Notes for Math 3137 - Real and Complex Analysis I

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Contents

Chapter 1

Real Numbers

1 Rational Numbers

Definition 1.1. The **natural numbers**, $\mathbb{N}$, is the set of all positive whole numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

The **integers**, $\mathbb{Z}$, is the set consisting of 0, all natural numbers, and all negative natural numbers:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

The **rational numbers**, $\mathbb{Q}$, is the set of all possible fractions of integers,

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

Any number which is not rational is **irrational**.

The main goal of the course is to explore the properties of what lies beyond $\mathbb{Q}$, that is: What are the real numbers, $\mathbb{R}$?

At a first glance, one thinks about $\mathbb{Q}$ as the set of numbers with possibly a **finite or repeating decimal expansion**, e.g.

$$\frac{1}{3} = 0.3333\dots, \quad \frac{7}{4} = 1.75, \quad \frac{5}{7} = 0.\overline{714285}\dots, \quad \frac{7}{6} = 1.16666\dots$$

Naturally, the **irrational numbers** should be those with infinite non-repeating decimal expansions. Are such numbers definable? Do they exist in some sense?

Example 1.1. The positive solution to $x^2 = 2$ is called $x = \sqrt{2}$. This number has a non-repeating decimal expansion,

$$\sqrt{2} = 1.41421356237\dots$$

Is this number

(a) Rational or Irrational?

(b) Definable?

The answer to (a) is Irrational. The answer to (b) comes from our construction of the **real numbers.** The proof of (a) will be our first "proof by contradiction".

Proposition 1.1. *$\sqrt{2}$ is irrational.*

Proof. Let $x = \sqrt{2}$. If x was rational, then we could write $x = \dfrac{p}{q}$ **in reduced form** i.e.

- $p, q \in \mathbb{Z}$
- $q \neq 0$
- p and q share no common factors

Then x would solve
$$2 = x^2 = \frac{p^2}{q^2},$$
Since $q \neq 0$ we can write
$$2q^2 = p^2.$$
This implies that the integer p^2 is even. Now p can only be even or odd, but if p was odd then p^2 would also be odd being the product of two odd numbers. Thus p must also be even, which means for some $n \in \mathbb{Z}$,
$$p = 2n.$$
Thus
$$2q^2 = p^2 = 4n^2 \implies q^2 = 2n^2.$$
By the same argument, q must also be even. This implies that q and p share a factor, 2, in common which is a contradiction of the assumption that p and q share no common factors. Therefore, the assumption that $\sqrt{2}$ is rational is false, proving the claim. $\square$

To go deeper into the mysteries surrounding numbers with non-repeating decimal expansions, we need some preliminaries.

Definition 1.2. If A is a set of **elements** (any type of object, not just numbers), we denote an element, x of A by saying
$$x \in A \text{ or } A \ni x$$
If x is not in A, then $x \notin A$. If B is a set whose elements are all elements of A, we say A is a **subset** of B, written
$$B \subseteq A \text{ or } A \supseteq B$$
If an element of A is not in B, then B is a **proper subset** of A. Otherwise $A = B$ and are improper subsets of each other. A common method of proving $A = B$ is to show that they are subsets of each other.

Example 1.2. If $A = \{\star, \triangle, \square, \circ\}$ and $B = \{\triangle, \circ\}$ then $\star \in A$, but $\star \notin B$. Also $B \subseteq A$ but $A \not\subseteq B$. If $C = \{\square, \circ, \star, \triangle\}$ then $C = A$.

2 Least Upper Bounds

The way we will define the **real numbers** is by constructing "least upper bounds" of subsets of $\mathbb{Q}$. Consider the infinite sequence consisting of truncated decimal expansions of $\sqrt{2}$, starting at the first decimal place and increasing:

$$1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ 1.414213,\ 1.4142135,\ 1.41421356,\ \dots \tag{2.1}$$

We will later formalize this, but the above sequence "tends to" $\sqrt{2}$ as a limit from calculus. Further thought makes clear that $\sqrt{2}$ is the **least upper bound** of the sequence. This is somewhat perplexing since this sequence consists only of rational numbers yet its "limit" is irrational. One might say this is the defining property of $\mathbb{R}$.

Before we define such things, we need to understand **ordering**

Definition 1.3. A set S is (totally) **ordered** if there is a binary relation $<$ on it (called an (total) **order**) with the following properties:

(i) If $x, y \in S$ then only one of the following can be true:

$$x < y, \quad x = y, \quad y < x.$$

(ii) If $x, y, z \in S$, $x < y$, and $y < z$ then $x < z$.

We say $x < y$ to mean "x is less than y" and $x \leq y$ for either $x < y$ or $x = y$. That is, $x \leq y$ is the logical opposite of $y < x$. Of course, these symbols work backwards, $x < y$ is the same as $y > x$.

Example 1.3. Define the order, $<$, on $\mathbb{Q}$ to mean $x < y$ if and only if $y - x$ is a positive, rational number. Then $<$ is the usual order on $\mathbb{Q}$.

In $\mathbb{R}^2$ define the order $<$ to mean $(x_1, x_2) < (y_1, y_2)$ if and only if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. This is the so-called **dictionary order** on $\mathbb{R}^2$.

Now we will define bounds.

Definition 1.4. Let S be an ordered set and $E \subseteq S$. We say E is **bounded above** if there is some $b \in S$ so that

$$x \leq b,\ \forall x \in E.$$

In this case, b is an **upper bound** of E. Similarly, E is **bounded below** if there is an $a \in S$ so that

$$a \leq x,\ \forall x \in E,$$

whence a is a **lower bound** of E.

Now we can define sets which have a "least" upper bound.

Definition 1.5. Once again suppose S is an ordered set and $E \subseteq S$, and additionally that E is bounded above. If $\beta \in S$ so that

(i) β is an upper bound of E, and

(ii) If $y < \beta$ then y is not an upper bound of E,

then β is called the **least upper bound** of E. Uniqueness of b follows from (ii). The least upper bound is also called the **supremum**, written

$$\beta = \sup E.$$

Similarly, if E is bounded below and $a \in S$ is a lower bound of E where $\alpha < y$ implies that y is not a lower bound of E, then α is the **greatest lower bound** of E. We call α the **infimum** of E,

$$\alpha = \inf E.$$

Example 1.4. Let A be the set of all truncated decimal expansions of $\sqrt{2}$ as described in (2.1). Then $1.4 = \inf A$ and

$$\sqrt{2} = \sup A.$$

Note that $\sqrt{2} \notin \mathbb{Q}$, but it is still the supremum of A by definition.

Let B be the sequence of fractions $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ that is,

$$B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

We have that $0 = \inf B$ and $1 = \sup B$.

