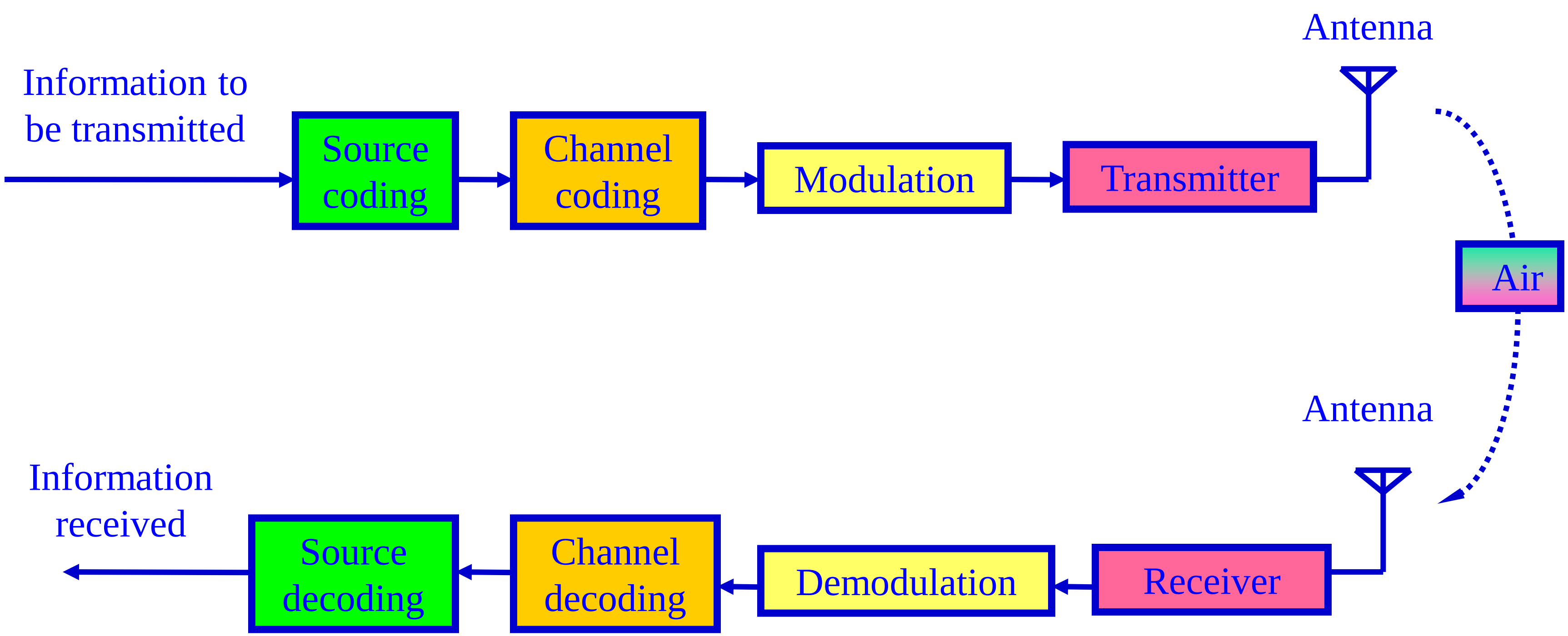
Chapter 4

Channel Coding and Error Control

Outline

- Introduction
- Block Codes
- Cyclic Codes
- CRC (Cyclic Redundancy Check)
- Convolutional Codes
- Interleaving
- Information Capacity Theorem
- Turbo Codes
- ARQ (Automatic Repeat Request)
  - Stop-and-wait ARQ
  - Go-back-N ARQ
  - Selective-repeat ARQ

Introduction



Forward Error Correction (FEC)

- The key idea of FEC is to transmit enough redundant data to allow receiver to recover from errors all by itself. No sender retransmission required
- A simple redundancy is to attach a parity bit

  1010110 0
- The major categories of FEC codes are
  - Block codes
  - Cyclic codes
  - Reed-Solomon codes (Not covered here)
  - Convolutional codes, and
  - Turbo codes, etc.

Linear Block Codes

- Information is divided into blocks of length k
- r parity bits or check bits are added to each block (total length $n = k + r$)
- Code rate $R = k/n$
- Decoder looks for codeword closest to received vector (code vector + error vector)
- Tradeoffs between
  - Efficiency
  - Reliability
  - Encoding/Decoding complexity
- Modulo 2 Addition

$$\begin{array}{cccc} 0 & 0 & 1 & 1 \\ \underline{0} & \underline{1} & \underline{0} & \underline{1} \\ 0 & 1 & 1 & \cancel{1}0 \end{array} \qquad \oplus$$

throw this away

Linear Block Codes: Example

Example: Find linear block code encoder G if code generator polynomial $g(x)=1+x+x^3$ for a (7, 4) code; n = Total number of bits = 7, k = Number of information bits = 4, r = Number of parity bits = n - k = 3

Coefficients of $x^0x^1x^2$

$$p_1 = \mathrm{Re}\left[\frac{x^3}{x^3+x+1}\right] = 1 + x \rightarrow [110]$$

$$\begin{array}{r}
1 \\
x^3+x+1 \;\big)\overline{\; x^3 \;} \\
x^3+x+1 \\
\hline
x+1
\end{array}$$

Coefficients of $x^0x^1x^2$ 110

$$p_4 = \mathrm{Re}\left[\frac{x^6}{x^3+x+1}\right] = 1 + x^2 \rightarrow [101]$$

$$\begin{array}{r}
x^3 + x + 1 \\
x^3+x+1 \;\big)\overline{\; x^6 \;} \\
x^6+x^4+x^3 \\
\hline
x^4+x^3 \\
x^4+x^2+x \\
\hline
x^3+x^2+x \\
x^3+x+1 \\
\hline
x^2+1
\end{array}$$

Coefficients of $x^0x^1x^2$ 101

Linear Block Codes: Example

Example: Find linear block code encoder G if code generator polynomial $g(x)=1+x+x^3$ for a (7, 4) code; n = Total number of bits = 7, k = Number of information bits = 4, r = Number of parity bits = n - k = 3

Coefficients of $x^0x^1x^2$

$$p_1 = \mathrm{Re}\left[\frac{x^3}{x^3+x+1}\right] = 1+x \rightarrow [110]$$

$$p_2 = \mathrm{Re}\left[\frac{x^4}{x^3+x+1}\right] = x+x^2 \rightarrow [011]$$

$$p_3 = \mathrm{Re}\left[\frac{x^5}{x^3+x+1}\right] = 1+x+x^2 \rightarrow [111]$$

$$p_4 = \mathrm{Re}\left[\frac{x^6}{x^3+x+1}\right] = 1+x^2 \rightarrow [101]$$

$$G = \begin{bmatrix} 1000 | 110 \\ 0100 | 011 \\ 0010 | 111 \\ 0001 | 101 \end{bmatrix} = [I \mid P]$$

I is the identity matrix

P is the parity matrix

Linear Block Codes: Example

The Generator Polynomial can be used to determine the Generator Matrix **G** that allows determination of parity bits for a given data bits of **m** by multiplying as follows:

$$m.G = \underset{\text{Data}}{[1011]}\begin{bmatrix} 1000110 \\ 0100011 \\ 0010111 \\ 0001101 \end{bmatrix} = [\underset{\text{Data}}{1011} \mid \underset{\text{Parity}}{100}]$$

Can be done for other combination of data bits, giving the code word **c**

Other combinations of **m** can be used to determine all other possible code words (generator polynomial $g(x)=1+x+x^3$)

Linear Block Codes

k- data and $r = n-k$ redundant bits

$$\begin{cases} c_1 = m_1 \\ c_2 = m_2 \\ \ldots \\ c_k = m_k \end{cases}$$

$$\begin{cases} c_{k+1} = m_1 p_{1(k+1)} \oplus m_2 p_{2(k+1)} \oplus \ldots \oplus m_k p_{k(k+1)} \\ \ldots \\ c_n = m_1 p_{1n} \oplus m_2 p_{2n} \oplus \ldots \oplus m_k p_{kn} \end{cases}$$

Linear Block Codes

- The uncoded k data bits be represented by the **m** vector:

  $$\mathbf{m} = (m_1, m_2, \ldots, m_k)$$

  The corresponding codeword be represented by the n-bit **c** vector:

  $$\mathbf{c} = (c_1, c_2, \ldots c_k, c_{k+1}, \ldots, c_{n-1}, c_n)$$

- Each parity bit consists of weighted modulo 2 sum of the data bits represented by $\oplus$ symbol for Exclusive OR or modulo 2 addition

