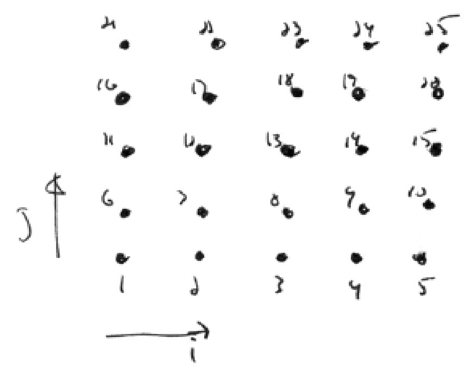
FDM Matrix Structure

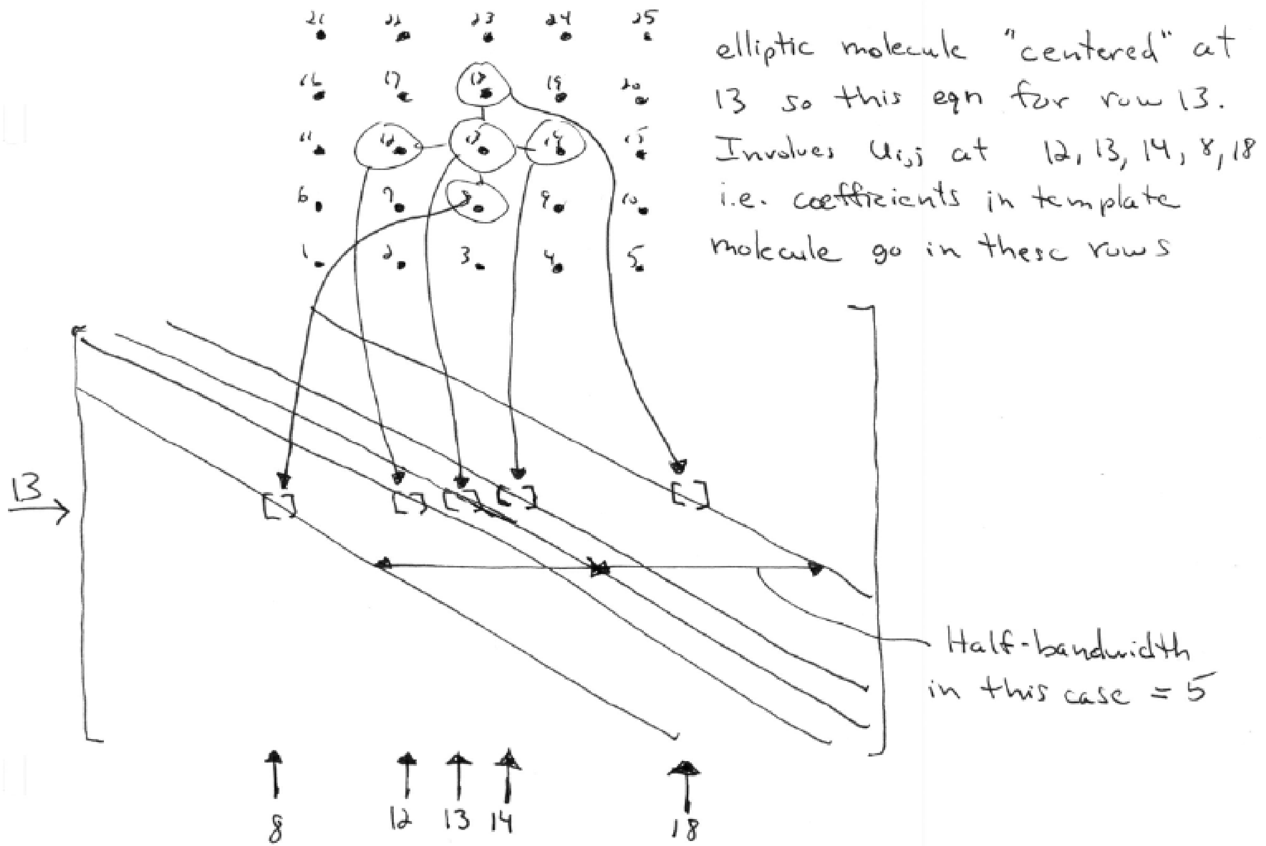
- Need a mapping between (i,j) template location and (i,j) matrix entry in A . . . assign a unique number to each mesh point . . . generates pentadiagonal structure provided some "natural" ordering is used

e.g



- each $u_{i,j}$ maps to a unique column in A
- each (i,j) template "center" maps to unique row in A

FDM Matrix Structure



Typical to store as banded matrix...