Let C be the closed interval $(-\infty, -1]$. We have that $\sup C = -1$, but $\inf C$ is not defined.

Please note that infs and sups need not be elements of their arguments.

The following property is what essentially defines the real numbers, from the rationals.

Definition 1.6. An ordered set S has the **least upper-bound property** if each non-empty subset E which is bounded above has a supremum in S.

Example 1.5. It is clear that $\mathbb{Q}$ does *not* have the least upper-bound property since $\sqrt{2}$ is the supremum of a set of rationals, yet $\sqrt{2} \notin \mathbb{Q}$. Discrete sets like $\mathbb{N}$ and $\mathbb{Z}$ trivially have the LUB property, as do finite subsets of $\mathbb{Q}$.

We will now show that we do not need a separate definitions for sets which contain their infimums - the least upper-bound property is equivalent to the "greatest lower-bound" property.

Theorem 1.1. *If a set S has the least upper-bound property, then every non-empty subset E which is bounded below has an infimum in S.*

Proof. Suppose $B \subseteq S$ and B is bounded below. Let L be the set of all $y \in S$ which are lower bounds of B,

$$L = \{y \in S : y \leq x, \forall x \in B\} \neq \varnothing.$$

By definition, each $x \in B$ is an upper bound of L, hence L is bounded above. Since S has the least upper-bound property and $L \subseteq S$, we can set

$$\alpha := \sup L \in S.$$

We will show that $\alpha = \inf B$ as follows:

(i) Let $x \in B$. If $x < \alpha$, then x is not an upper bound of L. This is impossible since every $x \in B$ is an upper bound of L, by definition. Thus $\alpha \leq x$ for all $x \in B$. This shows that α is a lower bound of B.

(ii) Now let $\alpha < y$. Since α is larger than every element of L, $y \notin L$ i.e. y is not a lower bound of B.

We have shown that α satisfies the definition of a greatest lower-bound of B, so we are done. $\square$

3 Fields

In this section, we establish the familiar rules of arithmetic. Afterwards, we will define the subject of the course: the real numbers, $\mathbb{R}$.

Definition 1.7. A **field** is a non-empty set, $\mathbb{F}$, endowed with the binary operations of addition, $+$, and multiplication, $\cdot$ which obey the following properties: Let x, y, z be any elements of $\mathbb{F}$, then

(i) (Closure under Addition) $(x + y) \in \mathbb{F}$

(ii) (Associativity of Addition) $x + (y + z) = (x + y) + z = x + y + z$.

(iii) (Commutativity of Addition) $x + y = y + x$.

(iv) (Additive Identity) There is an additive identity element called $0 \in \mathbb{F}$ so that $x + 0 = 0$.

(v) (Additive Inverse) To each $x \in \mathbb{F}$ there is an additive inverse element called $-x \in \mathbb{F}$ so that $x + (-x) = 0$.

(vi) (Closure under Multiplication) $x \cdot y = xy \in \mathbb{F}$

(vii) (Associativity of Multiplication) $x(yz) = (xy)z = xyz$.

(viii) (Commutativity of Multiplication) $xy = yx$.

(ix) (Multiplicative Identity) There is a multiplicative identity element called $1 \in \mathbb{F}$ where $1 \neq 0$, so that $1x = x$.

(x) (Multiplicative Inverse) If $x \neq 0$ then there is a multiplicative inverse element $x^{-1} \in \mathbb{F}$ so that $xx^{-1} = 1$. It is common to write $1/x$ in place of x^{-1}.

(xi) (Distributivity) $x(y + z) = xy + xz$.

These properties are called the **field axioms**.

Naturally, the rationals form a field.

Proposition 1.2. *$\mathbb{Q}$ is a field under the usual addition and multiplication.*

It will turn out that $\mathbb{R}$ (the reals) and $\mathbb{C}$ (the complex numbers) are fields too, so we will state arithmetic principles common to all fields first, then specific properties afterwards.

Example 1.6. The set $\{0, 1\}$ is a field under the usual addition and multiplication.

The set $\{0, 1, 2, 3, 4\}$ is also a field under "modular" addition and multiplication, w.r.t. the number 5. That is, if we consider two integers equal if and only if they have the same remainder after dividing by 5. For example, $0, 5, 10, -5, -10, \ldots$ are the "same" as are $2, 17, -3, \ldots$. So operations like

$$1 + 6 = 7 = 2 \text{ or } 2(4) = 8 = 3$$

make sense. It is clear that we have 0 and 1 as the usual additive and multiplicative identities, but what is different are the multiplicative inverses: 1 has 1, 2 has 3, and 4 has itself as one can verify.

The following proposition and subsequent proof serves as a reminder of how "low-level" we are getting in this course. We are truly starting from base principles and building the theory up from there.

Proposition 1.3. *In any field, the following cancellations hold:*

(a) If $x + y = x + z$ then $y = z$.

(b) If $xy = xz$ and $x \neq 0$ then $y = z$.

Proof. We will proceed by invoking exactly one field axiom or assumption in each equality. This level of detail is usually not necessary, but serves to illustrate how low-level we are working. For (a), we have

$$y = 0 + y = (-x + x) + y = -x + (x + y) = -x + (x + z) = (-x + x) + z = 0 + z = z.$$

For (b),

$$y = 1y = \left(\frac{x}{x}\right) y = \left(\frac{1}{x}\right) xy = \left(\frac{1}{x}\right) xz = \left(\frac{x}{x}\right) z = 1z = z.$$

□

Corollary 1.1. *The following can be proved directly via cancellation:*

(i) *If $x + y = x$ then $y = 0$ (The additive identity is unique)*

(ii) *If $x + y = 0$ then $y = -x$ (The additive inverse is unique)*

(iii) *$-(-x) = x$. (Reflexivity of the additive inverse)*

(iv) *If $x \neq 0$ and $xy = x$ then $y = 1$ (The multiplicative identity is unique)*

(v) *If $x \neq 0$ and $xy = 1$ then $y = 1/x$ (The multiplicative inverse is unique)*

(vi) *If $x \neq 0$ then $1/(1/x) = x$. (Reflexivity of the multiplicative inverse)*

Proof. Homework. □

Proposition 1.4. *In any field, the following properties hold.*

(a) *$0x = 0$*

(b) *$(-1)x = -x$*

(c) *If $xy = 0$ then either $x = 0$ or $y = 0$.*

Proof. For (a), we have $0x+0x = (0+0)x = 0x$. By cancellation (prop 1.3), we have $0x = 0$. For (b), we have

$$x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$$

The last equality follows from (a). By uniqueness of the additive inverse, (ii) from the previous Corollary, we have $(-1)x = -x$. To prove (c), suppose $xy = 0$ but $x \neq 0$ and $y \neq 0$. Then $1/x$ and $1/y$ are defined so that

$$1 = \left(\frac{x}{x}\right)\left(\frac{y}{y}\right) = \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)xy = \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)0.$$

□

Now that we have derived the basics of field arithmetic, we will include what happens when we induce a notion of *positivity* and *negativity* to a field, i.e. an **order** (see definition 1.3).

