Chapter 2

Probability, Statistics, and Traffic Theories

Outline

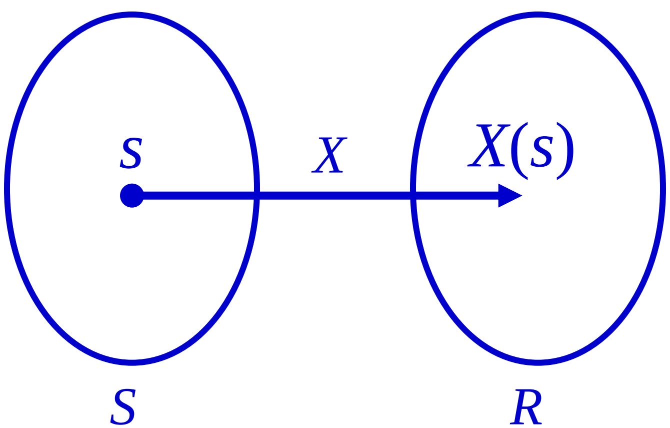
- Introduction
- Probability Theory and Statistics Theory
  - Random variables
  - Probability mass function (pmf)
  - Probability density function (pdf)
  - Cumulative distribution function (CDF)
  - Expected value, nth moment, nth central moment, and variance
  - Some important distributions
- Traffic Theory
  - Poisson arrival model, etc.
- Basic Queuing Systems
  - Little's law
  - Basic queuing models

Introduction

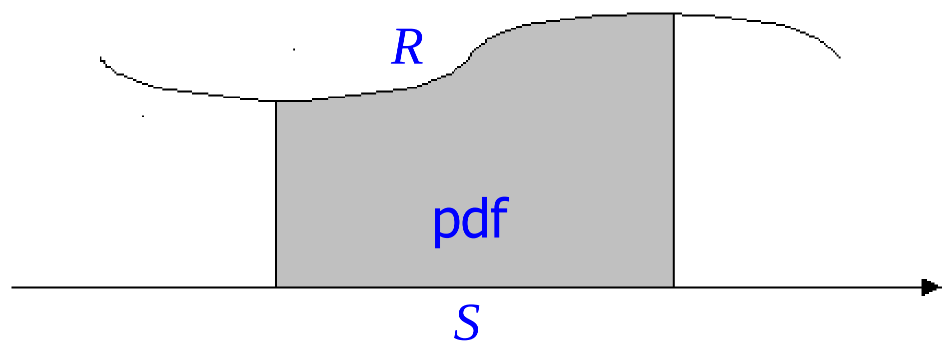
- Several factors influence the performance of wireless cellular systems:
  - Density of mobile users
  - Cell size
  - Moving direction and speed of users (Mobility models)
  - Call rate, call duration
  - Interference, etc.
- Probability, statistics theory and traffic patterns, help make these factors tractable
- It helps in implementing a system that could satisfy desired design parameters

Probability Theory and Statistics Theory

- Random Variables (RVs)
  - Let S be sample associated with experiment E
  - X is a function that associates a real number to each $s \in S$
  - RVs can be of two types: Discrete or Continuous
  - Discrete random variable => probability mass function (**pmf**)
  - Continuous random variable => probability density function (**pdf**)



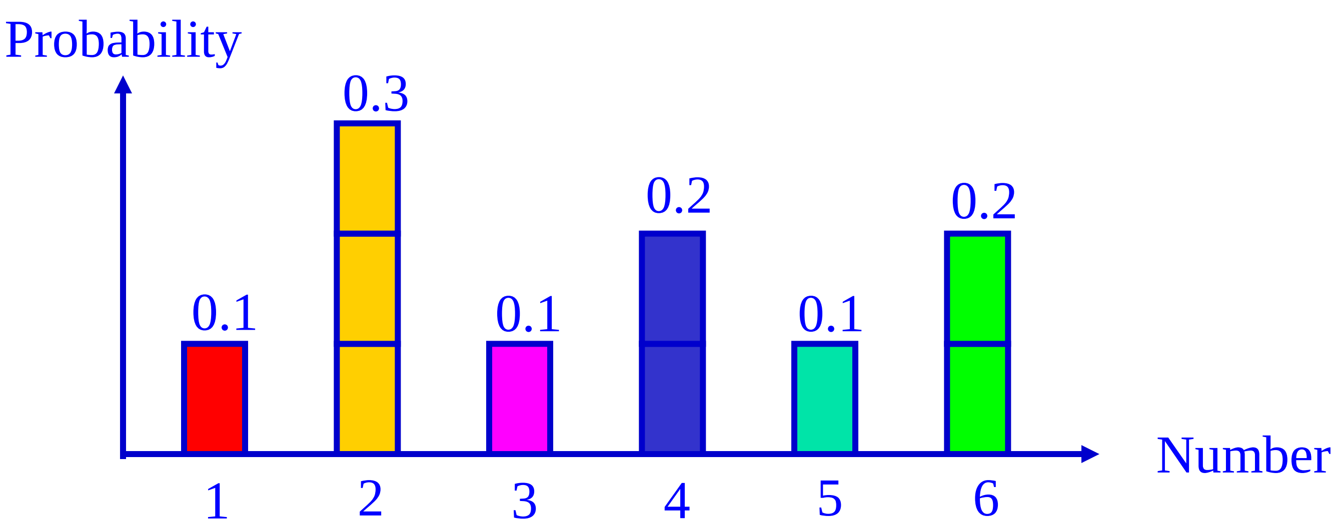
Discrete random variable



Continuous random variable

Discrete Random Variables

- In this case, $X(s)$ contains a finite or infinite number of values
  - The possible values of X can be enumerated
- **For Example:** Throw a 6 sided dice and calculate the probability of a particular number appearing



Discrete Random Variables

- The probability mass function (**pmf**) $p(k)$ of X is defined as:

  $$p(k) = p(X = k), \qquad \text{for } k = 0, 1, 2, \ldots$$

  where

  1. Probability of each state occurring

     $$0 \le p(k) \le 1, \text{ for every } k;$$

  2. Sum of all states

     $$\sum p(k) = 1, \text{ for all } k$$

Continuous Random Variables

- In this case, X contains an infinite number of values
- Mathematically, X is a continuous random variable if there is a function f, called probability density function (**pdf**) of X that satisfies the following criteria:
  1. $f(x) \geq 0$, for all x;
  2. $\int f(x)dx = 1$

Cumulative Distribution Function

- Applies to all random variables
- A cumulative distribution function (**CDF**) is defined as:
  - For discrete random variables:

$$P(k) = P(X \leq k) = \sum_{\text{all} \leq k} P(X = k)$$

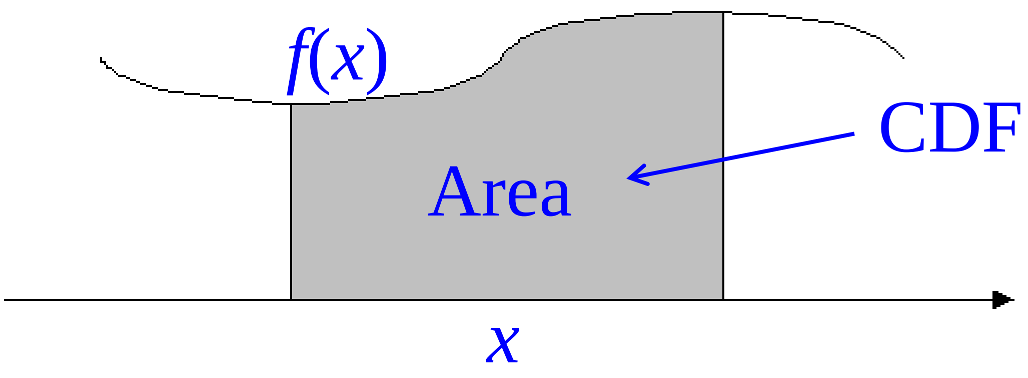
  - For continuous random variables:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Probability Density Function

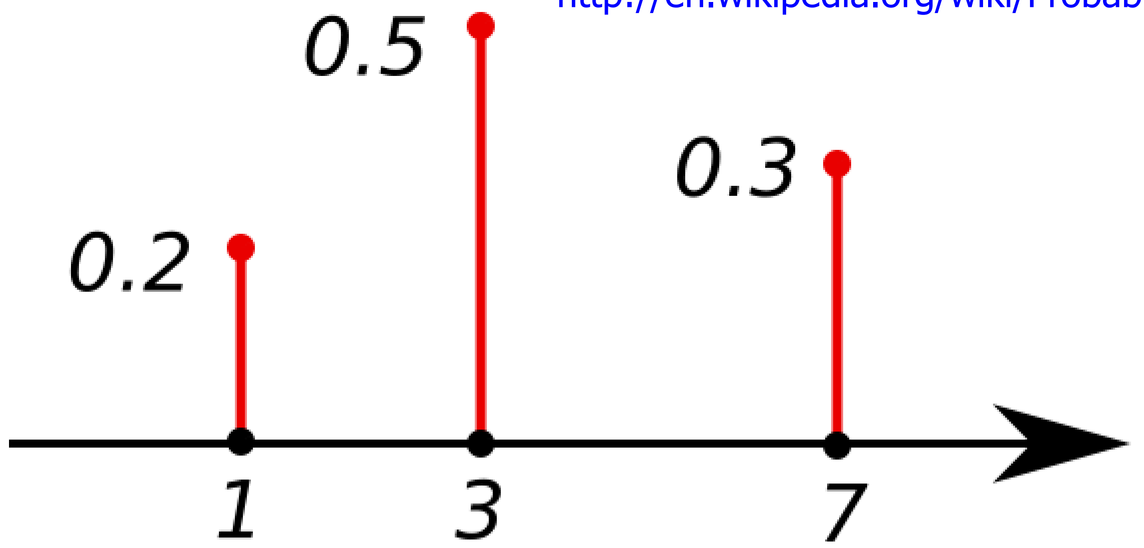
- The pdf $f(x)$ of a continuous random variable X is the derivative of the **CDF** $F(x)$, i.e.,

