EXISTENCE OF WEAK SOLUTIONS TO THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

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Abstract. The purpose of this note is to give a self–contained presentation of the proof of the existence of weak solutions to the three-dimensional Navier–Stokes equations, using the Galerkin method. The main reference for this note is [1].

Notation

Throughout this note, all function spaces are assumed to be real-valued. We will fix $\Omega \subset \mathbb{R}^3$ to be a bounded domain with a smooth boundary. When we refer to spaces of functions on Ω, we will usually omit the letter Ω in our notations; for example, we will simply write L^p for $L^p(\Omega)$. When we refer to L^2 norm, we will usually drop the subscript, writing $\|\cdot\|$ for $\|\cdot\|_{L^2(\Omega)}$. The bracket notation $\langle \cdot, \cdot \rangle$ means the real inner product in $L^2(\Omega)$.

1. Weak formulation

The initial and boundary problem of the Navier–Stokes equations is formulated as

$$\begin{cases} \partial_t u - \triangle u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } [0,T] \times \Omega, \\ \operatorname{div} u = 0 & \text{in } [0,T] \times \Omega, \\ u = 0 & \text{on } [0,T] \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $u_0 : \Omega \to \mathbb{R}^3$ is a given vector field. We will solve for the velocity $u : [0,T] \times \Omega \to \mathbb{R}^3$ and the pressure $p : [0,T] \times \Omega \to \mathbb{R}$.

In this section, we will deduce a weak formulation for (1). Notice that if a vector field ϕ is divergence-free, by integration by parts we can get

$$\langle \nabla p, \, \phi \rangle = - \langle p, \, \operatorname{div} \phi \rangle = 0.$$

So by applying divergence-free test functions, we can temporarily get rid of the pressure p. Thus, a nice space for the test functions to live in would be

$$\mathcal{D}_\sigma(\Omega) = \left\{ \phi \in C_c^\infty([0,\infty) \times \Omega)^3 : \, \operatorname{div} \phi(t) = 0 \text{ for all } t \geqslant 0 \right\}. \tag{2}$$

Let $\phi \in \mathcal{D}_\sigma$, and multiply the first equation of (1) by ϕ and integrate in space, we get

$$\langle \partial_t u, \, \phi \rangle + \langle \nabla u, \, \nabla \phi \rangle + \langle (u \cdot \nabla)u, \, \phi \rangle = 0.$$

Then integrate in time from 0 to s, we get

$$- \int_0^s \langle u, \, \partial_t \phi \rangle + \int_0^s \langle \nabla u, \, \nabla \phi \rangle + \int_0^s \langle (u \cdot \nabla)u, \, \phi \rangle = \langle u_0, \, \phi(0) \rangle - \langle u(s), \, \phi(s) \rangle. \tag{3}$$

The equation (3) could be considered as a weak form of the Navier–Stokes equations (1).

Next, We should determine a suitable space for the weak solutions to live in. Let

$$\begin{aligned}
&C_{c,\sigma}^{\infty}(\Omega) = \left\{\phi \in (C_c^{\infty}(\Omega))^3 : \operatorname{div}\phi = 0\right\},\\
&H(\Omega) = \text{The closure of } C_{c,\sigma}^{\infty}(\Omega) \text{ in } (L^2(\Omega))^3,\\
&V(\Omega) = H(\Omega) \cap (H_0^1(\Omega))^3;
\end{aligned}$$

here $V(\Omega)$ could be considered as the space of divergence-free functions that vanish on the boundary (in the weak sense). Hence, it's natural to require $u(t) \in V(\Omega)$ for a.e. $t > 0$.

For physical considerations, the weak solutions should also satisfy that for all $T > 0$,

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|u(t)\| < \infty, \quad \int_0^T \|\nabla u(s)\|^2 < \infty.$$

Hence, we can conclude that the weak solutions should live in

$$L^{\infty}(0,T;H) \cap L^2(0,T;V)$$

for all $T > 0$. Here we impose L^2 norm on H and H_0^1 norm on V (i.e. $\|u\|_V = \|\nabla u\|$).

Definition. A function u is called a **weak solution** of the Navier–Stokes equations (1) with initial condition $u_0 \in H$, if

(a) $u \in L^{\infty}(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$;

(b) For all $\phi \in \mathcal{D}_\sigma$ and a.e. $s > 0$, the equation (3) holds. Here $\mathcal{D}_\sigma$ is given by (2).