Linear Block Codes

- Linear Block Code

The block length C of the Linear Block Code is

$$\mathbf{C} = \mathbf{mG}$$

where m is the information code word block length, **G** is the generator matrix from a given generator polynomial g(x):

$$\mathbf{G} = [\mathrm{I}_k \mid \mathrm{P}]_{k \times n}$$

where P_i = Remainder of $[x^{n-k+i-1}/g(x)]$ i.e.,

$$c_p(x) = rem\left[\frac{m(x)x^{n-k}}{g(x)}\right]$$

for $i=1, 2, .., k$, and **I** is unit or identity matrix

- At the receiving end, with the error polynomial e(r), the syndrome s(x) becomes:

$$s(x) = rem\left[\frac{c(x)+e(x)}{g(x)}\right]$$

If there is no error, s(x)=0

Linear Block Codes

Example: Find linear block code encoder **G** if code generator polynomial $g(x)$ with k data bits, and r parity bits = n - k

$$\because \quad G = [I \mid P] = \begin{bmatrix} 1\,0 \cdots\cdots 0\,P^1 \\ 0\,1 \cdots\cdots 0\,P^2 \\ \cdots\cdots\cdots\cdots\cdots \\ 0\,0 \cdots\cdots 1\,P^k \end{bmatrix}$$

where

$$P^i = \mathrm{Re}\,mainder\ of\ \left[\frac{x^{n-k+i-1}}{g(x)}\right],\ for \quad i = 1, 2, \cdots, k$$

Linear Block Codes

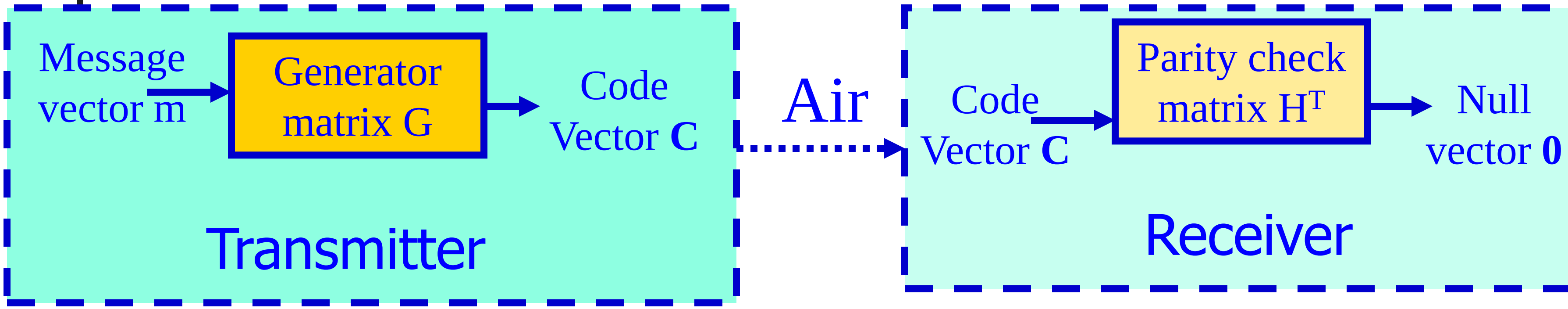
Example: The parity matrix **P** (*k* by *n*-*k* matrix) is given by:

$$P = \begin{bmatrix} p_{11} p_{12} \cdots p_{1(n-k)} \\ p_{21} p_{22} \cdots p_{2(n-k)} \\ \cdots\cdots\cdots\cdots\cdots \\ p_{k1} p_{k1} \cdots p_{k(n-k)} \end{bmatrix} = \begin{bmatrix} P^1 \\ P^2 \\ \cdots \\ P^k \end{bmatrix}$$

where

$$P^i = \mathrm{Re}\,mainder\ of \left[\frac{x^{n-k+i-1}}{g(x)}\right],\ for \quad i = 1, 2, \cdots, k$$

Block Codes: Linear Block Codes



Operations of the generator matrix and the parity check matrix

- Consider a (7, 4) linear block code, given by **G** as

$$G = \begin{bmatrix} 1000 \mid 111 \\ 0100 \mid 110 \\ 0010 \mid 101 \\ 0001 \mid 011 \end{bmatrix} = [I_k P]_{kxn} \qquad H^T = \begin{bmatrix} P \\ I_{n-k} \end{bmatrix} = \begin{bmatrix} 1110 \mid 100 \\ 1101 \mid 010 \\ 1011 \mid 001 \end{bmatrix}^T$$

For convenience, the code vector is expressed as

$\mathbf{c} = \left[\mathbf{m} \middle| \mathbf{c}_p \right]$ Where $\mathbf{c}_p = \mathbf{m}\mathbf{P}$ is an *(n-k)*-bit parity check vector

Block Codes: Linear Block Codes

Define matrix $\mathbf{H}^{\mathrm{T}}$ as $\mathrm{H}^{\mathrm{T}} = \begin{bmatrix} \mathrm{P} \\ \mathrm{I}_{n-k} \end{bmatrix}$

Received code vector $\mathrm{x} = \mathrm{c} \oplus \mathrm{e}$, here **e** is an error vector, the matrix $\mathbf{H}^{\mathrm{T}}$ has the property

$$\mathrm{cH}^{\mathrm{T}} = \begin{bmatrix} \mathrm{m} | \mathrm{c}_p \end{bmatrix} \begin{bmatrix} \mathrm{P} \\ \mathrm{I}_{n-k} \end{bmatrix}$$

$$= \mathrm{mP} \oplus \mathrm{c}_p = \mathrm{c}_p \oplus \mathrm{c}_p = 0$$

The transpose of matrix $\mathbf{H}^{\mathrm{T}}$ is

$$\mathbf{H}^{\mathrm{T}} = \begin{bmatrix} \mathbf{P}^{\mathrm{T}} \mathbf{I}_{n-k} \end{bmatrix}$$

Where $\mathbf{I}_{\text{n-k}}$ is a *n-k* by *n-k* unit matrix and $\mathbf{P}^{\mathrm{T}}$ is the transpose of parity matrix **P**.

H is called parity check matrix.

Compute syndrome as $\mathrm{s} = \mathrm{xH}^{\mathrm{T}} = (\mathrm{c} \oplus \mathrm{e}) \times \mathrm{H}^{\mathrm{T}} = \mathrm{cH}^{\mathrm{T}} \oplus \mathrm{eH}^{\mathrm{T}} = \mathrm{eH}^{\mathrm{T}}$

Linear Block Codes

If **S** is **0** then message is correct else there are errors in it, from common known error patterns the correct message can be decoded.

- For the (7, 4) linear block code, given by **G** as

$$G = \begin{bmatrix} 1000 \mid 111 \\ 0100 \mid 110 \\ 0010 \mid 101 \\ 0001 \mid 011 \end{bmatrix} \qquad H = \begin{bmatrix} 1110 \mid 100 \\ 1101 \mid 010 \\ 1011 \mid 001 \end{bmatrix}$$

- For **m** = [1 0 1 1] and **c** = **mG** = [1 0 1 1| 0 0 1]
  If there is no error, the received vector **x** = **c**, and
  s=**cH**^{T}=[0, 0, 0]

Linear Block Codes

Let **c** suffer an error such that the received vector

x=**c** $\oplus$**e**

=[1 0 1 1 0 0 1] $\oplus$ [0 0 1 0 0 0 0]

=[1 0 0 1 0 0 1]

Then,

Syndrome **s** = **xH**^{T} $= \left[1001 \,|\, 001\right] \begin{bmatrix} 111 \\ 110 \\ 101 \\ 011 \\ -- \\ 100 \\ 010 \\ 001 \end{bmatrix} = [101] = (eH^T)$

5th position

This indicates error position, giving the corrected vector as [1011001]

Example 4.1 on Linear Block Code

- A generator matrix for a (6, 3) block code is given.
- Find its corresponding parity check matrix **H**?

