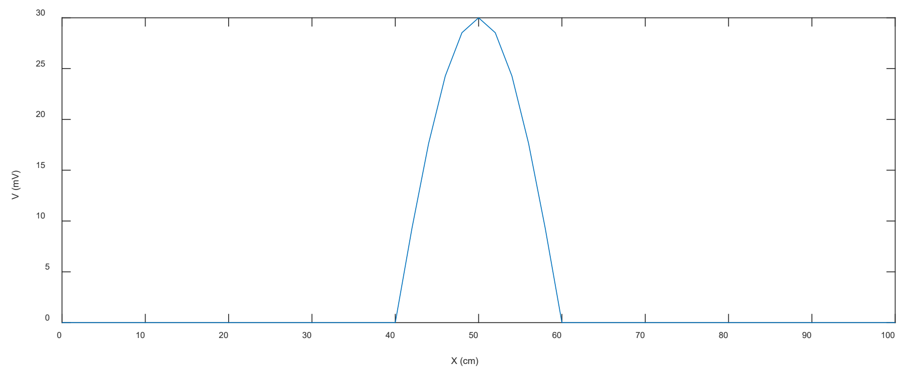
Next assignment…

$$\frac{\partial^2 U}{\partial t^2} + \tau \frac{\partial U}{\partial t} = c^2 \nabla^2 U$$

$$U = \theta\left(\frac{U^{l+1} + U^{l-1}}{2}\right) + (1-\theta)U^{l}$$

$$U(x,t=0) = 30\cos\left(\frac{\pi r}{2r_o}\right) \text{ for } r \le r_o, \text{and } U(x,t=0) = 0 \text{ for } r \ge r_o$$



BCs

$$\frac{\partial U}{\partial t} - c\frac{\partial U}{\partial x} + \frac{\tau U}{2} = 0 \quad \text{for } x = 0$$

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} + \frac{\tau U}{2} = 0 \quad \text{for } x = L_x$$

Let's consider…

$$\frac{\partial^2 U}{\partial t^2} + \tau \frac{\partial U}{\partial t} = c^2 \nabla^2 U$$

$$U = \theta\left(\frac{U^{l+1} + U^{l-1}}{2}\right) + (1-\theta)U^{l}$$

What is the approach?

Now what?

$$\frac{\partial U}{\partial t} = \frac{U_i^{l+1} - U_i^{l-1}}{2\Delta t}$$

$$c^2 \nabla^2 U$$

$$\frac{\partial^2 U}{\partial t^2} = \frac{U_i^{l+1} - 2U_i^{l} + U_i^{l-1}}{\Delta t^2}$$

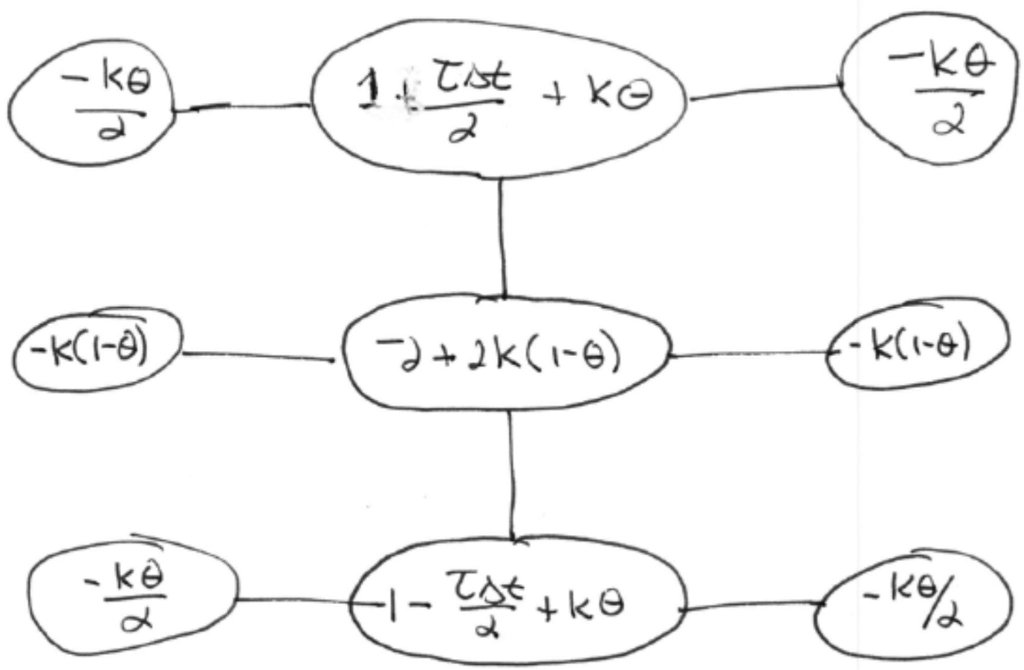
$$\theta\left(\frac{c^2\nabla^2 U^{l+1} + c^2\nabla^2 U^{l-1}}{2}\right) + (1-\theta)\left(c^2\nabla^2 U^{l}\right)$$

Continuing…

$$U_i^{l+1} - 2U_i^l + U_i^{l-1} + \frac{\tau\Delta t}{2}\left(U_i^{l+1} - U_i^{l-1}\right) - k\left(\theta\left(\frac{U_{i+1}^{l+1} - 2U_i^{l+1} + U_{i-1}^{l+1} + U_{i+1}^{l-1} - 2U_i^{l-1} + U_{i-1}^{l-1}}{2}\right) + (1-\theta)\left(U_{i+1}^l - 2U_i^l + U_{i-1}^l\right)\right) = 0$$

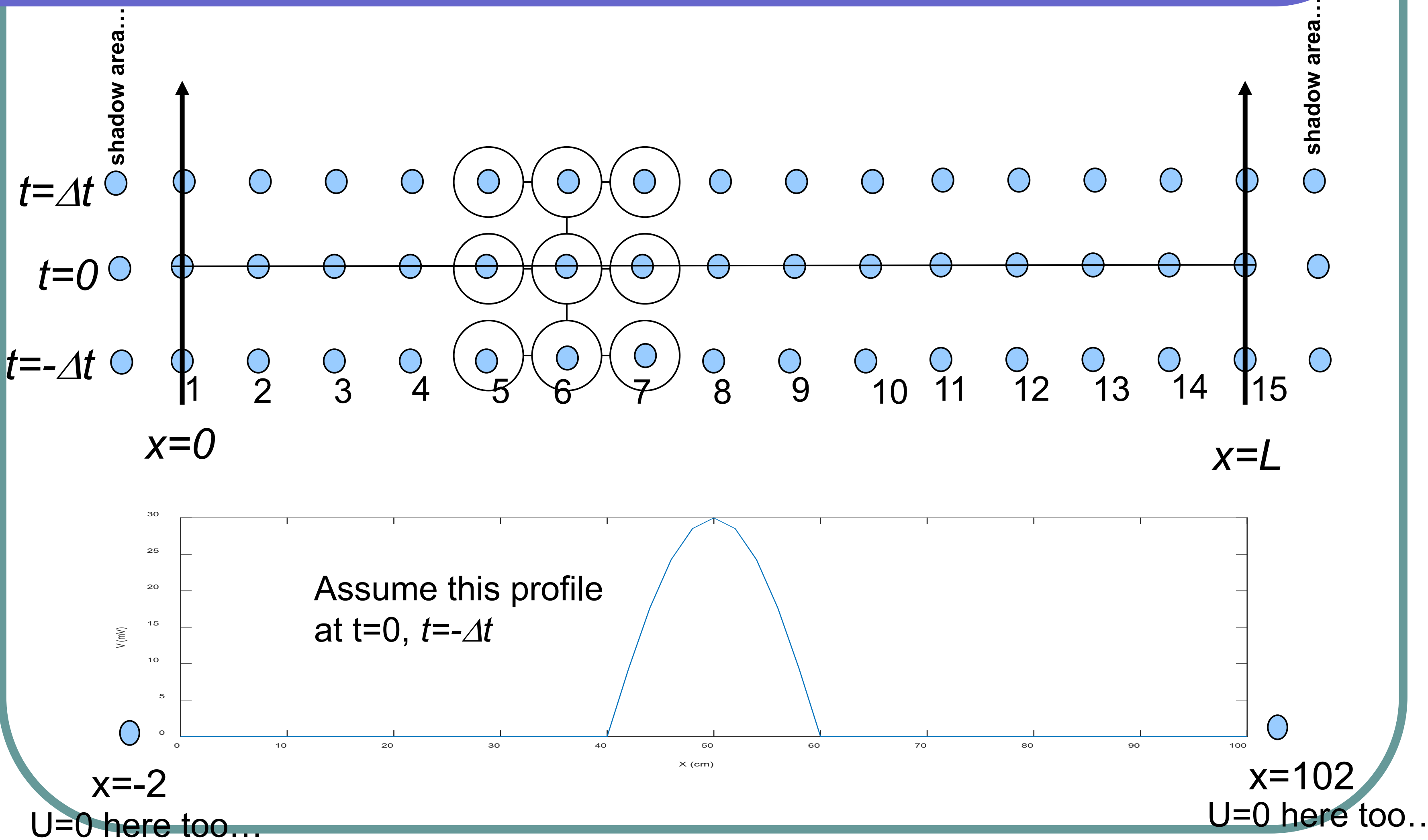
$$\left(u_i^{l+1} - 2u_i^l + u_i^{l-1}\right) + \frac{\tau\Delta t}{2}\left(u_i^{l+1} - u_i^{l-1}\right) - k\delta_x^2\left[\theta\left(\frac{u_i^{l+1} + u_i^{l-1}}{2}\right) + (1-\theta)u_i^l\right]$$

$k = c^2\Delta t^2/h^2$



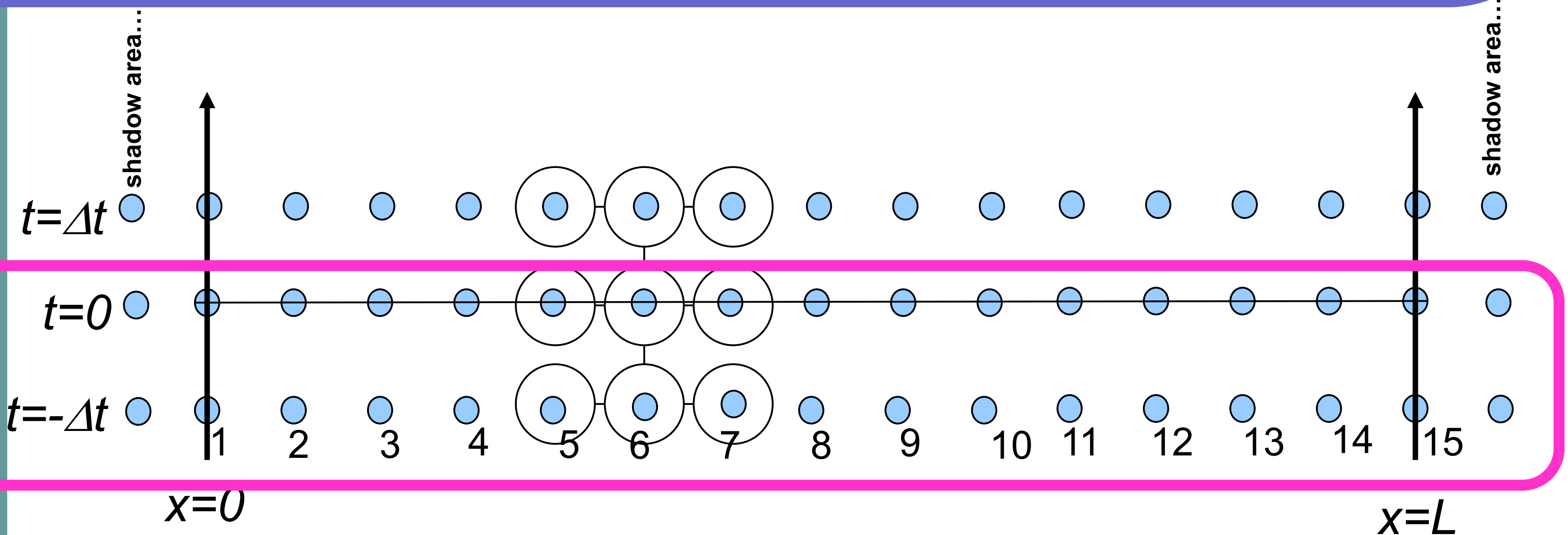
= 0

Continuing …



shadow area…
shadow area…
t=Δt
t=0
t=-Δt
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
x=0
x=L
Assume this profile
at t=0, t=-Δt
V (mV)
0 5 10 15 20 25 30
0 10 20 30 40 50 60 70 80 90 100
X (cm)
x=-2
U=0 here too…
x=102
U=0 here too…

