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Quantitative and Functional Imaging
Problem Set 1
Due Thursday, September 29, 2022

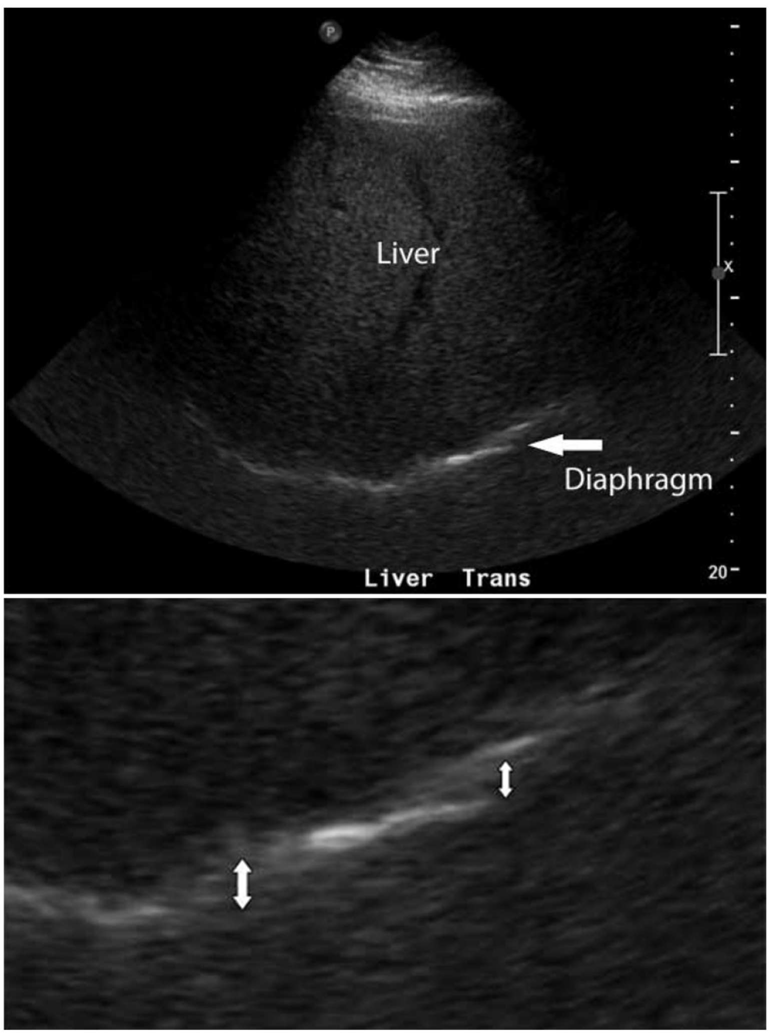
1. Ultrasound images are calculated assuming acoustic waves propagate at the same speed in all tissues (1540 m/s). Violation of this assumption can lead to image artifacts. The top panel of the figure below shows the liver and behind it, the diaphragm, which is normally a continuous sheet of muscle. In this image, the diaphragm appears to be discontinuous (see the bottom panel for more detail).

   A). Is this an abnormal diaphragm or an image artifact?

   Image artifact.

   B). What physical properties of the tissues could explain the appearance of the diaphragm?

   Since the liver overlaps with the diaphragm, this causes a difference in in return time due to the acoustic impedance / attenuation of the liver. Such attenuation violates the assumption in the common acoustic speed for all tissues (1540 m/s), causing the discontinuous artifact.



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2. Computed Tomography images are reconstructed from projections of x-ray attenuation information. An important assumption is that the attenuation coefficient for a given tissue is independent of the propagation direction of the beam. However, the beam contains x-rays with a range of energies and attenuation depends on photon energy.
   A). In the image below, strong streaking artifacts emanate from the bright object in the middle-left of the image. What physical properties do you think this bright object has that account for its appearance?
   Streaking artifact.
   B). What effect does the object have on the x-ray energies in the beam?
   Since the beam contains X-rays with a range of energies, the metal object expresses different attenuation coefficients for each photon energy.
   C). How does the object create the streaking artifacts?
   The effect is beam hardening, where low energy photons are absorbed, leaving high energy photons in the beam after passing through the metal object. The harder the beam, the less it gets attenuated by tissues behind the metal object as the beam proceeds. This causes the streaking artifacts.

