Last time …

Last time …numerical Integration

If choose a_i, x_i carefully ; expect that we should be able to integrate exactly a polynomial of degree $2N-1$

- Example: Consider case of 2 sample points

$$I = a_1 f(x_1) + a_2 f(x_2)$$

4 parameters ; expect to be able to integrate a cubic or lower exactly.

Recall: Trapezoidal rule only exact for linear

i.e. let
$$\left.\begin{aligned} f(x) &= 1 \\ &= x \\ &= x^2 \\ &= x^3 \end{aligned}\right\}$$
Integrate and solve 4 equations in 4 unknowns

Last time …numerical Integration

$$\int_{-1}^{1} dx = 2 = a_1 + a_2$$

$$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 = a_1 x_1 + a_2 x_2$$

$$\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = 2/3 = a_1 x_1^2 + a_2 x_2^2$$

$$\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0 = a_1 x_1^3 + a_2 x_2^3$$

$$I = a_1 f(x_1) + a_2 f(x_2)$$

$$I = a_1 \cdot 1 + a_2\, 1$$

$$I = a_1 \cdot x + a_2\, x$$

$$I = a_1 \cdot x^2 + a_2\, x^2$$

$$I = a_1 \cdot x^3 + a_2\, x^3$$

4 nonlinear equations in 4 unknowns… solve w/ Newton for systems (or by hand in this case)

Last time …numerical Integration

solution: $a_1 = a_2 = 1$; $-x_1 = x_2 = \sqrt{1/3}$

x_i's are called Gauss points

a_i's are called the weights

- can generalize to an N-pt formula : Exact for polynomials degree $2N-1$
- turns out that x_i's (Gauss Pts) are roots on Legendre Polynomials on $[-1, 1]$ with weights : $\int_{-1}^{1} L_{N,i}(x)\,dx = \omega_i$

N^{th} order Lagrange centered at each x_i

- Legendre polynomial is orthogonal set on $[-1, 1]$ $\Rightarrow$ key is these have a weight of unity

Last time …numerical Integration

- Fortunately; Gauss pts and weights are tabulated

- However, Gauss pts assume $\int_{-1}^{1} f(x)\,dx$ ∴ Need to transform general integral $\int_a^b f(y)\,dy$

let $y = \frac{a+b}{2} + \frac{b-a}{2} x$; $dy = \frac{b-a}{2} dx$

actual position on $[a,b]$

Gauss pt

If $x = -1 \Rightarrow y = a$

If $x = 1 \Rightarrow y = b$

so $$\int_a^b f(y)\,dy = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2} x\right) dx$$

Last time …numerical Integration

so
$$\int_a^b f(y)\,dy = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}x\right)dx$$

$$= \frac{b-a}{2}\sum_{i=1}^{N} w_i f(y_i)$$

where
$$y_i = \frac{a+b}{2} + \frac{b-a}{2}x_i$$

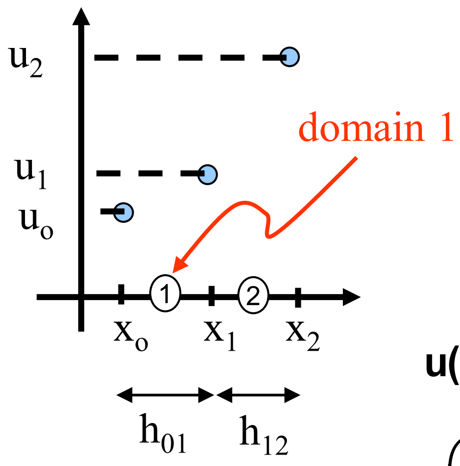
- See text for elegant proof of Gaussian Quadrature

→ Numerical Analysis by Burden and Faires

→ This form is often referred to as Gauss-Legendre Integration

Last time …

Last time … let's reinterpret integration!



Recall…

… so now I can express $\mathbf{u_i}$ anywhere in the domain of $\mathbf{0 \leq x_i \leq 1}$ using our Lagrange basis…

$$u(x) = \sum_{i=0}^{1} u_i L_{1,i}(x) = u_o L_{1,0}(x) + u_1 L_{1,1}(x)$$

$$u\left(x = x_o + \frac{h_{01}}{3}\right) = u_o L_{1,0}\left(x_o + \frac{h_{01}}{3}\right) + u_1 L_{1,1}\left(x_o + \frac{h_{01}}{3}\right)$$

$$u\left(x = x_o + \frac{h_{01}}{3}\right) = u_o \frac{\left(x_1 - \left(x_o + \frac{h_{01}}{3}\right)\right)}{h_{01}} + u_1 \frac{\left(x_o + \left(\frac{h_{01}}{3} - x_o\right)\right)}{h_{01}}$$

$$u\left(x = x_o + \frac{h_{01}}{3}\right) = \frac{2}{3} u_o + \frac{1}{3} u_1$$

Gives interpolated value in domain

Last time … FEM Spine - Numerical Integration

Strategy: $$\int_a^b f(x)dx \approx \sum_{i=0}^{N} a_i f(x_i)$$ i.e. a weighted sum of function evaluations

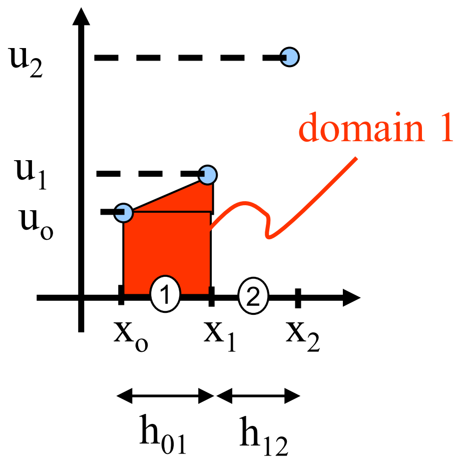
Like Calc: $$\int_a^b f(x)dx = \mathrm{Lim}_{\Delta x \to 0} \sum_{i=0}^{N} f(x_i)\Delta x_i \quad \text{but...}$$

Don't take limit: $$\int_a^b f(x)dx = \sum_{i=0}^{N} f(x_i)\Delta x_i \quad + \text{Error}$$

Could expand f(x) in a Lagrange basis:

$$f(x) = \sum_{i=0}^{N} f_i L_{N,i}(x) + \text{Error}$$

Last time … FEM Spine - Numerical Integration



$$\int_a^b f(x)dx = \int_a^b \sum_{i=0}^{N} f_i L_{N,i}(x) + \int_a^b E_{trunc}$$

$$\int_a^b f(x)dx = \sum_{i=0}^{N} f_i \underbrace{\int_a^b L_{N,i}(x)}_{a_i} + \text{error}$$

$$\int_a^b f(x)dx = f_o \int_a^b L_{1,0}(x)dx + f_1 \int_a^b L_{1,1}(x)dx$$

$$\int_a^b f(x)dx = f_o \int_{x_o}^{x_1} \frac{(x - x_1)}{(x_o - x_1)}dx + f_1 \int_{x_o}^{x_1} \frac{(x - x_o)}{(x_1 - x_o)}dx$$

$$\int_a^b f(x)dx = \frac{x_1 - x_o}{2}(f_o + f_1) + \text{error}$$

Trapezoidal rule!