2. Helmholtz–Weyl decomposition

Let $u \in (L^2)^3$, and n be the outer normal vector field on $\partial\Omega$. When $\operatorname{div} u \in L^2$, normal component $u\cdot n$ of u could be defined as a linear functional acting on the trace of H^1 functions, by a weak form of Gauss formula

$$\langle u \cdot n,\, v\rangle_{\partial\Omega} := \langle \operatorname{div} u,\, v\rangle + \langle u,\, \nabla v\rangle, \quad \forall v \in H^1(\Omega).$$

In this sense, we have the following characterization of the space H.

Lemma 1. *For $u \in (L^2)^3$, we have*

$$u \in H \iff \operatorname{div} u = 0 \textit{ in } \Omega,\ u \cdot n = 0 \textit{ on } \partial\Omega.$$

Remark. A function in H is divergence-free and has a zero normal component, but is not necessarily vanish on the boundary.

Theorem 2 (Helmholtz–Weyl)**.** *The orthogonal complement of H in $(L^2)^3$ is*

$$G := \left\{\nabla h : h \in H^1\right\}.$$

Proof. By integration by parts, we have $\langle \phi,\, \nabla g\rangle = 0$ for all $\phi \in C_{c,\sigma}^{\infty}$ and $g \in H^1$. Then by using the fact that $C_{c,\sigma}^{\infty}$ is dense in H, we conclude that G and H are orthogonal to each other.

Suppose $u \in (L^2)^3$. Since $\operatorname{div} u \in H^{-1}$, by the knowledge of the Poisson equation, the weak form of the Dirichlet problem

$$\begin{cases} \triangle h = \operatorname{div} u \quad \text{in } \Omega;\\ h = 0 \quad \text{on } \partial\Omega \end{cases}$$

is well-posed. The weak solution h will live in $H^1(\Omega)$. Let

$$v = u - \nabla h,$$

then v is divergence-free. But we do not necessarily have $v \cdot n = 0$ on $\partial\Omega$. To handle this issue, take $w \in H^1(\Omega)$ be the weak solution to the Neumann problem

$$\begin{cases} \triangle w = 0 & \text{in } \Omega; \\ \partial w / \partial n = v \cdot n & \text{on } \partial\Omega. \end{cases}$$

Then $v - \nabla w$ is divergence-free. Thus, we have

$$u = \nabla(h + w) + (v - \nabla w),$$

which is the desiring decomposition. □

Definition. Let $\mathbb{P} : (L^2)^3 \to H$ denote the orthogonal projection onto H, i.e.

$$\mathbb{P}u = v \iff u = v + \nabla h \text{ for some } h \in H^1.$$

Projection $\mathbb{P}$ is called the **Leray operator**.

Proposition 3. *$\mathbb{P} : (L^2)^3 \to H \subset (L^2)^3$ is self-adjoint, i.e.*

$$\langle \mathbb{P}u,\, v\rangle = \langle u,\, \mathbb{P}v\rangle \quad \textit{for all } u, v \in (L^2)^3.$$

Proof. Easy computation using the definition. □

If we apply $\mathbb{P}$ on both sides of the Navier–Stokes equation

$$\partial_t u - \triangle u + (u \cdot \nabla)u + \nabla p = 0,$$

where $u \in H$, we can get rid of pressure p and get the projected version of Navier–Stokes equation:

$$\partial_t u - \mathbb{P}\triangle u + \mathbb{P}[(u \cdot \nabla)u] = 0. \tag{4}$$

Equation (4) is a nice simplification to the original problem when we only care about the velocity field u in the Navier–Stokes equations.

3. The Stokes equation and Stokes operator

Next, we study the linear term $-\mathbb{P}\triangle u$ in (4).

Definition. The **Stokes operator** A is defined by

$$Au = -\mathbb{P}\triangle u \quad \text{for all } u \in D(A) := V \cap (H^2)^3.$$

By the definition of Leray projector, functions $u \in D(A)$ and $f \in H$ satisfy $Au = f$ if and only if

$$-\triangle u + \nabla p = f \quad \text{for some } p \in H^1. \tag{5}$$

The equation (5) is called **Stokes equation**, which could be viewed as the stationary linearized version of Navier–Stokes equation. Multiply (5) by $v \in V$ and integrate by parts, we can deduce the weak form of the Stokes equation:

$$\langle \nabla u,\, \nabla v\rangle = \langle f,\, v\rangle \quad \text{for all } v \in V. \tag{6}$$

Lemma 4 (existence and uniqueness of weak solutions). *For every $f \in V^*$, there is a unique $u \in V$ satisfies* (6).

