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Chapter 1

The Torsion Pendulum

The restoring forces responsible for harmonic motion in a conventional pendulum are provided by the local gravitational field. A grandfather clock, standing on the surface of the Moon, will run slow. A grandfather clock, floating in interplanetary space, will not 'tick' at all. Gravity on the Moon is reduced; in free space it is zero.

But if, by the term pendulum, we mean to suggest the rather general situation of a hanging mass executing periodic motion, then in principle different types of pendula could be created by devising configurations for which other forms of restoring force arise. For example, in a *torsion pendulum*, a mass-structure is suspended from a thin fiber that is anchored securely at its other end as shown in Fig. 1.1.

Any twist in the fiber results in a counter torque that leads to the desired back-and-forth oscillations (clockwise, counterclockwise, clockwise, etc.). Note that the plane in which this motion takes place is orthogonal to the direction of gravity, which therefore plays no role other than to give the pendulum weight and thus place the fiber under tension.

The torsion pendulum has proved to be an exceptionally sensitive and indespensible instrument in the determination of a surprisingly varied range of natural phenomena, and has provided (and continues to provide) the means to precisely measure some of the fundamental physical constants. The long and distinguished history of the torsion pendulum must begin with an accounting of its key component – the fiber.

1.1 Elasticity of the Fiber

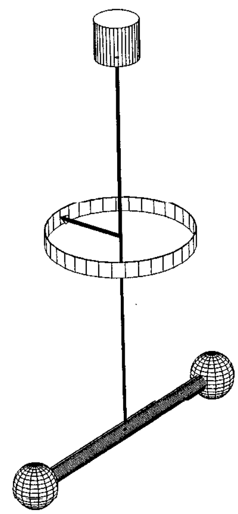

Figure 1.1: The elements of a torsion pendulum. As the apparatus turns, the fiber twists and the pointer rotates to indicate the angular shift.

The key element in this apparatus is the suspension fiber which can be regarded as a long, thin, solid rod. The hanging pendulum payload will cause the fiber to stretch (longitudinal strain). If a torque is applied at the payload, then the fiber also will twist (angular strain). Both of these types of deformation will now be discussed.

The process of stretching is illustrated in Fig. 1.2. With the application of forces F at the ends, the sample elongates by a net amount ΔL above its original length L. The strain ϵ_ℓ is defined as the relative elongation

$$\epsilon_\ell = \frac{\Delta L}{L}$$

If the cross-sectional area of the sample is A, then the applied stress is just the force per unit area F/A.

Perfectly elastic materials (an ideal closely approximated by many real solids) obey Hooke's Law which says that deformation is proportional to force – in other words stress and strain are linearly related. This also implies that when the stress is removed, the sample will return to its original shape

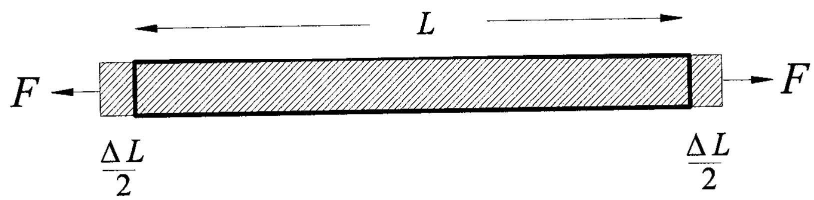

Figure 1.2: Stretching of a thin rod by an applied force-pair at the ends. The outlines of the original sample are in bold.

and size. In real materials, a limit exists (the *elastic limit*) beyond which a sample will not fully recover when the load is removed.

The proportionality constant relating stress and strain is called Young's modulus (or elastic modulus) Y; it is defined by

$$Y = \frac{\text{applied stress}}{\text{resulting strain}}$$

Alternately,

$$F = \left[Y\frac{A}{L}\right]\Delta L \tag{1.1}$$

Of course, as a wire stretches, it also tends to become thinned down. That is, its cross section decreases as suggested in Fig. 1.3.The transverse strain is defined by the expression

$$\epsilon_t = -\frac{\Delta w}{w}$$

The ratio of thinning to stretching is an important parameter known as

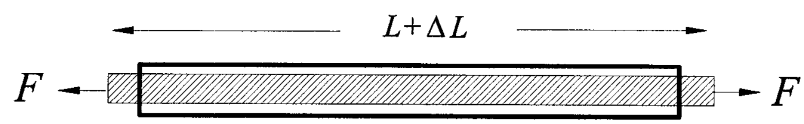

Figure 1.3: Stretching of a thin rod, showing the accompanying decrease in cross sectional dimension. The outlines of the original sample are shown in bold.

Poisson's Ratio σ

$$\sigma = -\frac{\epsilon_t}{\epsilon_\ell}$$

The twisting of a suspension fiber can be modelled by considering the response of a solid cylinder to an applied torque, as in Fig. 1.4.

The shaded sector is located between radii r and $r + dr$, and is bounded by two arc segments of length dx. In the drawing, the sectors happen to be shown at their outermost position, with $r = R$, where R denotes the cylinder radius. F is the force couple [1] acting on the shaded patches at the ends of the twisted column in the diagram.

The modulus of rigidity η is defined by the ratio of shearing stress to shearing angle, or

$$\eta = \frac{F/(dr \cdot dx)}{\phi}$$

[1]A couple is particular form of torque consisting of a pair of force vectors that are oppositely directed and symmetrically placed with repect to the axis of rotation. A couple thus induces no translation – only twisting.

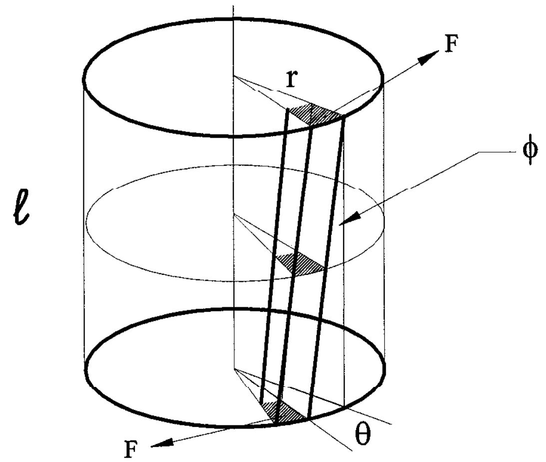

Figure 1.4: A solid cylinder of length ℓ twisted through an angle θ by the application of a torque.

In the differential limit, $\ell\phi = r\theta$, since each measures the length of the same arc segment. Thus,

$$\eta = \frac{F\ \ell}{dr\ dx\ r\ \theta}$$

The total applied torque Γ is calculated by integrating rF over the extent of an end face.

$$\Gamma = \frac{\eta\theta}{\ell} \iint r^2 dr\ dx$$

For a solid cylinder the limits on the integrals are $x : 0 \rightarrow 2\pi r$ and $r : 0 \rightarrow R$. Hence

$$\Gamma = \left[\eta \frac{\pi R^4}{2\ell}\right] \theta \tag{1.2}$$

This expression links the twist angle θ to the applied torque Γ through the dimensions of the cylinder and its modulus of rigidity, which is a material constant. The combined terms within bracket are a proportionality constant, called the torsional rigidity, which is generally defined by $\tau = \Gamma/\theta$. Note the similarity of the role played by η in the torque-twist relationship [Eq. (1.2)] to that of Young's modulus Y in the stress-strain relationship [Eq. (1.1)].

A significant consequence of Eq. (1.2) is that the amount of twist induced by any given torque varies as the inverse fourth power of the fiber radius. Therefore, suspension fibers become rapidly more sensitive as they are made thinner, a gain that is offset by their increasing fragility.

The following table [Newman 1961] gives elastic constants for some important and representative materials.

Material	Y	η	σ
steel	20.8×10^{11}	8.12×10^{11}	0.287
nickel	20.2×10^{11}	7.70×10^{11}	0.309
platinum	16.8×10^{11}	6.10×10^{11}	0.387
gold	8.0×10^{11}	2.77×10^{11}	0.422
aluminum	7.1×10^{11}	2.67×10^{11}	0.339
quartz	7.5×10^{11}	3.0×10^{11}	

The units of the constants Y and η are dynes per square centimeter; σ is a dimensionless quantity.

1.2 Statics and Dynamics

In the most common applications of the torsion pendulum (as will be discussed in detail later), the objects at the ends of the horizontal beam are acted upon either by gravitational forces produced by attraction to some other object(s), or by electrostatic attraction (or repulsion) produced by charges on the pendulum and on neighbouring objects. As depicted in Fig. 1.5, the resulting force-couple twists the pendulum beam.

The instantaneous deflection angle can be sensed by a so-called optical lever, as shown in the diagram. A small mirror attached to the apparatus will cause an incoming light beam to be deflected at twice the twist angle. A suitable photodetector then can determine the orientation of this reflected beam. If the distance from the mirror to detector is D, then the arc length of the beam-shift at the detector will be approximately $2D\theta$. A very small angular deflection can thus be "multiplied" into a considerably larger scale reading.

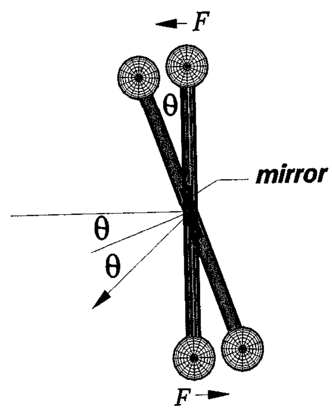

Figure 1.5: Top view of a torsion pendulum twisted through an angle θ by an external force couple F. An incoming light beam is deflected by 2θ from its original direction.