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3. Use your knowledge of CT, PET, MRI, or Ultrasound to give an example of how <u>one</u> of these imaging modalities violates shift invariance. Describe how the imperfect behavior could be corrected or mitigated.

   MRI can violate shift invariance when the magnetic field inside is encoded with varying gradients targeting specific imaging locations and the patient moves during the exam. Thus, we get distorted images.

   One way to mitigate is have the patient rest calmly or hibernate during the exam.

4. (below) Suppose you measure the signal-to-noise ratio (SNR) in an image and find that the value is lower than you'd like. One method to improve SNR is to repeat the scan and average the images. This problem deals with image averaging from both a theoretical and practical viewpoint.

   A). If you acquire some number, N, of images under the same conditions and average them, the signal in the average image will be

$$S = \frac{1}{N}\sum_{i=1}^{N} S_i$$

   where S_i is the signal in the i^{th} image. Use the propagation of errors to find the expected signal-to-noise ratio, SNR(N), in the average of N images, using the fact that S is a function of each of the measured quantities $S_1, S_2, \ldots, S_N$ (as given in the relation above). Assume the noise variance is the same in each image, but noise in different images is uncorrelated.

   B). How many images should you average in order to double the SNR? How many images would you need to average to increase the SNR by a factor of 100? Discuss the practical limitations of image averaging to improve SNR.

<u>Problem 5 is required for students enrolled in BME 7450 (extra credit for those in BME 4420)</u>

5. **(below)** Let $f(x)$ represent **the edge of an imaged object** (for simplicity we'll consider just one dimension):

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

   Suppose that the point spread function (PSF) of the imaging system is given by

$$h(x) = \begin{cases} 1 + \cos(2\pi x), & |x| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

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The image intensity as a function of position is given by the convolution

$$g(x) = \int_{-\infty}^{\infty} f(x') \cdot h(x - x') \, dx'$$

A). Evaluate this integral by breaking it into appropriate piecewise intervals.
B). Plot $f(x)$, $h(x)$, and your expression for $g(x)$ on the interval [-1, 1]. Describe in words the effect that the PSF has on image intensity near the edge.

4. A)

We have • $\langle S\rangle = \frac{1}{N}\sum_{i=1}^{N} S_i = f(S_1, S_2, \dots, S_N)$

• $\sigma = \sigma_1 = \sigma_2 = \dots = \sigma_N$ where σ_i is the variance of noise in image "i"

• Noise are uncorrelated so their covariance $= 0$

+ And we assume noise has zero mean

So our noise propagation model to the averaged image is:

$$\sigma_{\langle S\rangle}^2 = \sigma^2\left[\left(\frac{\partial\langle S\rangle}{\partial S_1}\right)^2 + \left(\frac{\partial\langle S\rangle}{\partial S_2}\right)^2 + \dots + \left(\frac{\partial\langle S\rangle}{\partial S_N}\right)^2\right]$$

with $\langle S\rangle = \frac{1}{N}\sum_{i=1}^{N} S_i$ hence $\frac{\partial\langle S\rangle}{\partial S_1} = \frac{1}{N}(\underbrace{1 + 0 + \dots + 0}_{N \text{ items}}) = \frac{1}{N}$ and same for all "S_i" where $i = \{1; 2; \dots; N\}$

Thus

$$\sigma_{\langle S\rangle}^2 = \sigma^2\left[\underbrace{\frac{1}{N^2} + \frac{1}{N^2} + \dots + \frac{1}{N^2}}_{N \text{ items}}\right]$$

$$= \frac{\sigma^2}{N}$$

So $\frac{1}{\sigma_{\langle S\rangle}} = \sqrt{N} \times \frac{1}{\sigma}$

Therefore: $\langle SNR(N)\rangle = \frac{\langle S\rangle}{\sigma_{\langle S\rangle}}$

