83673149492073$ and $\tilde{\mathbf{W}}^6(2000,80) \approx 5.836731494908671$

27 Abstract Algebra

- A linear mapping is a mapping $V \to W$ between two modules (including vector spaces) that preserves the operations of addition and scalar multiplication.

28 Recreational

A magic square is a square divided into smaller squares each containing a number, such that the sum in each row, column and diagonal is a constant. That constant is equal to:
$$\frac{n(n^2+1)}{2}$$

29 Physics

29.1 Classic Mechanics

- Διαφορά ενός μεγέθους X = Αρχική τιμή του μεγέθους - Τελική τιμή του μεγέθους = $X_{\text{αρχ}} - X_{\text{τελ}}$
- Μεταβολή ενός μεγέθους X = Τελική τιμή του μεγέθους - Αρχική τιμή του μεγέθους = $X_{\text{τελ}} - X_{\text{αρχ}}$
- Ρυθμός μεταβολής φυσικού μεγέθους Φ σχετικά με τον χρόνο $t = \dfrac{\Delta\Phi}{\Delta t}$
- $x = U \cdot t$: θέση αντικειμένου / οντότητας από την αρχή σημείου αναφοράς (m)
- $\Delta\vec{x} = \Delta\vec{U} \cdot \Delta t$: Μετατόπιση(m) [E.O.K (rectilinear)]
- $\Delta\vec{x} = x_{\text{τελ}} - x_{\text{αρχ}}$ (όταν η κατεύθυνση συμπίπτει με την κατεύθυνση της μετατόπισης)
- $p = \dfrac{m}{V}$: Πυκνότητα (kg/m^3)
- $\bar{U} = \dfrac{S}{t}(m/s) = \big(= \dfrac{|\Delta\vec{x}|}{\Delta t}\big)$: Μέση ταχύτητα

S = διάστημα (μόνο θετικό), ή απόσταση
- $p = \dfrac{m_{\text{ΟΛ}}}{V_{\text{ΟΛ}}}(kg/m^3)$: Μέση πυκνότητα (για σταθερές συνθήκες πίεσης και θερμοκρασίας)
- $\vec{a} = \dfrac{\Delta\vec{U}}{\Delta t}$: Επιτάχυνση ($\dfrac{m}{s^2}$ στην Ε.Ο.Μ. κίνηση. Έχει πάντα την ίδια κατεύθυνση με την μεταβολή της ταχύτητας $\Delta\vec{U} = \vec{U} - U_0$
- $\Delta\vec{x} = U_0 t + \dfrac{1}{2}\vec{a}t^2$ (m) : Μετατόπιση στην Ε.Ο.Μ. κίνηση
- 1ος Νόμος Νεύτωνα: Κάθε σώμα, που βρίσκεται μέσα σε ένα αδρανειακό σύστημα, διατηρεί την κατάσταση ηρεμίας, ή ευθυγραμμης και ομαλής κίνησης του, εφόσον καμία εξωτερική δύναμη δεν επιδρά για τη μεταβολή της, ή η συνισταμένη των δυνάμεων ισούται με 0. $\Sigma\vec{F}_{\text{εξ}} = 0 \leftrightarrow \vec{U} =$ σταθερή
- 2ος Νόμος Νεύτωνα: Περιγράφει τη συμπεριφορά του σώματος, όταν η συνισταμένη των δυνάμεων που ασκούνται σε αυτό δεν είναι μηδέν. Τότε η δύναμη που θα του ασκηθεί θα είναι: $\vec{F} = m \cdot \vec{a}$ $\quad (N = kg \cdot \dfrac{m}{s})$ (Ισχύει για σώμα σταθερής μάζας m)
- 3ος Νόμος Νεύτωνα: Όταν δύο σώματα αλληλεπιδρούν και το πρώτο ασκεί δύναμη F στο δεύτερο, τότε και το δεύτερο ασκεί δύναμη ίδιου μέτρου F και αντίθετης φοράς, δηλαδή ασκεί αντίθετη δύναμη $-F$ στο πρώτο.

- Συνθήκη ισορροπίας: $\Sigma F = 0$
- $W = m \cdot g$ (N) : Βάρος
- $F = -kx$: Νόμος του Hooke. k= σταθερά του ελατηρίου. Το αρνητικό πρόσημο υποδηλώνει ότι αυτή η δύναμη ασκείται σε αντίθετη κατεύθυνση από την κατεύθυνση τέντωσης, ή συμπίεσης του ελατηρίου
- $P = \dfrac{F}{A}\big(Pa = 1\text{Pascal} = \dfrac{N}{m^2}\big)$: Πίεση, που είναι το μέτρο της ολικής δύναμης που ασκείται κάθετα σε επιφάνεια εμβαδού A

- $P_{\text{1atm}} = 101,293Pa$: Πίεση μιας ατμόσφαιρας στην επιφάνεια της θάλασσας
- Αρχή του Pascal: Κάθε μεταβολή της πίεσης σε οποιοδήποτε σημείο ενός περιορισμένου ρευστού που είναι ακίνητο, προκαλεί ίση μεταβολή της πίεσης σε όλα τα σημεία του.
- $P = p \cdot g \cdot h$ (Pa): Υδροστατική πίεση, που είναι η πίεση που ασκεί ένα ρευστό σε αντικείμενο ή επιφάνεια που βρίσκεται μέσα σάυτό. p : πυκνότητα του ρευστού, h : βάθος στο οποίο βρίσκεται το αντικείμενο. Οφείλεται στο βάρος του ρευστού.
- Αρχή των συγκοινωνούντων δοχείων: Όταν αντικείμενα α, β διαφορετικά, βρίσκονται σε ίδιο βάθος εντός ενός ρευστού θα ισχύει: $P_a = P_\beta$
- $A = p \cdot g \cdot V_\Sigma (N)$: Άνωση, που δέχεται σώμα όγκου V_Σ : βυθισμένου σε ρευστό.
- $p_{\text{σώματος}} < P_{\text{ρευστού}} \rightarrow A = W_\Sigma$: Συνθήκη πλεύσης
- $W = \vec{F} \cdot \Delta\vec{x} \cos\theta$ $(J = N \cdot m)$: Έργο, δύναμης $\vec{F}$

θ : γωνία που σχηματίζει η δύναμη με την μετατόπιση που προκαλεί
Ότι είναι η επιταγή για το χρήμα, είναι το έργο για τη δύναμη •
$U_{\Delta\Upsilon\text{N}} = W \cdot h = m \cdot g \cdot h$ (J) : (Βαρυτική) Δυναμική ενέργεια

- $E_{\text{KIN}} = \frac{1}{2} m U^2$ (J) : Κινητική ενέργεια
- $E_{\text{MH}\Xi} = E_{\text{KIN}} + U_{\Delta\Upsilon\text{N}}$ (J)
- $E_{\text{MH}\Xi}^{\text{APX}} = E_{\text{MH}\Xi}^{\text{TE}\Lambda}$: Διατήρηση της Μηχανικής Ενέργειας
- $P = \frac{W}{t} = \frac{E}{t}$ $(W = J/s)$: Ισχύς, προκύπτει ότι: $P = F \cdot U$
- $n = \frac{E_{\text{χρήσιμη}}}{E_{\text{καταναλισκόμενη}}}$: Απόδοση μηχανής
- $Q = m \cdot c \cdot \Delta\theta$ (J) : Νόμος της θερμιδομετρίας

c : ειδική θερμότητα υλικού $(J/kg \cdot k)$ (specific heat capacity), $\Delta\theta =$ μεταβολή της θερμοκρασίας, Q : ποσότητα θερμότητας, ή θερμοχωρητικότητα (thermal capacity)

- $\Delta l = l_0 \cdot a_l \cdot \Delta\theta$ (m) : Μεταβολή μήκους επίμηκους σώματος - ράβδου, λόγω γραμμικής θερμικής διαστολής, ή συστολής, a_l : συντελεστής γραμμικής διαστολής υλικού της ράβδου, l_0 : αρχικό μήκος της ράβδου
- $\Delta V = V_0 \cdot a_v \cdot \Delta\theta$: Μεταβολή του όγκου ύγρου ή στερεού κατά τη διαστολή, ή συστολή του, V_0 : αρχικός όγκος, a_v : συντελεστής όγκου υλικού
- $Q = L_T \cdot m$ (J) : θερμότητα που μεταφέρεται σε στερεό σώμα κατά την τήκη του, L_T : λανθάνουσα θερμότητα τήξης, m : μάζα σώματος
- $Q = L_B \cdot m$ (J) : θερμότητα που μεταφέρεται σε υγρό σώμα κατά τον βρασμό, L_B : λανθάνουσα θερμότητα βρασμού.
- $\Delta U_\Delta = -W \leftrightarrow U_{\text{TE}\Lambda} - U_{\text{APX}} = -W$: Σχέση της μεταβολής της δυναμικής ενέργειας συστήματος σωμάτων συγκριτικά με το έργο των συντηρητικών δυνάμεων αλληλεπίδρασης.
- $\vec{F_c} = k \frac{q_1 q_2}{r^2}$ (N) : Νόμος Coulomb, $k = \frac{1}{4\pi\epsilon_0}$ σταθερά του Coulomb / ηλεκτρική σταθερά $\simeq 9 \cdot 10^9 N \cdot m^2/Cb^2$

Ο νόμος ισχύει για φορτισμένα σώματα των οποίων οι διαστάσεις είναι πολύ μικρές σε σχέση με τη μεταξύ τους απόσταση, ή για φορτισμένες σφαίρες

- Αριθμός ηλεκτρονίων $= \frac{\text{συνολικό φορτίο}}{\text{στοιχειώδες φορτίο} q_e}$

- $I = \frac{q}{t}\ (A)$: Ένταση του ηλεκτρικού ρεύματος που διαρρέει έναν αγωγό. Η σχέση ισχύει

μόνον όταν το ρεύμα είναι σταθερό, δηλαδή για απειροελάχιστα φορτία: $I = \frac{dq}{dt}\ (A = \frac{C}{s})$

- $\epsilon_{\text{ΠΗΓ}} = \frac{E_{\text{ΗΛ}}}{q}\ (V)\ \big(\text{Volt} = \frac{J}{C}\big)$: Ηλεκτρεγερτική δύναμη πηγής

ή $V = \frac{U_{\text{Ηλ, Σ}}}{q}\ (V)$, $U_{\text{ΗΛ, Σ}}$ ηλ. δυναμική ενέργεια του πεδίου που έχει το φορτίο q στη θέση Σ.

