Deep learning

3.5. Gradient descent

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We saw that training consists of finding the model parameters minimizing an empirical risk or loss, for instance the mean-squared error (MSE)

$$\mathscr{L}(w, b) = \frac{1}{N} \sum_n \left(f(x_n; w, b) - y_n\right)^2 .$$

Other losses are more fitting for classification, certain regression problems, or density estimation. We will come back to this.

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So far we minimized the loss either with an analytic solution for the MSE, or with *ad hoc* recipes for the empirical error rate (k-NN and perceptron).

There is generally no *ad hoc* method. The logistic regression for instance

$$P_w(Y = 1 \mid X = x) = \sigma(w \cdot x + b), \text{ with } \sigma(x) = \frac{1}{1 + e^{-x}}$$

leads to the loss

$$\mathscr{L}(w, b) = -\sum_n \log \sigma(y_n(w \cdot x_n + b))$$

which cannot be minimized analytically.

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which cannot be minimized analytically.

The general minimization method used in such a case is the **gradient descent**.

Given a functional

$$\begin{aligned} f : \mathbb{R}^D &\to \mathbb{R} \\ x &\mapsto f(x_1, \dots, x_D), \end{aligned}$$

its gradient is the mapping

$$\begin{aligned} \nabla f : \mathbb{R}^D &\to \mathbb{R}^D \\ x &\mapsto \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_D}(x) \right). \end{aligned}$$

To minimize a functional

$$\mathscr{L} : \mathbb{R}^D \to \mathbb{R}$$

the gradient descent uses local linear information to iteratively move toward a (local) minimum.

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For $w_0 \in \mathbb{R}^D$, consider an approximation of $\mathscr{L}$ around w_0

$$\tilde{\mathscr{L}}_{w_0}(w) = \mathscr{L}(w_0) + \nabla \mathscr{L}(w_0)^\top (w - w_0) + \frac{1}{2\eta} \|w - w_0\|^2.$$

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We have

$$\nabla \tilde{\mathscr{L}}_{w_0}(w) = \nabla \mathscr{L}(w_0) + \frac{1}{\eta}(w - w_0),$$

which leads to

$$\underset{w}{\operatorname{argmin}}\, \tilde{\mathscr{L}}_{w_0}(w) = w_0 - \eta \nabla \mathscr{L}(w_0).$$

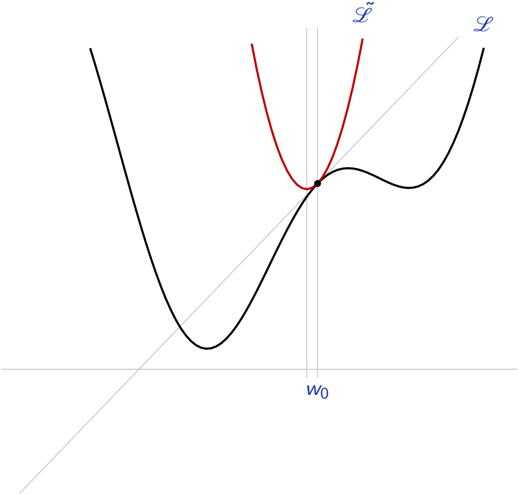
The resulting iterative rule, which goes to the minimum of the approximation at the current location, takes the form:

$$w_{t+1} = w_t - \eta \nabla \mathscr{L}(w_t),$$

which corresponds intuitively to "following the steepest descent".

This [most of the time] eventually ends up in a **local** minimum, and the choices of w_0 and η are important.

$\eta = 0.125$



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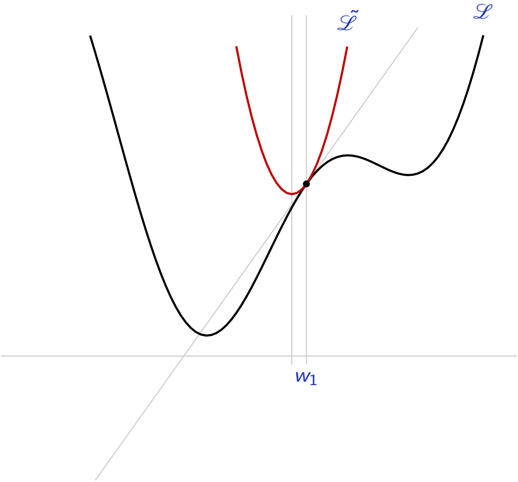

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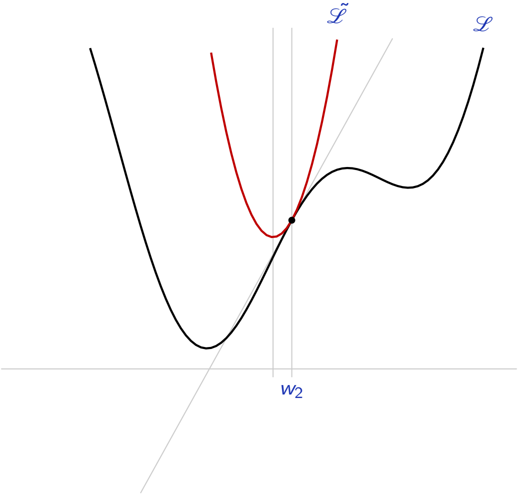

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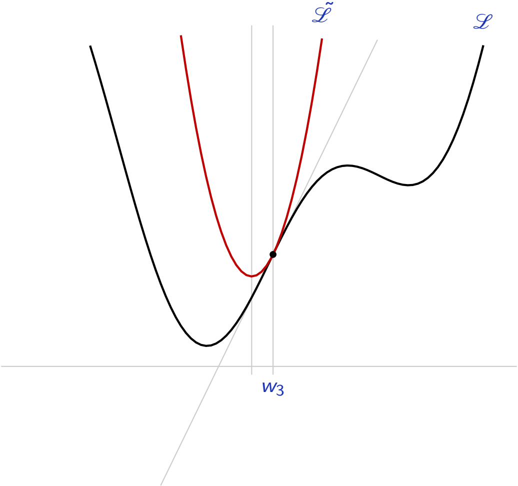

