4. A)

We have: $\langle S \rangle = \frac{1}{N}\sum_{i=1}^{N} S_i = f(S_1, S_2, \ldots, S_N)$

- $\sigma = \sigma_1 = \sigma_2 = \ldots = \sigma_N$ where σ_i is the variance of noise in image "i"
- Noise are uncorrelated so their covariance $= 0$
- And we assume noise has zero mean

So our noise propagation model to the avaraged image is:

$$\sigma^2_{\langle S \rangle} = \sigma^2\left[\left(\frac{\partial \langle S \rangle}{\partial S_1}\right)^2 + \left(\frac{\partial \langle S \rangle}{\partial S_2}\right)^2 + \ldots + \left(\frac{\partial \langle S \rangle}{\partial S_N}\right)^2\right]$$

with $\langle S \rangle = \frac{1}{N}\sum_{i=1}^{N} S_i$ hence $\frac{\partial \langle S \rangle}{\partial S_1} = \frac{1}{N}(\underbrace{1 + 0 + \ldots + 0}_{N \text{ items}}) = \frac{1}{N}$ and same for all "S_i" where $i = \{1; 2; \ldots; N\}$

Thus

$$\sigma^2_{\langle S \rangle} = \sigma^2\left[\underbrace{\frac{1}{N^2} + \frac{1}{N^2} + \ldots + \frac{1}{N^2}}_{N \text{ items}}\right]$$

$$= \frac{\sigma^2}{N}$$

So $\frac{1}{\sigma_{\langle S \rangle}} = \sqrt{N} \times \frac{1}{\sigma}$

Therefore: $\langle SNR(N) \rangle = \frac{\langle S \rangle}{\sigma_{\langle S \rangle}}$

$$= \sqrt{N} \cdot \frac{1}{\sigma} \cdot \langle S \rangle \quad (*)$$

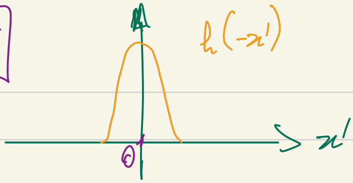
B) So to double $\langle SNR(N) \rangle$, we need to capture $N = 4$ images based on $(*)$

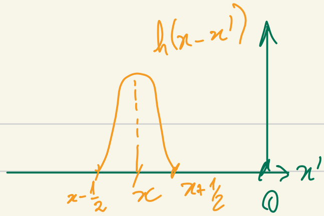
and to increase by factor of 100, we need $N = 10{,}000$

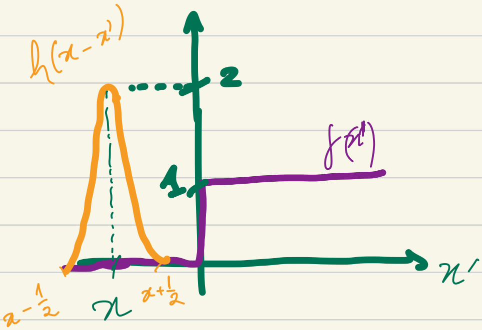
The limitations are: + higher patient doses in CT if we capture for example, 10,000 images.

+ longer scan time if we do MRI

$\boxed{5}$







For $x+\frac{1}{2}<0$ or $x<-\frac{1}{2}$

No overlap $g(x)=0$

For $-\frac{1}{2}\leq x\leq 0$:

$$g(x)=\int_0^{(x+1/2)} f(x')\,h(x-x')\,.dx'$$

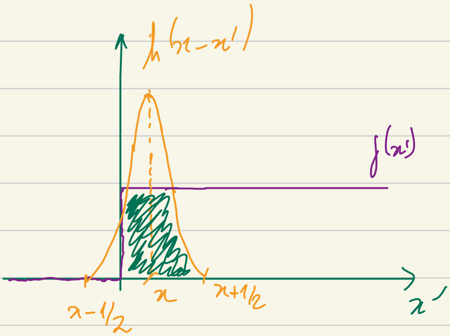
$$=\int_0^{(x+1/2)} 1.\left(1+\cos\left[2\pi(x-x')\right]\right).dx'$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\int_0^{(x+1/2)}\cos\left[2\pi(x-x')\right]d\left(2\pi(x-x')\right)$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\sin\left(2\pi(x-x')\right)\Big|_{x'=0}^{x'=(x+1/2)}$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\left(\sin(-\pi)-\sin 2\pi x\right)$$