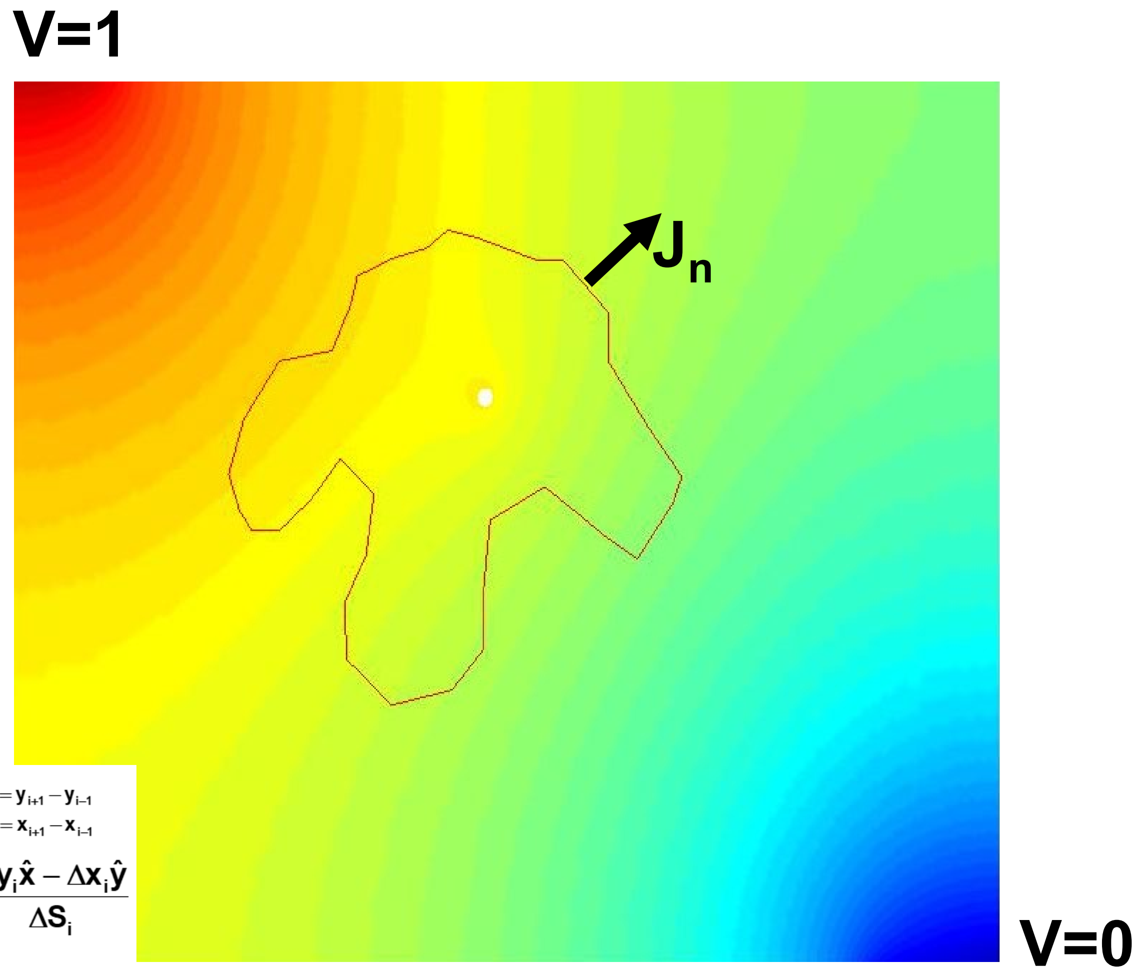
Last time …Polynomial Basis/Weighting Functions

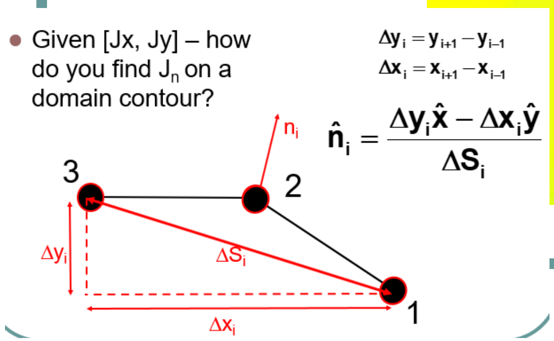
- Key point: When an integral has functions in the integrand represented by a basis expansion, the constant coefficients can slip outside the integral, and integration is applied to the user-prescribed basis function, in this case the Lagrange polynomial.
- Result: Integration becomes simple – it is easy to integrate the inter-nodal behavior function if it's just a polynomial!

Last time … now spatially…

Last time …numerical Integration

What if?





Last time …numerical integration

$$\iint_S F \bullet \hat{n} dS = \iiint_V \nabla \bullet F dV$$

Divergence Theorem

- Theorem connects surface integrals and volume integrals.. very powerful
- Although some may think it has its roots in electrostatics, it is independent of physics and is purely a mathematical concept

Last time … numerical integration

$$\iint_S J \bullet \hat{n}\, dS = \iiint_V \nabla \bullet J$$

So integrating this around a domain ……… tells me something about the content of that domain

Last time… numerical integration

How would I do this?

$$\iint_S J \bullet \hat{n} dS$$

- Given vectors ($[J_x + J_y]$) about a contour, how does one calculate in this case a line integral?
- Like everything, we want to break things down to a series of function evaluations that sum to approximate the value of the integral.

Last time…numerical integration

- Let's expand **J_n** as a series of function evaluations

$$J_n = \sum_{j=1}^{n} c_j \varphi(s)_j$$

- Alternatively, we could think of this as expanding the function in a linear basis with a function of a known polynomial form

Last time… numerical integration

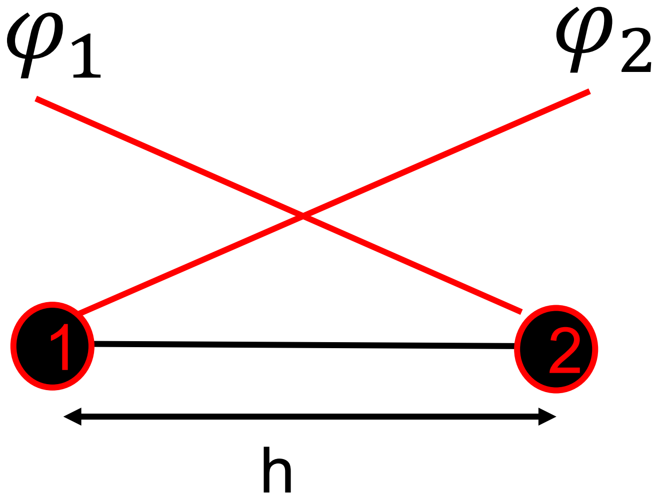
- How to select function?