Definition 1.8. An **ordered field** is a field, $\mathbb{F}$, which is also an ordered set so that

(i) $x + y < x + z$ for all $x, y, z \in \mathbb{F}$ with $y < z$, and

(ii) $xy > 0$ for all $x, y \in \mathbb{F}$ with $x > 0$ and $y > 0$.

If $x > 0$ we call it **positive** and if $x < 0$ we call it **negative**. These two properties ensure that cancellation holds as it does with equality and that multiplication of positive numbers yields a positive.

We now prove the convenient arithmetic properties of inequalities, as we did with equalities earlier.

Proposition 1.5. *In any ordered field, the following properties hold.*

(a) If $x > 0$ then $-x < 0$.

(b) If $x > 0$ and $y < z$ then $xy < xz$.

(c) If $x < 0$ and $y < z$ then $xy > xz$.

(d) If $x \neq 0$ then $x^2 > 0$.

(e) If $0 < x < y$ then $0 < 1/y < 1/x$.

Proof. For (a), if $x > 0$ then $-x = 0 + -x < x + -x = 0$ by property (i) in the definition. Transitivity of $<$ completes the proof of (a).

For (b), by definition of orders, $y < z$ is the same as saying $0 < z - y$. From property (ii) in the definition of an ordered field and the fact $x > 0$, we have $0 < x(z - y) = xz - xy$, which implies $xy < xz$.

For (c), note that $y < z$ implies $0 < z - y$ and from (a), $x < 0$ means $-x > 0$. So

$$0 < (-x)(z - y) = xy - xz \implies xz < xy.$$

For (d), if $x > 0$ then $(x)(x) = x^2 > 0$ by property (ii) again. If $x < 0$, then $-x > 0$ and

$$x^2 = (-x)(-x) > 0.$$

Thus $x^2 > 0$ for any $x \neq 0$.

For (e), if $0 < x < y$, then $1/x$ and $1/y$ are both positive, hence

$$\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) > 0.$$

From (b) we can deduce then that

$$0 < x\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y \iff 0 < \frac{1}{y} < \frac{1}{x}.$$

$\square$

4 The Real Numbers

Finally, we can construct the real numbers.

Theorem 1.2. *There exists an ordered field, $\mathbb{R}$, with the least upper-bound property that contains $\mathbb{Q}$ and inherits its order and field operations. We call $\mathbb{R}$ the* **real numbers***.*

Proof. We omit the proof, but you can view at least one version of it in our textbook. □

We have now defined the real numbers, $\mathbb{R}$. Perhaps this definition is underwhelming, as we do not include the proof that they exist. However, the idea of how they are constructed out of $\mathbb{Q}$ is more intuitive than the rigorous proof would lead a first-time student to believe. The real numbers essentially come out of the **completion** of $\mathbb{Q}$ in the least upper-bound sense, that is every real number is in **one-to-one** correspondence with the supremum of a bounded-above set of rational numbers. We already saw one case with the decimal approximations, that

$$\sqrt{2} = \sup\{1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, 1.41421356, \dots\}$$

Another, equivalent, construction accomplishes this task via **Cauchy Sequences**. The real numbers are the limits of sequences of rational numbers which are "arbitrarily close" to each other in a sense we will discuss later.

Either way, the real numbers serve to "fill the gaps" in $\mathbb{Q}$. The following theorem gives us two crucial properties of the reals, both of which follow from only the least upper-bound property and arithmetic.

Theorem 1.3. *Let $x, y \in \mathbb{R}$ be any real numbers.*

(a) (Archimedean Property) If $x > 0$ then there is an $n \in \mathbb{N}$ so that

$$nx > y.$$

(b) (Density of $\mathbb{Q}$ in $\mathbb{R}$) If $x < y$ then there is a $q \in \mathbb{Q}$ so that

$$x < q < y.$$

Proof. For (a), let $A = \{x, 2x, 3x, \dots\}$ be the set of all positive multiples of x. If the conclusion of (a) is false, then $y \geq nx$ for all $n \in \mathbb{N}$, i.e. y is an upper bound for A. Thus A has a least upper-bound in $\mathbb{R}$, call it $\alpha = \sup A$. Since $x > 0$, we have

$$\alpha - x < \alpha.$$

By the definition of supremum, $\alpha - x$ is not an upper bound of A, which means for some $m \in \mathbb{N}$,

$$\alpha - x < mx \implies \alpha < (m+1)x.$$

This implication follows from the arithmetic of equalities of inequalities we proved previously. The inequality on the right says that α, the supremum of A is bounded above by an element of A, which is a contradiction.

For (b), we have $x < y$, thus $0 < y - x$. From (a), we can find an $n \in \mathbb{N}$ so that

$$1 < n(y-x) \implies 1 + nx < ny. \tag{4.1}$$

We claim there is an integer m so that $nx < m \leq 1 + nx$, this integer is called the **floor** of $nx + 1$. To see this, (a) let E be the set of integers m_1 so that $nx < m_1$. Since E is a non-empty set of integers which is bounded below, it contains a least element, call it m. Because of this, $m - 1$ is not a member of this set. Combining these two facts, we have shown

$$m - 1 \leq nx < m \implies nx < m \leq 1 + nx.$$

Combining this with (4.1), we have

$$nx < m \leq 1 + nx < ny,$$

dividing we get

$$x < \frac{m}{n} < y,$$

which completes the proof. □

4.1 Epsilon Arguments

One of the central tools of real analysis is the **epsilon arugment**, which is an argument carried out by showing two numbers can be "arbitrarily close", and are thus equal. First we should show that this line of reasoning makes sense, we state it as a lemma.

Definition 1.9. The **absolute value** of a real number, x, called $|x|$, is the distance from 0 to x:

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}.$$

The distance between numbers x and y is $|x - y| = |y - x|$.

Lemma 1.1. *Let $x, y \in \mathbb{R}^+$. If for every $\epsilon > 0$, we have*

$$0 \leq |x - y| < \epsilon$$

then $x = y$.

Proof. Suppose $x \neq y$, then, without loss of generality, $0 < y - x$. Let $\epsilon = \frac{y-x}{2}$, then by hypothesis (and the definition of $|\cdot|$),

$$|x - y| = y - x < \frac{y - x}{2},$$

a contradition. □

We can equivalently characterize the inf and the sup by ϵ's.

