Overdetermined System of Equations

$$A\underline{x} = y$$

$A = n \times m$ with $n > m$

$\underline{x} = m \times 1$ and unknown

$y = n \times 1$ and known

Overdetermined means more equations than unknowns.

There are four possibilities for the solution, $\underline{x}$:

unique and exact
unique and approximate
non-unique and exact
non-unique and approximate

To figure out which of these applies we have to define

$$A_{avg} = [A \mid y]$$

Matlab: Aavg = [A y];

The rank of a matrix is the number of linearly independent rows and/or columns in the matrix.

By looking at the rank(A) and rank(Aavg) we can tell which of the four cases we have.

rank(A) is the Matlab command

Matrix A is full rank if

$\mathrm{rank}(A) = \min(n, m) = m$ in our case.

If $\underline{A}$ is not full rank, there is not a unique solution, rather it is non-unique (an infinite number of equivalent solutions).

If $\mathrm{rank}(A) = \mathrm{rank}(A_{aug})$ the solution is exact, i.e. $A\underline{x} = y$.

If $\mathrm{rank}(A) < \mathrm{rank}(A_{aug})$ the solution is approximate, i.e., $A\underline{x} \simeq y$.

Least Squares (LS) Solution

Define the errors

$$\underline{e} = A\underline{x} - \underline{y}$$

The sum of the squared errors is

$$E = \underline{e}^T\underline{e} = \underline{x}^T A^T A \underline{x} - \underline{x}^T A^T \underline{y} - \underline{y}^T A \underline{x} + \underline{y}^T \underline{y}$$

We want to find $\underline{x}$ to minimize E.

$$\nabla E = \begin{bmatrix} \frac{\partial E}{\partial x_1} \\ \vdots \\ \frac{\partial E}{\partial x_m} \end{bmatrix} = 2(A^T A)\underline{x} - 2A^T\underline{y} = \underline{0}$$

$$\boxed{\hat{\underline{x}}_{LS} = (A^T A)^{-1} A^T \underline{y}} \quad ☆☆☆$$

If $\underline{\text{rank}(A) < m}$ the $A^T A$ is $\underline{\text{singular}}$, i.e., $\underline{\text{not invertible}}$.

In this case we need to use the pseudoinverse, also called the Moore-Penrose inverse.

$$\hat{\underline{x}}_{LS} = \text{pinv}(A)\underline{y}$$

Alternate notation: $A^{\dagger}$

<u>Matlab</u>

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xhatls = pinv(A)*y;
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The result is the unique <u>minimum norm solution</u> for $\underline{x}$.