- $R = \frac{V}{I}\ (\Omega)\big(Ohm = \frac{V}{A} = \frac{J}{C} \cdot \frac{s}{J} = \frac{s}{C}\big)$: Ηλεκτρική αντίσταση διπόλου, ή αγωγού
- $I = \frac{V_{\text{ΠΗΓΗΣ}}}{R}$: Νόμος του Ohm, $V_{\text{ΠΗΓΗΣ}}$: τάση στους πόλους της πηγής
- $R = p_\theta \frac{l}{A}\ (\Omega)$: Αντίσταση του αγωγού, l : μήκος του αγωγού (l), A : εμβαδό διατομής

του αγωγού (m^2), p_θ : ειδική αντίσταση του υλικού του αγωγού ($\Omega \cdot m$), ως συνάρτηση της θερμοκρασίας

- Για $\theta \in (0°C, 100°C)$, είναι: $p_\theta p_0\,(1 + a\theta)\,(\Omega)$ ειδική αντίσταση του αγωγού σε θερμοκρασία ϑ.

p_0 : ειδική αντίσταση υλικού του αγωγού στους 0° C, a : θερμικός συντελεστής ειδικής αντίσταση που για τα περισσότερα καθαρά μέταλλα έχει τιμή $a = 1/273°$ C.

- $R_\theta = R_0(1 + a\theta)\ (\Omega)$ Αντίσταση αγωγού σε θερμοκρασία $\theta°$ C, $R_0 = p_0\frac{l}{A}$: αντίσταση αγωγού σε θερμοκρασία 0°.
- $Q_{\text{αντ.}} = I^2 \cdot R \cdot t\ (J)$: Νόμος του Joule (Joule effect), ή $Q_{\text{αντ.}} = a \cdot I^2 \cdot R \cdot t\ (J)$,

$a = 0.24(cal/J)$: ηλεκτρικό ισοδύναμο της θερμότητας

- 1 calorie = 1 cal = 4.184J
- $E_{\text{ηλ}} = V \cdot I \cdot t\ (J)$: Ηλεκτρική ενέργεια που χρησιμοποιεί μια ηλεκτρική συσκευή σε χρόνο t
- $P_{\text{ηλ}} = V \cdot I\ (W)$: Ισχύς που χρησιμοποιεί μια ηλεκτρική συσκευή
- Απόδοση: $n(\%) = \frac{\text{Ωφέλιμο ποσό}}{\text{Παρεχόμενο ποσό}} \cdot 100(\%)$
- $\Delta K = K_{\text{τελ}} - K_{\text{αρχ}} = \Sigma W_f = W_{F_{\text{ολ}}}$ ή $\Delta K = W_{F_{\text{ολ}}}$

<u>Θεώρημα μεταβολής κινητικής ενέργειας - Θ.Μ.Κ.Ε.</u> ή θεώρημα έργου - ενέργειας.

- <u>Θεώρημα Διατήρησης Μηχανικής Ενέργειας</u>: Όταν σ'ένα σώμα, ή σύστημα σωμάτων

δρουν μόνο διατηρητικές δυνάμεις, τότε η μηχανική ενέργεια διατηρείται, δηλαδή $E_{\text{ΜΗΧ}}^{\text{ΑΡΧ}} = E_{\text{ΜΗΧ}}^{\text{ΤΕΛ}}$. Προκύπτει ότι το άθροισμα της μεταβολής της Κ.Ε. και της Δ.Ε. είναι 0 ή $\Delta K + \Delta U = 0$.

- Θ.Μ.Κ.Ε.: Εφαρμόζεται για ένα σώμα, ισχύει πάντα.
- Α.Δ.Μ.Ε.: Εφαρμόζεται για σύστημα σωμάτων και ισχύει μόνον όταν όλες οι δυνάμεις που ασκούνται στο σύστημα είναι συντηρητικές.
- Α.Δ.Ε.: Ισχύει παντού και πάντοτε.
- $T_k = \mu_k \cdot N$: Τριβή ολίσθησης (kinetic friction) (N),

μ_k : συντελεστής τριβής ολίσθησης, N : κάθετη δύναμη με την οποία συμπιέζονται οι επιφάνειες

- $T_s = \mu_s \cdot N$ (N) : Στατική τριβή, όταν το σώμα παραμένει ακίνητο, μ_s : συντελεστής στατικής τριβής
- Οριζόντια βολή: $\begin{cases} \text{Κίνηση στον άξονα} x : \text{Ε.Ο.Κ., με } x = U_0 t \\ \text{Κίνηση στον άξονα} y : \text{Ελεύθεση πτώση} \end{cases}$
- <u>Αρχή ανεξαρτησίας των κινήσεων</u>: Οι δύο κινήσεις είτε εκτελούνται ανεξάρτητα, είτε διαδοχικά και διαρκούν χρόνο: $t = \sqrt{\dfrac{2h}{g}}$

  → $\vec{U} = \dfrac{S}{t}$ $(\dfrac{m}{s})$: Γραμμική ταχύτητα στην ομαλή κυκλική κίνηση, S : τόξο που διαγράφεται σε χρόνο t,

  → $\vec{\omega} = \dfrac{\theta}{t}$ $(\dfrac{rad}{s})$: Γωνιακή ταχύτητα στην Ο.Κ.Κ., με διεύθυνση κάθετη στο επίπεδο της τροχιάς και φορά περιστροφής του κινητού που συμπίπτει με την κατεύθυνση των υπόλοιπων δακτύλων (figure 44)
  Για $t = T$, είναι: $\omega = \dfrac{2\pi}{T}$

Figure 44: Μέθοδος του δεξιού χεριού (στη Φυσική)

- $U = \omega \cdot r$: Σχέση γραμμικής και γωνιακής ταχύτητας
- $f = \dfrac{N}{\Delta t}$ (1Hz = 1 rep / sec): Συχνότητα ταλάντωσης, N : αριθμός ταλαντώσεων σε χρονικό διάστημα Δt.

Για $N = 1$, $\Delta t = T$: περίοδος ταλάντωσης. Άρα $f = \dfrac{1}{T}$
- $\omega = 2\pi f$
- $a_c = \dfrac{U^2}{r}$ $(\dfrac{m}{s^2})$: κεντρομόλος επιτάχυνση (centripetal acceleration). Έχει κατεύθυνση προς το κέντρο της κυκλικής τροχιάς.
- $F_c = \dfrac{mU^2}{r}$ (N) : Κεντρομόλος δύναμη. Έχει κατεύθυνση προς το κέντρο της κυκλικής τροχιάς
- $F = G\dfrac{m_1 m_2}{r^2}$ (N) : Νόμος της παγκόσμιας έλξης

$G = 6.67 \cdot 10^{-11} (N \cdot m^2/kg^2)$: σταθερά της παγκόσμιας έλξης. Ισχύει μόνο για σωμάτια, ή ομογενή σφαιρικά σώματα r : απόσταση μεταξύ των κέντρων τους F : δύναμη βαρυτικής έλξης
- $\vec{g} = \dfrac{\vec{F}}{m} = G\dfrac{M}{r^2}$ $(\dfrac{N}{kg})$: Ένταση του βαρυτικού πεδίου $\vec{g}$ ίδια κατεύθυνση με το βάρος
- Η σχέση δεν ισχύει για $< r$, δηλαδή όταν το ελκούμενο σώμα βρίσκεται εντός του

σώματος που ¨δημιουργεί' το βαρυτικό πεδίο

- $U = \sqrt{G\frac{M}{r}}(\frac{m}{s})$: Ταχύτητα περιστροφής των δορυφόρων

- Επίσημη διατύπωση του 2ου Νόμου του Νεύτωνα: Η συνισταμένη των δυνάμεων που ασκούνται σε ένα σώμα, ισούται με το ρυθμό μεταβολής της ορμής του σώματος.