LT: Introduction to the Method of Weighted Residuals

Method of Weighted Residuals

- Consider $\nabla^2 \mathbf{u} + \mathbf{fu} = \mathbf{g} \rightarrow \underbrace{(\nabla^2 + \mathbf{f})}_{L}\mathbf{u} = \mathbf{g}$
- Finite Difference Approach
  - Approximate L w/ $L_{ij} \rightarrow \underbrace{(\delta^2 + \mathbf{f})}_{Lij}\mathbf{u}_{ij} = \mathbf{g}_{ij}$

    i.e. replace "differential" operator w/ "difference" operator $\rightarrow$ get "exact solution" to "approximate operator"
  - Limitations:
    - cumbersome on irregular meshes
    - curved boundaries difficult to handle
    - **u** only found (i,j) points – need interpolation strategy

Method of Weighted Residuals

- Weighted Residuals
  - Approximate **u** as $\hat{u} = \sum_{j=1}^{N} c_j \Phi_j(x, y, z)$

    coefficients

    known function
    - "basis" func.
    - "trial" func.
    - "expansion" func.
  - Define "Residual" $\rightarrow$ $R(\hat{u}) = L\hat{u} - g$
  - For exact solution: $R(\hat{u}) = 0$ everywhere then
  - Want $R(\hat{u}) = 0$ to vanish in average way, one way is in weighted integral sense

    $$\iiint R(\hat{u}) W(x, y) dxdydz = 0$$

    any function of position
  - " $R(\hat{u})$ orthogonal to all *W(x,y,z)*"

Method of Weighted Residuals

- So for $\hat{u}: \; R(\hat{u}) \neq 0 \; \ldots$ choose 'N' c_j's such that $\langle R(\hat{u}), W_i \rangle = 0$ for $i = 1,2,3,\ldots,N$

"Inner Product" $\Rightarrow \; \langle \mathbf{a},\mathbf{b} \rangle \equiv \iint \mathbf{a} \bullet \mathbf{b} \, \mathbf{dxdydz}$

- W_i's set of "weighting" functions → finite!

  local "testing" functions

- Use 'N' independent W_i's → generate 'N' equations in 'N' unknown c_j's

$$\langle R(\hat{u}), W_i \rangle = \sum_{j=1}^{N} c_j \langle L(\phi_j) W_i \rangle = \langle g, W_i \rangle \quad \text{for each } W_i(x,y) \;\; i = 1,2,\ldots N$$

- Necessary, but not sufficient for $\hat{u} = u$

Method of Weighted Residuals

- Continuous function must be zero if it is orthogonal to every member of a complete set $\therefore$ WRM can be thought of as a technique which enforces orthogonality between basis and weighting function sets
  - General idea is that the basis is a subset of a complete set that can represent any function, i.e. the true solution
  - As number of coefficients goes up, the approximation approaches the complete set

Method of Weighted Residuals

- Take home interpretation:
  - PDE is determined from first principles
  - Introduce the idea of a weighting function which is a known function of space – on its introduction, the physics behavior is prescribed to a region
  - Integration formulation
  - Introduction of basis as an approximation to the solution – on its introduction, it provides a description of how the solution will be treated behaviorally within in a local region

Method of Weighted Residuals

- Weighted Residual Methods Summary
  - L is typically a differential operator
  - W_i not complete in practice (N finite), but make "R(u) orthogonal to 1^{st} N members of a complete set"
  - "Approximate solution" which exactly satisfies "differential relations" in PDE
  - Is an "integral" formulation

Method of Weighted Residuals

- What's involved numerically?
  - Expand unknown solution as $\hat{\mathbf{u}} = \sum_{j=1}^{N} \mathbf{c}_j \phi_j(\mathbf{x}, \mathbf{y}, \mathbf{z})$
    - Finite sum
    - $\phi_j(\mathbf{x}, \mathbf{y}, \mathbf{z})$ known function of (x,y,z)
    - c_j's unknown coefficients to determine
  - Generate system of equations in unknowns

$$\mathbf{R}(\hat{\mathbf{u}}) = \langle (\mathbf{L}(\hat{\mathbf{u}}) - \mathbf{g}), \mathbf{W}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rangle = \mathbf{0} \text{ for } \mathbf{i} = 1,2,3,\ldots,\mathbf{N}$$

$$\sum_{j=1}^{N} \mathbf{c}_j \langle \mathbf{L}(\phi_j), \mathbf{W}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rangle = \langle \mathbf{g}, \mathbf{W}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rangle \text{ for } \mathbf{i} = 1,2,3,\ldots,\mathbf{N}$$

    - W_i(x,y,z) known functions of (x,y,z)
    - For each i, generate algebraic equation in c_j's
    - Creates 'N' equations in 'N' unknowns

Method of Weighted Residuals

- But analytic methods…
  - Require special knowledge of how to choose basis and weighting functions
  - Different choices are needed for different problems
  - Usually need an infinite # of them
  - Can't find them for many practical problems
- Numerically want …
  - Basis and Weighting functions to be simple … easy to integrate
  - Single choice suitable for many problems
  - Can only use finite #, but want convergence as number used increases

