Def) $\mathbb{Q} = \{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \}$ is the rational numbers.

Any number which is not rational is irrational.

1. Is $\sqrt{4}$ rational? Is $\sqrt{5}$ rational?

i) $\sqrt{4} = 2$ is rational with $p=2, q=1$.

ii) Claim) $\sqrt{5}$ is irrational.

($\because$) Suppose that $\sqrt{5}$ is rational number.

Then, $\exists\, p, q \in \mathbb{Z}, q \neq 0$ s.t $\sqrt{5} = \frac{p}{q}$
$\gcd(p,q) = 1$

$$\Rightarrow q\sqrt{5} = p$$

$$\Rightarrow (q\sqrt{5})^2 = p^2$$

$$\Rightarrow q^2 \cdot 5 = p^2$$

$$\Rightarrow 5q^2 = p^2$$

If p is even number, then by $5q^2=p^2$

q is also even

Then $\gcd(p,q)\neq 1$ $(\times)$

Similarly, if q is even, then p is also even.

Then $\gcd(p,q)\neq 1$ $(\times)$

Thus, p, q are odd numbers.

Let $p=2n+1$, $q=2m+1$ for some $n,m\in\mathbb{Z}$.

Since $5q^2=p^2$,

$$5(2m+1)^2=(2n+1)^2$$

$$\Rightarrow 5(4m^2+4m+1)=(4n^2+4n+1)$$

$$\Rightarrow 20m^2+20m+5=4n^2+4n+1$$

$\Rightarrow 20m^2 + 20m + 4 = 4n^2 + 4n$

$\Rightarrow 5m^2 + 5m + 1 = n^2 + n$

$\Rightarrow 5m(m+1) + 1 = n(n+1)$

$n(n+1)$ is always even number

similarly $m(m+1)$ is even number

$\Rightarrow 5m(m+1)$ is even

$\Rightarrow 5m(m+1)+1$ is odd

This contradicts to $5m(m+1)+1 = n(n+1)$

Therefore, $\sqrt{5}$ is irrational $\blacksquare$

2. (a) State the definition of the **greatest lower bound** of a set.

(b) Give an example of a set with all of the following properties or state that it's impossible: bounded above, isn't bounded below, and doesn't contain its supremum.

(c) Give an example of a set with all of the following properties or state that it's impossible: is not bounded above, is bounded below, and does contain its supremum.

(a) Let S be an ordered set and $E \subseteq S$

be a bounded below set.

Then $\alpha \in S$ is the greatest lower bound

of E

if ① α is a lower bound of E

② If $y > \alpha$, then y is not a lower bound

of E

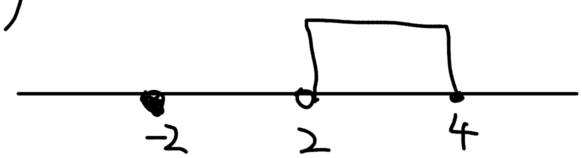
(There is no lower bound that is greater than α)

We call the greatest lower bound the infimum of E, $\alpha = \inf E$.

e.g) $\inf (0,1) = 0$

$\inf [0,1) = 0$

$\inf (\{-2\} \cup (2,4]) = -2$



(b) Give an example of a set with all of the following properties or state that it's impossible: bounded above, isn't bounded below, and doesn't contain its supremum.

The least upperbound

Let $E = (-\infty, 2) \subseteq \mathbb{R}$.

Then E is not bounded below, but is bouned above.

And $\sup E = 2 \notin E$.

(c) Give an example of a set with all of the following properties or state that it's impossible: is not bounded above, is bounded below, and does contain its supremum.

So) It's impossible

$(\because)$ If E is not bounded above,

then $\sup E = +\infty$.

$\Rightarrow +\infty \notin E$.

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$. $(\inf B \not> \sup B)$

(b) A finite set that contains its infimum but not its supremum. (impossible)

(c) A bounded subset of **Q** that contains its supremum but not its infimum.

(a) Let $B = \{1\}$. Then $\inf B = 1 = \sup B$.

(b) If B is a finite set,

then $B = \{b_1, b_2, \cdots, b_m\}$.

$\inf B = \min B$, $\sup B = \max B$

(c) Let $B = \{x \in \mathbb{Q} : 2 \leq x^2 \leq 4,\ x > 0\} \subseteq \mathbb{R}$

$(\Leftrightarrow \sqrt{2} \leq x \leq 2)$

$\inf B = \sqrt{2} \notin B$, $\sup B = 2 \in B$

4. Show that $\sqrt{28}$ is irrational number.

Pf) Suppose that $\sqrt{28}$ is rational number.

Then, $\exists\, p, q \in \mathbb{Z},\ q \neq 0$ s.t $\sqrt{28} = \dfrac{p}{q}$

$\gcd(p, q) = 1$

$$\Rightarrow q\sqrt{28} = p$$

$$\Rightarrow (q\sqrt{28})^2 = p^2$$

$$\Rightarrow 28q^2 = p^2$$

$\Rightarrow$ p is even, q is odd.

Let $p = 2n$, $q = 2m+1$ for some $n, m \in \mathbb{Z}$

Since $28q^2 = p^2$,

$$28(2m+1)^2 = (2n)^2 \Rightarrow \cancel{28}\,(2m+1)^2 = \cancel{4}n^2$$

$$\Rightarrow 7(2m+1)^2 = n^2$$

7 divides n. Let $n = 7l$.

Then, $7(2m+1)^2 = (7l)^2$

$$\Rightarrow \cancel{7}(2m+1)^2 = \cancel{49}l^2$$

$$\Rightarrow (2m+1)^2 = 7l^2$$

7 divides $(2m+1)^2 \Rightarrow$ 7 divides $\overset{q}{\overset{\shortparallel}{2m+1}}$

7 divides q, $p \Rightarrow \gcd(p,q) \neq 1$ $(\divideontimes)$.

Therefore, $\sqrt{28}$ is irrational $\blacksquare$