$\Sigma\vec{F} = \left(\frac{\Delta\vec{P}}{\Delta t}\right) = \frac{\delta\vec{P}}{\delta t} = \frac{d}{dt}(m\vec{U})$, δηλαδή: $\vec{P} = m\vec{U}(kg \cdot m/s)$: Ορμή(Momentum)

- $m = \frac{F}{a}\ (kg)$: Αδρανειακή μάζα
- $m = \frac{W}{g}\ (kg)$: Βαρυτική μάζα
- $\vec{F} = \frac{\vec{p}_{\text{τελ.}} - \vec{p}_{\text{αρχ.}}}{\Delta t}$
- Αρχή Διατήρησης Ορμής (Α.Δ.Ο.): $\vec{P}_{\text{ΟΛ}}^{\text{ΑΡΧ}} = \vec{P}_{\text{ΟΛ}}^{\text{ΤΕΛ}}$

Η συνολική ορμή ενός μεμονωμένου συστήματος σωμάτων διατηρείται σταθερή.

Νόμοι Αερίων

1. Νόμος του Boyle: $p \cdot V = k =$ σταθ. (ισόθερμη), όταν ο αριθμός των mol n και η θερμοκρασία T είναι σταθερά.
2. Νόμος Charles: $V \propto T$, όταν n, p σταθερά. (ισοβαρής)
3. Νόμος Gay-Lussac: $p \propto T$, όταν n, p σταθερά. Καταλήγουμε ότι: (1, 2, 3 →): $p \cdot V = n \cdot R \cdot T$: Καταστατική εξίσωση των ιδανικών αερίων. Η εξίσωση ισχύει και για αέρια μείγματα. Ιδανικό αέριο είναι αυτό για το οποίο ισχύει η καταστατική εξίσωση ακριβώς, σε όλες τις πιέχεις και θερμοκρασίες

- Επίσης, αν μιλάμε για συγκεκριμένο αέριο, μπορούμε από (1) να πούμε ότι: $p_1 \cdot V_1 = p_2 \cdot V_2$
- $U_{\text{ΕΣ}} = N \cdot \bar{K}$: Εσωτερική ενέργεια αερίου (ουσιαστικά θερμική ενέργεια)

$\bar{K} = \frac{(k_1 + k_2 + \ldots + k_n)}{N}$: μέση κινητική ενέργεια μορίων

- $Q = \Delta U$: Προσφερόμενη θερμότητα = Αύξηση εσωτερικής ενέγειας αερίου
- $W = P \cdot \Delta V$: Έργο που δημιουργεί η διαστολή αερίου, ΔV : αύξηση του όγκου αερίου κατά τη θέρμανση του
- Από τα δύο προηγούμενα, αναγόμαστε στο εξής: $Q = \Delta U + W$: Προσφερόμενη θερμότητα = Αύξηση Εσωτερικής Ενέργειας αερίου + Ενέργεια απαιτούμενη για την ανύψηση του εμβόλου. Η προηγούμενη σχέση ουσιαστικά αποτελεί την Α.Δ.Ε. για τα αέρια.
- Θερμική ενέργεια = Τροφοδοτούμενη εν. - Αποδιδόμενη εν.
- $U = \lambda \cdot f$: Θεμελιώδης Νόμος της Κυματικής
- $\frac{1}{p} + \frac{1}{p'} = \frac{1}{f}$, f : απόσταση εστίας - κορυφής, εστιακή απόσταση, ή απόσταση που συγκλίνουν οι ανακλώμενες ακτίνες και σχηματίζεται το είδωλο (figure 45)

p' : απόσταση ειδώλου μήκους $P'Q'$ από την κορυφή.

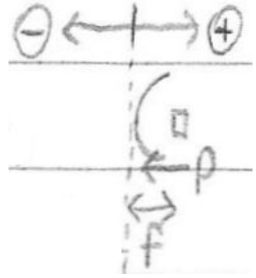


Figure 45: Αναπαράσταση του θεμελιώδης νόμου της κυματικής

- Για σφαιρικό καθρέπτη ισχύει $R = 2f$ (ακτίνα καμπυλότητας)
- $m = -\dfrac{p'}{p} = \dfrac{\overline{P'Q'}}{\overline{PQ}}$ (μεγέθυνση). Αν το είδωλο είναι ορθό η μεγέθυνση είναι θετική, ενώ

αν είναι αντεστραμμένο, η μεγέθυνση είναι αρνητική.

- $n = \dfrac{C_0}{U}$: δείκτης διάθλασης, ή $n = \dfrac{\sin(\theta_\pi)}{\sin(\theta_\delta)}$

U : ταχύτητα φωτός στο υλικό μέσο

Επίσης: $n = \dfrac{\lambda_0}{\lambda} \leftrightarrow \lambda = \dfrac{\lambda_0}{n}$, λ, n : χαρακτηριστικό υλικού.

- $\dfrac{\sin(\theta_\pi)}{\sin(\theta_\delta)} = \dfrac{n_2}{n_1} = \dfrac{U_1}{U_2}$ <u>Νόμος του Snell</u> (figure 46)

$\dfrac{\sin(\theta_\pi)}{\sin(\theta_\delta)} = $ σταθερό

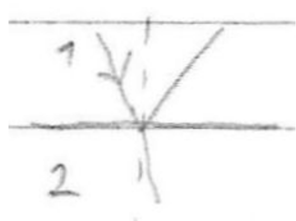


Figure 46: Ο Νόμος του Snell ή Νόμος της διάθλασης

- <u>Νόμος της Ανάκλασης</u> (του Νεύτωνα):

Γωνία πρόσπτωσης (θ_π) = Γωνία ανάκλασης (θ_a)

- $\sin(\theta_\pi = \dfrac{1}{n_{o\lambda}} \rightarrow \theta_\pi$ οριακή γωνία

- Νόμος Brewster: $\tan(\theta_\pi) = \dfrac{n_2}{n_1} \rightarrow \theta_\pi =$γωνία Brewster, ή γωνία ολικής πόλωσης.

Συμβαίνει όταν η ανακλώμενη και διαθλώμενη γωνία είναι κάθετες μεταξύ τους.

29.2 Electrostatic Field

- $\vec{E} = \dfrac{\vec{F}}{q} = k\dfrac{|Q|}{r^2} \ \left(\dfrac{N}{C}ή\dfrac{V}{m}\right)$ Ένταση (σε σημείο) ηλεκτροστατικού πεδίου, που δέχεται φορτίο q από φορτίο - πηγή Q.
- $U_\Sigma = k\dfrac{Qq}{r}\ (J)$: Ηλεκτρική Δυναμική Ενέργεια (electric potential energy); η οποία ανήκει και στα δύο φορτία -ανήκει στο σύστημα- κινούμενου φορτίου q σε σημείο Σ του πεδίου, r : απόσταση μεταξύ q και Q.
- $V_\Sigma = \dfrac{U_\Sigma}{q} = k\dfrac{Q}{r}\ (V)$: Δυναμικό ηλεκτροστατικού πεδίου (Coulomb), σε θέση X του πεδίου, r : απόσταση μεταξύ του σημείου Σ και του φορτίου Q που δημιουργεί το πεδίο.

Αντίστοιχα ισχύει ότι: $V_{\Sigma \mathrm{P}} = \dfrac{W_{\Sigma\rightarrow P}}{q}\ (V) = V_\Sigma - V_P$: Διαφορά Δυναμικού, ή τάση μεταξύ δύο σημείων Σ και Ρ του ηλεκτρικού πεδίου

- $C = \dfrac{Q}{V}\ (F)\ (Farad = \dfrac{Coulomb}{Volt}$: Χωρητικότητα του πυκνωτή, $Q =$ φορτίο πυκνωτή, V = διαφορά δυναμικού μεταξύ των οπλισμών του πυκνωτή. Μασ πληροφορεί για το φορτίο που μπορεί να αποθηκευτεί ανά μονάδα τάσης μεταξύ των οπλισμών του.
- $C = \epsilon \cdot \epsilon_0 \dfrac{S}{l}\ (F)$: Εξάρτηση χωρητικότητας επίπεδου πυκνωτή από σχετική διηεκτρική σταθερά ϵ του διηλεκτρικού που βρίσκεται μεταξύ των οπλισμών του, S : εμβαδό πλάκας πυκνωτή, l : απόσταση μεταξύ των πλακών του

$U = \dfrac{1}{2}CV^2 = \dfrac{1}{2}Q\cdot V = \dfrac{1}{2}\dfrac{Q^2}{C}(J)$: Ηλεκτρική δυναμική ενέργεια του πυκνωτή

- $E = \dfrac{V}{l}\ \dfrac{V}{m}$: Ένταση ομογενούς ηλεκτροστατικού πεδίου, άρα μόνο για επίπεδο πυκνωτή
- Αρχή Διατήρησης του Φορτίου: Όσο φορτίο διέρχεται από κάποια διατομή του αγωγού στη μονάδα του χρόνου
- $\Sigma I_{εισ.} = \Sigma I_{εξ.}$: 1ος κανόνας Κιρςηοφφ: Το άθροισμα των εντάσεων των ρευμάτων, που ¨εισέρχονται' σ'έναν κόμβο, ισούται με το αθροισμα των εντάσεων των ρευμάτων, που ¨εξέρχονται' από αυτόν, ή $\Sigma I = 0$ στο κόμβο αυτό
- $\Sigma(\Delta V) = 0$: 2ος κανόνας Kirchoff: Κατά μήκος μιας κλειστής διαδρομής σε ένα κύκλωμα, το αλγεβρικό άθροισμα των διαφορών δυναμικού είναι0.
- $V_{\Pi\mathrm{H}\Gamma} = \epsilon - Ir_{εσ}\ (V)$: Πολική τάση πηγής, ή τάση στους πόλους της πηγής, $I =$ ρεύμα κυκλώματος, $r_{εσ}$: Εσωτερική αντίσταση πηγής, $V_{\Pi\mathrm{H}\Gamma}$: τάση στους πόλους της πηγής.