$$J_n = \sum_{j=1}^{n} c_j \varphi(s)_j$$

- Choose one … some may be more straight-forward than others. If I choose a Lagrange Polynomial, it has an interesting behavior … it transforms the expansion

$$J_n = \sum_{j=1}^{n} J_{n_j} \varphi(s)_j$$

Last time… numerical integration



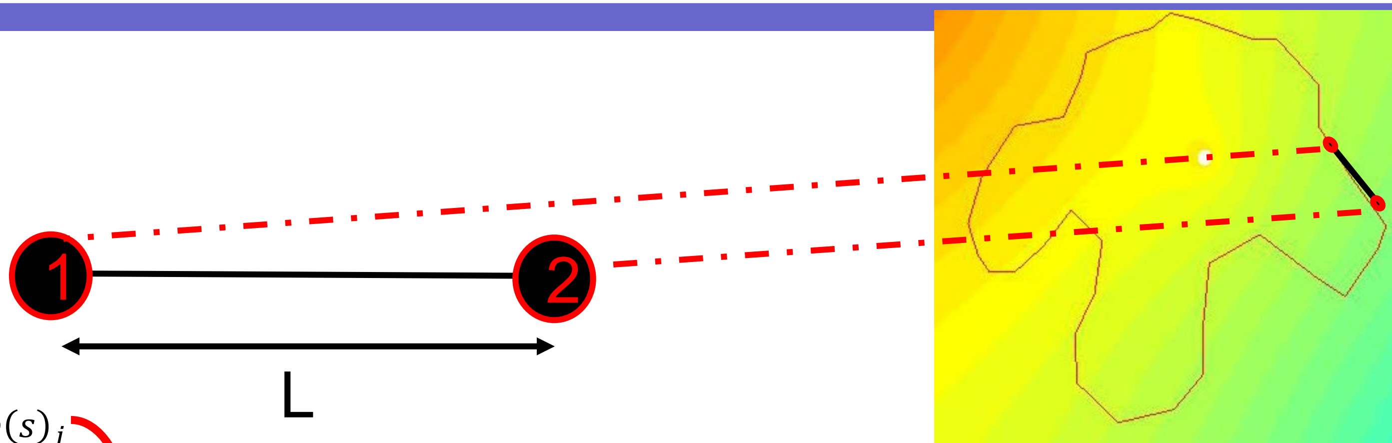
- Rooftop or chapeau function

$$\varphi_1 = \frac{x_2 - x}{h}, \varphi_2 = \frac{x - x_1}{h}$$

$$\int_{x_1}^{x_2} \frac{(x_2 - x)}{h} dx = \frac{x_2^2 - 2x_2x_1 + x_1^2}{2h} = \frac{(x_2 - x_1)^2}{2h}$$

$$\int_{x_1}^{x_2} \frac{(x_2 - x)}{h} dx = \frac{h}{2}$$

Last time… numerical integration



$$J_n = \sum_{j=1}^{n} J_{n_j}\varphi(s)_j$$

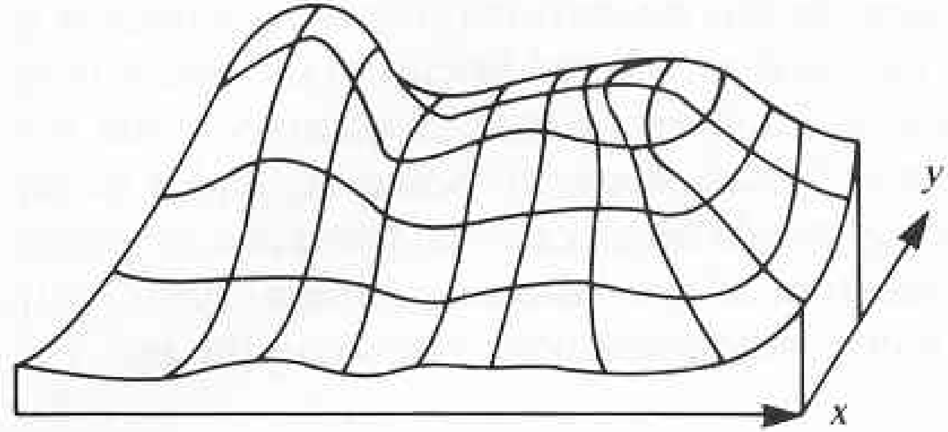
$$\iint_S J \bullet \hat{n}\, dS \overset{\text{3D}-\text{2D rep}}{\Rightarrow} \int_\ell J_n d\ell = J_{n1}\int_1^2 \varphi_1(\ell)d\ell + J_{n2}\int_1^2 \varphi_2(\ell)d\ell$$

$$\iint_S J \bullet \hat{n}\, dS \overset{\text{3D}-\text{2D rep}}{\Rightarrow} \int_\ell J_n d\ell = J_{n1}\frac{L}{2} + J_{n2}\frac{L}{2}$$

This represents the integration of $\mathbf{J_n}$ over this length. Just need to go length by length and sum up!

INTRODUCTION TO PDEs

Introduction



- Consider a real valued function

$$\mathbf{z} = \mathbf{f}(\mathbf{x}, \mathbf{y})$$

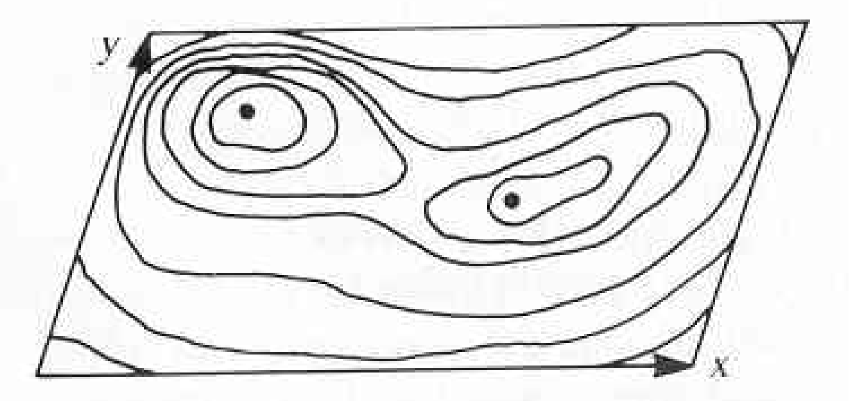
The function f will generally represent some spatially varying quantity…

e.g. -density of population at (x,y)

-concentration of substance at (x,y)

-temperature at (x,y)

Introduction (cont.)



- To visualize, think of function evaluations representing the height above a plane, i.e. R^3
- Regions of the functional space that have

$$\mathbf{z} = \mathbf{f}(\mathbf{x}, \mathbf{y}) = \text{constant}$$

represent contours of constant value

-e.g. isotherms - constant temperautres

Introduction (cont.)