If the pendulum is allowed to come to rest, then the condition of equilibrium dictates that there be no net torque,so

$$FL = \Gamma$$

with L denoting the overall length of the horizontal pendulum beam. Since the reaction torque from the twisted fiber is given by Eq. (1.2),

$$F = \left[\eta \frac{\pi R^4}{2\ell L}\right] \theta \tag{1.3}$$

Therefore, a determination of the equilibrium deflection angle becomes a measurement of the external force.

The sensitivity of this method of measuring force is given by

$$S = \frac{dF}{d\theta}$$

and so

$$S = \frac{\eta \pi R^4}{2\ell L} \tag{1.4}$$

Consider, as an example, the following specifications [2] for a possible torsion pendulum:

η	5×10^{11} dyne/cm^2
R	12 micron
ℓ	1 meter
L	10 cm

The first three numbers are determined by the choice of suspension fiber, while the final parameter is the length of the pendulum beam. Plugging these values into Eq. (1.4), one obtains for the force sensitivity $S \approx 1.6 \times 10^{-3}$ dynes per radian, or about 2.8×10^{-5} dynes per degree. Very small forces induce measurable rotations in the apparatus. To appreciate the smallness of such forces, the mass of a single grain of rice is on the order of a few milligrams, so its weight (force) would be around 10^{-5} N or one dyne.

As noted earlier, the sensitivity varies as R^4, so the best performance is achieved with ultra-fine suspension wires [see for example, [Gillies and Ritter 1993], page 293].

1.2.1 Free Oscillations Without External Forces

If one imagines a small mass m attached to an axle via a massless rod of length L, and if a force F acts on that mass in a direction orthogonal to both connecting rod and axle, then Newton's law of motion $F = m\frac{dv}{dt}$ can be seen to imply $FL = mL\frac{d}{dt}\left[L\frac{d\theta}{dt}\right]$. In this, $\frac{d\theta}{dt}$ is the angular velocity of the mass as it spins around the axle. The left hand side of this expression is just the torque Γ exerted by the applied force, and on the right, the term mL^2 is the moment of inertia I of the mass with respect to the axis of rotation. Hence

$$\Gamma = I\frac{d^2\theta}{dt^2}$$

This equation is actually quite general and may be applied to extended objects such as wheels, in which case I is the total moment of inertia of all rotating components and Γ is the net torque acting on the object.

A torsion pendulum consists of a light connecting rod of length L with equal masses m at its ends (figure 1.1), suspended from a fiber. The moment

[2] Rather than adopting standard mks units for all parameters, each parameter is quoted in its most commonly used dimensional form.

of inertia is $I = mL^2/2$. Using Eq. (1.2),

$$I\frac{d^2\theta}{dt^2} + \left[\eta\frac{\pi R^4}{2\ell}\right]\theta = 0 \tag{1.5}$$

This is the differential equation for a simple harmonic oscillator. If the initial conditions are $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$, then the solution is

$$\theta = \theta_0 \cos(\omega_0 t)$$

with the the natural frequency given by

$$\omega_0 = \sqrt{\frac{\eta\ \pi R^4}{2\ell\ I}} \tag{1.6}$$

The free oscillation period is therefore

$$T_0 = \frac{2}{R^2}\sqrt{\frac{\pi 2\ell I}{\eta}} \tag{1.7}$$

Using the parameters given in the previous example, and assuming the pendulum consists of 10 gram masses mounted at the two ends of the connecting rod of length L = 10 cm, Eq. (1.7) gives an oscillation period of about 1000 seconds (a frequency of 1 millihertz). This is very slow motion indeed. Note from Eq. (1.7) that thicker suspension fibers lead to shorter periods.

An application of this last exprcsion is immediately apparent, especially if it is rearranged in the form

$$\eta = \left[\frac{4\pi 2\ell I}{R^4}\right]\frac{1}{T_0^2} \tag{1.8}$$

All quantities on the right hand side are dimensional properties of the apparatus, except for the oscillation period T_0, which can be measured experimentally. Hence this first of many practical uses of the torsion pendulum is for determining the modulus of rigidity of the fiber material.

1.2.2 Free Oscillations With External Forces

Now suppose that the masses are acted upon by forces as depicted in Fig. 1.5. The equation of motion changes from Eq. (1.5) to

$$I\frac{d^2\theta}{dt^2} + \left[\eta\frac{\pi R^4}{2\ell}\right]\theta = \pm FL \tag{1.9}$$

where + in $\pm$ would be used when the constant external force F induces a torque in the same sense (clockwise or counterclockwise) as the convention adopted for positive angular coordinate. The negative sign would be used in the opposite case. The solution of this modified equation of motion is

$$\theta = \theta_0 \cos(\omega_0 t) \pm \frac{FL}{\omega_0^2 I}$$

Using the previous expression for ω_0,

$$\theta = \theta_0 \cos(\omega_0 t) \pm F \frac{2\ell L}{\eta \pi R^4}$$

The oscillations now have an offset or "dc" component

$$\theta_{dc} = F \frac{2\ell L}{\eta \pi R^4}$$

which is just the same as the earlier static result Eq. (1.3). Therefore, a measurement of this dc or time-averaged deflection becomes a measurement of the external force.

1.2.3 Damping

There are no lossless dynamics – in the macroscopic world motion is always accompanied by dissipation (the only exceptions being superconductivity and superfluidity). Therefore, in the absense of external forces, a more realistic description of the torsion pendulum might be

$$I \frac{d^2\theta}{dt^2} + \beta \frac{d\theta}{dt} + \left[\eta \frac{\pi R^4}{2\ell}\right] \theta = 0, \tag{1.10}$$

which can be compared to Eq. (1.5). Friction generates a resistive counter-torque that is proportional to the instantaneous rate of rotation, the proportionality specified by the constant β. This is the most common model for friction, but it should be said that in reality this is a complex phenomenon, and that bearing friction, air friction, etc. can depend to an extent on higher powers of the angular velocity. The dominant behaviour is, however, reasonably described by the linear relationship included in Eq. (1.10).

The general solution of this second-order differential equation has the form

$$\theta(t) = Ae^{r_1 t} + Be^{r_2 t}$$

where A and B are constants to be determined from initial conditions, and r_1 and r_2 are the two roots of the quadratic equation in the differential operator $D = d/dt$:

$$D^2 + \left(\frac{\beta}{I}\right) D + \left(\frac{\eta \pi R^4}{2\ell I}\right) = 0$$

$$r_1 = -\frac{\beta}{2I} + \sqrt{\left(\frac{\beta}{2I}\right)^2 - \omega_0^2} \quad (1.11)$$

$$r_2 = -\frac{\beta}{2I} - \sqrt{\left(\frac{\beta}{2I}\right)^2 - \omega_0^2} \quad (1.12)$$

The frequency of undamped oscillations ω_0 was given in Eq. (1.6). The strength of the damping now determines the time-dependent pendulum behaviour. Three distinct regimes exist.

Underdamped $\left(\frac{\beta}{2I}\right)^2 < \omega_0^2$

For this case, define a new frequency

$$\omega^2 = \omega_0^2 - \left(\frac{\beta}{2I}\right)^2 \quad (1.13)$$

Then the two roots are complex quantities which can be written

$$r_1 = -\frac{\beta}{2I} + i\omega$$

$$r_2 = -\frac{\beta}{2I} - i\omega$$

The negative real part of the roots implies exponential decay, while the imaginary parts ($\pm i\omega$) yield oscillations at a new frequency that is slightly smaller

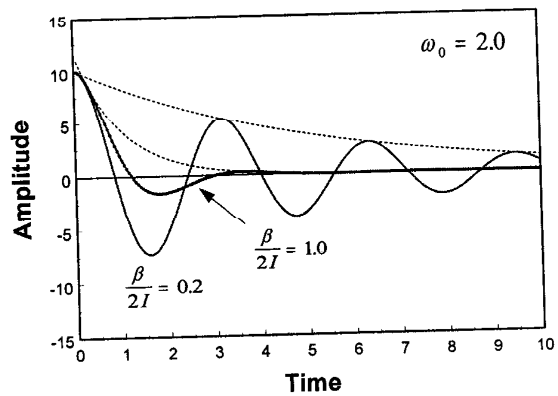

Figure 1.6: Illustration of an underdamped oscillation in which the amplitude slowly decays under control of the exponential prefactor. The pendulum is released from rest at an angle of 10 degrees. Two different damping values are shown.

than the undamped value ω_0. Combining these components, the equation of motion can be expressed in the form

$$\theta(t) = Ce^{-\frac{\beta}{2I}t}\left[\cos\left(\omega t + \alpha\right)\right] \tag{1.14}$$

The canonical solution given earlier contained two constants A and B. An illustration of this form of damped oscillation is given in Fig.1.6

Two constants remain in the new form of solution as expressed in Eq. (1.14) — one (C) defines the amplitude; the other (α) appears as a phase shift. As noted before, specification of initial conditions for the pendulum will explicitly determine C and α. For example, suppose it is known that at $t = 0$, $\theta = \theta_0$ and the pendulum is at rest. Then

$$\theta_0 = C\cos\left(\alpha\right)$$

and

$$\omega\sin\left(\alpha\right) + \frac{\beta}{2I}\cos\left(\alpha\right) = 0$$

From these two equations C and α are easily calculated.

