Chapter 2

Linear Time-Invariant Systems

We will begin with continuous-time systems (Chapter 2.2).

Recall, we can write $x(t)$ as a shifted and scaled sum of unit impulses.

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

Now consider a Linear Time-Invariant (LTI) system.

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t)$$

Suppose the input is $x(t) = \delta(t)$.
We will call the output $h(t)$, the impulse response.

$$\boxed{\delta(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow h(t)}$$

Since the system is time-invariant, if we shift $\delta(t)$ in ~~the~~ time, the output will also be shifted in time.

$$\boxed{\delta(t-t_0) \longrightarrow \underset{}{\overset{\text{LTI}}{\boxed{}}} \longrightarrow h(t-t_0)}$$

Since it is also linear, if we have scaled and shifted impulses added together we get:

$$\boxed{a\,\delta(t) + b\,\delta(t-t_0) \longrightarrow \overset{\text{LTI}}{\boxed{}} \longrightarrow a\,h(t) + b\,\delta(t-t_0)}$$

Basically, to get the output we replace the δ's with h's.

Thus, since we can write $x(t)$ as a bunch of shifted impulses $\delta(t-\tau)$ scaled by $x(\tau)$ and summed, the output in general is

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

Convolution Integral

The notation for this is

$$\boxed{y(t) = x(t) * h(t)}$$

$*$ mean <u>convolution</u>

<u>not</u> <u>multiplication</u>

Mathematically, the roles of x and h can be reversed in the integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

thus it is <u>commutative</u>

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

To compute the convolution we will use a graphical approach to guiding the integration.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Consider the integrand

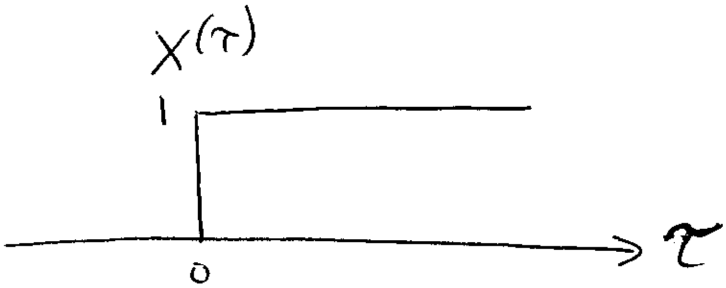
$$x(\tau)\, h(t-\tau)$$

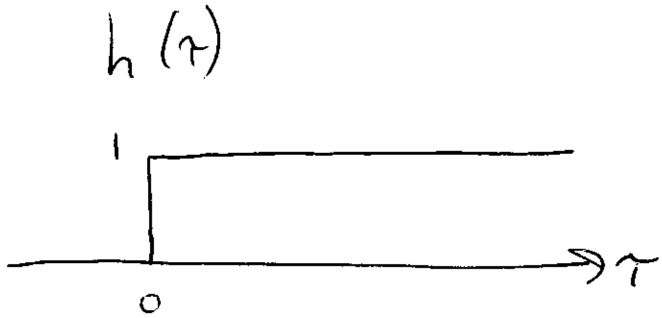
and note that the variable of integration is τ, not t.

With respect to τ, $x(\tau)$ is simply $x(t)$ with t replaced by τ.

However, $h(t-\tau)$ is a time-reversed (due to the $-\tau$ part) and time-shifted (due to t) signal. We will explore this with some examples.

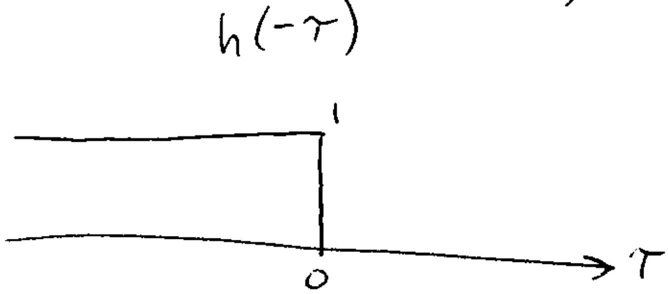
EX Consider $x(t) = u(t)$, $h(t) = u(t)$.



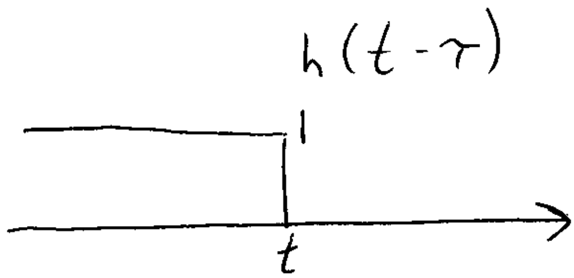


Now to form $h(t-\tau)$.

Start with $h(-\tau)$, a time-reverse.



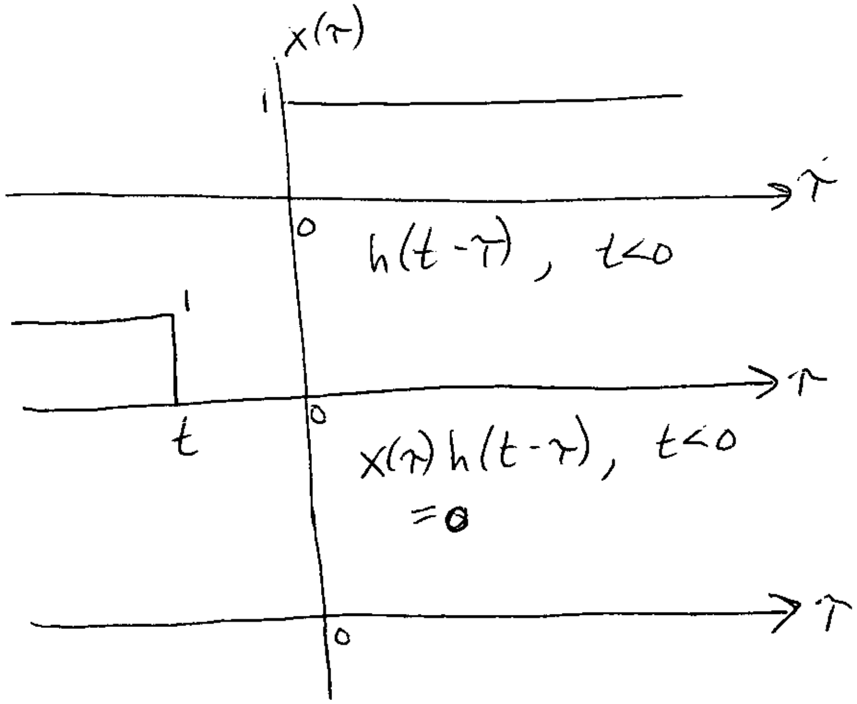
To get $h(t-\tau)$, add t to all the horizontal axis labels.



Note that t changes where the discontinuity occurs.