The corresponding parity check matrix **H** can be calculated by

$$P = \begin{bmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 1\,1\,0 \end{bmatrix}$$

Therefore, the transpose of parity matrix **P** is

$$P^T = \begin{bmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 1\,1\,0 \end{bmatrix}$$

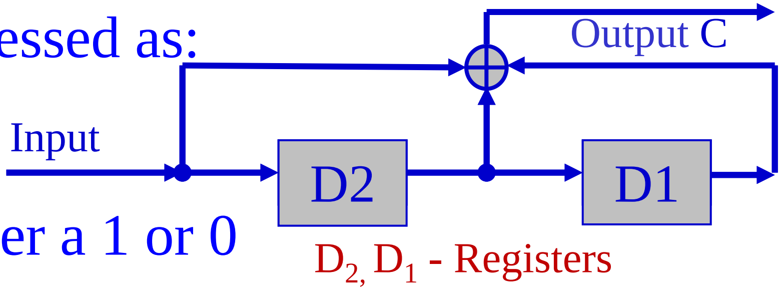
For **m** = [1 0 1], the code vector **c** can be calculated by Equation (4.4) as

$$H = \left[P^T I_{n-k}\right] = \left[\begin{array}{ccc:ccc} 1&0&1&1&0&0 \\ 0&1&1&0&1&0 \\ 1&1&0&0&0&1 \end{array}\right]$$

Cyclic Codes

It is a block code which uses a shift register to perform encoding and decoding the code word with n bits is expressed as:

$$c(x)=c_{n-1}x^{n-1}+c_{n-2}x^{n-2}\ldots\ldots+c_1$$



where each coefficient c_i ($i=n,n-1,..,2,1$) is either a 1 or 0

The code word can be expressed by the data polynomial $m(x)$ and the check polynomial $c_p(x)$ as

$$c(x) = m(x)\, x^{n-k} + c_p(x)$$

where $c_p(x)$ = remainder from dividing $m(x)\, x^{n-k}$ by generator $g(x)$

if the received signal is $c(x) + e(x)$ where $e(x)$ is the error

To check if received signal is error free, the remainder from dividing $c(x) + e(x)$ by $g(x)$ is obtained (syndrome). If this is 0 then the received signal is considered error free else error pattern is detected from known error syndromes

Cyclic Code: Example

Example: Find the code words $c(x)$ if $m(x) = x^2 + x + 1$ and $g(x) = x^3\ x^3 + x + 1$ for (7, 4) cyclic code

We have n = total number of bits = 7, k = number of information bits = 4, r = number of parity bits = $n - k = 3$

$$\because \quad c_p(x) = rem\left[\frac{m(x)x^{n-k}}{g(x)}\right]$$

$$= rem\left[\frac{x^5 + x^4 + x^3}{x^3 + x + 1}\right] = x$$

Then,

$$c(x) = m(x)x^{n-k} + c_p(x) = x^5 + x^4 + x^3 + x$$

Cyclic Redundancy Check (CRC)

- Cyclic Redundancy Code (CRC) is an error-checking code
- The transmitter appends an extra *n*-bit sequence to every frame called Frame Check Sequence (FCS). The FCS holds redundant information about the frame that helps the receivers detect errors in the frame
- CRC is based on polynomial manipulation using modulo arithmetic. Blocks of input bit as coefficient-sets for polynomials is called message polynomial. Polynomial with constant coefficients is called the generator polynomial

Cyclic Redundancy Check (CRC)

- Generator polynomial is divided into the message polynomial, giving quotient and remainder, the coefficients of the remainder form the bits of final CRC

- **Define:**

  Q – The original frame (*k* bits) to be transmitted

  F – The resulting frame check sequence (FCS) of *n*-*k* bits to be added to *Q* (usually *n* = 8, 16, 32)

  J – The cascading of *Q* and *F*

  P – The predefined CRC generating polynomial

The main idea in CRC algorithm is that the FCS is generated so that *J* should be exactly divisible by P

Cyclic Redundancy Check (CRC)

- The CRC creation process is defined as follows:
  - Get the block of raw message
  - Left shift the raw message by *n* bits and then divide it by *p*
  - Get the remainder *R* as FCS
  - Append the *R* to the raw message. The result *J* is the frame to be transmitted $J=Q.x^{n-k}+F$
  - *J* should be exactly divisible by *P*
- Dividing $Q.x^{n-k}$ by *P* gives $Q.x^{n-k}/P=Q+R/P$
  - Where *R* is the reminder
  - $J=Q.x^{n-k}+R$. This value of *J* should yield a zero reminder for *J/P*

Common CRC Codes

Code-parity check bits	Generator polynomial $g(x)$
CRC-12	$x^{12}+x^{11}+x^3+x^2+x+1$
CRC-16	$x^{16}+x^{15}+x^2+1$
CRC-CCITT (Comité Consultatif International Téléphonique et Télégraphique)	$x^{16}+x^{12}+x^5+1$
CRC-32	$x^{32}+x^{26}+x^{23}+x^{22}+x^{16}+x^{12}+x^{11}+x^{10}+x^8+x^7+x^5+x^4+x^2+x+1$

Code	Generator polynomial $g(x)$	Parity check bits
CRC-12	$x^{12}+x^{11}+x^3+x^2+x+1$	12
CRC-16	$x^{16}+x^{15}+x^2+1$	16
CRC-CCITT	$x^{16}+x^{15}+x^5+1$	16

Example 4.1 on Linear Block Code

- A generator matrix for a (6, 3) block code is given
- Find its code vector when message vector **m** = [1 0 1]?
- The code vector c for **m** = [1 0 1] can be obtained by:

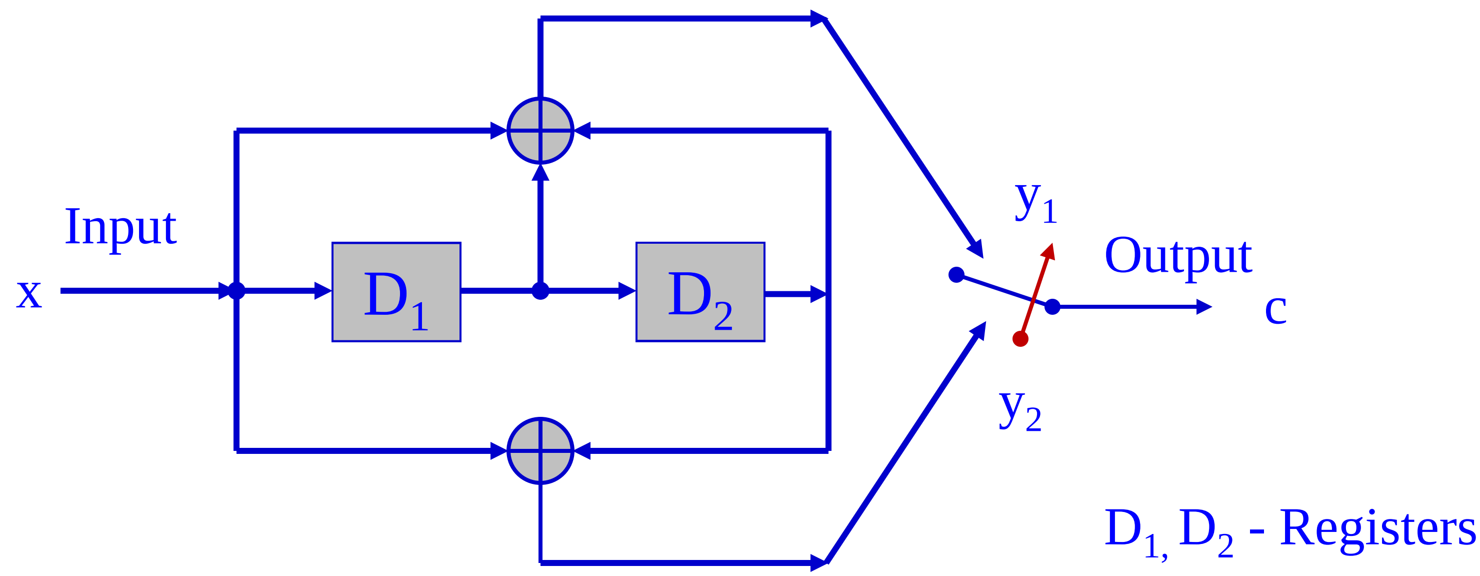
$$
\begin{aligned}
c &= mG \\
&= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100101 \\ 010011 \\ 001110 \end{bmatrix} \\
&= \begin{bmatrix} 101011 \end{bmatrix}
\end{aligned}
$$

Convolutional Codes

- Most widely used channel code
- Encoding of information stream rather than information blocks
- Decoding is mostly performed by the Viterbi Algorithm (not covered here)

- The constraint length K for a convolution code is defined as $K=M+1$ where M is the maximum number of stages in any shift register
- The code rate r is defined as $r = k/n$ where k is the number of parallel information bits and n is the the number of parallel output encoded bits at one time interval
- A convolution code encoder with $n=2$ and $k=1$ or code rate $r=1/2$ is shown next