Critically Damped $\left(\frac{\beta}{2I}\right)^2 = \omega_0^2$

In this special case, the two roots become equal and the method described above cannot be applied. The solution to the equation of motion is instead

$$\theta(t) = e^{-\frac{\beta}{2I}t}\left[A + Bt\right] \tag{1.15}$$

The two constants can be determined as before from given initial conditions. If at $t = 0$ the pendulum is at rest at a displacement θ_0, then

$$\theta_0 = A$$

and

$$B - A\frac{\beta}{2I} = 0$$

A critically damped response using the previous choice $\omega_0 = 2$ is shown in Fig. 1.7.

As can be seen in the figure, for critical damping the decay is optimum in the sense that the angle asymptotically approaches zero without overshoot (underdamping) [or undershoot (overdamping)].

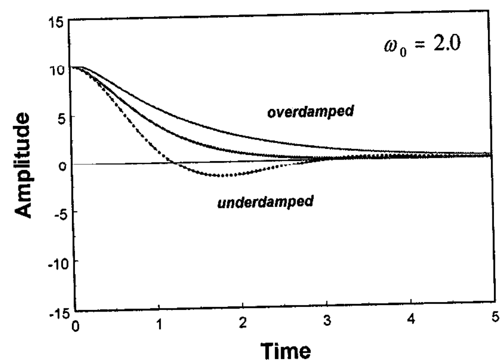

Figure 1.7: Decay of the pendulum for three regimes: underdamped (lower curve: $\frac{\beta}{2I} = 1$); critically damped (middle curve: $\frac{\beta}{2I} = 2$); overdamped (upper curve: $\frac{\beta}{2I} = 3$).

Overdamped $\left(\frac{\beta}{2I}\right)^2 > \omega_0^2$

For the overdamped case, both roots [Eqs. (1.11) and (1.12)] are real, so the decay is entirely exponential. With

$$k = \sqrt{\left(\frac{\beta}{2I}\right)^2 - \omega_0^2}$$

,which is of course a real quantity in this situation, the solution is

$$\theta(t) = e^{-\frac{\beta}{2I}t}\left[Ae^{kt} + Be^{-kt}\right] \tag{1.16}$$

As in the previous two cases, the constants A and B will be fixed by the initial conditions of the problem.

1.3 Two Historical Achievements

The torsion pendulum began its illustrious history with two profoundly important scientific applications that occurred late in the eighteenth century.

These nearly simultaneous developments were the creative products of the emerging technological cultures of France and England. Although, as is nearly always the case, many persons contributed to the final achievements, the most celebrated individuals were Charles Coulomb and Henry Cavendish

1.3.1 Coulomb and the Electrostatic Force

Charles Augustin Coulomb (1736-1806) was a French military engineer. He was a student at the engineering school at Mézières (now called Charleville-Mézières, a town north-east of Paris near the border with Belgium[3]), an establishment mostly reserved for minor nobility, but occasionally, as in Coulomb's case, open to other students with clear promise. After graduating in 1761 as an officer in the Corps of Engineers, he spent less than three years within France before being posted to the island of Martinique in February of 1764. There he was in charge of the design and construction of a new defensive fort at Port Royal. In spite of bouts of ill health, his stay lasted eight years. He returned to France, finally, in June1772.

Remarkably, Coulomb was able to pursue his interests in science and engineering during his protracted stay in the West Indies, so that by the spring of 1773 he was in a position to read two memoirs to the Academy in Paris. The subject of this first work was the application of statics to problems in architecture. For the next nine years following his return to France, Coulomb was posted to various military sites, ending in 1781 with a permanent transfer to Paris. His rank then allowed him a degree of freedom to pursue his interests in science, most particularly in physics, but also including public health and education, canals and navigation, structural engineering, and military topics.

Coulomb's accomplishments in electricity and magnetism ranged from his early (1777) memoir entitled "Investigations of the Best Method of Making Magnetic Needles", through a famous series of seven memoirs read to the Academy from 1785 through 1791. In the First (1785) and Second (1787) of these, the torsion balance was introduced and the force law for electrostatics was established. The Third Memoir treated charge leakage, while the remaining memoirs dealt with charge distributions on conductors of various shapes, and magnetism.

[3]Charleville was also the birthplace of the poet-prodigy Arthur Rimbaud (1854-91) whose writings in the brief period 1870-74 established him as a leader in the Symbolist movement and as one of the true originals in French poetry.

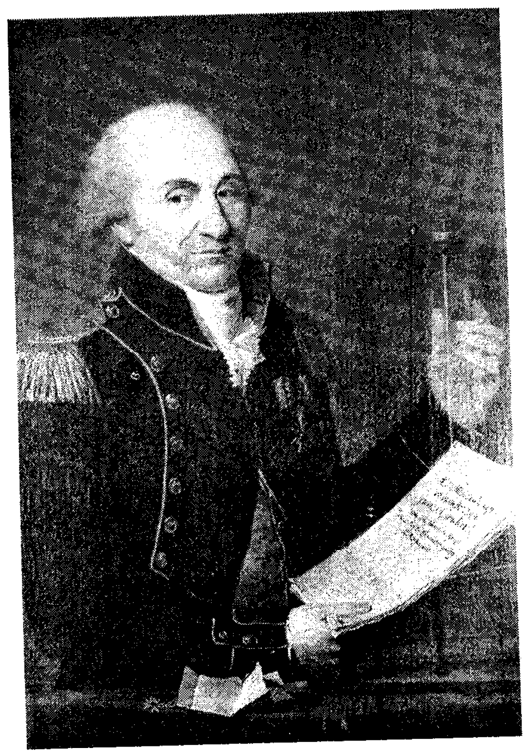

Figure 1.8: Portrait of Charles Augustin Coulomb, probably around 1804.

Coulomb resigned from military service on April 1, 1791. He lived in Paris during the period 1781 to July 1793 when he moved to his country house north of the city. Thus he escaped the nightmare of the Reign of Terror led by Robespierre in 1794. Soon after, for safety, he moved to a property near Blois where he remained until at last he returned to Paris around 1797. He died in his home on August 23, 1806.

The professional life of Charles Augustin Coulomb spanned some of France's most significant and turbulent years. The Bastille was stormed in July of 1789 and shortly thereafter Louis XVI and the royal family were imprisoned in the palace of the Tuileries. Louis was guillotined in the Place de la Concode on January 21, 1793 (about six months before Coulomb left the city). Napoleon rose to power in the interval 1795-1799 when he became First Consul, and then in 1804, Emperor of France. War with Britain was an ongoing affair, with Toulon besieged in 1793 and blockaded again by Nelson at the beginning of new hostilities in 1803, the Battle of the Nile in 1798, the Battle of Trafalgar in 1805. War with Russia and Austria culminated at the Battle of Austerlitz in December, 1805. Coulomb did not live to see Napoleon's forced abdication in 1814, nor his second and final defeat at Waterloo in 1815. It is remarkable that such scientific achievements were possible in a career woven through such upheavals and dangers (the great chemist Lavoisier was guillotined in May of 1794).

The theory of a torsion pendulum was presented earlier in this Chapter. Coulomb's version of a practical apparatus is shown in Fig. 1.9.

This instrument was used to determine the dependence of the period of the pendulum on the suspension wire material (iron and brass were tested), wire length and diameter, and tension. He concluded that the tension was not significant as an influence, but that the period varied as the square root of the length of the wire and as the square root of the inverse fourth power of its diameter. Thus he correctly deduced the functional form of the earlier expression Eq. (1.7).

The most famous achievement of Charles Augustin Coulomb was his application of the torsion pendulum to the problem of determining the nature of the law governing the attraction or repulsion of electrical charge. A proper perspective on this accomplishment must take account of what was known about charge in the late eighteenth century, or, more to the point, what was not known. Still ahead and unforseen were the discoveries of Michael Faraday during the 1820's and 1830's regarding electromagnetism and the law of electrolysis [Serway, Moses and Moyer 1989]. The identification of negative

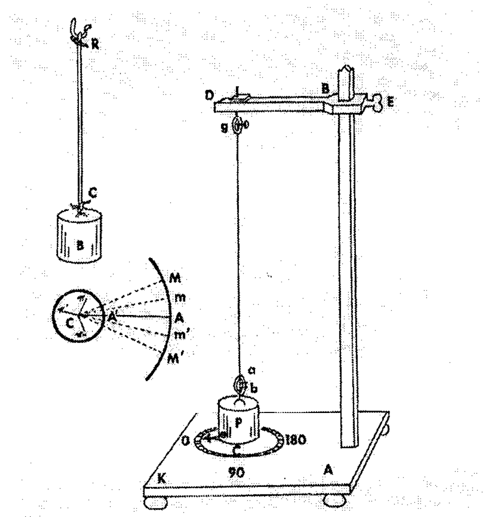

Figure 1.9: Coulomb's torsion pendulum from 1784.

charge with electrons and the measurement of the charge-to-mass ratio of the electron was the work of J.J. Thomson in 1897. The precise determination of the charge of the electron by means of the celebrated oil drop experiment was achieved of Robert Millikan in 1909. Thus it was to be more than one hundred years before the true nature of negative charge was determined. The fundamental particle carrying the basic unit of positive charge we now know to be the proton, but this understanding only emerged with the clear picture of the atom as a small massive and positively charged nucleus surrounded by orbiting electrons. This nuclear model was established by Ernest Rutherford in 1910. All of this lay ahead in the unknown future.

There was, nevertheless, wide interest in the phenomenon of static electricity. Benjamin Franklin's celebrated kite experiments in thunderstorms took place in 1752. Franklin conceived a two-fluid picture of electricity in which "positive" and "negative" charges either attracted or repelled one another. Static charge was usually produced by the rubbing of dissimilar materials[4]. The fortuitous discovery in about 1745 by Pieter van Musschen-

[4] *triboelectricity.* : "a charge of electricity generated by friction, e.g. by rubbing glass with silk", from Greek tribein-to rub The New Penguin English Dictionary, Penguin Books (London, 2000)

broek and by Georg von Kleist of what became known as the Leyden jar made it possible to accumulate and store charge. The Leyden jar was the forerunner of the modern capacitor.