Method of Weighted Residuals

- WRM … eighting Funct…
  - Su… N' small reg… equal to uni…
  - Co… residual exa…
  - Le… weighting fur…
  - Mc… nction
  - Ga… ction are the…
- We … Residual Meth…

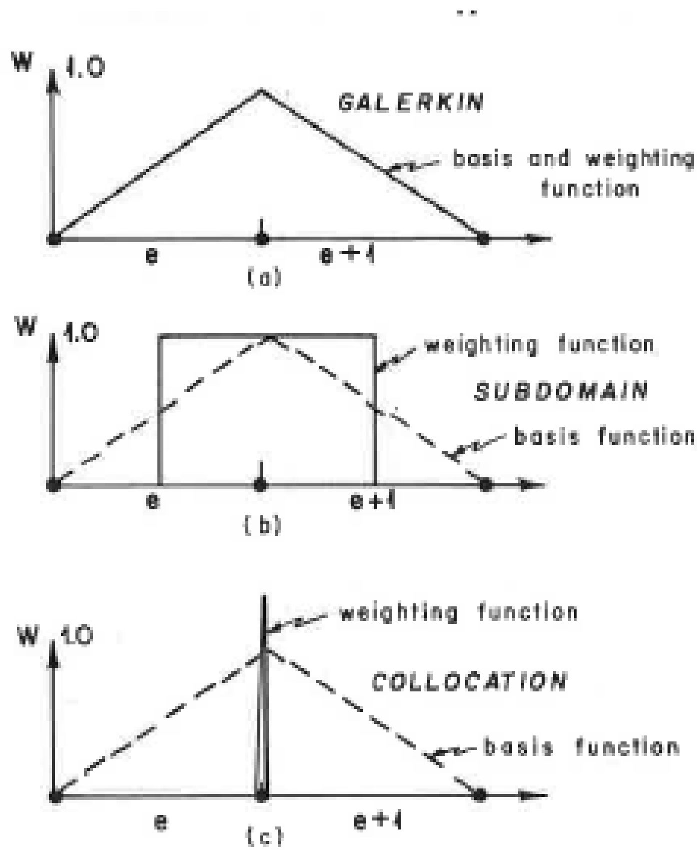


Figure 2.4. Schematic representation of the one-dimensional weighting functions for the Galerkin, subdomain and collocation methods. (It is assumed here that the chapeau function is used as a basis for all methods.)

Galerkin Weighted Residual Method

- Key Feature:
  - Basis function and weighting function are selected to be the same function form
- So what do we choose?
  - Polynomials may be a nice choice
  - Other’s exist

Polynomial Basis/Weighting Functions

- Why?
  - Easy to differentiate
  - Represent a complete set for continuous functions … e.g. Taylor Polynomial
- Lagrange Polynomials

$$\phi_i = \prod_{j \neq i} \frac{(x - x_i)}{(x_j - x_i)}$$

  - Nth order polynomial for N+1 pts
  - Easily automated
  - Handy on uneven grids
  - $\phi_i(x) = 0 \text{ at } x = x_i \text{ for } i \neq j$
  - $\phi_j(x) = 1 \text{ at } x = x_i \text{ for } i = j$

Polynomial Basis/Weighting Functions

- Lagrange Polynomials
  - $\phi_j(\mathbf{x}_j) = 1 \Rightarrow \mathbf{u}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{c}_j \phi_j(\mathbf{x}, \mathbf{y}) \Rightarrow \mathbf{u}(\mathbf{x}_i) = \mathbf{c}_i$
  - $\therefore \quad \mathbf{u}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{u}_j \phi_j(\mathbf{x}, \mathbf{y})$

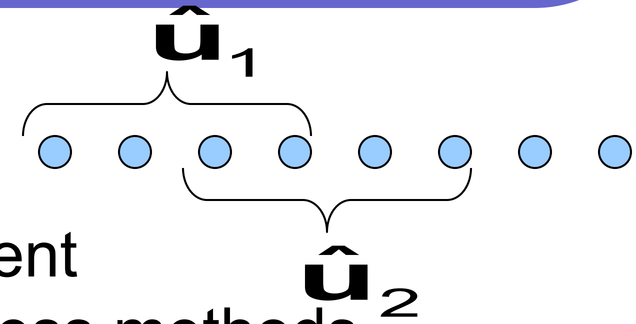
    Coefficients are solution at nodes … similar to FD but have functional form specified in between
  - everywhere $\sum_{j=1}^{N} \phi_j(\mathbf{x}, \mathbf{y}) = 1$
  - $\sum_{j=1}^{N} \frac{\partial \phi_j(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \sum_{j=1}^{N} \phi_j(\mathbf{x}, \mathbf{y}) = 0$