$$f(x) = \frac{dF_X(x)}{dx}$$

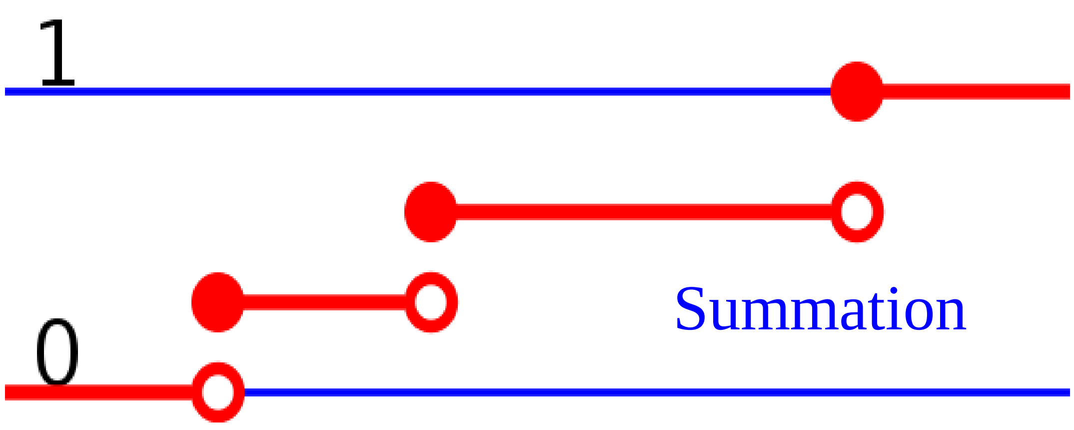


Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

http://en.wikipedia.org/wiki/Probability_distribution



The probability mass function of a discrete probability distribution. The probabilities of the **singletons** {1}, {3}, and {7} are respectively 0.2, 0.5, 0.3



The **pmf** of a discrete probability distribution, is summation of discrete probability distribution

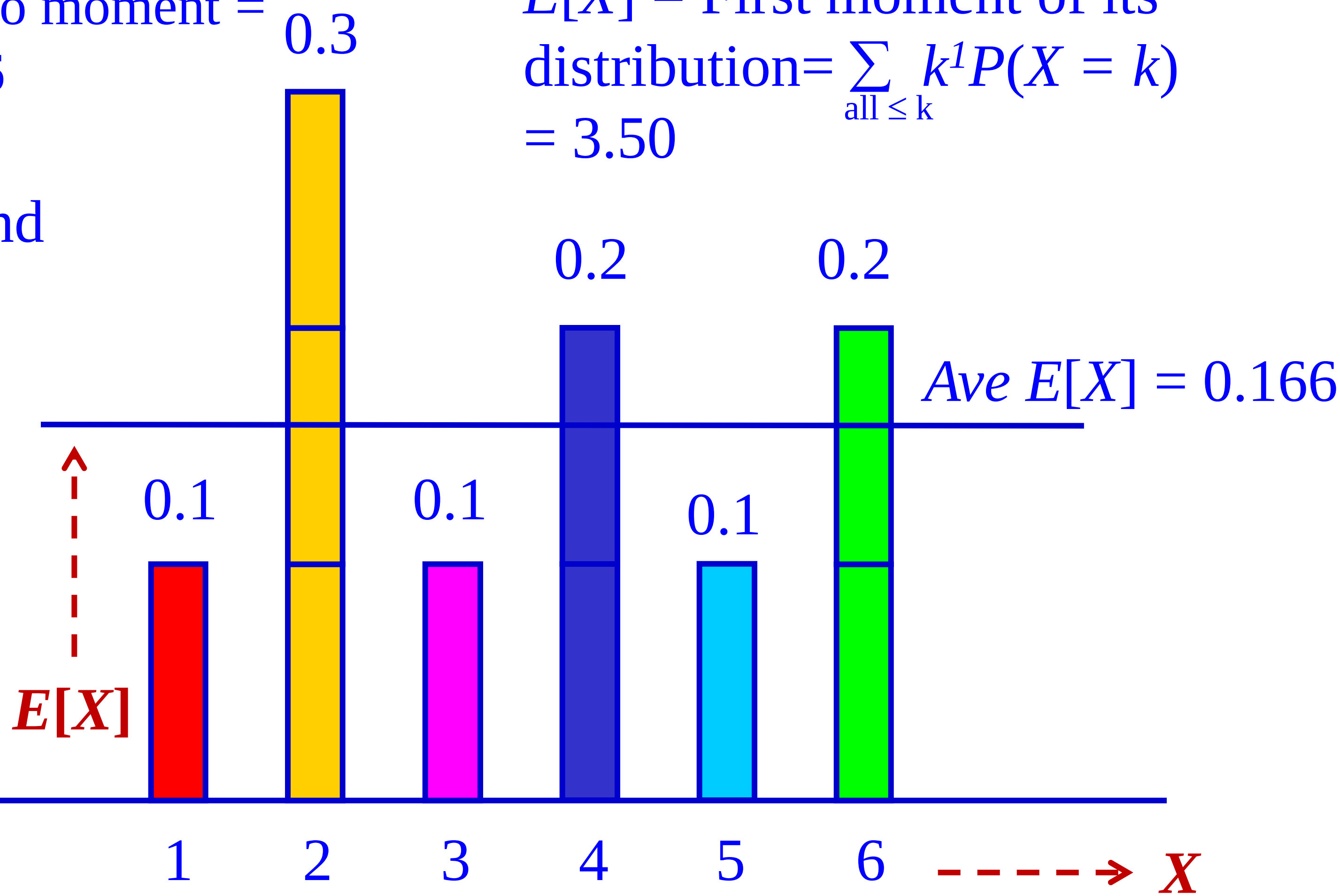
Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

Probability distribution =
Average $E[X]$ =zero moment =
$\sum_{\text{all} \leq k} k^0 P(X = k) = 1/6$

Average or Mean value of
$E[X]$ = First moment of its
distribution= $\sum_{\text{all} \leq k} k^1 P(X = k)$
= 3.50

Variance = Second
central moment

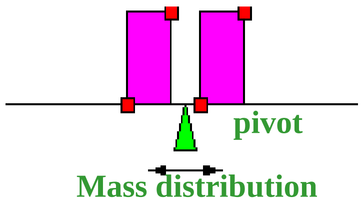
$\sigma^2 = Var(X) =$
$E[(X - E[X])^2] =$
$E[X^2] - (E[X])^2$



2nd Moment and Variance using Teeter-Totter (seesaw)

http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/moreApps/moments.html

1



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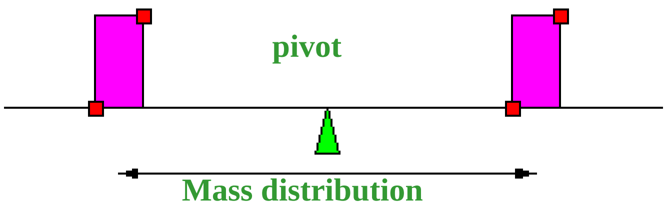
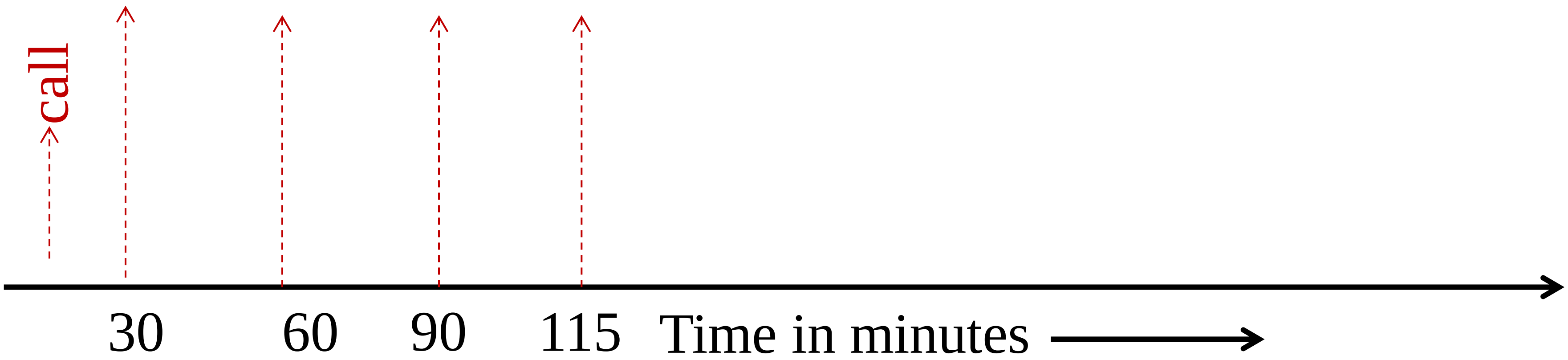


- Balanced in both situations
- Center of mass of the distribution is at the same place: precisely at the pivot point
- However, the mass is distributed very differently in these two cases
- In the first case, the mass is clustered close to the center, whereas in the second, it is distributed further away

- How a probability density distribution is distributed about its mean, need to consider moments higher than the first one
- **Variance** σ^2 is defined as the average value of the quantity: distance from mean (square σ^2 to avoid cancelling values to the left and right of the mean)