Proof. We try to apply the Lax-Milgram theorem. Consider the bilinear functional

$$h : (u, v) \in V \times V \mapsto \langle \nabla u,\, \nabla v \rangle \,.$$

Clearly

$$|h(u, v)| = |\langle \nabla u,\, \nabla v \rangle| \leqslant C \, \|\nabla u\| \, \|\nabla v\| = C \, \|u\|_V \, \|v\|_V$$

so $h : V \times V \to \mathbb{R}$ is bounded. Moreover,

$$h(u, u) = \|\nabla u\|^2 = \|u\|_V^2 \,,$$

Hence by the Lax-Milgram theorem, we can finish the proof. □

Lemma 5 (regularity of weak solutions). *Suppose $f \in L^2$ and $u \in V$ satisfies* (6)*, then $u \in H^2$ and $Au = f$.*

Proof. Note that the weak form 6 is a high-dimension analog of the weak form of the Laplace equation; by the standard techniques in elliptic PDE theory, one can prove $u \in H^2$. These standard techniques can be found in many textbooks, such as chapter 6 of [2].

By $u \in H^2$ we can integrate by parts in (6), and conclude that $f + \triangle u$ is L^2-orthogonal to V. Since V is dense in H, by Helmholtz–Weyl decomposition we see $f + \triangle u \in G$, which means u is also strong solution to Stokes equation 5 and hence $Au = f$. □

Proposition 6. *The Stokes operator A satisfies the following properties.*
(a) $A : D(A) \to H$ is bijective.
(b) $A^{-1} : H \to D(A) \subset H$ is self-adjoint and positive.

Proof. By lemma 5 and $H \subset V^*$, clearly A is surjective. Next, we prove A is injective by showing its null space is trivial. Suppose $Au = 0$ for some $u \in D(A)$, then there exists $p \in H^1$ s.t. $-\triangle u + \nabla p = 0$. Multiply this equation by u and integrate by parts, we get $\|\nabla u\|^2 = 0$, hence $u = 0$. Therefore, (a) is true.

Suppose $Au = f$ and $Av = g$ for $u, v \in D(A)$, by 6 we have

$$\left\langle g,\, A^{-1} f \right\rangle = \langle g,\, u \rangle = \langle \nabla v,\, \nabla u \rangle = \langle v,\, f \rangle = \left\langle A^{-1} g,\, f \right\rangle ,$$

so A^{-1} is self-adjoint. By taking $u = v$, one gets

$$0 \leqslant \langle \nabla u,\, \nabla u \rangle = \langle f,\, u \rangle = \left\langle f,\, A^{-1} f \right\rangle ,$$

hence A^{-1} is positive. □

Theorem 7. *(a) The Stokes operator A has a sequence of eigenvalues*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \quad \textit{s.t. } \lambda_j \to \infty \ (j \to \infty),$$

and corresponding eigenfunctions $a_j \in H$ s.t. $\{a_j\}$ is an orthonormal Schauder basis of H.
(b) Moreover, $\{a_j\}$ is also an orthogonal Schauder basis of V, and $\|a_j\|_V = \lambda_j^{1/2}$.
(c) All the a_j's belong to $C^\infty(\overline{\Omega})$.

Proof. The Sobolev embedding $H^1 \subset\subset L^2$ implies that $D(A) \subset\subset H$. Therefore, $A^{-1} : H \to H$ is a compact operator. Moreover, one can check that A^{-1} is self-adjoint and positive, therefore by spectral theorem, H has an orthonormal basis $\{a_j\}$ consisting of eigenfunctions of A^{-1}, where all the corresponding eigenvalues are a positive sequence descending to zero. These a_j's are also eigenfunctions of A, and the corresponding eigenvalues are a sequence increasing to infinity. Hence, we conclude that (a) is true.

Next, notice that

$$\begin{aligned} \langle a_j,\, a_k\rangle_V &= \langle \nabla a_j,\, a_k\rangle = -\langle a_j,\, \triangle a_k\rangle = -\langle \mathbb{P} a_j,\, \triangle a_k\rangle \\ &= \langle a_j,\, -\mathbb{P}\triangle a_k\rangle = \langle a_j,\, A a_k\rangle = \lambda_k \langle a_j,\, a_k\rangle . \end{aligned}$$

Then by the L^2 orthogonality, we can prove (b).

To prove (c), one can make use of lemma 5 and apply a bootstrapping argument. This is also standard in elliptic PDE theory. For details, one can refer to chapter 6 of [2]. □

4. Alternative space for test functions

Next, we will show that the eigenfunctions of the Stokes operator can be used to construct a more convenient space of test functions. Let

$$\tilde{\mathcal{D}}_\sigma = \left\{ \phi : \ \phi(t,x) = \sum_{k=1}^{n} c_k(t) a_k(x),\ c_k \in C_c^1([0,\infty)),\ n \in \mathbb{N} \right\},$$

where $\{a_j\}$ is the eigenfunctions of the Stokes operator which is also an orthonormal Schauder basis of H and V.

