1. If r is rational ($r \neq 0$) & x is irrational, prove $r+x$ and rx irrational

i) Proof: Suppose $r+x$ is rational, then $\exists\, p, q \in \mathbb{Z}, q \neq 0$ such that $r+x = \frac{p}{q}$, and since r is rational $r \neq 0$, there $\exists\, a, b \in \mathbb{Z}, b \neq 0, a \neq 0$ such that $r = \frac{a}{b}$

Then $x = x + r + (-r) = \frac{p}{q} - \frac{a}{b} = \frac{pb - aq}{qb}$

since $pb - aq$ and $qb \in \mathbb{Z}$, $qb \neq 0$, x is rational (contradiction!)

Thus $r+x$ is irrational.

ii) Similarly, suppose rx is rational and $rx = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ $q \neq 0$

Then $x = x \cdot \frac{r}{r}$ $(r \neq 0)$

$$= xr \cdot \frac{1}{r} = \frac{p}{q} \cdot \frac{1}{\left(\frac{a}{b}\right)} = \frac{p}{q} \cdot \frac{1}{\left(\frac{a}{b}\right)} \cdot \frac{b}{b} \quad (b \neq 0, a \neq 0)$$

$$= \frac{p}{q} \cdot \frac{b}{a} = \frac{pb}{qa}$$

Since pb and $qa \in \mathbb{Z}$ and $qa \neq 0$ ($q \neq 0$ and $a \neq 0$), x is rational (contradicts!)

Thus rx is irrational

Therefore if r is rational ($r \neq 0$) and x is irrational, $r+x$ and rx are irrational

4. Suppose $\alpha > \beta$. Since α is a LB of E and β is an UB of E,
$\exists x \in E$ such that $\alpha \le x$ and $x \le \beta$
Hence $\alpha \le x \le \beta$ or $\alpha \le \beta$ contradicts !

Therefore $\alpha \le \beta$ for α and β are LB and UB of E

5. Let A be a nonempty set of real numbers, bounded below. $-A$ be the set of all numbers $-x$, where $x \in A$. Prove

$$\inf A = -\sup(-A)$$

Proof: A is bounded below, let $\alpha = \inf A$, $\alpha \in \mathbb{R}$.
Then α is the GLB of A and

$\alpha \leq x$ for all $x \in A$. Then,

$\alpha + (-\alpha - x) \leq x + (-\alpha - x)$ or

$\alpha + (-\alpha) + (-x) \leq x + (-\alpha) + (-x)$

$-x \leq -\alpha$ for $\forall x \in A$

Let $y = -x$, then $y \in -A$

$y \leq -\alpha$ for $\forall y \in -A$

Since $\mathbb{R}$ is ordered, $\alpha \in \mathbb{R}$ then $(-\alpha) \in \mathbb{R}$, Hence there $\exists (-\alpha) \in \mathbb{R}$ s.t. $y \leq (-\alpha)$ for $\forall y \in -A$

Then $-\alpha$ is an UB for $(-A)$

and $-A$ is bounded above

Need to prove $(-\alpha)$ is LUB of $(-A)$

Suppose there $\exists \beta \in \mathbb{R}$ where $\beta < -\alpha$ and β is an UB for $(-A)$

Then $y \leq \beta < -\alpha$ $\forall y \in -A$

Then $-x \leq \beta < -\alpha$ $\forall x \in A$

or $x \geq -\beta > \alpha$ $\forall x \in A$

Then $-\beta$ is a LB for A, contradicts $\alpha = \inf A$

Therefore $-\alpha \leq \beta$, or $-\alpha$ is the LUB of $(-A)$

Thus $-\inf A = -\alpha = \sup(-A)$

or $\inf A = -\sup(-A)$ q.e.d.

2. For $x, y \in F$, we have:

(i) $y = y + 0$ (addictive identity)

$= y + x + (-x)$ (addictive inverse)

$= (y + x) + (-x)$ (associativity)

$= x + (-x)$

$= 0$

ii) $y = y + 0$ (addictive identity)

$= y + x + (-x)$ (addictive inverse)

$= 0 + (-x)$

$= -x$

iii) $y = 1 . y$ (multiplicative identity)

$= \frac{x}{x} y$ (multiplicative inverse $x \neq 0$)

$= \frac{1}{x}(xy)$ (associativity)

$= \frac{1}{x} x$

$= 1$

iv) $y = 1 . y$ (multiplicative identity)

$= \frac{x}{x} y$ (multiplicative inverse $x \neq 0$)

$= \frac{1}{x}(xy)$ (associativity)

$= \frac{1}{x} \cdot 1 = \frac{1}{x}$

3. a) Prove if $x+y < x+z$ then $y < z$

We have $y = y+0 = y+x+(-x) < z+x+(-x) = z$

, Hence $y < z$

b) $y = y \cdot 1 = y\left(\frac{x}{x}\right) = xy\left(\frac{1}{x}\right) < xz\left(\frac{1}{x}\right) = z \quad (x>0)$

Hence $y < z$

Now we can use all cancellation laws