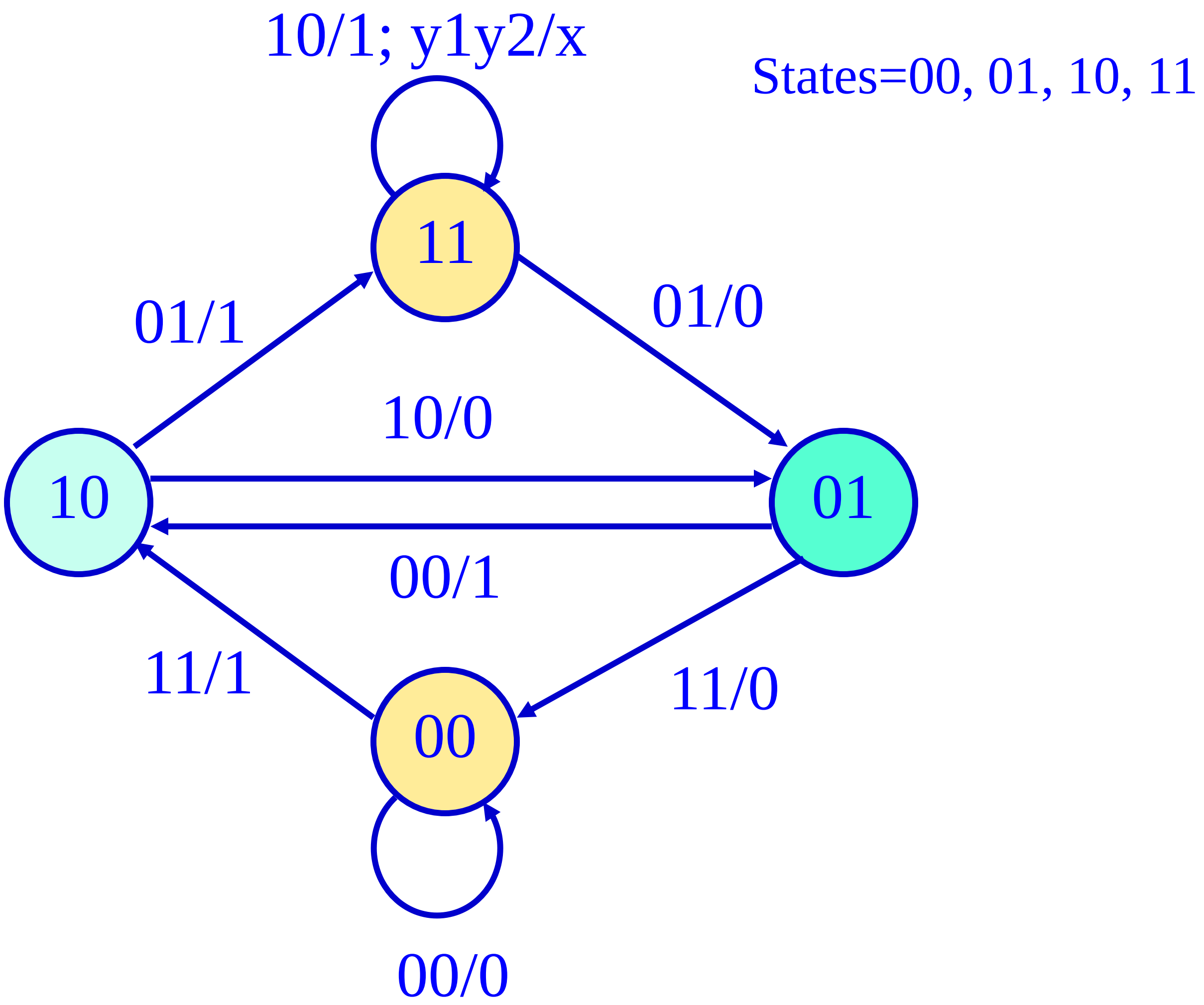
Convolutional Codes: (n=2, k=1, M=2) Encoder



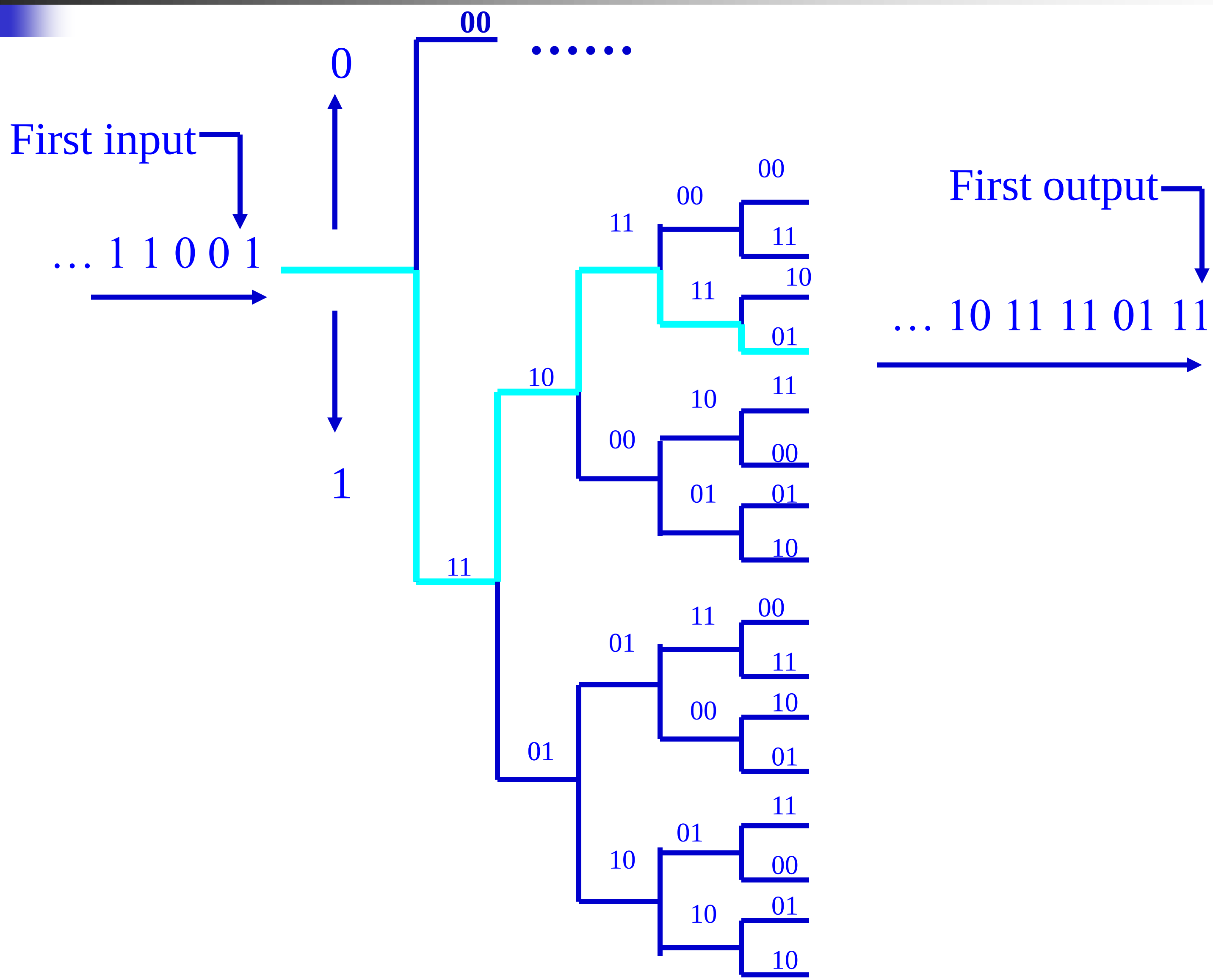
Initial value D1D2=00

Input x:	1	1	1	0	0	0	…
Output y_1y_2:	11	01	10	01	11	00	…
Input x:	1	0	1	0	0	0	…
Output y_1y_2:	11	10	00	10	11	00	…