Proposition 1.6. *If $A \subseteq \mathbb{R}$ is non-empty and bounded above, then $\alpha = \sup A$ if and only if α is an upper bound of A and for each $\epsilon > 0$, there is an $x \in A$ so that*

$$\alpha - \epsilon < x$$

Proof. Saying $\alpha = \sup A$ is equivalent to saying that it is an upper bound of A and each $y < \alpha$ is not an upper bound of A, i.e. for some $x \in A$,

$$y < x.$$

The density implies that every $\epsilon > 0$ uniquely represents one such y by the relation $y = \alpha - \epsilon$. Thus $\alpha = \sup A$ if and only if it is an upper bound of A and for each $\epsilon > 0$, there is some x so that

$$\alpha - \epsilon < x.$$

$\square$

One convenient result is the existence of n-th roots. Henceforth, we will use the notation $\mathbb{R}^+$ to denote the positive reals.

Theorem 1.4. *If $n \in \mathbb{N}$ and $x > 0$, there is a unique $y \in \mathbb{R}^+$ so that $y^n = x$. We call this the (positive) n-th root of x, $y = \sqrt[n]{x} = x^{1/n}$.*

Proof. Define the following sets:

$$A = \{t > 0 : t^n < x\} \text{ and } B = \{t^n : t \in A\}.$$

We will compute a supremum for A and show that it is the number we seek. A is non-empty since we can set, for example,

$$t = \frac{x}{1+x} < 1,$$

so that $t^n < t < x$. A is also bounded above since if $t > 1 + x$, then

$$x < t < t^n \implies t \notin A.$$

Since everything larger than $1 + x$ is outside of A, $1 + x$ must be an upper bound for A. We have shown A and thus B have supremums in $\mathbb{R}$. Let

$$y = \sup A = \sup\{t > 0 : t^n < x\}.$$

Note that $t < y$ implies $t^n \le y^n$, so y^n is an upper bound for B. Moreover,

$$0 \le y^n - t^n = (y - t)(y^{n-1} + y^{n-2}t + \cdots + yt^{n-2} + t^{n-1}) < n(y - t)y^{t-1}$$

By definition, for any $\epsilon > 0$ we have

$$0 \le y^n - t^n < ny^{t-1}\left(\frac{\epsilon}{ny^{t-1}}\right) = \epsilon.$$

Thus $y^n = \sup B$. We will show that $y^n = x$.

First assume $x \le y^n$, then for each $t \in A$, $t^n < x \le y^n$, implying

$$0 < x - t^n \le y^n - t^n$$

We know x is an upper bound for B, and for each $\epsilon > 0$, $y^n - t^n < \epsilon$. Thus $x = \sup B = y^n$. Now assume $y^n < x$ then for any $y > \delta > 0$, we have

$$(y + \delta)^n = y^n + \delta(y^{n-1} + \cdots + \delta^{n-1}) \le y^n + \delta n y^{n-1}. \tag{4.2}$$

By density, we can find a number q so that $y^n < q < \min\{x, (n+1)y^n\}$, so let

$$\delta = \frac{q - y^n}{n y^{n-1}}.$$

The upper bound on q ensures that $0 < \delta < y$. Plugging this into the (4.2) yields,

$$y^n < (y + \delta)^n \le q < x.$$

Thus $y + \delta$ is an upper bound of A, a contradiction. □

As a corollary, we have the usual rule for exponents of the type $1/n$.

Corollary 1.2. *If $a, b > 0$ and $n \in \mathbb{N}$, then*

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

We can set $\alpha = a^{1/n}$ and $\beta = b^{1/n}$ and compute $(\alpha\beta)^n = \alpha^n \beta^n = ab$. Thus $\alpha\beta = (ab)^{1/n}$.

Example 1.7. Every real number x has a decimal representation,

$$x = n_0.n_1 n_2 \ldots$$

where $n_0 \in \mathbb{Z}$ and each $n_i \in \{0, 1, \ldots, 9\}$ for $i = 1, 2, \ldots,$ and can be computed via induction. We start with

$$n_0 = \max\{n \in \mathbb{Z} : n \le x\},$$

and set (inductively) $n_1, \ldots, n_{k-1}$, so that

$$n_k = \max\left\{n \in \mathbb{Z} : n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x\right\}.$$

We have that

$$x = \sup\{n_0,\ n_0 + 10^{-1} n_1,\ n_0 + 10^{-1} n_1 + 10^{-2} n_2,\ \ldots\},$$

thus each decimal representation of x is unique up to a removal of "repeating 9s". For example, numbers like $x = 1.0499999\ldots$ would not appear from this construction, instead we would have the equal number $x = 1.05$.

5 The Complex Numbers

We will now construct the field of **complex numbers** without initial reference to an "imaginary unit", i being a square root of -1. Of course, this will come out after verifying the field axioms.

Definition 1.10. A **complex number** is an ordered pair (a, b) where $a, b \in \mathbb{R}$. We write this as $a + bi$. Equality is of course defined to mean $a + bi = c + di$ if and only if $a = c$ and $b = d$. Addition and multiplication are defined as one expects. Let $x = a+bi$ and $y = c+di$, then

$$x + y := (a + c) + (b + d)i \quad \text{and} \quad xy := (ac - bd) + (ad + bc)i.$$

The set of **complex numbers** is called $\mathbb{C}$.

Theorem 1.5. *With the operations defined above, $\mathbb{C}$ is a field with the roles of $0 = 0 + 0i$ and $1 = 1 + 0i$ inherited from $\mathbb{R}$.*

Proof. One can verify all the field axioms independently, we shall include only the proof of existence of the multiplicative inverse. Let $z = a+bi$, then through the formal computation,

$$\frac{1}{z} = \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}.$$

If we define $1/z$ to be this quantity on the RHS,

$$\frac{1}{z} := \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i,$$

then

$$z\left(\frac{1}{z}\right) = (a + bi)\left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i\right) = 1.$$

$\square$

Corollary 1.3. *The real numbers, $\mathbb{R}$ form a subfield of $\mathbb{C}$.*

Proof. We have that every real number $x \in \mathbb{R}$ has the form $x = x + 0i \in \mathbb{C}$, where complex $+$ and $\times$ are the same as they are in $\mathbb{R}$ when restricted from $\mathbb{C}$ to $\mathbb{R}$. $\square$

Theorem 1.6. $i^2 = -1$.