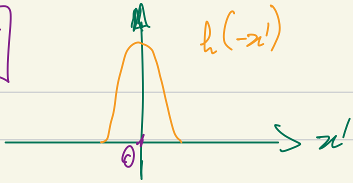
$$= \sqrt{N}\cdot\frac{1}{\sigma}\cdot\langle S\rangle \quad (*)$$

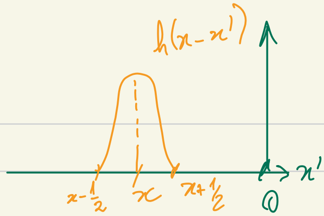
B) So to double $\langle SNR(N)\rangle$, we need to capture $N=4$ images based on $(*)$ and to increase by factor of 100, we need $N = 10{,}000$

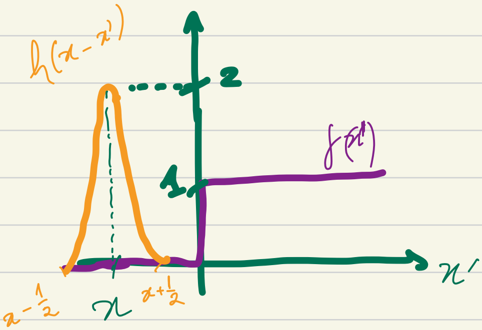
The limitations are: + higher patient doses in CT if we capture for example, 10,000 images.

+ longer scan time if we do MRI

$\boxed{5}$







For $x+\frac{1}{2}<0$ or $x<-\frac{1}{2}$

No overlap $g(x)=0$

For $-\frac{1}{2}\leq x\leq 0$:

$$g(x)=\int_0^{(x+1/2)} f(x')\,h(x-x')\,.dx'$$

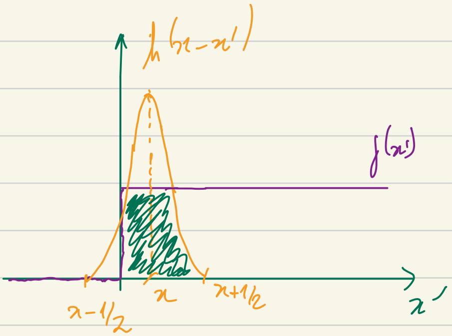
$$=\int_0^{(x+1/2)} 1.\left(1+\cos[2\pi(x-x')]\right).dx'$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\int_0^{(x+1/2)}\cos\left[2\pi(x-x')\right]d\left(2\pi(x-x')\right)$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\sin\left(2\pi(x-x')\right)\Big|_{x'=0}^{x'=(x+1/2)}$$

$$=\left(x+\frac{1}{2}\right)-\frac{1}{2\pi}\left(\sin(-\pi)-\sin 2\pi x\right)$$

$$=x+\frac{1}{2}+\frac{1}{2\pi}.\sin 2\pi x$$



For $0 \leq x \leq \frac{1}{2}$:

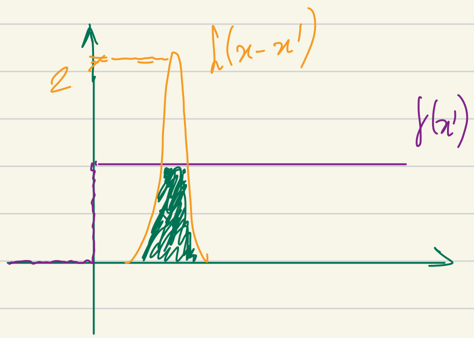
$$g(x) = \int_0^x \left(1+\cos\left[2\pi(x-x')\right]\right) dx' + \int_x^{x+\frac{1}{2}} \left(1+\cos\left[2\pi(x-x')\right]\right) dx'$$

