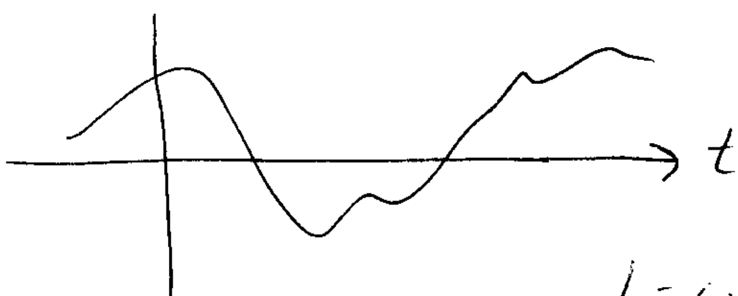
Chapter 1

Signals

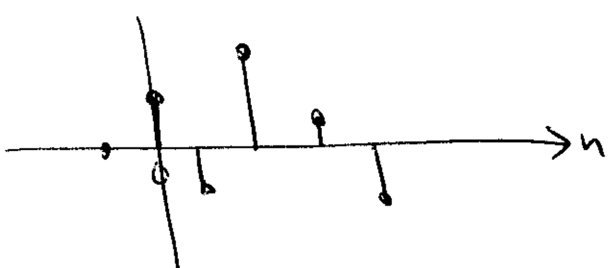
Continuous-Time and Discrete-Time

$x(t)$



t = continuous-time variable

$x[n]$



n = discrete-time variable (sample index)

Signal Energy

Continous-Time

$$\boxed{E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt}$$ ☆☆☆

Discrete-Time

$$\boxed{E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2}$$ ☆☆☆

Power (Average Power)

$$\boxed{P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt}$$ ☆☆☆

If $x(t)$ is periodic with period $= T_0$ the P_∞ is the average power over one period ☆☆☆

$$P_{AVG} = P_\infty = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt$$

$$\boxed{P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2}$$ ☆☆☆

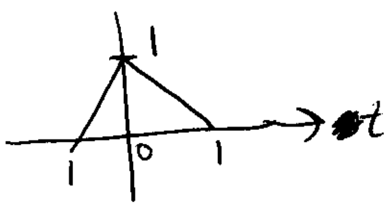
Time Shift

$y(t) = x(t - t_0)$

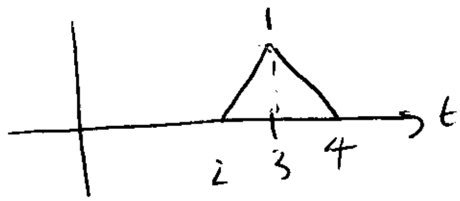
If $t_0 > 0 \Rightarrow$ shift $x(t)$ right by t_0.

If $t_0 < 0 \Rightarrow$ shift $x(t)$ left by t_0.

