



MATHEMATICAL RECREATIONS

by Ian Stewart

Election Fever in Blockvotia

The National Assembly of Blockvotia had just counted the votes for the Palmgreasing Slushfund Bill, and President Freebie Perks was not pleased. His secretary, Penelope, was doing her best to calm him down.

"Penny, you told me that four of the six districts were in favor, including the biggest. How did we lose?"

"It's the weighted voting system, sir. Each district has assigned to it some number of votes that is roughly proportional to its population. Here's a table showing the details. The total number of votes is 31, so any coalition that has 16 votes, or one more than half of that total, can decide the outcome.

"Sheepshire, Fiddlesex, Slurrey and the Porkney Isles voted for the bill. Four districts of six, as I said, including the biggest. But they have only 15 votes. The two who voted against have 16."

"The presidential election is coming up next month, and I don't want a repeat. If we got the Boundary Commission to give Sheepshire one more vote and Candlewick one fewer—"

Penny shook her head. "I wouldn't recommend that, sir. Richfolk and Candlewick both favor your reelection. Sheepshire is wavering, and the other

three are opposed. Richfolk and Candlewick can block a coalition formed by the other four, but not if you take a vote away from either one of them."

There was a knock on the door, and Charlie Hogg, the representative from the Porkneys, stormed in.

"Mr. President, your so-called democratic voting system is a farce. The Porkneys have no power!"

"But you have one vote, in proportion to your population. Slurrey, which has a larger population, also has one vote. You actually have more power than Slurrey."

"Nope. The outcome of any vote is decided entirely by the three largest districts. At least two of them will vote the same way, and their combined votes will be at least as great as those possessed by Richfolk and Candlewick, the second and third largest districts. That's 16 votes—a majority. You would get the same result in any vote even if the three smallest districts had none to cast!"

"I see. But what can I do about it?"

"Give us another vote! Then at least the three smallest districts could join with Sheepshire to produce a tie. If you gave Slurrey a second vote, too, we could form a winning coalition."

"I get it. The total number of votes

would then be 33," Penny said. "So 17 or more would win. A coalition of Fiddlesex, Slurrey, the Porkneys and Sheepshire could force a win."

"Yes! Any one of the smallest three districts could swing the vote."

The Boundary Commission liaison officer, Gerry Mander, walked in. "Gerry, can the Boundary Commission redraw the districts so that Slurrey and the Porkneys get an extra vote?" Perks asked.

Mander shook his head. "Might swing it for Slurrey. But the Porkneys are islands."

"My constituents won't be pleased," Hogg growled.

"No. But, as you said, that will have no effect, because your district is powerless," the president chirped. "Don't make threats you can't keep, Hogg."

"You can't be happy that three districts alone can put you out of office. There must be something you can do."

"I could give two votes to Sheepshire."

"But extra votes for the biggest district can't possibly help the smallest one gain a share of power!" Hogg wailed.

"On the contrary," Perks said. "If Sheepshire has two more votes, you get a share of the power."

"Yes," Penny said, looking over the numbers. "The same coalition musters 17 votes out of 33; again each of the three smallest districts can claim to hold the balance of power."

"That's weird," Hogg said. "You give more power to Sheepshire, and some of it miraculously rubs off on us."

"No, Hogg, we don't give them more power—we give them more votes," Penny sighed. "As you argued, those aren't the same at all."

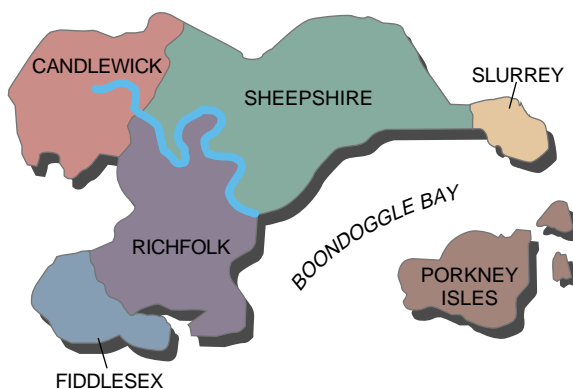
"Oh, but wait," Perks said. "If power isn't votes, what is it? I need to know. Power wins elections."

"I think you need the Banzhaf power index, sir," Penny said. "In 1965 lawyer and activist John F. Banzhaf III proposed a way to measure the power held by each member in a weighted voting system. The idea is that a representative can exercise power either by joining a losing coalition and turning it into a winning one or by leaving a winning coalition and turning it into a losing one."

"Aren't those the same thing?"

"That's correct, sir. When you join one coalition, you leave another formed by everybody else. So we need only to consider one case—say, creating a winning

MAP OF BLOCKVOTIA shows the size of the various districts. The charts list the current voting weights and two additional possibilities.



VOTING WEIGHTS FOR THE DISTRICTS OF BLOCKVOTIA		AMENDED VOTING WEIGHTS		THIRD-CHOICE VOTING WEIGHTS	
DISTRICT	NUMBER OF VOTES	DISTRICT	NUMBER OF VOTES	DISTRICT	NUMBER OF VOTES
SHEEPSHIRE	10	SHEEPSHIRE	10	SHEEPSHIRE	12
RICHFOLK	9	RICHFOLK	9	RICHFOLK	9
CANDLEWICK	7	CANDLEWICK	7	CANDLEWICK	7
FIDDLESEX	3	FIDDLESEX	3	FIDDLESEX	3
SLURREY	1	SLURREY	2	SLURREY	1
PORKNEY ISLES	1	PORKNEY ISLES	2	PORKNEY ISLES	1

