Chapter 5

Discrete-Time Fourier Transform

DTFT

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$\boxed{\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \end{aligned}}$$ ★★★

As in the continuous-time case the DTFT represents $x[n]$ as a sum of sinusoids, however, the frequency range covers only a 2π interval, $-\pi \leq \omega < +\pi$.

examples

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$$

$$\boxed{\delta[n] \longleftrightarrow 1}$$ ★★★

ex) $x[n] = a^n u[n] \qquad |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n$$

Recall, the summation identity ★★★

$$\boxed{\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1}$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

★★★

$$\boxed{\begin{array}{l} a^n u[n] \\ |a| < 1 \end{array} \longleftrightarrow \frac{1}{1 - a e^{-j\omega}}}$$

Periodicity of $X(e^{jw})$

Consider

$$e^{-jwn}$$

$$e^{-j(w+2\pi)n} = e^{-jwn} \underbrace{e^{-j2\pi n}}_{1}$$

$$= e^{-jwn}$$

$\boxed{\text{Thus } e^{-jwn} \text{ is periodic with respect to } w \text{, with period} = 2\pi}$ ★★★

As a result

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

is also periodic in w with period of 2π

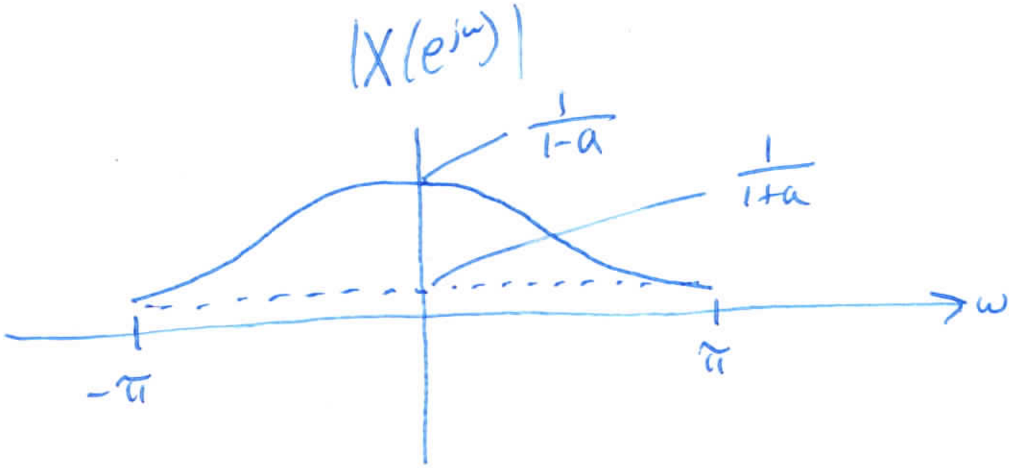
$$\boxed{X(e^{j(w+2\pi)}) = X(e^{jw})}$$ ★★★

Consider again

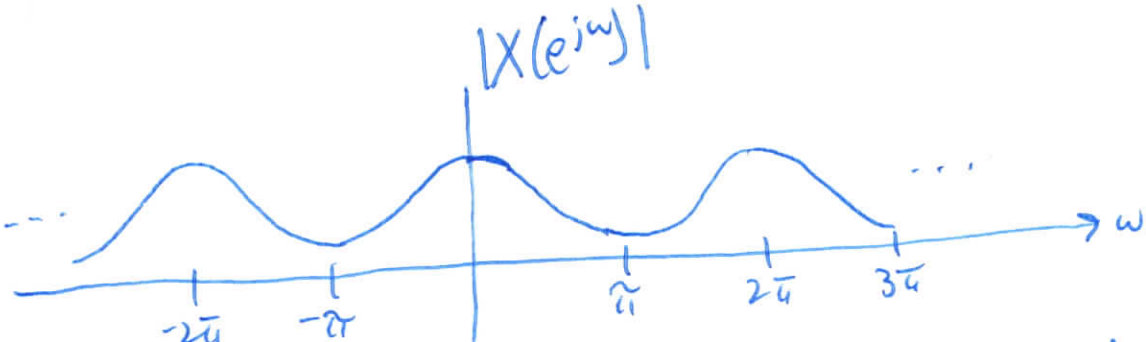
$$x[n] = a^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$|a| < 1$

