Discrete Time Fourier Transform (DTFT) and Discrete Fourier Transform (DFT)

Generally, the DTFT is defined for infinite length signals. However, in practice we only tend to actually observe finite length signals.

The finite length signal $x[n]$ is observed over *n* = 0, 1, ... , *N*-1 and is assumed to be zero everywhere else, thus it is a *finite length signal*. The Discrete Time Fourier Transform (DTFT) is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\omega n}$$

For our finite length signal this reduces to

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-\omega n}$$

Now, if we sample the frequencies N times evenly distributed from 0 to just below 2π according to

$$\omega_k = \frac{2\pi}{N}k \qquad k = 0, \dots, N-1$$

we obtain the Discrete Fourier Transform (DFT). Because of the sampling in time and frequency it is discrete in both domains. We express this discrete frequency behavior by indexing it by the frequency index, *k*.

$$X[k] = X\left(e^{j\omega_k}\right) \qquad k = 0, \dots, N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\left(\frac{2\pi}{N}k\right)n} \qquad k = 0, \dots, N-1$$

The Inverse DFT (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{\left(\frac{2\pi}{N}k\right)n} \qquad n = 0, \dots, N-1$$

However for *n* < 0 and *n* > *N*-1 we must enforce our initial assumption that the signal is zero. The IDFT equation will not do this for us, and it is *very important* to remember this.

If we just plug in values of n outside of the observed range *n* = 0, 1, ... , *N*-1 we will just get a periodic signal where the values from $x[n]$ repeat periodically with a period of *N*.

If we plug in values of *k* outside of the *k* = 0, 1, ... , *N*-1 into the DFT equation we also get a periodic replication of $X[k]$ with a period of *N*. However, this is actually correct, since the frequency domain of discrete-time signals is periodic in ω with a period of 2π, and we have chosen to sample this frequency domain using *N* samples over each interval of 2π.

Zero-Padding the DFT

Since we assume the signal $x[n]$ is zero outside of *n* = 0, 1, ... , *N*-1 there is no theoretical problem in including some of the zeros after *N*-1, thus increasing the length of the signal. Suppose we include zeros up to *n* = *NFFT*-1 (NFFT contains a reference to the Fast Fourier Transform, FFT, but we will discuss this later). Thus, by this new interpretation the signal is now *NFFT* points long. This process is called zero-padding.

To get the DFT of this longer signal we sample the frequencies according to

$$\omega_k = \frac{2\pi}{NFFT}k \qquad k = 0, \dots, NFFT - 1$$

Notice that even though we are still using the index *k* it has a different meaning now. It refers to samples that are closer together in frequency than before. Previously they were $2\pi/N$ radians apart but now they are $2\pi/NFFT$ radians apart. Since we can choose $NFFT$ to be as large as we want, we can make the frequency samples as close together as we want. Thus zero-padding can be used to get frequency samples as closely spaced as we want. The DFT after zero-padding becomes

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{2\pi}{NFFT}kn} \qquad k = 0, \dots, NFFT - 1$$

Notice the top of the summation symbol still has *N*-1, since $x[n]$ is still nonzero only over *n* = 0, 1, ... , *N*-1. Zero-padding doesn't change that. However, because of the new frequency sampling, the IDFT becomes

$$x[n] = \frac{1}{NFFT} \sum_{k=0}^{NFFT-1} X[k]e^{\frac{2\pi}{NFFT}kn} \qquad n = 0, \dots, NFFT - 1$$