- $E = h\cdot f\ (j)$: Ενέργεια φωτονίου
- $L = mUr$: Μέτρο στροφορμής του ηλεκτρονίου
- $\hbar = \dfrac{h}{2\pi}$: Μειωμένη σταθερά Planck. Από τα προηγούμενα συμπεραίνουμε ότι η

στροφορμή του ηλεκτρονίου μπορεί να πάρει τιμές ίσες με τα ακέραια πολλαπλάσια της ποσότητας $\hbar$. Ηλεκτρόνια με την ίδια στροφορμή κινούνται σε μία από τις επιτρεπόμενες τροχιές, ακτίνας r, δλδ.: $mUr = n\hbar$, $n \in \mathbb{N}$. Υποδηλώνεται έτσι η κβάντωση της στροφορμής.

- $E_{\text{αρχ}} - E_{\text{τελ}} = h \cdot f\ (J)$: Ενέργεια που παράγεται όταν ηλεκτρόνιο μεταπηδά από μία επιτρεπόμενη τροχιά σε άλλη μικρότερης ενέργειας.
- $E_{\text{ΟΛ}} = E_K + U + \Delta = k\dfrac{e^2}{2r} + \left(-k\dfrac{e^2}{4}\right) = -\dfrac{ke^2}{2r}$: Ολική ενέργεια ηλεκτρονίου του ατόμου του υδρογόνου (στο σύστημα ηλεκτρονίου ηλεκτρονίου - πυρήνα).
- $r_n = n^2 \cdot r_1\ (m)$: Ακτίνα επιτρεπόμενων τροχιών.

$n \in \mathbb{N}^*$: κύριος κβαντικός αριθμός. $r_1 = \dfrac{\hbar^2}{mkq_e^2} = 0.53 \cdot 10^{-10}m =$ ακτίνα Bohr

- $1\text{ev} = 1.6 \cdot 10^{-19}\ (J)$: Το ηλεκτρονιοβόλτ είναι η ενέργεια που μεταβιβάζεται σε ένα ηλεκτρόνιο όταν αυτό επιταχύνεται μέσω διαφοράς δυναμικού 1V.
- $E_n = \dfrac{E_1}{n^2}\ (J)$: Επιτρεπόμενες τιμές ενέργειας, $E_1 = -\dfrac{mk^2e^4}{2\hbar^2} = -13.6\text{eV}$
- $E_{\text{ιονισμού}} = E_\infty - E_1$: Ποσότητα ενέργειας που απαιτείται για την απομάκρυνση του ηλεκτρονίου εκτός του ηλεκτρικού πεδίου του πυρήνα.
- $\lambda_{MIN} = \dfrac{ch}{eV}\ (m)$: Μικρότερο μήκος κύματος της ακτινοβολίας που εκπέμπεται, όταν η ενέργεια του ηλεκτρονίου μετατρέπεται σε ενέργεια φωτονίου σε μία μόνο κρούση($k_{\text{τελ}} = 0$), V : τάση που επιταχύνει τη δέσμη ηλεκτρονίων

- Η <u>ατομική μονάδα μάζας</u> (atomic mass unit = amu) ορίζεται ως το 1/12ο της μάζας του ατόμου του άνθρακα-12 ^{12}C, $1\text{amu} = 1.66 \cdot 10^{-24}\text{g}$.
- <u>Σχετική ατομική μάζα</u>, ή <u>ατομικό βάρος</u> A_r λέγεται ο αριθμός που δείχνει πόσες φορές είναι μεγαλύτερη η μάζα του ατόμου του στοιχείου από το 1amu.

$A_r = \dfrac{m_{\text{ατόμου}}}{1.66 \cdot 10^{-27}\text{kg}}$

- Το mol είναι μονάδα ποσότητας ουσίας στο $S.I.$ και ορίζεται ως η ποσότητα της ύλης που περιέχει τόσες στοιχειώδεις οντότητες, όσος είναι ο αριθμός των ατόμων που υπάρχουν σε 12g του άνθρακα -12 ^{12}C. Ο αριθμός των ατόμων που υπάρχουν σε 12g του ^{12}C ονομάζεται αριθμός Avogadro (N_A) και υπολογίστηκε πειραματικά πως είναι ίσος με: $N_A \simeq 6.0252 \cdot 10^{-23}$ mol^{-1}. Με άλλα λόγια 1 mol είναι η ποσότητα μιας ουσίας που περιέχει N_A οντότητες.
- Ο αριθμός Avogadro εκφράζει τον αριθμό των ατόμων οποιουδήποτε στοιχείου που περιέχονεται σε μάζα τόσων γραμμαρίων όσο είναι η σχετική ατομική μάζα του. Μπορούμε να πούμε ότι 1mol ατόμων περιέχει N_A άτομα και ζυγίζει A_rg.

$(1\text{g} \simeq 1.66 \cdot 10^{-23}\ A_r)$

- Ο αριθμός Avogadro εκφράζει τον αριθμό των μορίων στοιχείου χημικής ένωσης που περιέχονται σε μάζα τόσων γραμμαρίων όσο είναι η σχετική ατομική μάζα τους. Δηλαδή, 1mol μορίων περιέχει N_Aμόρια και ζυγίζει M_r g.
- <u>Γραμμομοριακή μάζα</u> M ενός στοιχείου ή μίας χημικής ένωσης είναι η μάζα ενός mole μορίων της και μετριέται στο S.I. σε kg / mol. Η γραμμομοριακή μάζα (molar mass) είναι 1,000 φορές μικρότερη από τη σχετική μοριακή μάζα (μοριακή βάρος) M_r : $\therefore M = \dfrac{M_r}{1000}$

- <u>Γραμμομοριακός, ή μοριακός όγκος αερίου</u> (V_m) ονομάζεται ο όγκος που καταλαμβάνει 1mol αυτού, σε ορισμένες συνθήκες πίεσης και θερμοκρασίας. Σε πρότυπες συνθήκες πίεσης (1atm) και θερμοκρασίας ($0°C$) - S.T.P., ο γραμμομοριακός όγκος των αερίων βρέθηκε πειραματικά ότι είναι ίσος με 22.4L, δλδ.: $V_m = 22.4\frac{L}{\text{mol}}$ σε STP συνθήκες. (δλδ. $V \propto n$).
- N_A άτομα ↔ 1 mol ατόμων ↔ $A_r g$ μέσω του μοριακού τύπου
- N_A μόρια ↔ 1 mol μορίων ↔ $M_r g$ STP και μόνο για αέρια
- N_A οντότητες αέριας ουσίας ↔ 1 mol αέριας ουσίας ↔ V_m = 22.4L
- Αριθμός των mol ουσίας: $n = \frac{m_{\text{ΟΛ}}}{M}$, $m_{\text{ΟΛ}}$: ολική μάζα ουσίας
- Η μεταβολή στην οποία η θερμοκρασία παραμένει σταθερή ονομάζεται <u>ισόθερμη</u>.
- Η μεταβολή στην οποία ο όγκος παραμένει σταθερός ονομάζεται <u>ισόχωρη</u>.
- Η μεταβολή στην οποία η πίεσης παραμένει σταθερή ονομάζεται <u>ισοβαρής</u>.
- $p = \frac{1}{3}\frac{Nm\bar{U}^2}{V}$: Σχέση πίεσης με τη μέση τιμή των ταχυτήτων των μορίων του αερίου, m : μάζα κάθε μορίου, N : πλήθος μορίων, V : όγκος δοχείου.
- $U = \sqrt{\bar{U}^2} = \sqrt{\frac{3kT}{m}}$: Σχέση ταχύτητας (U) με τη θερμοκρασία (T) (των μορίων) του αερίου,

M : η γραμμομοριακή μάζα, U_π : πιθανότερη τιμή
- $\frac{1}{2}m\bar{U}^2 = \frac{3}{2}kT$: Η μέση Κ.Ε. των μορίων του ιδανικού αερίου είναι ανάλογη με την απόλυτη θερμοκρασία.
- Οι ταχύτητες των μορίων κάποιας ποσότητας αερίου σε θερμοκρασία T ακολουθούν την κατανομή Maxwell-Boltzmann.
- Έπειτα από n χρόνους υποδιπλασιασμού (ημιζωές) ($t_{1/2}$) ραδιενεργού ισοτόπου έχει απομείνει: $m = (\frac{1}{2})^n \cdot m_0$

m_0 : ποσότητα αρχικής, ραδιενεργούς ουσίας

Μονάδες Ραδιενέργειας:

