Lab Course Automatic Control I

Analysis of control loops using MATLAB/Simulink II

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1 Introduction

In the first MATLAB experiment, a closed loop control for *linear* systems with frequency domain methods was considered. Based on that, this experiment covers *nonlinear* time-invariant systems in the form of

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{1}$$

with *linear* frequency domain methods. The first step therefore is the linearization of these *nonlinear* systems around a suitable equilibrium $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ with

$$\boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}, \quad \boldsymbol{y}_e = \boldsymbol{h}(\boldsymbol{x}_e, \boldsymbol{u}_e). \tag{2}$$

The deviations $\Delta\boldsymbol{x}(t)$, $\Delta\boldsymbol{y}(t)$ from $\boldsymbol{x}_e$ and $\boldsymbol{y}_e$ can, for sufficiently small changes of $\Delta\boldsymbol{u}(t)$, $\Delta\boldsymbol{x}_0$ from $\boldsymbol{u}_e$ and $\boldsymbol{x}_0$, be described by the *linear* time-invariant system

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{u}, \quad \Delta\boldsymbol{x}(t_0) = \Delta\boldsymbol{x}_0 = \boldsymbol{x}_0 - \boldsymbol{x}_e \\ \Delta\boldsymbol{y} &= \boldsymbol{C}\Delta\boldsymbol{x} + \boldsymbol{D}\Delta\boldsymbol{u} \end{aligned} \tag{3}$$

with

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e, \boldsymbol{u}_e}, \quad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_e, \boldsymbol{u}_e}, \quad \boldsymbol{C} = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e, \boldsymbol{u}_e}, \quad \boldsymbol{D} = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_e, \boldsymbol{u}_e}. \tag{4}$$

The system (3) ist called *linearization* of (1) around the *equilibrium* $(\boldsymbol{x}_e, \boldsymbol{u}_e)$.

2 Preparation

The tasks in this section have to be done before the actual day of the experiment. Understanding the theoretical fundamentals is mandatory for the successful execution of the experiment. Make sure to approach the experiment in a systematic and structured way.

The *nonlinear* time-invariant system

$$\dot{x}_1 = -x_1^3 + x_2 \tag{5a}$$
$$\dot{x}_2 = x_1 x_2 - 1 + u \tag{5b}$$
$$y = x_1 \tag{5c}$$

with $\boldsymbol{x} \in \mathbb{R}^2$ and $u \in \mathbb{R}$ is considered.

(P1) *Determine the two equilibriums $\boldsymbol{x}_{e,1}$ and $\boldsymbol{x}_{e,2}$ of the system with $u_e = 0$.*

(P2) *Calculate the respective linearizations for both equlibriums of the system.*
Note: *Use the MATLAB symbolic math toolbox for calculating the partial derivations with the `jacobian` command.*

(P3) *Calculate the corresponding transfer function for each linearization. Then check the transfer functions for input-output stability.*
Note: *To calculate the poles of a transfer function in MATLAB the command `pole` can be used.*

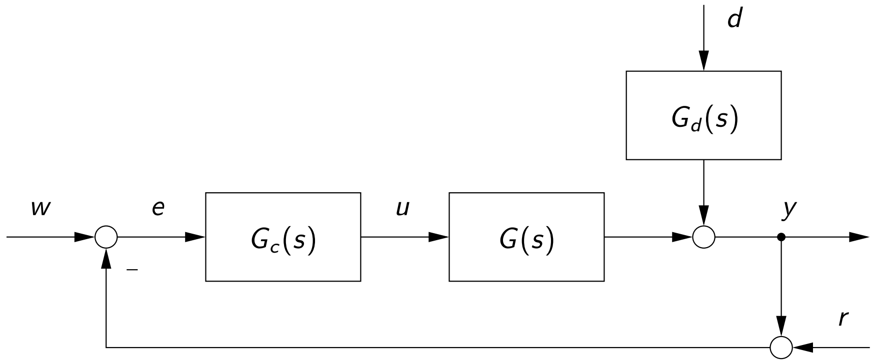

Figure 1: Default structure of a closed control loop.

Intermediate result: You should have found a stable & an unstable equilibrium.

Now the system shall be stabilized around the **unstable** equlibrium position with the help of a controller. The control loop structure from Figure 1 is used for this.

(P4) *Set up the disturbance transfer function (for $w(t) = r(t) = 0$) depending on a general controller $G_c(s)$.*

(P5) *Show that the system can be stabilized for $G_d(s) = 1$ with a P-controller. What condition must hold for the controller gain K to stabilize the system?*

(P6) *Test the P-controller from task **(P5)** simulatively in Simulink by perturbing the original **nonlinear** system in the unstable equilibrium with a constant (output) disturbance d. To do this, build the complete control loop in Simulink. Implement the nonlinear system equations as*

- `MATLAB Function`*-Block and as*
- `Level-2 MATLAB S-Function`*-Block (see* `msfunction_template.m`*).*

Can the system be stabilized against any size of disturbance? What is the reason for this? How are the initial values to be passed in both variants?