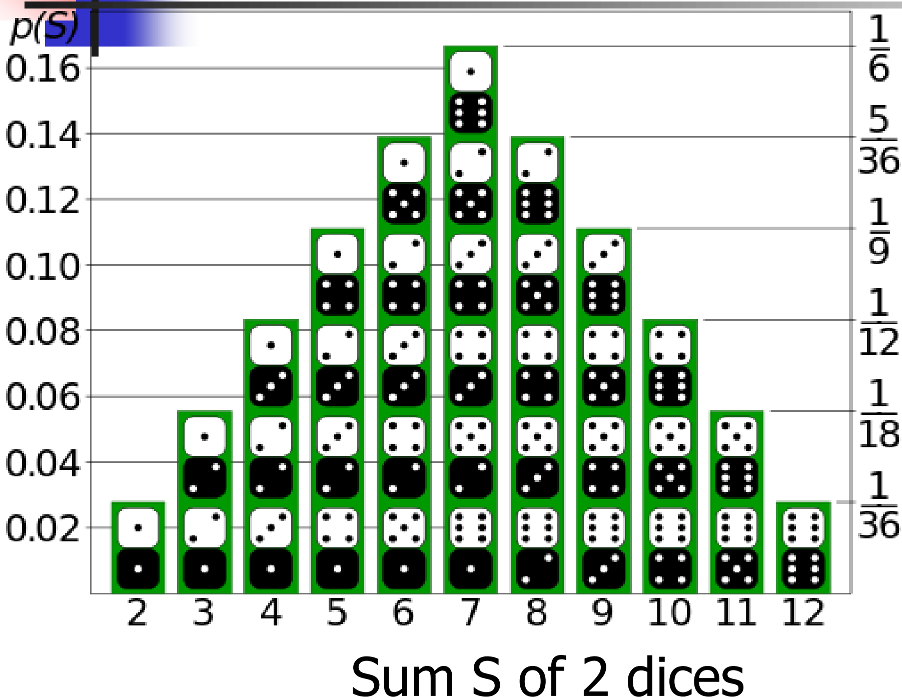
2^{nd} Moment and Variance in Cell Phones

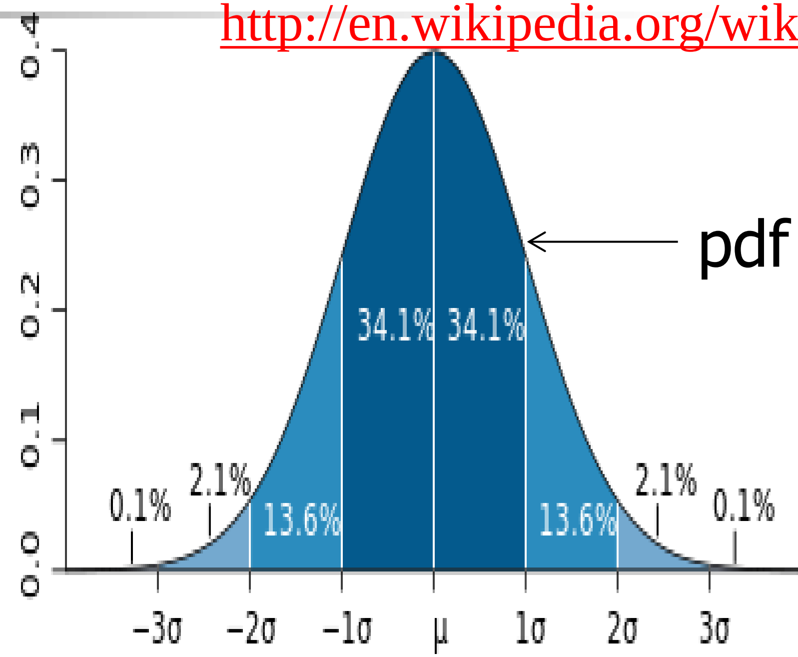


- On an average, receiving 2 call per hour (60 minutes)
- Probability of a call =average= 0^{th} moment= 2/60=1/30 per minute
- Mean or average calls/hour is 2 by obtaining average using first moment
- A new call may come after 30 minutes, or could be 29 or 31 minutes later; such variation from mean is variance and can be obtained from 2^{nd} moment
- To determine variation in variation, we need to take 3^{rd} moment

Some Important Discrete Random Distributions

http://en.wikipedia.org/wiki/





μ is the *mean,* σ is standard deviation

- The probability mass function (pmf) $p(S)$ specifies the probability distribution for the sum S of counts from two dices
- For example, the figure shows that $p(11) = 1/18$
- The pmf allows the computation of probabilities of events such as $P(S > 9) = 1/12 + 1/18 + 1/36 = 1/6$, and all other probabilities in the distribution

- The probability density function (pdf) of the normal distribution, also called Gaussian or "bell curve", the most important continuous random distribution
- The probabilities of intervals of values correspond to the area under the curve

Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance of Discrete Variables

- Discrete Random Variables
  - Expected value represented by E or weighted average of random variable

$$E[X] = \sum_{\text{all} \leq k} kP(X = k)$$

  - Variance = Second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - n^{th} moment

$$E[X^n] = \sum_{\text{all} \leq k} k^n P(X = k)$$

  - n^{th} central moment

$$E[(X - E[X])^n] = \sum_{\text{all} \leq k} (k - E[X])^n P(X = k)$$

Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance of Random Variables

- Continuous Random Variable
  - Expected value or mean value

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

  - Variance or the second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - n^{th} moment

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

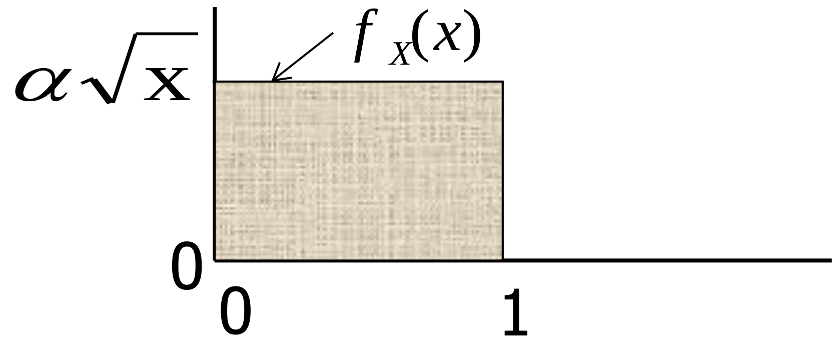
  - n^{th} central moment

$$E[(X - E[X])^n] = \int_{-\infty}^{+\infty} (x - E[X])^n f(x) dx$$

Example of Continuous Random Variable

- **Example 2.1**: The pdf of a continuous random variable X is

$$f_X(x) = \begin{cases} \alpha\sqrt{x}, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



- (1). Find constant *a*?

Based on the conditions of continuous random variables, we have the pdf as

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 \alpha\sqrt{x}dx = \alpha\left[\frac{x^{3/2}}{3/2}\right]_{x=0}^{x=1} = \frac{2\alpha}{3} = 1$$

Therefore, constant *a* is 3/2=1.5

pdf as

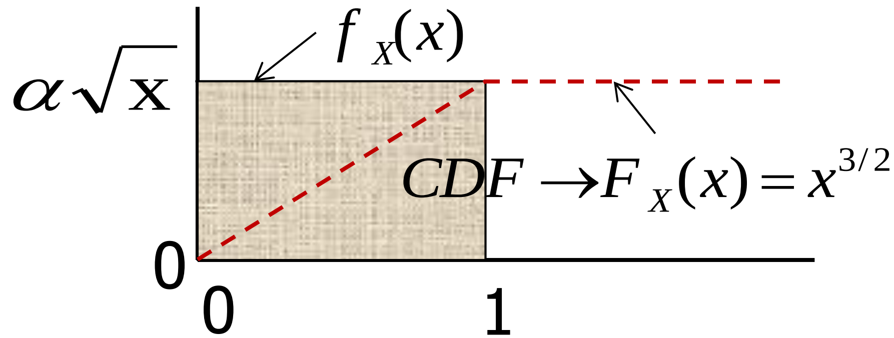
$$1.5\left[\frac{x^{3/2}}{3/2}\right]_{x=0}^{x=1} = \left[x^{3/2}\right]_{x=0}^{x=1}$$

Example of Continuous Random Variable

- **Example 2.1**: The **pdf** of a continuous random variable X is

$$f_X(x) = \begin{cases} \alpha\sqrt{x}, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

- Find (2). Find **pdf** of X ?



From Equation (2.6), the **pdf** of random variable X is the integral of its pdf ---- that is,

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)dx = \begin{cases} 0 & \text{for} \quad x < 0 \\ x^{3/2} & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad 1 \le x \end{cases}$$

Some Important Discrete Random Distributions

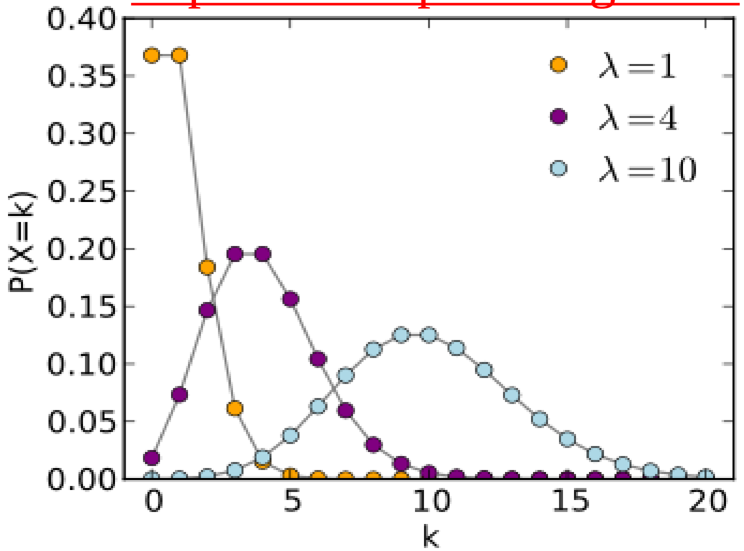
- Poisson

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k{=}0,1,2,\ldots,\text{ and }\lambda>0$$