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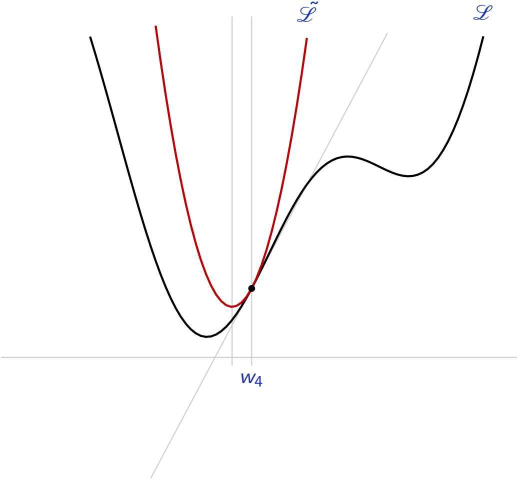

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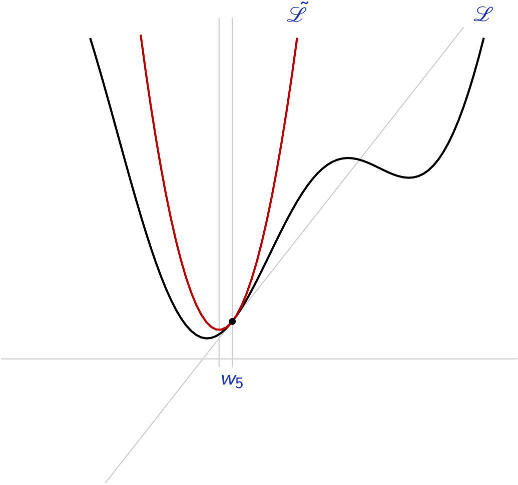

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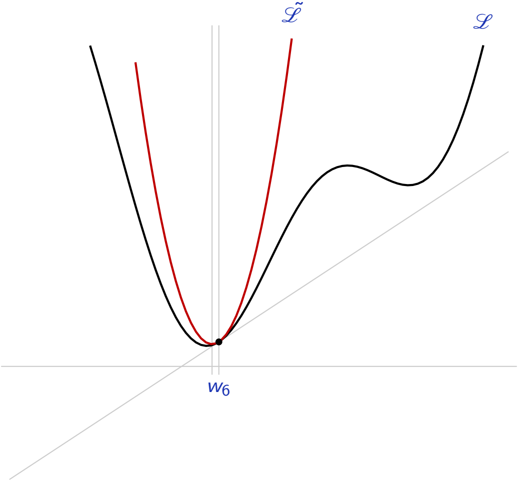

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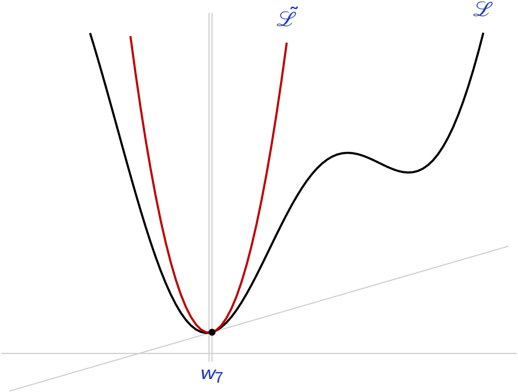

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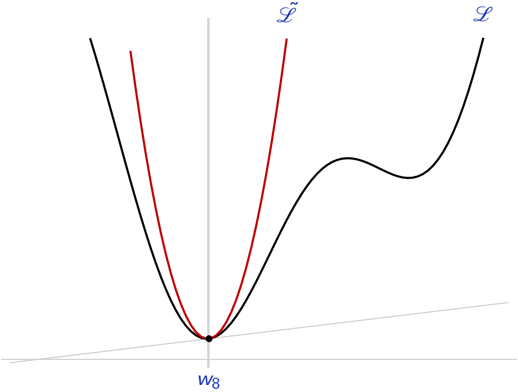

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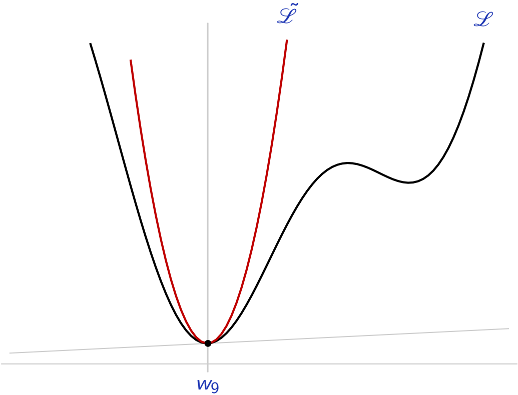

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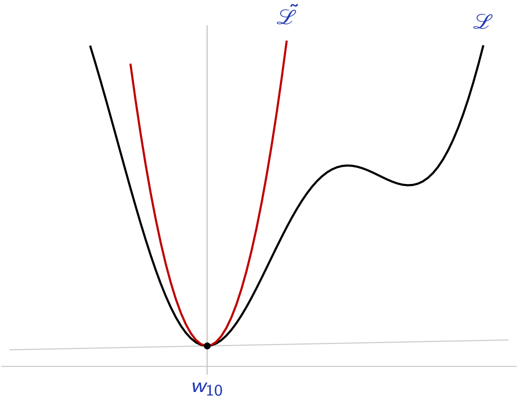