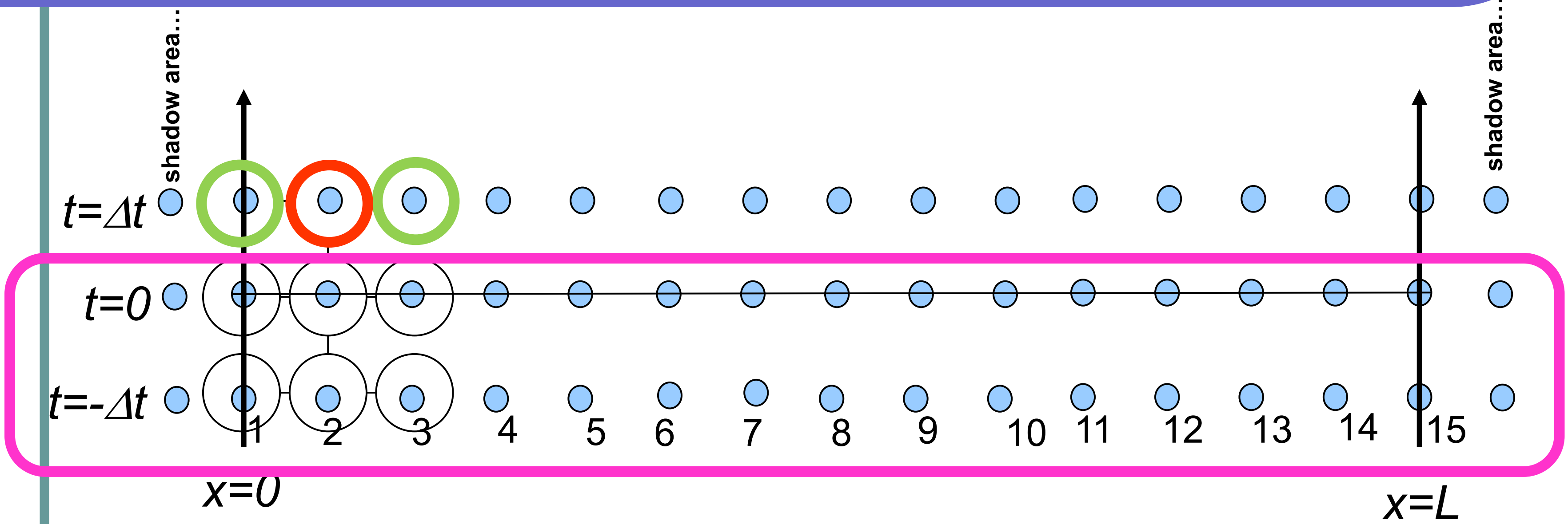
Continuing …



I know every u-value here at the start including the shadow nodes

My task is to solve for u-values at t=Δt, the l+1 level…

Continuing …



Let's assume Jacobi's method…. What am I doing?

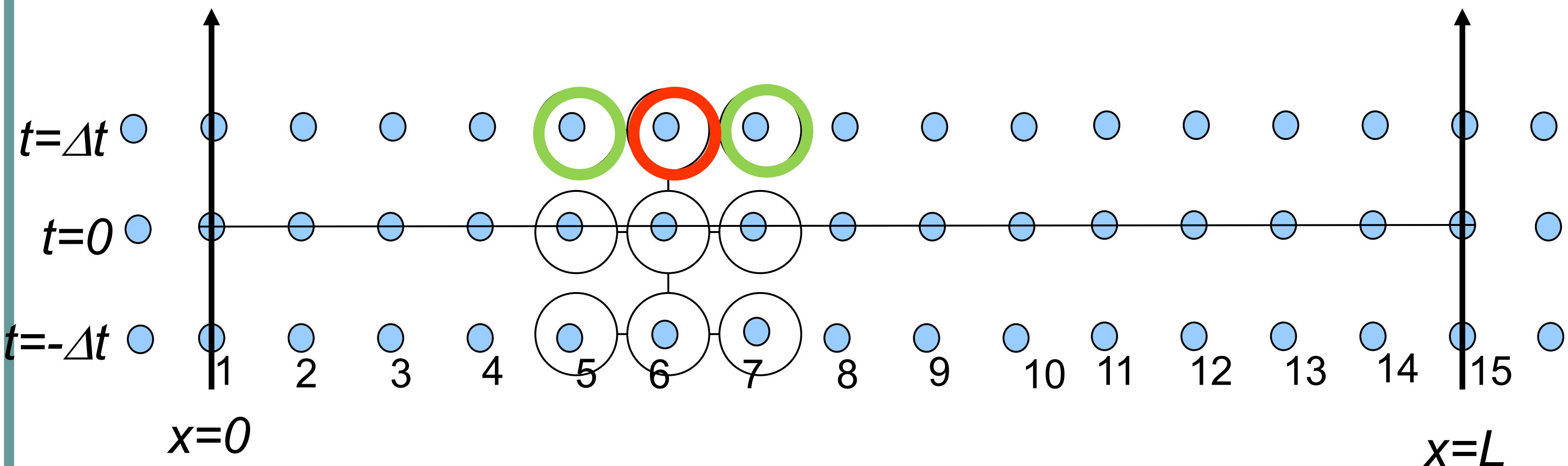
Assume a solution at u(t=Δt), call it u_{old}.

Magenta is all known from previous values

Green – use values from u_{old}

Solve for the Red Node Only

Step-by-Step Tick-Tock



$$\boxed{U_6^{l+1}} - \boxed{2U_6^l + U_6^{l-1}} + \frac{\tau\Delta t}{2}\left(\boxed{U_6^{l+1}} - \boxed{U_6^{l-1}}\right) - k\left(\theta\left(\frac{\boxed{U_7^{l+1}} - \boxed{2U_6^{l+1}} + \boxed{U_5^{l+1}} + \boxed{U_7^{l-1} - 2U_6^{l-1} + U_5^{l-1}}}{2}\right) + (1-\theta)\boxed{\left(U_7^l - 2U_6^l + U_5^l\right)}\right) = 0$$

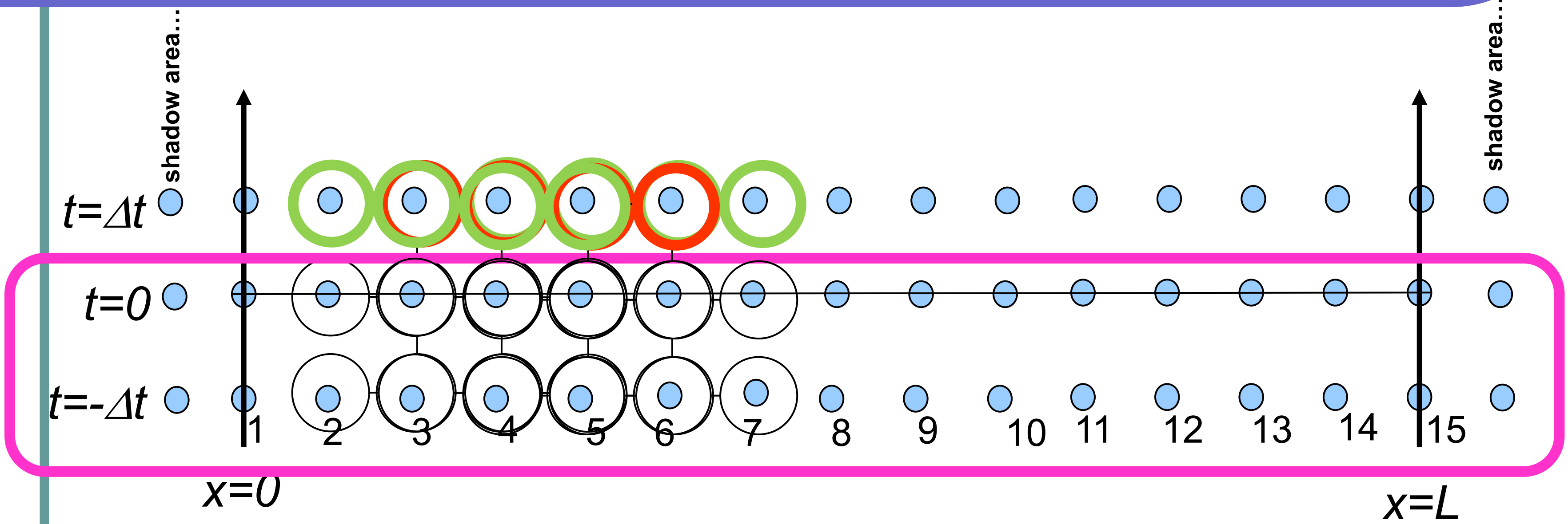
Have to group these right?

Have to use solution at previous iterate?

This is explicit info already known from previous time steps?

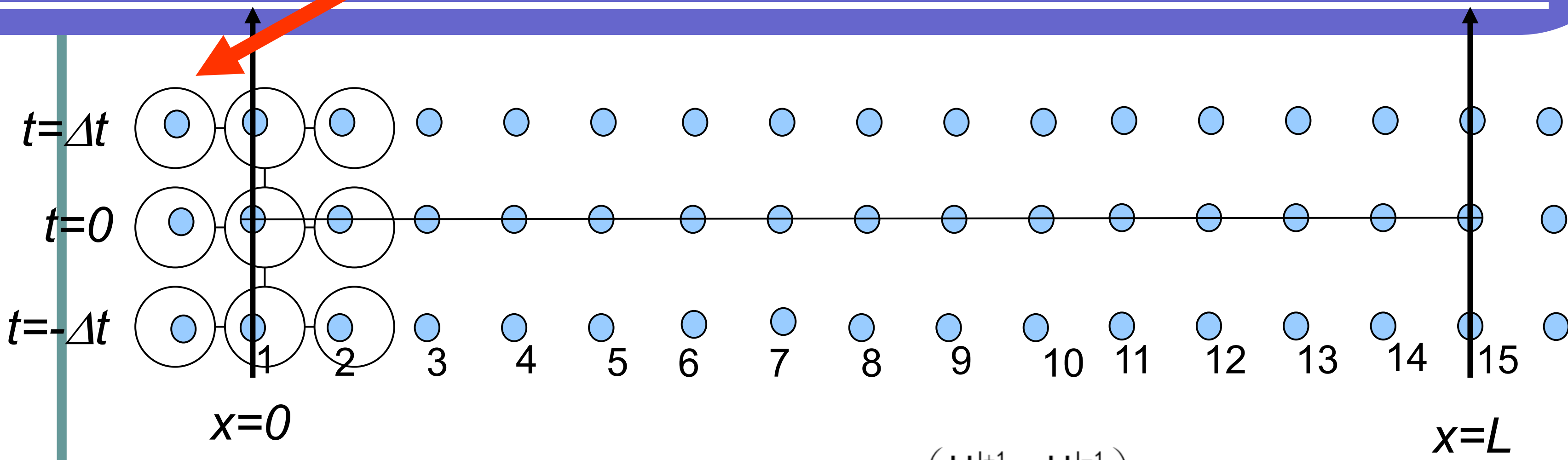
$$U_i^{l+1} - 2U_i^l + U_i^{l-1} + \frac{\tau\Delta t}{2}\left(U_i^{l+1} - U_i^{l-1}\right) - k\left(\theta\left(\frac{U_{i+1}^{l+1} - 2U_i^{l+1} + U_{i-1}^{l+1} + U_{i+1}^{l-1} - 2U_i^{l-1} + U_{i-1}^{l-1}}{2}\right) + (1-\theta)\left(U_{i+1}^l - 2U_i^l + U_{i-1}^l\right)\right) = 0 \qquad k = \frac{c^2\Delta t^2}{h^2}$$

Continuing …



Let's assume Jacobi's method…. What am I doing?
Assume a solution at u(t=Δt), call it u_{old}.
Magenta is all known from previous values
Green – use values from u_{old}
Solve for the Red Node Only
Move to the next internal node …