Theorem 8. *Suppose $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, $u_0 \in H$, then the following statements are equivalent:*

(a) u satisfy (3) for all $\phi \in \mathcal{D}_\sigma$.

(b) u satisfy (3) for all $\phi \in \tilde{\mathcal{D}}_\sigma$.

To prove the theorem 8, we first need a lemma to estimate the nonlinear term.

Lemma 9. *If $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$, then*

$$(u \cdot \nabla) u \in L^{4/3}(0,T;L^{6/5}).$$

Proof. Note that $5/6 = 1/2 + 1/3$, so by Hölder inequality we have

$$\|(u\cdot\nabla)u\|_{L^{6/5}} \leqslant \|u\|_{L^3} \|\nabla u\|_{L^2} .$$

Then by the Lebesgue interpolation

$$\|u\|_{L^3} \leqslant \|u\|_{L^2}^{1/2} \|u\|_{L^6}^{1/2} ,$$

and the Sobolev embedding $H^1 \hookrightarrow L^6$, we conclude that

$$\|(u\cdot\nabla)u\|_{L^{6/5}} \leqslant C \|u\|_{H^1}^{3/2} \|u\|_{L^2}^{1/2} .$$

Hence,

$$\int_0^T \|(u\cdot\nabla)u\|_{L^{6/5}}^{4/3} \leqslant C \int_0^T \|u\|_{H^1}^2 \|u\|_{L^2}^{2/3} \leqslant C \|u\|_{L^\infty(0,T;L^2)}^{2/3} \|u\|_{L^2(0,T;H^1)} < \infty.$$

□

Proof of theroem 8. First, we assume (a) is true and prove (b). It suffices to show that u satisfy (3) for every ϕ of the form $\phi(t,x) := c(t)\alpha(x)$, where $c \in C_c^1([0,\infty))$, $\alpha \in \{a_j\}$.

By mollifying, we can find $c_n \in C_c^\infty([0,s+1])$ s.t.

$$c_n \to c \text{ and } c_n' \to c_n' \quad \text{uniformly on } [0,s].$$

Using the density of $C_{c,\sigma}^\infty$ in V, we can find $\alpha_n \in C_{c,\sigma}^\infty$ s.t.

$$\alpha_n \to \alpha \quad \text{in } H^1 \text{ as } n \to \infty.$$

Let

$$\phi_n(t,x) = c_n(t)\alpha_n(x) \in \mathcal{D}_\sigma.$$

By (a) we have

$$-\int_0^s \langle u,\, \partial_t \phi_n \rangle + \int_0^s \langle \nabla u,\, \nabla \phi_n \rangle + \int_0^s \langle (u\cdot\nabla)u,\, \phi_n \rangle = \langle u_0,\, \phi_n(0) \rangle - \langle u(s),\, \phi_n(s) \rangle\,,$$

thus we only need to pass the limit $n \to \infty$ in the equality.

It's easy to see

$$\begin{aligned} \partial_t \phi_n \to \partial_t \phi \quad & \text{in } L^2(0,s;L^2); \\ \phi_n \to \phi \qquad & \text{in } C(0,s;H^1), \end{aligned}$$

so there's no problem to pass the limit in linear terms. But the nonlinear term does not necessarily belong to $L^2(0,s;L^2)$, so we need lemma 9:

$$\begin{aligned} \left| \int_0^s \langle (u\cdot\nabla)u,\, \phi_n - \phi \rangle \right| &\leqslant \int_0^s \|(u\cdot\nabla)u\|_{L^{6/5}} \|\phi_n - \phi\|_{L^6} \\ &\leqslant \|(u\cdot\nabla)u\|_{L^{4/3}(0,s;L^{6/5})} \|\phi_n - \phi\|_{L^4(0,s;L^6)} \\ &\to 0 \ (n \to \infty). \end{aligned}$$

Suppose (b) is true, next we show (a). Suppose $\phi \in \mathcal{D}_\sigma$. Then $\phi \in V$, so we can write

$$\phi(t,x) = \sum_{k=1}^\infty c_k(t) a_k(x),$$

where the sum is pointwise convergent in t, with respect to H^1 norm in x.