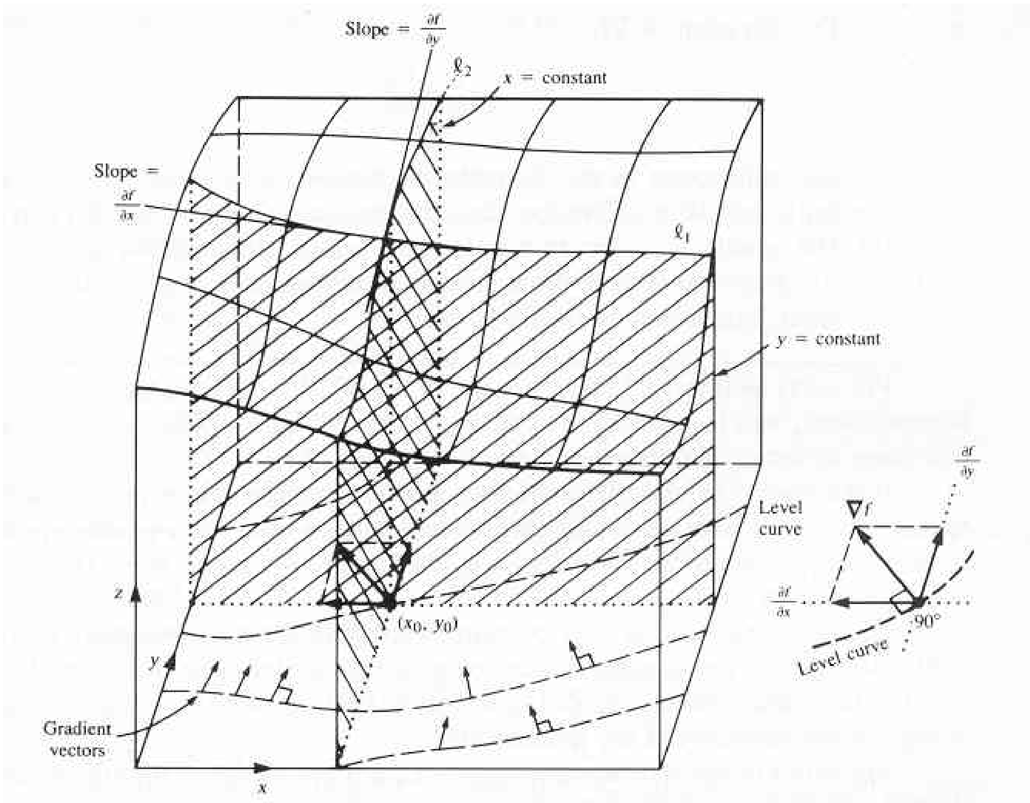
- For a spatially dependent function:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \lim_{\Delta \mathbf{x} \to 0} \frac{\mathbf{f}(\mathbf{x} + \Delta \mathbf{x}, \mathbf{y}) - \mathbf{f}(\mathbf{x}, \mathbf{y})}{\Delta \mathbf{x}}$$

- Similar expressions can be established for *x, y, z,* and *t*
  - Shorthand: >>> $\mathbf{f}_\mathbf{x} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \mathbf{f}_{\mathbf{yy}} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{y}^2}$
  - Can have mixed derivatives also

$$\mathbf{f}_{\mathbf{yx}} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right), \mathbf{f}_{\mathbf{yx}} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) = \mathbf{f}_{\mathbf{xy}}$$

Introduction (cont.)



- Physical meaning of partial derivative

Classification of PDEs

- Definitions:
  - ODE - differential equation with one independent variable
  - PDE - differential equation with more than one independent variable
  - Order - determined by highest derivative that appears
  - Degree - determined by the power of highest derivative
  - Homogeneous differential equation - no term involving only independent variables or constants

Classification (cont.)

- Linear - dependent variable and derivatives (thereof) appear only to 1st (or zero) power and no products of dependent variables and its derivatives are present
- Quasilinear - highest-order derivative appears linearly with respect to itself
- Nonlinear - anything else

Example...

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Conservation of what? Mass

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Conservation of what?

Momentum

What are these equations?

Navier-Stokes

ODE/PDE? PDE

Linear, quasi-linear, nonlinear? Quasi-linear

Order? 2nd order

Degree? 1st degree

Heterogeneous or homogeneous? homogeneous

Tensors

- Zero Order Tensor
  - Specified with magnitude only

Scalar → e.g. ????

- 1st Order Tensor
  - Specified with magnitude and direction

Vector → e.g. ????

- 2nd Order Tensor
  - Specified with magnitude, direction, and directional reference

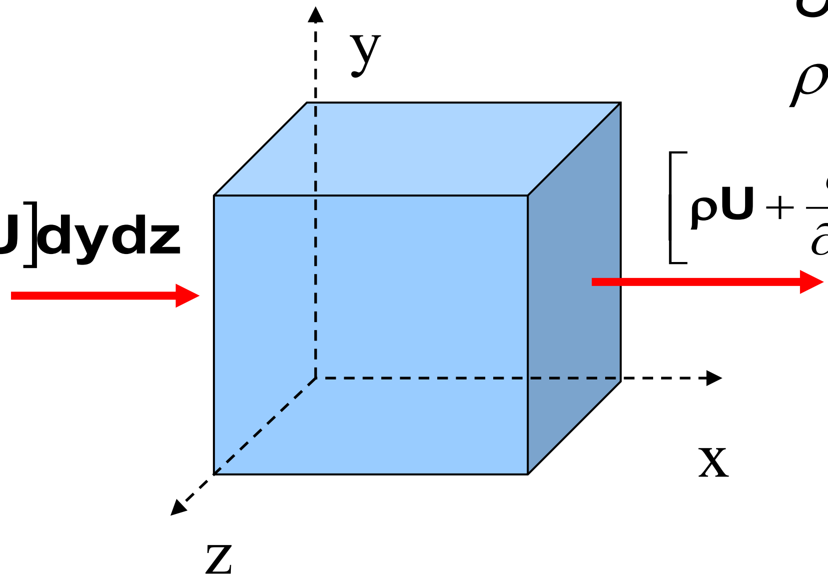
Tensor → e.g. ????

Conservation

U=velocity in x direction

ρ=density

$$[\rho U]dydz$$



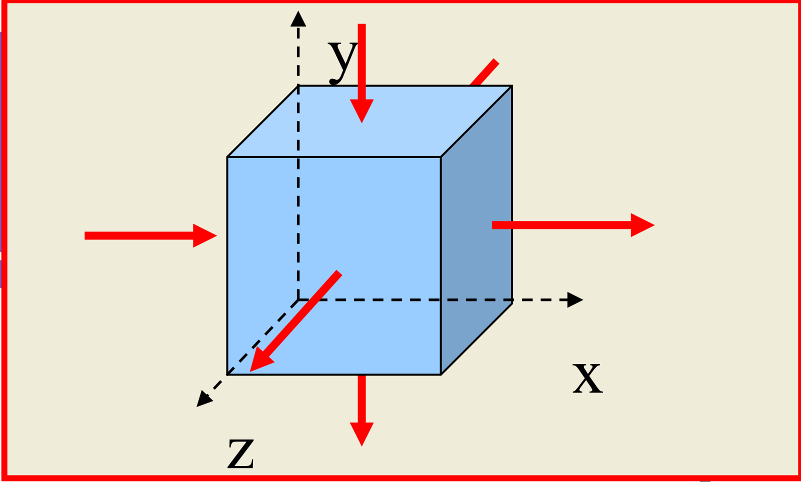
$$\left[\rho U + \frac{\partial}{\partial x}(\rho U)dx\right]dydz$$

$$Mass_{in} - Mass_{out} = \Delta M_x$$

$$[\rho U]dydz - \left[\rho U + \frac{\partial}{\partial x}(\rho U)dx\right]dydz = \Delta M_x$$

$$-\left[\frac{\partial}{\partial x}(\rho U)dx\right]dydz = \Delta M_x$$

Conservation (cont.)



$$\mathbf{Mass}(\mathbf{x},\mathbf{y},\mathbf{z})_{\mathbf{in}} - \mathbf{Mass}(\mathbf{x},\mathbf{y},\mathbf{z})_{\mathbf{out}} = \Delta\mathbf{M}_{\mathbf{x}} + \Delta\mathbf{M}_{\mathbf{y}} + \Delta\mathbf{M}_{\mathbf{z}}$$

$$-\left[\frac{\partial}{\partial \mathbf{x}}(\rho\mathbf{U})\mathbf{dx}\right]\mathbf{dydz}$$

$$+$$

$$-\left[\frac{\partial}{\partial \mathbf{y}}(\rho\mathbf{V})\mathbf{dy}\right]\mathbf{dxdz} = \Delta\mathbf{M}_{\mathbf{x}} + \Delta\mathbf{M}_{\mathbf{y}} + \Delta\mathbf{M}_{\mathbf{z}} = 0$$

$$+$$

$$-\left[\frac{\partial}{\partial \mathbf{z}}(\rho\mathbf{W})\mathbf{dz}\right]\mathbf{dxdy}$$