Now we will consider various values of t.

Consider $t < 0$:



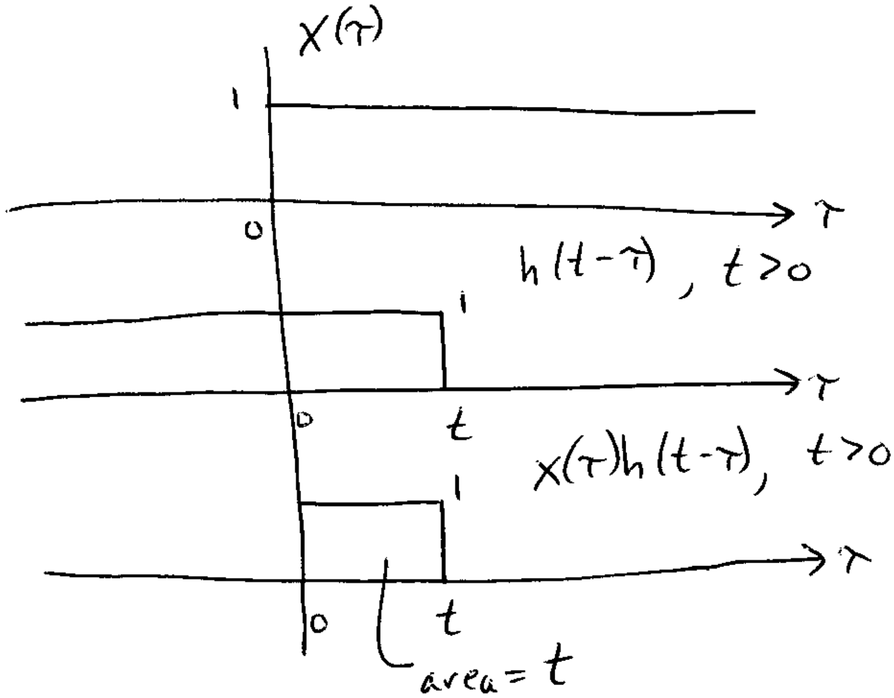
point-by-point multiply of $x(\tau)$ and $h(t-\tau)$

We integrate $x(\tau)h(t-\tau)$ over all τ which yields 0 in this case.

Thus

$$y(t) = 0 \quad t < 0$$

Now consider $t > 0$:

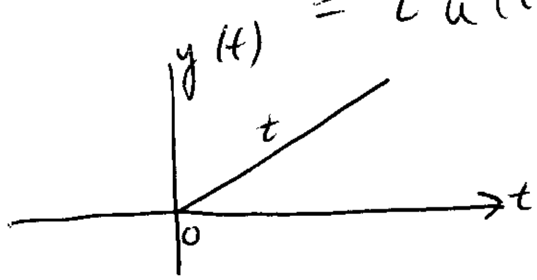


$$y(t) = t \qquad t > 0$$

So

$$y(t) = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= t\,u(t)$$



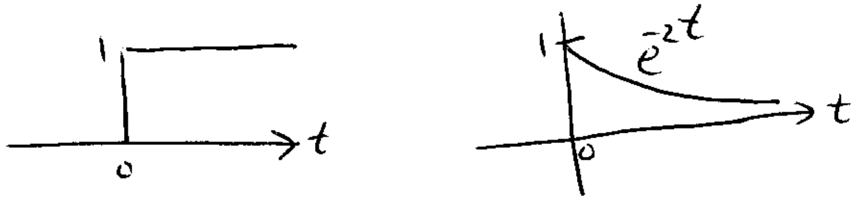
Note that $x(t)$ and $h(t)$ have signal changes that occur at $t=0$.

$h(t-\tau)$ is "$\underbrace{\text{flipped}}_{\text{(time-reversed)}}$" and "shifted"

and the signal change in $y(t)$ occurs when $h(t-\tau)$ is shifted past the signal change in $x(\tau)$ at the origin, $\tau=0$.

The purpose of the graphical approach is to keep track of where the signal changes occur.

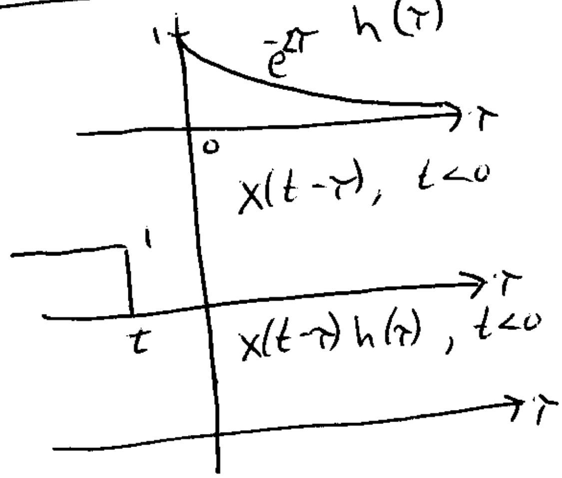
ex) $x(t) = u(t) \quad h(t) = e^{-2t} u(t)$



Since the roles of x and h can be swapped in the convolution integral we can choose either signal to be the one that is "flipped" and "shifted." In practice I recommend choosing the simplest one. This example has $x(t) = u(t)$ which I think is simplest.

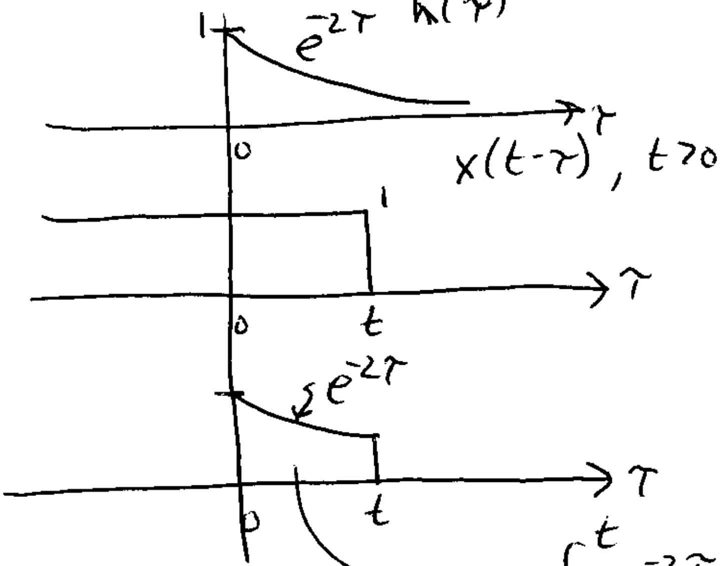
Flipping and shifting $u(t)$ is the same as in the previous example.