Let

$$\phi_n(t,x) = \sum_{k=1}^n c_k(t) a_k(x) \in \tilde{\mathcal{D}}_\sigma,$$

Since $c_k = \langle \phi,\, a_k \rangle$, we conclude that $\{c_k\} \subset C_c^\infty([0,\infty))$, and they are uniformly bounded on $[0,s]$. Therefore,

$$\phi_n \to \phi \quad \text{in } C(0,s;H^1).$$

Furthermore,

$$\partial_t \phi_n \to \partial_t \phi \quad \text{in } L^2(0,s;L^2),$$

thus we can pass the limit $n \to \infty$ in the same way as we've done in (a). $\square$

5. Existence of the weak solutions

In this section, we will prove the main result of this note, which is the following theorem.

Theorem 10 (Hopf)**.** *For every given $u_0 \in H(\Omega)$, there exists a global-in-time weak solution to the Navier–Stokes equations with initial condition u_0.*

The method by which we construct the weak solution is called **Galerkin method**, whose main idea is:

1. Construct a sequence of larger and larger finite dimensional space approximating the function space in which the original PDE is posed. Project the original PDE onto each finite-dimensional space.
2. Solve each projected equation by the existence theorem of ODE (since it's finite-dimensional) and get a sequence of approximate solutions.
3. Show all approximate solutions that exist globally in time.
4. Extract a subsequence of approximate solutions (in some suitable sense) and conclude that the limit is a solution to the original PDE.

Next, we will follow these steps and apply the method to Navier–Stokes equation.

Step 1: Project the equation onto some finite-dimensional spaces. Let $\{a_j\}$ be the eigenfunctions of the Stokes operator, which is an orthonormal Schauder basis of H. Let

$$P_n : (L^2)^3 \mapsto P_n H := \operatorname{span}\{a_1, \cdots, a_n\}, \quad u \mapsto \sum_{j=1}^{n} \langle u,\, a_j \rangle\, a_j$$

be the projection onto the space spanned by the first n eigenfunctions.

Proposition 11. *P_n is self-adjoint, i.e.*

$$\langle P_n u,\, v\rangle = \langle u,\, P_n v\rangle \quad \text{for all } u, v \in (L^2)^3.$$

Proof. Trivial computation. □

The operator P_n could be thought of as the finite-dimensional approximation of the Leray operator $\mathbb{P}$. From this perspective, we can define approximate solutions u_n of the Navier–Stokes equation.

Definition. The n-th **Galerkin approximation** u_n of the Navier Stokes equation with initial condition $u_0 \in H$ is defined by the problem

$$\partial_t u_n + A u_n + P_n[(u_n \cdot \nabla) u_n] = 0, \quad u_n(0) = P_n u_0. \tag{7}$$

Here (7) is called the n-th **Galerkin equation**.

The Galerkin equation (7) could be viewed as a finite-dimensional approximation of the projected Navier–Stokes equation (4).

Step 2: Solve the projected equations for approximate solutions. Since P_nH is finite-dimensional, the Galerkin equation could be viewed as an ODE in the space P_nH, and we naturally expect its solution exists, at least in a short time. We can write

$$u_n(t,x) = \sum_{j=1}^{n} c_{n,j}(t)a_j(x). \tag{8}$$

Taking inner product of (7) with a_k, substituting u_n with (8), we obtain

$$c'_{n,k}(t) = -\lambda_k c_{n,k}(t) - \sum_{i,j=1}^{n} \langle (a_i \cdot \nabla)a_j,\, a_k\rangle\, c_{n,i}(t)c_{n,j}(t), \quad c_{n,k}(0) = \langle u_0,\, a_k\rangle\,. \tag{9}$$

(9) is an ODE of the vector $c_n := (c_{n,k})_{k=1}^n$, and we could check that the right-hand side is a locally Lipschitz function of c_n. Hence, by existence and uniqueness theorem of ODE, we conclude that there exists $0 < T_n \leqslant \infty$ s.t. c_n is defined in time $[0, T_n)$, which means u_n exists in a short time.

Step 3: Show that the approximate solutions exist globally in time. Next, we will show that u_n is globally defined in time. Since

$$\|u_n(t)\|^2 = \sum_{k=1}^{n} |c_{n,k}(t)|^2\,,$$

to show that $c_{n,k}$ doesn't blow up in finite time, we only need to show $\|u_n\|$ will not blow up. Taking inner product of (7) with u_n, the nonlinear term vanish:

$$\langle P_n(u_n\cdot\nabla)u_n,\, u_n\rangle = \langle (u_n\cdot\nabla)u_n,\, Pu_n\rangle = \langle (u_n\cdot\nabla)u_n,\, u_n\rangle = 0;$$