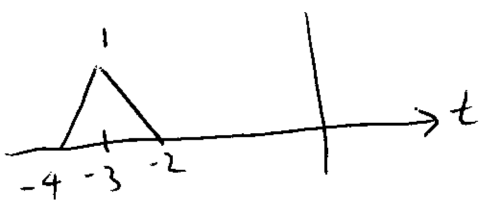
$x(t)$



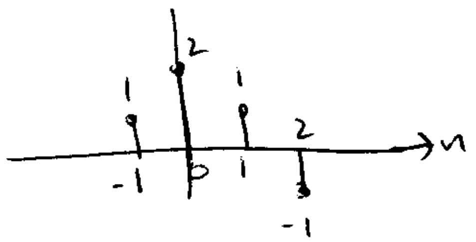
$x(t-3)$



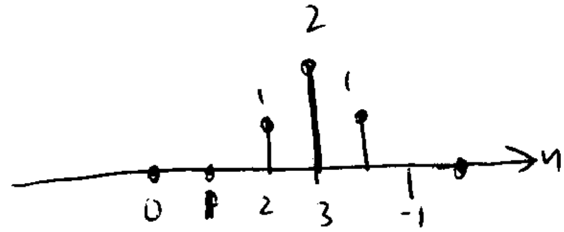
$x(t+3)$



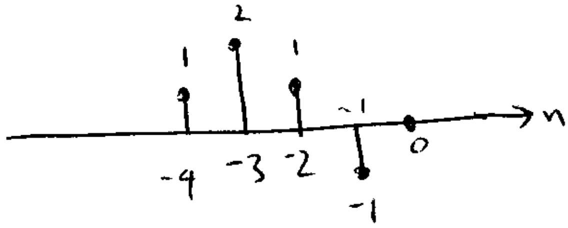
$x[n]$



$x[n-3]$



$x[n+3]$



Time Scaling

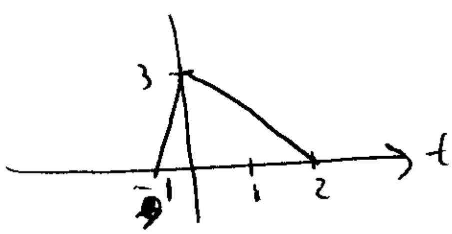
$y(t) = x(at)$

If $|a| > 1$ then the signal gets "squished".

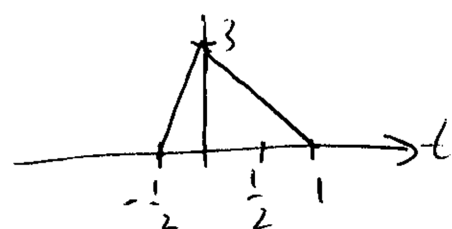
If $|a| < 1$ then the signal gets "stretched."

If $a < 0$ then the signal is also time-reversed (i.e, "flipped")

$x(t)$

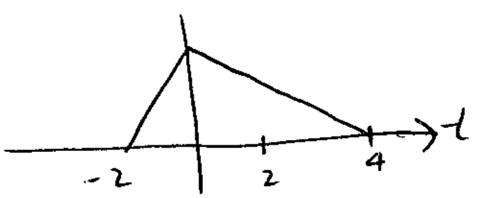


$x(2t)$



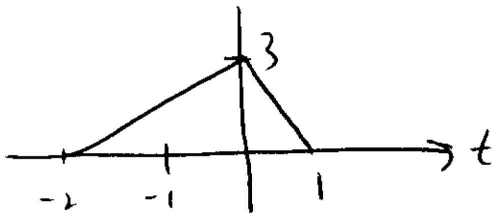
squished to $-\frac{1}{2}$ to 1

$x(\frac{1}{2}t)$



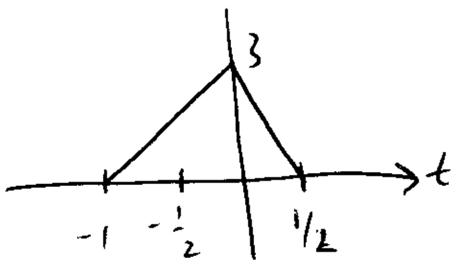
stretched to -2 to 4

$x(-t)$

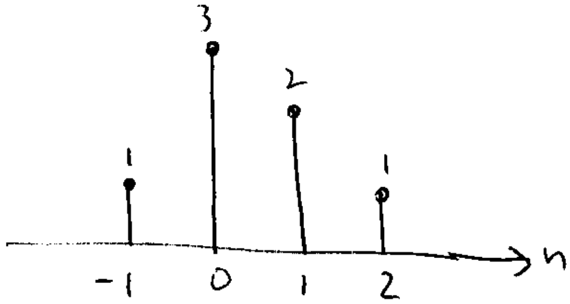


"flipped"