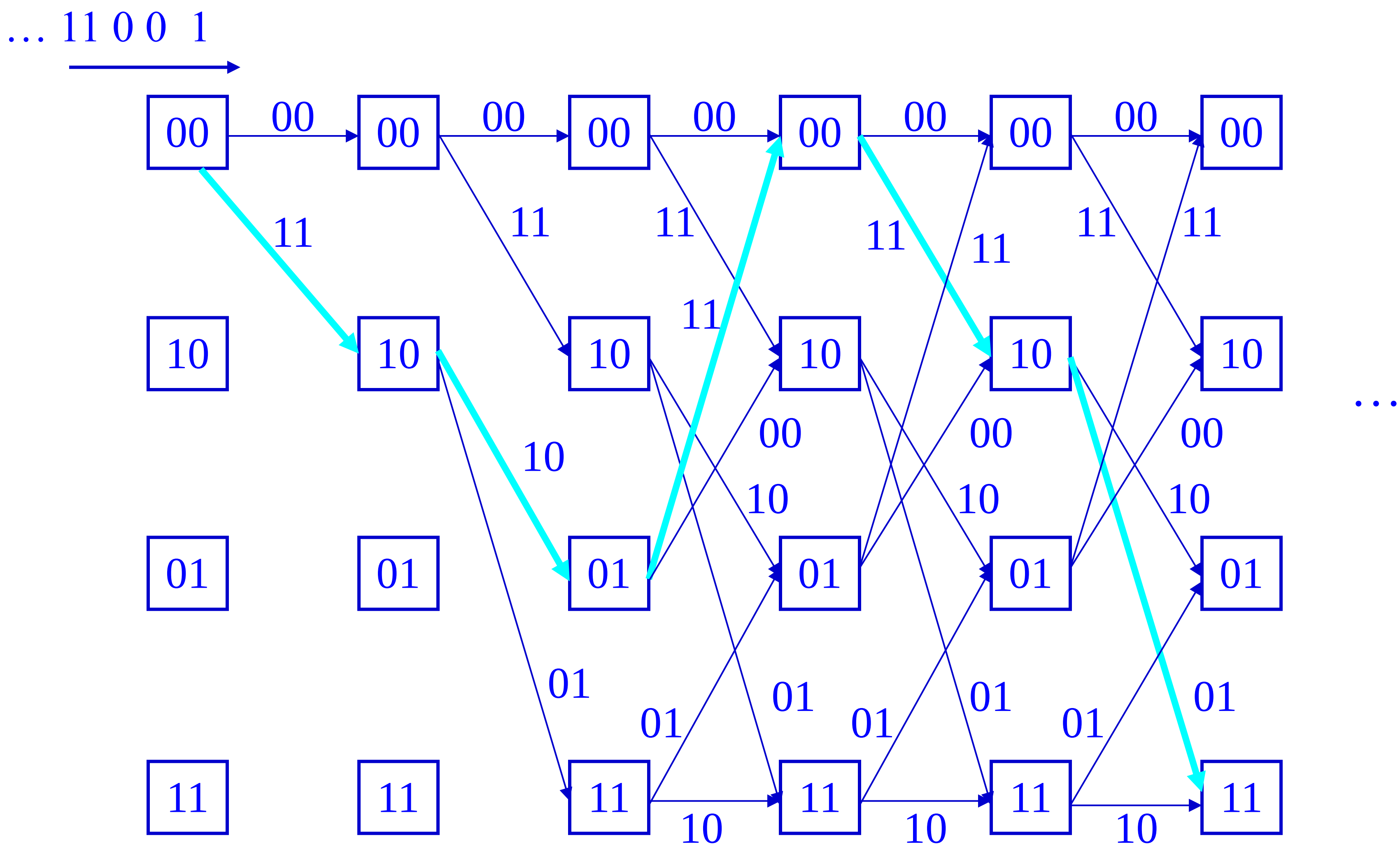
State Diagram



Tree Diagram



Trellis



Interleaving

Input data: a1, a2, a3, a4, a5, a6, a7, a8, a9, …

Interleaving (Write →, Read ↓)

a1,	a2,	a3,	a4
a5,	a6,	a7,	a8
a9,	a10,	a11,	a12
a13,	a14,	a15,	a16

Transmitting data: a1, a5, a9, a13, a2, a6, a10, a14, a3, …

Through Air

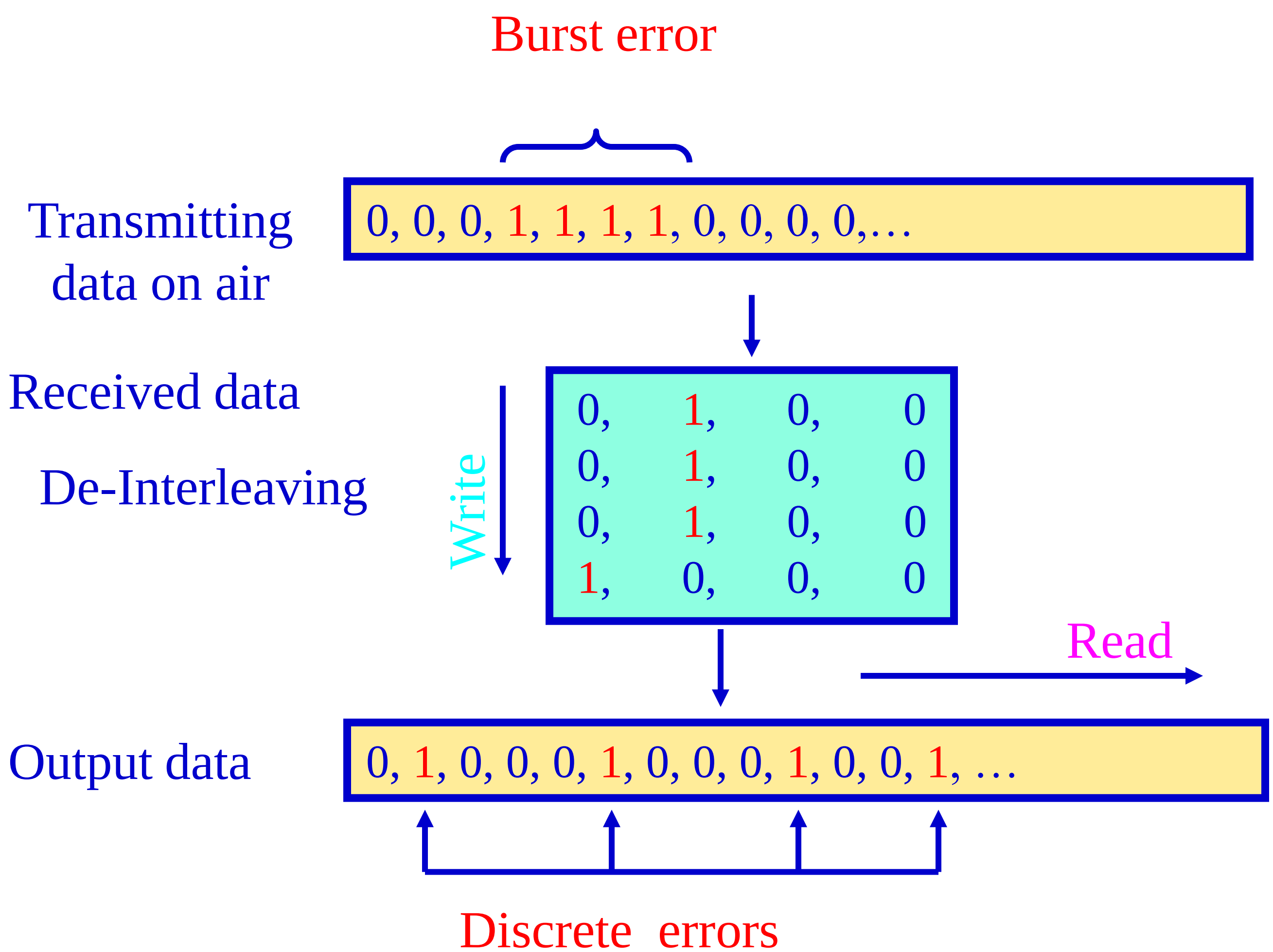
Received data: a1, a5, a9, a13, a2, a6, a10, a14, a3, …

De-Interleaving (Write ↓, Read →)

a1,	a2,	a3,	a4
a5,	a6,	a7,	a8
a9,	a10,	a11,	a12
a13,	a14,	a15,	a16

Output data: a1, a2, a3, a4, a5, a6, a7, a8, a9, …

Interleaving (Example)



Information Capacity Theorem (Shannon Limit)

- The information capacity (or channel capacity) C of a continuous channel with bandwidth B Hertz can be perturbed by additive Gaussian white noise of power spectral density $N_0/2$, provided bandwidth B satisfies

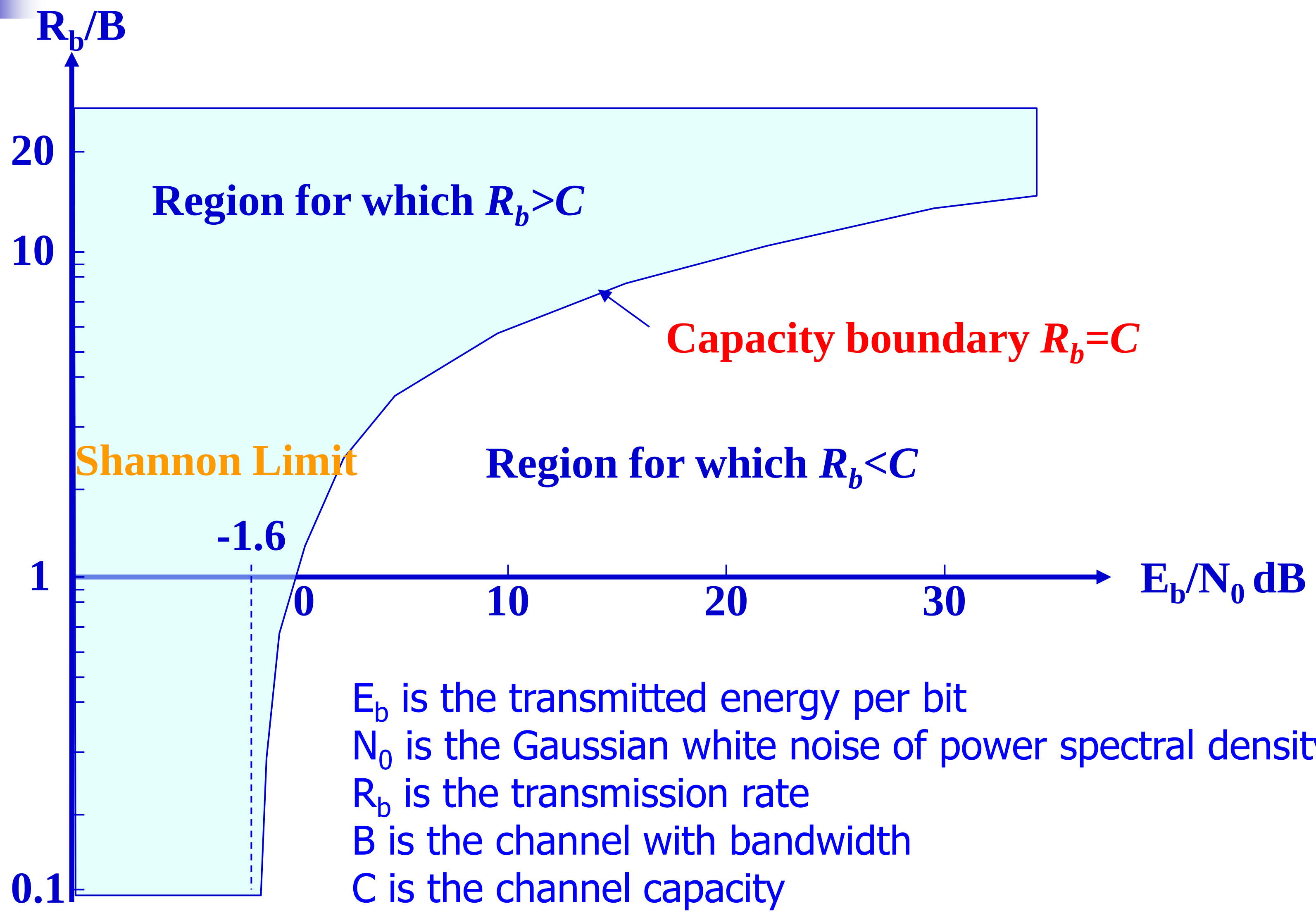
$$C = B\log_2\left(1+\frac{P}{N_0 B}\right) \quad bits/second$$