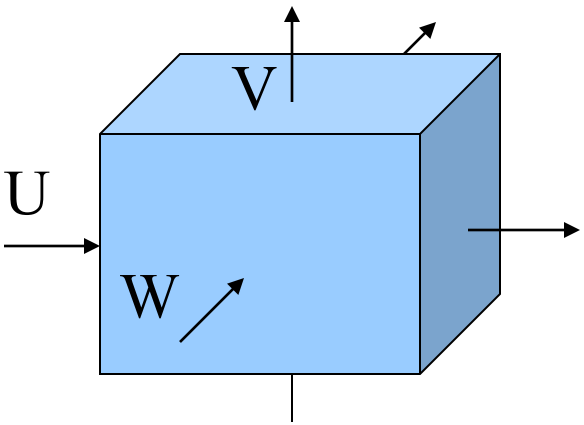
Volume, *dxdydz,* cancels

$$\frac{\partial}{\partial \mathbf{x}}(\rho\mathbf{U}) + \frac{\partial}{\partial \mathbf{y}}(\rho\mathbf{V}) + \frac{\partial}{\partial \mathbf{z}}(\rho\mathbf{W}) = 0$$

Divergence Operator

$$\frac{\partial}{\partial \mathbf{x}}(\rho \mathbf{U}) + \frac{\partial}{\partial \mathbf{y}}(\rho \mathbf{V}) + \frac{\partial}{\partial \mathbf{z}}(\rho \mathbf{W}) = 0$$

$$\nabla \bullet \left(\rho \vec{\mathbf{V}}\right) = 0$$

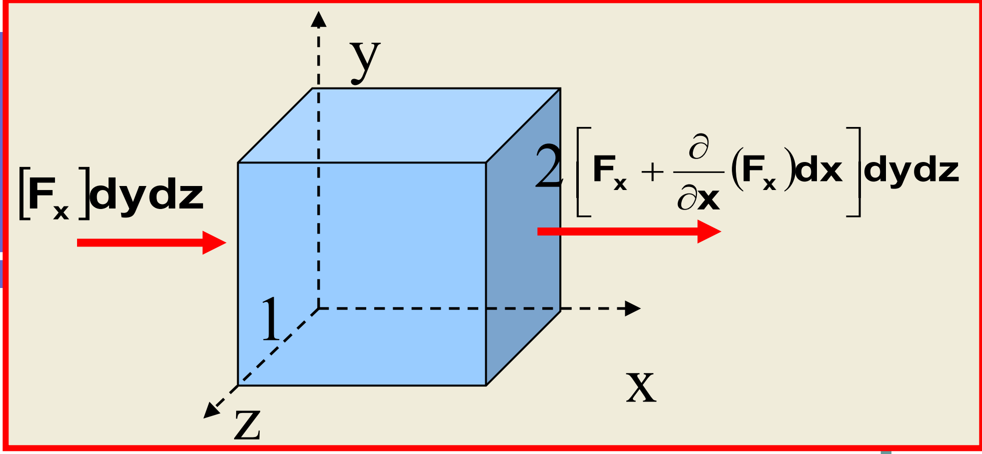


If density is constant …

$$\nabla \bullet \vec{\mathbf{V}} = \frac{\partial \mathbf{V_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{V_y}}{\partial \mathbf{y}} + \frac{\partial \mathbf{V_z}}{\partial \mathbf{z}} = 0$$

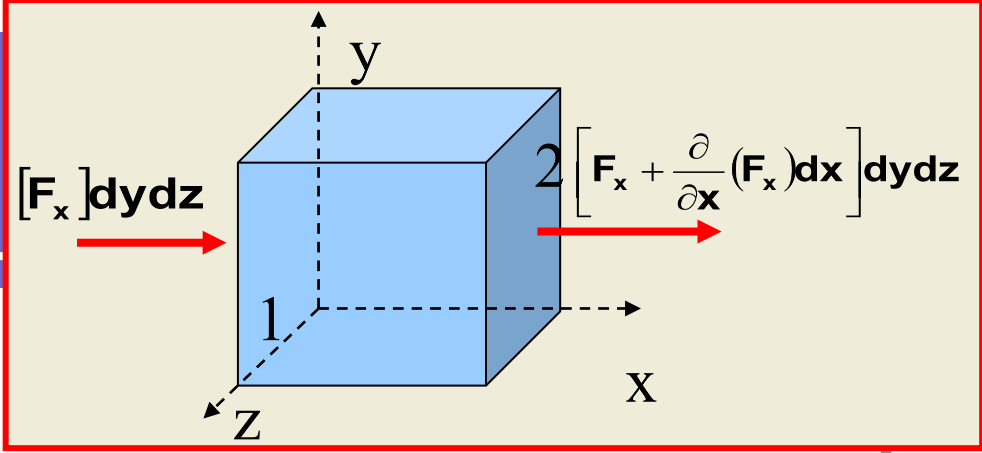
- Operates on vector/tensor and is principally used in *conservation* statements within physics

Surface to Volume Relationship



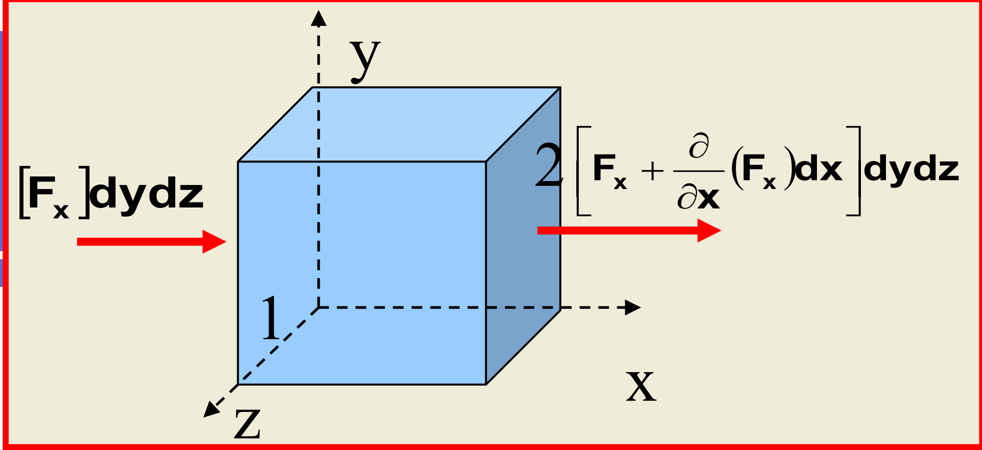
$$\frac{1}{\Delta V}\iint_S \mathbf{F} \bullet \hat{\mathbf{n}} dS =$$

Surface to Volume Relationship



$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}d\mathbf{S} = \frac{1}{dxdydz}(F_x\bullet\hat{n}_1)(dS_1) + \frac{1}{dxdydz}\left(\left(F_x + \frac{\partial}{\partial x}(F_x)dx\right)\bullet\hat{n}_2\right)(dS_2)$$