1. Μονάδες που εκφράζουν το επίπεδο ραδιενέργειας ενός υλικού.
   Συνηθέστερη μονάδα είναι το Curie(Ci), που είναι ποσότητα ουσίας που υφίσταται $3.7 \cdot 10^{10}$ ραδιενεργές διασπάσεις ανά δευτερόλεπτο.
   Στο S.I. μονάδα ραδιενέργειας είναι το Becquerel (Bq), που αντιστοιχεί σε μία ραδιενεργό διάσπαση ανά δευτερόλεπτο, δλδ. $1Ci = 37 \cdot 10^9 Bq$.
   Μονάδες που εκφράζουν την απορροφούμενη ακτινοβολία από έναν οργανισμό.
   Για ποσοτική εκτίμηση των αποτελεσμάτων της επίδρασης της ακτινοβολίας, θεσπίστηκες το RAD (Radiation Absorbed Dose) που εκφράζει δόση ακτινοβολίας, η οποία απελευθερώνει 10^{-2}J ενέργειας ανά kg βάρους του σώματος που την απορροφά.
   Στο SI μονάδα είναι το Gray(Gy)
   1Gy = 100 RAD

2. Μονάδες που εκφράζουν την απορροφούμενη ακτινοβολία από έναν οργανισμό σε σχέση

με τις βιολογικές επιπτώσεις που προκαλούν.
Το REM (Radiation Equivalent Man) είναι μια μονάδα ραδιενέργειας που δεν εξαρτάται από το είδος της ακτινοβολίας και εκφράζει τις βιολογικές καταστροφές που προκαλούνται στον άνθρωπο από την απορρόφηση των διαφόρων ακτινοβολιών. Δηλαδή 1rem είναι ποσότητα ακτινοβολίας, η οποία επιφέρει ένα συγκεκριμένο βιολογικό αποτέλεσμα.
1rem = 1rad ακτινών X ή γ. 1Gy ακτινοβολίας α προκαλεί 20 φορές μεγαλύτερη καταστροφή στους ανθρώπινους ιστούς από 1 Gy ακτινοβολίας γ.

29.3 Thermodynamics

- Όταν σ'ένα θερμοδυναμικό σύστημα οι θερμοδυναμικές μεταβλητές, δηλαδή η πίεση(p), η πυκνότητα(p) και η θερμοκρασία (T), που το περιγράφουν διατηρούνται σταθερές με το χρόνο, τότε το σύστημα βρίσκεται σε κατάσταση θερμοδυναμικής ισορροπίας. Η κατάσταση θερμοδ. ισορ. ενός συστήματος μπορεί να παρασταθεί γραφικά με ένα σημείο. Ένα σύστημα που δεν βρίσκεται σε ισορροπία δεν παριστάνεται γραφικά.
- $\Delta W = p\Delta V$: στοιχειώδες έργο (ΔW) της δύναμης που ασκεί το αέριο (πίεσης p) στο σώμα (π.χ. έμβολο) μετατοπίζοντας το, κατά όγκο ΔV.
- $U = \frac{3}{2}nRT\ (J)$: Η εσωτερική ενέργεια ορισμένης ποσότητας ιδανικού αερίου εξαρτάται μόνο από τη θερμοκρασία του.
- <u>1ος θερμοδυναμικός νόμος</u>: $Q = \Delta U + W$

Το ποσό θερμότητας (Q) που απορροφά ή αποβάλλει ένα θερμοδυναμικό σύστημα είναι ίσο με το αλγεβρικό άθροισμα της μεταβολής της εσωτερικής του ενέργειας και του έργου που παράγει ή δαπανά το σύστημα.
- Ειδική γραμμομοριακή θερμότητα αερίου υπό σταθερή πίεση (constant pressure specific heat): C_p

Ισχύει: $Q_p = nC_p\Delta T$: θερμότητα που απορροφά το αέριο όταν θερμαίνεται υπό σταθερό όγκο
- Ειδική γραμμομοριακή θερμότητα αερίου υπό σταθερό όγκο: C_v

Θα ισχύει: $Q_v = nC_v\Delta T$: θερμότητα που απορροφά το αέριο όταν θερμαίνεται υπό σταθερό όγκο
- Από τον 1ο Ν.Δ. προκύπτει ότι: $C_p - C_v + R$
- $\gamma = \dfrac{C_p}{C_v}$, $\gamma > 1$

Εφαρμογή του 1ου θερμοδυναμικού νόμου
- Α) Ισόθερμη Αντιστρεπτή μεταβολή

Έργο: $W = nRT\ln\left(\dfrac{V_{\text{τελ}}}{V_{\text{αρχ}}}\right)$, αφού $T = \text{σταθ.} \rightarrow Q = W$
- Στην ισόθερμη εκτόνωση όλο το ποσό θερμότητας που απορροφά το αέριο μετατρέπεται σε μηχανικό έργο.
- Β) Ισόχωρη Αντιστρεπτή μεταβολή

$Q = \Delta U$: Στην ισόχωρη θέρμανση όλο το ποσό θερμότητας που απορρόφησε το αέριο χρησιμοποιήθηκε για την αύξηση της εσωτερικής του ενέργειας.
- Γ) Ισοβαρής Αντιστρεπτή μεταβολή

Έργο: $W = p\,(V_{\text{τελ}} - V_{\text{αρχ}})$, Από τον 1ο Θ.Ν.: $Q = \Delta U + p\,(V_\tau - V_\alpha)$.
Στην ισοβαρή θέρμανση ένα μέρος από το ποσό θερμότητας που απορρόφησε το αέριο από το περιβάλλον χρησιμοποιήθηκε για την αύξηση της εσβτερικής του ενέργειας και το υπόλοιπο αποδόθηκε εκ νέου στο περιβάλλον υπό μορφή έργου.

- Δ) Αδιαβατική Μεταβολή

Μεταβολή κατά την οποία δε συντελείται μεταφορά θερμότητας από το περιβάλλον στο σύστημα και αντίστροφα.
$pV^\gamma =$ σταθ.: Νόμος του Poisson που διέπει τη μεταβολή.
Από τον 1ο Θ.Ν.: $0 = \Delta U + W \leftrightarrow W = -\Delta U$

- Ε) Κυκλική Αντιστρεπτή μεταβολή

Κυκλική ονομάζουμε τη μεταβολή στην οποία το σύστημα μετά από μια διεργασία επιστρέφει στην ίδια κατάσταση $\therefore Q = W$

Υπολογισμός μεγεθών κίνησης απ΄τα Διαγράμματα

- Διάγραμμα επιτάχυνσης - χρόνου: το εμβαδόν από το γράφημα μέχρι τον άξονα του χρόνου μας δίνει τη μεταβολή της ταχύτητας.
- Διάγραμμα ταχύτητας - χρόνου: το εμβαδόν από το γράφημα μέχρι τον άξονα του χρόνου μας δίνει τη μετατόπισηΔx : η κλίση της ευθείας μας δίνει την επιτάχυνση
- Διάγραμμα θέσης - χρόνου: η κλίση της ευθείας μας δίνει την ταχύτητα του σώματος

Ταλαντώσεις

- Το έργο δύναμη F βρίσκεται από το εμβαδόν της $F = f(x)$ μέχρι τον οριζόντιο άξονα x.
- Ιδιοσυχνότητα ελεύθερης ταλάντωσης: $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$,

m : σώμα που ταλαντώνεται, k : σταθερά ελατηρίου. Μιλάμε ιδανικά. Στην πραγματικότητα η f_0 θα είναι λίγο μικρότερη αφού απαιτείται επιπλέον δύναμη για να διατηρηθεί ως έχει.

- $T = F \cdot l\ (N \cdot m)$: Ροπή δύναμης F, l : (κάθετη) απόσταση μεταξύ άξονα περιστροφής και σημείου εφαρμογής της δύναμης. Η ροπή έχει τη διεύθυνση του άξονα περιστροφής και η φορά της δίνεται από τον κανόνα του δεξιού χεριού.
- $W = T \cdot \theta$ (J): Έργο κατά τη στροφική κίνηση που προκαλείται από στροφική δύναμη (ροπή). Για να το υπολογίσουμε χωρίζουμε τη γωνία θ σε απειροστά μικρές γωνίες $d\theta_1, d\theta_2, \ldots$ και αθροίζουμε τα αντίστοιχα έργα. Αν η ροπή της δύναμης παραμένει σταθερή το έργο δίνεται από τον παραπάνω τύπο.
- Προώθηση του πυραύλου: $U_{\text{τελ}} - U_{\text{αρχ}} = U_{\text{σχ. αερίων}} \ln\left(\dfrac{M_{\text{αρχ}}}{M_{\text{τελ}}}\right)$

$U_{\text{σχ. αερίων}} =$ σχετική ταχύτητα αερίων
Η προώθηση του στηρίζεται στην Α.Δ.Ο., επομένως με την εξίσωση αυτή, μελετάμε την προώθηση του στο διάστημα, μακριά από κάθε βαρυτική έλξη, όπου μπορούμε έτσι να θεωρήσουμε το σύστημα μονωμένο.

- Φαινόμενο Doppler: πηγή $= S$, παρατηρητής $= A$

  1. Ακίνητη πηγή - Ακίνητος παρατηρητής: $f_A = f_s = \dfrac{U}{\lambda}$,

όπου: U, λ η ταχύτητα και το μήκος του κύματος που εκπέμπει η πηγή

2. Ακίνητη πηγή - Κινούμενος παρατηρητής με ταχύτητα U_A
$f_A = \dfrac{U \pm U_A}{U} f_s$, +: παρατηρητής απομακρύνεται από την πηγή και -: όταν πλησιάζει σε αυτή.