  where P is the average transmitted power $P = E_b R_b$ (for an ideal system, $R_b = C$)

  E_b is the transmitted energy per bit

  R_b is transmission rate

Shannon Limit



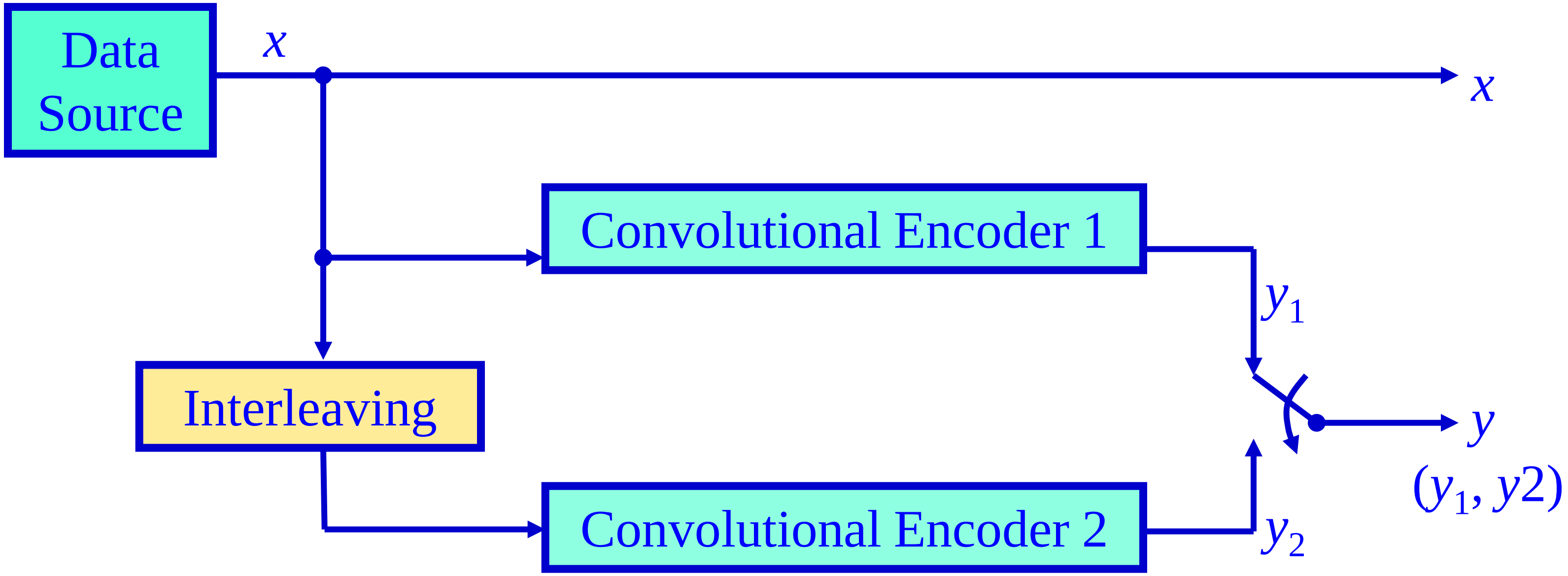
E_b is the transmitted energy per bit
N_0 is the Gaussian white noise of power spectral density
R_b is the transmission rate
B is the channel with bandwidth
C is the channel capacity

Turbo Codes

- A brief historic of turbo codes:

  The turbo code concept was first introduced by C. Berrou in 1993. Today, Turbo Codes are considered as the most efficient coding schemes for FEC
- Scheme with known components (simple convolutional or block codes, interleaver, soft-decision decoder, etc.)
- Performance close to the Shannon Limit ($E_b/N_0 = -1.6$ db if $R_b \rightarrow 0$) at modest complexity!
- Turbo codes have been proposed for low-power applications such as deep-space and satellite communications, as well as for interference limited applications such as third generation cellular, personal communication services, ad hoc and sensor networks