  - $E[X] = \lambda$, and $Var(X) = \lambda$

Poisson distribution specifies how likely it is that the number of calls in each hour will be 3, or 5, or 10, or any other number

http://en.wikipedia.org/wiki/

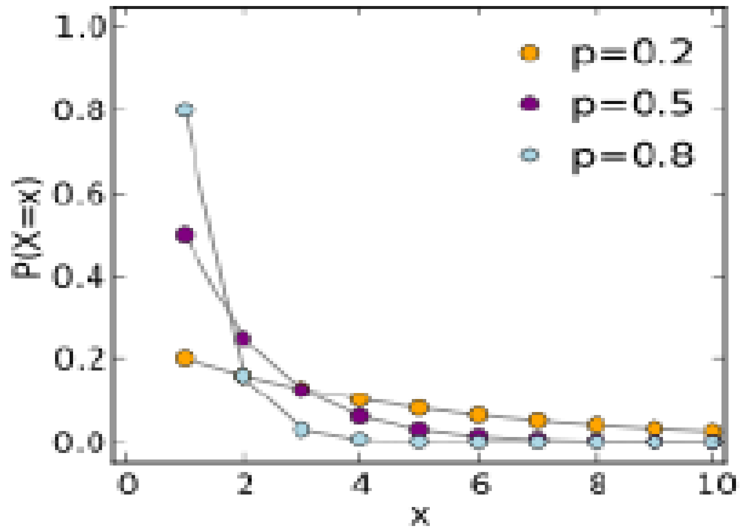


- Geometric

$$P(X = k) = p(1\text{-}p)^{k\text{-}1}\ ,$$

where p is success probability

  - $E[X] = 1/(1\text{-}p)$, and $Var(X) = p/(1\text{-}p)^2$



http://en.wikipedia.org/wiki/

Some Important Discrete Random Distributions

- Binomial

  Out of n dice, **exactly** k dice have the same value: probability p^k and $(n-k)$ dice have different values: probability$(1-p)^{n-k}$

where $\boldsymbol{p}$ is the probability of success.

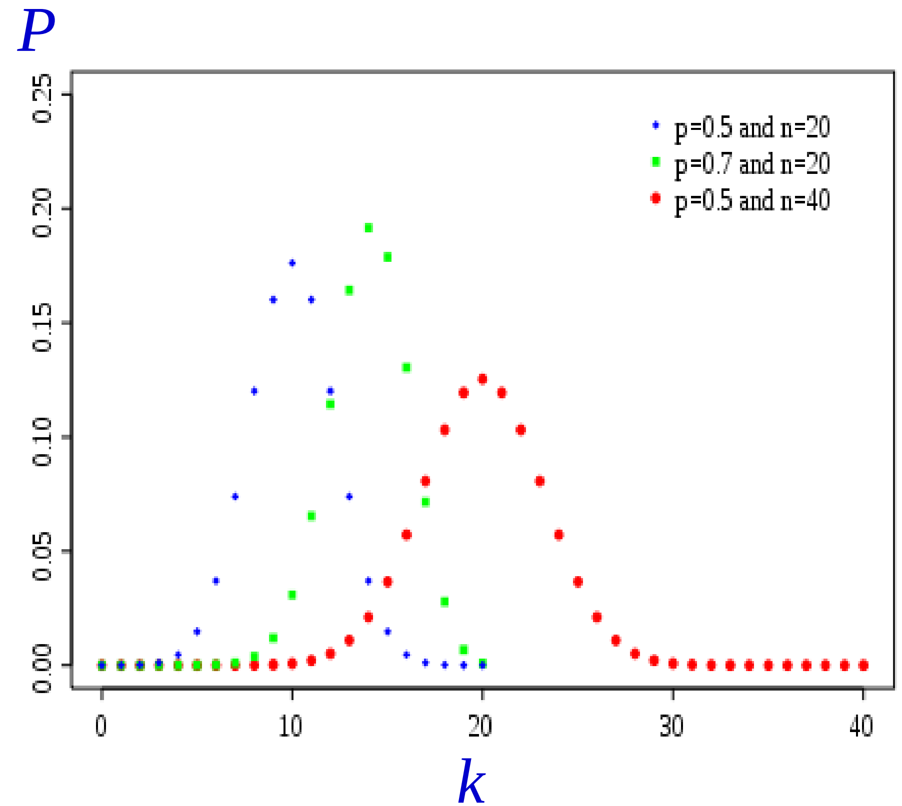
For any k dice out of n:

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k},$$

where,

$k=0,1,2,\ldots,n$; $n=0,1,2,\ldots$; p is the sucess probability, and

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$



http://en.wikipedia.org/wiki/

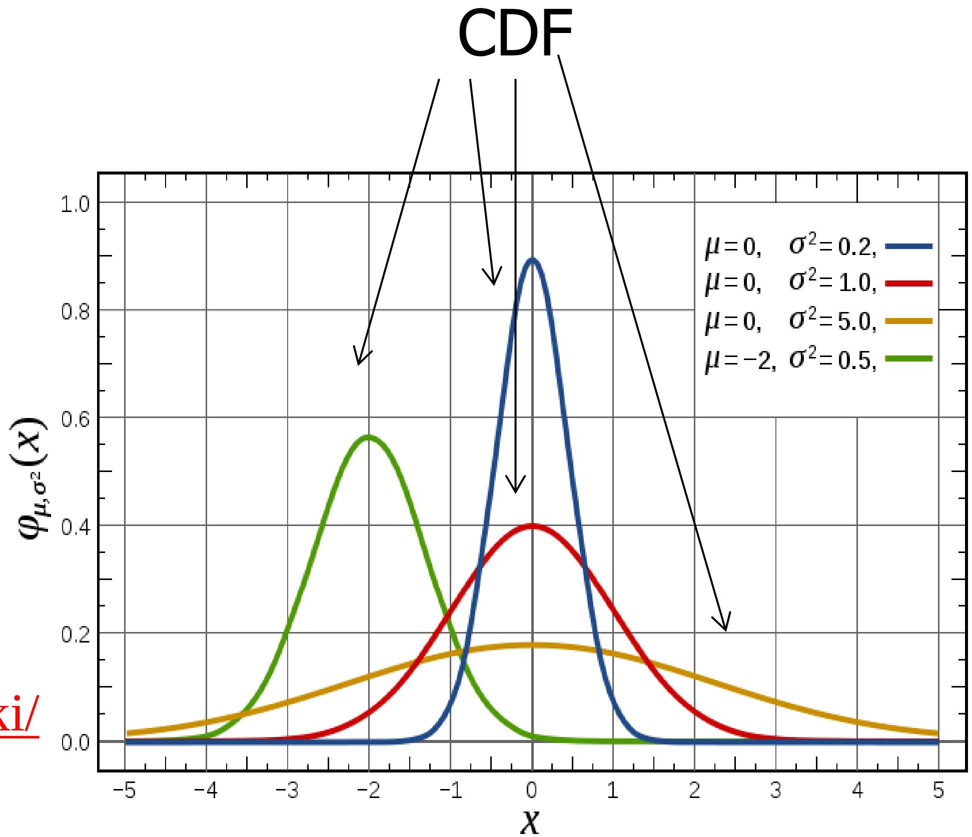
Some Important Continuous Random Distributions

- Normal: $E[X] = \mu$, and $Var(X) = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

and the cumulative distribution function can be obtained by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$



http://en.wikipedia.org/wiki/

Red curve is the Gaussian distribution

Some Important Continuous Random Distributions

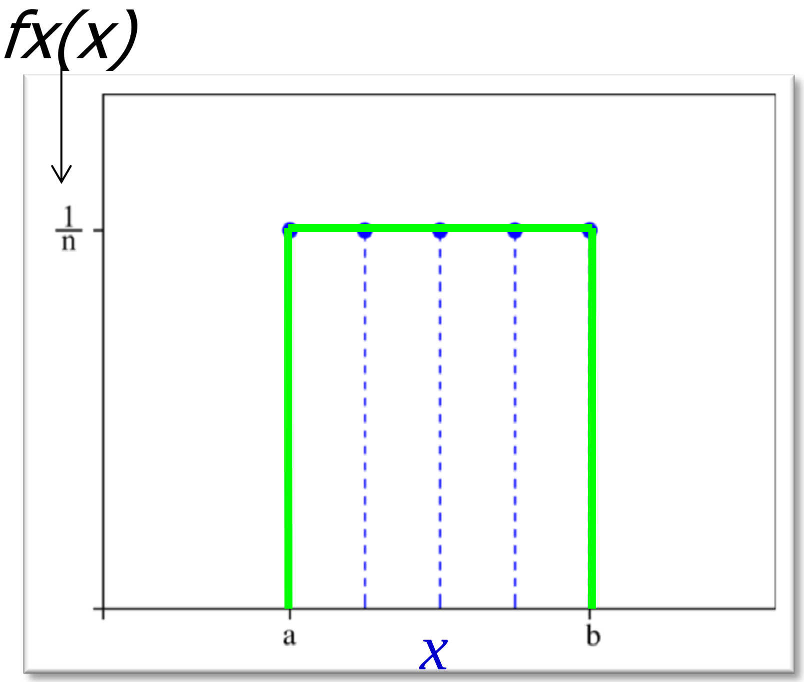
- Uniform

http://en.wikipedia.org/wiki/

$$f_X(x) = \begin{cases} \dfrac{d}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & \text{for } x < a \\ \dfrac{x-a}{b-a}, & \text{for } a \le x \le b \\ 1, & \text{for x > b} \end{cases}$$