For $t < 0$



$y(t) = 0$
for $t < 0$

For $t > 0$



$$\text{area} = \int_0^t e^{-2\tau}\, d\tau = -\frac{1}{2} e^{-2\tau}\Big|_0^t$$

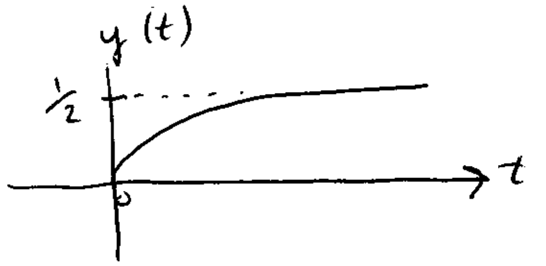
$$= -\frac{1}{2}\left(e^{-2t} - 1\right)$$

$$= \frac{1}{2}\left(1 - e^{-2t}\right)$$

$$= y(t)$$

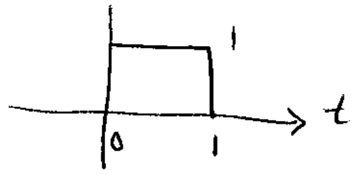
$$y(t) = \begin{cases} \frac{1}{2}\left(1 - e^{-2t}\right) & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \frac{1}{2}\left(1 - e^{-2t}\right) u(t)$$



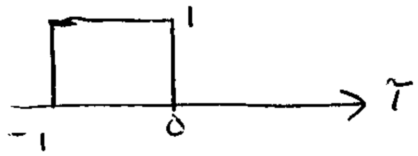
ex)