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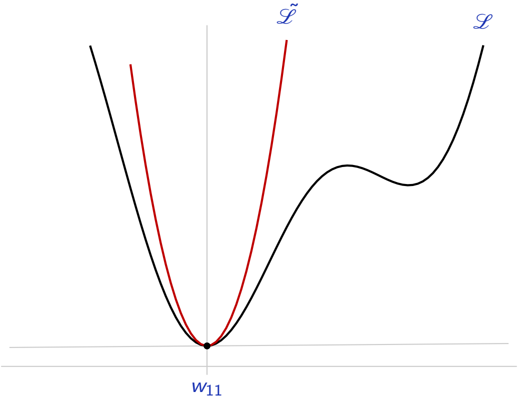

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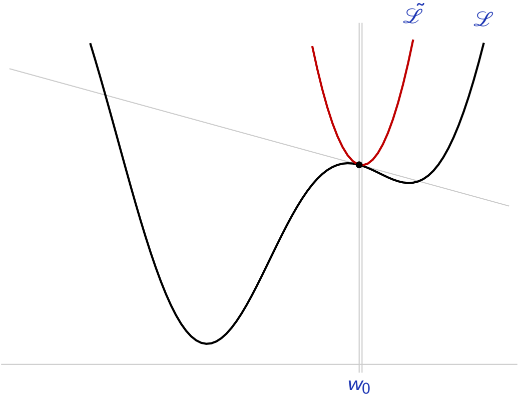

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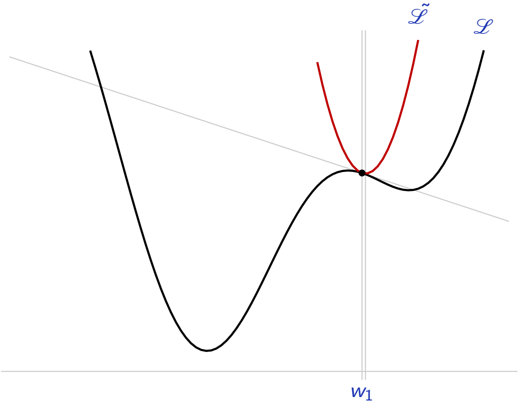

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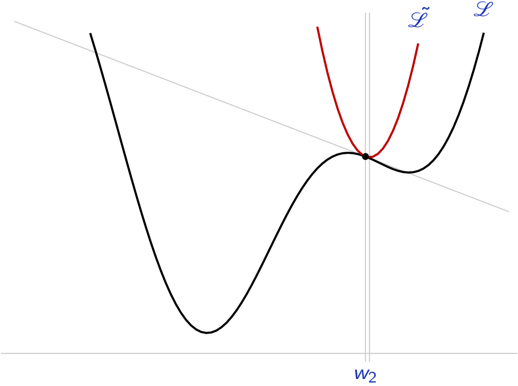

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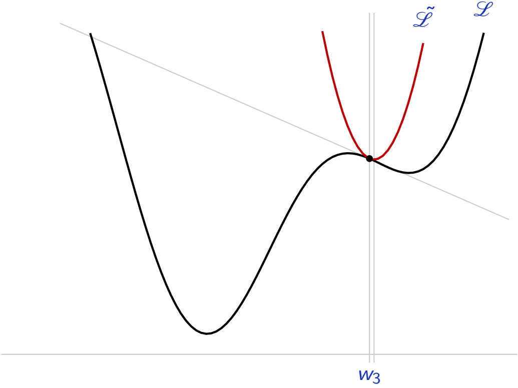

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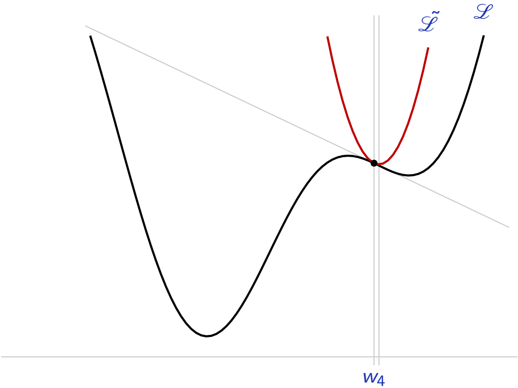

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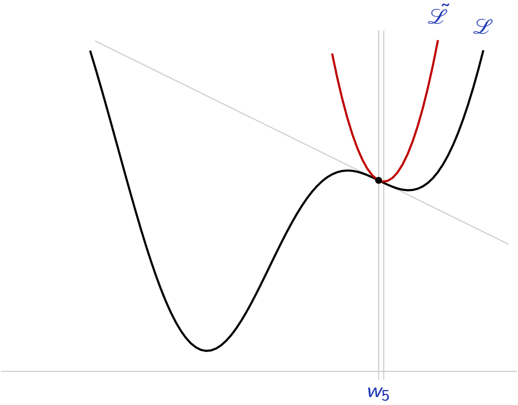

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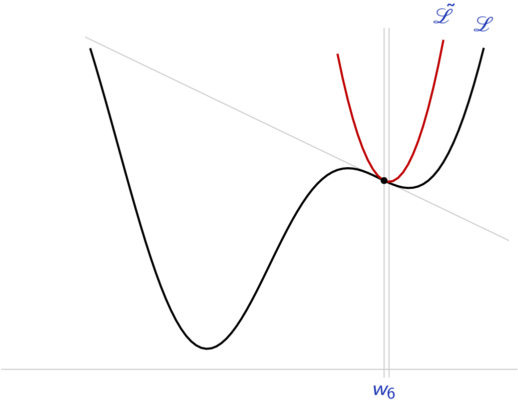