Continuing …



$$U = \theta\left(\frac{U^{l+1}+U^{l-1}}{2}\right) + (1-\theta)U^{l}$$

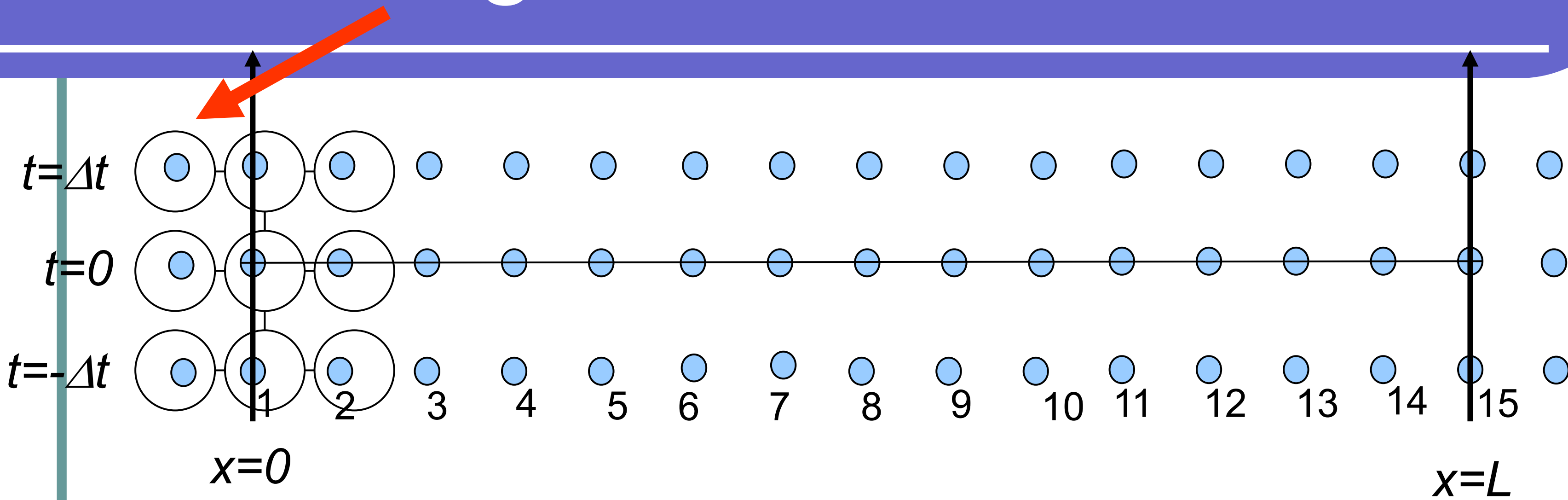
$$\frac{\partial U}{\partial t} - c\frac{\partial U}{\partial x} + \frac{\tau U}{2} = 0 \quad \text{for } x = 0$$

$$\frac{U_i^{l+1}-U_i^{l-1}}{2\Delta t} - c\left[\theta\left(\frac{\frac{U_{i+1}^{l+1}-U_{i-1}^{l+1}}{2h}+\frac{U_{i+1}^{l-1}-U_{i-1}^{l-1}}{2h}}{2}\right) + (1-\theta)\left(\frac{U_{i+1}^{l}-U_{i-1}^{l}}{2h}\right)\right] + \frac{\tau}{2}\left[\theta\left(\frac{U_i^{l+1}+U_i^{l-1}}{2}\right) + (1-\theta)\left(U_i^{l}\right)\right]=0$$

Solve for the shadow node…

Remember… U_{i-1}^{l+1} is solved for in terms of other term …

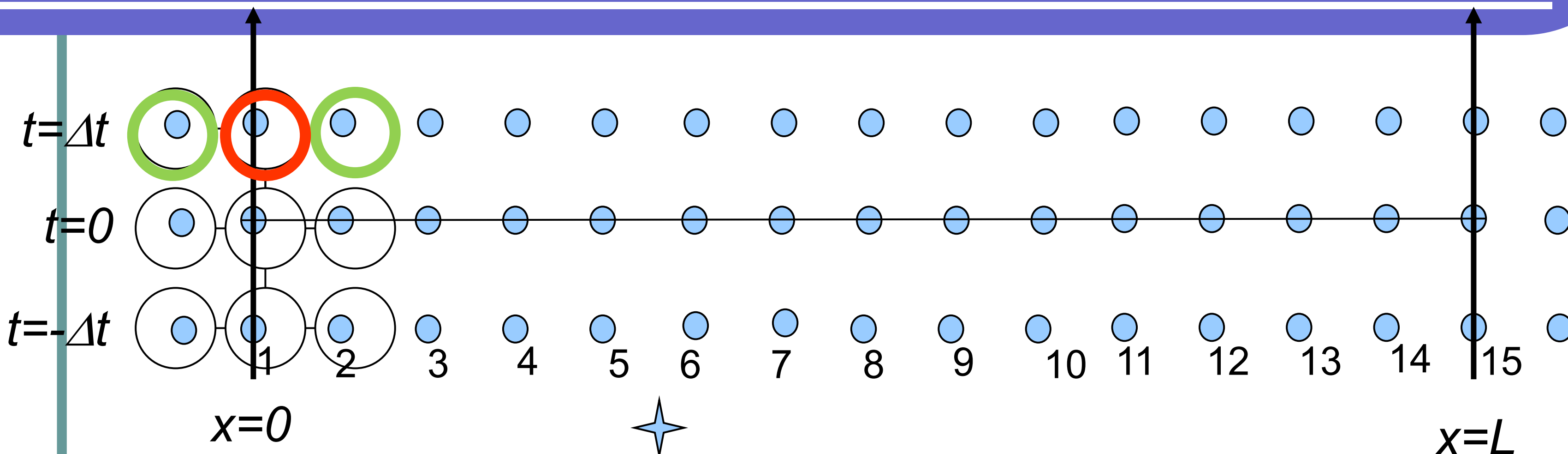
Continuing …



$$\frac{U_i^{l+1}-U_i^{l-1}}{2\Delta t}-c\left[\theta\left(\frac{\frac{U_{i+1}^{l+1}-U_{i-1}^{l+1}}{2h}+\frac{U_{i+1}^{l-1}-U_{i-1}^{l-1}}{2h}}{2}\right)+(1-\theta)\left(\frac{U_{i+1}^{l}-U_{i-1}^{l}}{2h}\right)\right]+\frac{\tau}{2}\left[\theta\left(\frac{U_i^{l+1}+U_i^{l-1}}{2}\right)+(1-\theta)\left(U_i^{l}\right)\right]=0$$

$$U_{i-1}^{l+1}=f_1+f_2=fcn\left(U_i^{l+1}\right)+fcn\left(U_{i+1}^{l+1},U_{i-1}^{l},U_i^{l},U_{i+1}^{l},U_{i-1}^{l-1},U_i^{l-1},U_{i+1}^{l-1}\right)$$

Continuing …



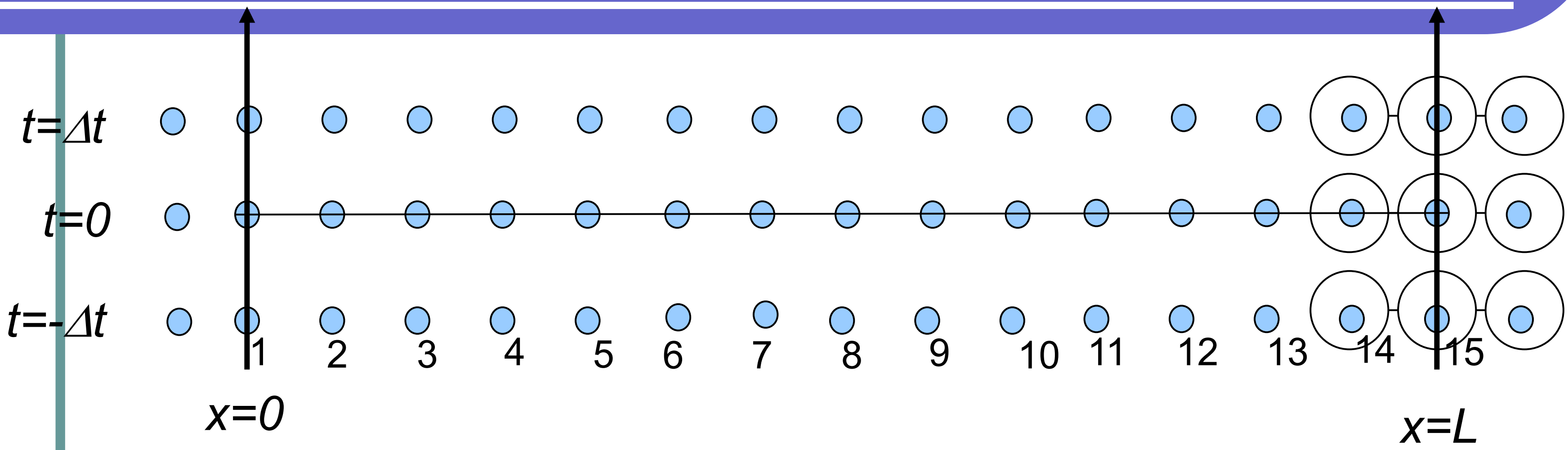
$$\mathrm{U}_{i-1}^{l+1} = \boxed{f_1} + \boxed{f_2}$$

$$k = \frac{c^2 \Delta t^2}{h^2}$$

$$\boxed{\mathrm{U}_1^{l+1}} \boxed{-2\mathrm{U}_1^{l} + \mathrm{U}_1^{l-1}} + \frac{\tau \Delta t}{2}\left(\boxed{\mathrm{U}_1^{l+1}} - \boxed{\mathrm{U}_1^{l-1}}\right) - k\left(\theta\left(\frac{\boxed{\mathrm{U}_2^{l+1}} - 2\boxed{\mathrm{U}_1^{l+1}} + \mathrm{U}_{i-1}^{l+1} + \boxed{\mathrm{U}_2^{l-1} - 2\mathrm{U}_1^{l-1} + \mathrm{U}_{shadow}^{l-1}}}{2}\right) + (1-\theta)\boxed{\left(\mathrm{U}_2^{l} - 2\mathrm{U}_1^{l} + \mathrm{U}_{shadow}^{l}\right)}\right) = 0$$

For Jacobi iteration update at node 1, group these right? $\boxed{}$

Continuing …



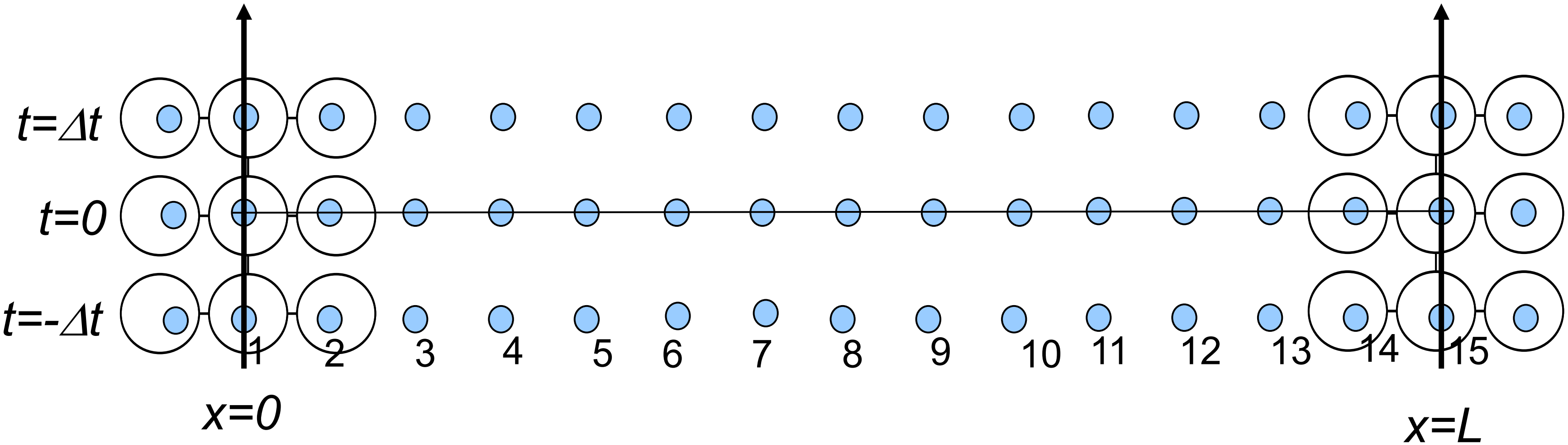
Solve for the shadow node…

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} + \frac{\tau U}{2} = 0 \quad \text{for } x = L$$

$$\frac{U_i^{l+1} - U_i^{l-1}}{2\Delta t} + c\left[\theta\left(\frac{\frac{U_{i+1}^{l+1} - U_{i-1}^{l+1}}{2h} + \frac{U_{i+1}^{l-1} - U_{i-1}^{l-1}}{2h}}{2}\right) + (1-\theta)\left(\frac{U_{i+1}^{l} - U_{i-1}^{l}}{2h}\right)\right] + \frac{\tau}{2}\left[\theta\left(\frac{U_i^{l+1} + U_i^{l-1}}{2}\right) + (1-\theta)\left(U_i^{l}\right)\right]$$

Do the same on this side….

