The Discrete Fourier Transform (DFT), the practical transform.

Suppose you have a finite length observation of $x[n]$:

$x[n]$ observed over $n = 0, 1, \ldots, N-1$

and we assume it is 0 everywhere else.

The DTFT would be

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

The DTFT is still not computationally practical since ω is a continuous variable. It is not possible to compute the DTFT for all values of ω.

Suppose we sample the ω variable from 0 to 2π.

$$\boxed{\omega_k = \frac{2\pi}{N} k \qquad k = 0, 1, \ldots, N-1}$$

frequency samples

Now we have

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} \quad , \; k = 0, 1, \ldots, N-1$$

<u>Note</u>: This will repeat periodically for values of k outside the 0 to $N-1$ range.

This sampled form of the DTFT is practical in that it can be computed for $k = 0, 1, \ldots, N-1$.

However, it is a sampled form of the DTFT, but not actually the DTFT any more.

If it is a sampled form of the DTFT, why should we care if it is not exactly the DTFT?

The reason is the inverse transform.

The DFT is

$$X[k] = X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$k = 0, 1, \ldots, N-1$$

The inverse DFT is

$$x[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} & n = 0, 1, \ldots, N-1 \\ 0 & \text{else} \end{cases}$$

Notice that the inverse transform equation

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

is only valid over $n = 0, 1, \ldots, N-1$.

We have to force the result to be zero outside this range.

If we do not force the result to be zero and just blindly use the inverse equation, it will periodically repeat $x[n]$ with a period of N.

This periodicity is a result of sampling in the frequency domain.

So what can we observe?

1) If you only want to examine or analyze the frequency domain information about a signal, the DFT as a sampled form of the DTFT is fine and you really don't need to take any special care in doing this.

2) If you want to do processing of the frequency domain information and then inverse transform this back to the time domain then it may be important to consider the fact that the inverse equation implicitly creates a periodic time-domain signal.

Another difference between the DFT and the DTFT is that the DFT is a <u>computational</u> tool. We don't need to learn any transform tables or use partial fraction expansion. We compute the DFT and also can compute the inverse DFT (IDFT).

<u>ex</u>

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1\} \qquad N=2$$

$$X[k] = \sum_{n=0}^{1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad k=0,1$$

note: $\frac{2\pi}{N} = \frac{2\pi}{2} = \pi$

$$e^{-j\frac{2\pi}{N}kn} = e^{-j\pi kn}$$

$$e^{\pm j\pi} = -1$$

So

$$X[k] = \sum_{n=0}^{1} x[n] \left(e^{-j\pi}\right)^{kn}$$

$$= \sum_{n=0}^{1} x[n] (-1)^{kn} \qquad k=0,1$$

$$X[0] = 1+1 = 2$$

$$X[1] = 1-1 = 0$$

What does this tell us?

First the k doesn't mean much by itself. It is a pointer or index of a frequency.

$$\boxed{\omega_k = \frac{2\pi}{N} k \qquad k = 0, 1, \ldots, N-1}$$

k as a frequency index

So for our example

$N = 2$

$k = 0 \rightarrow \omega_k = 0$

$k = 1 \rightarrow \omega_k = \pi$

So we only have samples at $\omega = 0$ and $\omega = \pi$. That's a very sparse sampling of the frequencies. What about the rest of the frequencies?

There is a nice trick called <u>zero-padding</u> that gives us a lot more frequencies.

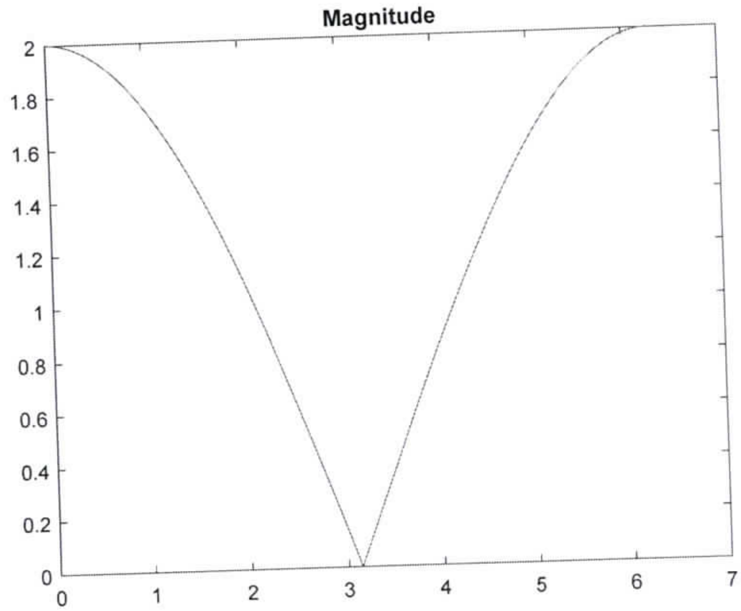
You add a bunch of zeroes onto the end of the signal, thus increasing the size of N. This gives us more samples of ω.

In matlab we use the command, fft, for computing the DFT. The fast fourier transform (FFT) is a very fast algorithm for computing the DFT. Additionally, the FFT revolutionized computational signal processing.

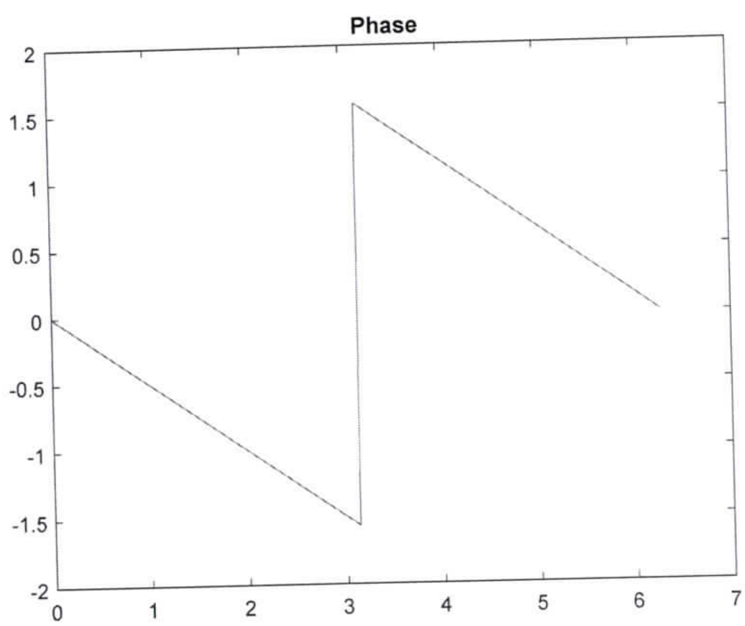
For our example:

Set <u>NFFT = 1024</u> to zero-pad so that the length of the padded signal is 1024. We use $1024 = 2^{10}$ because it is a power of two, which is optimum for the FFT.

```
x=[1 1];
X= fft(x, NFFT);
k=0:NFFT-1;
wk= 2*pi/NFFT*k;
figure(1), plot(wk, abs(X))
figure(2), plot(wk, angle(X))
```



ω



ω

Note: The DFT/FFT gives results over the $0 \leq \omega < 2\pi$ interval.
The second half (right half) corresponds to a replication of the negative frequencies