One hundred years earlier, in the Principia of 1687, Isaac Newton had established the foundations of mechanics and had discovered the universal law of gravitation

$$F = G\frac{m_1 m_2}{r^2} \tag{1.17}$$

which states that the force between masses m_1 and m_2 is proportional to the product of the masses and is inversely proportional to the square of their separation. Thus it would have been quite natural to wonder if the law governing electrostatic attraction or repulsion might be in some way similar. Of course gravity can only produce mutual attraction – there are no "positive" or "negative" masses.

Coulomb adapted his torsion pendulum to the task. The basic idea is illustrated in Fig. 1.10.

An equal charge is placed on two spheres, one fixed and the other located at the end of an arm that is suspended from a thin fiber [5]. The electrostatic repulsion causes a twist which is then measured. According to the earlier equation (1.3), the force is then determined:

$$F = \left[\eta\frac{\pi R^4}{\ell L}\right]\theta, \tag{1.18}$$

where a factor of 2 has been introduced to account for the fact that the pair of repelling spheres is only at one end of the suspended beam.

The apparatus employed by Coulomb is depicted in Fig. 1.11.

Another drawing is given in Fig. 1.12. Overall dimensions of the apparutus were about 12 inches in diameter by about 36 inches in height.

In the first of these depictions, the suspension fiber (silver, copper, or silk) is seen to be attached to the horizontal beam at (P). The ends of the beam were occupied by a paper-disc counterweight (g) which also served as an air-damper, and a pith-ball (a)[6]. A second pith-ball was located at the

[5] If the two spheres are identical in size, this is easily done: a static charge is placed on one sphere after which the pair is brought into contact. The initial charge is equally divided between the pair, which can then be separated again.

[6] Pith ball (pith, as in a pith helmet, originally was dried material from the spongewood tree of Bengal or other similar plants). Coulomb employed gilded elderwood spheres (about 1/6 inch diameter) for this purpose.

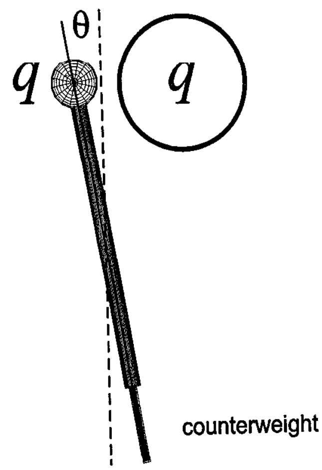

Figure 1.10: Basic principle of Coulomb's torsion balance (seen looking down the suspension axis). When the two spheres have equal charges q, the spheres repel, as shown.

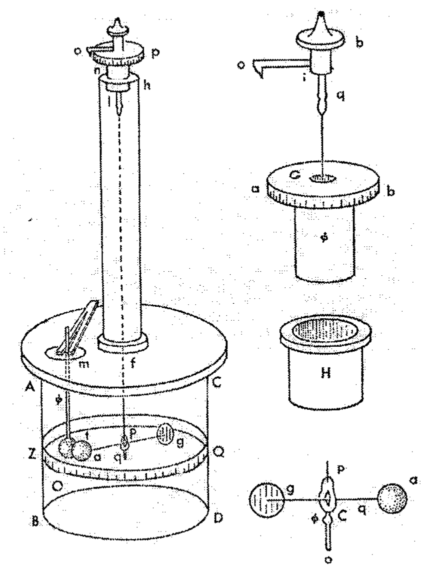

Figure 1.11: Coulomb's torsion balance (1785) used for determining the law of electrostatic repulsion.

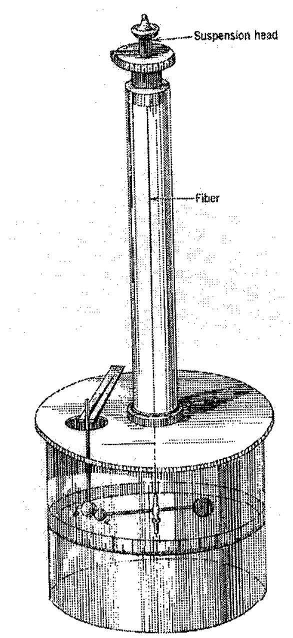

Figure 1.12: Another rendering of the electrostatic apparatus from 1785.

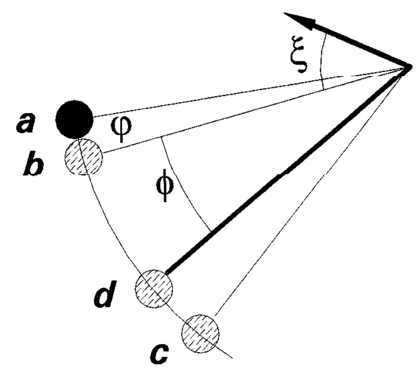

Figure 1.13: Geometry of the Coulomb apparatus. The object shown with solid shading (a) is the fixed pith ball; the hatched sphere is the moveable pith ball shown at its uncharged (b) and charged (c) positions. When the clamp holding the torsion fiber is rotated by an angle ξ, the charged pith ball will shift to position (d).

lower end of an insulated rod that extended down through the top plate at (m). Two angular scales were engraved on the apparatus, one around the wall of the large glass cylinder, and the other as part of the fiber attachment at the top.

To begin, the suspension head was then rotated, turning the fiber and beam until the two pith-balls were just in contact. By contacting the (temporarily joined) pair of pith-balls to a Leyden jar, equal charge became distributed on each, causing a force of repulsion and leading to a twisting away of the beam. The amount of charge was of course, unknown.

In equilibrium, the angular deflection of the beam (ϕ) could be read from the scale attached to the side of the glass cylinder. Now if the suspension head had been manually rotated by some amount (ξ), as depicted in Fig. 1.13, then the net twist of the fiber θ would be the sum $(\phi + \xi)$. Some reflection shows that these readings are related according to

$$\begin{aligned} \text{separation proportional to} &: \quad (\phi + \varphi) \\ \text{force proportional to} &: \quad (\phi + \xi) \end{aligned}$$

where φ is the angular diameter of a pith-ball. This must be included to account for the fact that in the uncharged, undeflected reference position, the balls are in contact with their centres still a diameter apart. For Coulomb's apparatus, this amounted to about 1.8 degrees. In [Gillmor 1971], the following data values[7] are quoted (p. 185):

experiment	#1	#2	#3
ϕ	36°	18°	8.5°
ξ	0°	126°	567°

In Fig. 1.14, the effective force which is proportional to $\theta = (\phi+\xi)$ is plotted versus angular separation $(\phi + \varphi)$. An inverse square law for electrostatic repulsion would be expressed in the form

$$y = \frac{A}{(\phi + \varphi)^2}$$

For the figure, the value of the single adjustable parameter was chosen to give agreement with Coulomb's middle data point ($A = 56624$). The continuous curve passes through the remaining two experimental points, showing that the inverse law was clearly demonstrated.

An especially elegant feature of this classic experiment is that the form of the law was precisely demonstrated without the need of specific quantitive information on the apparatus (e.g., suspension fiber parameters), or on the amount of charge actually involved.

How many Coulombs did Coulomb have?

In modern terminology, the electrostatic force law is

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{1.19}$$

where q_1, q_2 are two charges whose separation is r; and $\epsilon_0 = 8.854187817 \times 10^{-12}$ Farad/meter is the permittivity of free space. Taking the first experiment above, with $\phi = 36$ degrees, and using the sensitivity of Coulomb's apparatus (quoted in [Gillmor 1971]) $\approx 4.3 \times 10^{-4}$ dyne per degree, it can be deduced that the experimental charge on the pith-balls was about 10^{-10} coul. – a total indicating that approximately one billion electrons resided on each sphere..

[7] The surprisingly large value of ξ in the third trial is correct; a soft suspension fiber can require twists of more than a full turn.

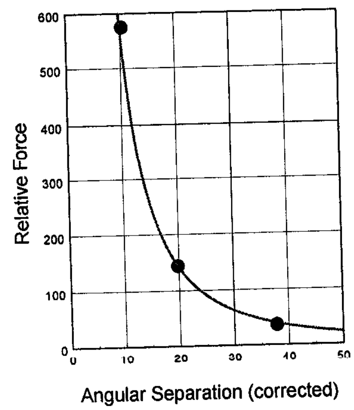

Figure 1.14: Three data points from Coulomb's original experiment, and a fitted inverse square law relationship.

Note that Coulomb's experiment confirmed only the inverse-square separation dependence in Eq. (1.19). The factor in the numerator – the dependence on the *product* of the charge magnitudes – was not addressed.

Modern methods and experiments have verified the inverse square dependence in Coulomb's law to very high precision[8]. If the separation dependence was expressed in the form r^{-n}, then one now can say that n is known to be almost exactly 2, within an uncertainty of less than a few times 10^{-16}.

1.3.2 Cavendish and the Gravitational Force

About Henry Cavendish, the Funk and Wagnalls New Encyclopedia entry reads

> Cavendish, Henry (1731-1810)
>
> British physicist and chemist, born of British parents in Nice, France, and educated at Peterhouse College, University of Cam-

[8] These methods are not based on the torsion pendulum. Instead they test Gauss's Law, from which Coulomb's law follows, and its implication that static charge must reside on the outside of conductors.