Surface to Volume Relationship

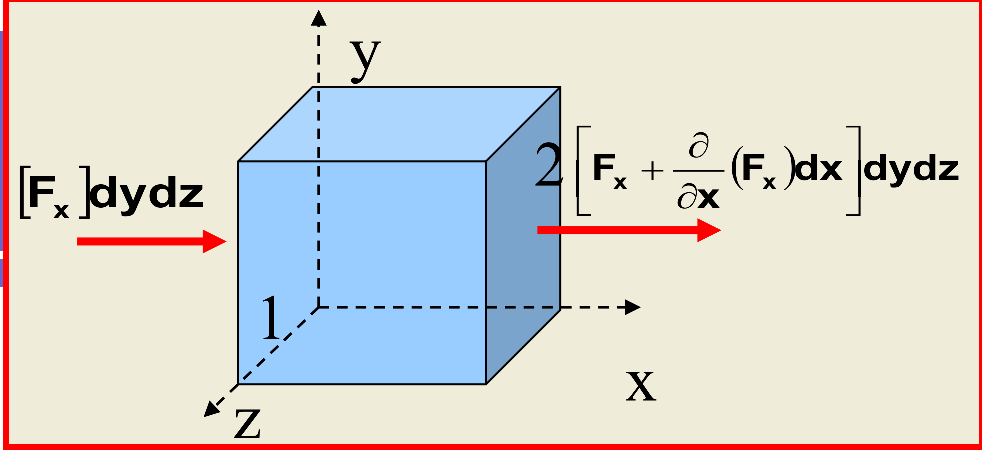


$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}dS = \frac{1}{dxdydz}\left(F_x \bullet \hat{n}_1\right)\left(dS_1\right) + \frac{1}{dxdydz}\left(\left(F_x + \frac{\partial}{\partial x}(F_x)dx\right)\bullet \hat{n}_2\right)\left(dS_2\right)$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}dS = \frac{1}{dxdydz}\left(F_x \bullet (-\hat{x})\right)(dydz) +$$

$$\frac{1}{dxdydz}\left(F_x \bullet (\hat{x}) + \frac{\partial}{\partial x}\left(F_x \bullet (\hat{x})\right)dx\right)(dydz)$$

Surface to Volume Relationship



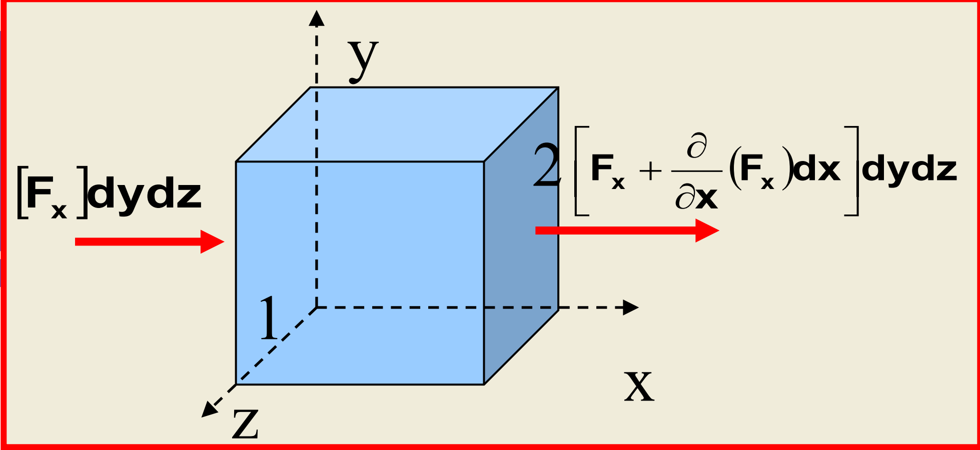
$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}dS = \frac{1}{dxdydz}(F_x \bullet \hat{n}_1)(dS_1) + \frac{1}{dxdydz}\left(\left(F_x + \frac{\partial}{\partial x}(F_x)dx\right)\bullet\hat{n}_2\right)(dS_2)$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}dS = \frac{1}{dxdydz}(F_x \bullet (-\hat{x}))(dydz) +$$

$$\frac{1}{dxdydz}\left(F_x \bullet (\hat{x}) + \frac{\partial}{\partial x}(F_x \bullet (\hat{x}))dx\right)(dydz)$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}dS = \frac{1}{dxdydz}\left(\frac{\partial}{\partial x}F_x dx\right)(dydz)\ldots\ldots \xrightarrow{\text{multi-dim}} \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Surface to Volume Relationship



$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}\,d\mathbf{S} = \frac{1}{dxdydz}\left(\mathbf{F_x}\bullet\hat{\mathbf{n}}_1\right)\left(d\mathbf{S}_1\right) + \frac{1}{dxdydz}\left(\left(\mathbf{F_x} + \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{F_x}\right)dx\right)\bullet\hat{\mathbf{n}}_2\right)\left(d\mathbf{S}_2\right)$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}\,d\mathbf{S} = \frac{1}{dxdydz}\left(\mathbf{F_x}\bullet(-\hat{\mathbf{x}})\right)\left(dydz\right) +$$

$$\frac{1}{dxdydz}\left(\mathbf{F_x}\bullet(\hat{\mathbf{x}}) + \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{F_x}\bullet(\hat{\mathbf{x}})\right)dx\right)\left(dydz\right)$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}\,d\mathbf{S} = \frac{1}{dxdydz}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{F_x}dx\right)\left(dydz\right)\ldots\ldots \xrightarrow{\text{multi-dim}} \frac{\partial \mathbf{F_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F_y}}{\partial \mathbf{y}} + \frac{\partial \mathbf{F_z}}{\partial \mathbf{z}}$$

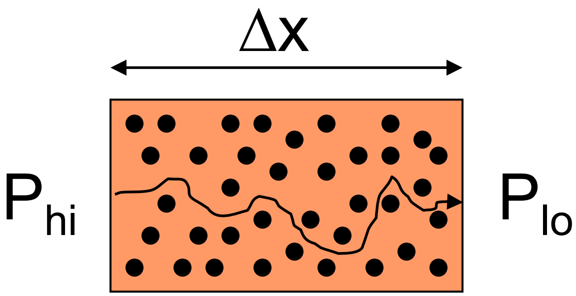
$$\lim_{\Delta V \to 0}\frac{1}{\Delta V}\iint_S \mathbf{F}\bullet\hat{\mathbf{n}}\,d\mathbf{S} = \nabla\bullet\mathbf{F}$$

Divergence Theorem

$$\iint_S \mathbf{F} \bullet \hat{\mathbf{n}} d\mathbf{S} = \iiint_V \nabla \bullet \mathbf{F} dV$$ Divergence Theorem

- Theorem connects surface integrals and volume integrals.. very powerful
- Although some may think it has its roots in electrostatics, it is independent of physics and is purely a mathematical concept DivF represent the flix thru a surface of vector field F

Gradient Operator



What if ???