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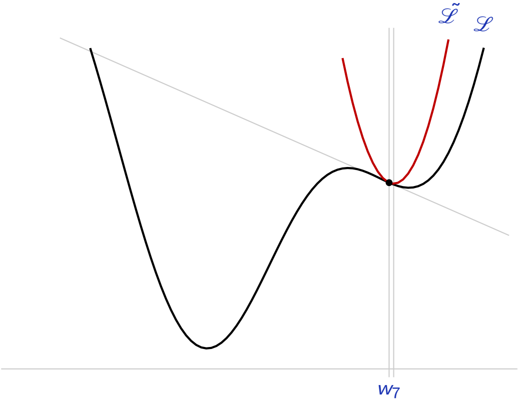

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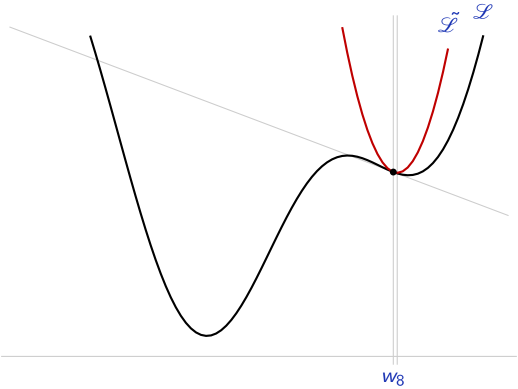

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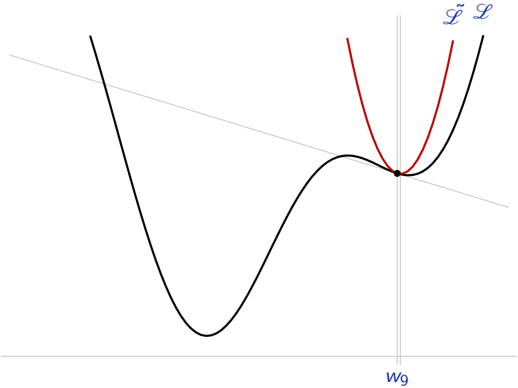

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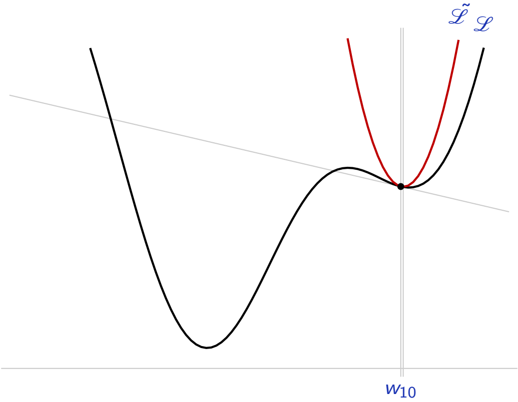

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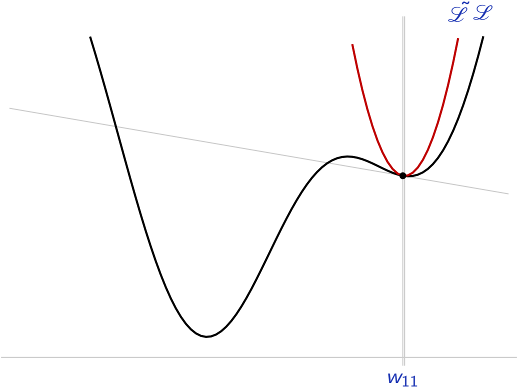

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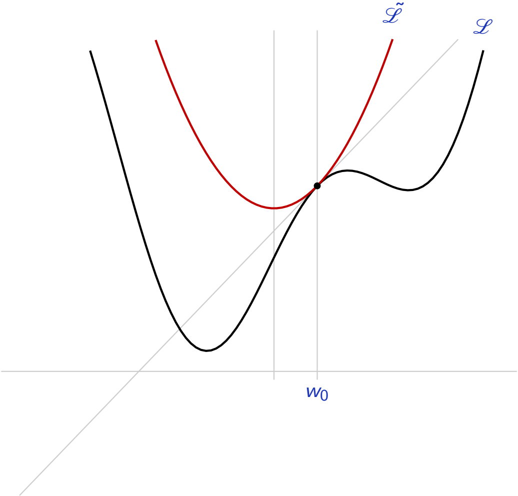

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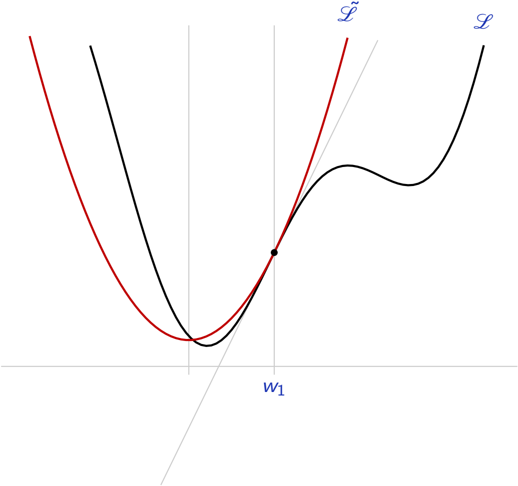

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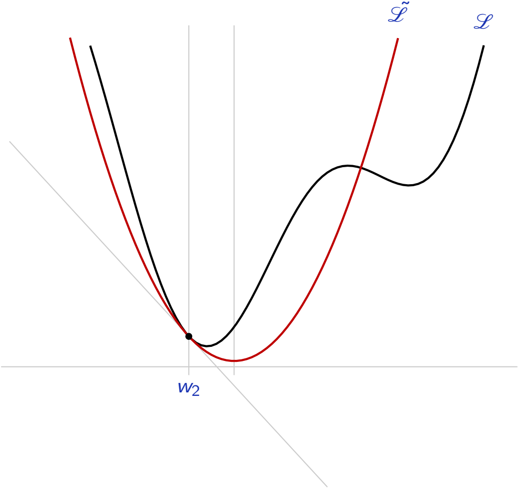