and the other two terms become

$$\begin{aligned}
\langle \partial_t u_n,\, u_n\rangle &= \frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2\,,\\
\langle Au_n,\, u_n\rangle &= \langle -\mathbb{P}\triangle u_n,\, u_n\rangle = \langle -\triangle u_n,\, \mathbb{P}u_n\rangle = \langle -\triangle u_n,\, u_n\rangle = \|\nabla u_n\|^2\,.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2 = -\|\nabla u_n(t)\|^2 \leqslant 0,$$

which implies $\|u_n\|$ is non-increasing in time, and thus will not blow up. Moreover, by integrating in time we conclude that for a.e. $s > 0$,

$$\frac{1}{2}\|u_n(s)\|^2 + \int_0^s \|\nabla u_n(t)\|^2\,dt = \frac{1}{2}\|u_n(0)\|^2 \leqslant \frac{1}{2}\|u_0\|^2\,. \tag{10}$$

Step 4: Extract a convergent subsequence of approximate solutions. For a test function $\phi(t,x) = \sum_{j=1}^{N} c_j(t)a_j(x) \in \tilde{\mathcal{D}}_\sigma$, by the definition of u_n we can check that if $n > N$, we have

$$-\int_0^s \langle u_n,\, \partial_t\phi\rangle + \int_0^s \langle \nabla u_n,\, \nabla\phi\rangle + \int_0^s \langle (u_n\cdot\nabla)u_n,\, \phi\rangle = \langle u_n(0),\, \phi(0)\rangle - \langle u_n(s),\, \phi(s)\rangle\,. \tag{11}$$

Hence, if we extract a convergent subsequence of $\{u_n\}$ (in a suitable sense) and let $n \to \infty$ in (11), then by theorem 8 we conclude that the limit of u_n is a weak solution to Navier–Stokes equation. To figure out what kind of convergence can enable us to pass the limit, next we will examine (11) term by term.

Fixing $T > 0$. Suppose we can extract a subsequence $\{u_n\}$ (which we relabel) s.t.

$$u_n \to u \quad \text{in } L^2(0,T;L^2). \tag{12}$$

Moreover, if (12) holds, we can extract a further subsequence $\{u_n\}$ (relabel again) s.t.

$$u_n(t) \to u_n(t) \quad \text{in } L^2 \text{ for a.e. } t \in (0,T),$$

we conclude that for a.e. $s \in (0,T)$,

$$\langle u_n(0),\, \phi(0)\rangle - \langle u_n(s),\, \phi(s)\rangle \to \langle u_0,\, \phi(0)\rangle - \langle u(s),\, u(s)\rangle \quad (n \to \infty).$$

Furthermore, by (10), the sequence $\{\nabla u_n\}$ is bounded in $L^2(0,T;L^2)$ for all $T > 0$. By reflexivity, we can extract a subsequence $\{\nabla u_n\}$ (which we relabel) s.t.

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^2(0,T;L^2) \text{ for all } T > 0. \tag{13}$$

Thus, for a.e. $s > 0$,

$$\int_0^s \langle \nabla u_n,\, \nabla \phi\rangle \to \int_0^s \langle \nabla u,\, \phi\rangle\,.$$

To summarize, the condition (12) suffices for the linear part of (11) to converge.

Next, we seek conditions enabling the convergence of the nonlinear term in (11). Suppose (12) holds. Notice that

$$(u_n \cdot \nabla u_n)u_n - (u \cdot \nabla u)u = (u_n - u) \cdot \nabla u_n - u \cdot \nabla (u_n - u).$$

Hence, for $s \in (0,T)$, we have

$$\begin{aligned}
\left| \int_0^s \langle (u_n - u) \cdot \nabla u_n,\, \phi\rangle \right| &\leqslant C(\phi) \int_0^s \|u_n - u\|\, \|\nabla u_n\| \\
&\leqslant C(\phi) \left(\int_0^s \|u_n - u\|^2 \right)^{1/2} \left(\int_0^s \|\nabla u_n\|^2 \right)^{1/2} \\
&\leqslant C(\phi) \left(\frac{1}{2} \|u_0\|^2 \right)^{1/2} \left(\int_0^s \|u_n - u\|^2 \right),
\end{aligned}$$

(using (10) for the last inequality). On the other hand, by the weak converge of gradients in (13), we conclude that

$$u \cdot \nabla (u_n - u) \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;L^2),$$

thus for $s \in (0,T)$,

$$\int_0^s \langle u \cdot \nabla (u_n - u),\, \phi\rangle \to 0.$$

Therefore, the nonlinear terms also satisfy

$$\int_0^s \langle (u_n \cdot \nabla) u_n,\, \phi\rangle \to \int_0^s \langle (u \cdot \nabla) u,\, \phi\rangle$$

for $s \in (0,T)$.