$x(t)$

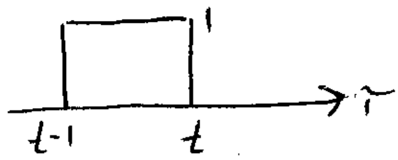


$h(t)$

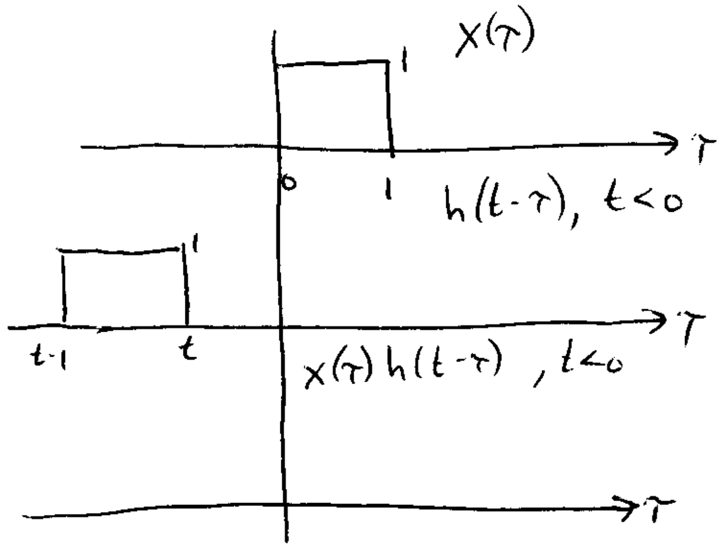
$h(-\tau)$



$h(t-\tau)$

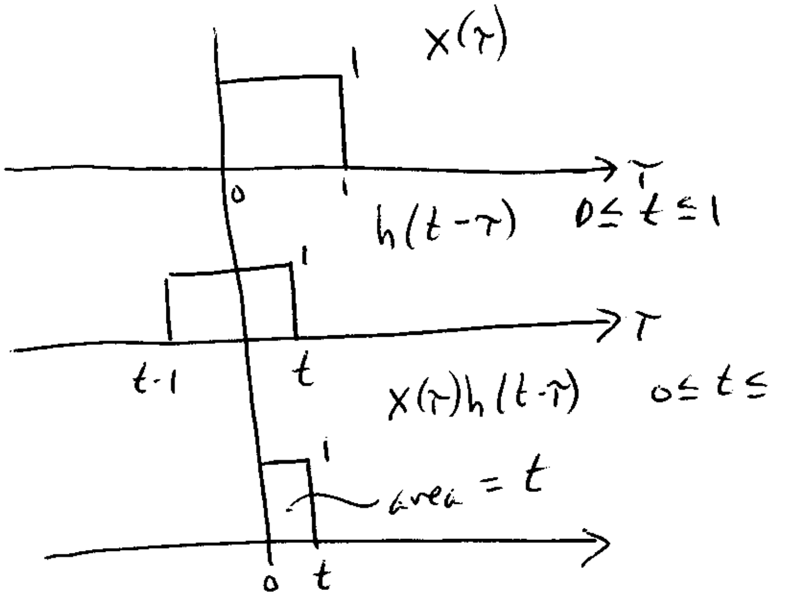


$t<0$



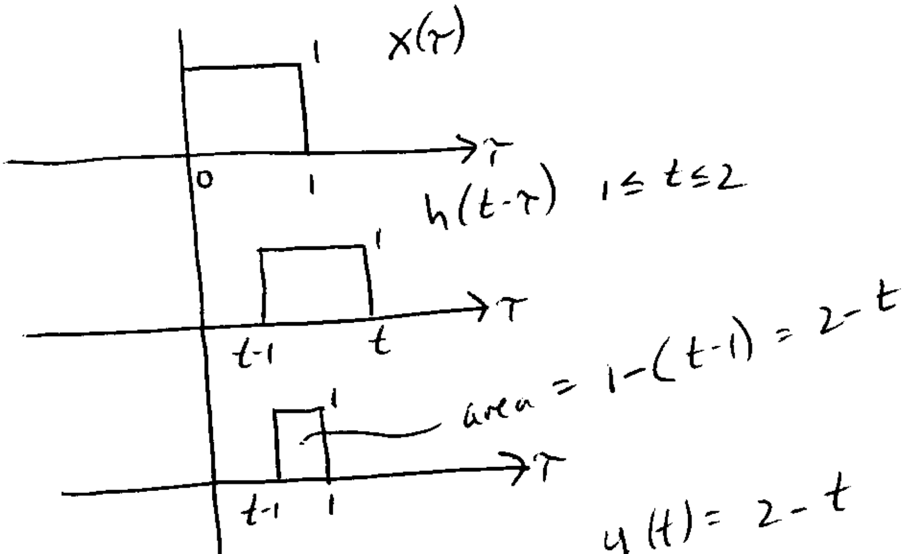
$y(t)=0 \quad t<0$

$0 \le t \le 1$



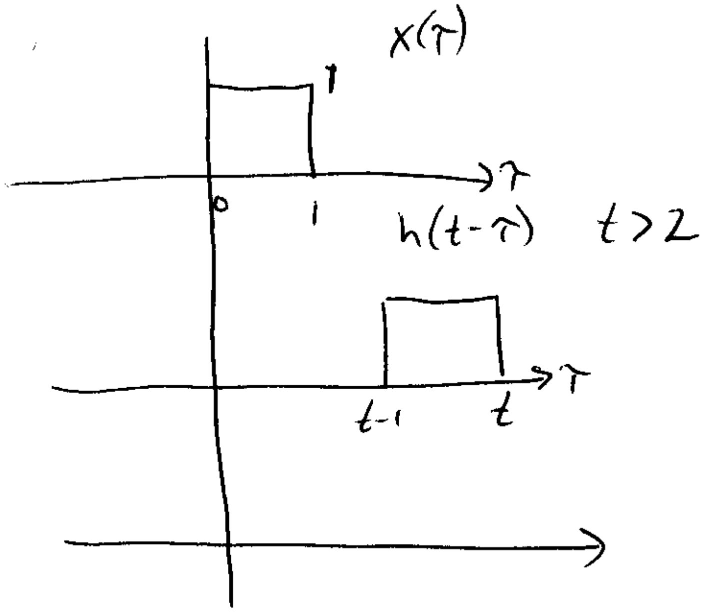
$y(t) = t \quad 0 \le t \le 1$

$0 \le t-1 \le 1 \rightarrow \quad 1 \le t \le 2$



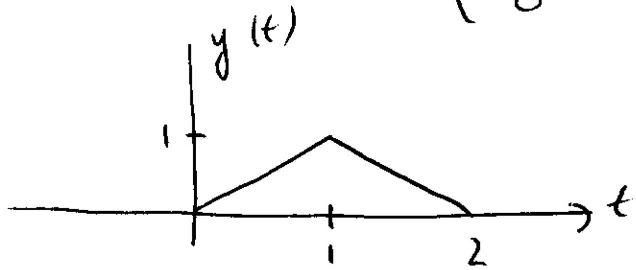
$y(t) = 2 - t \quad 1 \le t \le 2$

$t > 0$



$y(t) = 0 \quad t > 2$

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & t > 2 \end{cases}$$



Note that the result $y(t)$ is wider and more smoothed out than either $x(t)$ or $h(t)$.

Special Cases

$h(t) = \delta(t)$

$$x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow y(t) = x(t) * \delta(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$= x(t)$$

$$\boxed{x(t) * \delta(t) = x(t)}$$

$h(t) = \delta(t - t_0)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,\underbrace{\delta(t-\tau-t_0)}_{\text{"fires" at } \tau = t - t_0}\,d\tau$$

$$= x(t - t_0)$$

$$\boxed{x(t) * \delta(t - t_0) = x(t - t_0)}$$

Discrete Time Convolution

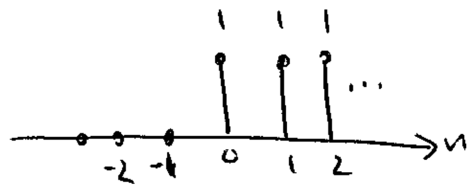
$$x[n] = \delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n] = \underline{\text{impulse response}}$$

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

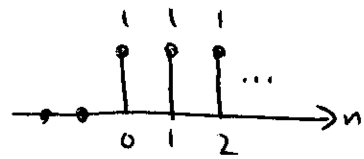
$$\boxed{\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m]\,h[n-m] = \sum_{m=-\infty}^{\infty} h[m]\,x[n-m] \\ &= x[n] * h[n] = h[n] * x[n] \end{aligned}}$$

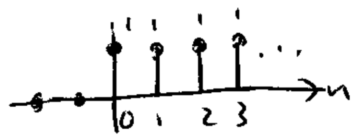
This has a graphical visual interpretation similar to what we saw with the C.T. case.

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



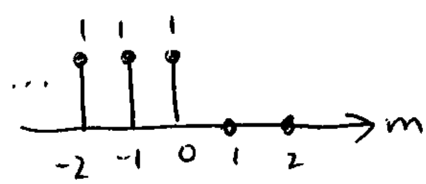
ex) $x[n] = u[n]$ $\qquad$ $h[n] = u[n]$