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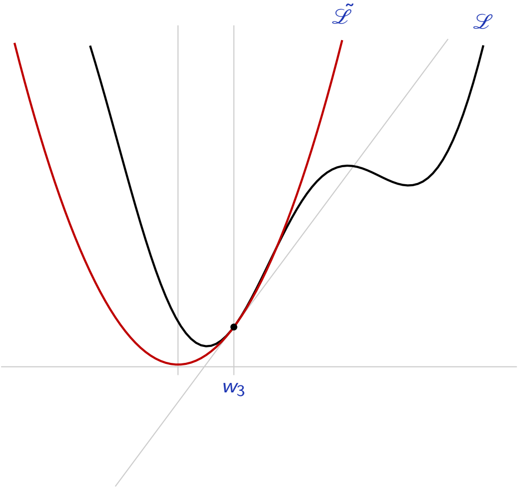

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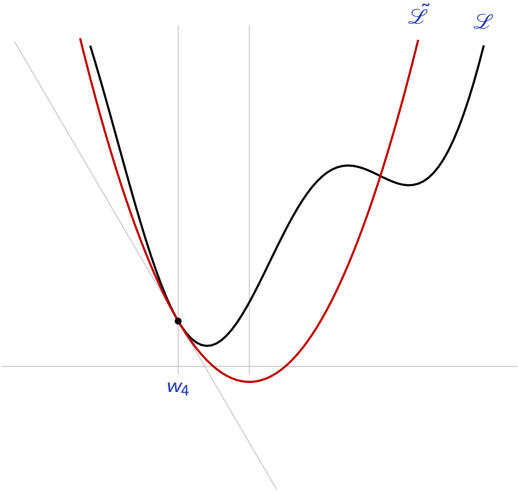

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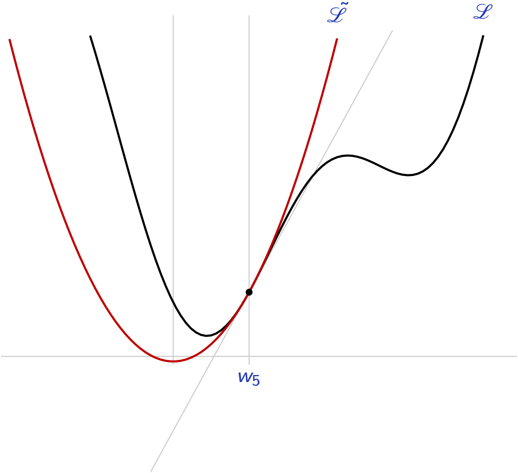

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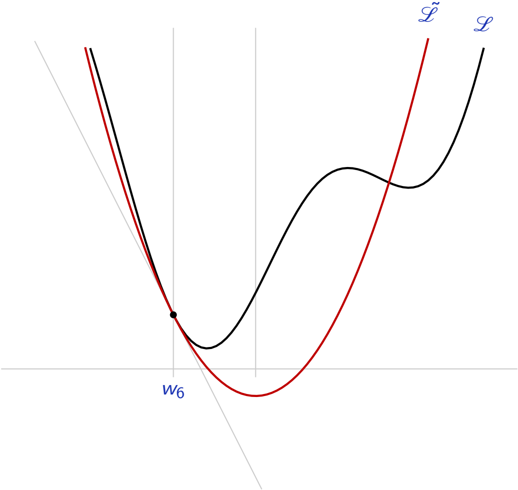

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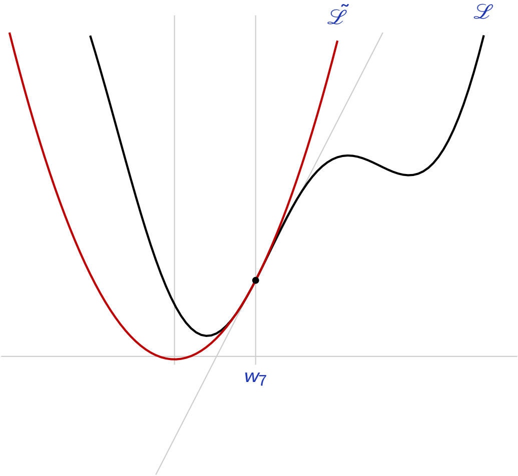

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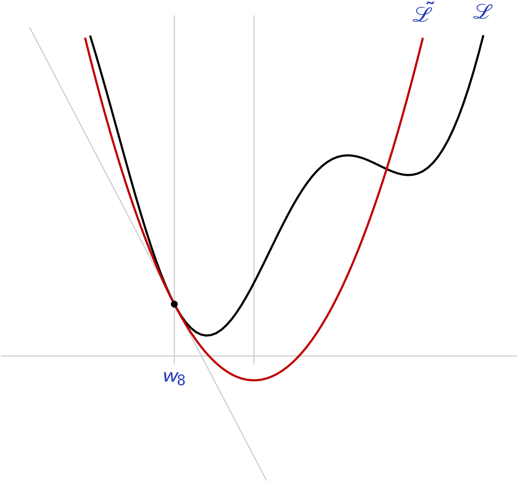

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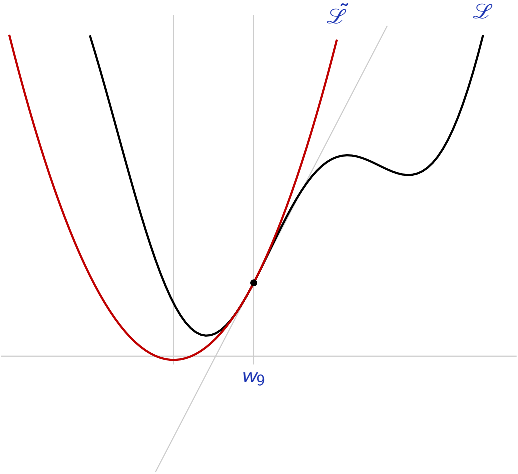