3. Κινούμενη πηγή με ταχύτητα U_s - Ακίνητος παρατηρητής: $f_A = \dfrac{U}{U \pm U_s} f_s$, +: πηγή απομακρύνεται από τον παρατήρητη, ενώ -: πλησιάζει

4. Κινούμενη πηγή με ταχύτητα U_s - Κινούμενος παρατηρητής με ταχύτητα U_A :
$f_A = \dfrac{U \pm U_A}{U \mp U_s} f_s$, $\dfrac{+}{-}$: όταν πλησιάζουν, $\dfrac{-}{+}$: όταν απομακρύνονται

- $e = \dfrac{W}{Q_h}$: Συντελεστής απόδοσης οποιασδήποτε μηχανής

Q_n : θερμότητα με την οποία τροφοδοτούμε τη μηχανή

- Στην κυκλική μεταβολή το έργο που παράγει το αέριο ισούται με το καθαρό ποσό θερμότητας που απορροφά, το οποίο είναι ίσο με το ποσό θερμότητας που τροφοδοτείται μείον το ποσό θερμότητας που αποβάλλει Q_c.

$e = 1 - \dfrac{|Q_c|}{Q_h}$

- Μηχανή Carnot: $e_{\text{carnot}} = 1 - \dfrac{T_c}{T_h}$

T_c : θερμοκρασία ψυχρής δεξαμενής
T_h : θερμοκρασία θερμής δεξαμενής
Δεν μπορεί να υπάρξει θερμική μηχανή που να έχει μεγαλύτερη απόδοση από μια μηχανή Carnot η οποία λειτουργεί ανάμεσα στις δύο θερμοκρασίες.

- 2ος θερμοδυναμικός νόμος
- Είναι αδύνατο να κατασκευαστεί μηχανή που να μετατρέπει εξ ολοκλήρου τη θερμότητα σε ωφέλιμο έργο.
- Είναι αδύνατο να κατασκευαστεί μηχανή που να μεταφέρει θερμότητα από ένα ψυχρό σώμα σε ένα θερμότερο, χωρίς να δαπανάται ενέργεια για τη λειτουργία της.
- $\Delta S = \int \dfrac{\Delta Q}{T}$ (J/k): Η μεταβολή της εντροπίας (ΔS) συστήματος κατά τη διάρκεια μιας πολύ μικρής αντιστρεπτής μεταβολής, τόσο μικρής ώστε η θερμοκρασία του συστήματος να μπορεί να θεωρηθεί σταθερή.

Όταν σε μια αντιστρεπτή μεταβολή το ΔQ είναι θετικό όταν το σύστημα απορροφά θερμότητα, επομένως η εντροπία αυξάνεται. Ισχύει και το αντίθετο.

- Από μακροσκοπική άποψη η αύξηση της εντροπίας οδηγεί σε μείωση της ικανότητας του συστήματος να παράγει ωφέλιμο έργο, ενώ από μικροσκοπική άποψη η αύξηση της εντροπίας οδηγεί σε αύξηση της αταξίας του συστήματος

Περίπτωσεις υπολογισμού μεταβολής της εντροπίας

- Αδιαβατική αντιστρεπτή μεταβολή: $\Delta S = 0$
- Ισόθερμη αντιστρεπτή μεταβολή: $\Delta S = \dfrac{Q}{T}$
- Κυκλική μεταβολή: $\Delta S_{\text{ΟΛ}} = 0$
- Ελεύθερη εκτόνωση: $\Delta S = nR\,\left(\dfrac{V_B}{V_A}\right)$

- Το έργο στην αδιαβατική αντιστρεπτή μεταβολή είναι:

$W = \dfrac{p_{\text{τελ}} V_{\text{τελ}} - p_{\text{αρχ}} V_{\text{αρχ}}}{1-\gamma}$

- Η μεταβολή στην εσωτερική ενέργεια ενός αερίου δίνεται από τη σχέση: $\Delta U = nC_v \Delta T$
- Θερμικές μηχανές ονομάζουμε αυτές που μετατρέπουν τη θερμότητα σε μηχανικό έργο.

29.4 Electric Field

$\vec{\Phi}_E = \vec{E} \cdot A\cos(\theta)\ (N \cdot m^2/C)$: Ηλεκτρική ροή που διέρχεται από μια επίπεδη επιφάνεια, εμβαδού Α, η οποία βρίσκεται μέσα σε ομογενές ηλεκτρικό πεδίο έντασης Ε.
θ : η γωνία που σχηματίζει το κάθετο στην επιφάνεια διάνυσμα Α με τη διεύθυνση των δυναμικών γραμμών

- Στη γενικότερη περίπτωση όπου η επιφάνεια δεν είναι επίπεδη και βρίσκεται μέσα σε ανομοιογενές ηλ. πεδίο:

$\vec{\Phi}_E = \sum_{i=1}^{n} \vec{E}_i \Delta A_i \cos(\theta_i)$, όπου n το πλήθος των τμήσεων της επιφάνειας Α σε στοιχειώδεις επιφάνειες επίπεδες και σταθερής έντασης.

- $\Phi_E = \dfrac{Q_{\text{εγκ}}}{\epsilon_0}$: Νόμος του Γκάους για το ηλ. πεδίο

Η ηλεκτρική ροή που διέρχεται από μια κλειστή επιφάνεια ισούται με το πηλίκο του ολικού φορτίου που περικλείει η επιφάνεια, προς τη σταθερά ϵ_0.

- Δυναμική ενέργεια ισούται με το πηλίκο του ολικού φορτίου που περικλείει η επιφάνεια, προς τη σταθερά ϵ_0.
- Δυναμική ενέργεια πολλών (έστωn) σημειακών φορτίων

$U = k\dfrac{q_1 q_2}{r_1} + k\dfrac{q_1 q_3}{r_2} + k\dfrac{q_2 q_3}{r_3} + k\dfrac{q_1 q_4}{r_4} + \ldots k\dfrac{q_{n-1} q_n}{r_n}$

δλδ. η ενέργεια του συστήματος είναι το άθροισμα των ενεργειών που έχουν τα φορτία ανά ζεύγη.

- $K = \dfrac{C}{C_0} > 1$: διηλεκτρική σταθερά του υλικού

C : χωρητικόητα του πυκνωτή χωρίς το διηλεκτρικό

29.5 Magnetic Field

- $\vec{g} = \dfrac{\vec{F}}{m}$ (N/kg): Ένταση πεδίου βαρύτητας σε ένα του σημείο, που ταυτίζεται με την επιτάχυνση a που θα αποκτήσει το σώμα, εάν αφεθεί ελεύθερο σε εκείνο το σημείο, δλδ.: $\vec{a} = \vec{g}\ (\dfrac{m}{s^2})$
- $V_A = \dfrac{W_{A\to\infty}}{m}$ (J/kg): Δυναμικό του πεδίου βαρύτητας.
- $V_{AB} = V_A - V_B = \dfrac{W_{A\to\infty}}{m}$ (J/kg): Διαφορά δυναμικού μεταξύ δύο σημείων του πεδίου βαρύτητας
- $g = G\dfrac{M}{r^2}$ (N/kg): ένταση βαρυτικού πεδίου που παράγεται από μάζα M σε σημείο που βρίσκεται μάζα m που απέχει απόσταση r από το υλικό σημείο
- $U = -G\dfrac{m_1 m_2}{r}$ (J): Δυναμική ενέργεια συστήματος δύο υλικών σημείων με μάζες m_1, m_2 που απέχουν μεταξύ τους απόσταση r. Το αρνητικό πρόσημο υποδηλώνει ότι πρέπει να προσφέρουμε ενέργεια για να κάνουμε άπειρη την απόσταση των δύο μαζών
- Με ικανοποιητική προσέγγιση, μπορούμε να θεωρήσουμε ότι για τη Γη, ή οποιοδήποτε γεωειδές ουράνιο σώμα, θα ισχύει:

$g = G\dfrac{M_\Gamma}{(R_\Gamma + h)^2}$ και $V = -G\dfrac{M_\Gamma}{R_\Gamma + h}$

Για $h = 0 : g = G\dfrac{M_\Gamma}{R_\Gamma^2} \simeq 9.8\ (m/s^2)$

- $\vec{U}_\delta = \sqrt{\dfrac{2GM}{R+h}}\ (m/s)$: Ταχύτητα διαφυγής σώματος, από ουράνιο σώμα μάζας M ακτίνας P όταν το σημείο εκτόξευσης βρίσκεται σε ύψος h από την επιφάνεια.

