Part 1:

WTS: $\exp\left[-\frac{1}{2}\left(\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right]\right. \propto \exp\left(-\frac{1}{2}(\tau+n\phi)\left(\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right)^2\right)$

$$\exp\left[-\frac{1}{2}\left(\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right]\right.$$

$$=\exp\left[-\frac{1}{2}\left(\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2\right)\right]$$

$$=\exp\left[-\frac{1}{2}\left(\left(\phi\sum_{i=1}^{n}(x_i^2-2x_i\theta+\theta^2)\right)+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2\right)\right]$$

$$=\exp\left[-\frac{1}{2}\left(\left(\phi\left(\sum_{i=1}^{n}x_i^2-2\theta\sum_{i=1}^{n}x_i+n\theta^2\right)\right)+\tau\theta^2-2\tau\theta_0\theta+\tau\theta_0^2\right)\right]$$

$$=\exp\left[-\frac{1}{2}\left(\phi\sum_{i=1}^{n}x_i^2-2\left(\phi\sum_{i=1}^{n}x_i+\tau\theta_0\right)\theta+(\phi n+\tau)\theta^2+\tau\theta_0^2\right)\right]$$

$$=\exp\left[-\frac{1}{2}\left((\tau+n\phi)\theta^2-2\left(\phi\sum_{i=1}^{n}x_i+\tau\theta_0\right)\theta+\left(\tau\theta_0^2+\phi\sum_{i=1}^{n}x_i^2\right)\right)\right]$$

$$=\exp\left[-\frac{1}{2}\left((\tau+n\phi)\theta^2-2\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\theta+\left(\tau\theta_0^2+\phi\sum_{i=1}^{n}x_i^2\right)\right)\right]$$

$$=\exp\left[-\frac{1}{2}(\tau+n\phi)\left(\theta^2-2\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\theta+\frac{1}{\tau+n\phi}\left(\tau\theta_0^2+\phi\sum_{i=1}^{n}x_i^2\right)\right)\right]$$

Then we complete the square for θ

$$=\exp\left[-\frac{1}{2}(\tau+n\phi)\left(\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right)^2\right]$$

Since they are equal when $\alpha=1$, thus we finished the proof