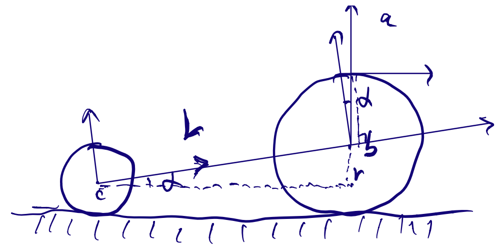
1.

$${}_aT_c = {}_bT_c * {}_aT_b$$



$\alpha = \arcsin \frac{r}{L}$

for x: $-D$

for y: $\sin\alpha \cdot 2r$

for z: $\cos\alpha \cdot 2r$

if b & a looks like $\uparrow$

Rotation around the x-axis:

$${}_bT_c = \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & \cos\alpha & -\sin\alpha & \sin\alpha \cdot 2r \\ 0 & \sin\alpha & \cos\alpha & \cos\alpha \cdot 2r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around the x-axis:

$d = 2\pi r : \theta$

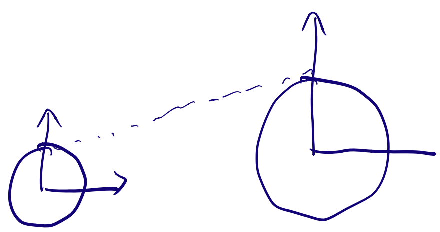
Same d, $r_b = 2r_a \Rightarrow \theta_b = \frac{1}{2}\theta_a$

$${}_aT_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & L\cos 2\theta \\ 0 & \sin\theta & \cos\theta & L\sin 2\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}_aT_c = {}_bT_c * {}_aT_b$$

$$= \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & \cos\theta\cos\alpha - \sin\theta\sin\alpha & -\cos\theta\sin\alpha - \sin\theta\cos\alpha & 2r\cos\theta\sin\alpha - 2r\sin\theta\cos\alpha + L\cos 2\theta \\ 0 & \sin\theta\cos\alpha + \cos\theta\sin\alpha & -\sin\theta\sin\alpha + \cos\theta\cos\alpha & 2r\sin\theta\sin\alpha + 2r\cos\theta\cos\alpha + L\sin 2\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if b & a looks like:



then ${}_bT_c = \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2r \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$${}_aT_c = {}_bT_c * {}_aT_b = \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & \cos\theta & -\sin\theta & L\cos 2\theta - 2r\sin\theta \\ 0 & \sin\theta & \cos\theta & 2r\cos\theta + L\sin 2\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.

From so2 to SO2

$$so2 = \{\hat{\theta} \in R^{2\times 2} \mid \theta \in R\}.$$

$$R = \exp(\hat{\theta}) = I + \hat{\theta} + \frac{1}{2!}\hat{\theta}^2 + \frac{1}{3!}\hat{\theta}^3 + \cdots \qquad \theta = \log(R)^\vee$$

For so2:

$$G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \hat{\theta} = \theta \cdot G = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \in so2.$$

$$\exp(\hat{\theta}) = I + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots & -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots \\ 0 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO_2.$$

From SO_2 to so2.

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in SO_2.$$

$$\theta = \log(R)^\vee = \arctan\left(\frac{R_{21}}{R_{11}}\right) \text{ or } \frac{\|\theta\|}{2\sin\|\theta\|}(R - R^T)$$

From se2 to SE2.:

$$se2 = \left\{ \hat{\delta} = \begin{bmatrix} \hat{\theta} & \rho \\ 0 & 0 \end{bmatrix} \middle| \delta = \begin{bmatrix} \rho \\ \theta \end{bmatrix} \right\}$$

$$T = \exp(\hat{\delta}) = \begin{bmatrix} \exp(\hat{\theta}) & J_L(\theta)\rho \\ 0^T & 1 \end{bmatrix} \quad J_L(\theta)\rho = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\hat{\theta})^n$$

$$= I + \hat{\delta} + \frac{1}{2!}\hat{\delta}^2 + \frac{1}{3!}\hat{\delta}^3 + \cdots$$

$$= I + \begin{bmatrix} \hat{\theta} & \rho \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \hat{\theta}^2 & \rho\hat{\theta}^2 \\ 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} \hat{\theta}^3 & \rho\hat{\theta}^3 \\ 0 & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} I + \hat{\theta} + \frac{1}{2!}\hat{\theta}^2 + \cdots & \rho(I + \hat{\theta}^2 + \hat{\theta}^3 + \cdots) \\ 0 & 1 \end{bmatrix}$$