Polynomial Basis/Weighting Functions

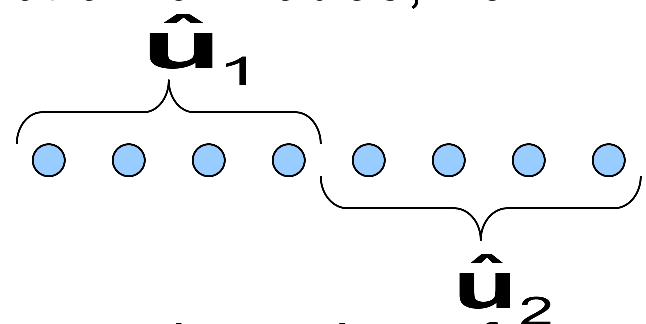
- Global Polynomials
  - Potential for disaster … "polynomial wiggle"
  - Can have N zeros
  - Sensitive to all u_j - "Global Support"
  - Can have large variation between points
- Local Interpolation
  - Use subset of nodes to represent solution over localized areas – so-called finite elements
  - Possibilities : Overlapping Domains, Non-overlapping Domains

Polynomial Basis/Weighting Functions

- Overlapping Domains
  - Nonunique
  - Somewhat difficult to implement
  - Further study of this in meshless methods



- Non-overlapping Domains
  - Idea of finite "element" as collection of nodes, i.e. unit of local interpolation
  - Unequal spacing of nodes, i.e. conformity of shape
  - Basic building block of FEM
  - Typically use same type of element throughout for programming ease (unless reason not to do so)



Polynomial Basis/Weighting Functions

- Continuity of $\hat{\mathbf{u}}$: In 1D, if N+1 nodes/element
  - $\hat{\mathbf{u}}$ is locally N^{th} order polynomial
  - On element **interior** … 1^{st} N derivatives continuous
  - … but, at element boundaries only $\hat{\mathbf{u}}$ is continuous; $\partial\hat{\mathbf{u}}/\partial\mathbf{x}$ changes abruptly
  - "C^o" continuity → continuous in 0^{th} derivative
- Higher order continuity possible

  e.g. "C^1" continuity … need Hermite polynomial
  - Simplest local unit: Hermite cubic
  - $\partial\hat{\mathbf{u}}/\partial\mathbf{x}$ becomes "nodal parameter"

FEM Steps – 1D Example Problem

FEM Steps

- Determine PDE expression for physics
- On Paper
  - Weight each expression with “trial” function
  - Integrate the expression
  - Approximate solution as expansion containing coefficients and “basis” functions
  - Select the “weighting” function
- In Code
  - Assemble matrices by a process of multi-dimensional numerical integration
  - Apply boundary conditions
  - Solve for unknown expansion coefficients
  - Construct approximate solution
  - Perhaps construct derived quantities

1D FEM Example

Example: $\frac{d^2u}{dx^2} + fu = g$ $u(0)=1$, $\frac{du}{dx}(L)=5$

constant

Step 1: Generate weighted residual equation

$$\underbrace{\left\langle \frac{d^2u}{dx^2}, w_i \right\rangle}_{\text{common to integrate by parts}} + \langle fu, w_i \rangle = \langle g, w_i \rangle$$

$$\frac{du}{dx} w_i \Big|_0^L - \left\langle \frac{du}{dx} \frac{dw_i}{dx} \right\rangle + \langle fu, w_i \rangle = \langle g\, w_i \rangle$$

1-D FEM Example

$$\left\langle \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\mathbf{du}}{\mathbf{dx}}\right), \phi \right\rangle$$

$$\int \mathbf{udv} = \mathbf{uv} - \int \mathbf{vdu}$$

$$\mathbf{u} = \phi \qquad \mathbf{dv} = \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\mathbf{du}}{\mathbf{dx}}\right)\mathbf{dx}$$

$$\mathbf{du} = \frac{\mathbf{d}\phi}{\mathbf{dx}}\mathbf{dx} \qquad v = \left.\frac{du}{dx}\right|_0^{\mathbf{L}}$$

so

$$\left\langle \frac{\mathbf{d}^2\mathbf{u}}{\mathbf{dx}^2}, \phi \right\rangle = \phi \left.\frac{du}{dx}\right|_0^{\mathbf{L}} - \left\langle \frac{du}{dx} \frac{\mathbf{d}\phi}{\mathbf{dx}} \right\rangle$$

OR

$$\underbrace{\left\langle \mathbf{f}\nabla^2\mathbf{g} + \nabla\mathbf{f} \bullet \nabla\mathbf{g} \right\rangle = \oint \mathbf{f}\nabla\mathbf{g} \bullet \hat{\mathbf{n}}\mathbf{dS}}_{\text{First form of Green's theorem}}$$

let

$$\mathbf{f} = \phi, \quad g = u$$

$$\left\langle \phi \frac{\mathbf{d}^2\mathbf{u}}{\mathbf{dx}^2} \right\rangle = \phi \left.\frac{du}{dx}\right|_0^{\mathbf{L}} - \left\langle \frac{du}{dx} \frac{\mathbf{d}\phi}{\mathbf{dx}} \right\rangle$$

