Ch.1

#7. (a) For any positive integer n, $b^n - 1 \geq n(b-1)$

Prove by induction:

Base case: $n=1$, then

$b^n - 1 = b - 1$

$n(b-1) = b-1$

Hence $b^n - 1 \geq n(b-1)$, statement holds for $n=1$

Assume statement holds for $n=k$, $b^k - 1 \geq k(b-1)$

Then $b^{k+1} - 1 = (b-1)\underbrace{(b^k + b^{k-1} + \ldots + 1)}_{k+1 \text{ items}}$

$\geq (b-1)(k+1)$

Thus the statement holds for $n = k+1$

Therefore $b^n - 1 \geq n(b-1)$ for any positive integer n

(b) With $b-1 = (b^{1/n})^n - 1$, let $m = b^{1/n}$ we can apply (a):

$$b-1 = (b^{1/n})^n - 1 = m^n - 1 \geq n(m-1) = n(b^{1/n}-1)$$

(c) If $t > 1$ and $n > \frac{(b-1)}{t-1}$, then $b^{1/n} < t$

Proof: Suppose $n > \frac{b-1}{t-1}$ and $t > 1$ or $t-1 > 0$, then using (b):

$n > \frac{b-1}{t-1} \geq \frac{n(b^{1/n}-1)}{t-1}$. Since $n>0$ and $t-1>0$, $t-1 > (b^{1/n}-1)$ or $t > b^{1/n}$

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

This is similar to proving, $\exists a \in \mathbb{R}$ s.t. P is true $\forall n \geq a$ where P is $b^{w+(1/n)} < y$

Proof: Suppose $y > b^w$.
Take $t = y \cdot b^{-w}$ Then $t > b^w \cdot b^{-w} = 1$.
Assume $\exists n \in \mathbb{Z}, n > (b-1)/(t-1)$, then results from (c) can be used, we have:

$t > b^{1/n}$

$y \cdot b^{-w} > b^{1/n}$

$y > b^{1/n + w}$

Thus, with $n > (b-1)/(t-1)$, $n \in \mathbb{Z}$ and $t = y \cdot b^{-w} > 1$, the statement holds.

(e) Suppose $b^w > y$. Take $t = y^{-1} \cdot b^w$, then $t > y^{-1} \cdot y = 1$
Assume $\exists n \in \mathbb{Z}, n > (b-1)/(t-1)$, then results from (c) can be used.

$t > b^{1/n}$

$y^{-1} b^w > b^{1/n}$

$y^{-1} > b^{-(w - 1/n)}$

$y < b^{w - 1/n}$

Thus with $n > (b-1)/(t-1)$, $n \in \mathbb{Z}$ and $t = y^{-1} \cdot b^w > 1$, the statement holds

(f) Let $b^x = y$, then $b^w < y = b^x$ or $w < x$ for $\forall w \in A$
Hence x is an UB of A, have to prove x is the least.

Suppose m is an upper bound of A, then $b^w < b^m$ for $\forall w \in A$
Case 1: $b^m > y$, clearly $b^m > b^x$ or $m > x$ for x is the LUB or $x = \sup A$
Case 2: $b^m < y$, then $m \in A$. This implies there exists n sufficiently large st.

$b^{m+1/n} < y$. Hence $< m < m + 1/n$ which says m, is no longer an upper bound.

Hence $m = x$ is the LUB of A, or $x = \sup A$.

g) prove that x is unique

Suppose A has 2 LUB x_1, x_2. Since a LUB is basically an UB, let x_1 be LUB and x_2 be UB.

Hence $x_1 \leq x_2$. Now, x_2 is also a LUB and x_1 is an UB $x_2 \leq x_1$.

Thus, we can conclude $x_1 = x_2$ or x is unique.

#13. $|x-y|^2 = (x-y)(\overline{x-y})$
$= (x-y)(\bar{x}-\bar{y})$
$= x.\bar{x} - x\bar{y} - y\bar{x} + y\bar{y}$
$= |x|^2 - 2Re(x\bar{y}) + |y|^2$
$\geq |x|^2 - 2|x.\bar{y}| + |y|^2$ $\qquad |Re(x\bar{y})| \leq |x\bar{y}|$
$= |x|^2 - 2|x|.|y| + |y^2|$ $\qquad |y| = |\bar{y}|$
$= (|x| - |y|)^2$
$= ||x| - |y||^2$

Hence $|x-y| \geq ||x| - |y||$ q.e.d.

Ch2#2. We will prove the set of algebraic numbers $\mathbb{A}$ being countable by first proving the set of polynomials with integer coefficients $\mathbb{P}$ countable.
Define $N^* = \{0\} \cup \mathbb{N}$, and $\mathbb{P}_n$ is the set of polynomials degree n.
Let $p(z) \in \mathbb{P}_n$ be written as:
$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \qquad a_i \in \mathbb{Q}, a_n > 0$
Suppose there is a mapping $f: \mathbb{P}_n \to \underbrace{N^* \times N^* \times N^* \ldots \times N^*}_{n \text{ items}}: f(p) = (a_0, a_1, \ldots, a_n)$
Let $q(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0$

Suppose $f(p(z)) = f(q(z))$, then
$(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$, which implies
$p(z) = q(z)$
Hence f is injective
Also, since range $f = \underbrace{N^* \times N^* \times N^* \ldots \times N^*}_{n \text{ items}}$, f is bijective

Now, since $N^* \times N^* \times \ldots \times N^*$ is countable, and $f: \mathbb{P}_n \to \underbrace{N^* \times N^* \ldots \times N^*}_{n \text{ items}}$ is bijective, $\mathbb{P}_n$ is countable
Thus $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$ is also countable.