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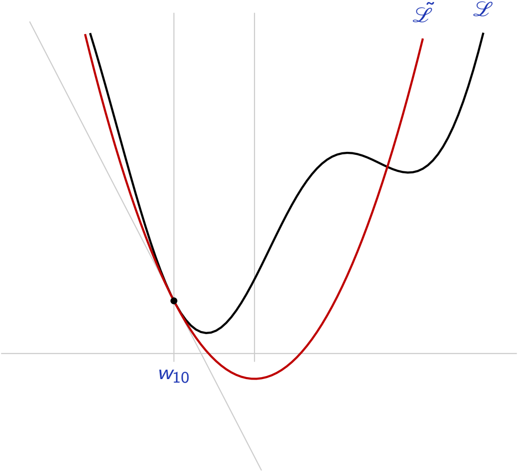

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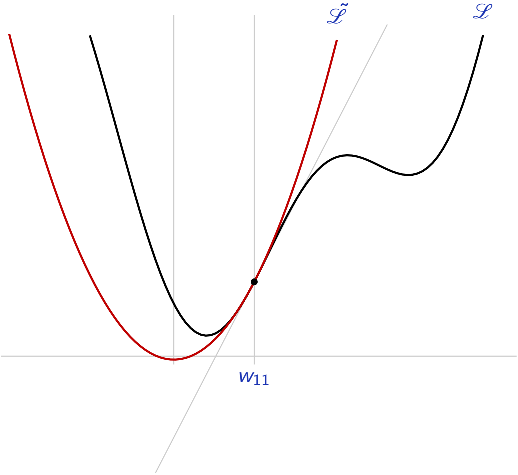
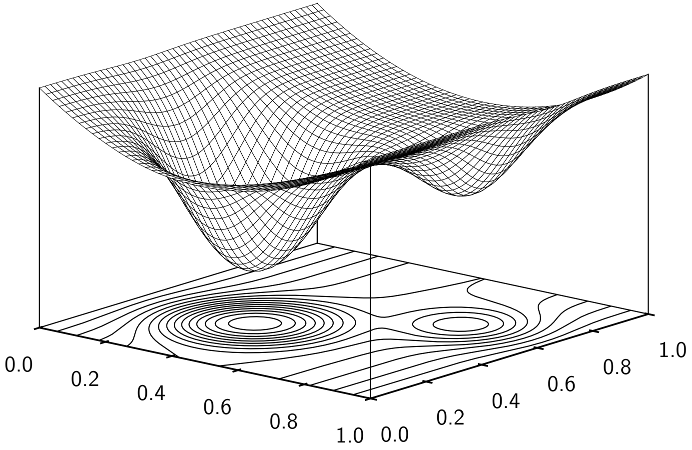
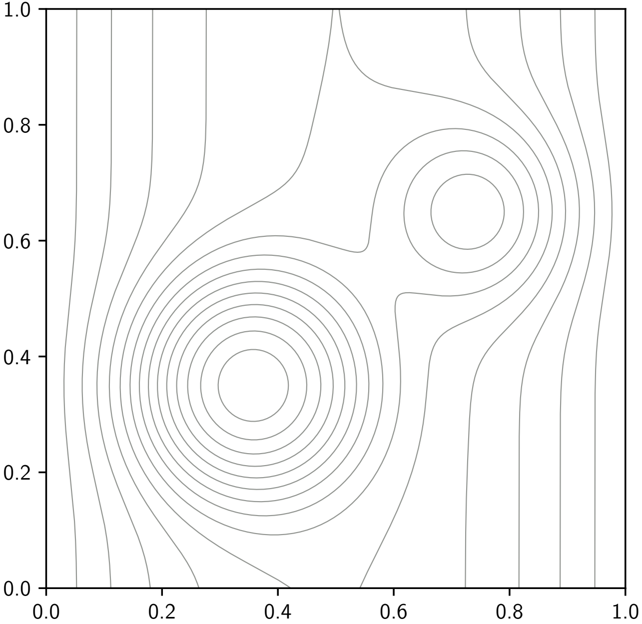
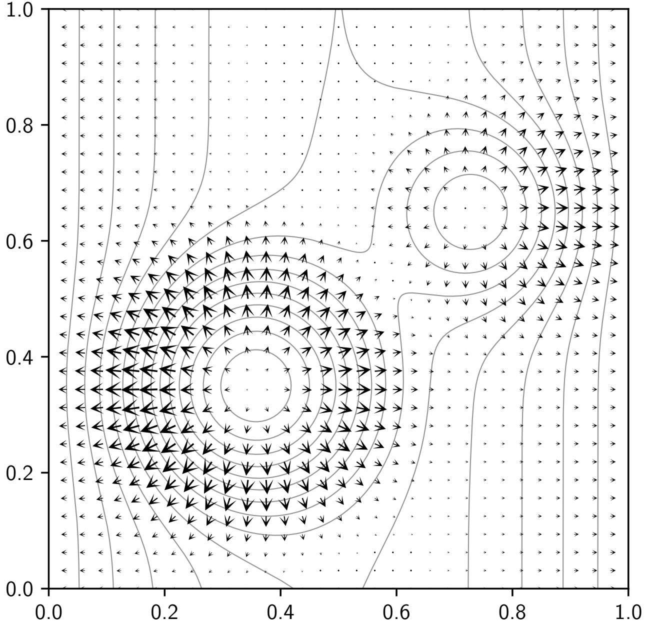
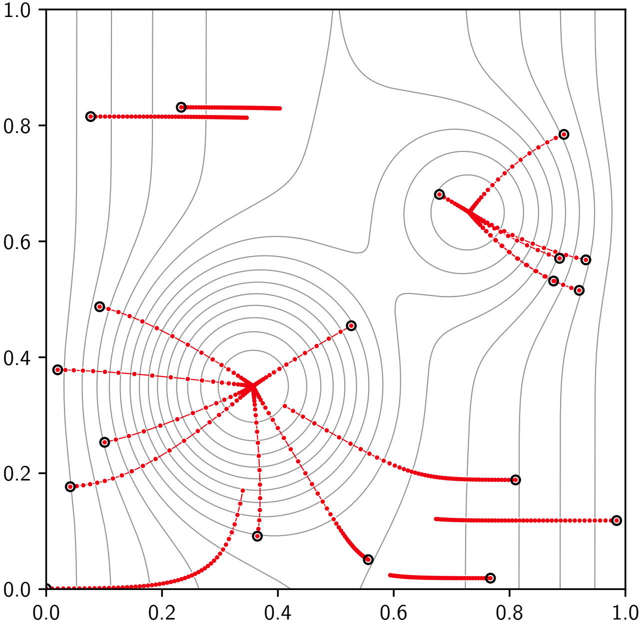