1-D FEM Example

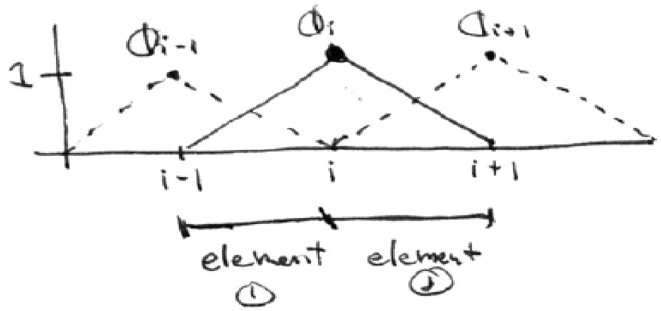
Why... reduces continuity requirements needed on ϕ_j, w_i ... integrand may be piecewise discontinuous with finite discontinuities.

Another sense in which the approach is "weak"

1-D FEM Example

Step 2: choose Basis for $\hat{u}$

- Try simplest Lagrange polynomial ⇒ Linear
- C^0 continuity... ok for this PDE
- has local support



"chapeau"

"Hat"

"Roof-top"

Step 3: Choose weighting function; $w_i = \Phi_i$; Galerkin

1-D FEM Example

Step 4: Assemble matrices (evaluate coefficients)

- each row of matrix is $\langle R, \phi_i \rangle = 0$

But $\phi_i = 0$ over most of x (local)

$\phi_i \neq 0$ only on two elements

$$\int (\quad)\, \phi_i\, dx = \underbrace{\int_{①} (\quad)\, \phi_i\, dx}_{\text{integration over element ①}} + \underbrace{\int_{②} (\quad)\, \phi_i\, dx}_{\text{integration over element ②}}$$

$$\Rightarrow -\left\langle \frac{du}{dx}\frac{dw_i}{dx} - f u w_i \right\rangle = \langle g, w_i \rangle - \frac{du}{dx} w_i \Big|_0^L$$

1-D FEM Example

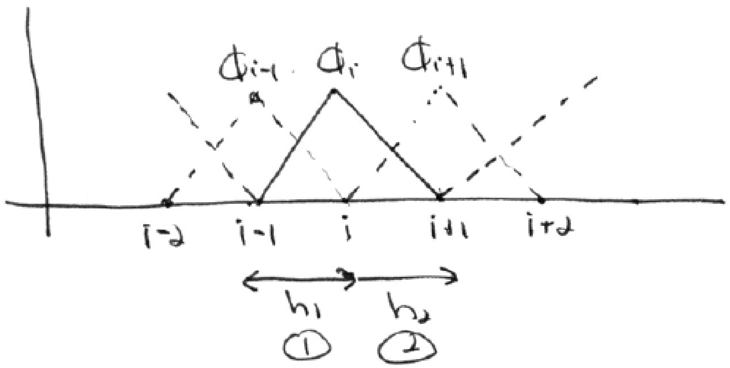
$$-\sum_{j=1}^{N} u_j \left\langle \frac{d\phi_j}{dx}\frac{d\phi_i}{dx} - f\,\phi_j\phi_i \right\rangle = \langle g, \phi_i \rangle - \frac{du}{dx}\phi_i \Big|_0^L$$

Do this for each $i = 1, 2, \ldots N$; becomes system

$$[A][u] = \{b\} \quad w/ \quad a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\,\phi_i\phi_j \right\rangle$$

$$b_i = \langle g\,\phi_i \rangle - \frac{du}{dx}\phi_i \Big|_0^L$$

1-D FEM Example



$$a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\,\phi_i\phi_j \right\rangle$$

for $j = 1, 2, \ldots N$ but only $j = i-1, i, i+1$ contribute

basis — weighting

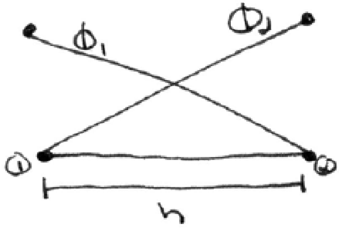
$\phi_{i-2} = 0$ on 1 + 2

so:

$$j = i-1: \quad \left\langle \frac{d\phi_{i-1}}{dx}\frac{d\phi_i}{dx} \right\rangle = \overset{1}{-\frac{1}{h_1}\frac{1}{h_1}h_1} + \overset{2}{0} = -1/h_1$$

$$j = i: \quad \left\langle \frac{d\phi_i}{dx}\frac{d\phi_i}{dx} \right\rangle = \overset{1}{\frac{1}{h_1}\frac{1}{h_1}h_1} + \overset{2}{\frac{-1}{h_2}\frac{-1}{h_2}h_2} = \frac{1}{h_1} + \frac{1}{h_1}$$