Let $p \in \mathbb{P}$ be a polynomial of any degree and define

$g: g(p) = \{z_1, z_2, \dots, z_n\}$ where z_i is a root of "p".

By the Fundamental Theorem of algebra, a polynomial of degree "n" can have a maximum of "n" roots.

Thus each $g(p)$ is finite, and $\mathbb{P}$ is also countable.

This implies $A = \bigcup_{p \in P} g(p)$

#6. Let E' be the set of all limit points of E. Prove that E' is closed.

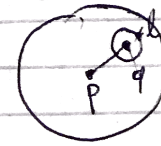
i) Proof: Let p is a limit point of E', need to prove $p \in E'$

With p being a limit point of E', there exists $q \in B_r(p), q \neq p$

such that $q \in E'$. This also say $d(p,q) \leq r$

Then q is a limit point of E, There exists $l \in B_\varepsilon(q), l \neq q$

such that $l \in E$, choose $\varepsilon = r - d(p,q)$



Then $d(p,l) \leq d(p,q) + d(q,l)$

$\leq d(p,q) + r - d(p,q) = r$

This implies l is also in the $B_r(p)$ neighborhood of l.

Hence for a point p, $B_r(p)$ contains a point $l \in E$, $l \neq p$.

This proves p is a limit point of E, meaning $p \in E'$.

Therefore E' contains its limit points, or E' is closed

ii) Let D be the set of limit points of E'
$\overline{D}$ be the set of limit points of $\overline{E}$

Prove: $D \subseteq \overline{D}$
Let $d \in D$, then d is a limit point of E'

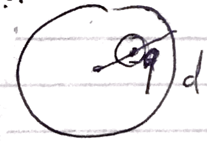
Also, since d is a limit point of E', there exist $q \in B_r(d)$, $q \neq d$ and $q \in E'$, Hence $q \in E \cup E'$

This implies there exist $q \in B_r(d)$, $q \neq d$ and $q \in E \cup E'$. Thus d is a limit point of $\overline{E} = E \cup E'$.
Therefore $D \subset \overline{D}$ (1)

Prove: $\overline{D} \subset D$.
Let $d \in \overline{D}$, or d is a limit point of $\overline{E}$. Then there exists $q \in B_r(d)$, $q \neq d$ and $q \in \overline{E}$, or $q \in E \cup E'$

- If $q \in E'$, then d will be a limit point of E'!
Hence $\overline{D} \subset D$, done.

- If $q \in E$, choose $r < d(q,d)$ so that $N_r(q) \subseteq N_R(d)$.



Hence there exists $h \in N_r(q)$, $h \neq q$ and $h \in E \cup E'$
If $h \in E$, then $q \in E'$, and d will be the limit point of E'
If $h \in E'$, then this automatically conclude "d is a limit point of E'"

Thus $\overline{D} \subset D$ (2)
Therefore, from (1) (2) $D = \overline{D}$

iii) Let $E = \{ \frac{1}{a} \mid a \in \mathbb{N}, a \geq 1 \}$. Then $E' = \{0\}$ doesn't have any limit point

#7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$ for $n = 1, 2, 3, \ldots$

Proof: We have $\overline{B_n} = B_n \cup B_n'$. (1)

Need to prove $B_n' = \bigcup_{i=1}^{n} A_i'$.

$B_n' \subseteq \bigcup_{i=1}^{n} A_i'$:

Let $p \in B_n'$, then p is a limit point of B_n. Then there exists $q \in N_R(p)$ st. $q \in B_n$, $q \neq p$

Hence $q \in \bigcup_{i=1}^{n} A_i$, which implies there is A_i s.t. $q \in A_i$

So, there exist a point $q \in N_r(p) \mid q \neq p$ s.t. $q \in A_i$, then p is a limit point of A_i or $p \in A_i'$

Hence $p \in \bigcup_{i=1}^{n} A_i'$ or $B_n' \subseteq \bigcup_{i=1}^{n} A_i'$ (2)

$\bigcup_{i=1}^{n} A_i' \subseteq B_n'$:

Let $p \in \bigcup_{i=1}^{n} A_i'$, then there exist A_i' s.t. $p \in A_i'$. Hence p is a limit point of A_i, which implies there is a $q \in N(p)$ st. $q \in A_i$, $q \neq p$

Hence $q \in \bigcup_{i=1}^{n} A_i$ or $q \in B_n$.

This proves that $p \neq q \in B_n$ is a limit point of B_n, or $p \in B_n'$

Thus $\bigcup_{i=1}^{n} A_i' \subseteq B_n'$ (3)

(2)(3), we can conclude $B_n' = \bigcup_{i=1}^{n} A_i'$, together with (1)

$$\overline{B_n} = B_n \cup B_n' = \bigcup_{i=1}^{n} A_i \cup \bigcup_{i=1}^{n} A_i' = \bigcup_{i=1}^{n} \overline{A_i}$$

b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$

The proof is similar, using (3) from part A, then

$$B'_{\infty} \supseteq \bigcup_{i=1}^{\infty} A_i'$$

$$B_{\infty} \cup B'_{\infty} \supseteq B_{\infty} \cup \bigcup_{i=1}^{\infty} A_i'$$

$$\overline{B_{\infty}} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$$

This is ~~proper~~ in case $A_i = \{1/a; a \in \mathbb{N}, a > 1\}$ (same as #6)

$B = \bigcup_{a=1}^{\infty} \{1/a\}$ and $\overline{B} = \bigcup_{a=1}^{\infty} \{1/a\} \cup \{0\}$

$\bigcup_{i=1}^{\infty} \overline{A_i} = \{1/a : a \in \mathbb{N}, a > 1\}$