For simplicity consider a to be real and positive



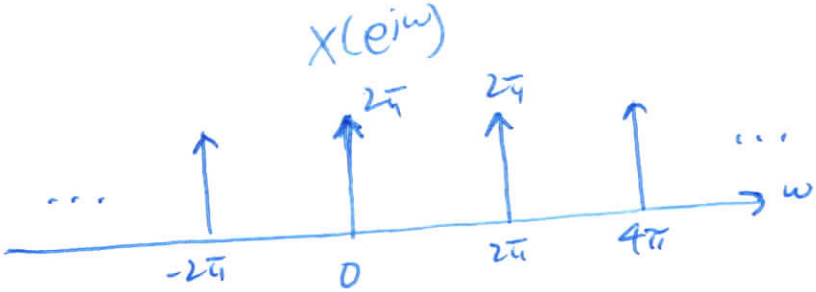
If we expand the range we see the periodic behavior.



$\angle X(e^{j\omega})$ also shows a periodic behavior ex) Consider

$$X(e^{j\omega}) = 2\pi\delta(\omega) \quad \text{over} \quad -\pi \le \omega < \pi$$

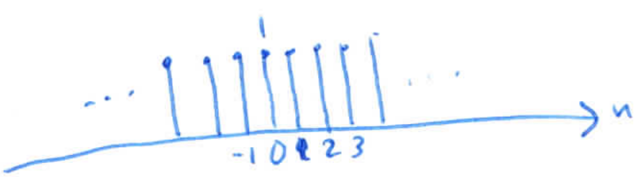
Of course, it is periodic so outside this range we get a periodic replication. This is why I specified it over one period.



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n}\, d\omega$$

$$= 1 \quad \text{for all } n$$



$$\boxed{1 \longleftrightarrow 2\pi\delta(\omega) \quad -\pi \le \omega < \pi \text{ and repeats with period} = 2\pi}$$

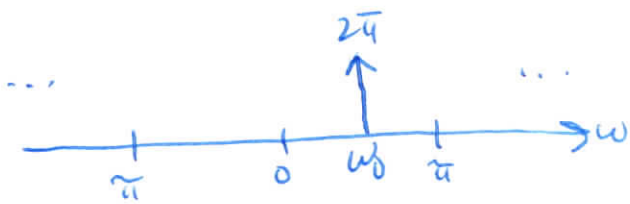
Similarly, if ω_0 is between $-\pi$ and π,

$$-\pi \le \omega_0 < \pi$$

then

$$X(e^{j\omega}) = 2\pi\,\delta(\omega-\omega_0) \qquad -\pi \le \omega < \pi$$

and repeats periodically



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega$$

$$= e^{+j\omega_0 n} \qquad \text{for all } n$$

$$\boxed{e^{j\omega_0 n} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0) \quad -\pi \le \omega_0 < \pi \text{ and repeats periodically}}$$

And as we saw in the continuous-time case

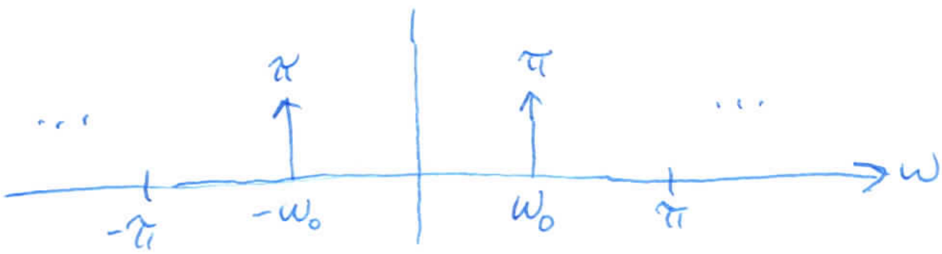
$$\cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$ ★★★