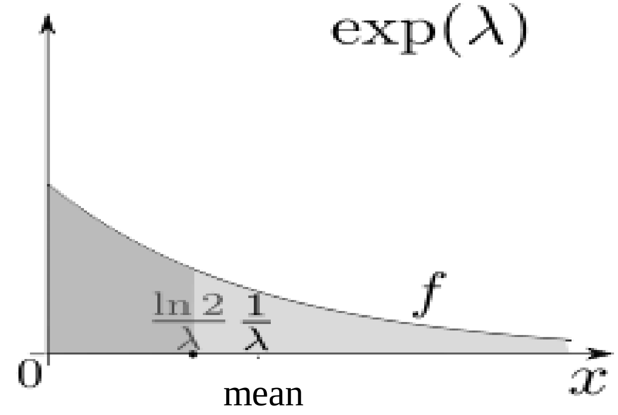


- $E[X] = (a+b)/2$, and $Var(X) = (b-a)^2/12$

Some Important Continuous Random Distributions

- Exponential

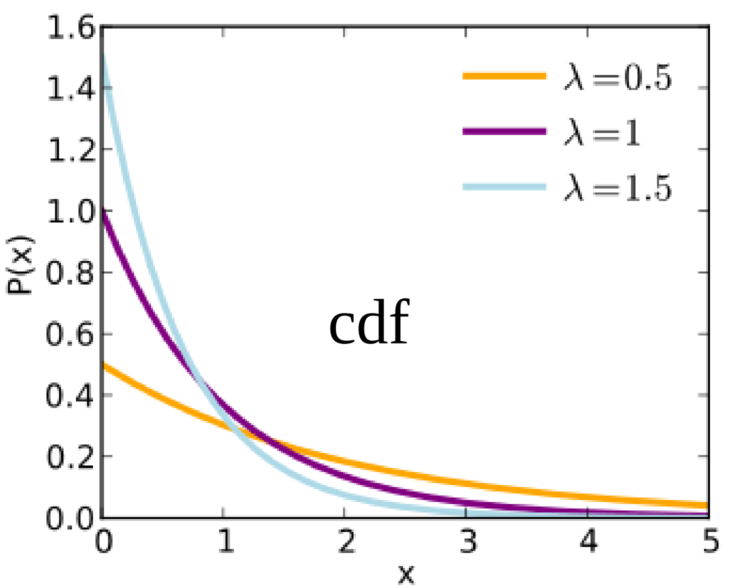
$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$



and the cumulative distribution function is

http://en.wikipedia.org/wiki/

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$



Mean $E[X] = 1/\lambda$, and $Var(X) = 1/\lambda^2$

Poisson Distributions

- A random variable X for k successive events follow Poisson distribution with parameter λ,

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,2,\ldots,\ \text{and}\ \lambda>0$$

- Thus, the value of variance $Var(X)$ is given by

$$E[X]=\sum_{k=0}^{\infty}kP(X=k)\ =\sum_{k=0}^{\infty}k\frac{\lambda^k}{k!}e^{-\lambda}$$

$$=\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

- Therefore, we can see that the expected value and variance of a random variable X which follows Poisson distribution have same value λ

Poisson Distributions

- A random variable X for k successive events follow Poisson distribution with parameter λ,

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,2,\ldots,\ \text{and}\ \lambda>0$$

- From Equation (2.13), the second moment $E[X^2]$ can be calculated by

$$E[X^2]=\sum_{k=0}^{\infty}k^2P(X=k)=\sum_{k=0}^{\infty}k^2\frac{\lambda^k}{k!}e^{-\lambda}$$

$$=\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\lambda^{k-1}}{(k-1)!}$$

$$=\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{(k+1)\lambda^{k}}{k!}$$

$$=\lambda e^{-\lambda}\left[\sum_{k=1}^{\infty}\frac{\lambda^{k}-1}{(k-1)!}+\sum_{k=0}^{\infty}\frac{\lambda^{k}}{k!}\right]$$

$$=\lambda^2+\lambda$$

Poisson Distributions

- A random variable *X* for k successive events follow Poisson distribution with parameter λ,

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k=0, 1, 2,..., \text{and } \lambda > 0$$

- Thus, the value of variance *Var*(*X*) is given by

$$Var(X) = E\left[(X - E[X])^2\right]$$
$$= E\left[X^2\right] - E[X])^2 = \lambda$$

- Therefore, we can see that the expected value and variance of a random variable *X* which follows Poisson distribution have the same value λ.

Multiple Random Variables

- There are cases where the result of one experiment determines the values of several random variables
- The joint probabilities of these variables are:
  - Discrete variables:

    $p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$
  - Continuous variables:

    CDF: $F_{x1x2\ldots xn}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$

    pdf: $f_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n) = \dfrac{\partial^n F_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$

Independence and Conditional Probability

- Independence: The random variables are said to be independent of each other when the occurrence of one does not affect the other. The pmf for discrete random variables in such a case is given by:

  $p(x_1,x_2,\ldots x_n)=P(X_1=x_1)P(X_2=x_2)\ldots P(X_3=x_3)$ and for continuous random variables as:

  $F_{X1,X2,\ldots Xn} = F_{X1}(x_1)F_{X2}(x_2)\ldots F_{Xn}(x_n)$

- Conditional probability: is the probability that $X_1= x_1$ given that $X_2= x_2$. Then for discrete random variables the probability becomes:

$$P(X_1 = x_1 \mid X_2 = x_2,\ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2,\ldots, X_n = x_n)}{P(X_2 = x_2,\ldots, X_n = x_n)}$$

and for continuous random variables it is:

$$P(X_1 \le x_1 \mid X_2 \le x_2,\ldots, X_n \le x_n) = \frac{P(X_1 \le x_1, X_2 \le x_2,\ldots, X_n \le x_n)}{P(X_2 \le x_2,\ldots X_n \le x_n)}$$

Bayes Theorem

- A theorem concerning conditional probabilities of the form $P(X|Y)$ (read: the probability of X, given Y) is

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

where $P(X)$ and $P(Y)$ are the unconditional probabilities of X and Y, respectively

Important Properties of Random Variables

- Sum property of the expected value
  - Expected value of the sum of random variables:

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$

- Product property of the expected value
  - Expected value of product of stochastically independent random variables

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

Important Properties of Random Variables

- Sum property of the variance
  - Variance of the sum of random variables is

$$Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \operatorname{cov}[X_i, X_j]$$

where $cov[X_i,X_j]$ is the covariance of random variables X_i and X_j and

$$\begin{aligned}\operatorname{cov}[X_i, X_j] &= E[(X_i - E[X_i])(X_j - E[X_j])] \\ &= E[X_i X_j] - E[X_i]E[X_j]\end{aligned}$$

If random variables are independent of each other, i.e., $cov[X_i,X_j]=0$, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Important Properties of Random Variables

- **Distribution of sum -** For continuous random variables with joint pdf $f_{XY}(x, y)$ and if $Z = \Phi(X,Y)$, the distribution of Z may be written as

$$F_Z(z) = P(Z \le z) = \int_{\phi Z} f_{XY}(x, y)dxdy$$

where Φ_Z *is a subset of* Z.

- For a special case $Z= X+Y$

$$Fz(z) = \iint_{\phi Z} f_{XY}(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy$$

- If X and Y are independent variables, the $f_{XY}(x,y)= f_X(x)f_Y(y)$

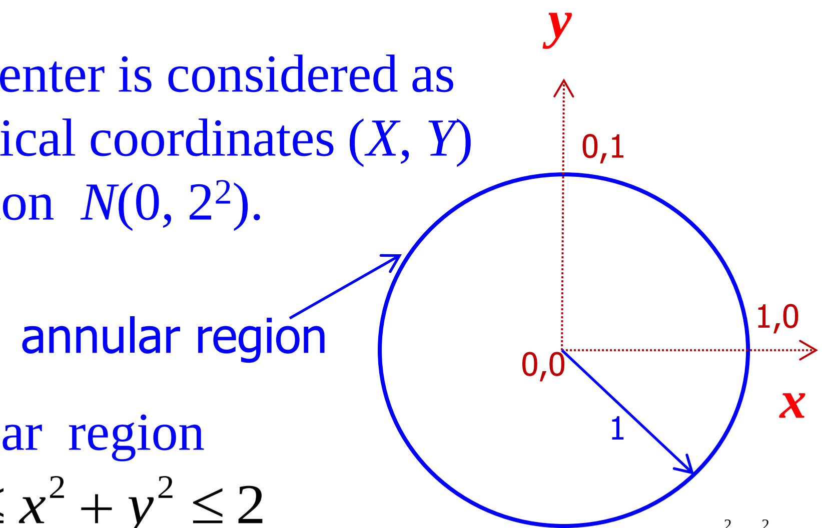
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

- If both X and Y are non negative random variables, then pdf is the convolution of the individual pdfs, $f_X(x)$ and $f_Y(y)$

$$f_Z(z) = \int_{0}^{z} f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

Random Variables

- **Example 2.3:** Shoot a target, and the target center is considered as the origin of coordinates. Horizontal and vertical coordinates (X, Y) are independent, and follow normal distribution $N(0, 2^2)$.