0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0



1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

We saw that the minimum of the logistic regression loss

$$\mathscr{L}(w, b) = -\sum_n \log \sigma(y_n(w \cdot x_n + b))$$

does not have an analytic form.

We can derive

$$\frac{\partial \mathscr{L}}{\partial b} = -\sum_n \underbrace{y_n\,\sigma(-y_n(w \cdot x_n + b))}_{u_n},$$

$$\forall d,\ \frac{\partial \mathscr{L}}{\partial w_d} = -\sum_n \underbrace{x_{n,d}\,y_n\,\sigma(-y_n(w \cdot x_n + b))}_{v_{n,d}},$$

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$$\forall d, \; \frac{\partial \mathscr{L}}{\partial w_d} = -\sum_n \underbrace{x_{n,d} \, y_n \, \sigma(-y_n(w \cdot x_n + b))}_{v_{n,d}},$$

which can be implemented as

```
def gradient(x, y, w, b):
    u = y * ( - y * (x @ w + b)).sigmoid()
    v = x * u.view(-1, 1) # Broadcasting
    return - v.sum(0), - u.sum(0)
```

We can derive

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$$\forall d,\ \frac{\partial \mathscr{L}}{\partial w_d} = -\sum_n \underbrace{x_{n,d}\, y_n\,\sigma(-y_n(w \cdot x_n + b))}_{v_{n,d}},$$

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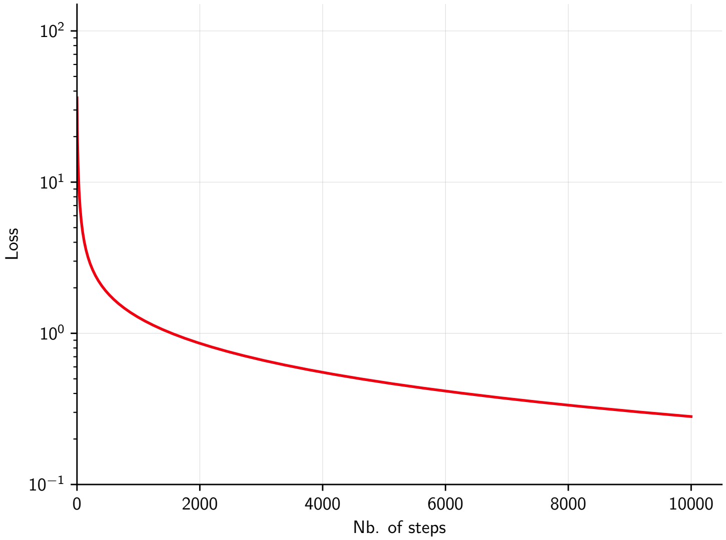
```

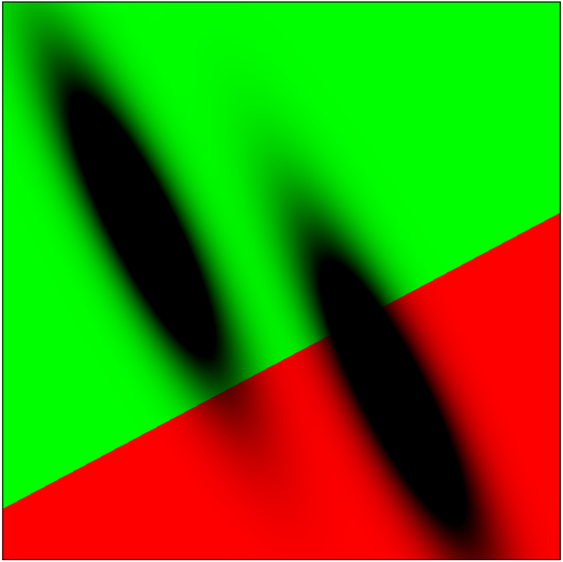
and the gradient descent as