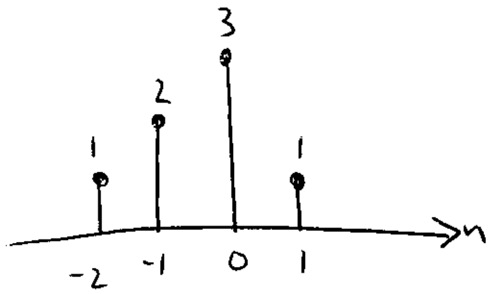
$x(-2t)$



$x[n]$



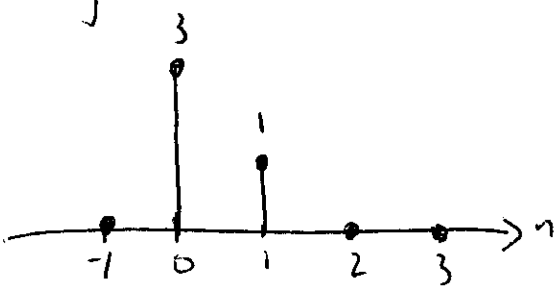
$x[-n]$



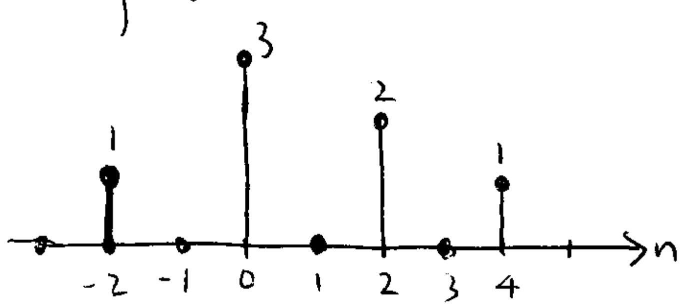
$y[n] = x[an]$

For each value of n, $y[n] = x[an]$ only if "an" is an integer.

$y[n] = x[2n]$



$y[n] = x[\frac{1}{2}n]$



Systems

We will focus on single-input single-output (SISO) systems.

C.T.

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

D.T.

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

A system takes the input signal, transforms it in some way to produce the output signal.

Basic System Properties

Memory

If the output at time t (C.T.) or n (D.T.) is dependent only on the input at time t or n, then it is memoryless. If it depends on the input at any other time (past or future) then it has memory.

$y(t) = (x(t))^2$ memoryless

$y[n] = 10x[n]$ memoryless

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ has memory

$y[n] = x[n-1]$ has memory

Causality

C.T.

If the output at time t $\left(\begin{array}{c}\text{for all}\\ \text{time } t\end{array}\right)$ depends only on the values of the input at time t or any time before t (the past) and does not depend on knowing the input at a future time, then it is a <u>causal</u> <u>system</u>.

It means you never need to know a future value of the input in order to compute the current output.

This must be true for <u>all</u> choices of time t.

D.T.

The definition is the same, just replace t with n.

Note: <u>Memoryless systems</u> are <u>causal</u>.

<u>ex</u>

$y(t) = (x(t))^2$ memoryless, causal

$y[n] = x[n-1]$ causal (depends on a past value)

$y[n] = x[n+1]$ <u>non-causal</u> (depends on a <u>future</u> value)

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ <u>causal</u>

$y[n] = \sum_{m=-\infty}^{n+5} x[m]$ <u>non-causal</u>

Stability

A signal is bounded if there exists a finite positive value M such that

$$-M \le |x(t)| \le +M$$

$$-M \le |x[n]| \le +M$$

A system is Bounded Input Bounded Output Stable (BIBO) if for every bounded input ($x(t)$ or $x[n]$) the resulting output ($y(t)$ or $y[n]$) is also bounded.

Note: The value of the bound, M, can be different for the input and output signals. It only matters that a bound exists.

examples

$y(t) = 10\,x(t)$ BIBO stable

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ unstable

Consider $x(t) = 1$ for all t.

Then $y(t) \to \infty$

$y[n] = n\,x[n]$ unstable

if $x[n] = 1$ for all n then

$n\,x[n] \to \infty$ as $n \to \infty$

$y[n] = x[n-1]$ stable

Time - Invariance

Consider put $x_1(t)$ as the input to a system. The output we will call $y_1(t)$.

$$x_1(t) \longrightarrow \boxed{} \longrightarrow y_1(t)$$

Now define $x_2(t)$ as a time-shifted version of $x_1(t)$.

$$x_2(t) = x_1(t - t_0)$$

Put $x_2(t)$ as the input to the same system. The output will be $y_2(t)$.

$$x_2(t) \longrightarrow \boxed{} \longrightarrow y_2(t)$$

If, for <u>all</u> choices of $x(t)$ and t_0

$$y_2(t) = y_1(t - t_0)$$

then the system is <u>time-invariant</u>. Otherwise it is <u>time-varying</u>.