Summing up all the discussions above, we conclude that extracting a subsequence $\{u_n\}$ satisfying (12) is sufficient to complete the proof of theorem 10. To prove the existence of such a subsequence, we need the following lemma.

Lemma 12 (Aubin–Lions)**.** *Suppose $p, q > 1$, $T > 0$. If a sequence $\{u_n\}$ satisfies*

$$\|u_n\|_{L^q(0,T;V)} + \|\partial_t u_n\|_{L^p(0,T;V^*)} \leqslant C \quad \text{for all } n \in \mathbb{N}, \tag{14}$$

where C is independent of n, then $\{u_n\}$ has a subsequence converging strongly in $L^q(0,T;H)$.

Proof. 1. Consider the finite-dimensional projector

$$P_k u := \sum_{j=1}^{k} \langle u,\, a_j \rangle\, a_j.$$

For each k, we next try to extract a subsequence from $\{P_k u_n\}_{n\in\mathbb{N}}$ which converges in $C(0,T;H)$. We will do this by the Ascoli-Arzela theorem.

Notice that each $\langle u_n,\, a_j \rangle$ is absolutely continuous in time, with $\partial_t \langle u_n,\, a_j \rangle = \langle \partial_t u_n,\, a_j \rangle$ (here ∂_t is taken in weak sense). Hence, for each u_n and a_j, there exists $s_0 \in [0,T]$ s.t.

$$\langle u_n(s_0),\, a_j \rangle = \frac{1}{T} \int_0^T \langle \partial_t u_n,\, a_j \rangle\, dt.$$

Therefore, for every $s \in [0,T]$, we have

$$\begin{aligned}
|\langle u(s),\, a_j \rangle| &\leqslant |\langle u(s_0),\, a_j \rangle| + \left| \int_{s_0}^{s} \langle \partial_t u,\, a_j \rangle\, dt \right| \\
&\leqslant \frac{1}{T} \int_0^T \|u(s)\|\, \|a_j\|\, dt + \int_0^T \|\partial_t u\|_{V^*} \|a_j\|_V \\
&\leqslant \frac{1}{T} \|u(s)\|_{L^q(0,T;H)}\, T^{1-1/q} + \|\partial_t u\|_{L^p(0,T;V^*)}\, \lambda_j^{1/2} \\
&\leqslant C(1 + \lambda_j^{1/2}),
\end{aligned}$$

where C is independent of s. Therefore,

$$\|P_k u_n(s)\| \leqslant \sum_{j=1}^{k} |\langle u_n(s),\, a_j \rangle| \leqslant C \sum_{j=1}^{k} (1 + \lambda_j^{1/2}),$$

which says $\{P_k u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C(0,T;H)$ for every $k \in \mathbb{N}$.

Next, we prove equicontinuity. For every $s, s' \in [0,T]$, we have

$$\begin{aligned}
\|P_k u_n(s) - P_k u_n(s')\| &\leqslant \sum_{j=1}^{k} \int_{s'}^{s} |\langle \partial_t u_n,\, a_j \rangle|\, dt \\
&\leqslant \sum_{j=1}^{k} \int_{s'}^{s} \|\partial_t u_n\|_{V^*} \|a_j\|_V\, dt \\
&\leqslant \sum_{j=1}^{k} \lambda_j^{1/2} \left(\int_{s'}^{s} \|\partial_t u_n\|_{V^*}^p\, dt \right)^{1/p} \left(\int_{s'}^{s} dt \right)^{1-1/p} \\
&\leqslant C_k\, |s - s'|^{1-1/p},
\end{aligned}$$

where we used (14) in the last line. Therefore, the sequence $\{P_k u_n\}_{n\in\mathbb{N}}$ is semicontinuous. Now we can use Ascoli-Arzela to extract a subsequence from $\{P_k u_n\}_{n\in\mathbb{N}}$ which converges in $C(0,T;H)$.

2. By standard diagonal argument, we can extract a subsequence $\{u_n\}$ (which we relabel) s.t. $\{P_k u_n\}_{n\in\mathbb{N}}$ converges for every $k\in\mathbb{N}$. Next we will show this $\{u_n\}$ is a Cauchy sequence in $L^q(0,T;H)$. Let

$$Q_k u := \sum_{j=k+1}^{\infty} \langle u,\, a_j\rangle\, a_j. \tag{15}$$

By (14) and the fact that $\{\lambda_j\}$ is increasing,

$$\begin{aligned} C &\geqslant \int_0^T \|Q_k u_n\|_V^q \, dt = \int_0^T \left(\sum_{j=k+1}^{\infty} \lambda_j \langle u_n,\, a_j\rangle^2\right)^{q/2} dt \\ &\geqslant \lambda_k^{q/2} \int_0^T \left(\sum_{j=k+1}^{\infty} \langle u_n,\, a_j\rangle^2\right)^{q/2} dt = \lambda_k^{q/2} \|Q_k u_n\|_{L^q(0,T;H)}^q . \end{aligned}$$