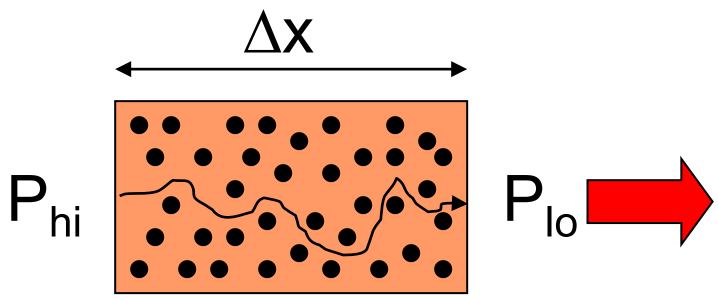
$$V_x = -k \frac{dp}{dx} \Rightarrow p = f(x,y,z)$$

Velocity ….
a vector, right !!

$$\left\{ \begin{aligned} V_x &= -k \frac{\partial p}{\partial x} \\ V_y &= -k \frac{\partial p}{\partial y} \\ V_z &= -k \frac{\partial p}{\partial z} \end{aligned} \right. \Rightarrow \vec{V} = -k \nabla p$$

What if I dotted this into the unit vector normal to a surface, i.e. perpendicular to the surface?

Porous-Media Example



$$\vec{\mathbf{V}} = -\mathbf{k}\nabla\mathbf{p} \quad \text{where} \quad \mathbf{p} = \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})$$

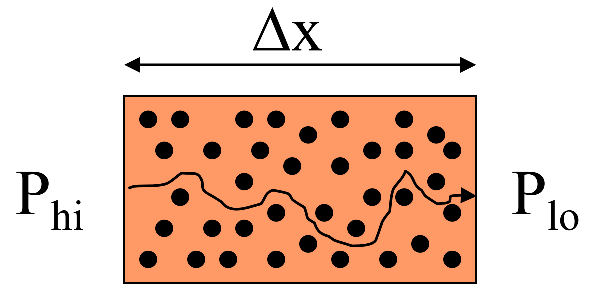
•What if I wanted to conserve mass within an infinitesimal control volume

Recall…

$$\nabla \bullet \vec{\mathbf{V}} = \frac{\partial \mathbf{V_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{V_y}}{\partial \mathbf{y}} + \frac{\partial \mathbf{V_z}}{\partial \mathbf{z}} = 0$$

$$\nabla \bullet \vec{\mathbf{V}} = \nabla \bullet \left(-\mathbf{k}\nabla\mathbf{p}\right) = 0$$

Porous-Media Example



$$\nabla \bullet \vec{\mathbf{V}} = \nabla \bullet \left(-\mathbf{k} \nabla \mathbf{p} \right) = 0$$

What am I assuming?

$$\frac{\partial}{\partial \mathbf{x}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{x}} \right) + \frac{\partial}{\partial \mathbf{y}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{y}} \right) + \frac{\partial}{\partial \mathbf{z}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{z}} \right) = 0$$

Assume homogeneous constant hydraulic conductivity…

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{x}} \right) + \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{y}} \right) + \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{z}} \right) = 0$$

$$\frac{\partial^2 \mathbf{p}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{p}}{\partial \mathbf{y}^2} + \frac{\partial^2 \mathbf{p}}{\partial \mathbf{z}^2} = 0$$

Laplacian Operator

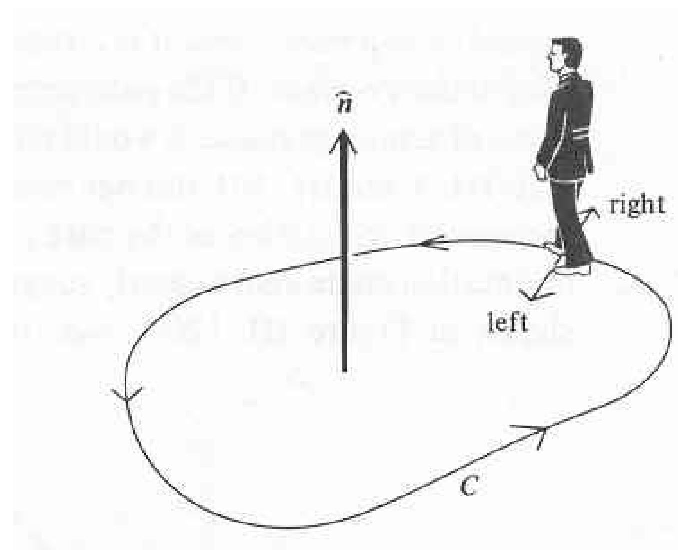
$$\nabla^2 \mathbf{p} = 0$$

Laplace's Equation

Curl Operator

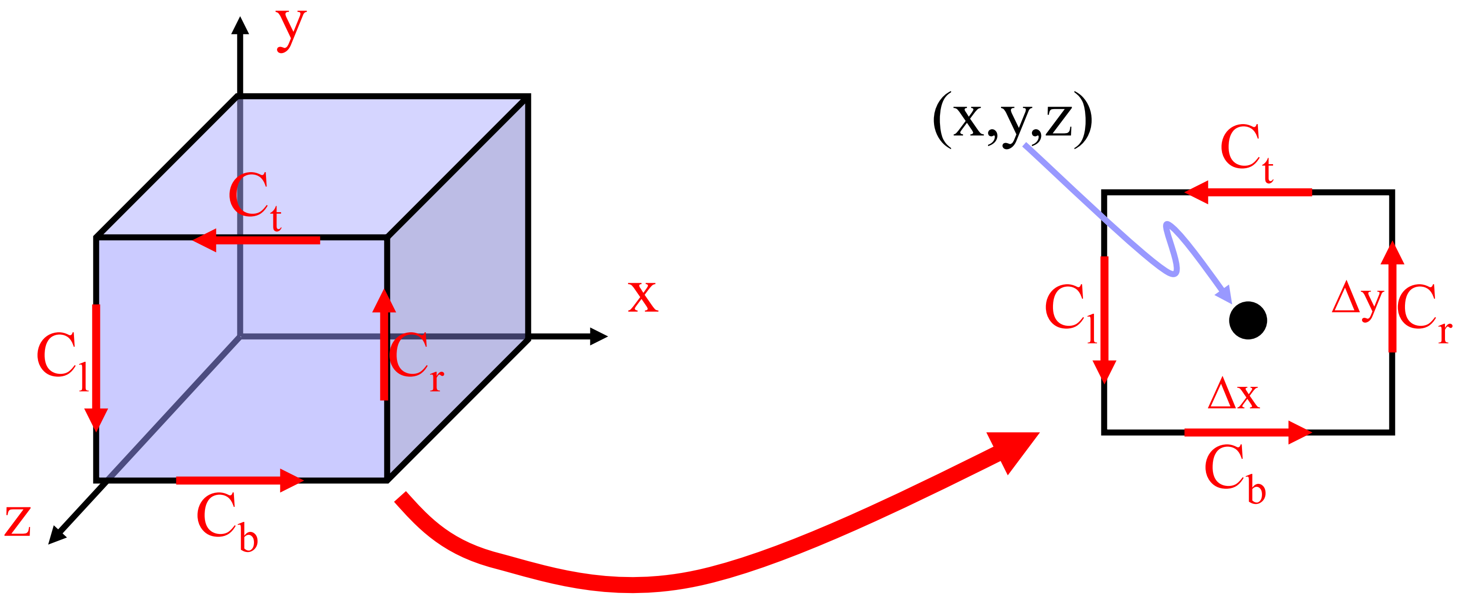
- Curl originates from a line integral around a closed path (is concerned with “swirling”)

$$\hat{\mathbf{n}} \bullet \nabla \times \mathbf{F} = \lim_{\Delta \mathbf{S} \to 0} \frac{1}{\Delta \mathbf{S}} \oint \mathbf{F} \bullet \hat{\mathbf{t}}\, \mathbf{ds}$$