> bridge. His earliest experiments involved the specific heats of substances. In 1766 he discovered the properties of the element hydrogen and determined its specific gravity. His most celebrated work was the discovery of the composition of water; he stated that "water consists of dephlogisticated air (oxygen) with phlogiston (hydrogen)." By what is now known as the Cavendish experiment, he determined that the density of the earth was 5.45 times as great as the density of water, a calculation very close to the 5.5268 established by modern techniques. Cavendish also determined the density of the atmosphere and made important investigations of electrical currents.

Cavendish and Coulomb were almost exact contemporaries. Just as what we now regard as Coulomb's law was in historical truth not entirely what Coulomb established, so too with the work of Cavendish. Coulomb confirmed the inverse square dependence in Eq. (1.19), but did not address the proportionality constant that we now write as $(4\pi\epsilon_0)^{-1}$. It is a widely held notion that Cavendish experimentally verified Newton's gravitational law Eq. (1.17), but as the biographical item above hints, such was not the case. In the first place, Newton himself did not write his "law" in the now familiar form, nor did he introduce the "universal" constant G. One hundred years had to pass before the the modern idea of "G" was to appear in a paper entitled "On the Newtonian Constant of Gravitation" by C.V. Boys [Boys 1894]. It is also worth keeping in mind that our familiar system of units was not established at the time of Cavendish. For example, the dyne was not introduced until 1873.

A preoccupation of eighteenth century scientists was a determination of the relative *density of the earth*, and it was to this task that Cavendish turned his attention.

The basic idea of the Cavendish experiment is illustrated in Fig. 1.15.

In this view down the axis of the suspension fiber, it can be seen that two small masses are fixed to the ends of a horizontal beam of length L. Two large masses are positioned at either pair of sites numbered $1, 1$ or $2, 2$. Gravitational attraction will rotate the beam and twist the support fiber through a counterclockwise angle θ when the large masses are in the first position, and through a clockwise angle θ when they are in the second position. The experiment consists of determining the total change in angle 2θ that occurs as the mass positions are switched.

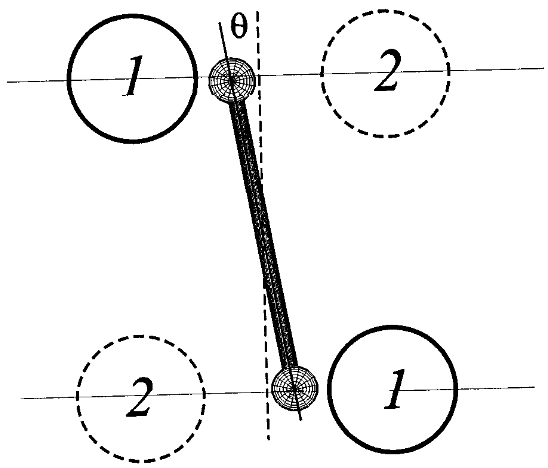

Figure 1.15: Arrangement of a Cavendish torsion balance for determining the gravitational constant G by the static method. The view is looking down the suspension fiber.

The equilibrium position specified by the angle θ is determined by the balance of the torque produced by the gravitational force of attraction on the small masses towards their fixed large-mass partners, and the opposing torque from the twisted suspension fiber. This is illustrated in Fig. 1.16.

A key point to note is that the fiber torque is a linear function of the twist θ, whereas the gravitationally induced torques vary as the square of the inverse distance between small and large masses, i.e., approximately as $[\text{constant} - \theta]^{-2}$. Also, even for zero twist when the fiber is unstrained, the gravitational forces, and thus torques, are non-zero as illustrated in the figure. For a fiber of sufficient stiffness, the linear function can be seen to intersect the nonlinear gravitational curve in two places. As pointed out in the caption, the lower crossing point is the stable equilibrium for the system. For a soft suspension, there may be no crossing point, in which case the gravitational torque will bring the spheres into contact and hold them there. However, as pointed out in the next section, this gravitational "glue" is exceedingly weak and tiny perturbations such as air currents may be enough to break the bond

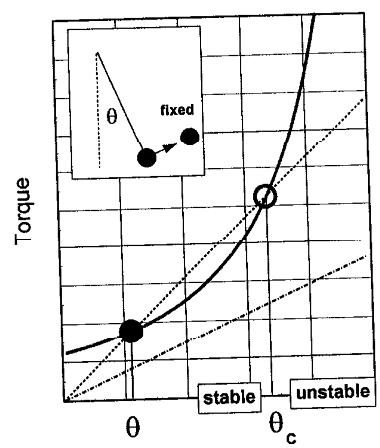

Figure 1.16: Behaviour of the attractive torque (solid curve) for two masses as a function of their relative position, and the counter torque from a twisting suspension fiber. Two possible vaues of fiber stiffness are illustrated (straight lines). For the stiffer suspension, if the movable ball is released from an angle less than θ_C, the system will reach equilibrium at the final angle θ (solid circle). If the release is from an angle greater than θ_C, then the attractive force dominates and causes the balls to come into contact. For the less-stiff fiber, no stable separation exists.

Modern versions of the experiment use a deflected light beam for readout [see Fig. 1.5]. At equilibrium, using Eq. (1.2)

$$G\frac{mM}{r^2}L = \left[\eta\frac{\pi R^4}{2\ell}\right]\theta \tag{1.20}$$

Equation (1.6) shows that the fiber stiffness can be determined from the frequency of free oscillations of the system (in the absence of the large masses), and hence

$$G = \left[I\omega_0^2\right]\left[\frac{r^2}{mML}\right]\theta \tag{1.21}$$

The moment of inertia of a massless beam with m at each end is given by $I = mL^2/2$ and so

$$G = \frac{2\pi^2 Lr^2}{MT_0^2}\theta \tag{1.22}$$

Therefore an observation of the angular deflection together with a knowledge of the large mass value, the separation between large and small masses, and the free oscillation frequency will yield the universal gravitational constant G.

From G one can deduce the density of the earth (or vice versa). Let the radius and mass of the earth be R_E and M_E, respectively. The acceleration of gravity at the surface of the earth, $g = GM_E/R_E^2$, is known from free-fall experiments to have the value be 9.80665 m/sec^2, the density of the earth

$$\rho_E = \frac{M_E}{(4/3)\pi R_E^3}$$

is thus

$$\rho_E = \frac{g}{(4/3)\pi G R_E} \tag{1.23}$$

Weak Forces

In the light of what we now know about the gravitational force law and the constant $G = 6.67259 \times 10^{-11}$ m^3/kg·s, it is remarkable that Cavendish was able to obtain meaningful results. Consider the difficulty of the task.

Suppose two one kilogram spheres, each of radius two centimeters, are placed in contact. Then according to Newton's gravitational law Eq. (1.17), the mutual attractive force between them would be about 4×10^{-8} N. Interestingly, if two similar spheres were charged to 10^{-10} coul, the Coulomb force turns out to be almost exactly the same, 5×10^{-8} N. As noted earlier, the weight of a single grain of rice is around 10^{-5} N, a magnitude that is nearly one thousand times larger than the force to be measured in the Cavendish experiment. Fortunately, the torsion balance provides sufficient sensitivity for the task.

The Cavendish Apparatus

It is generally acknowledged that Cavendish improved upon an original concept by the Reverend John Michell (1724-93). Figure 1.17 shows the torsion beam built by Cavendish [Clotfelter 1987]. It was quite large, with a beam 73.3 inches in length and large masses that were lead spheres one foot in diameter weighing 348 pounds. The small spheres were 2 inches in diameter and the center separation distance r was 8.85 inches. The copper suspension wire was 39.25 inches in length. As suggested in the drawing, the entire apparatus was housed in a room provided with illuminating lanterns and sighting telescopes so that the experimenter remained outside and air currents were reduced. From the historical perspective of Cavendish, for whom there was no "G", Eqs. (1.22,1.23) could more meaningfully be written

$$\rho_E = \frac{3g}{8\pi^3 R_E}\left(\frac{MT_0^2}{Lr^2\theta}\right) \tag{1.24}$$

An additional useful conversion can be obtained by recalling that a simple pendulum whose length is chosen to be $L/2$ would have a small-amplitude period

$$T_P = 2\pi\sqrt{L/2g}$$

Therefore Eq. (1.24) can be written

$$\rho_E = \frac{3}{4\pi R_E}\left(\frac{MT_0^2}{r^2T_P^2\theta}\right) \tag{1.25}$$

Now all required numerical values are at hand. The radius of the earth was known to eighteenth century scientists; the remaining terms are parameters

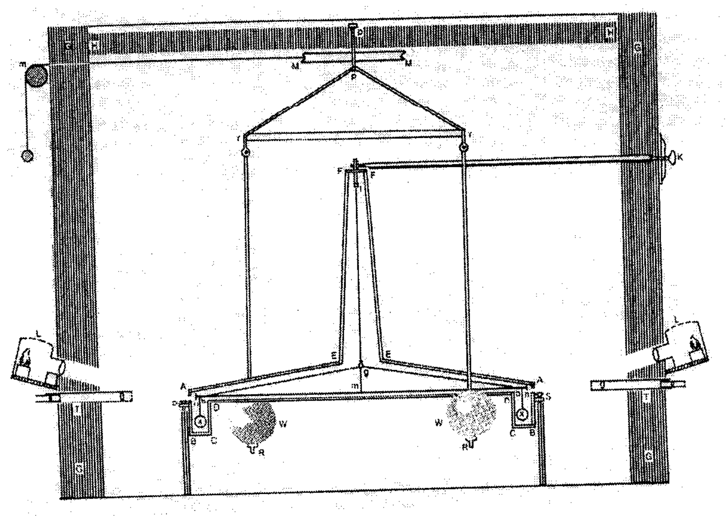

Figure 1.17: Original apparatus used by Henry Cavendish in 1798 to measure the density of the Earth.

of the experimental apparatus. Three measured quantities are required: T_P, T_0, θ. Cavendish determined the time for a single swing (what we would now identify as half a period) and obtained 0.97 sec for the simple pendulum and 424 sec for the torsion apparatus. The gravitational deflection was $\theta \approx 0.0039$ radian (0.22 degrees). From these the mean density is calculated to be 5.72 gm/cm^3. The presently accepted value for the mean density of the earth is 5.5268 gm/cm^3.

1.3.3 Scaling the Apparatus

The Coulomb apparatus was significantly smaller than that of Cavendish. Obviously the logic in Cavendish's thinking was to make the gravitational torque as large as practically possible by choosing large masses and a long (and heavy) beam. This, however, necessitated a strong, and therefore thick, suspension wire. As Eq. (1.4) showed, the sensitivity of a suspension fiber varies as the fourth power of its radius – very thin fibers are best. But the load-carrying capacity of a wire goes up as the square of the radius – very thick fibers are best. So Cavendish traded sensitivity for increased force.

This raises the obvious question about optimum designs. It turns out that Coulomb was on the better track. The Cavendish experiment was subsequently improved by making it smaller and using other suspension fibers such as quartz to increase torsional sensitivity. The list of developments includes those of Cornu and Baille (1878), C.V. Boys (1894), Eötvös (1896), Burgess (1901) and Heyl (1930). A typical design would now have large-masses of a few kg, sphere separations of a few cm., beam lengths of 10 or 20 cm., and suspension fibers with torsion constants of around 10^{-8} N·m/rad.

1.4 Modern Applications

1.4.1 Ballistic Galvanometer

In its conventional form, a galvanometer measures steady current. The current is made to flow through a mult-turn coil, usually rectangular in shape, which is suspended in the field of a permanent magnet. The interaction of the magnetic field with the moving charges results in Lorentz forces ($\vec{F} = q\vec{v}\times\vec{B}$) that collectively then act on the each side of the coil, producing a twisting torque. In equilibrium the resting orientation of the coil is proportional to the current circulating through it. A pointer then provides the conventional readout against an annotated scale.

A ballistic galvanometer is an adaptation so that a quantity of charge, rather than current, is sensed. The basic design remains the same, except for one important factor: the time during which the charge to be measured passes through the instrument must be short compared to the basic time constant of the device.

For a galvanometer coil of area A consisting of N turns carrying curent i, positioned in a magnetic field B that is normal to the plane of the coil, the resulting torque will be $\Gamma = NiAB$. Newton's law: $\Gamma = I\ddot{\theta}$ can be expressed in equivalent form: $\Gamma dt = I\ d\omega$. This says that impulsive torque equals change in angular momentum, much as the usual expression of Newton's law $F = ma$ can be stated: impulse equals change in linear momentum: $Fdt = m\ dv$. Therefore, $NiAB\ dt = I\ d\omega$, and so

$$NAB\int_0^{t_0} idt = I\int_0^{\omega_f} d\omega = I\omega_f$$

The integral on the left simply yields the total charge passing through the coil in the short interval during which the angular velocity of the coil changes from rest to a final value ω_f. Hence $NABQ = I\omega_f$. If at the end of this initial phase there is no additional charge flowing, then the impulsive Lorentz torque will return to zero. With the clock restarted, and new initial conditions $\theta = 0$, $\dot{\theta} = \omega_f$, the coil now oscillates clockwise and counterclockwise on the end of the suspension fiber according to the usual equation Eq. (1.5)

$$I\frac{d^2\theta}{dt^2} + \left[\eta\frac{\pi R^4}{2\ell}\right]\theta = 0$$

when there is no damping. The solution is

$$\theta = \frac{\omega_f}{\omega_0}\sin\omega_0 t$$

The amplitude of the oscillations, which can be observed experimentally, then gives with the aid of the expression for ω_f

$$\theta_{\max} = \frac{NABQ}{I}\left[\eta\frac{\pi R^4}{2\ell I}\right]^{-1/2}$$

or

$$Q = \frac{\theta_{\max}}{NAB}\left[\eta\frac{\pi R^4 I}{2\ell}\right]^{1/2} \tag{1.26}$$

Thus the observed maximum angular deflection is a measure of the total charge which passes through the galvanometer.

When small damping is present, the above treatment must be modified along the lines developed in the earlier section Damping. The complication now is that the undamped maximum $\theta_{\max}$ is required for the determination of the charge Q, but the amplitude of the observed back and forth galvanometer oscillations is decaying exponentially. Thus one can determine a sequence of numbers: maximum left angular throw, subsequent smaller maximum right angular throw, subsequent smaller-still maximum left angular throw, etc. From this sequence, it is possible to deduce the required $\theta_{\max}$and so obtain the value of Q.

Figure 1.18 is an illustration of a commercial[9] wall-mounted ballistic galvanometer.

[9]Model 2239, Leeds and Northrup, Philadelphia, PA.

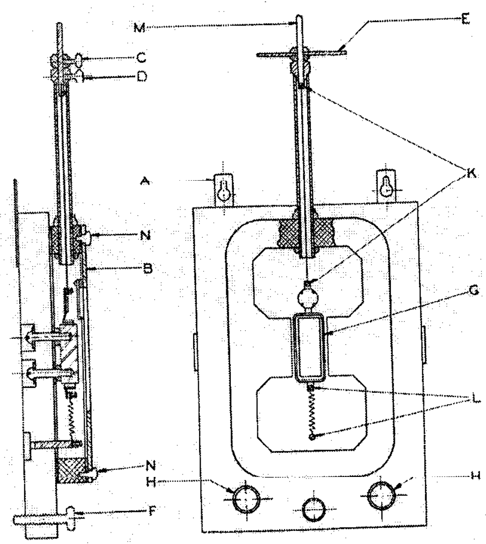

Figure 1.18: Side and front views of a wall-mounted ballistic galvanometer. Shown are the current coil (G), mirror, and suspension fiber extending between points (K).

The rectangular coil (G) hangs freely between pole faces of the permanent magnet in the shape of a large "8". A very fine suspension wire is clamped at the top (K) and just above a small mirror that is in turn fixed to the coil. Not shown is a telescope assembly and large curved scale that allow the rotation of the coil to be determined by sighting at the co-rotating mirror. The terminals for external connections are marked H. Typical specifications for this type of instrument are: period 2–4 sec., ballistic constant 0.03 = 0.04 microcoulomb per cm.

1.4.2 Universal Gravitational Constant

It was noted earlier that particular notion of a "Universal Gravitational Constant" was not included in either 17^{th} century (Newton) or 18^{th} century (Cavendish) formalisms. However by the 19^{th} century, modern mathematical language and the idea of "fields" – electromagnetic or gravitational – had taken root. Fundamental constants of nature such as Avagadro's number, assumed a certain pre-eminence. After more than two hundred years, the torsion pendulum remains the definitive instrument for determining G.

As Eq. (1.22) shows, the Cavendish experiment can give a value for the universal gravitational constant from a *static* measurement, that is from the observed angle of twist θ. Another scheme is now favoured for this task – the *dynamic* time of swing method.

Time-of Swing Method.

Consider first an arrangement as shown in Fig. 1.19.Clearly there is a restoring component F_x of the net attractive force between the m and M pairs. For small oscillations and neglecting the fiber stiffness in this instance,

$$\frac{F_x}{F} = \frac{x}{\sqrt{x^2 + d^2}} \approx \frac{x}{d}$$

But

$$x = \frac{L}{2} \tan\theta \approx \frac{L\theta}{2}$$

The equation of motion is then

$$-F_x L = -F\frac{x}{d}L = -F\frac{L^2}{2d}\theta = I\ddot{\theta}$$

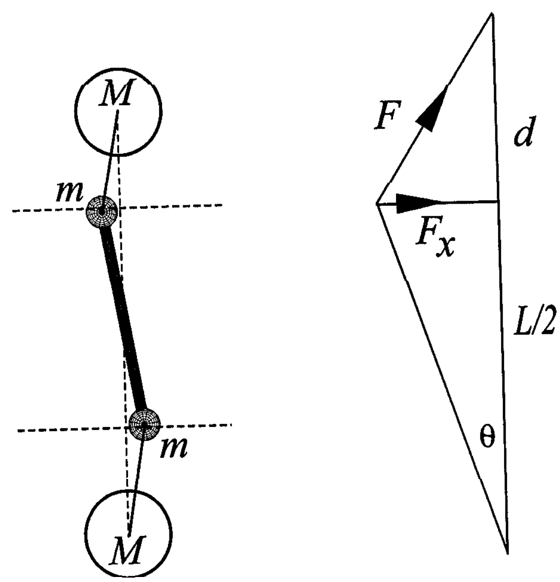

Figure 1.19: Alternate arrangement of the torsion pendulum for determining the gravitational constant G by the time-of-swing method.

Which describes harmonic oscillation at a frequency

$$\omega = \sqrt{\frac{FL^2}{2dI}}$$

or period

$$T = 2\pi\sqrt{\frac{2dI}{FL^2}}$$

Assuming an inverse square law force, this would predict the following relationship between separation and period $T \propto d^{3/2}$, a relationship which could be used as the basis of an experimental test.

In the event that the fiber stiffness is also significant (almost always true), then it can be seen that the restoring component of the gravitational torque effectively adds extra stiffness to the suspension, increasing the natural frequency. Suppose the beam was allowed to oscillate first with and then without the attracting masses M present. In the first case the oscillation frequency would be

$$\omega_1 = \sqrt{\left(\frac{\eta\pi R^4}{2\ell I} + G\frac{mM}{Id^2}\frac{L^2}{2d}\right)}$$

while in the second, the frequency would be

$$\omega_0 = \sqrt{\left(\frac{\eta\pi R^4}{2\ell I}\right)}$$

or

$$\omega_1^2 = \omega_0^2 + G\frac{mML^2}{2d^3 I} \tag{1.27}$$

Therefore a pair of frequency measurements (or, equivalently, periods) can give the gravitational constant G if the masses and apparatus dimensions are known.

The basic principle of the time-of-swing method has been applied to recent very precise determinations of G. The apparatus shown in Fig. 1.20 was used by Bagley and Luther[Bagley 1997] to measure the universal gravitational constant.

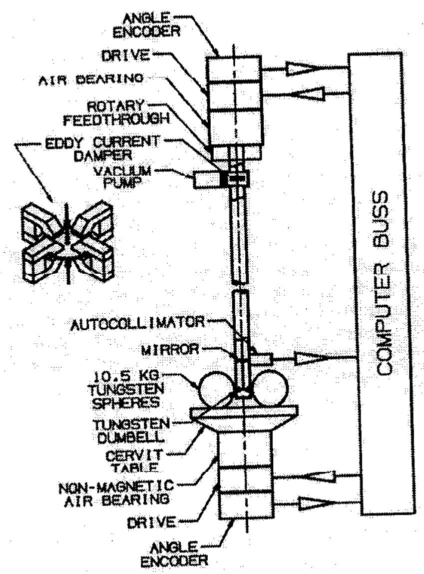

Figure 1.20: Apparatus of Bagley and Luther used to determine the Newtonian Gravitational Constant.

The two large masses were 10.489980 and 10.490250 kg respectively. The small-mass "dumbell" was composed of a pair of tungsten discs situated at the ends of a 1.0347 mm diameter rod 28 mm in length. The period of free oscillations of the torsion pendulum with a 12 μm suspension fiber was 210 sec. The pendulum was enclosed within a vacuum system and the entire apparatus was housed in an isolated building located on a mesa in New Mexico. In this version of the experiment, oscillation frequencies were measured not with and without the large masses, but instead with the large tungsten masses in either of two positions relative to the dumbell: in-line and perpendicular. Switching of positions was performed at half-hour intervals with the experimental runs lasting for as long as forty hours. The reported value of G was $(6.6740 \pm 0.0007) \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$.

Another, similar determination of G was reported by Luo et al.[Luo et al. 1998]. For this experiment, the two frequencies were measured with and without large attracting masses, which were 6.2513 and 6.2505 kg respectively and made from nonmagnetic stainless steel. The geometry was somewhat different from the classic Cavendish arrangement – these large masses were positioned on either side of one of the small masses, with the line of their centers orthogonal to the dumbell axis. The small mass at the other end of the dumbell served simply as a counterweight. The two small copper masses were 32.2560 and 32.2858 gm separated by 200 mm. The torsion fiber was 513 mm long 25 μm tungsten wire. The entire apparatus was enclosed in a high vacuum container and was located in room mounted on a shock-proof platform weighing 24 tons which was in turn supported on 16 very large springs. The laboratory itself was in the center of Yu-Jia Mountain in Wuhan, People's Republic of China.. The measured periods were 4441 sec and 3484 sec, with and without the attracting masses, respectively, yielding a final value for the gravtational constant of $(6.6699 \pm 0.0007) \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$.

It is rather interesting to note that despite these very well engineered designs that make use of the latest techniques in instrumentation, and the almost exteme measures taken to assure accuracy, still the overall precision is only about a few parts in ten thousand. Thus G remains the least accurately known of the universal natural constants [10].

Quite recently, a new determination was made using a very elegant torsion

[10] see Table II : The 1986 recommended values of the fundamental physical constants, from The 1986 CODATA Recommended Values of the Fundamental Physical Constants, Journal of Research of the National Bureau of Standards.

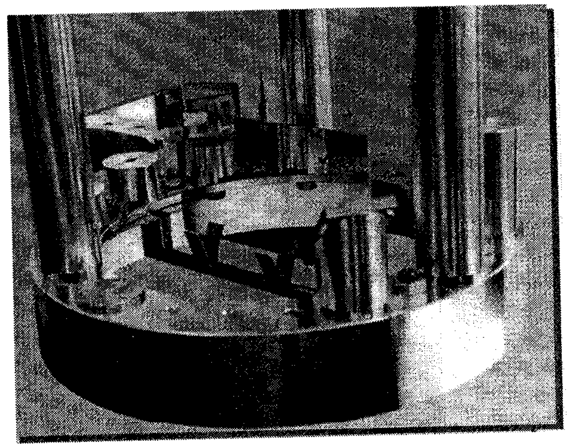

Figure 1.21: Torsion pendulum for determining the universal gravitational constant G. The pendulum proper is the gold-coated mirror-finished rectangular object in the center. The fiber attachment can be made out at its top edge.

pendulum as shown in Fig. 1.21.

This design by Jens Gundlach and Stephen Merkowitz at the University of Washington employs a simple Pyrex slab in place of the usual dumbell. External to the fully enclosed pendulum was an array of of 8 kg steel balls. The entire pendulum was placed on a rotating turntable. As the pendulum then turned past the steel balls, the suspension fiber would exhibit periodic twists. A further design refinement added feedback control to the turntable so as to accelerate and decelerate the rotation to minimize fiber twist. In the spring of 2000 they reported the best current value for G: $(6.674215 \pm 0.000092) \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$.

1.4.3 Universality of free fall: Equivalence of Gravitational and Inertial Mass

Newton's law of gravitation describes the mutual attraction of a pair of masses m_1 and m_2. No specification of material substance is provided and in fact (so far as we know) it does not matter if m_1 and m_2 are composed of copper, lead, hydrogen, or anything else. Thus, for example, a pair of one kilogram glass spheres will experience the same attracting force as a pair of one kilogram steel spheres.

This fact has a direct consequence in experiments dealing with falling bodies. At the surface of the earth, the force of attraction of the earth (mass M, radius R) acting on a test mass m is of course

$$F = G\frac{mM}{R^2}$$

neglecting the height of release above the surface in comparison to the comparatively huge value of R. The resulting acceleration downward is thus

$$a = \frac{F}{m} = \frac{GMm}{mR^2} = \frac{GM}{R^2}$$

which is usually assigned the special symbol g [11]. The seemingly trivial final step in which m is cancelled depends on the 'equivalence' of the gravitational mass in the numerator and the inertial mass in the denominator; this equivalence is not a matter of small importance. Notice that this acceleration g does not depend either on the test object's own mass, or on it's material composition. As Galileo surmised four hundred years ago, all objects fall at the same rate. This principle can appear to be violated in common experience, but when the experiment is performed in a vacuum, the prediction is confirmed – as was vividly demonstrated when an astronaut released an eagle feather and hammer on the surface of the moon. The fact that material composition is absent from such phenomena suggested to Einstein that gravity must be a function of space itself rather than the particular matter experiencing the force.

There has always been speculation on the issue of absolute correctness of the equivalence principle. From time to time ingenious experiments have been devised to test the hypothesis of material independence. One of the most celebrated experiementers was Baron Roland von Eötvös of Hungary. He used a torsion balance consisting of a 40 *cm* long beam suspended from a platinum-iridium wire. Weights were placed at the ends of the beam and the apparatus, enclosed so as to reduce air currents, was aligned in an east-west direction. Twisting of the fiber was detected by means of the usual optical lever, that is a light beam reflected from a mirror attached to the fiber/beam.

[11] The well known value for the acceleration of free fall at sea level at latitude 45 degrees is 980.616 *cm* sec^{-2}, or 32.17 feet sec^{-2}.

Since the apparatus was fixed to the surface of the spinning earth, the two beam masses were subject to gravitational forces from the earth and centrifugal forces generated by the rotating reference frame of the laboratory. Any small difference between these two forces as they were experienced by the beam masses (which were made from different materials) would have resulted in a net torque on the torsion pendulum. In experiments carried out between 1889 and 1908, no torque was observed to within the sensitivity limits of the apparatus [Eotvos 1922]. This famous null result confirmed the material independence of inertial and gravitational forces, thereby verifying the equivalence principle.

Since this early work, fresh attempts have been made to examine the question. In the early 1960's, R.H. Dicke [Dicke 1961] and others at Princeton improved on the original Eötvös torsion pendulum design, using a triangular balance beam with two copper weights and a third mass consisting of a glass container with lead chloride. Copper was chosen because of its almost equal numbers of protons and neutrons (29 and 34), while lead offered a contrast with 82 and 125, the idea being to see if neutron rich matter behaves differently. Great care was taken to minimize convection currents by using a high vacuum enclosure, and by eliminating magnetic impurities in the sample masses because of the possibility of torques caused by interaction with the earth's magnetic field. The need for such care is illustrated by Dicke's remark: "If one of the suspended weights in Eötvös' apparatus contained a single strongly magnetized iron fragment weighing only a few millionths of a gram, it would produce a deflection in the balance 1,000 times greater than the probable error given by Eötvös". This apparatus was placed in a twelve foot deep pit near the Princeton football stadium. In the end, this experiment improved the precision over Eötvös by a factor of 50, and gave once more a null result.

The newest test of universal free fall [Su et al. 1994] involved the torsion pendulum shown in Fig. 1.22.

This pendulum was located in a laboratory situated in a hillside on the campus of the University of Washington. The 20 μm suspension fiber was a gold coated tungsten wire 0.8 m long. The free oscillation period of the pendulum was 695 sec. As the diagram illustrates, the entire pendulum was placed on a turntable that continuously and slowly rotated about the vertical axis.

The business end of the pendulum is shown more clearly in Fig. 1.23.

Note that, in contrast to Eötvös who employed two masses, and Dicke

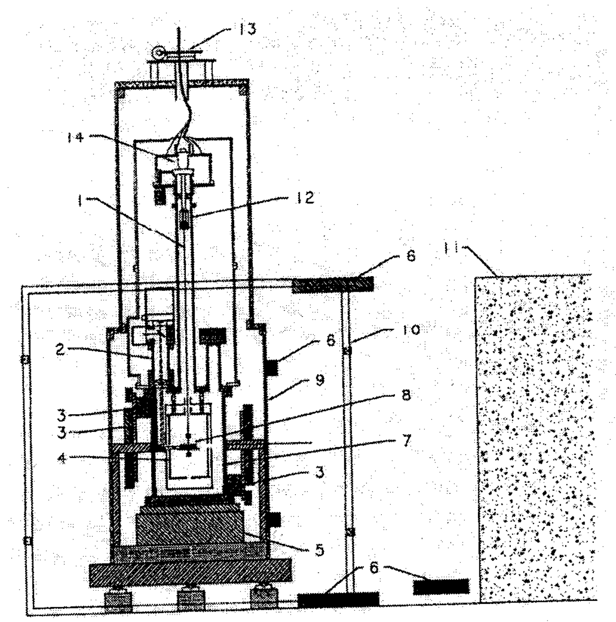

Figure 1.22: The Eöt-Walsh torsion pendulum. Principal components include the suspension fiber (1), the turntable (5), and the pendulum itself (8). The concrete block (11) is 1.23 meters high.

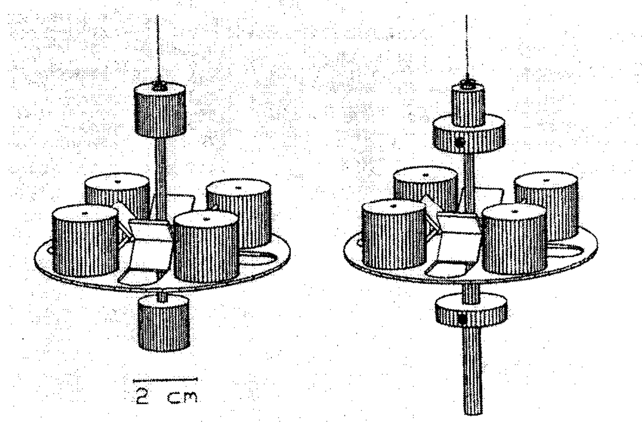

Figure 1.23: Configuration of the pendulum masses for two versions of the experiment.

who used three, this system has four. Aluminum, beryllium, and copper objects were used. The details of this experiment are quite complex, but the measurements again confirmed the equivalence principle.

The push to improve torsion pendulum performance is also illustrated by the apparatus[Bantel and Newman 2000] shown in Fig. 1.24.

The figure shows only the payload end of the system. This was located at the bottom of a large dewar which could store liquid nitrogen and liquid helium; its capacity was 95 liters. With liquid N_2 the system temperture can be reduced to 77 K; liquid He can go much lower – to just a few degrees kelvin. The low temperature environment reduces both thermal noise effects and thermal sensitivity. For the most precise work, even thermal fluctuations become troublesome because of their effect on the mechanical properties of mirrors used optical lever readouts [Heidmann et al. 1999].

The torsion fiber was 25 μm, 260 mm long hardened aluminum. The pendulum itself was a 17 gm, 16 mm tall aluminum octagon with mirror finished faces. The period of the torsion oscillations was 80 sec at room temperature, and 76 sec at low tmperature (77 K). Large source masses were placed outside the dewar.

As in most of the gravity pendulums already discussed, the location of this apparatus was slightly exotic. The system was placed in an abandoned

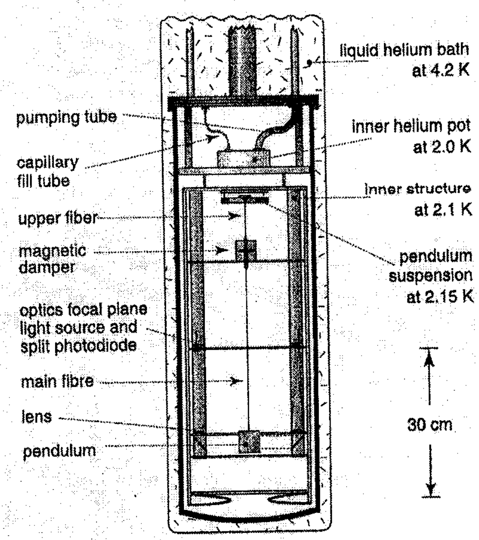

Figure 1.24: Cryogenic torsion pendulum for gravitational experiments. This drawing shows only the lower part of the complete apparatus which consists of a dewar several meters high.

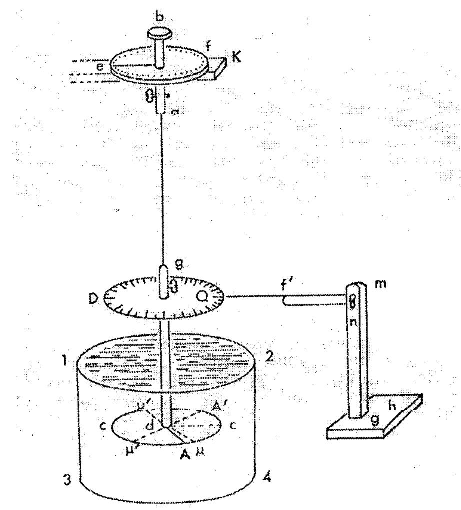

Figure 1.25: Coulomb's torsion pendulum used for viscosity measurements.

Nike missile bunker in a federal land preserve near Richland, in eastern Washington state. This isolation offered much reduced seismic noise.

1.4.4 Viscosity Measurements and Granular Media

It is possible to adapt the torsion pendulum to quite a different kind of practical measurement. In fact, Coulomb appreciated the potential of the torsion pendulum for determining the viscosity of fluids. As illustrated in Fig. 1.25, an oscillating disc is placed in the fluid of interest.

In the absence of viscous drag, the disc will simply oscillate at the end of the torsion fiber. The effect of viscosity is to damp these oscillations with resulting motion as described earlier [Eq. (??)]. For the usual case of underdamping, the solution will be of the form Eq.(1.14).

By observing experimentally the decrease in the magnitude of successive maxima in the left and right deflection angles of the twisting disc, the so-called logarithmic decrement λ can be measured. From two such measurements, one in air and the other in the fluid, the viscosity η can be deduced from theoretical expressions such as (p.232 [Newman 1961]):

$$\eta = \frac{16I^2}{\pi\rho t_0 \left(R^4 + 2R^3\delta\right)^2} \left[\frac{\lambda_1 - \lambda_0}{\pi} + \left(\frac{\lambda_1 - \lambda_0}{\pi}\right)^2\right]$$

where I is the total moment of inertia of the system, ρ is the density of the fluid, t_0 is the period of oscillation in air, R and δ are the radius and thickness of the disc, and λ_0 and λ_1 are the log decrements in air and the fluid, respectively.

A slightly different version of this idea was used in 1946 by Elevter Andronikashvili to determine the fraction of superfluid in liquid helium as a function of temperature. In the experiment a stack of closely spaced disks was attached to the end of the suspension fiber. When oscillating, a normal fluid penetrating the space between the disks would induce a drag torque. In contrast a superfluid, having zero viscosity, induces no drag.

At lower and lower temperatures, a given volume of liquid helium becomes composed of a larger fraction of superfluid as compared to normal fluid. Therefore the amount of drag torque becomes a measure of the ratio of normal fluid to super fluid. In turn the drag torque is reflected in the resonant frequency of the system. Andronikashvili's experiment measured this resonant frequency versus temperature.

A similar approach can be taken to the determination of properties of granular media such as fine powders. Figure 1.26 shows a forced torsion pendulum [D'Anna 2000] that ends in a probe which is immersed in the granular medium.

By means of the small permanent magnet, a harmonic torque could be applied to the pendulum by sinusoidally exciting a pair of external coils. The natural frequency of the system (30 Hz) was higher than the forcing frequency (1 Hz). The experiment consisted of measuring the loss factor (by monitoring the time dependent angular displacement $\alpha(t)$ via the usual optical lever readout) as a function of the amplitude of the applied ac torque. Various granualr materials were studied in this way; the list included large and small glass beads with diameters of 1.1 mm and 70 μm, sand with grain size 160 μm, and even snow!

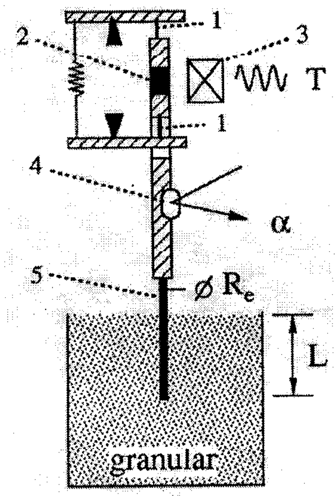

Figure 1.26: Torsion pendulum immersed in a granular medium. Components are: suspension wires (1); permanent magnet (2); external coils (3); mirror (4); probe (5).

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