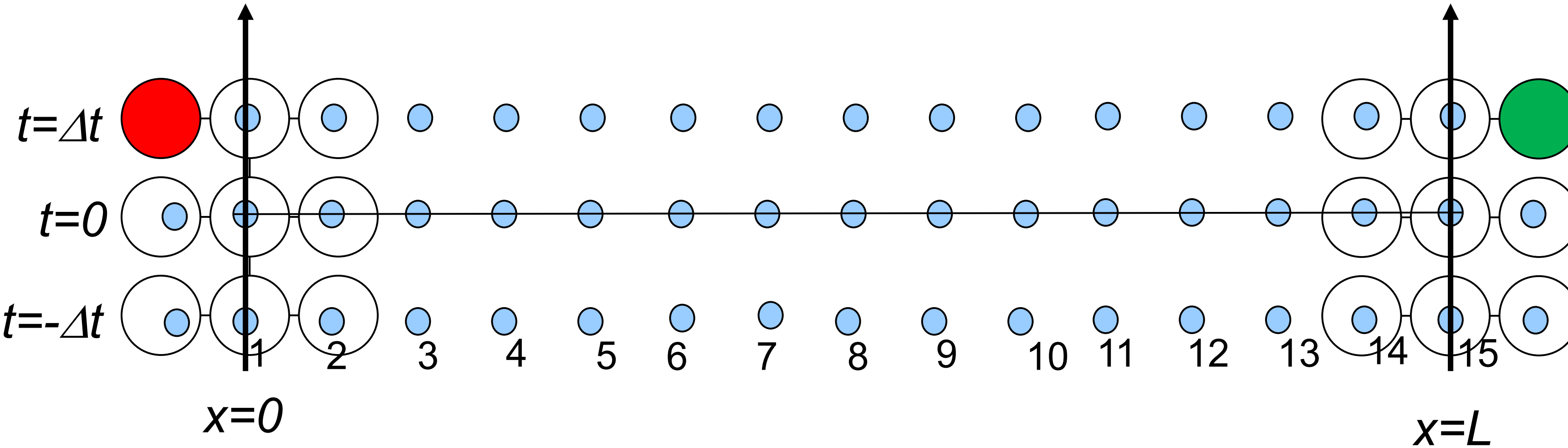
Continuing …



You can now update the solution at each node for one Jacobi iteration… keep doing it until you converge.

Point iterative method

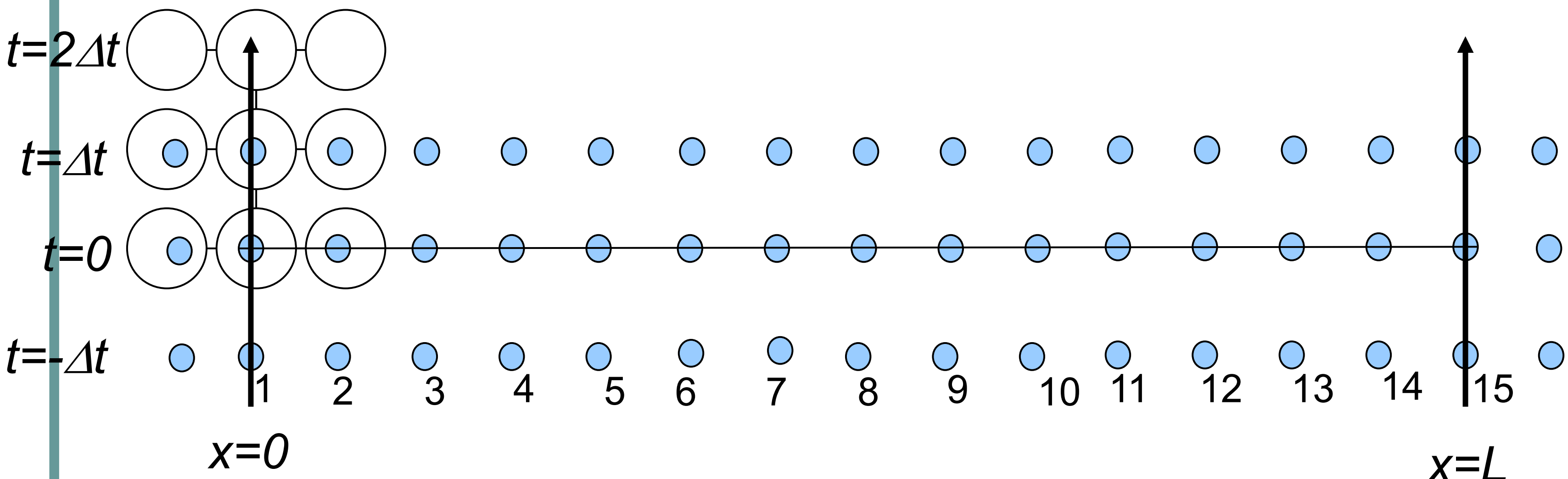
Continuing …



Once you converged…. Take your solution and calculate the shadow nodes at t=Δt

- $$U_{i-1}^{l+1} = f_1 + f_2 = fcn\left(U_i^{l+1}, U_{i+1}^{l+1}, U_{i-1}^{l}, U_i^{l}, U_{i+1}^{l}, U_{i-1}^{l-1}, U_i^{l-1}, U_{i+1}^{l-1}\right)$$
- $$U_{i+1}^{l+1} = f_1 + f_2 = fcn\left(U_i^{l+1}, U_{i-1}^{l+1}, U_{i-1}^{l}, U_i^{l}, U_{i+1}^{l}, U_{i-1}^{l-1}, U_i^{l-1}, U_{i+1}^{l-1}\right)$$

Continuing …



Moving forward? …

With shadow nodes known at $t=\Delta t$, the process for solving $t=2\Delta t$ is identical…

Review

(a.)

$$P_N = \tfrac{1}{3} P_{N-1} \quad \text{EXACT}$$

$$\hat{P}_N = \tfrac{1}{3} \hat{P}_{N-1} \quad \text{COMPUTER}$$

$$\underbrace{P_N - \hat{P}_N} = \tfrac{1}{3} \underbrace{\left(P_{N-1} - \hat{P}_{N-1} \right)}$$

$$\epsilon_N = \tfrac{1}{3} \epsilon_{N-1}$$

$$P_N = \tfrac{10}{3} P_{N-1} - P_{N-2} \quad \text{w/} \quad P_0 = 1 ,\ P_1 = \tfrac{1}{3}$$

then $\epsilon_N = \tfrac{10}{3} \epsilon_{N-1} - \epsilon_{N-2}$ $\Rightarrow$ difference eqn. with constant coefficients

$$\epsilon_N = C_1 (1/3)^N + C_2 \underbrace{(3)^N}$$

HW 1

$$w_{i+1} = w_i + \frac{\Delta t}{2}(-2k * w_i + k^2 \Delta t * w_i) \qquad (3)$$

$$w_N = \left(1 - \Delta t k + \frac{\Delta t^2 k^2}{2}\right)^N$$

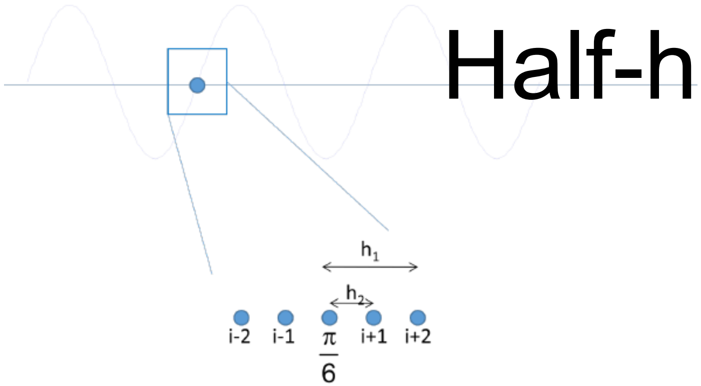
Center Difference Expression: $\dfrac{du}{dt} = \dfrac{u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}}{12h}$ Order (h^4)

Backward Difference Expression: $\dfrac{du}{dt} = \dfrac{u_{i-2} - 4u_{i-1} + 3u_i}{2h}$ Order (h^2)

Forward Difference Expression: $\dfrac{du}{dt} = \dfrac{u_{i+1} - u_i}{h}$ Order (h)

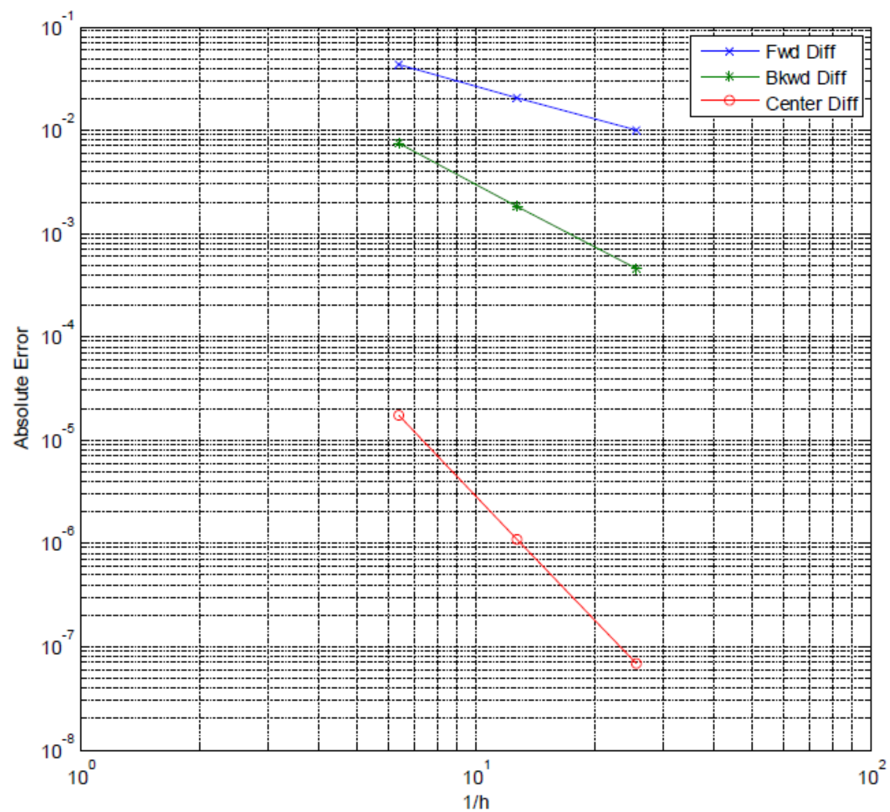
$$u_i' = \frac{du_i}{dx} = \frac{u_{i+1} - u_i}{h} - \frac{h}{2!} u_i'' + \ldots$$

Half-h error cuts -1/2, 1/4, 1/16



$$m = \frac{\log(6.864e-8) - \log(1.7523e-5)}{\log(80/pi) - \log(20/pi)}$$

m = -4



Slope of the order

Expr.	cos(π/6)	h_1 val.	h_2 val.	h_3 val.	Error for h_1	Error for h_2	Error for h_3
Cntr	0.866025403784	0.866007880594	0.866024306169	0.866025335145717	1.75231895e-05	1.09761485e-06	6.8638721706371e-08
Back	0.866025403784	0.873569309894	0.867862752074	0.866477904830691	0.007543906109	0.001837348290	4.5250104625227e-04
Fwd	0.866025403784	0.823279179126	0.845510468697	0.855986618817722	0.042746224658	0.020514935086	0.010038784966716

$$\vdots$$

$$= \sum_{j=1}^{N} \left[\frac{h}{2} \left(f_j + f_{j-1} \right) - \frac{h^3}{12} f''(\xi_j) \right] \qquad \xi_j \in [x_{j-1}, x_j]$$

$$= \frac{h}{2} \left[f_0 + f_N + 2 \sum_{j=1}^{N-1} f_j \right] - \underbrace{\frac{h^3}{12} \sum_{j=1}^{N} f''(\xi_j)}_{N \overline{f''(\xi_j)}}$$