Note: *For complex systems, an* `S-Function` *written in C code can be used in Simulink to increase computational efficiency. This will not be discussed in this experiment, but it should be noted that it is structurally equivalent to a* `MATLAB S-Function`*.*
More info on `MATLAB's S-Function` *can be found at* `https://mathworks.com/help/simulink/sfg/what-is-an-s-function.html` *and in Matlab help.*

(P7) *Test the P-controller from task **(P5)** simulatively in Simulink by perturbing the **linearized system** in the unstable equilibrium with a constant (output) disturbance d. To do this, construct the complete control loop in Simulink. The linear system equations can be implemented with a* `state-space` *block. Verify the controller gain condition found in task **(P5)**.*

Now a controller for an operating point change for the system linearized around the **stable** equilibrium shall be designed. Again, the control loop structure from Figure 1 is used.

(P8) *Establish the reference transfer function as a function of a general PI controller $G_c(s) = K(1 + \frac{1}{T_I s})$.*

(P9) *Design a PI controller for the linearized system. Let the rise time be $T_{on} = 2s$, the overshoot $e_{\mathrm{max}} = 10\%$ and the steady state control deviation $e_{\infty} = 0$.*

(P10) *Test the designed controller simulatively in Simulink for the* ***nonlinear*** *original system by applying various steps in the reference variable from the interval $[0.01, \ldots, 1]$. Are the specifications for arbitrary steps met? What is the reason for this?*

3 Experiment

This experiment uses the pendulum on a cart system from Figure 2 with the masses M and m of the cart and pendulum, the rod length l and the applied force $u = F$.

A model of the system dynamics is needed for the controller design. With the help of the Lagrange formalism the following nonlinear dynamic model can be derived:

$$\ddot{\phi} = \frac{6g(m+M)\sin(\phi) - 3\cos(\phi)(2u + lm\dot{\phi}^2\sin(\phi))}{l\left(m + 4M + 3m\sin^2(\phi)\right)} \tag{6a}$$

$$\ddot{x}_{\text{cart}} = \frac{8u + 4lm\dot{\phi}^2\sin(\phi) - 3mg\sin(2\phi)}{5m + 8M - 3m\cos(2\phi)}. \tag{6b}$$

The output y is the pendulum angle ϕ, described by the output equation

$$y = h(\boldsymbol{x}) = \phi\,. \tag{7}$$

(E1) *Convert the system into the state space form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$ with the state vector $\boldsymbol{x} = [\phi, \dot{\phi}, x_{\text{cart}}, \dot{x}_{\text{cart}}]^T$.*

The goal of the experiment is to stabilize the *nonlinear* system using *linear* frequency domain methods. For this purpose, the system shall be linearized around its equilibriums. For the considered pendulum on a cart system, the equilibriums are obtained for any cart position x_{cart} when the pendulum arm is oriented vertically up or down.

(E2) *Linearize the system around the upper equilibrium $\boldsymbol{x}_{e,o} = [0, 0, x_{\text{e,cart}}, 0]^T$ and the lower equilibrium $\boldsymbol{x}_{e,u} = [\pi, 0, x_{\text{e,cart}}, 0]^T$ with $x_{\text{e,cart}} = u_e = 0$ for general parameter values.*

(E3) *Calculate the corresponding transfer function for each linearization. Then examine the transfer functions for input-output stability. Use the parameters $M = 0.5\,\text{kg}$, $m = 0.5\,\text{kg}$, $l = 1\,\text{m}$ and $g = 9.81\,\text{m/s}^2$.*

Now the system shall be stabilized around the upper equilibrium $\boldsymbol{x}_{e,o}$ with the help of a controller. The control loop structure from Figure 1 is used for this.

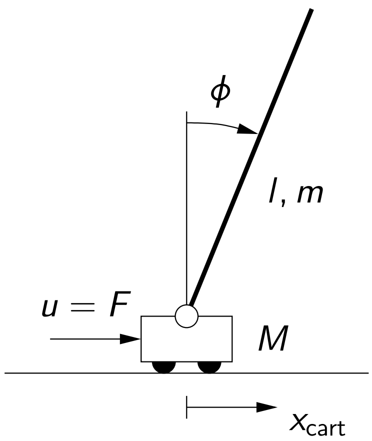

Figure 2: Cart with inverse pendulum.