$$f_{xy} = \frac{1}{2\pi.4} e^{-\frac{x^2+y^2}{8}}$$

- Find: (1). The probability of hitting the annular region

$$D = \{(x, y)\ for\ 1 \le x^2 + y^2 \le 2$$

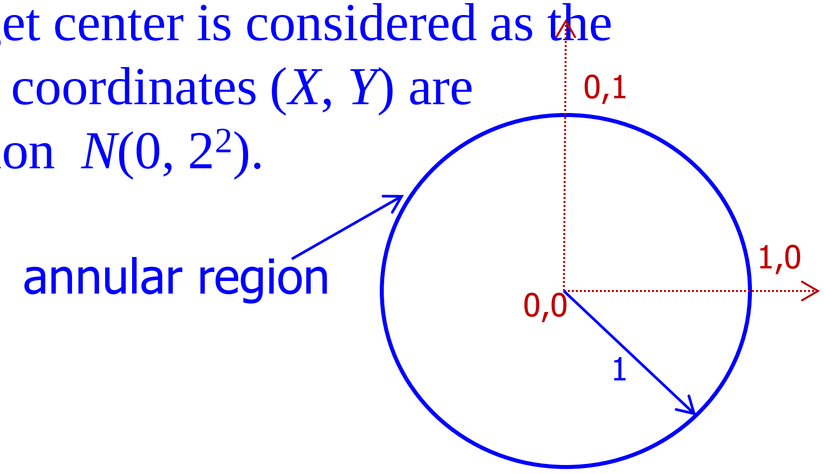
- Finding the probability of hitting the annular region means calculating the distribution of sum of the normal random N (0, 2^2). As Equation (2.42) shows that, the **distribution** of sum can be calculated as:

$$P[(X,Y) \in D] = \iint_D f_{xy}(x,y)dxdy = \iint_D \frac{1}{2\pi.4} e^{-\frac{x^2+y^2}{8}} dxdy$$

$$= \frac{1}{8\pi} \int_{\theta=0}^{2\pi} \int_{r=1}^{2} e^{-\frac{r^2}{8}} rdrd\theta = -\int_{r=1}^{2} e^{-\frac{r^2}{8}} d[-\frac{r^2}{8}] = e^{-\frac{1}{8}} - e^{-\frac{1}{2}}$$

Random Variables

- **Example 2.3:** Shoot a target, and the target center is considered as the coordinate origin. Horizontal and vertical coordinates (X, Y) are independent, and follow normal distribution $N(0, 2^2)$.



- Find: (2). The expected value $E[Z]$ of the distance $Z=\sqrt{X^2+Y^2}$ between the hitting point and the center point (0, 0).
- According to the concept of joint probability distribution, expected value $E[Z]$ can be calculated as:

$$E[Z]=E[\sqrt{X^2+Y^2}]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{X^2+Y^2}\,\frac{1}{8\pi}e^{-\frac{X^2+Y^2}{8}}dxdy$$

$$=\frac{1}{8\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}re^{-\frac{r^2}{8}}rdrd\theta=\frac{1}{4}\int_{r=0}^{\infty}r^2e^{-\frac{r^2}{8}}dr=\sqrt{2\pi}$$

Central Limit Theorem

The *Central Limit Theorem* states that whenever a random sample (X_1, X_2,.. X_n) of size n is taken from any distribution with expected value $E[X_i] = \mu$ and variance $Var(X_i) = \sigma^2$, where i =1,2,..,n, then their arithmetic mean is defined by

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Central Limit Theorem

- Mean of a sufficiently large number of independent random variables, each with finite mean and variance, will be approximately normally distributed
- The sample mean is approximated to a normal distribution with
  - $E[S_n] = \mu$, and
  - $Var(S_n) = \sigma^2 / n$
- The larger the value of the sample size n, the better the approximation to the normal
- This is very useful when inference between signals needs to be considered

Poisson Arrival Model

- Events occur continuously and independently of one another
- A Poisson process is a sequence of events "randomly spaced in time"
- For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
- The rate λ of a Poisson process is the average number of events per unit time (over a long time)

Properties of a Poisson Process

- Properties of a Poisson process
  - For a time interval $[0, t]$, the probability of n arrivals in t units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

  - For two disjoint (non overlapping) intervals (t_1, t_2) and (t_3, t_4), (i.e. , $t_1 < t_2 < t_3 < t_4$), the number of arrivals in (t_1, t_2) is independent of arrivals in (t_3, t_4)

Interarrival Times of Poisson Process

- Interarrival times of a Poisson process
  - We pick an arbitrary starting point t_0 in time . Let T_1 be the time until the next arrival. We have

    $$P(T_1 > t) = P_0(t) = e^{-\lambda t}$$

  - Thus the cumulative distribution function of T_1 is given by

    $$F_{T1}(t) = P(T_1 \leq t) = 1 - e^{-\lambda t}$$

  - The pdf of T_1 is given by

    $$f_{T1}(t) = \lambda e^{-\lambda t}$$

  Therefore, T_1 has an exponential distribution with mean rate λ

Exponential Distribution

- Similarly T_2 is the time between first and second arrivals, we define T_3 as the time between the second and third arrivals, T_4 as the time between the third and fourth arrivals and so on
- The random variables T_1, T_2, T_3… are called the interarrival times of the Poisson process
- T_1, T_2, T_3,… are independent of each other and each has the same exponential distribution with mean arrival rate λ

Memoryless and Merging Properties

- Memoryless property
  - A random variable X has the property that "the future is independent of the past" i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen
- Merging property
  - If we merge n Poisson processes with distributions for the inter arrival times

$$1- e^{-\lambda i t} \quad \text{for } i = 1, 2, \ldots, n$$

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution $1- e^{-\lambda t}$ with mean

$$\lambda = \lambda_1 + \lambda_2 + .. + \lambda_n$$

Basic Queuing Systems

- What is queuing theory?
  - Queuing theory is the study of queues (sometimes called waiting lines)
  - Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems
- Basic queuing theory

  Queuing theory is divided into 3 main sections:
  - Traffic flow
  - Scheduling
  - Facility design and employee allocation

Kendall's Notation

- D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types. Accordingly the systems were described by the notation A/B/C/D/E where:

		For cell phones
A	Distribution of inter arrival times of customers	Time between 2 successive calls
B	Distribution of service times	Time for a service call
C	Number of servers	# of Channels
D	Maximum number of customers in the system	# of registered users
E	Calling population size	Total population

Kendall's notation

A and B can take any of the following distributions types:

M	Exponential distribution (Markovian)
D	Degenerate (or deterministic) distribution
E_k	Erlang distribution (k = shape parameter)
H_k	Hyper exponential with parameter k

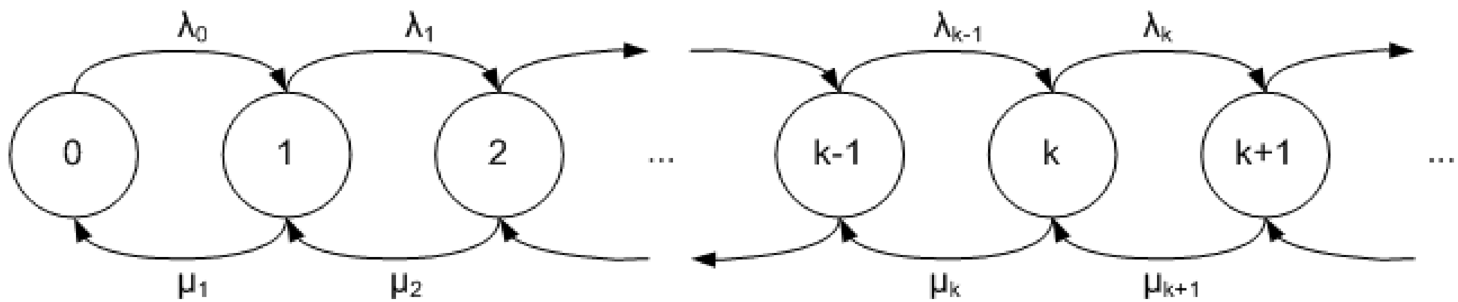
Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
  - λ – the mean steady state consumer arrival
  - N – the average no. of customers in the system
  - T – the mean time spent by each customer in the system

  which gives

$$N = \lambda T$$

Markov Process



- A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process
- The knowledge of the current state and the transition probabilities from this state allows us to predict the next state

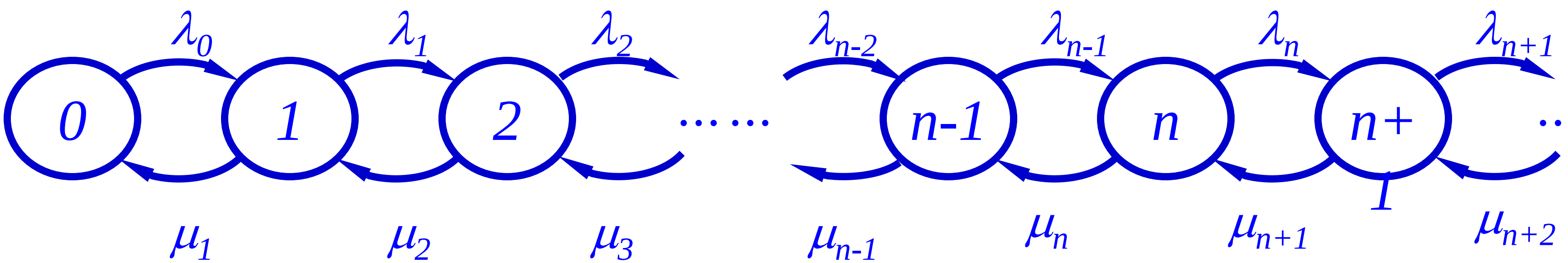
Birth-Death Process

- Special type of Markov process
- Often used to model a population (or, number of jobs in a queue)
- If, at some time, the population has *n* entities (*n* jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to n+1
- On the other hand, a death (a job removed from the queue for service) would cause the state to change to n-1
- Any state transitions can be made only to one of the two neighboring states

State Transition Diagram

State: # living people

State: # people being served



P(0) P(1) P(2) P(n-1) P(n) P(n+1)