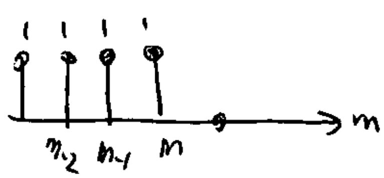


$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

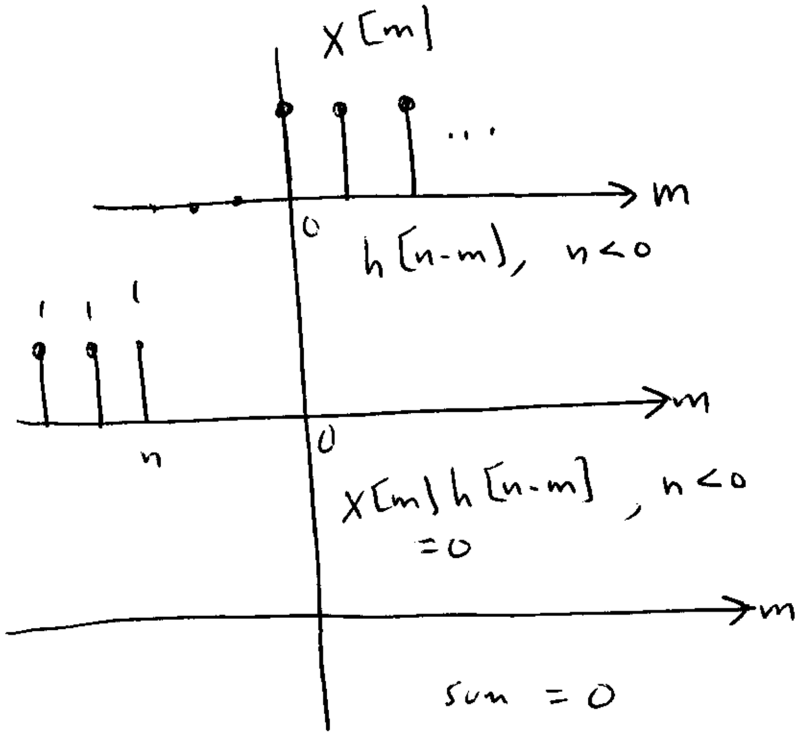
$h[-m]$



$h[n-m]$ $\qquad$ add n to the labels on the m axis

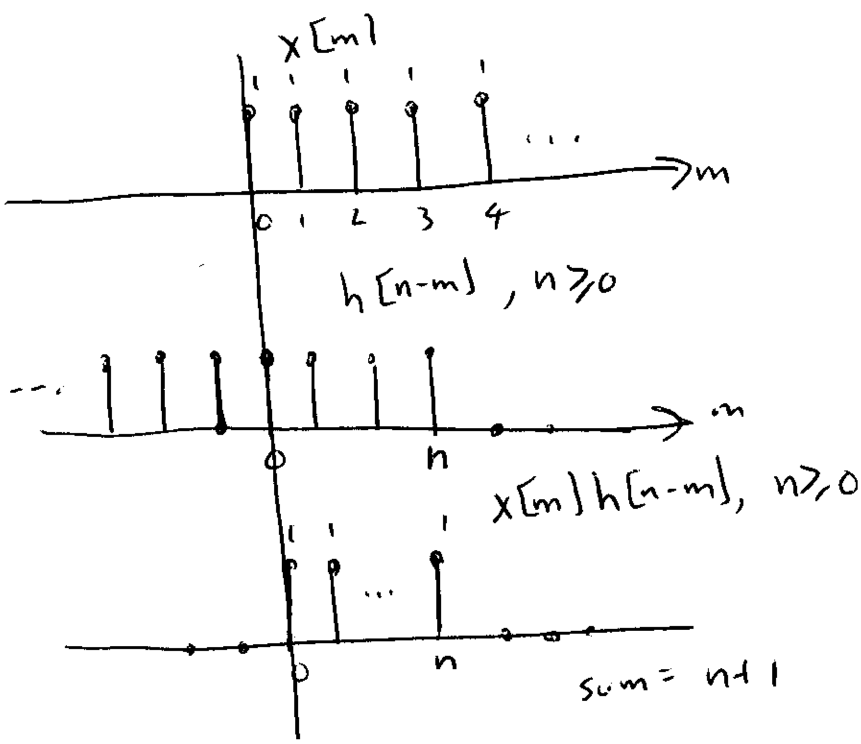


$n < 0$



$y[n] = 0 \quad n<0$

$n \geq 0$

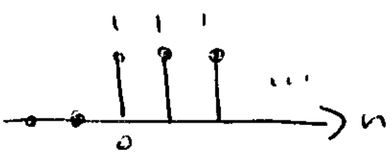


$y[n] = (n+1)$, $n \geq 0$

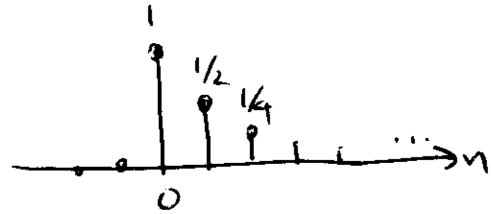
Thus $y[n] = u[n] * u[n] = (n+1)u[n]$

EX) $x[n] = u[n] \quad h[n] = \left(\frac{1}{2}\right)^n u[n]$

$x[n]$



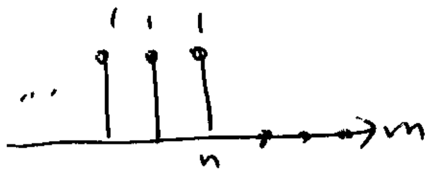
$h[n]$



$x[n] = u[n]$ is simpler so we will "flip and shift" $x[n]$.

as before

$x[n-m]$

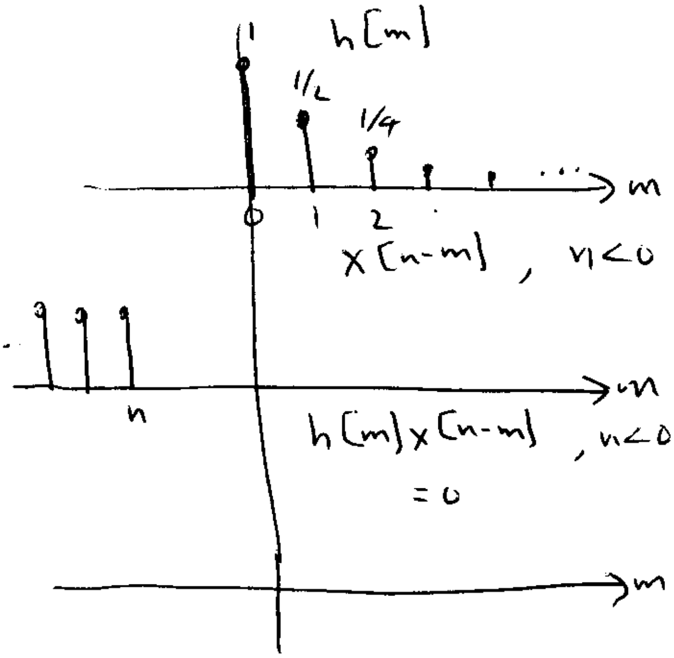


Note:

Sumation Identity

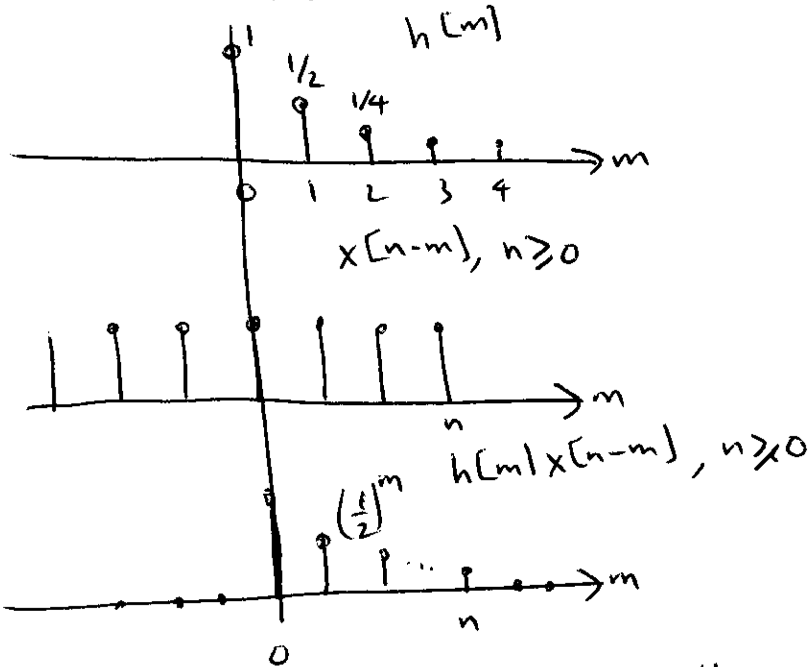
$$\sum_{m=0}^{N-1} \alpha^n = \begin{cases} N & \text{if } \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \end{cases}$$

$\underline{n<0}$



$$y[n] = 0 \qquad n<0$$

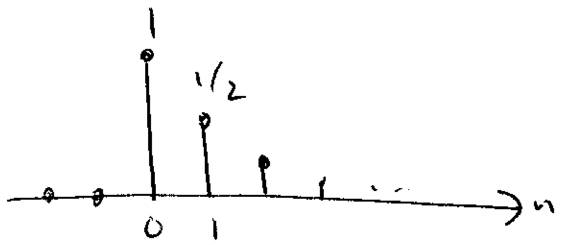
$n \geq 0$

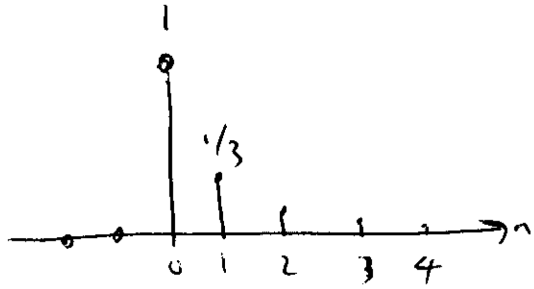


$$y[n] = \sum_{m=0}^{n} \left(\tfrac{1}{2}\right)^n = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} \qquad n \geq 0$$