Proof. By definition, $i^2 = (0 + 1i)^2 = -1 + 0i = -1$. $\square$

Definition 1.11. The **conjugate** of a complex number, $z = a + bi$, is the complex number $\overline{z} := a - bi$. The **real** and **imaginary** parts of z are defined to be the real numbers

$$\text{Re}(z) = a \text{ and } \text{Im}(z) = b.$$

A complex number and its conjugate has a unique relation with the geometry of $\mathbb{C}$.

Definition 1.12. The **magnitude** of a complex number $z = a + bi$ is its Pythagorean distance from 0 in the complex plane:

$$|z| = \sqrt{a^2 + b^2}.$$

One can verify immediately that

$$|z| = (z\overline{z})^{1/2} \in \mathbb{R}.$$

Moreover, the **distance** between $z = a + bi$ and $w = c + di$ is the magnitude of their difference,

$$\operatorname{dist}(z, w) = |z - w| = \sqrt{(a-c)^2 + (b-d)^2}.$$

Students of linear algebra will recognize immediately that this definition makes $\mathbb{C}$ a **Euclidean space** identifiable with $\mathbb{R}^2$, and also a 2-dimensional vector space over $\mathbb{R}$. $\mathbb{C}$ is actually more than a vector space, it is an **algebra** with some invalauble arithmetic properties that most vector spaces do not have.

Theorem 1.7. *Suppose $z, w \in \mathbb{C}$. Then*

(a) $\overline{z + w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = (\overline{z})(\overline{w})$.

(c) $z + \overline{z} = 2\operatorname{Re}(z)$, *and* $z - \overline{z} = 2i\operatorname{Im}(z)$.

(d) $|z| \geq 0$ *with equality if and only if* $z = 0$.

(e) $|\overline{z}| = |z|$.

(f) $|zw| = |z|\ |w|$.

(g) $|\operatorname{Re} z| \leq |z|$.

(h) $|z + w| \leq |z| + |w|$.

Proof. We omit most of the proofs except that of (h). This comes from a typical argument that students of linear algebra will recognize from inner products and an application of (g),

$$\begin{aligned} |z+w|^2 = (z+w)(\overline{z}+\overline{w}) &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z|\ |\overline{w}| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

□

Definition 1.13. If $x_1, \ldots, x_n \in \mathbb{C}$, we write

$$x_1 + x_2 + \cdots + x_n := \sum_{j=1}^{n} x_j.$$

We finish with a crucial fact about inner product spaces, the **Cauchy-Schwarz inequality**. We use a typical proof method, one that may not be easy to make up from scratch but a good method to know for these types of arguments.

Theorem 1.8. *If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers then*

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right). \tag{5.1}$$

Proof. Let $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, and $C = \sum_{j=1}^{n} a_j \bar{b}_j$. We want to prove $C^2 \leq AB$. If $B = 0$, then each $b_j = 0$, so $C = 0$ and the inequality trivially holds. So assume $B > 0$, then, taking all sums from $j = 1$ to n,

$$\begin{aligned} 0 \leq \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - BC \sum \overline{a_j} b_j - B\overline{C} \sum a_j \overline{b_j} + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 - B|C|^2 + |C|^2 B \\ &= B(AB - |C|^2). \end{aligned}$$

Thus $|C|^2 \leq AB$. □

6 Euclidean Space

We will not do much with Euclidean space in this course, but it is occasionally useful in examples.

Definition 1.14. The set of ordered n-tuples of real numbers is called $\mathbb{R}^n$,

$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, \ i = 1, \dots, n\}.$$

Elements $x \in \mathbb{R}^n$ are often refered to as *points*, *vectors*, or *n-vectors*. The operations of scalar multiplication and vector addition turn $\mathbb{R}^n$ into a vector space, writing $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$,

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \text{ and } \alpha x = (\alpha x_1, \dots, \alpha x_n), \ \alpha \in \mathbb{R}.$$

The space $\mathbb{R}^n$ is naturally endowed with a **scalar product** also called the *dot product* or *Euclidean inner product*. This is the operation

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j.$$

This, in turn, defines the *Euclidean norm* on $\mathbb{R}^n$,

$$|x| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

In a first course in linear algebra, one proves that any vector space over $\mathbb{R}$ of *finite dimension* with the inner product structure above is linearly isomorphic to $\mathbb{R}^n$, and we refer to all such spaces as **Euclidean spaces**.

Example 1.8. $\mathbb{C}$ is a Euclidean space when identified with $\mathbb{R}^2$, since any complex number $z = x + iy$ depends on $x, y \in \mathbb{R}$.

One can verify the following properties.

Proposition 1.7. *For any $x, y, z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,*

(a) $|x| \geq 0$ *with* $|x| = 0$ *if and only if* $x = \mathbf{0} = (0, \ldots, 0)$.

(b) $|\alpha x| = |\alpha||x|$.

(c) $|x \cdot y| \leq |x||y|$.

(d) $|x + y| \leq |x| + |y|$.

(e) $|x - y| \leq |x - z| + |y - z|$.

Chapter 2

Metric Spaces

1 Sets and Countability

Set theory and its notation serves to ease the burden of doing mathematics. The following definitions allow us to describe the basics of sets and relations between them.

Definition 2.1. A function (also called a **map**), f, with **domain** A and **codomain** B is written

$$f : A \to B.$$

The domain of f is the set of all possible inputs to f, the codomain is a set containing (but not necessarily equal to) the set of all values of f. The **range** of f is the set of all possible values of f. If $E \subseteq A$, we call the **image of** E **under** f the set of all values of f given inputs in E. We write it

$$f(E) = \{f(x) : x \in E\}.$$

Thus $f(A)$ is another way to write the range of f. If $U \subseteq B$ we say the **pre-image** of U is the set of all $x \in A$ that f maps to U,

$$f^{-1}(U) = \{x \in A : f(x) \in U\}.$$

For $y \in B$ we write $f^{-1}(y)$ for $f^{-1}(\{y\})$.

Example 2.1. Suppose $f : [0, 1] \to \mathbb{R}$ where $f(x) = x^2 + 1$. One can check, for example, that the range of f is $[1, 2]$, and that $f^{-1}([-1, 1.5]) = [0, 1/\sqrt{2}]$.

The following distinctions are crucial properties about sets and functions between them. They allow us to estimate the"sizes" of sets relative to each other.

Definition 2.2. Suppose $f : A \to B$. We say f is **injective** or **one-one** if each $x \in A$ corresponds uniquely to its value $f(x) \in B$. More concretely,

$$x \neq y \implies f(x) \neq f(y).$$

If f is one-one, then every value in its range cannot have more than one corresponding input in the domain. Another way to say this is that $f^{-1}(y)$ has only one element for each y in the range.