$$=x+\frac{1}{2}+\frac{1}{2\pi}.\sin 2\pi x$$



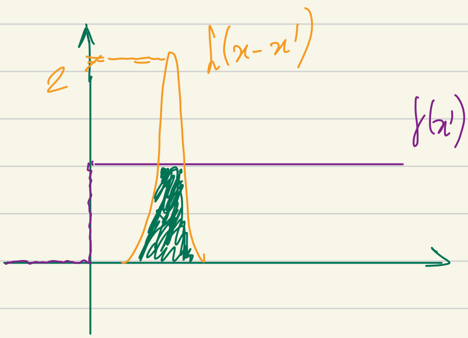
For $0 \leq x \leq \frac{1}{2}$:

$$g(x) = \int_0^x \left(1+\cos[2\pi(x-x')]\right) dx' + \int_x^{x+\frac{1}{2}} \left(1+\cos[2\pi(x-x')]\right) dx'$$

$$= x - \frac{1}{2\pi}\int_{x'=0}^{x'=x} \cos[2\pi(x-x')]\, d(2\pi(x-x')) + \frac{1}{2} - \frac{1}{2\pi}\int_{x'=x}^{x'=x+1/2} \cos[2\pi(x-x')]\, d(2\pi(x-x'))$$

$$= x - \frac{1}{2\pi}\left[0 - \sin 2\pi x\right] + \frac{1}{2} - \frac{1}{2\pi}\left[0 - 0\right]$$

$$= x + \frac{1}{2} + \frac{1}{2\pi}\sin 2\pi x$$



For $x > \frac{1}{2}$:

$$g(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(x')\cdot h(x-x')\cdot dx'$$

$$= 2\int_x^{x+1/2} 1\cdot\left(1+\cos[2\pi(x-x')]\right) dx'$$

$$= 2\left(x + 1/2 - x\right) + 2\left(\frac{-1}{2\pi}\right)\int_{x'=x}^{x'=(x+\frac{1}{2})} \cos\left[2\pi(x-x')\right] d\left(2\pi(x-x')\right)$$

$$= 1 - \frac{1}{\pi} \times \sin 2\pi(x-x')\Big|_{x}^{x+1/2}$$

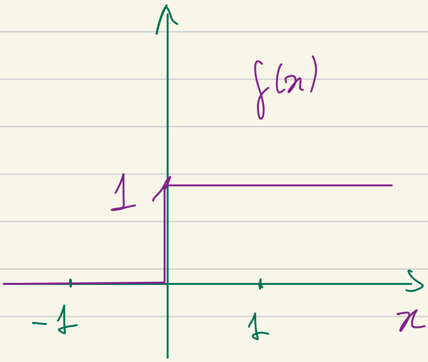
$$= 1 - \frac{1}{\pi} \times (0 - 0)$$

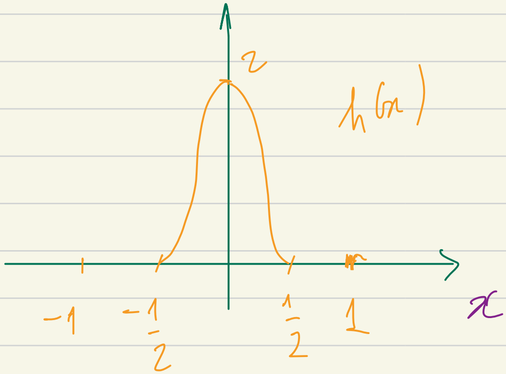
$$= 1$$

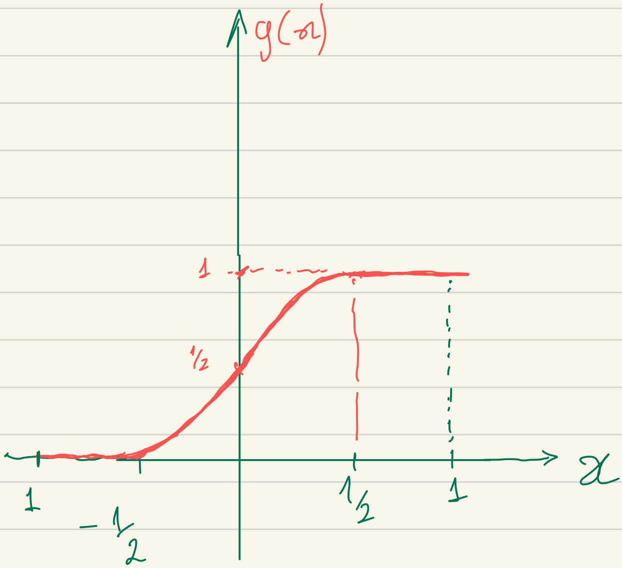
Therefore:

$$g(x) = \begin{cases} x + \frac{1}{2} + \frac{1}{2\pi}\sin 2\pi x & \frac{-1}{2} \leq x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

B)







The PSF smoothen the edge function, or make the image blur.