1-D FEM Example



$$\Phi_1 = \frac{x_2 - x}{h} \qquad \frac{d\Phi_1}{dx} = \frac{-1}{h}$$

$$\Phi_2 = \frac{x - x_1}{h} \qquad \frac{d\Phi_2}{dx} = \frac{1}{h}$$

e.g. $$\langle \Phi_{i-1}\, \Phi_i \rangle = \int_{x_{i-1}}^{x} \left(\frac{x - x_i}{x_{i-1} - x_i}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) dx$$

$$= -\frac{1}{h_i^2} \int_{x_{i-1}}^{x_i} (x - x_i)(x - x_{i-1})\, dx$$

$$= -\frac{1}{h_i^2} \int_{x_{i-1}}^{x_i} x^2 - x(x_i + x_{i-1}) + x_i x_{i-1} \; dx$$

$$= -\frac{1}{h_i^2} \left[\frac{x^3}{3} - \frac{x^2}{2}(x_i + x_{i-1}) + x\, x_i\, x_{i-1} \right]_{x_{i-1}}^{x_i}$$

w/o loss of generality ... pick $x_{i-1} = 0$

$$= -\frac{1}{h_i^2}\left[\frac{x_i^3}{3} - \frac{x_i^3}{2}\right] = \frac{1}{h_i^2}\left[\frac{h_i^3}{2} - \frac{h_i^3}{3}\right] = \frac{h_i}{6}$$

1-D FEM Example

$$= -\frac{1}{h_i^2}\left[\frac{x_i^3}{3} - \frac{x_i^3}{2}\right] = \frac{1}{h_i^2}\left[\frac{h_i^3}{2} - \frac{h_i^3}{3}\right] = \frac{h_i}{6}$$

... conclude we have exact integration/differentiation in our WR method ... approximation is in the assumption of linear variation of solution between nodes

1-D FEM Example

Assembly of Row i:

Now @ row i, equation i, we have looked at all the elements involved and consequently all nodes involved:

$\Rightarrow$ from $j = i-1, i, i+1$ of the first term of our WR equation we get $\hookrightarrow \left\langle -\frac{d\alpha_j}{dx}\frac{d\alpha_i}{dx}\right\rangle$

$$\underbrace{\frac{1}{h_1}u_{i-1}}_{j=i-1} \underbrace{-\left(\frac{1}{h_1}+\frac{1}{h_2}\right)u_i}_{j=i} + \underbrace{\frac{1}{h_2}u_{i+1}}_{j=i+1}$$ respectively

1-D FEM Example

and from our second term of our WR expression

$\swarrow \langle f\phi_s \, \phi_i \rangle$

$$+ \frac{fh_1}{6} u_{i-1} + f\left(\frac{h_1+h_2}{3}\right) u_i + \frac{fh_2}{6} u_{i+1}$$

and our RHS terms become:

$$= \underbrace{g_{i-1}\frac{h_1}{6} + g_i\left(\frac{h_1+h_2}{3}\right) + g_{i+1}\frac{h_2}{6}}_{g = \sum g_k \phi_k}$$

1-D FEM Example

Uniform h:

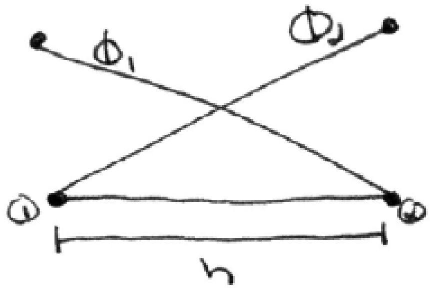
$$\frac{1}{h^2}\underbrace{\left(u_{i-1}-2u_i+u_{i+1}\right)}_{\delta^2 u_i} + \underbrace{\frac{f}{6}\left(u_{i-1}+4u_i+u_{i+1}\right)}_{\text{Simpson's Rule}}$$

$$= \frac{1}{6}\left(g_{i-1}+4g_i+g_{i+1}\right)$$

Very similar to FD method!!