We say f is **surjective** or **onto** if its codomain is its range, that is if $f(A) = B$.

We say f is **bijective** if it is both one-one and onto.

Now we illustrate the "size" relation for sets of finitely many elements.

Proposition 2.1. *Suppose A is a set of n elements and B is a set of m elements and $f : A \to B$.*

(a) If f is injective, then $n \leq m$.

(b) If f is surjective, then $n \geq m$.

(c) If f is bijective, then $n = m$.

Proof. The crux of each proof is about the relation between the size of A and the range of f, $f(A)$. Suppose $f(A)$ has *more* elements than A. Enumerate A as

$$A = \{x_1, \ldots, x_n\}$$

and write

$$f(A) = \{f(x_1), \ldots, f(x_n), y_1, \ldots, y_k\}.$$

Each y_j for $j = 1, \ldots, k$ has to have an $x \in x_1, \ldots, x_n$ so that $y_j = f(x)$, which is impossible since each $f(x)$ is different from y_j. Thus $f(A)$ at most as many elements as A.

For (a), suppose f is injective. Then each $y \in f(A)$ has exactly one $x \in A$ so that $f(x) = y$. Since A is enumerated as $A = \{x_1, \ldots, x_n\}$, we can thus enumerate $f(A) = \{y_1, \ldots, y_n\}$ where each $f(x_i) = y_i$. Thus $f(A)$ has n elements and since $f(A) \subseteq B$, $n \leq m$.

For (b), f surjective implies that $f(A) = B$ and thus $m \leq n$.

For (c), if f is bijective combine (a) and (b) to see that $n = m$. □

This presents a useful property of bijections: they tell us when two sets have the same "size". In the case that a set contains infinitely many elements, the notion of "size" becomes problematic while the notion of bijection remains untouched. We will thus use bijections to describe "sizes" of sets in the event a set is infinite. To cement this further, bijections are a form of *equivalence relation*.

Definition 2.3. Let $A \sim B$ mean that there exists a bijection $f : A \to B$. Rudin calls this relation **1-1 correspondence**. This is an equivalence relation, i.e.

- (Reflexivity) $A \sim A$
- (Symmetry) $A \sim B$ if and only if $B \sim A$
- (Transitivity) If $A \sim B$ and $B \sim C$ then $A \sim C$.

We say that A and B have the same **cardinality** if $A \sim B$. Occasionally you may see this written as $\#A = \#B$.

Proposition 2.2. *The relation $\sim$ defined above is an equivalence relation.*

Proof. For reflexivity, the identity map

$$\begin{aligned} I : A &\to A \\ x &\mapsto x \end{aligned}$$

is a bijection. For symmetry, if $A \sim B$ then there is some bijection $f : A \to B$. The inverse $f^{-1} : B \to A$ is also a bijection. For transitivity, suppose $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection. □

Now we define "countability" keeping in mind what bijections tell us about "size".

Definition 2.4. Let A be a non-empty set. We say

- A is **finite** if $A \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$.
- A is **infinite** if A is not finite.
- A is **countably infinite** if $A \sim \mathbb{N}$.
- A is **uncountable** if it is not finite or countabily infinite.

Rudin uses the term **countable** to describe only countably infinite sets, while other authors use **countable** to mean either finite or countably infinite. For clarity, we will adopt the latter usage of the word. Ruding uses "at most countable" for this distinction, but we are going with the common usage of the term.

Proposition 2.3. *Finite sets have the same number of elements if and only if they are in 1-1 correspondence.*

The term "countable" is chosen for a reason. It means any countable set can be exhaustively "counted" by the natural numbers: 1, 2, 3, and so on. More precisely, any countable set can be enumerated and indexed by $\mathbb{N}$. If A is countable, then

$$A = \{x_1, x_2, x_3, \dots\}.$$

The next few facts show which familiar sets are countable, and also how to construct countable sets out of others.

Proposition 2.4. *The integers, $\mathbb{Z}$, are countably infinite.*

Proof. Let $f(1) = 0$ and for any $n \in \mathbb{N}$, $f(2n) = n$ and $f(2n+1) = -n$. Then $f : \mathbb{N} \to \mathbb{Z}$ is a bijection. Since $\sim$ is symmetric, this proves $\mathbb{Z}$ is countably infinite. □

This fact may be surprising, why should $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality? Doesn't $\mathbb{Z}$ have twice as many elements as $\mathbb{N}$? This is a consequence of allowing sets to contain infinitely many elements. One important fact about countable sets is their representation as **sequences**.

Definition 2.5. A **sequence** of elements is the result of assigning a countable set, A, an **indexing function**, $f : \mathbb{N} \to A$ and writing $f(i) = x_i$. This indexing function need not be one-one so elements may be repeated, but we can write A as a list

$$A = \{x_1, x_2, x_3, \dots\}$$

If we want, we can find an indexing function which is one-one by eliminating repetitions. We call the integer subscript, i, the **index** of the term in the sequence.

We can use this idea to illustrate a key fact about countability: it is the "smallest" infinity.

Theorem 2.1. *Every infinite subset E of a countable set A is countable.*

Proof. If A is finite, there is nothing to prove, so we assume A is countably infinite. Let $E \subseteq A$ be infinite, and enumerate A without repetition as

$$A = \{x_1, x_2, x_3, \dots\}$$

$n_1 \in \mathbb{N}$ be the smallest number so that $x_{n_1} \in E$. Having chosen $n_1, \dots, n_{k-1}$, define n_k to be the smallest number so that

$$n_1 < n_2 < \cdots < n_{k-1} < n_k, \text{ and } x_{n_k} \in E.$$

Since every $x \in E$ is indexed as $x_i \in A$, there is a $k \in \mathbb{N}$ so that $i = n_k$ and thus x is an element of the sequence,

$$x \in \{x_{n_1}, x_{n_2}, \dots\}.$$

This shows that $E = \{x_{n_1}, x_{n_2}, \dots\}$, thus E is countable. □

Now we will show what the effects of unions and intersections are on countability.

Definition 2.6. If A and B are two sets, their **union**is the set of all elements in either A, B, or both, written

$$A \cap B = \{x : x \in A \text{ or } x \in B\}.$$

If J is any index set (not necessarily countable), and $\{E_\alpha\}_{\alpha \in J}$ is a family of sets indexed by J we can form the union of all E_α,

$$\bigcup_{\alpha \in J} E_\alpha = \{x : x \in E_\alpha \text{ for some } \alpha \in J\}.$$

Likewise, the **intersection** of two sets A and B is the set of all elements common to both, written
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$
Again, if J is an index set and $\{E_\alpha\}_{\alpha \in J}$ is a family of sets, then their intersection is
$$\bigcap_{\alpha \in J} E_\alpha = \{x : x \in E_\alpha \text{ for all } \alpha \in J\}.$$
If $J = \{1, \ldots, n\}$ or $\mathbb{N}$, we write
$$\bigcup_{j=1}^{n} E_j \text{ and } \bigcap_{j=1}^{n} E_j, \text{ or } \bigcup_{j=1}^{\infty} E_j \text{ and } \bigcap_{j=1}^{\infty} E_j$$
respectively.