29.6 Fundamental Constants

- $\epsilon_0 = 8.85 \cdot 10^{-12}\ C^2/N \cdot m^2$: Απόλυτη διηλεκτρική σταθερά του κενού
- $q_p = 1.6 \cdot 10^{-19}\ C$: Φορτίο πρωτονίου
- $q_e = -1.6 \cdot 10^{-19}\ C$: Φορτίο ηλεκτρονίου
- $m_p = 1.672631 \cdot 10^{-27}\ kg = 938.27231\ MeV/C^2$: Μάζα πρωτονίου
- $m_e = 9.1093897 \cdot 10^{-31}\ kg = 0.51099906 MeV/C^2$: Μάζα ηλεκτρονίου
- $m_n = 1.6749286 \cdot 10^{-27}\ kg = 939.56563\ MeV/C^2$: Μάζα νετρονίου
- $K = 8.987552 \cdot 10^9\ N \cdot m^2/C^2 = 1/4\pi\epsilon_0$: Ηλεκτρική σταθερά ή σταθερά του Coulomb $\simeq 9 \cdot 10^9\ Nm^2/C$
- $G = 6.67259 \cdot 10^{-11}\ \dfrac{m^3}{kg \cdot s^2} \simeq 6.67 \cdot 10^{-11}\ m^3 kg^{-1} s^{-2}$: Σταθερά της παγκόσμιας έλξης
- $C = 2.99792458 \cdot 10^8\ m/s \simeq 3 \cdot 10^8 m/s$: Ταχύτητα του φωτός
- $h = 4.1356692 \cdot 10^{-15}\ eV \cdot s = 6.63 \cdot 10^{-34}\ J \cdot s$: σταθερά του Planck
- $u = 1.6605402 \cdot 10^{-27} kg = 931.49432\ MeV/C^2 = 1 a.m.u.$: Ατομική μονάδα μάζας

- $N_A = 6.0221367 \cdot 10^{23}\ \dfrac{\text{οντότητες}}{\text{mol}}$: Σταθερά Avogadro.
- 1eV $= 1.6 \cdot 10^{-19}$ J: Ηλεκτρονιοβόλτ
- 1atm $= 101.325\ Pa$: Μία τυπική ατμόσφαιρα
- $\hbar = \dfrac{h}{2\pi} = 1.0545727 \cdot 10^{-34}\ J \cdot s$: Μειωμένη σταθερά Planck
- $F = e \cdot N_A = 96,485.309\ C/\text{mol}$: Σταθερά Faraday = ποσότητα θεμελιώδους φορτίου e^- / ανά mol
- $a_0 = 5.29177249 \cdot 10^{-11}\ m$: Ακτίνα του Bohr
- $g \stackrel{\triangle}{=} 9.80665\ m/s^2$: Επιτάχυνση βαρύτητας της Γης
- $\mu_0 = 4\pi \cdot 10^{-7} \simeq 1.256637 \cdot 10^{-6}\ N/A^2$ ή $T \cdot m$ ή $Wb/A \cdot m$): Μαγνητική διαπερατότητα του κενού

30 Chemistry

- pH: Μέτρο της οξύτητας ενός διαλύματος pH $= -\log[H^+]$

$[H^+]$: η συγκέντρωση των H^+ σε γραμμοιόντα ανά λίτρο
- Σύσταση χημικής ένωσης $= \dfrac{\text{μάζα χημ. στοιχείου 1}}{\text{μάζα χημ. στοιχείου 2}} =$ σταθερό
- Μάζα αντιδρώντων = Μάζα προιόντων
- Για κάθε άτομο ισχύει: $A = Z + N$
- οξύ + βάση → άλας + νερό
- Polyatomic / Compound ions: carbonate: CO_3^{2-}

hydroxide: OH^-, sulfate: SO_4^{2-}, nitrate: NO_3^-, ammonium: NH_4^+

- 1L = 1dm^3 = 1,000cm^3
- K=C+273 Kelvin ↔ Celcius • Ατομικότητα είναι ο αριθμός που μας δείχνει από πόσα άτομα αποτελείται το μόριο ενός στοιχείου
- Τα ανόργανα οξέα κατά Arrhenius έχουν το γενικό τύπο: H_xA όπου, A : αμέταλλο, ή ομάδα ατόμων (ρίζα π.χ. SO_4), x : ο αριθμός οξείδωσης του A
- Οι ανόργανες βάσεις κατά Arrhenius έχουν το γενικό τύπο: $M(OH)_x$, όπου, M : μέταλλο, x : ο αριθμός οξείδωσης του M.
- Τα περισσότερα οξείδια έχουν το γενικό τύπο: Σ_2O_x, όπου, x ο αριθμός οξείδωσης του στοιχείου Σ
- Τα άλατα είναι ιοντικές ενώσεις που περιέχουν κατιόν M (μέταλλο, ή θετικό πολυατομικό ιόν, π.χ. NH_4^+) και ανιόν A (αμέταλλο εκτός O, ή αρνητικό πολυατομικό ιόν, π.χ. CO_3^{2-}). Έτσι, ο γενικός τους τύπος είναι: $M_\psi A_x$, όπου x και ψ δείχνουν την αναλογία ανιόντων και κατιόντων αντίστοιχα στην ιοντική ένωση

- Συγκέντρωση, ή μοριακότητα κατ΄όγκο (molarity) διαλύματος: $c = \dfrac{n}{v}$ (M = 1 mol/L).

n : ποσότητα διαλυμένης ουσίας (mol), V : όγκος διαλύματος (L)
- Κατά την αραίωση διαλύματος ισχύει ο τύπος: $c_{\text{αρχ}} \cdot V_{\text{αρχ}} = c_{\text{τελ}} \cdot V_{\text{τελ}}$
- Κατά την ανάμειξη δύο, ή περισσότερων διαλυμάτων ισχύει η σχέση:

$c_{\text{αρχ}1} \cdot V_{\text{αρχ}1} + c_{\text{αρχ}2} \cdot V_{\text{αρχ}2} + \ldots + c_{\text{αρχν}} \cdot V_{\text{αρχν}} = c_{\text{τελ}} \cdot V_{\text{τελ}}$

όπου, 1,2, ..., n: πλήθος αρχικών διαλυμάτων

και $V_{\text{τελ}} = V_{\text{αρχ}1} + V_{\text{αρχ}2} + V_{\text{αρχ}3} + \ldots + V_{\text{αρχν}}$
- Χημική αντίδραση καύσης αλκανίων (για την έναρξη της απαιτείται σπινθήρας):

$C_vH_{2v+2} + \dfrac{3v+1}{2}O_2 \to vCO_2 + (v+1)H_2O$
- Χημική αντίδραση πλήρους καύσης αλκενίων:

$C_vH_{2v} + \dfrac{3v}{2}O_2 \to vCO_2 + vH_2O$
- Χημική αντίδραση πλήρους καύσης κορεσμένων μονοσθενών αλκοολών:

$C_vH_{2v+1}OH + \dfrac{3v}{2}O_2 \to vCO_2 + (v+1)H_20$
- Αντίδραση φωτοσύνθεσης:

$xCO_2 + \psi H_2O +$ ηλιακή ενέργεια $\to C_x(H_2O)_\psi + xO_2$

- Αντίδραση παραγωγής τριφωσφορικής αδενοσίνης:

ADP + $\text{P}_{\text{ανόργανα φωσφορικά άλατα}}$+ ενέργεια → ATP

30.1 Useful Chemical Substances

- $C_6H_{12}O_6$: Γλυκόζη
- $Ca_3(PO_4)_2$: Ανθρακικό άλας
- $CuSO_45H_2O$: Ένυδρο άλας / γαλαζόπετρα
- $CaSO_42H_2O$: Γύψος
- NH_4 : Αμμωνία
- C_2H_6O : Οινόπνευμα / αιθανόλη
- $C_{12}H_{22}O_{11}$: Ζάχαρη / σακχαρόζη
- CO_2 : Διοξείδιο του άνθρακα
- $Ca_3(PO_4)_2$: Φωσφορικό ασβέστιο
- H_2SO_4 : Θειικό οξύ / Βιτριόλι
- H_3PO_4 : Φωσφορικό οξύ
- HCN : Υδροκυάνιο
- $NaCl$: Χλωριούχο νάτριο / αλάτι
- HNO_3 : Νικτρικό οξύ / ακουαφόρτε
- KOH : Υδροξείδιο του καλίου / καυστική ποτάσα
- $NaOH$: Υδροξείδιο του νατρίου / καυστική σόδα
- $NaHCO_3$: Ανθρακικό νάτριο / σόδα
- Al_2O_3 : Οξείδιο του αργιλίου / ζαφείρι
- C : Άνθρακας, Διαμάντι, Γραφίτης
- $Al_2O_3 : C_r$: Οξείδιο αργιλίου - χρωμίου / ρουμπίνι (ρυβψ)
- $Be_3Al_2(S_iO_3)_6$: Σμαράγδι (εμεραλδ)
- $C_6H_8O_7$: Κιτρικό οξύ (figure 47)
- $(CH_3)_2CO$: Acetone / Propanone

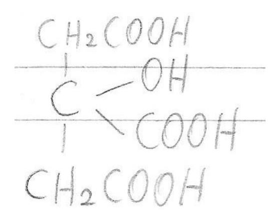


Figure 47: Χημική εξίσωση Κιτρικού οξέως

- $CaCO_3$: Ανθρακικό ασβέστιο / ασβεστόλιθος / μάρμαρο
- CH_4N_2O : Ουρία
- $C_2H_4O_2$ (CH_3COOH) : Αιθανικό / οξικό οξύ
- $C_9H_8O_4$: Ακετυλοσαλικιλικό οξύ / ασπιρίνη

