Sheet (4)

Convolution

Q1)

2.2. Show that

(a) $x(t) * \delta(t) = x(t)$ (2.58)

(b) $x(t) * \delta(t - t_0) = x(t - t_0)$ (2.59)

(c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (2.60)

(d) $x(t) * u(t - t_0) = \int_{-\infty}^{t - t_0} x(\tau)\, d\tau$ (2.61)

(a) By definition (2.6) and Eq. (1.22) we have

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau = x(\tau)|_{\tau = t} = x(t)$$

(b) By Eqs. (2.7) and (1.22) we have

$$x(t) * \delta(t - t_0) = \delta(t - t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - \tau)\, d\tau$$

$$= x(t - \tau)|_{\tau = t_0} = x(t - t_0)$$

(c) By Eqs. (2.6) and (1.19) we have

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t - \tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

since $u(t - \tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$.

(d) In a similar manner, we have

$$x(t) * u(t - t_0) = \int_{-\infty}^{\infty} x(\tau) u(t - \tau - t_0)\, d\tau = \int_{-\infty}^{t - t_0} x(\tau)\, d\tau$$

since $u(t - \tau - t_0) = \begin{cases} 1 & \tau < t - t_0 \\ 0 & \tau > t - t_0 \end{cases}$.

Q2)

2.6. Evaluate $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are shown in Fig. 2-6, (a) by an analytical technique, and (b) by a graphical method.

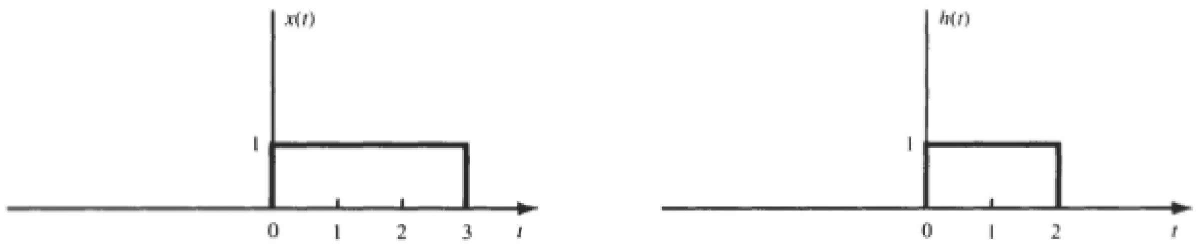

Fig. 2-6

(a) We first express $x(t)$ and $h(t)$ in functional form:

$$x(t) = u(t) - u(t-3) \qquad h(t) = u(t) - u(t-2)$$

Then, by Eq. (2.6) we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 3)][u(t-\tau) - u(t-\tau-2)]\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\, d\tau - \int_{-\infty}^{\infty} u(\tau)u(t-2-\tau)\, d\tau$$

$$- \int_{-\infty}^{\infty} u(\tau-3)u(t-\tau)\, d\tau + \int_{-\infty}^{\infty} u(\tau-3)u(t-2-\tau)\, d\tau$$

Since

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0 < \tau < t,\ t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau)u(t-2-\tau) = \begin{cases} 1 & 0 < \tau < t-2,\ t > 2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau-3)u(t-\tau) = \begin{cases} 1 & 3 < \tau < t,\ t > 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau-3)u(t-2-\tau) = \begin{cases} 1 & 3 < \tau < t-2,\ t > 5 \\ 0 & \text{otherwise} \end{cases}$$

we can express $y(t)$ as

$$y(t) = \left(\int_0^t d\tau\right)u(t) - \left(\int_0^{t-2} d\tau\right)u(t-2)$$

$$-\left(\int_3^t d\tau\right)u(t-3) + \left(\int_3^{t-2} d\tau\right)u(t-5)$$

$$= tu(t) - (t-2)u(t-2) - (t-3)u(t-3) + (t-5)u(t-5)$$

which is plotted in Fig. 2-7.