An important collection of sets are the real intervals.

Definition 2.7. If $a < b \in \mathbb{R}$, we have

Open Intervals: The **open interval** is $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Closed Intervals: The **closed interval** is $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Half-open/closed: The **half-open or half-closed intervals** are $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

Note that in the case of (a, b) and $(a, b]$, we may have $a = -\infty$ and in the case of (a, b) and $[a, b)$, $b = \infty$.

Example 2.2. For $n \in \mathbb{N}$, consider the half-open intervals
$$E_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$$
One can verify that
$$\bigcup_{n=1}^{\infty} E_n = (1/2, 1] \cup (1/3, 1/2] \cup \cdots = (0, 1],$$
and
$$\bigcap_{n=1}^{\infty} E_n = \varnothing.$$
Consider $A_\alpha = [0, \alpha)$ for all real $\alpha > 0$. Then
$$\bigcup_{\alpha > 0} A_\alpha = [0, \infty) \text{ and } \bigcap_{\alpha > 0} A_\alpha = \{0\}.$$

Next we introduce a pivotal argument in the development of modern mathematics: Georg Cantor's *diagonal argument.*

Theorem 2.2. *Countable unions of countable sets are countable.*

Proof. Let $\{E_n\}$ be a family of countably infinite sets, for $n = 1, 2, 3, \ldots$. Enumerate each as

$$E_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots\}.$$

We can visualize each in an *array*,

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{21} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

If we visually rotate this to form a triangular array,

$$\begin{array}{ccccccccc} & & & & x_{11} & & & & \\ & & & x_{21} & & x_{12} & & & \\ & & x_{31} & & x_{22} & & x_{13} & & \\ & x_{41} & & x_{32} & & x_{23} & & x_{14} & \\ \cdot^{\cdot^{\cdot}} & & & & & & & & \ddots \end{array}$$

one can assign $y_1, y_2, y_3, \ldots$ to be

$$\begin{array}{ccccccccc} & & & & y_1 & & & & \\ & & & y_2 & & y_3 & & & \\ & & y_4 & & y_5 & & y_6 & & \\ & y_7 & & y_8 & & y_9 & & y_{10} & \\ \cdot^{\cdot^{\cdot}} & & & & & & & & \ddots \end{array}$$

Letting $S = \bigcup_{n=1}^{\infty} E_n$, we have that every $y \in S$ is equal to some y_i. In fact, since this triangular enumeration may contain repetitions, S is in 1-1 correspondence with an infinite subset of $\mathbb{N}$. Thus, S is countable. Note that if one or more of the E_n is instead finite, S is still countable. □

This theorem and argument lead to some very important consequences concering the structure of $\mathbb{Q}$ and $\mathbb{R}$.

Definition 2.8. A **cartesian product** of sets A and B is the set of all ordered pairs, (a, b) where $a \in A$ and $b \in B$ and called

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

The ordering of the pairs means that $A \times B \neq B \times A$ if $B \neq A$. The n-fold cartesian product of sets $A_1, \ldots, A_n$ is the set of ordered n-tuples,

$$A_1 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) : a_i \in A_i,\ i = 1, \ldots, n\}.$$

We have that products of countable sets are indeed countable.

Theorem 2.3. *If A and B are countable, then so is $A \times B$. Moreover, if $A_1, \dots, A_n$ are countable, then so is $A_1 \times A_n$.*

Proof. Enumerate $A = \{a_1, a_2, \dots\}$ and define for each $i = 1, 2, \dots,$

$$E_i = \{a_i\} \times B = \{(a_i, b) : b \in B\}.$$

We have that $E_i \sim B$ by considering the projection

$$\begin{aligned} f : E_i &\to B \\ (a_i, b) &\mapsto b \end{aligned}$$

and observing that f is a bijection. This shows that each E_i is countable, so by the previous theorem

$$A \times B = \bigcup_{i=1}^{\infty} E_i$$

is countable.

We know $A_1 \times A_2$ is countable, thus (the equals sign below is ever-so-slightly misleading, but we allow this abuse of notation)

$$(A_1 \times A_2) \times A_3 = A_1 \times A_2 \times A_3$$

is also countable. If $A_1 \times A_{n-1}$ is countable, then so is

$$A_1 \times \cdots \times A_{n-1} \times A_n,$$

so by induction we are done. □

Now we can show a maybe surprising fact.

Theorem 2.4. *$\mathbb{Q}$ is countable.*

Proof. We know that $\mathbb{Q}$ is the set of all fractions a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$, in fact we can restrict b to $\mathbb{N}$. Consider the product of countable sets $\mathbb{Z} \times \mathbb{N}$, this set is countable by the previous theorem. If we identify each $a/b \in \mathbb{Q}$ as $(a, b) \in \mathbb{Z} \times \mathbb{N}$ and discard repetitions, we see that $\mathbb{Q}$ is in 1-1 correspondence with an infinite subset of $\mathbb{Z} \times \mathbb{N}$. Thus $\mathbb{Q}$ is countable. □

Now we can prove that $\mathbb{R}$ is uncountable.

Theorem 2.5. *The real interval $I = (0, 1)$ is uncountable.*

Proof. Suppose I is countable, then we can enumerate it, $I = \{x_1, x_2, \dots\}$. Write each x_i in its decimal representation, $x_i = 0.n_{i1}n_{i2}n_{i3}$ and list them,

$$\begin{aligned} x_1 &= 0.n_{11}n_{12}n_{13}\dots \\ x_2 &= 0.n_{21}n_{22}n_{23}\dots \\ x_3 &= 0.n_{31}n_{32}n_{33}\dots \\ &\vdots \end{aligned}$$

Now let $x = 0.m_1m_2m_3\dots$ where

1. $m_i \neq n_{ii}$, and
2. For every $N \in \mathbb{N}$, there is some $i > N$ so that $m_i \neq 9$.