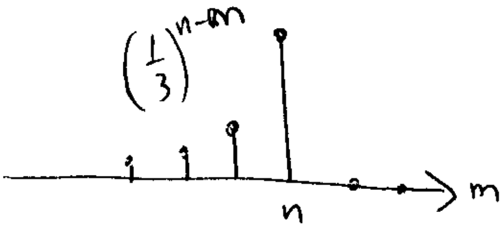
ex)

$x[n] = \left(\frac{1}{2}\right)^n u[n]$ $\qquad$ $h[n] = \left(\frac{1}{3}\right)^n u[n]$





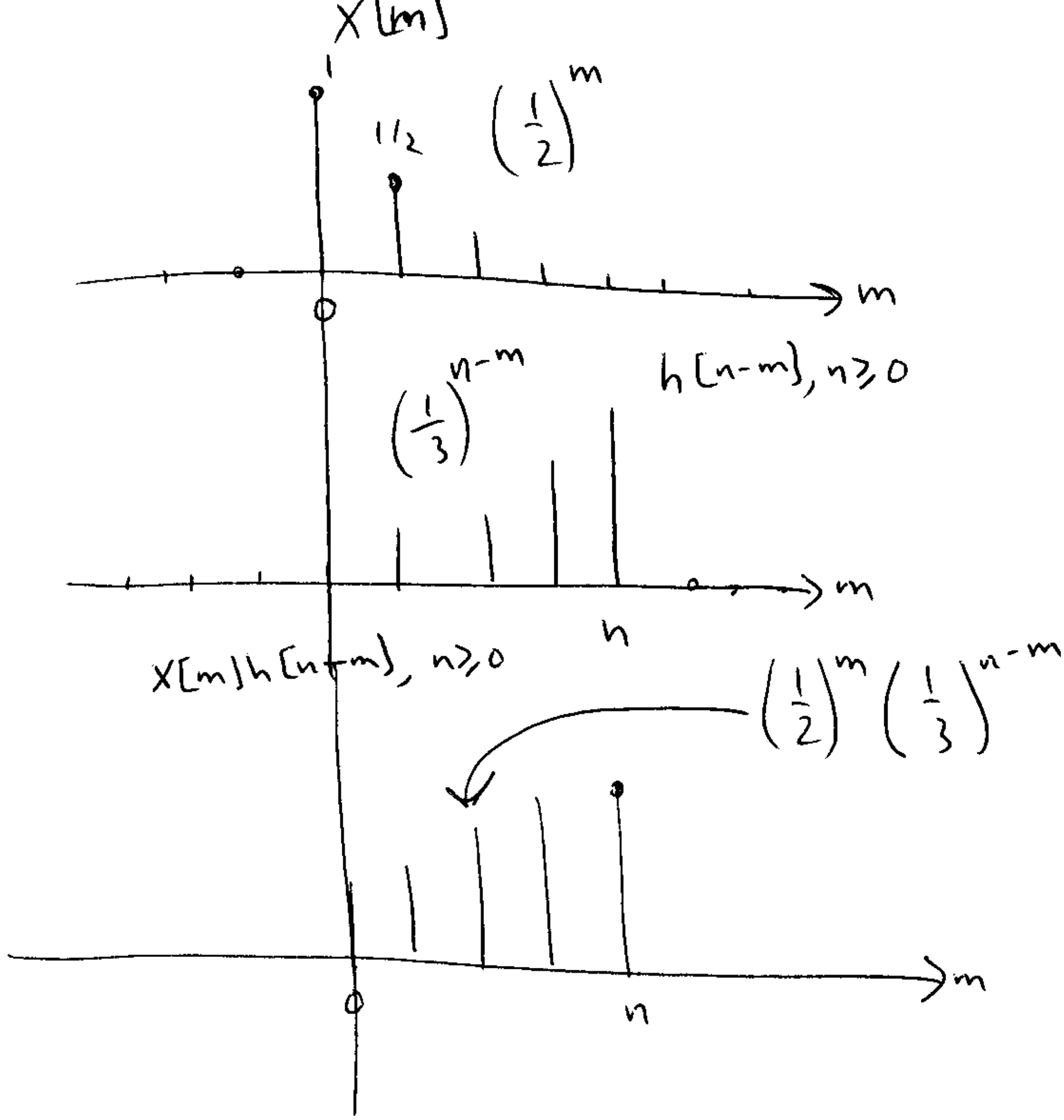
$h[n-m]$



$n<0 \rightarrow$ as we have seen before

this will yield $\underline{y[n]=0}$, $n<0$

$n \geq 0$



$$y[n] = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{3}\right)^{n-m}$$

$$= \left(\frac{1}{3}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2} \cdot \left(\frac{1}{3}\right)^{-1}\right)^m$$

$$= \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{1}{2} \cdot \left(\frac{1}{3}\right)^{-1}\right)^{n+1}}{1 - \left(\frac{1}{2} \cdot \left(\frac{1}{3}\right)^{-1}\right)} \qquad n \geq 0$$

This can be simplified but we will see a better approach later.

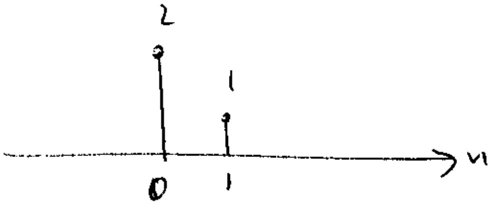
Also as we saw with C.T. convolution:

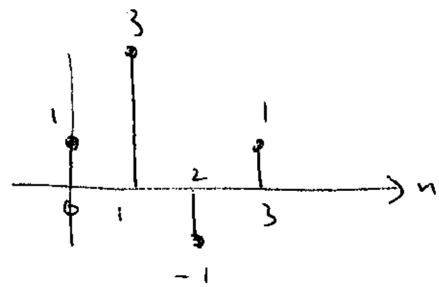
$$\boxed{\begin{aligned} x[n] * \delta[n] &= x[n] \\ x[n] * \delta[n-n_0] &= x[n-n_0] \end{aligned}}$$