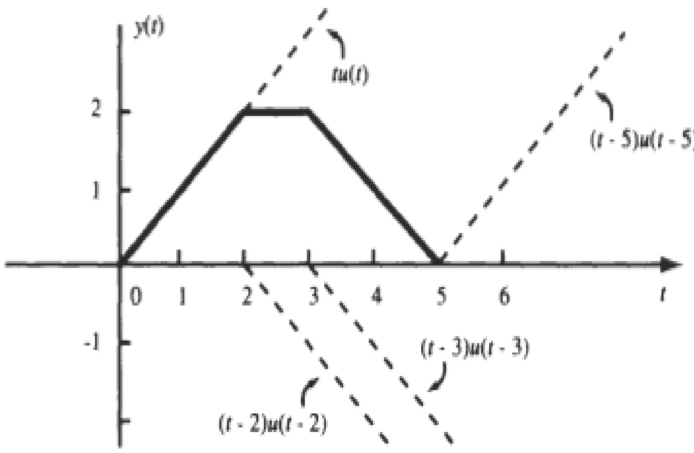

Fig. 2-7

(b) Functions $h(\tau)$, $x(\tau)$ and $h(t-\tau)$, $x(\tau)h(t-\tau)$ for different values of t are sketched in Fig. 2-8. From Fig. 2-8 we see that $x(\tau)$ and $h(t-\tau)$ do not overlap for $t < 0$ and $t > 5$, and hence $y(t) = 0$ for $t < 0$ and $t > 5$. For the other intervals, $x(\tau)$ and $h(t-\tau)$ overlap. Thus, computing the area under the rectangular pulses for these intervals, we obtain

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5 - t & 3 < t \le 5 \\ 0 & 5 < t \end{cases}$$

which is plotted in Fig. 2-9.