Then $x \neq x_i$ for any i, since the i-th digit in the decimal representation of each x_i differs with x and each decimal representation is unique. This shows that $x \notin I$, yet as defined, $0 < x < 1$ which means $x \in I$, a contradiction. Thus I is uncountable. □

Corollary 2.1. *$\mathbb{R}$ is uncountable.*

2 Motivation for Topology

One goal of this course is to develop a rigorous understanding of **continuous functions** and also **differentiable functions**. This will be accomplished in part by studying what types of domains such functions can have, and what effects they have on the behavior of the functions themselves. Thus we must start with a study of sets, which is where the subject of "topology" comes into play. The **standard topology** (also called the **metric topology** or the **order topology**) on $\mathbb{R}$ consists of all "open sets" in $\mathbb{R}$, including in this the empty set and $\mathbb{R}$ itself. As one expects, the "open sets" consist of the open intervals (a, b) but also all *arbitrary unions* and *finite intersections* of open intervals. The **closed sets** are the complements of the open sets, again including $\varnothing$ and $\mathbb{R}$. Both open and closed sets have important properties.

Example 2.3. Given an open interval (a, b) every point $x \in (a, b)$ can be shifted to the left/right a small amount and remain within the set. If $a < x < b$, take the midpoints

$$\frac{x-a}{2} \text{ or } \frac{b-x}{2}.$$

For any x in (a, b) one can verify that both of these numbers are also in (a, b). This is a crucial property and it carries over to general open sets. One valuable useage of this property is in *approximation*: if x is a "guess" then we have an open interval around x,

$$(x-\epsilon, x+\epsilon) \subseteq (a, b)$$

which one can think of as a "margin of error". In fact, this is where the "Epsilon arugment" comes from. See Ch 1, section 4.1. The property (or definition if you'd like) that open sets properly contain a smaller open interval around every element in the set is a *topological property*. It doesnt actually depend on the space itself, merely the fact that we have a "topology" on it. This property is common to every topological space.

Example 2.4. Closed intervals, $[a, b]$ also have important properties. For one, if we have a **continuous function**, $f : [a, b] \to \mathbb{R}$ then the range of f is also a closed, bounded interval, $[c, d]$. You may recall this as a consequence of the *intermediate value theorem*, but in fact this does not depend on the space $\mathbb{R}$ at all! This is a result of a topological property called **compactness** and also of continuous functions, who can be defined purely by topological means rather than limits in $\mathbb{R}$, like we are used to.

In general, a closed set in $\mathbb{R}$ is an arbitrary intersection or finite union of either closed intervals or infinite half-closed intervals like $(-\infty, a]$ or $[a, \infty)$. Closed sets have the property that they "contain their limits", that any point which is arbitrarily close (think distance zero) to a closed set must be an element of it. In fact, this is another way to define closed sets. Again, this is a purely topological property and does not depend on the fact that we are in $\mathbb{R}$ or that there is a notion of "distance" which not all topological spaces have.

Another topological space of immediate interest is $\mathbb{C}$. However, $\mathbb{C}$ is topologically different from $\mathbb{R}$ - the open sets in $\mathbb{R}$ are not "open" in $\mathbb{C}$ since $\mathbb{C}$ has the geometric structure of $\mathbb{R}^2$. Our notion of "open" in $\mathbb{R}^2$ changes partly due to the added dimension.

Example 2.5. Consider the "open" interval in $\mathbb{R}^2$,

$$I = \{(x, 0) : x \in (-1, 1)\}.$$

One can construct another one-dimensional interval containing the origin, say

$$J = \{(0, y) : y \in (-1, 1)\}.$$

It is clear that no subset of J lies in a subset of I and vice versa besides the singleton $\{(0, 0)\}$. This conflicts with the topological notion of "open set" since every point in an open set must have a smaller open set (called a neighborhood) both containing the point and properly contained within the larger open set. With this in mind, I and J cannot both be open in $\mathbb{R}^2$.

To correct for this, we can consider "open rectangles" to be open sets, that is

$$R = (a, b) \times (c, d) = \{(x_1, x_2) : x_1 \in (a, b),\ x_2 \in (c, d)\}.$$

Indeed these sets are open and form what is called a **base** for the standard topology of $\mathbb{R}^2$. Geometrically speaking, these are not the only types of open sets in $\mathbb{R}^2$. More uniformly defined, we have the **open balls**:

$$B_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}.$$

These are literally the disks of radius $r > 0$ centered at the point $a \in \mathbb{R}^2$, and since the inequality $|x - a| < r$ is strict, the boundary circle of radius r is not included. These open balls are the 2-dimensional analogue of the symmetric open intervals $(x - \epsilon, x + \epsilon) \subseteq \mathbb{R}$ with r playing the role of ϵ. Generally, the open sets of $\mathbb{R}^2$ are any set which results from an aribtrary union or closed intersection of these open balls. Closed sets in $\mathbb{R}^2$ can be defined similarly, but can also take on a different geometric structure than one may think. All that is required for a closed set (in any topology) is that it contains its "limit points", or all points which are arbitrarily close to it. Of course, the closed balls,

$$\overline{B_r}(a) = \{x \in \mathbb{R}^2 : |x - a| \leq r\}$$

are considered closed sets in $\mathbb{R}^2$, since the boundary circle consists of points which are "distance zero" to the ball. However, different structures can also be closed, such as **graphs** of continuous functions:

Example 2.6. The graph of the function $f(x) = 1/x^2$ for $x \neq 0$,

$$A = \{(t, 1/t^2) : t \in \mathbb{R} \setminus \{0\}\},$$

is considered closed in $\mathbb{R}^2$ even though it is a "one-dimensional" structure. More perplexing perhaps is the fact that A has, as a set in $\mathbb{R}^2$, distance zero to the y-axis, yet no point of the y-axis is in A.

Hopefully by now you are understanding at least one or two differences between $\mathbb{R}$ and $\mathbb{R}^2$. Perhaps on your mind is still the question "why are we defining open and closed sets in such a way?" The goals in both spaces are the same: open sets serve to "leave room for approximation" and closed sets serve to "fully contain their limits". Instead of studying just $\mathbb{R}$ and $\mathbb{R}^2$, we will study more general topological spaces and jump down to $\mathbb{R}$ and $\mathbb{R}^n$ when we need to for examples or for specific properties. The more

3 Metric Space Topology

4 Compact Sets

5 Connected Sets

Chapter 3

Sequences and Series

1 Convergent Sequences

2 Cauchy Sequences

3 Series of Complex Numbers

4 Power Series

5 Absolute Convergence

Chapter 4

Limits and Continuity

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2 Compactness

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4 Uniform Continuity

5 Discontinuities

Chapter 5

Derivatives in $\mathbb{R}$

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Sequences and Series of Functions

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3 Uniform Convergence and Limit Operations

4 Equicontinuity

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Chapter 7

Integrals in $\mathbb{R}$

1 Definition

2 Properties

3 Differentiating Under the Integral

4 Summation Under the Integral