$-\pi \leq \omega_0 < \pi$

transforms to

$$\pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

and repeats periodically



and

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$$\sin(\omega_0 t) \longleftrightarrow \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

$-\pi \leq \omega_0 < \pi$

$$= -j\pi\,\delta(\omega - \omega_0) + j\pi\,\delta(\omega + \omega_0)$$

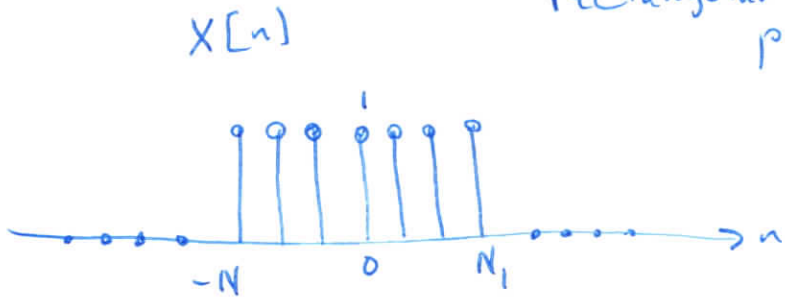
and repeats periodically

$$u[n] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} + \pi\,\delta(\omega)$$ ★★★

and repeats periodically

ex 5.3

rectangular pulse



$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$= \sum_{n=-N_1}^{N_1} e^{-jwn}$$

Recall summation identity

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \text{if } \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \end{cases}$$

also

$$\sum_{n=N_1}^{N_2-1} \alpha^n = \begin{cases} N_2 - N_1 & \alpha = 1 \\ \dfrac{\alpha^{N_1} - \alpha^{N_2}}{1-\alpha} & \alpha \neq 1 \end{cases}$$

Using the second one,

$$X(e^{jw}) = \frac{e^{jwN_1} - e^{jw(N_1+1)}}{1 - e^{jw}} \qquad w \neq 0$$

at $w=0$
$e^{-jw} = 1$

Now factor out $e^{-jw/2}$ from the numerator and the denominator. (A Trick)

$$X(e^{jw}) = \frac{e^{-jw/2}\left(e^{jw(N_1+\frac{1}{2})} - e^{-jw(N_1+\frac{1}{2})}\right)}{e^{-jw/2}\left(e^{jw/2} - e^{-jw/2}\right)}$$

The $e^{-jw/2}$ terms cancel.

$$X(e^{jw}) = \frac{e^{jw(N_1+\frac{1}{2})} - e^{-jw(N_1+\frac{1}{2})}}{e^{jw/2} - e^{-jw/2}}, \quad w \neq 0$$

Recall $2j\sin\theta = e^{j\theta} - e^{-j\theta}$

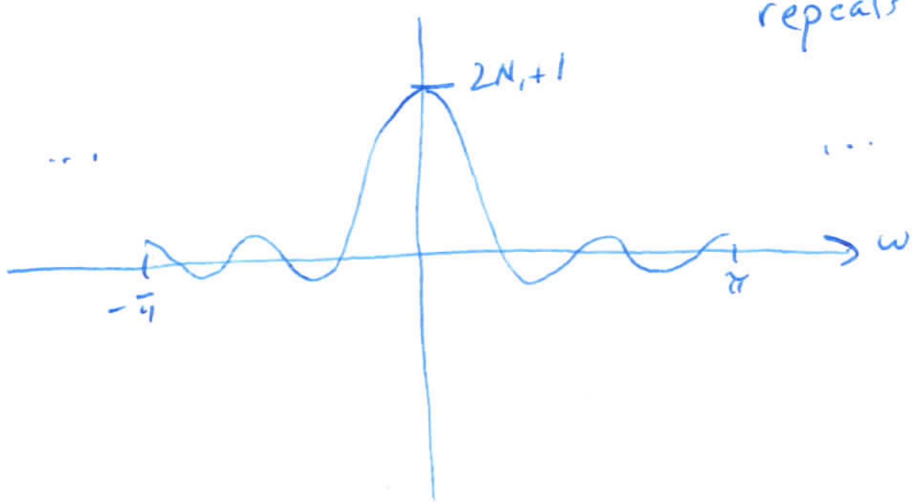
because $\sin\theta = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$