$$= x - \frac{1}{2\pi}\int_{x'=0}^{x'=x} \cos\left[2\pi(x-x')\right] d\left(2\pi(x-x')\right)$$

$$+ \frac{1}{2} - \frac{1}{2\pi}\int_{x'=x}^{x'=x+1/2} \cos\left[2\pi(x-x')\right] d\left(2\pi(x-x')\right)$$

$$= x - \frac{1}{2\pi}\left[0 - \sin 2\pi x\right] + \frac{1}{2} - \frac{1}{2\pi}\left[0-0\right]$$

$$= x + \frac{1}{2} + \frac{1}{2\pi}\sin 2\pi x$$



For $x > \frac{1}{2}$:

$$g(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(x') . h(x-x') . dx'$$

$$= 2\int_x^{x+1/2} 1 . \left(1+\cos\left[2\pi(x-x')\right]\right) dx'$$

$$= 2\left(x + 1/2 - x\right) + 2\left(\frac{-1}{2\pi}\right)\int_{x'=x}^{x'=(x+\frac{1}{2})} \cos\left[2\pi(x-x')\right] d\left(2\pi(x-x')\right)$$

$$= 1 - \frac{1}{\pi} \times \sin 2\pi(x-x')\Big|_{x}^{x+1/2}$$

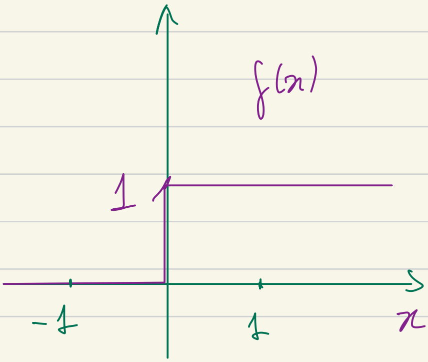
$$= 1 - \frac{1}{\pi} \times (0 - 0)$$

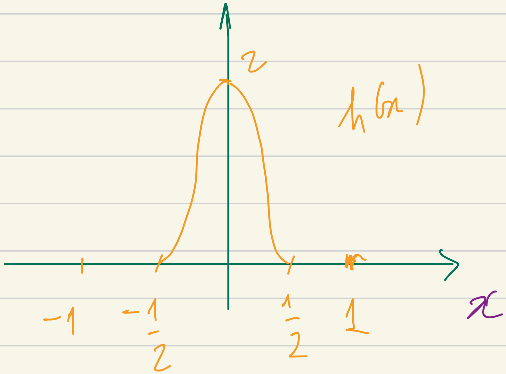
$$= 1$$

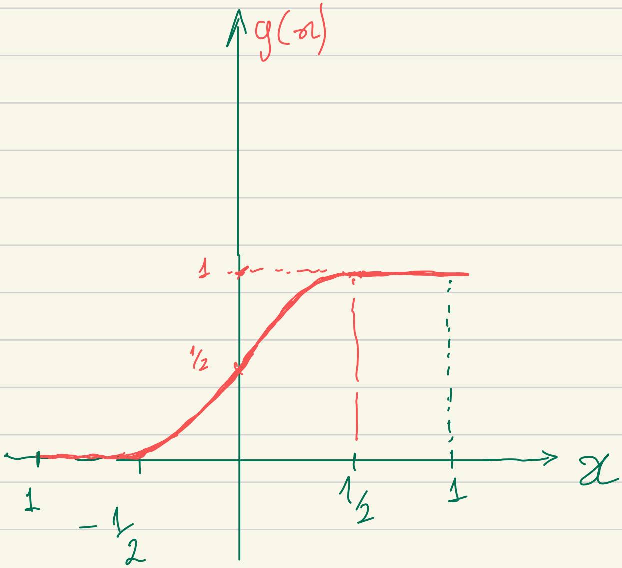
Therefore:

$$g(x) = \begin{cases} x + \frac{1}{2} + \frac{1}{2\pi}\sin 2\pi x & \frac{-1}{2} \leq x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

B)







The PSF smoothen the edge function, or make the image blur.