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office Hour: 11:00 AM - 12:00 PM on Tuesday.

Def) (Propositional Logic)

A proposition is a collection of statements that is either true or false.

e.g) "$12+9=3-2$" is false proposition.

"192 is large number" is not proposition

Connectives)

Five Connectives:

- or ($\vee$)
- And ($\wedge$)
- Negation / Not ($\neg$)
- Implication / if-then ($\rightarrow$) (when this is true, we use "$\Rightarrow$" instead of "$\rightarrow$")
- If and only if ($\leftrightarrow$) ($\Leftrightarrow$)

(First)
Mathematical Induction is a Mathematical proof technique.

(It is used to prove that a statement $P(n)$ holds for every $n \in \mathbb{N}$)

Let $P(n)$ be a statement for $n \in \mathbb{N}$.

① Show that $P(1)$ is true. (Base case)

② Suppose that for $k \in \mathbb{N}$, $P(k)$ is true.
Prove that $P(k+1)$ is true.

Then, the Mathematical Induction gives that

for every

$P(n)$ holds $\forall n \in \mathbb{N}$.

e.g) Prove that $1+2+\cdots+n=\frac{n(n+1)}{2}$, $\forall n\in\mathbb{N}$.

pf) Step 1) If $n=1$, $(\text{L.H})=1$

$$(\text{R.H})=\frac{1\cdot 2}{2}=1$$

It holds for $n=1$.

Step 2) Suppose that for $k\in\mathbb{N}$, $1+2+\cdots+k=\frac{k(k+1)}{2}$.

Need to show that $1+2+\cdots+k+1=\frac{(k+1)(k+2)}{2}$.

$$1+2+\cdots+k+1=\underbrace{(1+2+\cdots+k)}^{=\frac{k(k+1)}{2}}+k+1$$

$$=\frac{k(k+1)}{2}+k+1 \quad \text{because of } (\ast) \text{ assumption}$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}.$$

$$1+2+3+\cdots+k+1 = \frac{(k+1)(k+2)}{2}.$$

By the Mathematical Induction,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N} \quad \blacksquare$$

Strong Induction (Second Mathematical Induction)

Let $P(n)$ be a Statement, for $n \in \mathbb{N}$.

① Show that $P(1)$ is true.

② Suppose that for $k \in \mathbb{N}$, $P(1), P(2), \cdots, P(k)$ are true.

Prove that $P(k+1)$ is true.

Then, the Strong Induction gives that $P(n)$ holds, $\forall n \in \mathbb{N}$.

e.g) Show that every natural number $n > 1$ is the product of prime powers.

pf) Step 1) 2 is already prime. (Base case)

It holds for $n=2$.

Step 2) Suppose that for $k \in \mathbb{N}$, $2, 3, 4, \cdots, k$ are the product of prime powers.

Need to show

N.T.S) $k+1$ is the product of prime powers.

(Case 1) $k+1$ is already prime number.

Done!

(Case 2) $k+1$ is composite.

Then, $k+1 = a \cdot b$ for some $1 < a, b < k+1$

By the assumption, a, b are the product of prime powers.

This implies that $a \cdot b$ is the product of prime powers

implies

$\Rightarrow$ $k+1$ is the product of prime powers

By strong induction, every natural number $n > 1$ is the product of prime numbers $\boxtimes$

Well-ordering principle (axiom)

Every non-empty set of positive integers contains a least element.

Proof by Contradiction

(Mathematical proof technique)

Suppose that the proposition is false.

If this leads a contradiction, then the proposition is true.

e.g.) Show that there are infinitely many prime numbers.

pf) Suppose that there are finitely many prime numbers.

Let $P = \{p_1, p_2, \cdots, p_n\}$ be a collection of prime numbers.

Consider $q = (p_1 \cdot p_2 \cdots p_n) + 1$.

Since $q \notin P$, q is not prime number.

This implies that there exists $p' \in P$

such that p' divides q.

Since $p' \in P$, p' divides $p_1 \cdot p_2 \cdots p_n$

So, p' divides q and $p_1 \cdot p_2 \cdots p_n$

Then, p' divides $q - (p_1 \cdot p_2 \cdot p_3 \cdots p_{n-1} \cdot p_n)$

$$\begin{gathered} q - (p_1 \cdot p_2 \cdot p_3 \cdots p_{n-1} \cdot p_n) \\ \| \\ \cancel{(p_1 \cdot p_2 \cdot p_3 \cdots p_{n-1} \cdot p_n)} + 1 - \cancel{(p_1 \cdot p_2 \cdot p_3 \cdots p_{n-1} \cdot p_n)} \\ \| \\ 1 \end{gathered}$$

$\Rightarrow p'$ divides 1

$\Rightarrow p' = 1$ $(\ast)$ $\rightarrow$ Contradiction

Therefore, there are infinitely many prime numbers. $\square$

Set operations.

Let A, B be sets.

$a \in A \iff a$ is an element of A

$A \subseteq B \iff a \in A \Rightarrow a \in B$

Subset

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$

Union

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$.

Intersection