The same is true for discrete-time. Just replace t with n and t_0 with n_0.

ex) $y(t) = 10\, x(t)$

$$x_1(t) \rightarrow \boxed{} \rightarrow y_1(t) = 10\, x_1(t)$$

$$x_2(t) = x_1(t-t_0) \rightarrow \boxed{} \rightarrow y_2(t) = 10\, x_2(t)$$
$$= 10\, x_1(t-t_0)$$

$$y_1(t-t_0) = y_1(t)\Big|_{t=t-t_0}$$
$$= 10\, x_1(t-t_0)$$
$$= y_2(t)$$

$\therefore$ Time - Invariant

ex) $y[n] = n\, x[n]$

$y_1[n] = n\, x_1[n]$

$x_2[n] = x_1[n-n_0]$

$y_2[n] = n\, x_2[n] = n\, x_1[n-n_0]$

$y_1[n-n_0] = y_1[n]\Big|_{n=n-n_0} = (n-n_0)\, x_1[n-n_0]$

$\neq y_2[n]$

$\therefore$ time-varying

ex)
$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)\, d\tau$$

$$x_2(t) = x_1(t - t_0)$$

$$y_2(t) = \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x_1(\tau - t_0)\, d\tau$$

$$y_1(t - t_0) = y_1(t)\Big|_{t = t - t_0} = \int_{-\infty}^{t - t_0} x_1(\tau)\, d\tau$$

define $\alpha = \tau + t_0$

then $\tau = \alpha - t_0$

Change the variable of integration:

$$y_1(t - t_0) = \int_{-\infty}^{(t - t_0) + t_0} x_1(\alpha - t_0)\, d\alpha$$

$$= \int_{-\infty}^{t} x_1(\alpha - t_0)\, d\alpha$$

$$= y_2(t)$$

$\therefore$ Time-Invariant

ex)

$$y[n] = x[n-1]$$

$$y_1[n] = x_1[n-1]$$

$$x_2[n] = x_1[n-n_0]$$

$$y_2[n] = x_2[n-1] = x_1[n-1-n_0]$$

$$y_1[n-n_0] = y_1[n]\Big|_{n=n-n_0}$$

$$= x_1[n-n_0-1]$$

$$= y_2[n]$$

$\therefore$ Time-Invariant

Linearity

Linearity is made up of two simpler properties:

(1) homogeneity (scaling)

(2) superposition (adding)

Homogeneity

$$x_1(t) \longrightarrow \boxed{} \longrightarrow y_1(t)$$

$$x_2(t) = a\,x_1(t) \longrightarrow \boxed{} \longrightarrow y_2(t)$$

"a" is a scalar, real or complex

> If $y_2(t) = a\,y_1(t)$ for all choices of a and $x_1(t)$, then the system <u>obeys homogeneity</u>, others it does not.

That is

$$a\,x(t) \longrightarrow a\,y(t)$$

Superposition

$$x_1(t) \longrightarrow \boxed{} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{} \longrightarrow y_2(t)$$

Thus $x_1(t)$ yields $y_1(t)$ as its output, and $x_2(t)$ yields $y_2(t)$.

Now form

$$x_3(t) = x_1(t) + x_2(t)$$

$$x_3(t) \longrightarrow \boxed{} \longrightarrow y_3(t)$$

If $y_3(t) = y_1(t) + y_2(t)$
for all choices of $x_1(t)$ and $x_2(t)$,
then the system obeys <u>superposition</u>.

If a system obeys both homogeneity and superposition then it is linear, otherwise it is nonlinear.

The discete-time case is the same.

ex)

$$y(t) = 10\,x(t)$$

$$x_1(t) \longrightarrow y_1(t) = 10x_1(t) \quad \left(\begin{array}{l}\text{i.e., } x_1(t) \\ \text{yields } y_1(t) = 10x_1(t)\end{array}\right)$$

$$x_2(t) = a x_1(t) \longrightarrow y_2(t) = 10\,x_2(t) = a\,10\,x_1(t)$$
$$= a\,y_1(t)$$

$\therefore$ obeys homogeity

$$x_1(t) \longrightarrow y_1(t) = 10x_1(t) \qquad x_2(t) \longrightarrow y_2(t) = 10x_2(t)$$

$$x_3(t) = x_1(t) + x_2(t) \longrightarrow y_3(t) = 10\,x_3(t)$$
$$= 10x_1(t) + 10x_2(t)$$
$$= y_1(t) + y_2(t)$$

$\therefore$ Obeys Superposition

$\therefore$ Linear

ex) $y[n] = (x[n])^2$

$$x_1[n] \longrightarrow y_1[n] = (x_1[n])^2$$

$$x_2[n] = a\,x_1[n] \longrightarrow y_2[n] = (x_2[n])^2$$

$$= a^2\,(x_1[n])^2$$

$$\neq a\,y_1[n]$$

$\therefore$ Fails homogeneity

$\therefore$ Nonlinear