$\leftarrow$ average value

$Nh = b - a$

So

$$I(b) = \underbrace{\frac{h}{2} \left[f_a + f_b + 2 \sum_{j=1}^{N-1} f_j \right]}_{\text{Composite Trap Rule } b} - \underbrace{\left(\frac{b-a}{12} \right) h^2 \overline{f''(\xi_j)}}_{\text{Error term}}$$

Composite Trap Rule b

Error term one order lower in h relative to single application

Also talked in the context of integration and quadrature

Better (easier) to use concepts of a basis function

i.e. $P_N(x) = \sum_{j=0}^{N} \alpha_j b_j(x)$

HW 2

Since you have seen that $\int_{\ell} \left(\vec{J} \bullet \hat{n}\right) d\ell = J_{n1} \int \phi_1 d\ell + J_{n2} \int \phi_2 d\ell$

$$\int_1^2 \left(\vec{J} \bullet \vec{n}\right) \phi_1 d\ell = J_{n1} \int \phi_1 \phi_1 d\ell + J_{n2} \int \phi_2 \phi_1 d\ell$$

$\int \phi_n \phi_n = L/3$ and $\int \phi_m \phi_n = L/6$

then for the first scenario we obtain

$J_{n1}(L/3)+J_{n2}(L/6)$

Likewise,

$\int_1^2 \left(\vec{J} \bullet \vec{n}\right) \phi_2 d\ell$ evaluates to

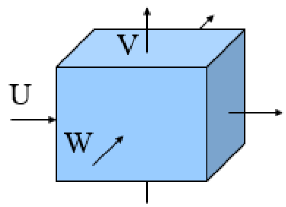
$J_{n1}(L/6)+J_{n2}(L/3)$

HW 2

$$\iint_S \mathbf{F} \bullet \hat{\mathbf{n}} \, d\mathbf{S} = \iiint_V \nabla \bullet \mathbf{F} \, d\mathbf{V}$$

$$\frac{\partial}{\partial \mathbf{x}}(\rho \mathbf{U}) + \frac{\partial}{\partial \mathbf{y}}(\rho \mathbf{V}) + \frac{\partial}{\partial \mathbf{z}}(\rho \mathbf{W}) = 0$$

$$\nabla \bullet (\rho \vec{\mathbf{V}}) = 0$$



If density is constant …

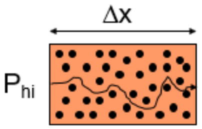
$$\nabla \bullet \vec{\mathbf{V}} = \frac{\partial \mathbf{V_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{V_y}}{\partial \mathbf{y}} + \frac{\partial \mathbf{V_z}}{\partial \mathbf{z}} = 0$$

- Operates on vector/tensor and is principally used in *conservation* statements within physics

Divergence op (+) , gradient op (no +, just deriv of vector)

(conservation of mass)

Gradient Operator



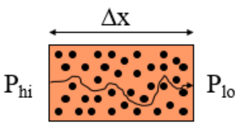
$$V_x = ?\, k \frac{dp}{dx}$$ → $p = f(x,y,z)$ What if ???

Velocity …. a vector, right !!

$$\begin{cases} V_x = -k \dfrac{\partial p}{\partial x} \\ V_y = -k \dfrac{\partial p}{\partial y} \\ V_z = -k \dfrac{\partial p}{\partial z} \end{cases} \rightarrow \vec{V} = -k \nabla p$$

What if I dotted this into the unit vector normal to a surface, i.e. perpendicular to the surface?

Porous-Media Example



$$\nabla \bullet \vec{\mathbf{V}} = \nabla \bullet (-\mathbf{k} \nabla \mathbf{p}) = 0$$

What am I assuming?

$$\frac{\partial}{\partial \mathbf{x}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right) + \frac{\partial}{\partial \mathbf{y}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{y}}\right) + \frac{\partial}{\partial \mathbf{z}}\left(-\mathbf{k}\frac{\partial \mathbf{p}}{\partial \mathbf{z}}\right) = 0$$

Assume homogeneous constant hydraulic conductivity…

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right) + \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{y}}\right) + \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial \mathbf{p}}{\partial \mathbf{z}}\right) = 0$$

$$\frac{\partial^2 \mathbf{p}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{p}}{\partial \mathbf{y}^2} + \frac{\partial^2 \mathbf{p}}{\partial \mathbf{z}^2} = 0$$

Laplacian Operator → $\nabla^2 \mathbf{p} = 0$ Laplace's Equation

(conservation of volume) = laplacian of pressure

curl = limit of circulation of a path

HW 2

$$\nabla \cdot (-\sigma \nabla V) = 0$$

expand

$$\nabla \cdot \left(-\sigma \begin{bmatrix} \frac{\partial V}{\partial x} \hat{i} \\ \frac{\partial V}{\partial y} \hat{j} \end{bmatrix} \right) = 0$$

$$\frac{\partial}{\partial x} \Big(\underbrace{-\sigma \frac{\partial V}{\partial x}}_{\text{let} = F} \Big) + \frac{\partial}{\partial y} \Big(\underbrace{-\sigma \frac{\partial V}{\partial y}}_{\text{let} = G} \Big) = 0$$

$$\frac{\partial}{\partial x} \Big(\underbrace{-\sigma \frac{\partial V}{\partial x}}_{\text{let} = F} \Big) + \frac{\partial}{\partial y} \Big(\underbrace{-\sigma \frac{\partial V}{\partial y}}_{\text{let} = G} \Big) = 0$$

expand using FD technique @ half-grid point

$$\frac{F_{i+1/2} - F_{i-1/2}}{h} + \frac{G_{j+1/2} - G_{j-1/2}}{k} = 0$$

$$\frac{\left(-\sigma_{i+1/2} \frac{\partial V}{\partial x}\Big|_{i+1/2}\right) - \left(-\sigma \frac{\partial V}{\partial x}\Big|_{i-1/2}\right)}{h} + \frac{\left(-\sigma_{j+1/2} \frac{\partial V}{\partial y}\Big|_{j+1/2}\right) - \left(-\sigma_{j-1/2} \frac{\partial V}{\partial y}\Big|_{j-1/2}\right)}{k} = 0$$

HW 2

$$\frac{\left(-\theta_{i+\frac{1}{2},j}\left(\frac{V_{i+1,j}-V_{i,j}}{h}\right)\right)-\left(-\theta_{i-\frac{1}{2},j}\left(\frac{V_{i,j}-V_{i-1,j}}{h}\right)\right)}{h}$$

$$+$$

$$\frac{\left(-\theta_{i,j+\frac{1}{2}}\left(\frac{V_{i,j+1}-V_{i,j}}{k}\right)\right)-\left(-\theta_{i,j-\frac{1}{2}}\left(\frac{V_{i,j}-V_{i,j-1}}{k}\right)\right)}{k}=0$$

simplify

$$\frac{-\theta_{i+\frac{1}{2},j}V_{i+1,j}+(\theta_{i+\frac{1}{2},j}+\theta_{i-\frac{1}{2},j})V_{ij}-\theta_{i-\frac{1}{2},j}V_{i-1,j}}{h^2}$$

$$+\qquad\qquad =0$$

$$\frac{-\theta_{i,j+\frac{1}{2}}V_{i,j+1}+(\theta_{i,j+\frac{1}{2}}+\theta_{i,j-\frac{1}{2}})V_{ij}-\theta_{i,j-\frac{1}{2}}V_{i,j-1}}{k^2}$$

on equal grid, $h=k$, multiply through by spatial step size..

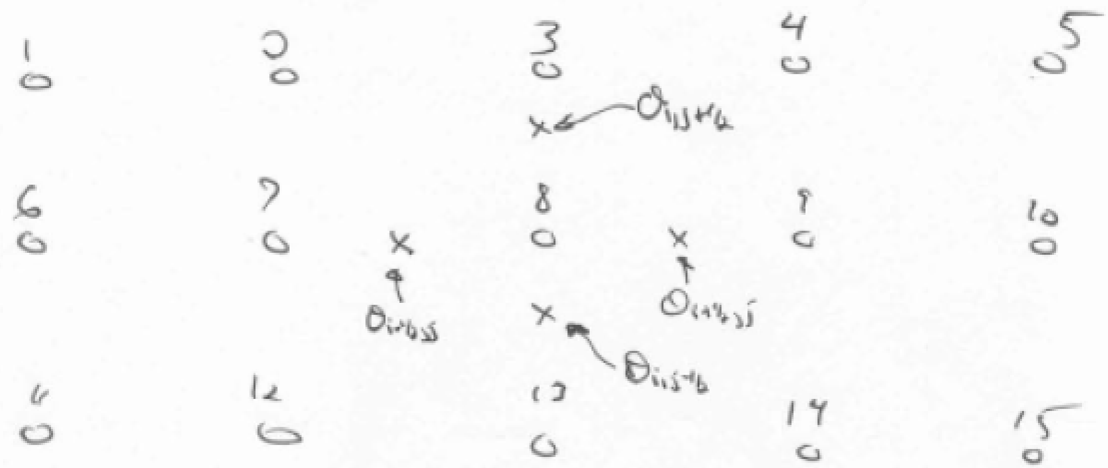
$$-\theta_{i+\frac{1}{2},j}V_{i+1,j}-\theta_{i-\frac{1}{2},j}V_{i-1,j}-\theta_{i,j+\frac{1}{2}}V_{i,j+1}-\theta_{i,j-\frac{1}{2}}V_{i,j-1}$$

$$+\qquad\qquad =0$$

$$(\theta_{i+\frac{1}{2},j}+\theta_{i-\frac{1}{2},j}+\theta_{i,j+\frac{1}{2}}+\theta_{i,j-\frac{1}{2}})V_{ij}$$

HW 2

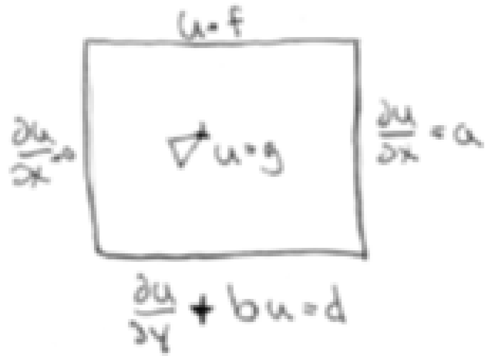
GRID



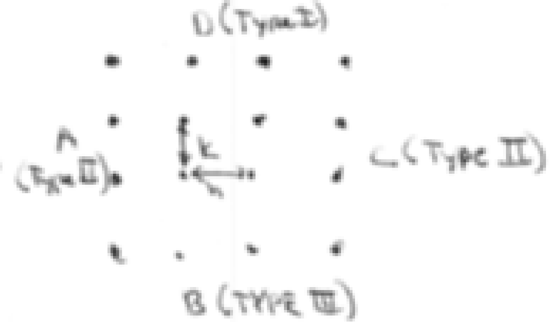
So for equation (8)

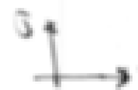
$$-\theta_{i+1/2,j} V_9 - \theta_{i-1/2,j} V_7 - \theta_{i,j+1/2} V_3 - \theta_{i,j-1/2} V_{13} + \left(\theta_{i+1/2,j} + \theta_{i-1/2,j} + \theta_{i,j+1/2} + \theta_{i,j-1/2}\right) V_8 = 0$$

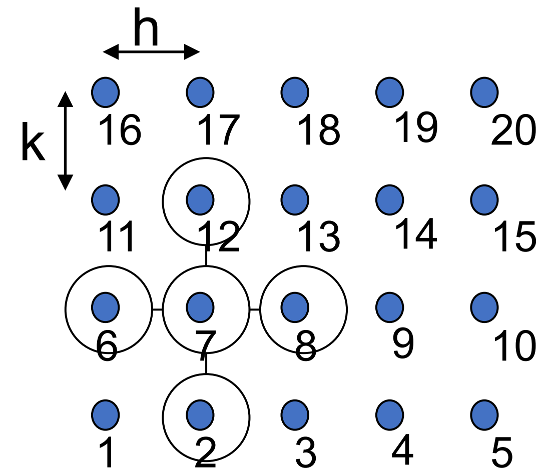
Ex. Consider :



$\Rightarrow$







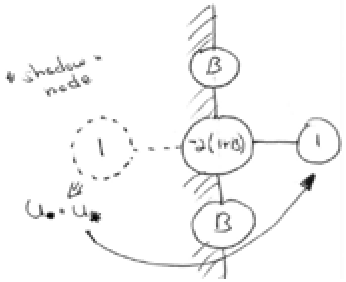
PDE : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g$