```
w, b = torch.randn(x.size(1)), 0
eta = 1e-1

for k in range(nb_iterations):
    print(k, loss(x, y, w, b))
    dw, db = gradient(x, y, w, b)
    w -= eta * dw
    b -= eta * db
```

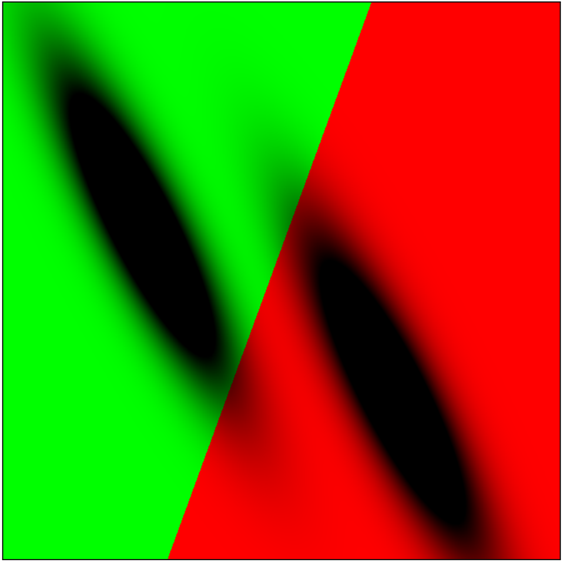

10^2
10^1
10^0
10^{-1}
Loss
0
2000
4000
6000
8000
10000
Nb. of steps

With 100 training points and $\eta = 10^{-1}$.



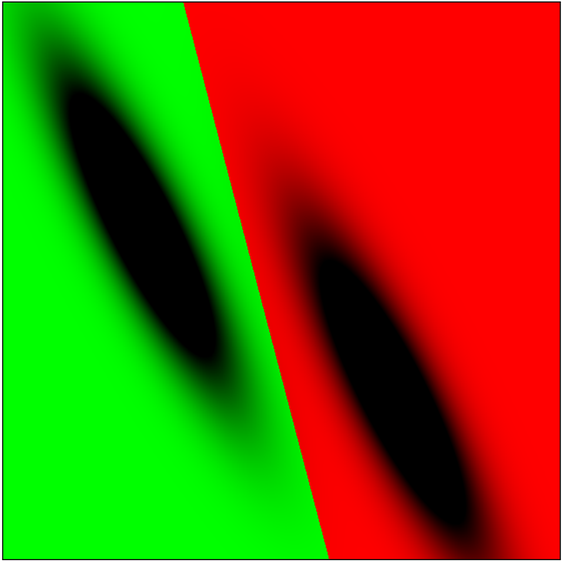
$n = 0$

With 100 training points and $\eta = 10^{-1}$.



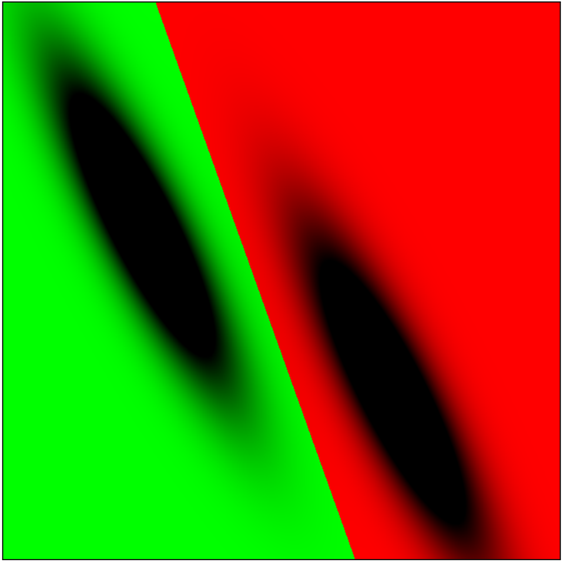
$n = 10$

With 100 training points and $\eta = 10^{-1}$.



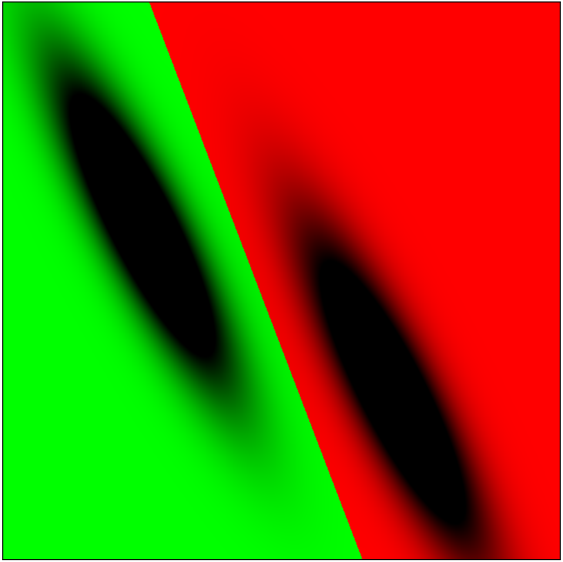
$n = 10^2$

With 100 training points and $\eta = 10^{-1}$.

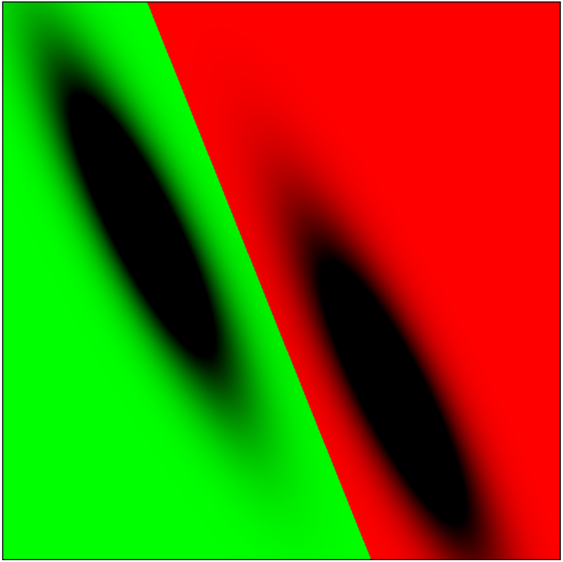


$n = 10^3$

With 100 training points and $\eta = 10^{-1}$.



$n = 10^4$



LDA

The end