- $CHCl_3$: Τριχλωρομεθάνιο / Χλωροφόρμιο
- CaC_2 : Ανθρακασβέστιο
- $Ca(OH)_2$: Ασβεστόνερο
- $HCHO$: Μεθανάλη / Φορμαλδεΰδη
- 2-υδροξυπροπανικό οξύ / Γαλακτικό οξύ (figure 48)

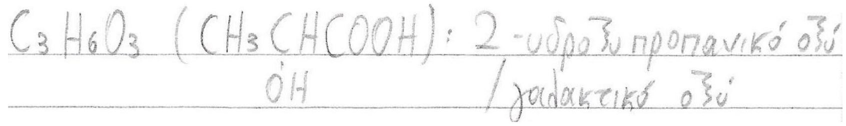


Figure 48: Χημική εξίσωση Γαλακτικού οξέως

- C_6H_5COOH : Βενζοικό οξύ / Ε120(συντηρητικό τροφίμων

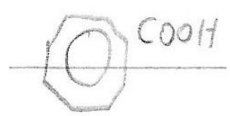


Figure 49: Χημικός τύπος Καρβοξυλικού οξέος - COOH - Βενζοικός δακτύλιος

- $C_4H_6O_6$: Χημικός τύπος Τρυγικού οξέως (απαντάται στο κρασί & σε αναψυκτικά) (figure 50)

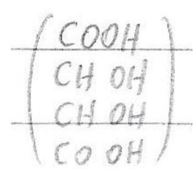


Figure 50: Τρυγικό οξύ - Κρασί

- $CH_2O_2/HCOOH$: Μεθανικό οξύ / Μυρμηκικό οξύ
- $C_4H_8O_2$: Βουτανικό / Βουτυρικό οξύ
- Πυροσταφιλικό οξύ (figure 51)

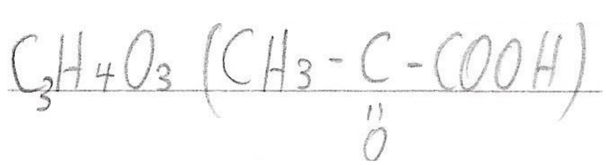


Figure 51: Τρυγικό οξύ - Κρασί

31 Economics

- $a_\nu = a(1+\tau)^\nu$: Τύπος του ανατοκισμού Αν καταθέσει κάποιος στη τράπεζα μετρητά, μετά από ν χρόνια θα εισπράξει a_ν μετρητά

α: κεφάλαιο, $\tau = \dfrac{\epsilon(\%)}{100}$: τόκος τους ενός ευρώ σε 1 χρόνο

ν: χρόνια, ε: επιτόκιο (%)

- $\tau = \dfrac{\epsilon}{100} \cdot \alpha$: τόκος τ που αποδίδει το κεφάλαιο α με επιτόκιο ε
- $\Sigma = \alpha(1+\tau)\dfrac{(1+\tau)^\nu - 1}{\tau}$: τύπος των ίσων καταθέσεων
- Κόστος ευκαιρίας του αγαθού $Y = \dfrac{\text{Μονάδες του αγαθού Χ που θυσιάζονται}}{\text{Μονάδες του αγαθού Υ που παράγονται}}$ σε όρους του αγαθού Χ

ή $KE_y = \dfrac{\Delta X}{\Delta Y}$: οι μονάδες του αγαθού Χ που θυσιάστηκαν για την παραγωγή μιας επιπλέον μονάδας του Υ

- $E_D = \dfrac{\Delta Q}{\Delta P} \cdot \dfrac{P_1}{Q_1}$: Ελαστικότητα της ζήτησης στο σημείο που αντιστοιχεί σε τιμή P_1 και ζητούμενη ποσότητα Q_1.

$|E_D| > 1 \rightarrow \left|\dfrac{\Delta Q}{Q}\right| > \left|\dfrac{\Delta P}{P}\right|$: Ελαστική ζήτηση

$|E_D| < 1 \rightarrow \left|\dfrac{\Delta Q}{Q}\right| < \left|\dfrac{\Delta P}{P}\right|$: Ανελαστική ζήτηση

- Μέσο προιόν$(AP) = \dfrac{\text{Συνολικό προιόν}(Q)}{\text{Ποσότητα μεταβλητού συντελεστή}}$
- Οριακό προιόν$(MP) = \dfrac{\text{Μεταβολή συνολικού προιόντος}(\Delta Q)}{\text{Μεταβολή ποσότητας μεταβλητού συντελεστής}}$
- Μέσο σταθερό κόστος$(AFC) = \dfrac{\text{Σταθ. Κόστος}(FC)}{\text{Ποσότητα παραγωγής}(Q)}$
- Μέσο μεταβλητό κόστος$(AVC) = \dfrac{\text{Μεταβλητό κόστος}(VC)}{\text{Ποσότητα παραγωγής}(Q)}$
- Μέσο συνολικό κόστος$(ATC) = \dfrac{\text{Συνολικό κόστος}(TC)}{\text{Ποσότητα παραγωγής}(Q)}$
- $ATC = AFC + AVC$
- Οριακό κόστος$(MC) = \dfrac{\text{Μεταβολή συνολικού κόστους}[\Delta(TC)]}{\text{Μεταβολή του προιόντος}[\Delta Q]} = \dfrac{\text{Μεταβολή μεταβλητού κόστους}[\Delta CVC]}{\text{Μεταβολή του προιόντος}[\Delta Q]}$

Δείχνει το ρυθμό με τον οποίο μεταβάλλεται το συνολικό κόστος, όταν μεταβάλλεται η παραγωγή κατά μια μονάδα • Οι παραπάνω 5 τύποι αφορούν τη βραχυχρόνια περίοδο. Κατά τη μακροχρόνια περίοδο όλοι οι παραγωγικοί συντελεστές δύναται να μεταβληθούν

- $E_s = \dfrac{\Delta Q}{\Delta P} \cdot \dfrac{P_1}{Q_1}$: Ελαστικότητα της προσφοράς, όπου: ΔQ : μεταβολή προσφερόμενης ποσότητας, ΔP : μεταβολή τιμής, P_1 : αρχική τιμή, Q_1 : αρχική ποσότητα
- Συνάρτηση ζήτησης: $Q_D = f(P)$ έχει αρνητική κλίση (η καμπύλη D). Συνάρτηση προσφοράς $Q_s = f(P)$ έχει θετική κλίση (η καμπύλη S).
- Total Revenue $(TR) = P \cdot Q$: Συνολικά έσοδα (επιχείρησης)

P : τιμή, Q : πωλούμενη ποσότητα

- Average Revenue $(AR) = \dfrac{RT}{Q}$: Μέσο έσοδο
- Marginal Revenue $(MR) = \dfrac{\Delta(P \cdot Q)}{\Delta Q}$: Επιπλέον έσοδο από την πώληδη μιας επιπλέον ποσότητας προιόντος
- $K = TR - TC = (AR - ATC) \cdot Q$: Κέρδος (ή ζημία)
- Καθαρή επένδυση = Ακαθάριστη ιδιωτική επένδυση - Αποσβέσεις
- Α.Ε.Π. = Ιδιώτικη κατανάλωση + Ακαθάριστη ιδιωτική επένδυση + Κρατική ή Δημόσια δαπάνη + (Εξαγωγές - Εισαγωγές)
- Εξαγωγές - Εισαγωγές = Καθαρό εισόδημα από το εξωτερικό
- Α.Ε.Π. = Μισθοί + Πρόσοδοι περιουσίας + Τόκοι + Κέρδη + Αποσβέσεις + Έμμεσοι φόροι - Κρατικές επιδοτήσεις - Τόκοι Δημοσίου χρέους - Καθαρό εισόδημα από το εξωτερικό
- Α.Ε.Θ.Π. = Α.Ε.Π. + Καθαρό εισόδημα από το εξωτερικό
- Κ.Ε.Θ.Π. = Α.Ε.Θ.Π. - Αποσβέσεις
- Καθαροί έμμεσοι φόροι = Έμμεσοι φόροι - Επιδοτήσεις
- Εθνικό εισόδημα = Κ.Ε.Θ.Π. - Καθαροί έμμεσοι φόροι
- Διαθέσιμο εισόδημα = Εθνικό εισόδημα + Μεταβιβαστικές πληρωμές + Τόκοι του δημόσιου χρέους - Αδιανέμητα κέρδη - Άμεσοι φόροι
- Αποταμίευση = Διαθέσιμο εισόδημα - Κατανάλωση
- Κατά κεφαλήν πραγματικό Α.Ε.Π. = $\dfrac{\text{Πραγματικό Α.Ε.Π.}}{\text{Πληθυσμός}}$
- Ποσοστό ρευστών διαθεσίμων = Ποσοστό χρημάτων που η τράπεζα διατηρεί αποθηκευμένο στα ταμεία της για κάθε 100 ευρώ φυσικού συναλλάγματος που διαθέτει
- Πραγματικό εισόδημα = $\dfrac{\text{Ονομαστικό Εισόδημα}}{\text{Επίπεδο τιμών}} \cdot 100\ (\%)$
- Ποσοστό ανεργίας = $\dfrac{\text{Αριθμός ανέργων}}{\text{Εργατικό δυναμικό}} \cdot 100\ (\%)$