D.T. Finite-length signals and convolution

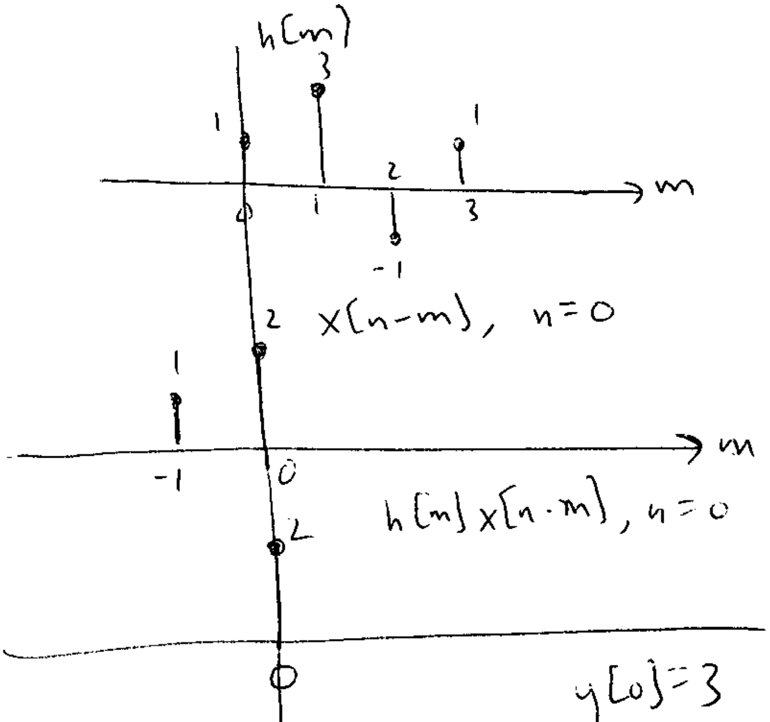
Consider two finite length signals.

$x[n] = \{2, 1\}$ $\qquad$ $h[n] = \{1, 3, -1, 1\}$





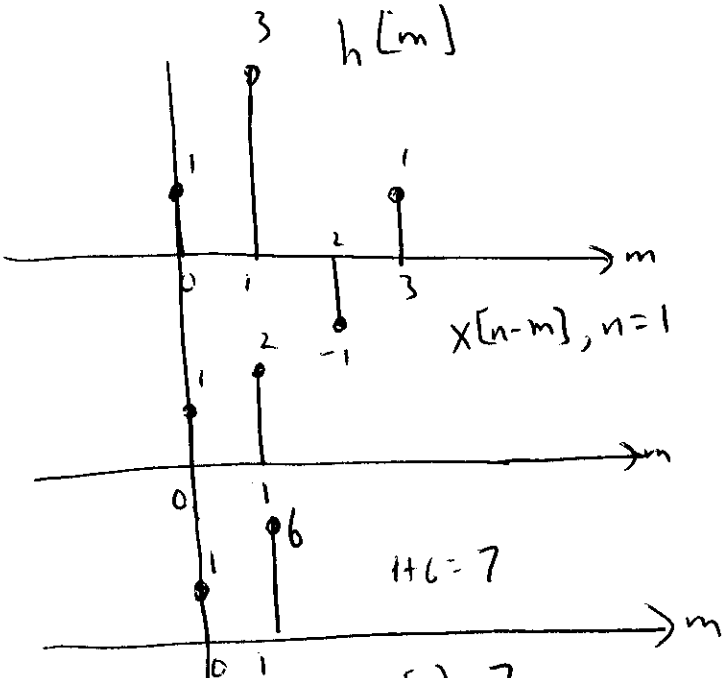
We will flip and shift $x[n]$.



$y[0]=3$

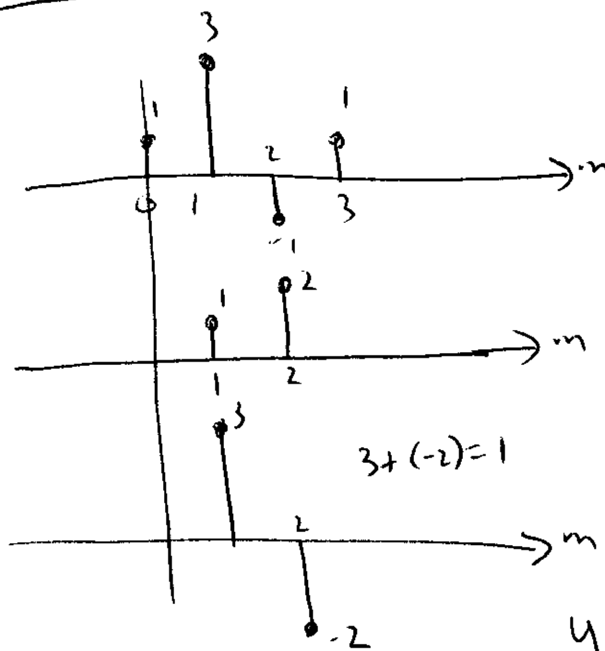
Note: for $n<0$ there is no overlap and $y[n]$ will be 0.

For $n=1$, just slide $x[n-m]$ one point to the right.