- Definition is the limit of circulation to area as the area tends to zero

Curl Operator



$$\int_{C_b} \mathbf{F} \bullet \hat{\mathbf{t}} ds = \int_{C_b} F_x dx \approx F_x\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x$$

$$\int_{C_t} \mathbf{F} \bullet \hat{\mathbf{t}} ds = \int_{C_t} F_x dx \approx -F_x\left(x, y + \frac{\Delta y}{2}, z\right)\Delta x$$

Curl Operator (cont.)

$$\int_{C_t+C_b} \mathbf{F} \bullet \hat{\mathbf{t}} ds = \int_{C_t+C_b} F_x dx \approx -\left[F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)\right]\Delta x$$

$$\int_{C_t+C_b} \mathbf{F} \bullet \hat{\mathbf{t}} ds = \int_{C_t+C_b} F_x dx \approx -\frac{\left[F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)\right]}{\Delta y}\Delta x \Delta y$$

$$\frac{1}{\Delta S}\int_{C_t+C_b} \mathbf{F} \bullet \hat{\mathbf{t}} ds = -\frac{\left[F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)\right]}{\Delta y}$$

and …. in the limit, as $\Delta S \rightarrow 0$

-
-
-

$$\iint_S \hat{\mathbf{n}} \bullet \nabla \times \mathbf{F} dS = \oint_C \mathbf{F} \bullet \hat{\mathbf{t}} ds$$

Stokes' Theorem

$$\hat{\mathbf{n}} \bullet \nabla \times \mathbf{F} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \mathbf{F} \bullet \hat{\mathbf{t}} ds$$

Types of PDEs

- General PDE (for 2 independent variables)

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

  - Elliptic

    $B^2-4AC<0$

    Laplace's eqn.

    $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

    $$B^2-4AC=0^2-4(1)(1)=-4$$

  - Parabolic

    $B^2-4AC=0$

    Diffusion eqn.

    $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

    $$B^2-4AC=0^2-4(1)(0)=0$$

  - Hyperbolic

    $B^2-4AC>0$

    Wave eqn.

    $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

    $$B^2-4AC=0^2-4(1)(-1)=4$$

Famous Elliptic PDEs & Examples

- Laplace's Equation
  - Potential flow, electro-potential flow, interstitial fluid flow

$$\nabla \bullet \left(-\mathbf{k} \nabla \mathbf{p} \right) = 0$$

- Poisson's Equation
  - Same as Laplace's except add a source/sink, e.g. epileptic/brain function source, or perhaps edema for interstitial pressure in brain

$$\nabla \bullet \left(-\boldsymbol{\sigma} \nabla \boldsymbol{\phi} \right) = \Omega$$

- Helmholtz's Equation
  - Harmonic waves (electrical, mechanical, etc.)

$$\mathbf{E} \nabla^2 \mathbf{u} + \rho \omega^2 \mathbf{u} = 0$$

Famous Parabolic PDEs & Examples

- Diffusion
  - Tumor growth

$$\frac{\partial C}{\partial t} = \nabla \bullet (D \nabla C) + \Omega$$

- Diffusion-Perfusion
  - Thermal ablation or hyperthermia

$$\rho c \frac{\partial T}{\partial t} - \nabla \bullet k \nabla T + mT = \sigma |E|^2$$

- Diffusion-convection
  - Convection chemotherapy

$$\frac{\partial \mathbf{c}}{\partial \mathbf{t}} = \nabla \bullet (\mathbf{D} \nabla \mathbf{c}) - \vec{\mathbf{v}} \bullet \nabla \mathbf{c}$$

Famous Hyperbolic PDEs & Examples

- Classic wave equation with wave speed 'c'
  - Pressure waves

$$\frac{\partial^2 \mathbf{p}}{\partial \mathbf{t}^2} = \mathbf{c}^2 \nabla^2 \mathbf{p}$$

- Elastodynamic wave equation
  - Elastography

$$\boldsymbol{\rho} \frac{\partial^2 \vec{\mathbf{u}}}{\partial \mathbf{t}^2} = \nabla \bullet \tilde{\boldsymbol{\sigma}}$$

Multi-Physics

- Convective chemotherapy

$$\nabla \bullet \sigma - \alpha \nabla p_f = F$$

$$\nabla \bullet \kappa \nabla p_f - \alpha \frac{\partial \varepsilon}{\partial t} - \frac{1}{S} \frac{\partial p_f}{\partial t} = \psi$$

$$\vec{v} = -k \nabla p_f$$

$$\frac{\partial c}{\partial t} = \nabla \bullet \left(D \nabla c \right) - \vec{v} \bullet \nabla c$$

Methods of Solution for PDEs

- Analytic
  - Separation of Variables
  - Laplace and Fourier Transforms
  - Eigenfunction Expansions
  - Method of Characteristics
- Numerical
  - Monte Carlo Methods
  - Spectral Methods
  - Boundary Element Methods
  - Finite Difference Methods
  - Finite Element Methods

Differentiation viewpoint

Integration viewpoint

Let's look here now…..

Not a complete list….

Finite Difference Method

Finite Difference Method

Examples:

$\nabla^2 u = 0$ Laplace's equation

- Potential flow
- Electrical potential distribution
- Pressure distribution

$\nabla^2 u + \lambda^2 u = 0$ Helmholtz's equation

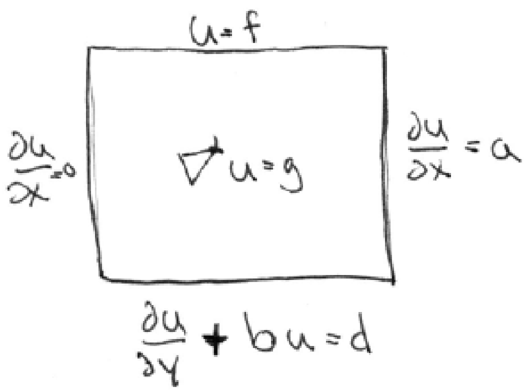
- Harmonic elasticity
- Harmonic acoustic waves

$\nabla^2 u = k$ Poisson's equation

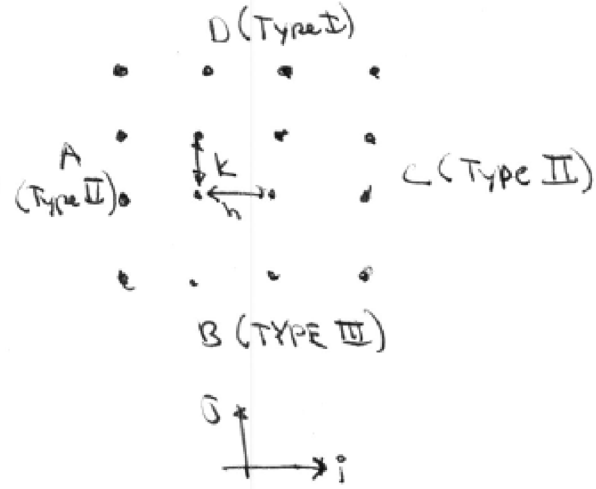
- Electrical Potential distribution in the presence of dipole
- Sink / source modeling

Finite Difference Method

Ex. Consider:



$\Rightarrow$



PDE: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g$

$\Rightarrow$ want second-order, centered FD expressions: