Hyperbolic Equations

Hyperbolic Equations

- classic 2nd order Form .... elliptic in space
  2nd derivative in time

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot c^2 \nabla u = f\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \ldots\right)$$

- Prototype Equation to study:

$$\frac{\partial^2 u}{\partial t^2} + \tilde{\tau} \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{"Telegraph Eqn"}$$

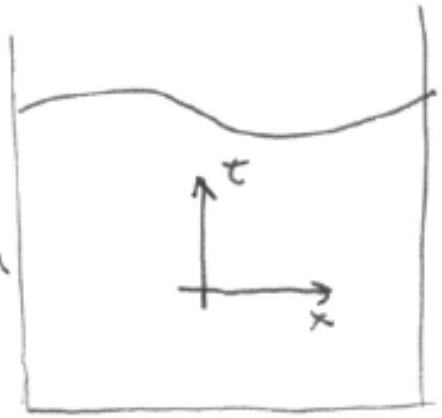
c - "wave speed" ; $\tilde{\tau}$ - dissipation or "loss factor"

$\tilde{\tau} = 0 \Rightarrow$ "Wave Equation"

Hyperbolic Equations

- Recall BCs + ICs needed:

u or $\frac{\partial u}{\partial x}$ or $a\frac{\partial u}{\partial x} + bu$



u or $\frac{\partial u}{\partial x}$ or $a\frac{\partial u}{\partial x} + bu$

u and $\frac{\partial u}{\partial t}$

Hyperbolic Equations

- can always decompose into coupled set of 1st order PDEs

(1) $\frac{\partial u}{\partial t} = c_1 \frac{\partial v}{\partial x}$

(2) $\frac{\partial v}{\partial t} + \tau v = c_2 \frac{\partial u}{\partial x}$

$\Rightarrow \underbrace{\frac{\partial^2 v}{\partial t^2} + \tau \frac{\partial v}{\partial t} - c^2 \frac{\partial^2 v}{\partial x^2} = 0}$

Differentiate (2) in time and substitute (1)

Or: Differentiate (1) in time and substitute (2)

$$\frac{\partial^2 u}{\partial t^2} = c_1 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = -c_1 \underbrace{\tau \frac{\partial v}{\partial x}}_{\frac{1}{c_1}\frac{\partial u}{\partial t}} + c^2 \frac{\partial^2 u}{\partial x^2}$$

So: $\frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial u}{\partial x} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Hyperbolic Equations

- BCs + ICs for coupled system...

Hyperbolic Equations

Time Stepping Strategies

- Consider 2nd order System
- Obvious approach ... replace $\frac{\partial^2 u}{\partial t^2}$ w/ centered 2nd order approximation ... requires 3 levels in time!

a) Explicit: $\frac{\delta_t^2 u_i^l}{\Delta t^2} + \frac{\tau}{2\Delta t}\delta_t u_i^l - c^2 \frac{\delta_x^2 u_i^l}{h^2} = 0$

$$\frac{u_i^{l+1} - 2u_i^l + u_i^{l-1}}{\Delta t^2} + \frac{\tau}{2\Delta t}\left(u_i^{l+1} - u_i^{l-1}\right) - \frac{c^2}{h^2}\left(u_{i+1}^l - 2u_i^l + u_{i-1}^l\right) = 0$$

Hyperbolic Equations

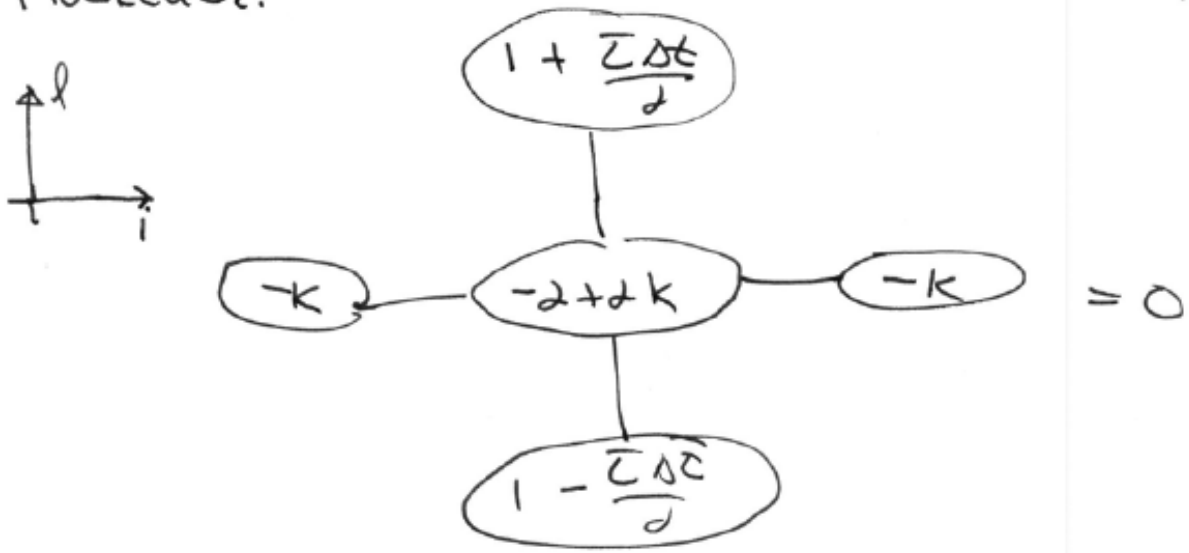
multiply through by Δt^2:

$$u_i^{l+1} - 2u_i^l + u_i^{l-1} + \frac{\tau \Delta t}{\alpha}\left(u_i^{l+1} - u_i^{l-1}\right) - \frac{c^2\Delta t^2}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

$$K = \frac{c^2\Delta t^2}{h^2}$$

↳ "Courant #"

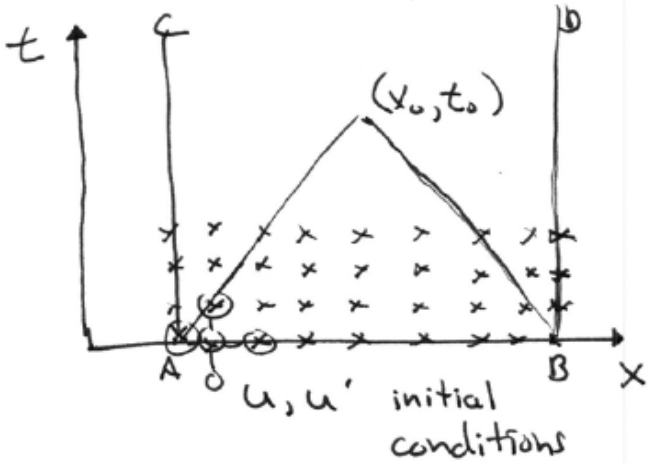
MOLECULE:



$= 0$

Hyperbolic Equations

- Features
  - centered in $x + t \Rightarrow O(h^2 + \Delta t^2)$
  - No matrices
  - Pointwise propagation ... experience w/ Parabolic suggests stability constraint
- Recall characteristic Family for wave equation



Hyperbolic Equations

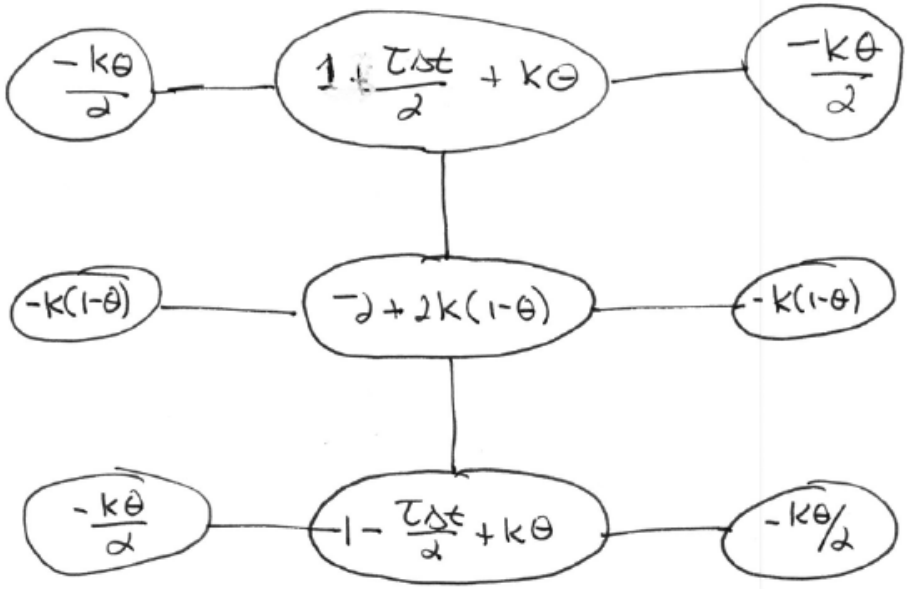
- Knowing initial conditions, can get to (x_0, t_0) without influence of AC or BD....
  suspect stability limits similar to previous work!
- Turns out: don't get convergence as $h, \Delta t \to 0$
  for $\frac{\Delta t}{h} > 1/c$

Hyperbolic Equations

2nd order Eqn (cont)

b) Implicit

$$\left(u_i^{\ell+1} - 2u_i^{\ell} + u_i^{\ell-1}\right) + \frac{\tau\Delta t}{2}\left(u_i^{\ell+1} - u_i^{\ell-1}\right) - k\delta_x^2\left[\theta\left(\frac{u_i^{\ell+1}+u_i^{\ell-1}}{2}\right) + (1-\theta)u_i^{\ell}\right] = 0$$



Hyperbolic Equations

- solution of system of Equations per-time step
- centered in $(x, t) \Rightarrow O(h^2 + \Delta t^2)$
- stability: $\theta \geq 1/2$ unconditional

  $k < 1/(1-2\theta)$ otherwise

- Shortest waves propagate + damp (phase distortions possible)

Hyperbolic Equations

- Courant, Friedrichs and Lewy Condition
  - ↳ convergence requires numerical characteristics (i.e. domain of dependence of FD) to have slopes less than $1/c$ (i.e. must include domain of dependence of PDE)

$$\therefore \quad \frac{\Delta t}{h} < 1/c \quad \Rightarrow \quad \underbrace{\frac{c\Delta t}{h} < 1}$$

This is essentially a stability requirement

Hyperbolic Equations

- Lets look at formal stability analysis

$$u_i^{l+1} = \gamma_0 u_i^l \; ; \; u_{i+1}^l = u_i^l e^{j\sigma h}$$

$$u_i^{l+1} - 2u_i^l + u_i^{l-1} + \frac{\tau\Delta t}{2}\left(u_i^{l+1} - u_i^{l-1}\right) + k\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right) = 0$$

$$\gamma_0 - 2 + \frac{1}{\gamma_0} + \frac{\tau\Delta t}{2}\left(\gamma_0 - \frac{1}{\gamma_0}\right) - k\left(2\cos\sigma h - 2\right) = 0$$

$$\gamma_0^2\left(1 + \frac{\tau\Delta t}{2}\right) + \gamma_0\left(2k(1-\cos\sigma h) - 2\right) + \left(1 - \frac{\tau\Delta t}{2}\right) = 0$$

Need $|\gamma_0| \le 1$; requires $\frac{c}{a} \le 1$; $|b| \le a+c$

Hyperbolic Equations

$$\rightarrow \quad \frac{c}{a} \leq 1 \Rightarrow \quad \frac{1 - \frac{2\Delta t}{J}}{1 + \frac{2\Delta t}{J}} < 1 \quad \text{always!}$$

$$\rightarrow |b| \leq a + c$$

$$|2k(1-\cos\theta h) - 2| \leq 2 \Rightarrow |k(1-\cos\theta h) - 1| \leq 1$$

i.e. $\quad -1 \leq k(1-\cos\theta h) - 1 \leq 1$

consider left side: $\quad -1 \leq k(1-\cos\theta h) - 1$

always true since $k > 0$

$1 - \cos\theta h > 0$

Hyperbolic Equations

right side: $K(1-\cos\theta h) - 1 \leq 1$

$$K \leq 2/1-\cos\theta h$$

most restrictive limit is stability restriction! as usual ... short waves are potentially most unstable .... i.e. $\theta h = \pi$

CONCLUDE: $\boxed{K \leq 1 \;;\; \frac{c^2 \Delta t^2}{h^2} \leq 1 \text{ OR } \frac{c\Delta t}{h} \leq 1}$

"courant" condition

Hyperbolic Equations

- So what kind of errors do we expect w/ explicit scheme...

- Experience tells us "short waves" are the most difficult to handle ... look at γ_0 ($\theta h = \pi$)

Recall: $\gamma_0^2\left(1 + \frac{\tau\Delta t}{2}\right) + \gamma_0\left(2k(1-\cos\theta h) - 2\right) + \left(1 - \frac{\tau\Delta t}{2}\right) = 0$

at $\theta h = \pi$

$$\gamma_0^2\left(1 + \frac{\tau\Delta t}{2}\right) + \gamma_0\left(4k - 2\right) + \left(1 - \frac{\tau\Delta t}{2}\right) = 0$$

Hyperbolic Equations

$$\gamma_0 = \frac{2-4K \pm \sqrt{(4K-2)^2 - 4\left(1-\left(\frac{\Delta t \tau}{2}\right)^2\right)}}{2(1+\tau\Delta t/2)}$$

$$= \frac{2-4K \pm j\left(4\left(1-\left(\frac{\Delta t \tau}{2}\right)^2\right) - (4K-2)^2\right)^{1/2}}{2(1+\tau\Delta t/2)}$$

generally get decay (Good News), but also phase distortion !!
(BAD NEWS)

Hyperbolic Equations

e.g. $k = 1/4$:

$$\gamma_0 = \frac{1 \pm j\sqrt{4\left(1-\left(\frac{\Delta t\, c}{2}\right)^2\right)-1}}{2(1+c\Delta t/2)}$$

let $c = .015$
$h = 5$
$\Delta t = .25$

$$\gamma_0 = \frac{1 \pm j\sqrt{4(1-(1.875e\text{-}3)^2)-1}}{2(1+1.875e\text{-}3)}$$

$$= \frac{1 \pm 1.732j}{2.00375} \Rightarrow |\gamma_0| < 1$$

$$Arg(\gamma_0) \approx \pm \pi/3$$

Amplitude damped; expect phase errors

FD Conservation

FD Conservation

- Conservation Law + Constitutive Relation ⇒ PDE

$\nabla \cdot q = \sigma$ $\quad q = -k\nabla u$ $\quad \Rightarrow \nabla \bullet (-k\nabla u) = \sigma$

Flux; source / volume; scalar surrogate for q

e.g.

q	Conserved	u	σ
Heat Flux	Thermal Energy	T	Heating rate
Diffusion Flux	Molecules of species	C	Reaction rate
Mass Flux	Fluid mass	ρ	Evaporation rate
⋮	⋮	⋮	⋮

FD Conservation

Note: θ need not be constant

- radioactive decay: $\theta = -\kappa C$

storage: $\theta = -\frac{\partial C}{\partial t}$

$$\nabla \bullet \left(-D\nabla c\right) = \underbrace{-\frac{\partial C}{\partial t} - \kappa C}_{\sigma}$$

(Parabolic - more later)

FD Conservation

- PDE is local conservation statement

- Global Conservation:

$$\int (\text{PDE})\, dV \Rightarrow \int \nabla \cdot q \, dV = \int \sigma \, dV$$

Diversence Theorem: $\oint q \cdot \hat{n} \, ds = \int \sigma \, dV$

Rate of escape = sum of sources

or

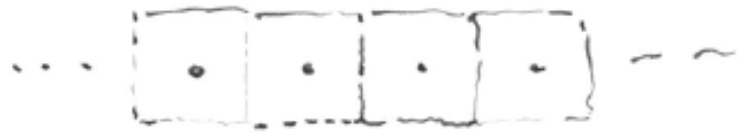
$$\oint -K \nabla u \cdot \hat{n} \, ds = \int \sigma \, dV$$

$$\oint -K \frac{\partial u}{\partial n} \, ds = \int \sigma \, dV$$

FD Conservation

- Numerical Conservation $\Rightarrow \int (\,)\,dV = \sum (\,)_i \Delta V_i$

e.g. 1-D .... $\sum_i \int_{i-1/2}^{i+1/2} (\,)_i \, dx$



$\frac{\partial}{\partial x} K \frac{\partial u}{\partial x} = -\sigma$ For 1 box:

FD Conservation

$$\int_{i-1/2}^{i+1/2} \left(\frac{\partial}{\partial x} k \frac{\partial u}{\partial x}\right)_i dx = -\int_{i-1/2}^{i+1/2} \sigma \, dx$$

$$\underbrace{k \frac{\partial u}{\partial x}\Big|_{i+1/2}}_{-q_{i+1/2}} - \underbrace{k \frac{\partial u}{\partial x}\Big|_{i-1/2}}_{+q_{i-1/2}} = -\sigma_i h$$

so σ_i represents all sources in the box associated w/ Node i

$$\sigma_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \sigma \, dx \Big/ (x_{i+1/2} - x_{i-1/2})$$

FD Conservation

Now:

$$-q_{i+1/2} + q_{i-1/2} = k_{i+1/2}\left(\frac{u_{i+1} - u_i}{h}\right) - k_{i-1/2}\left(\frac{u_i - u_{i-1}}{h}\right)$$

$$= -\theta_i h$$

Or

$$\frac{k_{i+1/2}\left(\frac{u_{i+1} - u_i}{h}\right) - k_{i-1/2}\left(\frac{u_i - u_{i-1}}{h}\right)}{h} = -\theta_i$$

FD Conservation

But left-hand side is exactly our FD approximation at node i ... so to conserve we must view θ_i as average of sources in box

i.e. $$\theta_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \theta \, dx \Big/ (x_{i+1/2} - x_{i-1/2})$$

Then $\sum_{i=1}^{N}$ (FD equation # i):

$$-q_{N+1/2} + q_{1/2} = -\sum_{i=1}^{N} \theta_i h$$

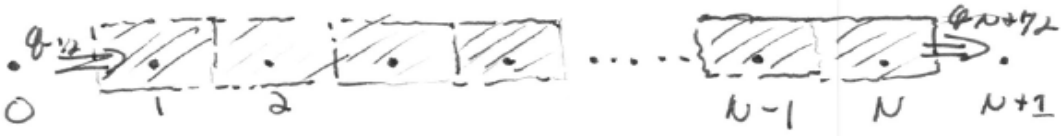
Internal $q_{i+1/2}$ cancel

FD Conservation

Then $\sum_{i=1}^{N}$ (FD equation # i):

$$-q_{N+1/2} + q_{1/2} = -\sum_{i=1}^{N} \theta_i h$$

Internal $q_{i+1/2}$ cancel



- Type I BCs at $0, N+1$ $\Rightarrow$ can solve ...
  - ↳ Conservation boundaries not at $0, N+1$
  - ↳ Conservation independent of $\theta_0 + \theta_{N+1}$

- Type II + Type III ... bring PDE molecule to boundary

FD Conservation

e.g. $-K\frac{\partial u}{\partial x} = q_0$ specified ... then

FD molecule at node 0:

$$K_{1/2}\left(\frac{u_{-1} - 2u_0 + u_1}{h}\right) = -q_0 h \qquad (K_{1/2} = K_{-1/2})$$

But BC says: $K_0 \frac{u_{-1} - u_1}{2h} = q_0$

$$\Rightarrow u_{-1} = \frac{2hq_0}{K_0} + u_1 = \frac{2hq_0}{K_{1/2}} + u_1 \qquad \left(K_0 = \frac{K_{1/2} + K_{-1/2}}{2}\right.$$

FD Conservation

So we get $2q_0 + 2\left(\frac{u_1 - u_0}{n}\right) k_{1/2} = -\theta_0 h$

or $q_0 + \underbrace{k_{1/2}\left(\frac{u_1 - u_0}{n}\right)}_{-q_{1/2}} = -\theta_0 h/2$

This provides the missing half box!

Similarly at node $N+1$: $-q_{N+1} + q_{N+1/2} = -\theta h/2$

FD Conservation

So For Global conservation add in the two boundary molecules ...

$$-q_{N+1} + q_0 = -\sum_{i=1}^{N} \theta_i h - \frac{\theta_0 h}{2} - \frac{\theta_{N+1} h}{2}$$

$$= -\sum_{i=0}^{N} (\theta_i + \theta_{i+1}) h/2$$

Trapezoidal Rule Integration

This is the FD conservation statement!

FD Conservation

- At Type I boundaries need to use all molecules to conserve... use the unused (in solution) molecules at boundary to compute fluxes

$$q_0 = -K_{1/2}\left(\frac{u_1 - u_0}{h}\right) - \frac{\sigma_0 h}{2}$$

This is the intuitive FD expression... looks $O(h)$

correction term to account for half box makes everything $O(h^2)$

similar at node $N+1$:

$$-q_{N+1} = K_{N+1/2}\left(\frac{u_{N+1} - u_N}{h}\right) - \sigma_{N+1} h/2$$

FD Conservation

In 2-D:

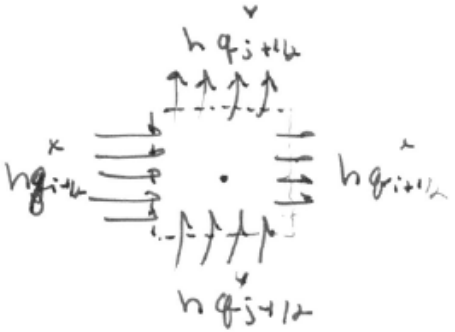
$$\frac{1}{h}\left[k_{i+1/2}\left(\frac{u_{i+1,j}-u_{ij}}{h}\right)-k_{i-1/2}\left(\frac{u_{ij}-u_{i-1,j}}{h}\right)\right]$$

$$+\frac{1}{h}\left[k_{j+1/2}\left(\frac{u_{ij+1}-u_{ij}}{h}\right)-k_{j-1/2}\left(\frac{u_{ij}-u_{ij-1}}{h}\right)\right]$$

$$=-\theta_{ij}$$

$$-h\,q^{x}_{i+1/2}+h\,q^{x}_{i-1/2}-h\,q^{y}_{j+1/2}+h\,q^{y}_{j-1/2}=-\theta_{ij}h^2$$

FD Conservation



Θ_{ij} is 2D average

i.e.

$$\Theta_{ij} = \frac{\int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} \Theta \, dx \, dy}{(x_{i+1/2} - x_{i-1/2})(y_{j+1/2} - y_{j-1/2})}$$

- Note: if $\Theta = Q\delta(x - x_0, y - y_0)$ then for

Node i,j corresponding to (x_0, y_0)

$$\Theta_{ij} = \frac{\iint Q\delta(x - x_0, y - y_0)\,dx\,dy}{h^2} = \underbrace{Q/h^2}$$

source strength divided by area of 1 box!!

Introduction to the Method of Weighted Residuals

Method of Weighted Residuals

- Consider $\nabla^2 \mathbf{u} + \mathbf{f}\mathbf{u} = \mathbf{g} \rightarrow \underbrace{\left(\nabla^2 + \mathbf{f}\right)}_{L}\mathbf{u} = \mathbf{g}$
- Finite Difference Approach
  - Approximate L w/ $L_{ij} \rightarrow \underbrace{\left(\delta^2 + \mathbf{f}\right)}_{Lij}\mathbf{u}_{ij} = \mathbf{g}_{ij}$

    i.e. replace "differential" operator w/ "difference" operator → get "exact solution" to "approximate operator"
  - Limitations:
    - cumbersome on irregular meshes
    - curved boundaries difficult to handle
    - **u** only found (i,j) points – need interpolation strategy

Method of Weighted Residuals

- Weighted Residuals
  - Approximate **u** as $\hat{u} = \sum_{j=1}^{N} c_j \Phi_j(x,y,z)$

    coefficients

    known function
    - "basis" func.
    - "trial" func.
    - "expansion" func.
  - Define "Residual" $\rightarrow$ $R(\hat{u}) = L\hat{u} - g$
  - For exact solution: $R(\hat{u}) = 0$ everywhere then
  - Want $R(\hat{u}) = 0$ to vanish in average way, one way is in weighted integral sense

    $$\iiint R(\hat{u}) W(x,y) dxdydz = 0$$

    any function of position
  - " $R(\hat{u})$ orthogonal to all *W(x,y,z)*"

Method of Weighted Residuals

- So for $\hat{u}: R(\hat{u}) \neq 0$ … choose 'N' c_j's such that $\langle R(\hat{u}), W_i \rangle = 0$ for $i = 1,2,3,\ldots,N$

"Inner Product" $\Rightarrow \langle \mathbf{a}, \mathbf{b} \rangle \equiv \iint \mathbf{a} \bullet \mathbf{b}\, dxdydz$

  - W_i's set of "weighting" functions → finite!
    local "testing" functions
  - Use 'N' independent W_i's → generate 'N' equations in 'N' unknown c_j's

$$\langle R(\hat{u}), W_i \rangle = \sum_{j=1}^{N} c_j \langle L(\phi_j) W_i \rangle = \langle g, W_i \rangle \quad \text{for each } W_i(x,y) \quad i = 1,2,\ldots N$$

  - Necessary, but not sufficient for $\hat{u} = u$

Method of Weighted Residuals

- Continuous function must be zero if it is orthogonal to every member of a complete set ∴ WRM can be thought of as a technique which enforces orthogonality between basis and weighting function sets
  - General idea is that the basis is a subset of a complete set that can represent any function, i.e. the true solution
  - As number of coefficients goes up, the approximation approaches the complete set

Method of Weighted Residuals

- Take home interpretation:
  - PDE is determined from first principles
  - Introduce the idea of a weighting function which is a known function of space – on its introduction, the physics behavior is prescribed to a region
  - Integration formulation
  - Introduction of basis as an approximation to the solution – on its introduction, it provides a description of how the solution will be treated behaviorally within in a local region

Method of Weighted Residuals

- Weighted Residual Methods Summary
  - L is typically a differential operator
  - W_i not complete in practice (N finite), but make "R(u) orthogonal to 1st N members of a complete set"
  - "Approximate solution" which exactly satisfies "differential relations" in PDE
  - Is an "integral" formulation

Method of Weighted Residuals

- What's involved numerically?
  - Expand unknown solution as $\hat{u} = \sum_{j=1}^{N} c_j \phi_j(x,y,z)$
    - Finite sum
    - $\phi_j(x,y,z)$ known function of (x,y,z)
    - c_j's unknown coefficients to determine
  - Generate system of equations in unknowns

$$R(\hat{u}) = \langle (L(\hat{u}) - g), W_i(x,y,z) \rangle = 0 \text{ for } i = 1,2,3,\ldots,N$$

$$\sum_{j=1}^{N} c_j \langle L(\phi_j), W_i(x,y,z) \rangle = \langle g, W_i(x,y,z) \rangle \text{ for } i = 1,2,3,\ldots,N$$

    - $W_i(x,y,z)$ known functions of (x,y,z)
    - For each i, generate algebraic equation in c_j's
    - Creates 'N' equations in 'N' unknowns

Method of Weighted Residuals

- But analytic methods…
  - Require special knowledge of how to choose basis and weighting functions
  - Different choices are needed for different problems
  - Usually need an infinite # of them
  - Can't find them for many practical problems
- Numerically want …
  - Basis and Weighting functions to be simple … easy to integrate
  - Single choice suitable for many problems
  - Can only use finite #, but want convergence as number used increases

Method of Weighted Residuals

- WRM … eighting Funct…
  - Su… N' small reg… equal to uni…
  - Co… residual exa…
  - Le… weighting fur…
  - Mc… nction
  - Ga… ction are the…
- We … Residual Meth…

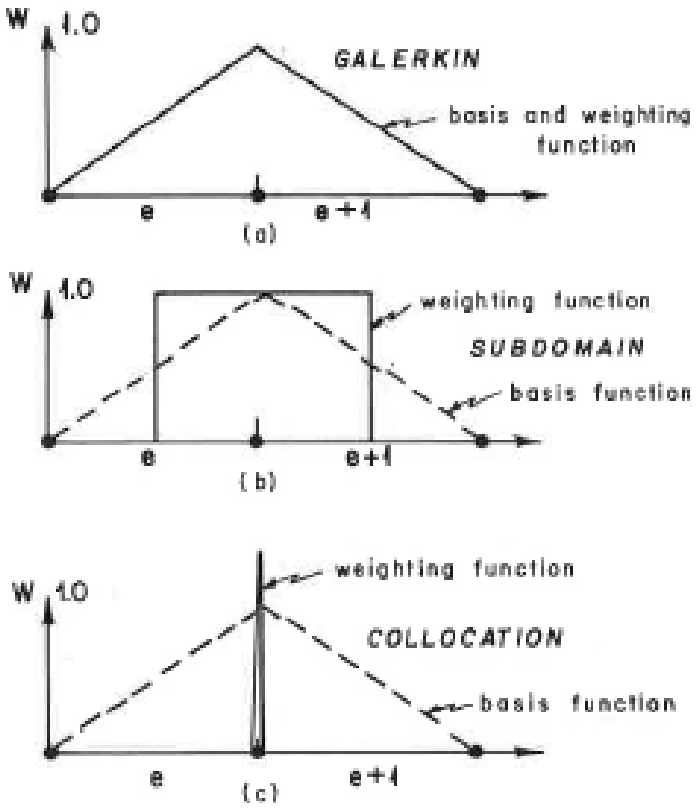


Figure 2.4. Schematic representation of the one-dimensional weighting functions for the Galerkin, subdomain and collocation methods. (It is assumed here that the chapeau function is used as a basis for all methods.)

Galerkin Weighted Residual Method

- Key Feature:
  - Basis function and weighting function are selected to be the same function form
- So what do we choose?
  - Polynomials may be a nice choice
  - Other's exist

Polynomial Basis/Weighting Functions

- Why?
  - Easy to differentiate
  - Represent a complete set for continuous functions … e.g. Taylor Polynomial
- Lagrange Polynomials
  - Nth order polynomial for N+1 pts
  - Easily automated
  - Handy on uneven grids
  - $\phi_i(x) = 0 \text{ at } x = x_i \text{ for } i \neq j$
  - $\phi_j(x) = 1 \text{ at } x = x_i \text{ for } i = j$

$$\phi_i = \prod_{j \neq i} \frac{(x - x_i)}{(x_j - x_i)}$$

Polynomial Basis/Weighting Functions

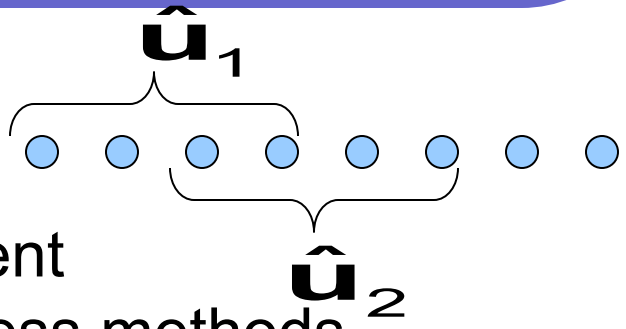
- Lagrange Polynomials
  - $\phi_j(\mathbf{x}_j) = 1 \Rightarrow \mathbf{u}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{c}_j \phi_j(\mathbf{x},\mathbf{y}) \Rightarrow \mathbf{u}(\mathbf{x}_i) = \mathbf{c}_i$
  - $\therefore \quad \mathbf{u}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{u}_j \phi_j(\mathbf{x},\mathbf{y})$
  - Coefficients are solution at nodes … similar to FD but have functional form specified in between
  - everywhere $\sum_{j=1}^{N} \phi_j(\mathbf{x},\mathbf{y}) = 1$
  - $\sum_{j=1}^{N} \frac{\partial \phi_j(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \sum_{j=1}^{N} \phi_j(\mathbf{x},\mathbf{y}) = 0$

Polynomial Basis/Weighting Functions

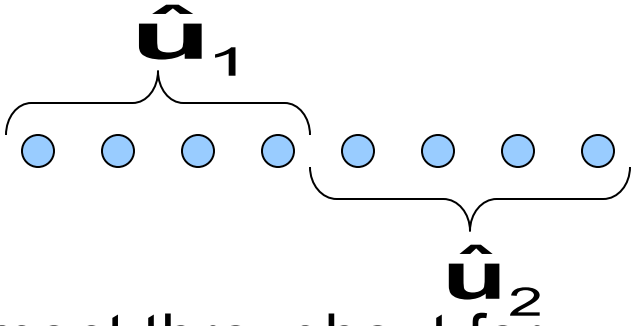
- Global Polynomials
  - Potential for disaster … “polynomial wiggle”
  - Can have N zeros
  - Sensitive to all u_j - “Global Support”
  - Can have large variation between points
- Local Interpolation
  - Use subset of nodes to represent solution over localized areas – so-called finite elements
  - Possibilities : Overlapping Domains, Non-overlapping Domains

Polynomial Basis/Weighting Functions

- Overlapping Domains
  - Nonunique
  - Somewhat difficult to implement
  - Further study of this in meshless methods



- Non-overlapping Domains
  - Idea of finite “element” as collection of nodes, i.e. unit of local interpolation
  - Unequal spacing of nodes, i.e. conformity of shape
  - Basic building block of FEM
  - Typically use same type of element throughout for programming ease (unless reason not to do so)



Polynomial Basis/Weighting Functions

- Continuity of $\hat{\mathbf{u}}$: In 1D, if N+1 nodes/element
  - $\hat{\mathbf{u}}$ is locally N^{th} order polynomial
  - On element **interior** … 1^{st} N derivatives continuous
  - … but, at element boundaries only $\hat{\mathbf{u}}$ is continuous; $\partial\hat{\mathbf{u}}/\partial\mathbf{x}$ changes abruptly
  - "C^o" continuity → continuous in 0^{th} derivative
- Higher order continuity possible

  e.g. "C^1" continuity … need Hermite polynomial
  - Simplest local unit: Hermite cubic
  - $\partial\hat{\mathbf{u}}/\partial\mathbf{x}$ becomes "nodal parameter"

A little bit of review

Topics Covered

- Difference equations an error
- Polynomials, Taylor, Lagrange, Splines, and basis functions
- Finite differences, truncation error, round-off error
- Numerical integration, quadrature, Gauss pts, Gaussian quadrature, truncation error, round-off error

Topics Covered

- Boundary integrals, Div, Grad, Curl
- PDE types, Anisotropy, Divergence Theorem
- Finite Difference Method for solving PDEs
- Internal FDM equations, and boundary conditions (Type 1, 2 , 3)
- Recovering flux from unused molecules

Topics Covered

- Modeling physics with FDM, incorporating heterogeneity, understanding matrix properties
- Iterative solution methods (Jacobi, Gauss seidel, SOR, Line Methods)
- Convergence behavior of iterative methods
- Cross derivatives in FDM

Topics Covered

- Advective-diffusive problems
- Parabolic (diffusion equation)
  - Explicit, Semi-implicit, Fully implicit
  - Potential use of higher order methods
- Stability analysis (Fourier, and Matrix Methods)
- Hyperbolic models (telegraph equation)
- FD Conservation