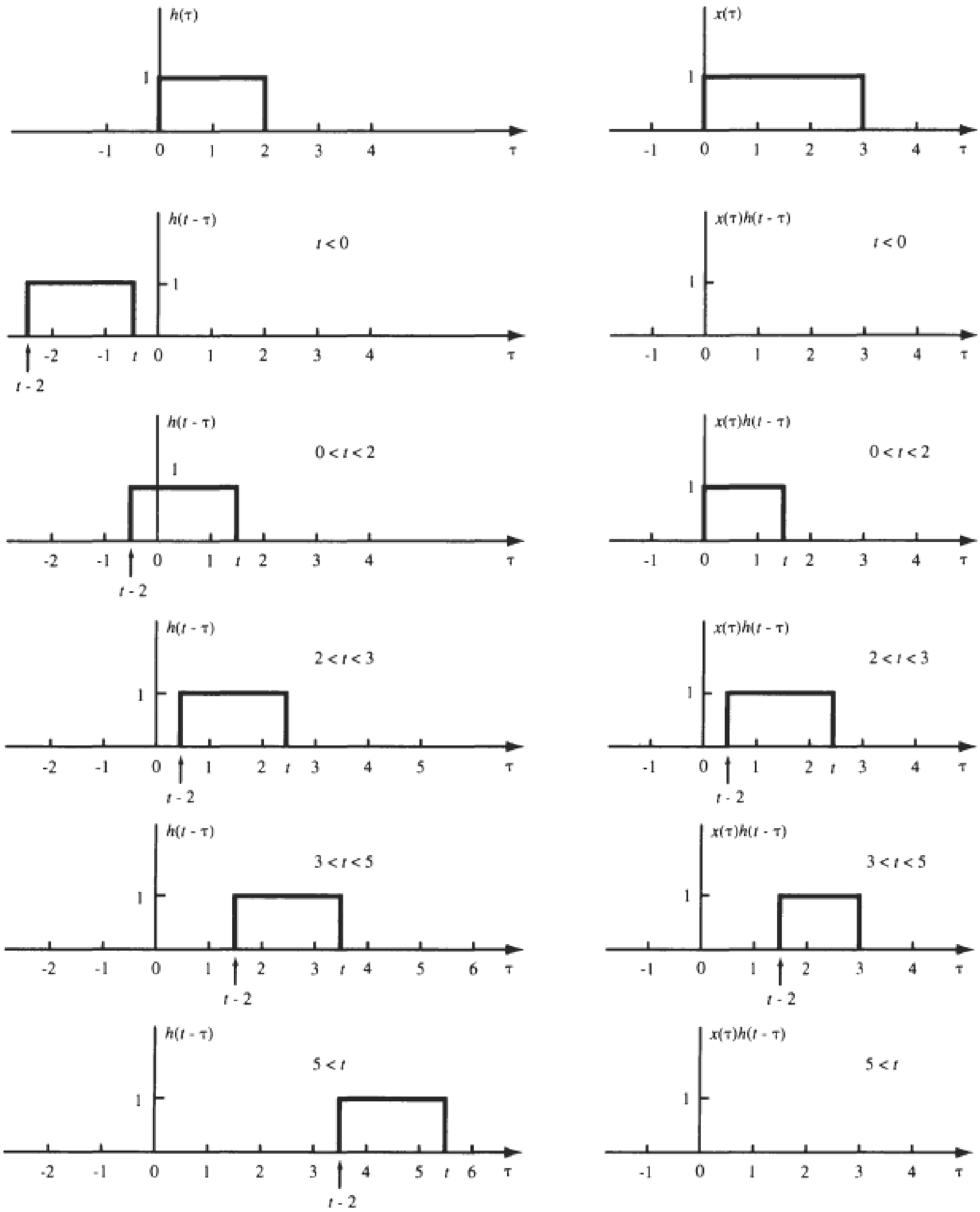

Fig. 2-8

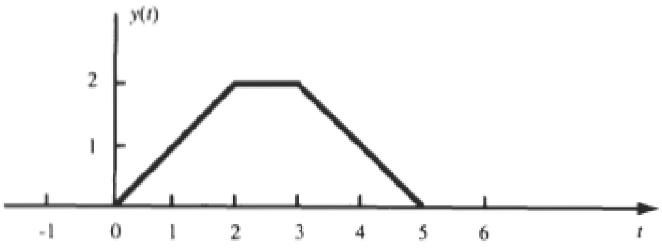

Fig. 2-9

Q3)

2.30. Evaluate $y[n] = x[n] * h[n]$, where $x[n]$ and $h[n]$ are shown in Fig. 2-23, (a) by an analytical technique, and (b) by a graphical method.

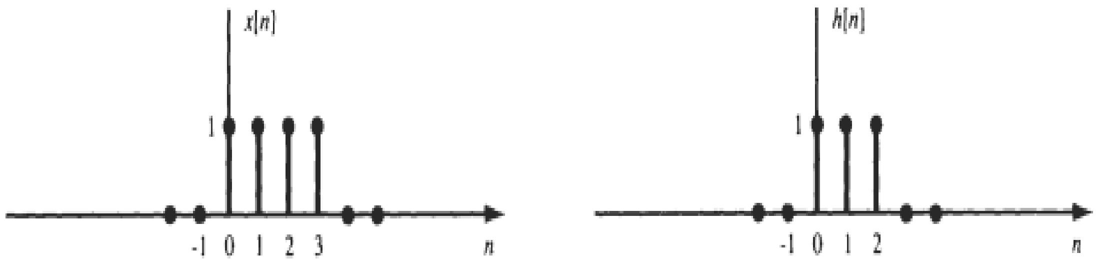

Fig. 2-23

(a) Note that $x[n]$ and $h[n]$ can be expressed as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Now, using Eqs. (2.38), (2.130), and (2.131), we have

$$x[n] * h[n] = x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2]\}$$

$$= x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2]\}$$

$$= x[n] + x[n-1] + x[n-2]$$

Thus,

$$y[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$
$$+ \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$
$$+ \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