(E4) *Set up the disturbance transfer function (for $w(t) = r(t) = 0$) as a function of a general controller $G_c(s)$.*

(E5) *Show that the system can be stabilized for $G_d(s) = 1$ with a PD controller. What conditions must hold for the controller parameters to stabilize the system?*

(E6) *Test the PD controller from task **(E5)** simulatively in Simulink by perturbing the **non-linear** original system in the upper equilibrium. To do this, implement the nonlinear system equations in Simulink as a `Level-2 MATLAB S-Function` block and build the control loop. Here, the parameters of the cart pendulum system should be passed to the `MATLAB S-Function` as parameter vector $\boldsymbol{p} = [m, M, g, l]^\mathsf{T}$. A typical perturbation of the pendulum on a cart system is that the phase angle changes abruptly due to a nudge. This type of perturbation can be realized in the simulation during initialization of the initial values by using an initial value of the pendulum angle that differs from the equilibrium.*
Can arbitrarily large initial disturbances of the pendulum angle be compensated? What is the reason for this?

Note: *Since the PD element is non causal and thus not realizable, a so-called realization term must be added. Thus, one obtains a PDT_1 element, which is also called lead element in control engineering.*

(E7) *So far, only the pendulum angle has been considered as the system output. What problem do you recognize when you take a closer look at the other system states for stabilization from task **(E6)**?*

To solve this problem, the x-position of the pendulum tip, described by the output equation

$$y = h(\boldsymbol{x}) = x_\text{cart} + l\sin(\phi) \tag{8}$$

is considered. This depends on both the cart position x_cart and the pendulum angle ϕ.

(E8) *For the position of the pendulum tip as the system output, calculate the associated transfer function for linearization around the upper equilibrium $\boldsymbol{x}_{e,o}$. Then examine the transfer functions for input-output stability.*

(E9) *Show that it is not possible to stabilize the system for $G_d(s) = 1$ with a classical PID controller.*

In order to stabilize the position of the pendulum tip nevertheless, a state feedback controller will now be investigated as an alternative to the frequency domain methods, using the modified control loop structure from Figure 3. The feedback vector $\boldsymbol{k}^\mathsf{T}$ can be chosen in various ways, for example by eigenvalue specification.

A different approach to the choice of $\boldsymbol{k}^\mathsf{T}$ is provided by (linear) optimal control, where the control performance is given by a quadratic cost functional of the form

$$J(\boldsymbol{u}, \boldsymbol{x}_0) = \frac{1}{2}\int_0^\infty \boldsymbol{x}(t)^\mathsf{T}\boldsymbol{Q}\,\boldsymbol{x}(t) + \boldsymbol{u}(t)^\mathsf{T}\boldsymbol{R}\,\boldsymbol{u}(t)\,\mathrm{d}t\,, \tag{9}$$

with the positive semi-definite diagonal matrix $\boldsymbol{Q}$ and the positive definite diagonal matrix $\boldsymbol{R}$. The minimization of (9) is done by the feedback law $u_e = \boldsymbol{k}^\mathsf{T}\boldsymbol{x}$. Due to the quadratic cost

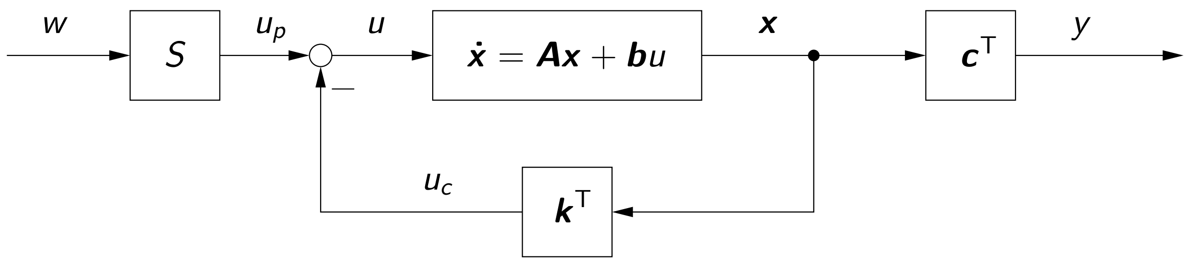

Figure 3: Control loop structure for state feedback controllers in the SISO case.

functional and the linear system dynamics, in the literature this problem is also referred to as the LQR problem (Linear Quadratic Regulator). The feedback gain $\boldsymbol{k}^{\mathsf{T}}$ can be calculated in MATLAB by the `lqr` command.

(E10) *Design an LQR controller for the upper equilibrium. To do this, select appropriate weighting matrices and calculate the resulting controller gain using the* `lqr` *command. Then test your controller in Simulink for an initial disturbance of the pendulum angle as well as the cart position. Select $w = 0$ and $S = 1$ for this, as only the stabilization of one operating point is considered.*
Can the position of the pendulum tip be stabilized successfully?