Let's have $V = I + \hat{\theta}^2 + \hat{\theta}^3 + \cdots$

then we have:

$$\begin{bmatrix} \exp(\hat{\theta}) & \rho V \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} \exp(\hat{\theta}) & J_2(\theta)\rho \\ 0 & 1 \end{bmatrix}$$

SE2 to se2

$$\xi = \begin{bmatrix} \rho \\ \theta \end{bmatrix} = \log(T)^\vee = \log \begin{bmatrix} R & \rho \\ 0 & 1 \end{bmatrix} \in se2.$$

3.

$$S_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \tau \begin{bmatrix} \cos\theta_t & 0 \\ \sin\theta_t & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} v_t \\ w_t \end{bmatrix} + w_t \right).$$

$$z_t = \begin{bmatrix} v_t \\ \phi_t \end{bmatrix} = \begin{bmatrix} \sqrt{x_t^2 + y_t^2} \\ \operatorname{atan2}(-y_t - x_t) - \theta_t \end{bmatrix} + \eta_t.$$

$\downarrow$

G noise.

a. Prediction

$\mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$.

$$F = \frac{df}{dx}(\mu_{t|t}, u_t, 0) = \begin{bmatrix} 1 & 0 & -\sin\theta_t \cdot \tau v_t \\ 0 & 1 & \cos\theta_t \cdot \tau v_t \\ 0 & 0 & 1 \end{bmatrix}.$$

$$Q = \frac{df}{dw}(\mu_{t|t}, u_t, 0) = \begin{bmatrix} \tau\cos\theta_t & 0 \\ \tau\sin\theta_t & 0 \\ 0 & \tau \end{bmatrix}.$$

$$\mu_{t+1|t} = \begin{bmatrix} x_t + \tau v_t \cos\theta_t \\ y_t + \tau v_t \sin\theta_t \\ \theta_t + \tau w_t \end{bmatrix}.$$

$$\Sigma_{t+1|t} = F \Sigma_{t|t} F_t^T + Q_t W Q_t^T \quad \text{(lecture 14: 7)}$$

b. Update:

$$\mathcal{N}(\mu_{t+1|t+1}, \Sigma_{t+1|t+1}).$$

$$H_{t+1} = \frac{dh}{dx}(\mu_{t+1|t}, 0)$$

$$= \begin{bmatrix} \frac{x_{t+1}}{\sqrt{x_{t+1}^2 + y_{t+1}^2}} & \frac{y_{t+1}}{\sqrt{x_{t+1}^2 + y_{t+1}^2}} & 0 \\ \frac{-y_{t+1}}{x_{t+1}^2 + y_{t+1}^2} & \frac{x_{t+1}}{x_{t+1}^2 + y_{t+1}^2} & -1 \end{bmatrix}.$$

$$R_{t+1} = \frac{dh}{dv}(\mu_{t+1|t}, 0). = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$m_{t+1|t} = h(\mu_{t+1|t}, 0).$$

$$S_{t+1|t} = H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + R_{t+1} V R_{t+1}^T$$

$$C_{t+1|t} = \Sigma_{t+1|t} H_{t+1}^T$$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})$$

$$= \mu_{t+1|t} + K_{t+1|t} \cdot \eta_{t+1}$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T$$

c. $\Omega_{t|t} = \Sigma_{t|t}^{-1}$,

predict $\Omega_{t+1|t}$ & update $\Omega_{t+1|t+1}$.

Prediction:

$$M_{t+1} = F_{t+1}^{-1} \Sigma_{t|t} F_t^{-1^T}$$
$$C_{k+1} = (M_{t+1} + Q_{t+1}^{-1})^{-1}$$
$$L_{t+1} = I - C_{k+1}$$

let $r_{t|t} = \Sigma^{-1}\mu$

$$r = L_{t+1} (F_{t+1}^{-1})^T r_{t|t}$$

$$\Omega_{t+1|t} = L_{t+1} M_{t+1} L_{t+1}^T + C_{t+1} Q_{t+1}^{-1} C_{t+1}^T$$

Update:

$$\Omega_{t+1|t+1} = \Omega_{t+1|t} + I_t \quad \text{where } I_t = H_t^T R_t^{-1} H_t$$
$$r_{t+1|t+1} = r_{t+1|t} + i_t \quad \text{where } i_t = H_t^T R_t^{-1} z_t$$