JOHNNY JOHNSON

**BOARD OF REPRESENTATIVES,
TOMPKINS COUNTY, NEW YORK, 1982**

[illegible]

MUNICIPALITY	POP.	WEIGHT	POWER INDEX	POWER/POP.
LANSING	8,317	404	4,747	0.571
DRYDEN EAST	7,604	333	4,402	0.579
ENFIELD & NEWFIELD	6,776	306	3,934	0.581
ITHACA WARD 3	6,550	298	3,806	0.581
ITHACA WARD 4	6,002	274	3,474	0.579
ITHACA SOUTHEAST	5,932	270	3,418	0.576
ITHACA WARD 1	5,630	261	3,218	0.572
ITHACA WARD 2	5,378	246	3,094	0.575
ITHACA NORTHEAST	5,235	241	3,022	0.577
GROTON	5,213	240	3,006	0.577
CAROLINE & DANBY	5,203	240	3,006	0.578
ITHACA WARD 5	5,172	238	2,978	0.576
ITHACA WEST	4,855	224	2,798	0.576
ULYSSES	4,666	214	2,666	0.571
DRYDEN WEST	4,552	210	2,622	0.576

1, 1] system. The Porkneys are pivotal in any coalition that contains them and has precisely 17 votes. There are exactly two, so the Porkneys' power index is 2."

“Our original voting system was a [16; 10, 9, 7, 3, 1, 1] system. The vote needed for a majority is 16; the districts have weights of 10, 9, 7, 3, 1 and 1. The Porkneys could play a pivotal role only in those coalitions having exactly 16 votes. If the group had more votes, it would make no difference if the Porkneys defected. If it had fewer, it wouldn't be a winning coalition. But there are no such coalitions and so the Porkneys' power index is 0. With the president's new proposal, we have a [17; 12, 9, 7, 3,

"Their population would be 12 times ours, but their power would be only nine times greater," Hogg exulted.

“Well, it’s best done by computer for large systems,” Penny said. “For small systems such as our own, however, there’s a nice graphic method. Suppose the system is [3; 2, 1, 1]. That is, there are three voters: A, B and C. A has two votes, B and C have one, and three makes a majority.

“First, you draw a lattice diagram that shows all the possible coalitions and links them by an edge if they differ by just one member. Label each edge with the member they do not have in common. Then mark every pivotal edge (*red*)—where the total vote changes from being below the majority to being equal to it or above it. The power index of any given member is the number of pivotal edges bearing its label. Here member A appears on

"I'd be happy if everyone had a power index roughly in line with their population," Hogg said.

"It's not so easy," Penny said. "Let me show you how it worked out for the Board of Representatives of Tompkins County, New York, in 1982. The power index was almost exactly proportional to the population."

"We could try to do something similar here," Hogg suggested.

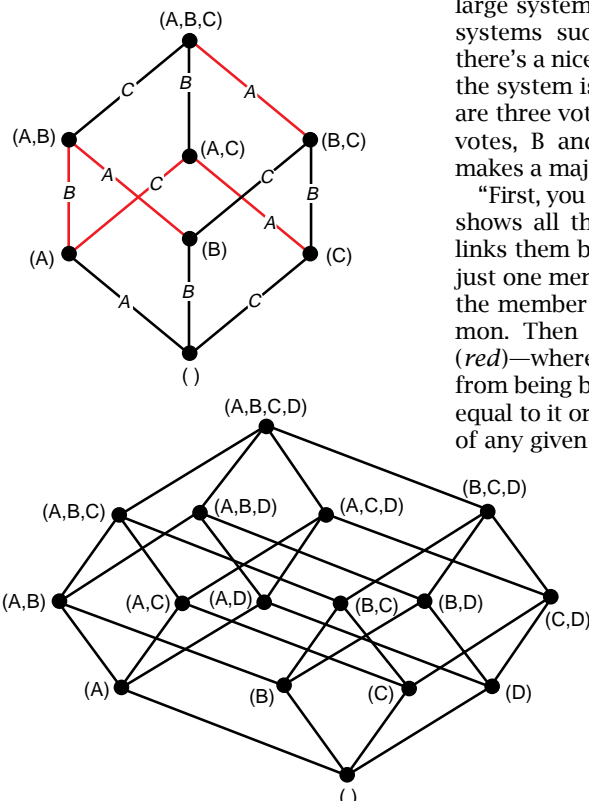
"Maybe," the president said slowly. "Do you have any studies of the power index of the U.S. president, Penny?"

"Yes, sir. He has a power index 40 times greater than that of a senator and 175 times greater than that of a member of the House of Representatives."

"That sounds wonderful."

"But the U.S. legislative body as a whole holds roughly two and a half times more power than the president."

Freebie Perks stared at her for a moment, then he looked Hogg firmly in the eye. "I think we'll stick to the present system."



LATTICE DIAGRAMS show a three-member voting system (top) and a four-member system (bottom).

FURTHER READING

ONE MAN, 3,312 VOTES: A MATHEMATICAL ANALYSIS OF THE ELECTORAL COLLEGE. John F. Banzhaf III in *Villanova Law Review*, Vol. 13, pages 304-346; Winter 1968. Rebuttals in Vol. 14, pages 86-96; Fall 1968.

GAME THEORY AND RELATED APPROACHES TO SOCIAL BEHAVIOUR:
SELECTIONS. Edited by Martin Shubik. R. E. Krieger, 1975.

FOR ALL PRACTICAL PURPOSES: INTRODUCTION TO CONTEMPORARY MATHEMATICS. Second edition. Edited by Lynn Arthur Steen. W. H. Freeman and Company, 1991.