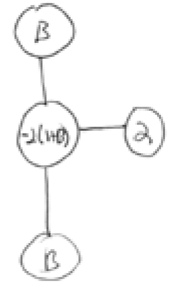
$$\Rightarrow \frac{\delta_x^2 u_{ij}}{h^2} + \frac{\delta_y^2 u_{i,j}}{k^2} = g_{ij}$$

$$\frac{u_8 - 2u_7 + u_6}{h^2} + \frac{u_{12} - 2u_7 + u_2}{k^2} = g_7$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = g_{ij}$$

⇒ for nodes on Boundary A, the molecule becomes:

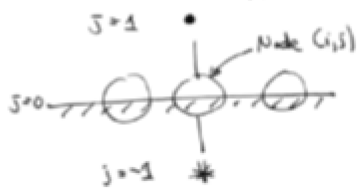




$= g_{i,j} h^2$

Boundary B:

$\left.\frac{\partial u}{\partial y}\right|_{j=0} + bu = d$

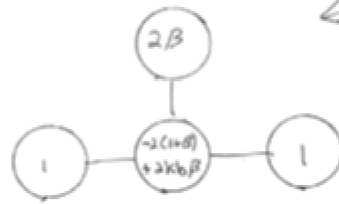


$\Rightarrow \frac{u_\circ - u_*}{2k} + b u_{i,j} = d$

solve for u_*

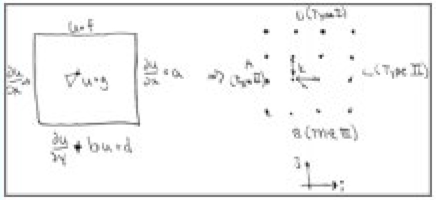
↳ $u_* = 2kb\, u_{i,j} - 2kd + u_\circ$

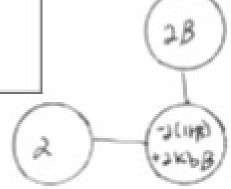
becomes:



$= g_{i,j} h^2 + 2kd\beta$

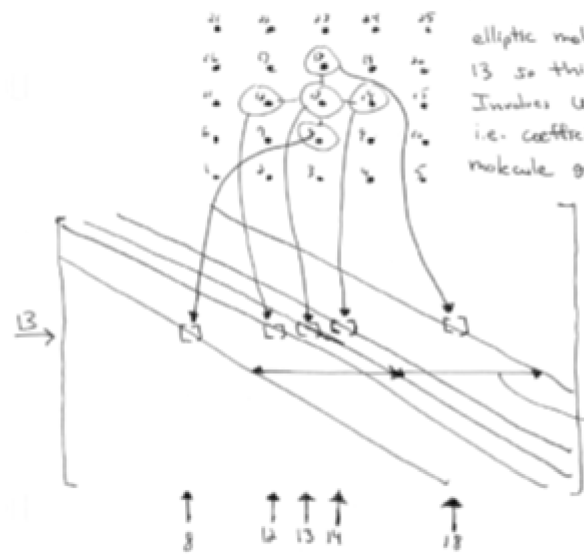
Corners: e.g





$$= g_{i,j}h^2 - 2ah + 2k\beta d$$

Basic Rule:
- Type I BC do not use PDE (i.e. let type 1 override)
- Type II, III, USE PDE PLUS BC together (i.e. use BC to eliminate shadow node)



elliptic molecule "centered" at 13 so this eqn for row 13. Involves $u_{i,j}$ at 12, 13, 14, 8, 18 i.e. coefficients in template molecule go in these rows

Half-bandwidth in this case = 5

Typical to store as banded matrix...

Classical Point Iteration Methods
(For Solving System $Au=b$)

- Need stopping criterion

Typical : $\| \underset{\sim}{u}^{\ell+1} - \underset{\sim}{u}^{\ell} \| < \epsilon$ Absolute

Better : $\dfrac{\| \underset{\sim}{u}^{\ell+1} - \underset{\sim}{u}^{\ell} \|}{\| \underset{\sim}{u}^{\ell+1} \|} < \epsilon$ Relative

- Determining $\rho(G)$ in practice
  - can compute w/ Power Method requires us to actually construct G → (see B+F, Num. An.)
  - estimate during iteration

$$\rho \approx \frac{\| \delta^{\ell} \|}{\| \delta^{\ell-1} \|} \quad \text{where } \delta^{\ell} = u^{\ell} - u^{\ell-1}$$

↳ so for error reduction by factor K

$$\| e \|^{\ell+m} = \rho^m \| e \|^{\ell}, \quad K = \rho^m \Rightarrow m > \frac{\log K}{\log(\rho)}$$

- Jacobi

$$u_{i,j}^{\ell+1} = \frac{1}{\beta_0}\left[\beta_1 u^{\ell}_{i+1,j} + \beta_2 u^{\ell}_{i-1,j} + \beta_3 u^{\ell}_{i,j+1} + \beta_4 u^{\ell}_{i,j-1} - h^2 g_{i,j} \right]$$

ℓ = iteration # ; $u^0_{i,j}$ "initial guess"

$$u_{i,j}^{\ell+1} = \frac{1}{\beta_0}\left[\beta_1 u^{\ell}_{i+1,j} + \beta_2 u^{\ell+1}_{i-1,j} + \beta_3 u^{\ell}_{i,j+1} + \beta_4 u^{\ell+1}_{i,j-1} - h^2 g_{i,j} \right]$$

- SOR

$$u_{i,j}^{\ell+1} = \frac{\omega}{\beta_0}\left[\beta_1 u^{\ell}_{i+1,j} + \beta_2 u^{\ell+1}_{i-1,j} + \beta_3 u^{\ell}_{i,j+1} + \beta_4 u^{\ell+1}_{i,j-1} - h^2 g_{i,j} \right] + (1-\omega) u^{\ell}_{i,j}$$

ω = relaxation parameter ($\omega = 1$ is Gauss Seidel)

Summary

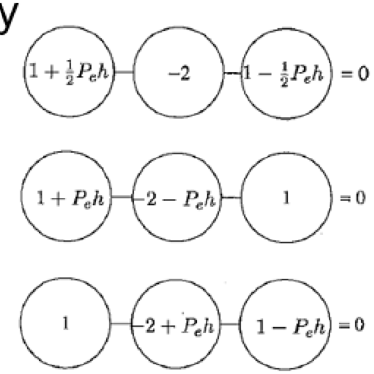
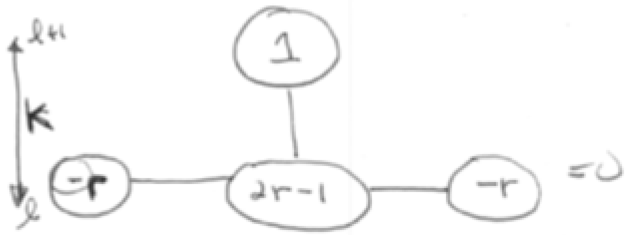


Figure 3.8: Centered, Upstream, and Downstream difference approximations to the Advective-Diffusive equation.

$$u_i^{l+1} - u_i^l - \frac{Dk}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right) = 0$$

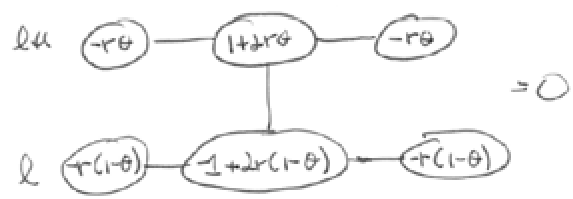
let $r = \frac{Dk}{h^2}$



- "Explicit"
- pointwise propagation
- $O(k+h^2)$
- Forward Difference in t
  - ↳ Euler integration
- conditional stability

2. SEMI-IMPLICIT MOLECULE

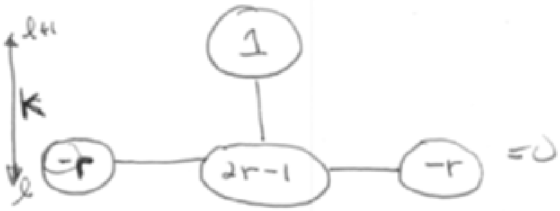
Generalization: $u_i^{l+1} - u_i^l = \underbrace{r\theta\delta_x^2 u_i^{l+1} + r(1-\theta)\delta_x^2 u_i^l}_{\text{averaging of } u_i^l + u_i^{l+1} \text{ weighted by } \theta}$



$\theta = 1$ fully implicit
$\theta = 0$ explicit
$\theta = .5$ "Crank-Nicolson"
(molecule centered @ $l+\frac{1}{2}$
$O(k^2+h^2)$)
$\theta \geq .5$ unconditional stability
$< .5$ conditional stability

$$u_i^{l+1} - u_i^l - \frac{Dk}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right) = 0$$

let $r = \frac{Dk}{h^2}$



- "Explicit"
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2. SEMI-IMPLICIT MOLECULE

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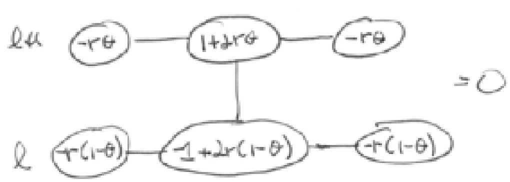
averaging of u_i^l + u_i^{l+1} weighted by θ



$\theta = 1$ fully implicit

$\theta = 0$ explicit

$\theta = .5$ "Crank-Nicolson"

(molecule centered @ $l+\frac{1}{2}$, $O(k^2+h^2)$)

$\theta \geq .5$ unconditional stability

$< .5$ conditional stability

1.) Convergence: $u_i^l \rightarrow U(x_i, t_l)$ as $h, k \rightarrow 0$

Exact solution to FD eqns

Exact solution to PDE

- requires Discretization error → 0 as meshlengths → 0

2.) Consistency: $L_i \rightarrow L$ as $h, k \rightarrow 0$

FD molecule

PDE operator

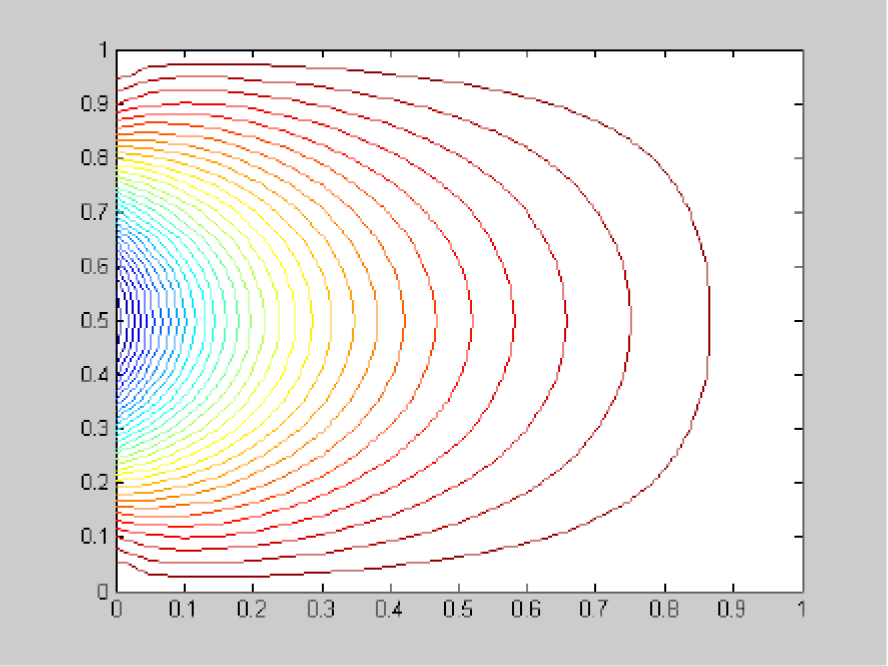
- weaker than convergence, but easier to show
- requires Truncation error → 0 as mesh lengths → 0

3.) Stability: u_i^l bounded for bounded BCs, ICs and forcing

Jacobi # of iterations: 582
Spectral radius estimate: 0.9877
Analytic spectral radius: $1-\pi^2h^2/2=0.98766$
$\Phi(0.7,0.7)=0.86971$

Gauss-Seidel # of iterations: 322
Spectral radius estimate: 0.97553
Analytic spectral radius: $(\rho_{jacobi})^2=0.97547$
$\Phi(0.7,0.7)=0.869996$

$\omega_{opt}=2/(1-\sqrt{(1-\rho_{Gauss\text{-}Seidel})})=1.7293$
$\rho_{SOR}=1.7293-1=0.7293$
$\rho_{estimate}=0.8339$
SOR # of iterations = 49
$\Phi(0.7,0.7)=0.870227$