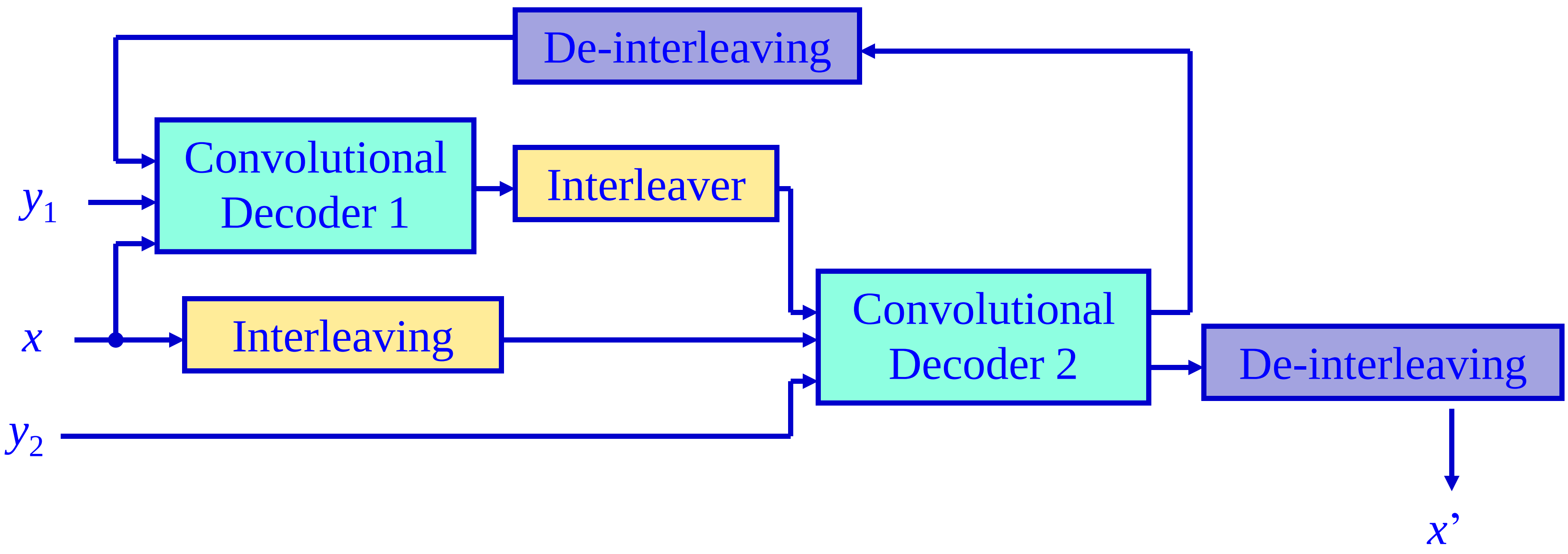
Turbo Codes: Encoder



x: Information

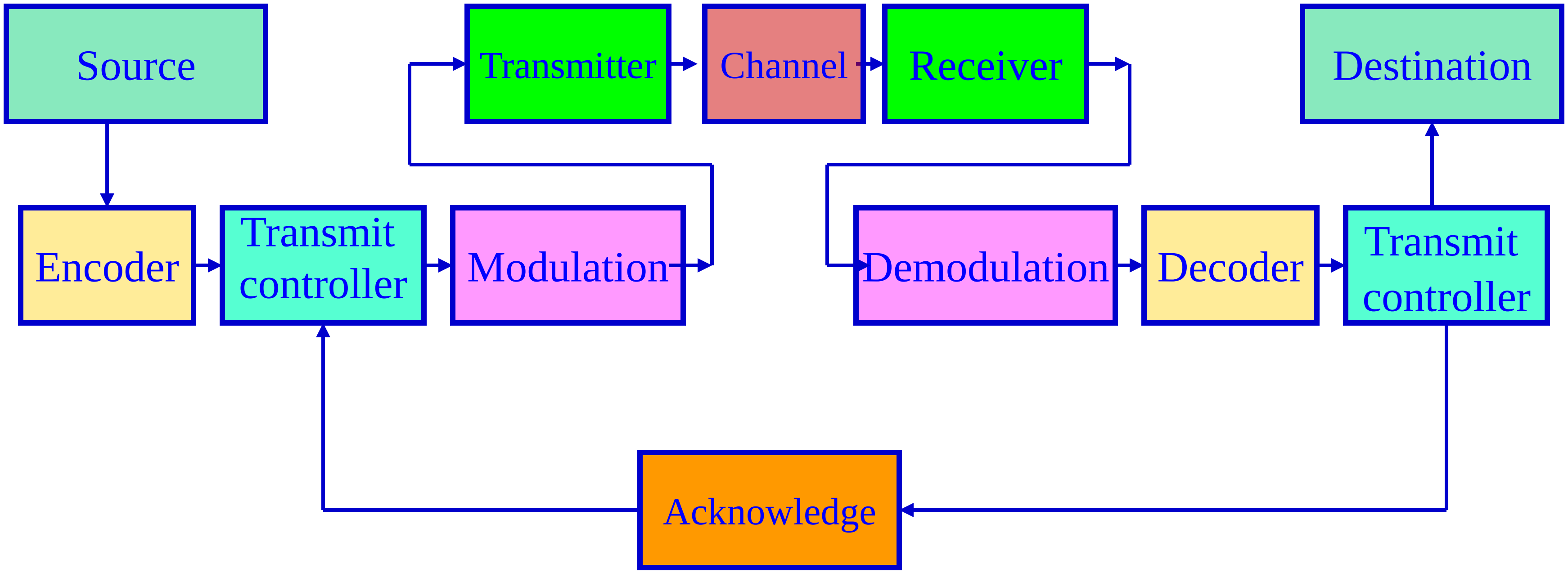
y_i: Redundancy Information

Turbo Codes: Decoder

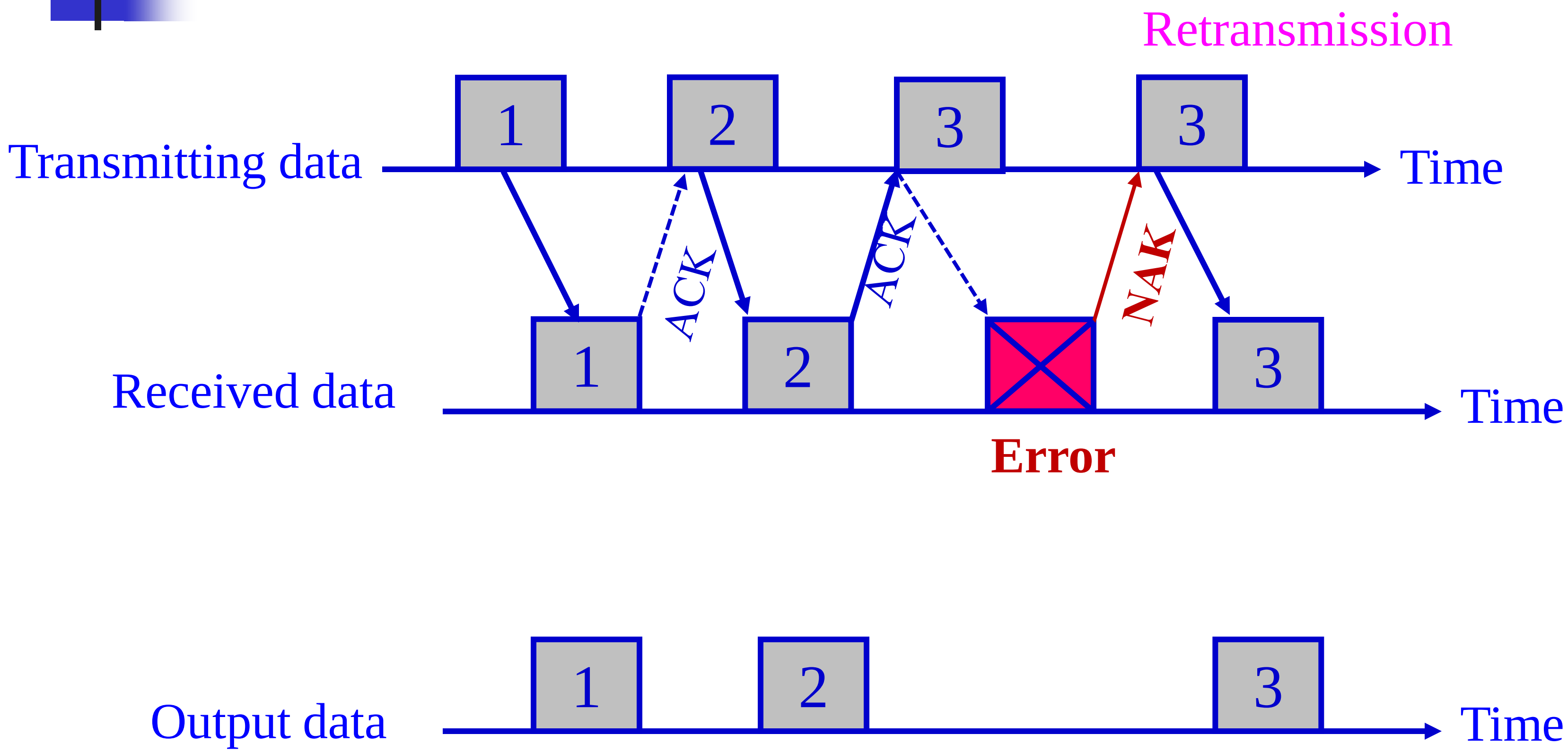


x': Decoded Information

Automatic Repeat Request (ARQ)



Stop-And-Wait ARQ (SAW ARQ)



ACK: Acknowledge

NAK: Negative ACK

Stop-And-Wait ARQ (SAW ARQ)

Given n = number of bits in a block, k = number of information bits in a block, D = round trip delay, R_b= bit rate, P_b = BER of the channel, and $P_{ACK} \approx (1- P_b)^n$

Throughput:

$$S_{SAW} = (1/T_{SAW}) \,.\, (k/n) = [(1- P_b)^n \;/\; (1 + D * R_b/ n)] * (k/n)$$