or

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

or

$$y[n] = \{1, 2, 3, 3, 2, 1\}$$

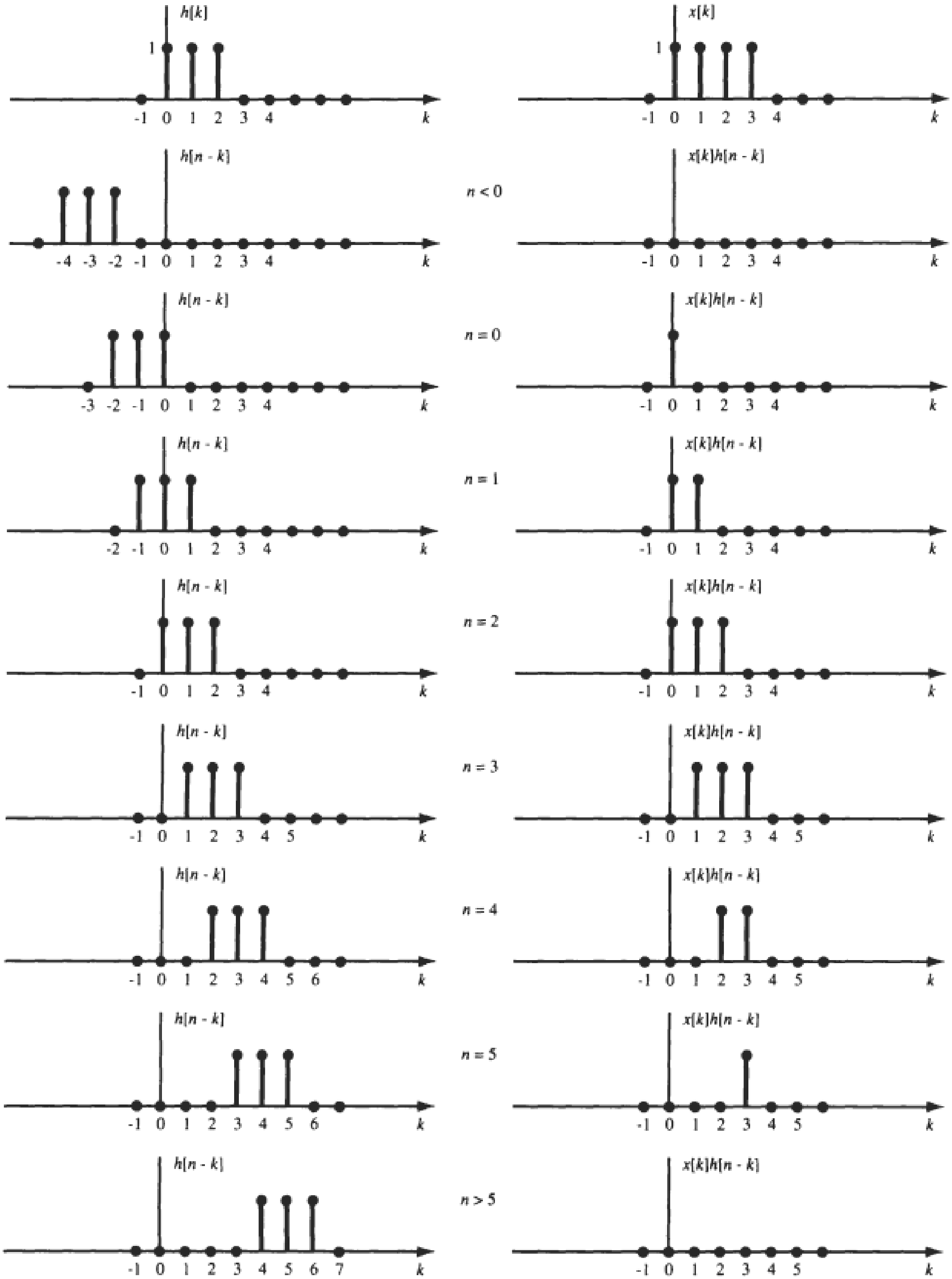
h[k]
1
-1 0 1 2 3 4
k
x[k]
1
-1 0 1 2 3 4
k
h[n - k]
-4 -3 -2 -1 0 1 2 3 4
k
n < 0
x[k]h[n - k]
-1 0 1 2 3 4
k
h[n - k]
-3 -2 -1 0 1 2 3 4
k
n = 0
x[k]h[n - k]
-1 0 1 2 3 4
k
h[n - k]
-2 -1 0 1 2 3 4
k
n = 1
x[k]h[n - k]
-1 0 1 2 3 4
k
h[n - k]
-1 0 1 2 3 4
k
n = 2
x[k]h[n - k]
-1 0 1 2 3 4
k
h[n - k]
-1 0 1 2 3 4 5
k
n = 3
x[k]h[n - k]
-1 0 1 2 3 4 5
k
h[n - k]
-1 0 1 2 3 4 5 6
k
n = 4
x[k]h[n - k]
-1 0 1 2 3 4 5
k
h[n - k]
-1 0 1 2 3 4 5 6
k
n = 5
x[k]h[n - k]
-1 0 1 2 3 4 5
k
h[n - k]
-1 0 1 2 3 4 5 6 7
k
n > 5
x[k]h[n - k]
-1 0 1 2 3 4 5
k

Fig. 2-24