Probability in a state

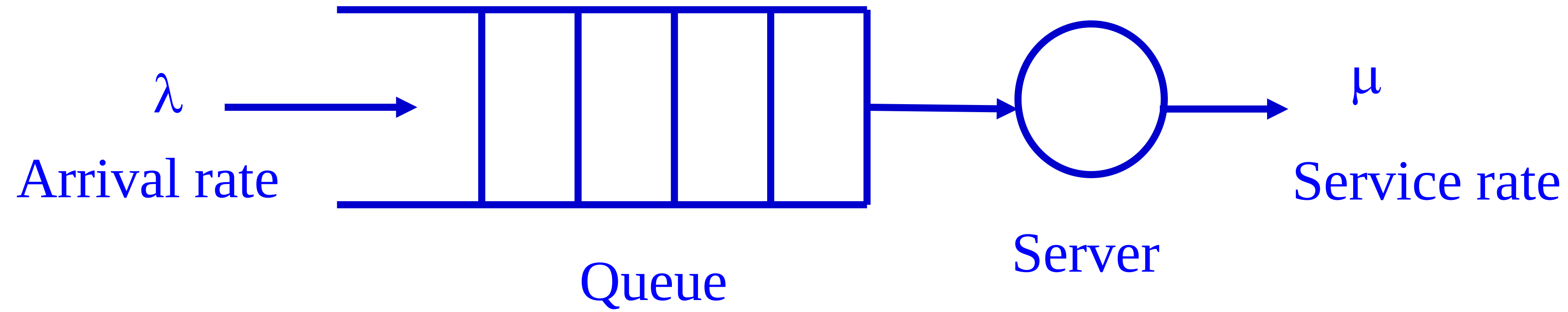
The state transition diagram of the continuous birth-death process

State transition is equally applicable to number of cell phone users

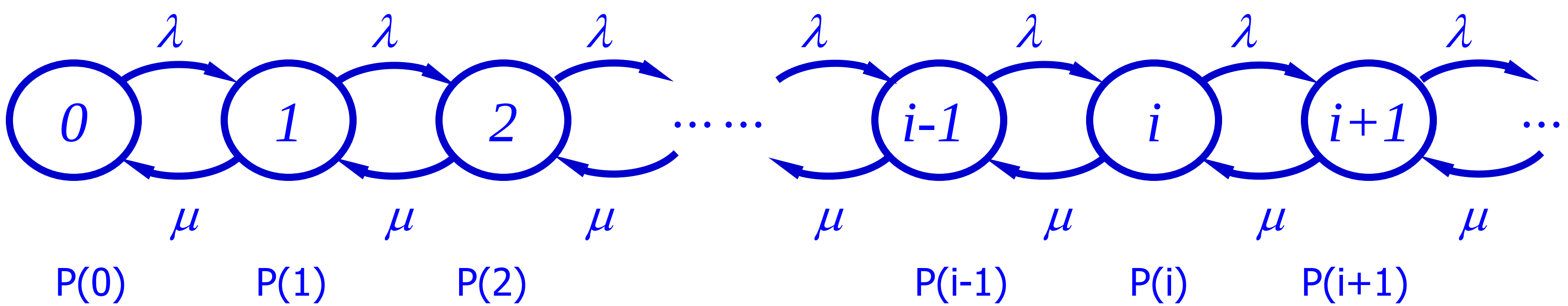
M/M/1/∞ or M/M/1 Queuing System

- Distribution of inter arrival times of customers: M
- Distribution of service times: M
- Number of servers: 1
- Maximum number of customers in the system: ∞
- When a customer arrives in this system it will be served if the server is free, otherwise the customer is queued
- In this system, customers arrive according to a Poisson distribution and compete for the service in a FIFO (first in first out) manner
- Service times are independent identically distributed (IID) random variables, the common distribution being exponential

Queuing Model and State Transition Diagram



The M/M/1/∞ queuing model

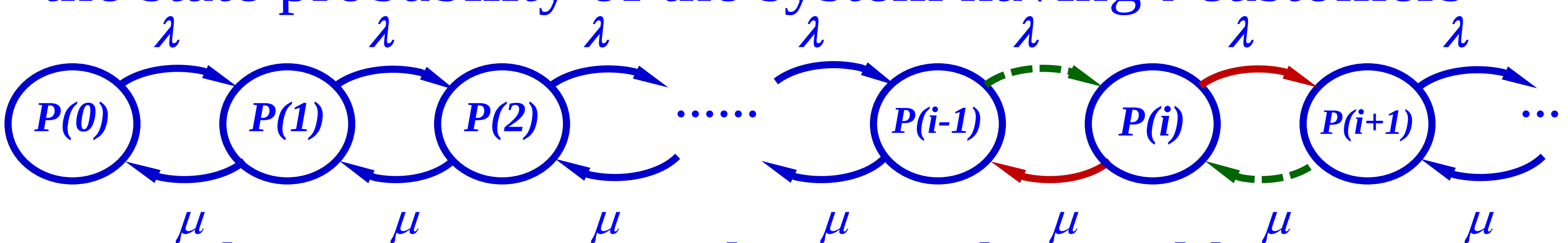


Probability in a state

The state transition diagram of the M/M/1/∞ queuing system

Equilibrium State Equations

- If mean arrival rate is λ and mean service rate is μ, $i = 0$, 1, 2 be the number of customers in the system and $P(i)$ be the state probability of the system having i customers



- From the state transition diagram, the equilibrium state equations are given by:

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$\lambda P(i-1) + \mu P(i+1) = \lambda P(i) + \mu P(i), \qquad i \geq 1$$

$$(\lambda + \mu) P(i) = \lambda P(i-1) + \mu P(i+1), \qquad i \geq 1$$

$$P(i+1) = \left(\frac{\lambda}{\mu} + 1\right) P(i) - \frac{\lambda}{\mu} P(i-1), \qquad i \geq 1$$

Traffic Intensity

$$P(1) = \frac{\lambda}{\mu} P(0)$$

$$P(2) = (\frac{\lambda}{\mu} + 1)P(1) - \frac{\lambda}{\mu} P(0) \quad = (\frac{\lambda}{\mu})^2 P(0)$$

$$P(3) = (\frac{\lambda}{\mu} + 1)P(2) - \frac{\lambda}{\mu} P(1) \quad = (\frac{\lambda}{\mu})^3 P(0)$$

Recursively substituting, we can get:

$$P(i) = (\tfrac{\lambda}{\mu})^i P(0), \quad i \geq 1$$

- We know that the *P*(*0*) is the probability of server being free. Since *P*(*0*) > 0, the necessary condition for a system being in steady state is,

$$\rho = \frac{\lambda}{\mu} < 1$$ Called traffic intensity

  This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time

Traffic Intensity

As summation of all states =1, we have the following: $\sum_{i=0}^{\infty} P(i) = 1$

$$As \quad \sum_{i=0}^{\infty} \rho^i P(0) = P(0)/(1-\rho) = \tfrac{P(0)}{1-\rho} = 1$$

$$Gives \quad P(0) = 1-\rho$$

$$P(i) = (\tfrac{\lambda}{\mu})^i P(0) = \rho^i (1-\rho), \quad i \geq 1$$

- The average number of customers in the system is = No. of users x probability in that state

$$L_s = \sum_{i=0}^{\infty} iP(i) = \rho(1-\rho)\sum_{i=0}^{\infty} i\rho^{i-1}$$

$$= \rho(1-\rho)(\frac{\rho}{1-\rho})^i = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

Traffic Intensity

- The average dwell time of customers is

$$W_S = \frac{L_S}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$$

The average queue length is

$$L_q = \sum_{i=1}^{\infty}(i-1)P(i) = \frac{\rho^2}{(1-\rho)} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

The average wait time is

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

Queuing System Metrics

Example 2.4: A repair shop has only one mechanic; r customers arrive at the shop following a Poisson distribution with a mean arrival rate of 3 per minute. The time it takes to repair for each customer follows a negative exponential distribution and the mean repair time is 10 seconds.

1. Find the probability of the server being free.

It can be seen that this is a M/M/1/ queuing system with λ = 3/m, and the service rate μ = 6/m. Therefore, the offered load $\rho = \lambda / \mu = 0.5$

Assume $P(0)$ is the probability of the server being free. From Equation(2.62), we have:

$P(0) = 1- \rho$

$=1-3/6=0.5$

Queuing System Metrics

Example 2.4: A repair shop has only one mechanic; r customers arrive at the shop following a Poisson distribution with a mean arrival rate of 3 per minute. The time it takes to repair for each customer follows a negative exponential distribution and the mean repair time is 10 seconds.

2. Find M/M/1/ queuing system with $\lambda = 3/m$, and the service rate $\mu = 6/m$. Therefore, the offered load $\rho = \lambda / \mu = 0.5$.

Assume $P(i)$ is the steady state probability of the system having i customers. From Equation(2.63), we have

$P(4) = (1-\rho)\rho^4 = (1-1/2)(1/2)^4 = 1/32 = 0.03125$

Queuing System Metrics

Example 2.4: A repair shop has only one mechanic; r customers arrive at the shop following a Poisson distribution with a mean arrival rate of 3 per minute. The time it takes to repair for each customer follows a negative exponential distribution and the mean repair time is 10 seconds.

3. Find the probability that there is at least 1 customer in the repair shop.

$$P(i \geq 1) = 1 - P(0) = 1 - \rho = 1 - \frac{1}{2} = 0.5$$

4. Find the average number of customers in the repair shop.

According to the probabilities $P(i)$, the average number of customers in the system is:

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{3}{6-3} = 1$$

Queuing System Metrics

Example 2.4: A repair shop has only one mechanic; r customers arrive at the shop following a Poisson distribution with a mean arrival rate of 3 per minute. The time it takes to repair for each customer follows a negative exponential distribution and the mean repair time is 10 minutes.

5. Find the average number of customers who are waiting for service.

It can be seen that the system is M/M/1/ queuing system with λ = 3/m, and the service rate μ = 6/m. Therefore, the offered load $\rho = \lambda/\mu$ =0.5.