where T_{SAW} is the average transmission time in terms of a block duration

$$T_{SAW} = (1 + D \,.R_b/ n).\, P_{ACK} + 2\,(1 + D\, R_b/ n)\; P_{ACK}\,(1- P_{ACK})$$

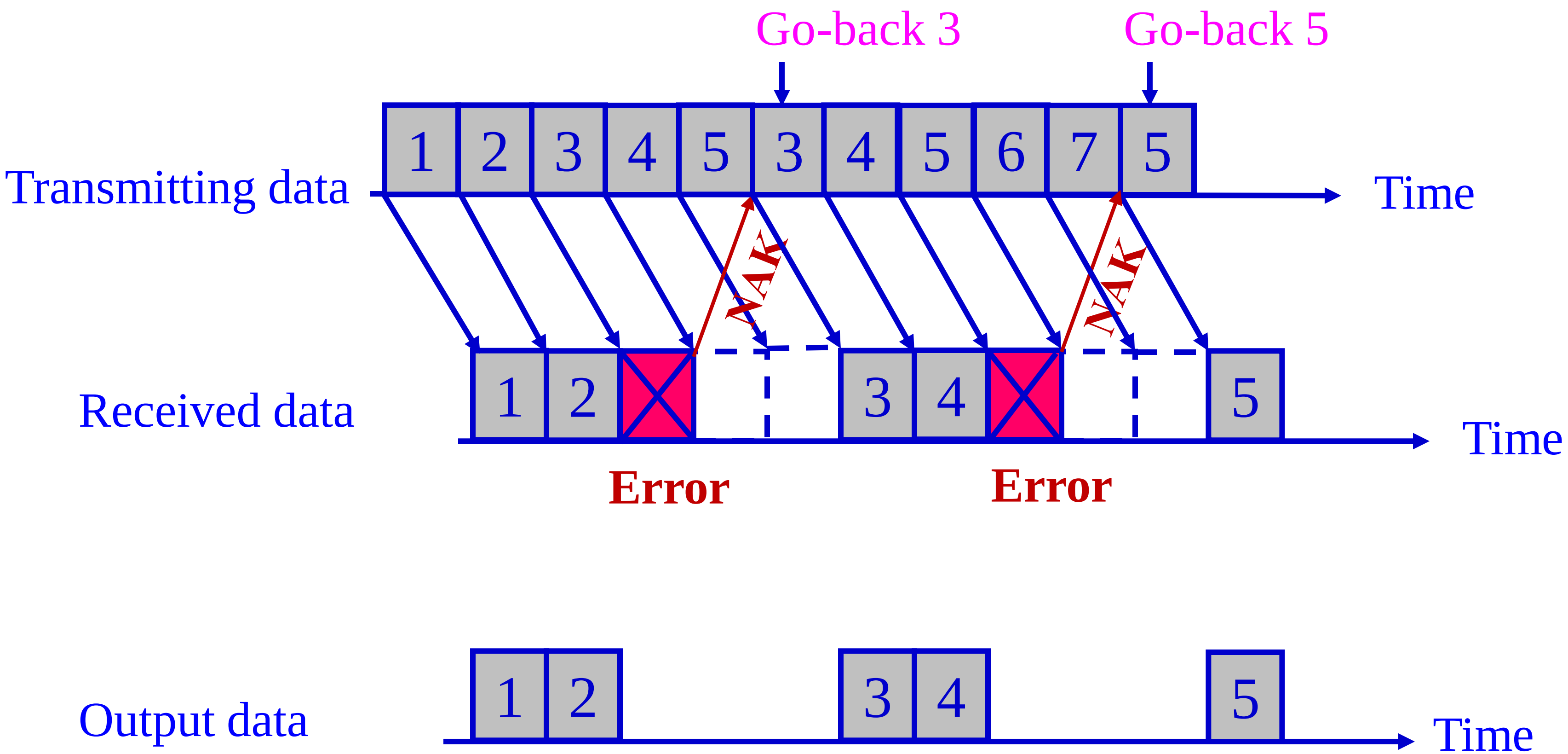
$$+ 3\;(1 + D\;\; R_b/ n)\, P_{ACK}\,(1- P_{ACK})^2 + \ldots..$$

$$= (1+ D\;\; R_b/n)\;\; P_{ACK} \sum_{i=1}^{\infty} i\;\; (1-P_{ACK})^{i-1}$$

$$= (1+ D\;\; R_b/n)\;\;\; P_{ACK}/[1-(1-P_{ACK})]^2$$

$$= (1 + D\;\; R_b/ n) \,/\, P_{ACK}$$

Go-Back-N ARQ (GBN ARQ)



Go-Back-N ARQ (GBN ARQ)

Throughput

$$S_{\mathrm{GBN}} = (1/T_{\mathrm{GBN}})\ (k/n)$$

$$= [(1- P_b)^n\ /\ ((1- P_b)^n + N\ \ (1-(1- P_b)^n\ )\ )]\ (k/n)$$

where

$$T_{\mathrm{GBN}} = 1\ .\ P_{\mathrm{ACK}} + (N+1)\ .\ P_{\mathrm{ACK}}\ .\ (1- P_{\mathrm{ACK}}) +2\ .\ (N+1)\ .\ \mathrm{P}_{\mathrm{ACK}}\ .$$

$$(1- P_{\mathrm{ACK}})^2 + \ldots.$$

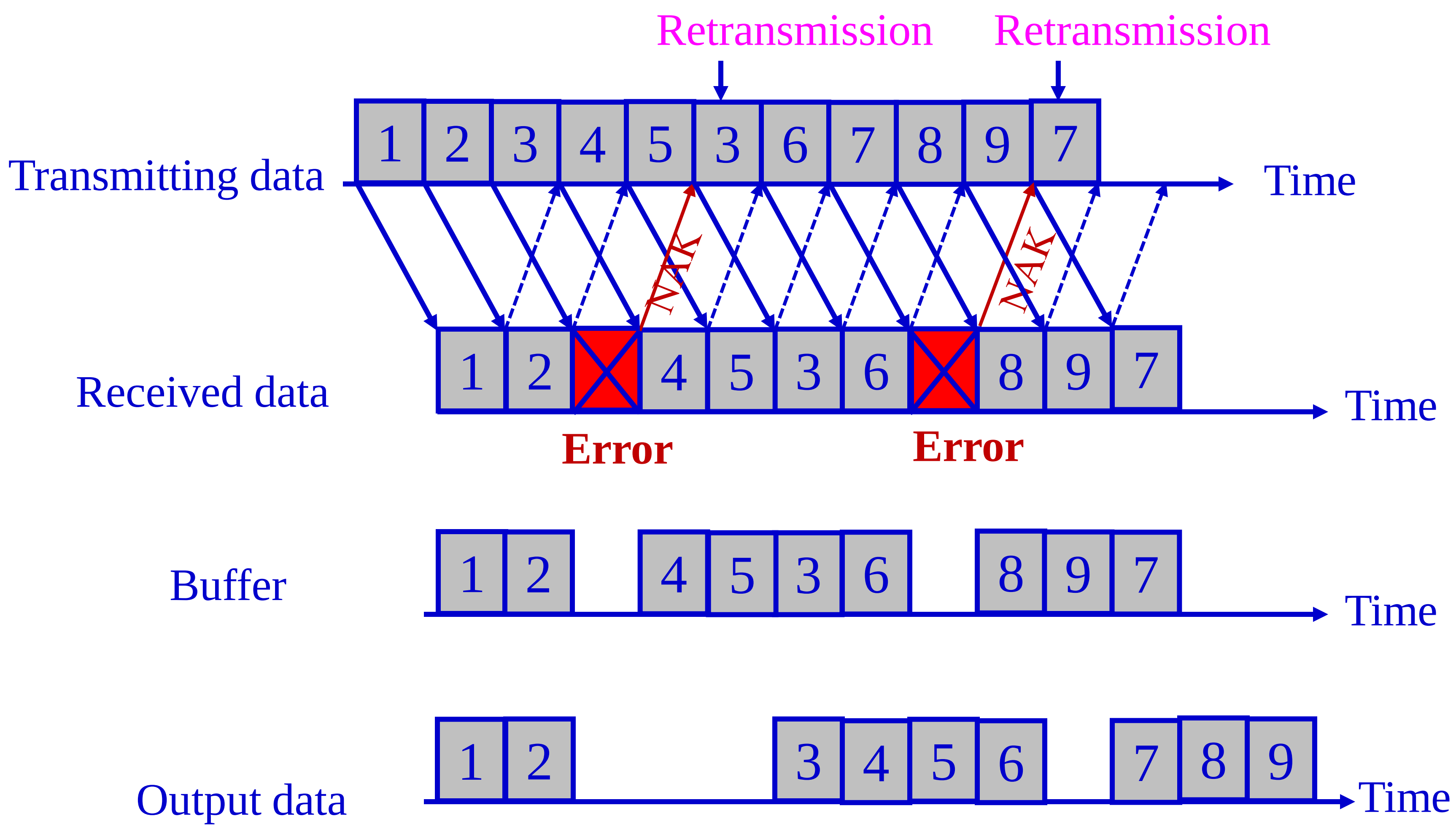
$$= P_{\mathrm{ACK}}+P_{\mathrm{ACK}}\ \ [(1-P_{\mathrm{ACK}})+(1-P_{\mathrm{ACK}}\ )^2 +(1-P_{\mathrm{ACK}})^3+\ldots]+$$

$$P_{\mathrm{ACK}}[N\ \ (1-P_{\mathrm{ACK}})+2\ \ N\ \ (1-P_{\mathrm{ACK}}\ )^2 +3\ \ N\ \ (1-P_{\mathrm{ACK}})^3+\ldots]$$

$$= P_{\mathrm{ACK}}+P_{\mathrm{ACK}}\ [(1-P_{\mathrm{ACK}})/\{1-(1-P_{\mathrm{ACK}}\ )\}+ N\ \ (1-P_{\mathrm{ACK}})/\{1-(1-P_{\mathrm{ACK}})\}^2]$$

$$= 1+ N(1-P_{\mathrm{ACK}})/P_{\mathrm{ACK}} \approx 1 + (N\ \ [1 - (1- P_b)^n\ ])/\ (1- P_b)^n$$

Selective-Repeat ARQ (SR ARQ)



Selective-Repeat ARQ (SR ARQ)

Throughput

$$S_{SR} = (1/T_{SR}) * (k/n)$$

$$= (1- P_b)^n * (k/n)$$

where

$$T_{SR} = 1 .P_{ACK} + 2\ P_{ACK}\ (1- P_{ACK}) + 3\ P_{ACK}\ (1- P_{ACK})^2$$

$$+ \ldots.$$

$$= P_{ACK} \sum_{i=1}^{\infty} i\ (1-P_{ACK})^{i-1}$$

$$= P_{ACK}/[1-(1-P_{ACK})]^2$$

$$= 1/(1- P_b)^n \qquad \text{where } P_{ACK} \approx (1-P_{\mathrm{b}})^n$$