Gauss-Seidel converges approximately twice as fast as Jacobi according to theory. Here we see that the reduction of iterations is approximately 2 with Jacobi taking 582 iterations verus 322 for Gauss-Seidel

In reducing our tolerance from 10^{-5} to 10^{-6} we expect the following number of additional iterations:

$$N_{Jacobi\ @\ 10^{-6}}=N_{Jacobi\ @\ 10^{-5}}+\log(0.1)/\log(\rho_{Jacobi})=582+\log(0.1)/\log(0.9877)=768$$
$$N_{Gauss\text{-}Seidel\ @\ 10^{-6}}=N_{Gauss\text{-}Seidel\ @\ 10^{-5}}+\log(0.1)/\log(\rho_{Gauss\text{-}Seidel})=322+\log(0.1)/\log(0.97553)=415$$

$\hookrightarrow$ so for error reduction by factor K

$\|e\|^{\ell+m} = \rho^m \|e\|^\ell$, $K = \rho^m \Rightarrow m > \frac{\log K}{\log(\rho)}$

In reducing our tolerance from 10^{-5} to 10^{-6} we expect the following number of additional iterations:

$$N_{Jacobi\ @\ 10^{-6}} = N_{Jacobi\ @\ 10^{-5}} + \log(0.1)/\log(\rho_{Jacobi}) = 582 + \log(0.1)/\log(0.9877) = 768$$

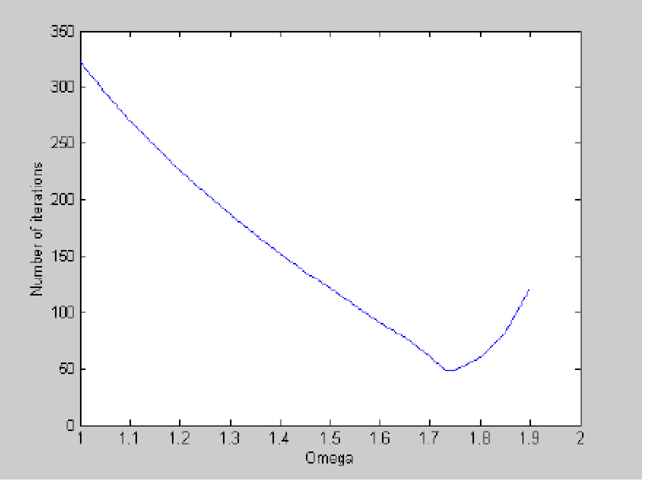
$$N_{Gauss\text{-}Seidel\ @\ 10^{-6}} = N_{Gauss\text{-}Seidel\ @\ 10^{-5}} + \log(0.1)/\log(\rho_{Gauss\text{-}Seidel}) = 322 + \log(0.1)/\log(0.97553) = 415$$

Rerunning the simulations at a tolerance of 10-6, produced the following:

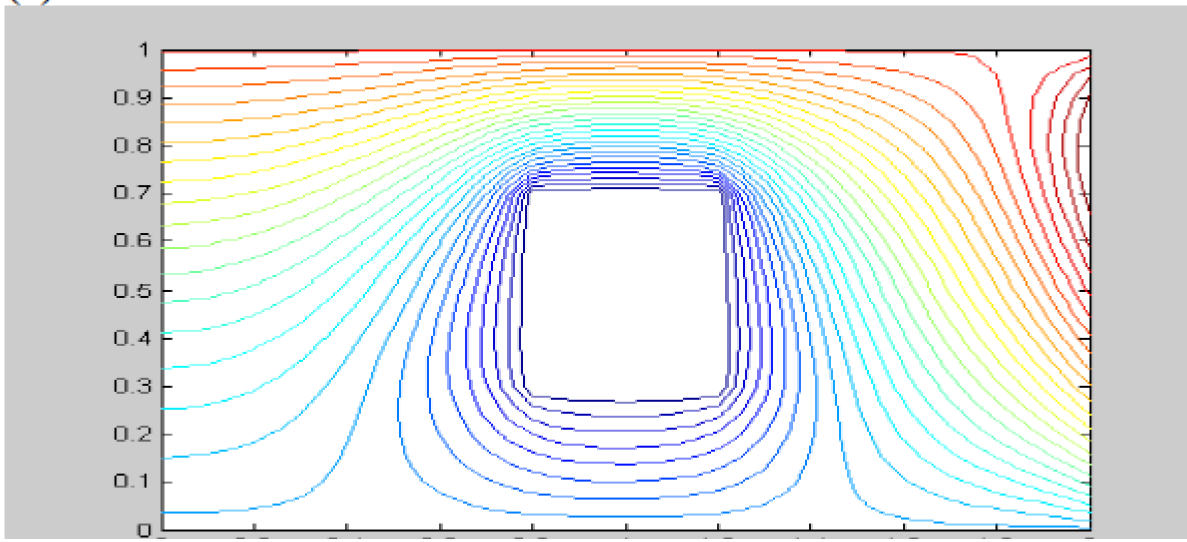
$N_{Jacobi\ @\ 10^{-6}} = 768$

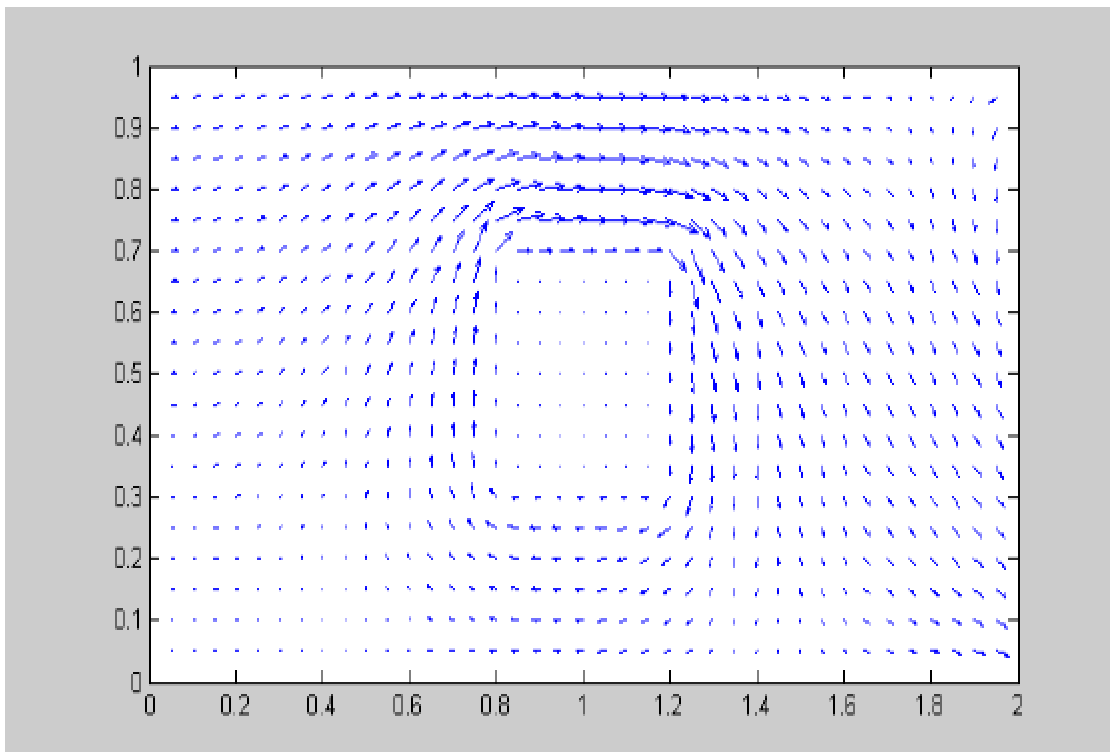
$N_{Gauss\text{-}Seidel\ @\ 10^{-6}} = 415$

The estimated increase in the number of iterations for the new tolerance is the same as the numerical output.



(a)

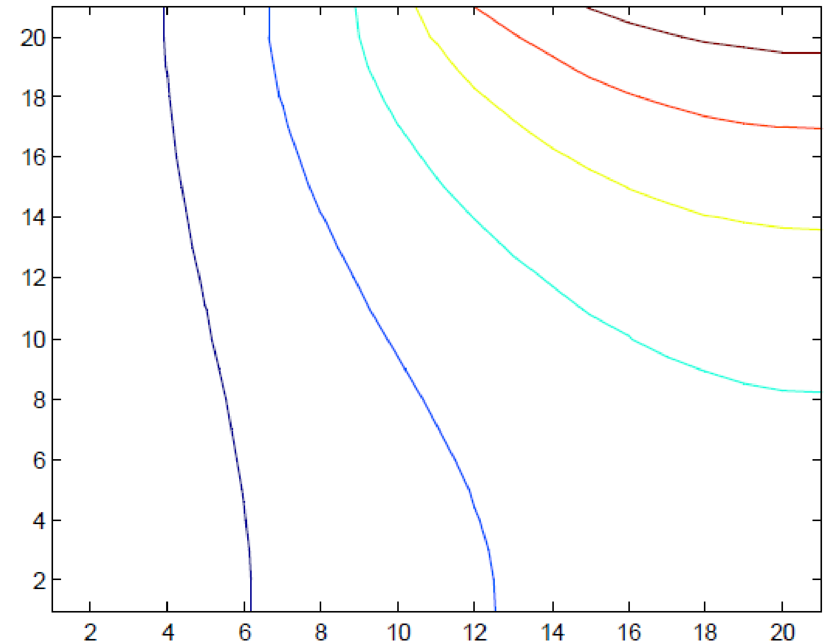

0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

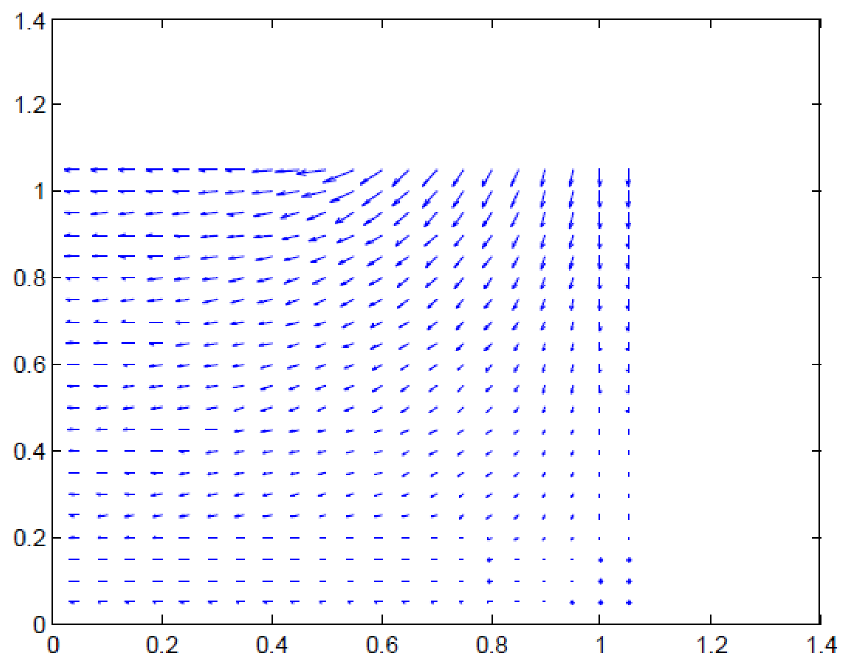

0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

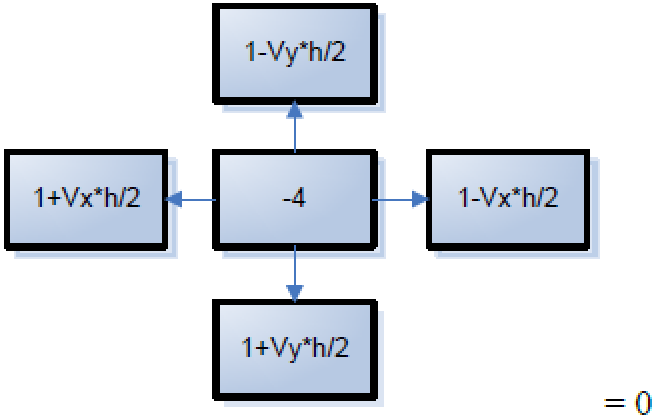
D

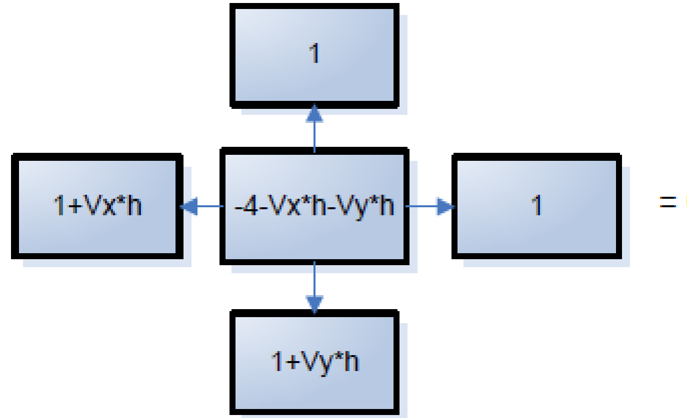
The velocity 'potential' isocontours (problem #1 physics) are orthogonal to the streamlines, i.e. if we were to be solving the potential problem of #1 with the boundary conditions of problem #2, the quiver plot shown above would be crossing potential lines orthogonally in the potential solution. As we can see in problem #2 streamlines, the stream function follows the vectors nicely. So streamlines actually indicate flow direction and where streamlines are concentrated such as at the top of the obstruction, this indicates large flow.