The average queue length is

$$L_S = \frac{\rho\lambda}{\mu-\lambda} = \frac{\frac{1}{2} X 3}{6-3} = 0.5$$

6. Find the average dwell time of a customer in the shop.

Using Little's law, the average dwell time of a customer in the repair shop is

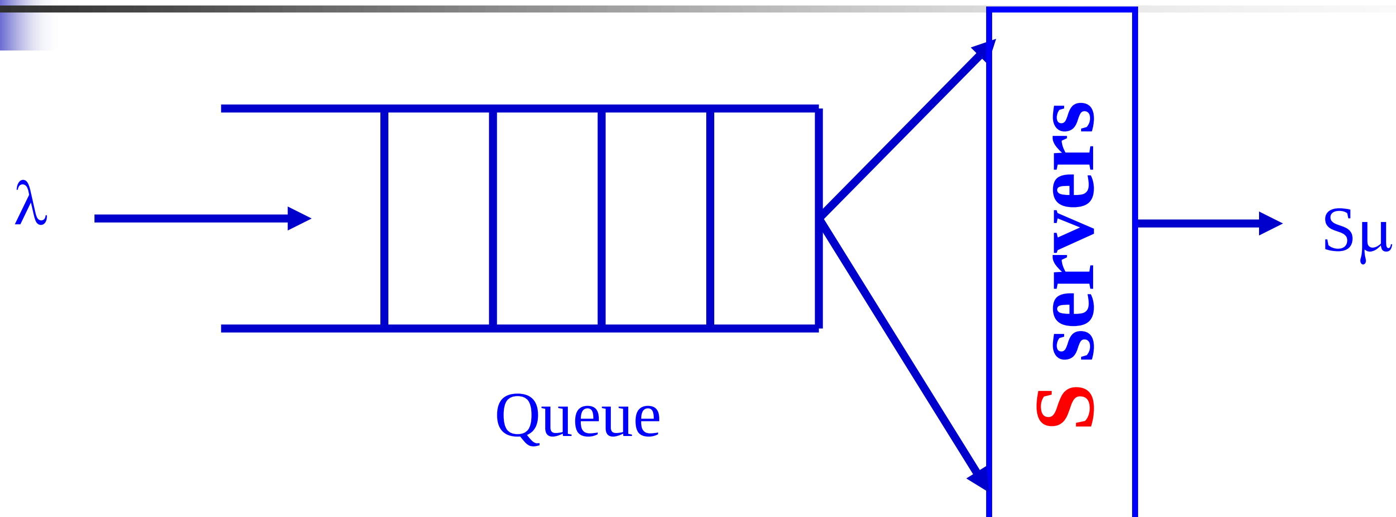
$$W_S = \frac{1}{\mu-\lambda} = \frac{1}{6-3} = \frac{1}{3} hrs$$

7. Find the average waiting time of a customer.

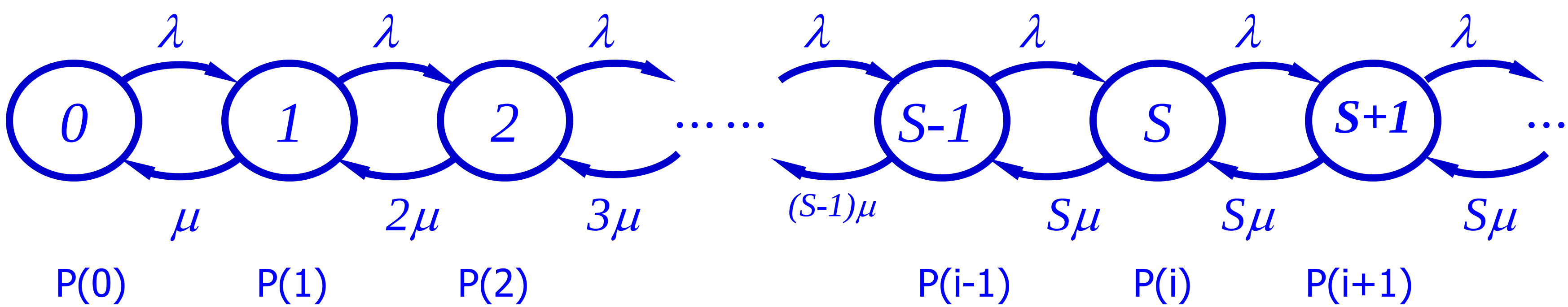
The average queuing waiting time of customers is

$$W_q = \rho W_S = \frac{1}{6} hrs$$

Queuing Model and State Transition Diagram with S server



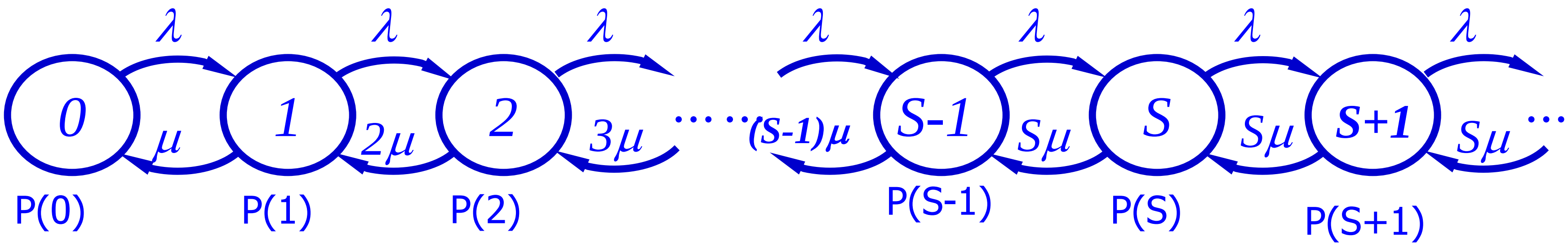
The M/M/S/∞ queuing model



Probability in a state

The state transition diagram of the M/M/S/∞ queuing system

Equilibrium State Equations



Probability in a state

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$(\lambda + i\mu)P(i) = \lambda P(i-1) + (i+1)\mu P(i+1), \qquad 1 \le i \le S$$

$$(\lambda + S\mu)P(i) = \lambda P(i-1) + S\mu P(i+1), \qquad S \le i$$

$$P(i) = \frac{\alpha^i}{i!} P(0) \qquad i < S \qquad\qquad \alpha = \frac{\lambda}{\mu}$$

$$P(i) = \frac{\alpha^S}{S!} \left(\frac{\alpha}{S}\right)^{i-S} P(0) \qquad S \le i$$

Queuing System Metrics

As summation of all states =1, we have the following:

$$\sum_{i=0}^{S-1} P(i) = \left[\sum_{i=0}^{S-1} \frac{\alpha}{i!}^{i} + \frac{\alpha}{S!}^{S} \sum_{i=0}^{\infty} (\tfrac{\alpha}{S})^{i} \right] P(0) = 1$$

We have:

$$P(0) = \left[\sum_{i=0}^{S-1} \frac{\alpha}{i!}^{i} + \frac{\alpha}{S!}^{S} \sum_{i=0}^{\infty} (\tfrac{\alpha}{S})^{i} \right]^{-1}$$

$$If \ \ \alpha < S, we\ have \quad \sum_{i=0}^{\infty} (\tfrac{\alpha}{S})^{i} = \frac{S}{S-\alpha}$$

$$Thus \quad P(0) = \left[\sum_{i=0}^{S-1} \frac{\alpha}{i!}^{i} + \frac{\alpha}{S!}^{S} \frac{S}{S-\alpha} \right] \quad = \left[\sum_{i=0}^{S-1} \frac{\alpha}{i!}^{i} + \frac{\alpha}{S!}^{S} \frac{1}{1-\alpha} \right]$$

Here $\rho=\alpha/S$ is called utilization factor and need to be larger than 1 for queue to be stable

Queuing System Metrics

- The average number of customers in the system is

$$L_s = \sum_{i=0}^{\infty} iP(i) = \alpha + \frac{\rho\alpha^s P(0)}{S!(1-\rho)^2}$$

- The average dwell time of customers is

$$W_s = \frac{L_S}{\lambda} = \frac{1}{\mu} + \frac{\alpha^s P(0)}{S\mu * S!(1-\rho)^2}$$

The average queue length is

$$L_q = \sum_{i=S}^{\infty} (i-S)P(i) = \frac{\alpha^{s+1} P(0)}{(S-1)(S-\alpha)^2}$$

The average wait time is

$$W_q = \frac{L_q}{\lambda} = \frac{\alpha^s P(0)}{S\mu * S!(1-\rho)^2}$$

Queuing System Metrics

$$When\ t \to \infty,\ we\ have$$

- The average queuing length is

$$L_q = \sum_{i=1}^{\infty}(i-S)P(i) = \frac{\alpha^{S+1}P(0)}{(S-1)(S-\alpha)^2}$$

- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\alpha^S P(0)}{S\mu * S!(1-\rho)^2}$$

M/G/1/∞ Queuing Model

- We consider a single server queuing system whose arrival process is Poisson with mean arrival rate λ
- Service times are independent and identically distributed with distribution function F_B and pdf f_b
- Jobs are scheduled for service as FIFO

Basic Queuing Model

- Let $N(t)$ denote the number of jobs in the system (those in queue plus in service) at time t.
- Let t_n (n= 1, 2,..) be the time of departure of the n^{th} job and X_n be the number of jobs in the system at time t_n, so that

$$X_n = N(t_n), \quad \text{for} \;\; n = 1, 2, ..$$

- The stochastic process can be modeled as a discrete Markov chain known as imbedded Markov chain, which helps convert a non-Markovian problem into a Markovian one.

Queuing System Metrics

- The average number of jobs in the system, in the steady state is

$$E[N] = \rho + \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$

- The average dwell time of customers in the system is

$$Ws = \frac{E[N]}{\lambda} = \frac{1}{\mu} + \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average waiting time of customers in the queue is

$$E[N] = \lambda W_q + \rho$$

- Average waiting time of customers in the queue is

$$W_q = \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average queue length is

$$L_q = \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$