1-D FEM Example

Integration Formulas for 1D Elements



$$\langle (\quad) \rangle = \int_{x_1}^{x_2} (\quad)\, dx$$

$$\Phi_1 = \frac{x_2 - x}{h} \qquad \frac{d\Phi_1}{dx} = \frac{-1}{h}$$

$$\Phi_2 = \frac{x - x_1}{h} \qquad \frac{d\Phi_2}{dx} = \frac{1}{h}$$

1-D FEM Example

$$\langle 1 \rangle = h$$

$$\langle \phi_1 \rangle = \langle \phi_2 \rangle = h/2$$

$$\langle \phi_1 \phi_2 \rangle = h/6$$

$$\langle \phi_1 \phi_1 \rangle = \langle \phi_2 \phi_2 \rangle = h/3$$

$$\langle \phi_1^3 \rangle = \langle \phi_2^3 \rangle = h/4$$

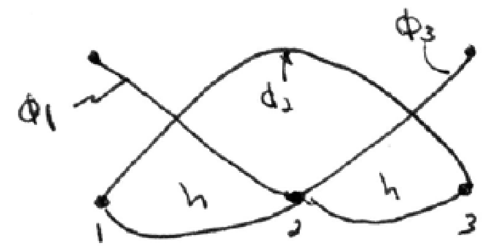
$$\langle \phi_1^2 \phi_2 \rangle = \langle \phi_2^2 \phi_1 \rangle = h/12$$

$$\left\langle a(x) \frac{d\phi_1}{dx} \right\rangle = -\frac{1}{h} \langle a(x) \rangle$$

$$\left\langle a(x) \frac{d\phi_2}{dx} \right\rangle = \frac{1}{h} \langle a(x) \rangle$$

1-D FEM Example

Integration Formulas for 1-D Quadratic Elements
w/ Equally spaced nodes



$$x_2 = 0$$

$$\langle (\quad) \rangle \equiv \int_{x_1=-h}^{x_3=h} (\quad)\, dx$$

$$\dot{\Phi}_i = \frac{d\Phi_i}{dx}$$

$$\Phi_1 = \frac{x(x-h)}{2h^2} \qquad \dot{\Phi}_1 = \frac{(2x-h)}{2h^2}$$

$$\Phi_2 = \frac{-(x+h)(x-h)}{h^2} \qquad \dot{\Phi}_2 = \frac{-2x}{h^2}$$

$$\Phi_3 = \frac{(x+h)x}{2h^2} \qquad \dot{\Phi}_3 = \frac{(2x+h)}{2h^2}$$

1-D FEM Example

$$\langle \phi_1 \rangle = h/3$$
$$\langle \phi_2 \rangle = 4h/3$$
$$\langle \phi_3 \rangle = h/3$$

$$\langle \phi_1 \phi_2 \rangle = 2h/15$$
$$\langle \phi_2 \phi_3 \rangle = 2h/15$$
$$\langle \phi_3 \phi_1 \rangle = -h/15$$

$$\langle \phi_1 \phi_1 \rangle = 4h/15$$
$$\langle \phi_2 \phi_2 \rangle = 16h/15$$
$$\langle \phi_3 \phi_3 \rangle = 4h/15$$

$$\langle \dot\phi_1 \rangle = -1$$
$$\langle \dot\phi_2 \rangle = 0$$
$$\langle \dot\phi_3 \rangle = 1$$

$$\langle \dot\phi_1 \dot\phi_2 \rangle = -4/3h$$
$$\langle \dot\phi_2 \dot\phi_3 \rangle = -4/3h$$
$$\langle \dot\phi_1 \dot\phi_3 \rangle = 1/6h$$

$$\langle \dot\phi_1 \dot\phi_1 \rangle = 7/6h$$
$$\langle \dot\phi_2 \dot\phi_2 \rangle = 8/3h$$
$$\langle \dot\phi_3 \dot\phi_3 \rangle = 7/6h$$

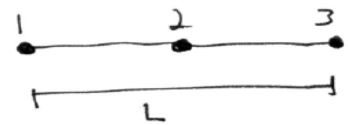
$$\langle \phi_1 \dot\phi_1 \rangle = -1/2$$
$$\langle \phi_1 \dot\phi_2 \rangle = 2/3$$
$$\langle \phi_1 \dot\phi_3 \rangle = -1/6$$

$$\langle \phi_2 \dot\phi_1 \rangle = -2/3$$
$$\langle \phi_2 \dot\phi_2 \rangle = 0$$
$$\langle \phi_2 \dot\phi_3 \rangle = 2/3$$

$$\langle \phi_3 \dot\phi_1 \rangle = 1/6$$
$$\langle \phi_3 \dot\phi_2 \rangle = -2/3$$
$$\langle \phi_3 \dot\phi_3 \rangle = 1/2$$

1-D FEM Example

MORE INTEGRATION FORMULAS FOR QUADRATIC ELEMENTS



$$\langle \phi_1^3 \rangle = \langle \phi_3^3 \rangle = \frac{39L}{420}$$

$$\langle \phi_1^2 \phi_2 \rangle = \langle \phi_2 \phi_3^2 \rangle = \frac{20L}{420}$$

$$\langle \phi_1^2 \phi_3 \rangle = \langle \phi_1 \phi_3^2 \rangle = -3L/420$$

$$\langle \phi_1 \phi_2^2 \rangle = \langle \phi_2^2 \phi_3 \rangle = 16L/420$$

$$\langle \phi_1 \phi_2 \phi_3 \rangle = -8L/420$$

$$\langle \phi_2^3 \rangle = 192L/420$$

Matrix Assembly

Matrix Assembly

- In FD common to proceed "molecule-by-molecule" with each representing single difference equation
- In FE more natural to proceed "element-by-element" need to integrate over problem domain ⇒ union of non overlapping elements ...