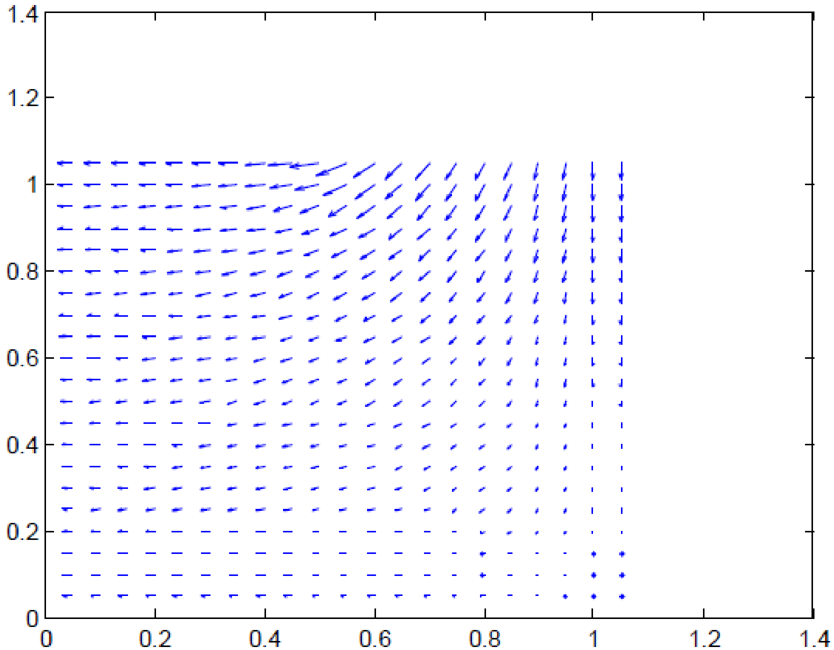
More specifically, as opposed to the potential where concentrated contours indicate rapid flow across the contours, concentrated contours in the stream function indicate rapid flow along the contours.

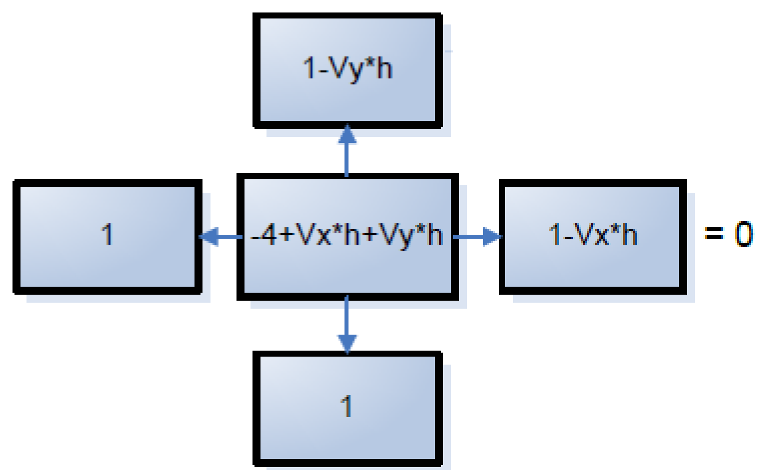

20
18
16
14
12
10
8
6
4
2
2 4 6 8 10 12 14 16 18 20

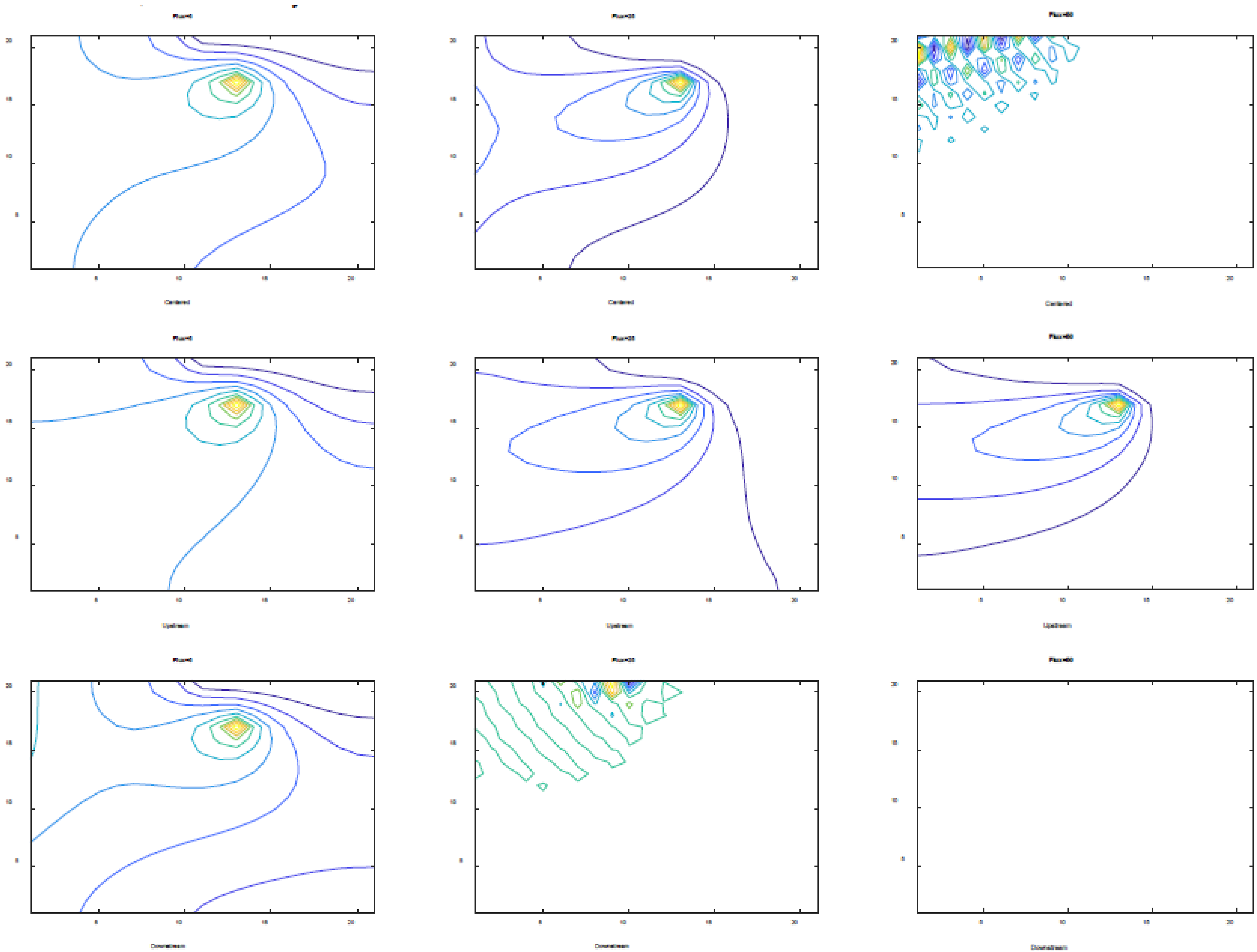

1.4
1.2
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1 1.2 1.4


1-Vy*h/2
1+Vx*h/2
-4
1-Vx*h/2
1+Vy*h/2
= 0


1
1+Vx*h
-4-Vx*h-Vy*h
1
1+Vy*h
= 0

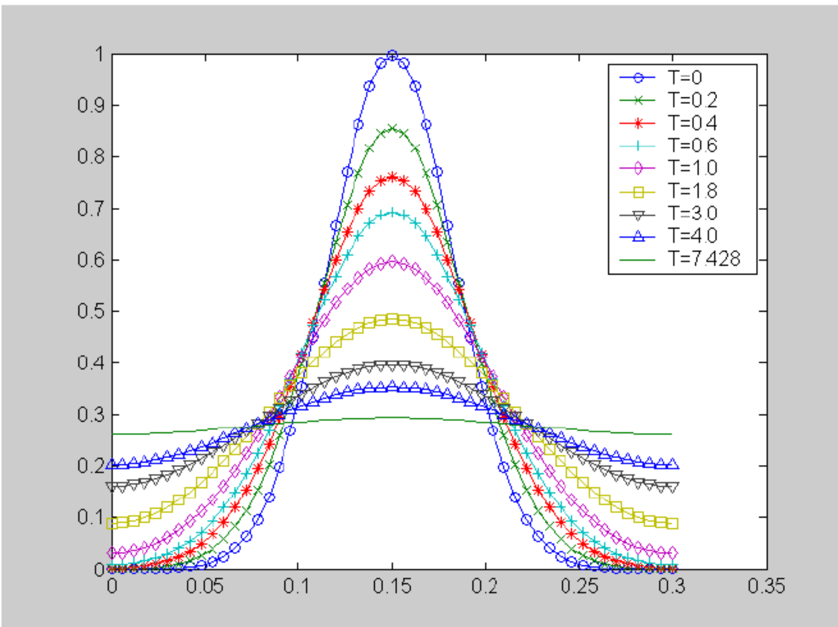

1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4

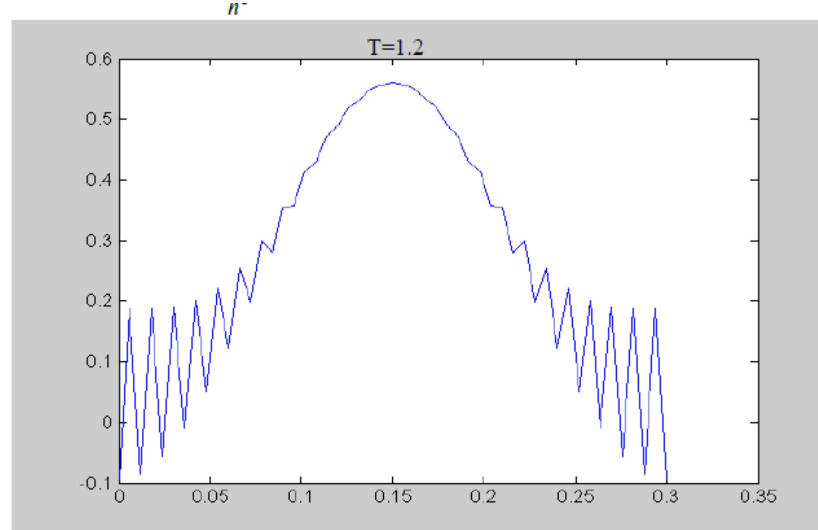

1-Vy*h
1
-4+Vx*h+Vy*h
1-Vx*h
1
= 0



LTE for RHS $= -D\frac{h^2}{12}\frac{\partial^4 C_i(\xi)}{\partial x^4}$ $\quad \xi \in [-h,h]$

LTE for LHS $= -\frac{\Delta t}{2}\frac{\partial^2 C(\xi)}{\partial t^2}$ $\quad \xi \in [0,\Delta t]$





(c) Now, investigate your numerical algorithm from part **(b)**, specifically, does your model go unstable? if so, can you determine a threshold for when instability occurs empirically. Now go back the notes from class, write out the Von Neumann analysis yourself step by step for this model. How do your findings with the Von Neumann analysis compare to your empirical analysis?

According to the Von Neumann analysis in the notes, if the quantity $\frac{D\Delta t}{h^2} < \frac{1}{4}$ we have a stable system. At $\Delta t = 0.0089$, the quantity is 0.2472