Since $\lambda_k \to \infty$, for every $\varepsilon > 0$ we can choose k sufficiently large s.t.

$$\sup_n \|Q_k u_n\|_{L^q(0,T;H)} < \varepsilon.$$

Since $\{P_k u_n\}_{n\in\mathbb{N}}$ converges in $C(0,T;H)$ and hence in $L^q(0,T;H)$, we could find $N>0$ s.t.

$$\sup_{n,m>N} \|P_k u_n - P_k u_m\|_{L^q(0,T;H)} < \varepsilon.$$

Therefore, for every $n,m>N$ we have

$$\|u_n - u_m\|_{L^q(0,T;H)} \leqslant \|P_k u_n - P_k u_m\|_{L^q(0,T;H)} + \|Q_k u_n - Q_k u_m\|_{L^q(0,T;H)} \leqslant 3\varepsilon,$$

which says $\{u_n\}$ is Cauchy in $L^q(0,T;H)$. □

Now we attempt to use the lemma 12 with $q=2$. By (10) we have

$$\|u_n\|_{L^2(0,T;V)} \leqslant C.$$

Next we try to estimate $\|\partial_t u_n\|_{L^p(0,T;V^*)}$ for some suitable p.

Suppose $v\in V$. Taking inner product of (7) with v we obtain

$$\langle \partial_t u_n,\, v\rangle = -\langle A u_n,\, v\rangle - \langle P_n[(u_n\cdot\nabla)u_n],\, v\rangle .$$

The first term on right-hand side is easy to estimate:

$$\begin{aligned} |\langle A u_n,\, v\rangle| &= |\langle -\mathbb{P}\triangle u_n,\, v\rangle| = |\langle -\triangle u_n,\, \mathbb{P}v\rangle| \\ &= |\langle -\triangle u_n,\, v\rangle| = |\langle \nabla u_n,\, \nabla v\rangle| \leqslant \|\nabla u_n\|\, \|v\|_V . \end{aligned}$$

By L^2-L^3-L^6 Hölder inequality, Lebesgue interpolation $\|u\|_{L^3} \leqslant \|u\|_{L^2}^{1/2} \|u\|_{L^6}^{1/2}$ and Sobolev embedding $H_0^1 \hookrightarrow L^6$, we have

$$\begin{aligned}
|\langle P_n[(u_n \cdot \nabla)u_n],\, v\rangle| &= |\langle (u_n \cdot \nabla)u_n,\, P_n v\rangle| \\
&\leqslant \|u_n\|_{L^3} \cdot \|\nabla u_n\|_{L^2} \cdot \|P_n v\|_{L^6} \\
&\leqslant C \|u_n\|_{L^2}^{1/2} \cdot \|u_n\|_{L^6}^{1/2} \cdot \|\nabla u_n\|_{L^2} \cdot \|P_n v\|_V \\
&\leqslant C \|u_n\|_{L^2}^{1/2} \cdot \|\nabla u_n\|_{L^2}^{3/2} \cdot \|v\|_V .
\end{aligned}$$

Therefore,

$$\|\partial_t u_n\|_{V^*} \leqslant \|\nabla u_n\| + C \|u_n\|^{1/2} \cdot \|\nabla u_n\|^{3/2} .$$

In order to take advantage of the inequality (10), we try to estimate $\|\partial_t u_n\|_{L^{4/3}(0,T;V^*)}$:

$$\begin{aligned}
\int_0^T \|\partial_t u_n\|_{V^*}^{4/3} &\leqslant C \int_0^T \|\nabla u_n\|^{4/3} + C \int_0^T \|u_n\|^{2/3} \cdot \|\nabla u_n\|^2 \\
&\leqslant C T^{1/3} \|\nabla u_n\|_{L^2(0,T;L^2)} + C \|u_n\|_{L^\infty(0,T;L^2)}^{2/3} \|\nabla u_n\|_{L^2(0,T;L^2)} \\
&\leqslant C T^{1/3} \|u_0\|^{4/3} + C \|u_0\|^{8/3} .
\end{aligned}$$

Therefore, we can apply Aubin–Lions lemma with $q = 2,\ p = 4/3$ and extract a subsequence from $\{u_n\}$ which converges in $L^2(0,T;L^2)$, and finally complete the proof of theorem 10.

References

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