e.g. $[A]\{u\}=\{b\}$

where $$a_{ij}=\left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{\partial x}+f\phi_i\phi_j\right\rangle ;\ b_j=\langle g\phi_i\rangle-\frac{du}{dx}\phi_i\Big|_0^L$$

Matrix Assembly

then $\langle (\ ) \rangle = \int (\ )\,dx = \underbrace{\sum_{\text{elements}} \int_e (\ )\,dx}_{\text{sum of elements comprises entire domain}} = \sum_e \underbrace{\langle\ \rangle^e}_{\text{element contribution to inner product}}$

so $[A] = \sum_e \underbrace{[A]^e}$

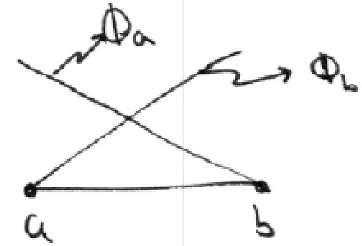
"element matrix" ... i.e contains <u>all</u> contributions to $[A]$ for a given element

likewise $\{b\} = \sum_e \{b\}^e$

Matrix Assembly

- to get $[A]^e$ need to determine contributions from a general element

  e.g. 1D linear (Chapeau)



- only 2 "active" (i.e. nonzero) in any element

- each weighting function constitutes a single equation in unknown coefficients ... 2 possible weighting functions : Φ_a, Φ_b

  a given row in [A]

- each basis function has a corresponding unknown coefficient ... 2 possible basis functions : Φ_a, Φ_b

  a given column in [A]

Matrix Assembly

All possible combinations...

Weighting function (Row)	Basis function (Column)
Φ_a	Φ_a
Φ_a	Φ_b
Φ_b	Φ_a
Φ_b	Φ_b

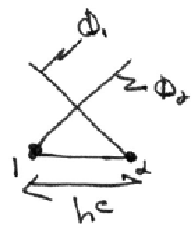
So $[A]^e$ has only 4 nonzero coefficients: $a^e_{aa}, a^e_{ab}, a^e_{ba}, a^e_{bb}$

$$\begin{array}{c} a \\ b \end{array} \begin{bmatrix} a^e_{aa} & a^e_{ab} \\ a^e_{ba} & a^e_{bb} \end{bmatrix}$$

Matrix Assembly

so $[A]^e$ may be stored as a 2x2 submatrix where we use a "local" node numbering scheme

e.g.



↳ depends on element

$$\Rightarrow \begin{bmatrix} a_{11}^e & a_{12}^e \\ a_{21}^e & a_{22}^e \end{bmatrix}$$

structure the same regardless of PDE for linear 1D element... only details of cofficients differ

$$a_{11} = \left\langle -\frac{d\Phi_1}{dx}\frac{d\Phi_1}{dx} + f\Phi_1\Phi_1 \right\rangle = \frac{-1}{h^e} + \frac{fh^e}{3}$$

$$a_{12} = \left\langle -\frac{d\Phi_2}{dx}\frac{d\Phi_1}{dx} + f\Phi_2\Phi_1 \right\rangle = \frac{1}{h^e} + \frac{fh^e}{6}$$

$$a_{22} = \left\langle -\frac{d\Phi_2}{dx}\frac{d\Phi_1}{dx} + f\Phi_2\Phi_2 \right\rangle = -\frac{1}{h^e} + \frac{fh^e}{3}$$

$$a_{21} = \left\langle -\frac{d\Phi_1}{dx}\frac{d\Phi_2}{dx} + f\Phi_1\Phi_2 \right\rangle = \frac{1}{h^e} + \frac{fh^e}{6}$$

Matrix Assembly

$$b_1^e = \langle g\,\phi_1 \rangle^e = \frac{gh^e}{\alpha}$$

$$b_2^e = \langle g\,\phi_2 \rangle = \frac{gh^e}{\alpha}$$

(assume g constant ... and neglecting boundary term for the moment...)

so

$$[A]^e = \begin{bmatrix} -\frac{1}{h^e} + \frac{fh^e}{3} & \frac{1}{h^e} + \frac{fh^e}{6} \\ \frac{1}{h^e} + \frac{fh^e}{6} & -\frac{1}{h^e} + \frac{fh^e}{3} \end{bmatrix} ; \{b\}^e = \begin{Bmatrix} \frac{gh^e}{\alpha} \\ \frac{gh^e}{\alpha} \end{Bmatrix}$$