$$X(e^{jw}) = \frac{2j\sin(w(N_1+\frac{1}{2}))}{2j\sin(w/2)}, \quad w \neq 0$$

$$\boxed{X(e^{jw}) = \begin{cases} \dfrac{\sin(w(N_1+\frac{1}{2}))}{\sin(w/2)}, & w \neq 0 \\ 2N_1+1, & w = 0 \end{cases}}$$

This is a sort of periodically repeating sinc function

$X(e^{j\omega})$

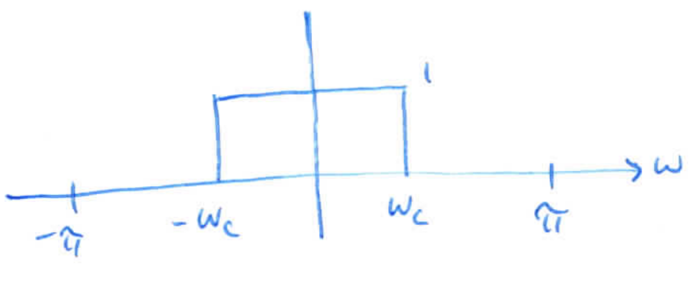


repeats with period of 2π

ex) Ideal lowpass

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{else} \end{cases} \quad , 0 \le \omega_c \le \pi$$

and repeats periodically



$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \frac{1}{jn} e^{j\omega n} \Big|_{-\omega_c}^{\omega_c} \quad , n \neq 0$$

$$= \frac{1}{j2\pi n} \underbrace{\left(e^{j\omega_c n} - e^{-j\omega_c n} \right)}_{2j\sin(\omega_c n)}$$

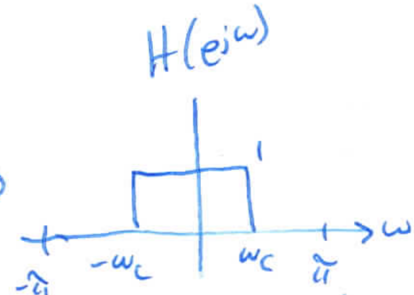
$$= \frac{\sin(\omega_c n)}{\pi n} \qquad n \neq 0$$

$n = 0$

$$h[0] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \, d\omega$$

$$= \frac{\omega_c}{\pi}$$

$$h[n] = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n} & , n \neq 0 \end{cases} \longleftrightarrow$$



and repeat periodically

Properties of the DTFT

Periodicity

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})}$$ ☆☆☆

Linearity ☆☆☆

$$\boxed{a x_1[n] + b x_2[n] \longleftrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})}$$

Time Shift ☆☆☆

$$\boxed{x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})}$$

Frequency Shift ☆☆☆

$$\boxed{e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})}$$

Conjugation ☆☆☆

$$\boxed{x^*[n] \longleftrightarrow X^*(e^{-j\omega})}$$

Conjugate Symmetry

If $x[n]$ is real-valued then

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

Thus if $x[n]$ is real-valued

$$|X(e^{-j\omega})| = |X(e^{j\omega})|$$ magnitude is even

$$\angle X(e^{-j\omega}) = -\angle X(e^{j\omega})$$ phase is odd

First Difference

$$\underbrace{x[n] - x[n-1]}_{\text{First difference}} \longleftrightarrow (1 - e^{-j\omega}) X(e^{j\omega})$$

Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \delta(\omega)$$

and repeats periodically

Time Reversal

$$\boxed{x[-n] \longleftrightarrow X(e^{-j\omega})}$$ ★★★

If $x[n]$ is real valued ★★★

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

thus

$$x[-n] \longleftrightarrow X^*(e^{j\omega})$$

Time Expansion

★★★

Define

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{else} \end{cases}$$

$$\boxed{x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})}$$ ★★★

Multiplication by n / Differentiation in Frequency

$$\boxed{n\,x[n] \longleftrightarrow j\,\frac{dX(e^{j\omega})}{d\omega}}$$ ☆☆☆

Parseval's Relation

$$\boxed{E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\,d\omega}$$ ☆☆☆

Convolution

$$\boxed{x[n] * h[n] \longleftrightarrow X(e^{j\omega})H(e^{j\omega})}$$ ☆☆☆

ex $\quad x[n] = \left(\frac{1}{2}\right)^n u[n] \qquad h[n] = \left(\frac{1}{4}\right)^n u[n]$

$$y[n] = x[n] * h[n]$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \; ; \; H(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

PFE for DTFT

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

Two methods:

(1) Cross Multiply

$$Y(e^{j\omega})(1-\tfrac{1}{2}e^{-j\omega})(1-\tfrac{1}{4}e^{-j\omega}) = 1 = A(1-\tfrac{1}{4}e^{-j\omega}) + B(1-\tfrac{1}{2}e^{-j\omega})$$

constant terms

$$1 = A + B \Rightarrow B = 1 - A$$

$e^{-j\omega}$ terms

$$0 = \left(-\tfrac{1}{4}\right)A + B\left(-\tfrac{1}{2}\right)$$

$$0 = -\tfrac{1}{4}A - \tfrac{1}{2}(1-A) = \tfrac{1}{4}A - \tfrac{1}{2}$$

$A = 2$ $\qquad B = -1$

(2) Cover-up

For A, cover up the $(1-\frac{1}{2}e^{-j\omega})$ term

and substitute $e^{j\omega} = \frac{1}{2}$

$$A = \left.\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right|_{e^{j\omega}=\frac{1}{2}} = \frac{1}{1-\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{-1}} = \underline{\underline{2}}$$

$$B = \left.\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right|_{e^{j\omega}=\frac{1}{4}} = \frac{1}{1-\frac{1}{2}(4)} = \underline{\underline{-1}}$$

$$y[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Multiplication Property

$$x_1[n]\,x_2[n] \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\,X_2(e^{j(\omega-\theta)})\,d\theta$$

$$= \frac{1}{2\pi} X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$$

periodic convolution

Systems Characterized by Linear Constant-Coefficient Difference Equations

LCCDE form:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m\, x[n-m]$$

$a_0 = 1$

Take DTFT

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{m=0}^{M} b_m e^{-j\omega m} X(e^{j\omega})$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \qquad a_0 = 1}$$

ex)

$$y[n] - a y[n-1] = x[n]$$

$$Y(e^{j\omega}) - a e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$\Updownarrow$$

$$h[n] = a^n u[n]$$

ex

$$H(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

find the difference equation

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

$$Y(e^{j\omega})\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\right) = X(e^{j\omega})\left(2 - e^{-j\omega}\right)$$

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] - x[n-1]$$

Now find $h[n]$

$$H(e^{jw}) = \frac{2 - e^{-jw}}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}}$$

$$= \frac{2 - e^{-jw}}{\left(1-\frac{1}{2}e^{-jw}\right)\left(1-\frac{1}{4}e^{-jw}\right)} = \frac{A}{1-\frac{1}{2}e^{-jw}} + \frac{B}{1-\frac{1}{4}e^{-jw}}$$

$$A = 0 \qquad B = \frac{-2}{1-2} = 2$$

$$\underline{\underline{h[n] = 2\left(\frac{1}{4}\right)^n u[n]}}$$