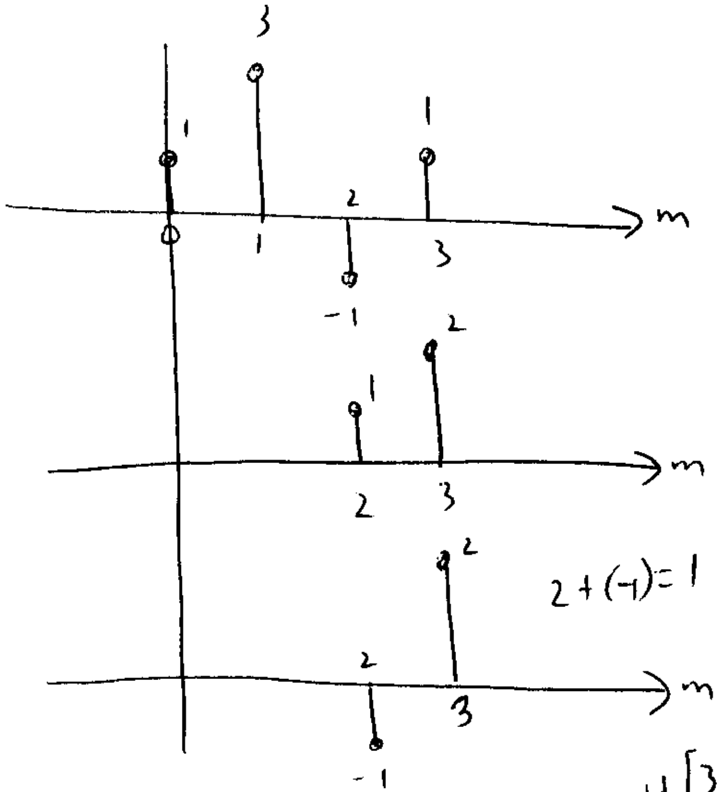
$y[1] = 7$

$n = 2$



$y[2] = 1$

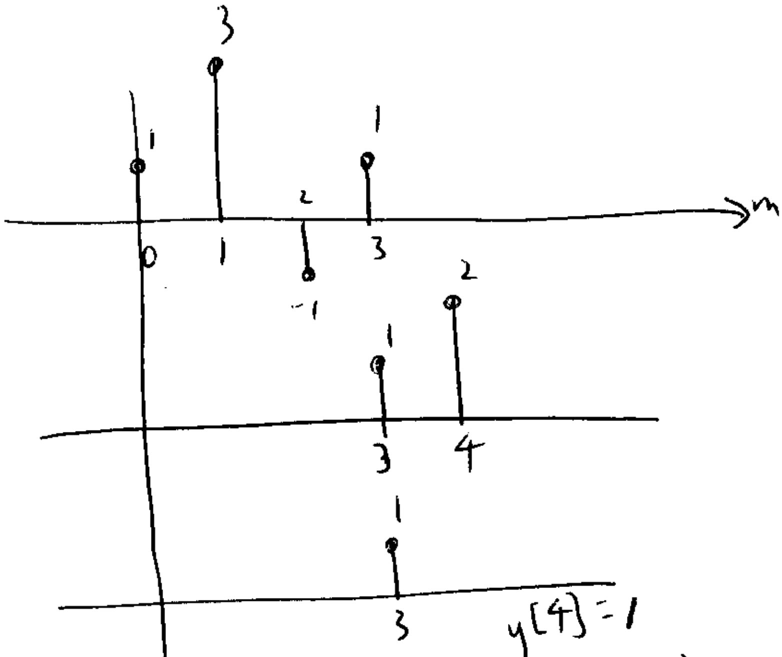
$n=3$



$2+(-1)=1$

$y[3] = 1$

$n=4$



$y[4]=1$

$y[n] = \{3, 7, 1, 1, 1\}$

$y[n] = 0$ for $n>4$

Matlab

```
h = [1  3  -1  1];
x = [2  1];
y = conv(h, x);
    y = [2  7  1  1  1]
```

☆☆☆

$x[n]$ first point is at $n_{x,start}$

last point is at $n_{x,stop}$

$h[n]$ first point is at $n_{h,start}$

last point is at $n_{h,stop}$

$y[n] = x[n] * h[n]$

first point is at $n_{y,start} = n_{x,start} + n_{h,start}$

last point is at $n_{y,stop} = n_{x,stop} + n_{h,stop}$

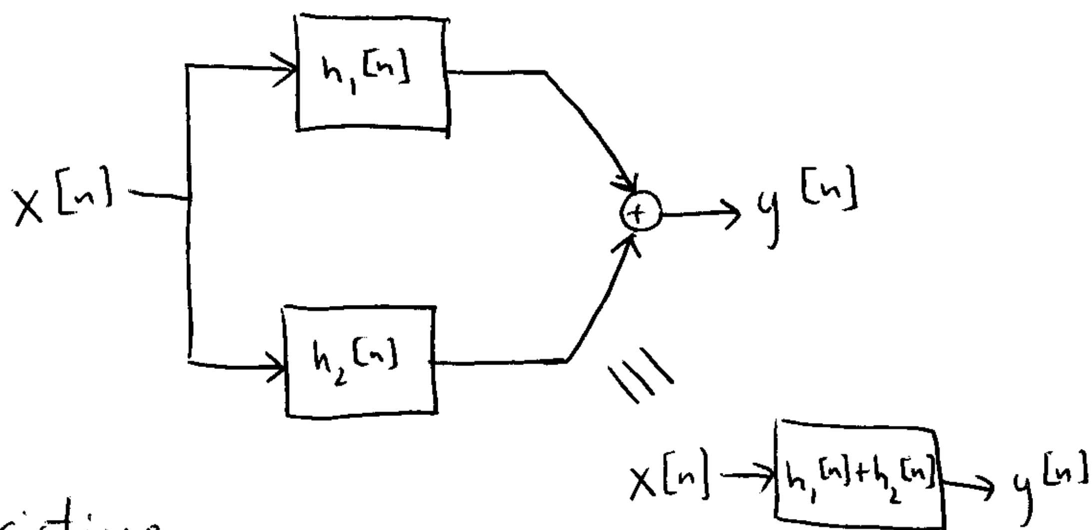
Properties of LTI Systems

Distributive

$$\boxed{\begin{aligned} y[n] &= x[n] * \left(h_1[n] + h_2[n]\right) \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}}$$ ☆☆☆

same for C.T. systems

parallel systems



Associative

$$y[n] = x[n] * h_1[n] * h_2[n]$$

$$= \left(x[n] * h_1[n]\right) * h_2[n]$$

$$= x[n] * \left(h_1[n] * h_2[n]\right)$$

same for C.T.

Cascade Systems

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

$$\equiv$$

$$x[n] \longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow y[n]$$

Same for C.T.

Memoryless LTI System

The impulse response of a memoryless LTI system is

$$\boxed{h[n] = h[0]\,\delta[n] = K\,\delta[n]}$$

$$\boxed{y[n] = x[n] * \left(K\,\delta[n]\right) = K\,x[n]}$$

$$\boxed{\begin{aligned} h(t) &= K\,\delta(t) \\ y(t) &= K\,x(t) \end{aligned}}$$

Causality

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \cdots + h[-2]\,x[n+2] + h[-1]\,x[n+1] + h[0]\,x[n] + h[1]\,x[n-1] + \cdots$$

An LTI system is causal

iff

$h[n] = 0$ for all $n < 0$

$h(t) = 0$ for all $t < 0$

Stability

An LTI system is stable

iff

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Step Response

If $x[n] = u[n]$ $\big(x(t) = u(t)\big)$

the step response is

$$\boxed{\begin{aligned} s[n] &= h[n] * u[n] \\ s(t) &= h(t) * u(t) \\ s[n] &= \sum_{m=-\infty}^{n} h[m] \\ s(t) &= \int_{-\infty}^{t} h(\tau)\, d\tau \end{aligned}}$$

Linear Constant Coefficient Differential Eq.s

(LCCDE)

This type of differential equation is associated with LTI systems.

$$a_N y^{(N)}(t) + \cdots + a_1 y'(t) + y(t) = b_M x^{(M)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

$$a_0 = 1$$

We will <u>wait</u> to study the <u>solution methods</u> until we cover the <u>Laplace transform</u>.

Linear Constant Coefficient Difference Eq.s

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{m=0}^{M} b_0 \, x[n-m]$$

$$a_0 = 1$$

We will study the solution methods later when we cover the <u>Z-Transform</u>.

<u>FIR system</u> $(N=0)$

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$h[n] = \{b_0, b_1, b_2, \ldots, b_M\}$$

$$h[n] = \begin{cases} b_n & n = 0, \ldots, M \\ 0 & \text{else} \end{cases}$$