Linear Algebra

For students of the first year

Computer Science Department
Faculty of Computers and Information
Damietta University

Matrix Algebra

BASIC DEFINITIONS

A **matrix** is a rectangular pattern or array of numbers.

For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 4 & 0.5 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

are all matrices. Note that we usually use a capital letter to denote a matrix, and enclose the array of numbers in brackets. To describe the size of a matrix we quote its number of rows and columns in that order so, for example, an $r \times s$ matrix has r rows and s columns. We say the matrix has **order** $r \times s$.

An $r \times s$ matrix has r rows and s columns.

Example 1

Describe the sizes of the matrices A, B and C at the start of this section, and give examples of matrices of order 3×1, 3×2 and 4×2.

Solution A has order 3×3, B has order 2×3 and C has order 1×3.

$$\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \text{ is a } 3 \times 1 \text{ matrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \text{ is a } 3 \times 2 \text{ matrix}$$

and

$$\begin{pmatrix} -1 & -1 \\ -1 & 2 \\ 2 & -0.5 \\ 1 & 0 \end{pmatrix} \text{ is a } 4 \times 2 \text{ matrix}$$

More generally, if the matrix A has m rows and n columns we can write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} represents the number or **element** in the ith row and jth column. A matrix with a single column can also be regarded as a column vector.

The operations of addition, subtraction and **multiplication** are defined upon matrices and these are explained in Section 8.3.

ADDITION, SUBTRACTION AND MULTIPLICATION

Matrix addition and subtraction

Two matrices can be added (or subtracted) if they have the same shape and size, that is the same order. Their sum (or difference) is found by adding (or subtracting) corresponding elements as the following example shows.

Example 2

If

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix}$$

find $A + B$ and $A - B$.

Solution

$$A + B = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 7 & -2 \\ 2 & 2 & 5 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ 4 & 0 & -3 \end{pmatrix}$$

On the other hand, the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ cannot be added or subtracted because they have different orders.

Example 3

If $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ show that $C + D = D + C$.

Solution

$$C + D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$D + C = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix}$$

Now $a + e$ is exactly the same as $e + a$ because addition of numbers is **commutative**. The same observation can be made of $b + f$, $c + g$ and $d + h$. Hence $C + D = D + C$. The addition of these matrices is therefore commutative. This may seem an obvious statement but we shall shortly meet matrix multiplication which is not commutative, so in general commutativity should not be simply assumed.

The result obtained in Example 3 is true more generally:

> Matrix addition is commutative, that is
>
> $A + B = B + A$

It is also easy to show that

> Matrix addition is **associative**, that is
>
> $A + (B + C) = (A + B) + C$

Scalar multiplication

Given any matrix A, we can multiply it by a number, that is a scalar, to form a new matrix of the same order as A. This multiplication is performed by multiplying every element of A by the number.

Example 4

If

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix}$$

find $2A$, $-3A$ and $\frac{1}{2}A$.

Solution

$$2A = 2\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -4 & 2 \\ 0 & 2 \end{pmatrix}$$

$$-3A = -3\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -9 \\ 6 & -3 \\ 0 & -3 \end{pmatrix}$$

and

$$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

In general we have

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

Matrix multiplication

Matrix multiplication is defined in a special way which at first seems strange but is in fact very useful. If A is a $p \times q$ matrix and B is an $r \times s$ matrix we can form the product AB only if $q = r$; that is, only if the number of columns in A is the same as the number of rows in B. The product is then a $p \times s$ matrix C, that is

$C = AB$ where A is $p \times q$
B is $q \times s$
C is $p \times s$

Example 5

Given $A = \begin{pmatrix} 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 7 & 6 \\ 5 & 2 & -1 \end{pmatrix}$ can the product AB be formed?

Solution A has size 1×2
B has size 2×3

Because the number of columns in A is the same as the number of rows in B, we can form the product AB. The resulting matrix will have size 1×3 because there is one row in A and three columns in B.

Suppose we wish to find AB when $A = \begin{pmatrix} 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. A has size 1×2 and B has size 2×1 and so we can form the product AB. The result will be a 1×1 matrix, that is a single number. We perform the calculation as follows:

$$AB = \begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = 4 \times 3 + 2 \times 7 = 12 + 14 = 26$$

Note that we have multiplied elements in the row of A with corresponding elements in the column of B, and added the results together.

Example 6

Find CD when $C = \begin{pmatrix} 1 & 9 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix}$.

Solution

$$CD = \begin{pmatrix} 1 & 9 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} = 1 \times 2 + 9 \times 6 + 2 \times 8 = 2 + 54 + 16 = 72$$

Example 7

If $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ find, if possible, the matrix C where $C = AB$.

Solution We can form the product

$$C = AB = \underset{\substack{\uparrow \\ 2 \times 2}}{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}} \underset{\substack{\uparrow \\ 2 \times 1}}{\begin{pmatrix} 5 \\ -3 \end{pmatrix}}$$

because the number of columns in A, that is 2, is the same as the number of rows in B. The size of the product is found by inspecting the number of rows in the first matrix, which is 2, and the number of columns in the second, which is 1. These numbers give the number of rows and columns respectively in C. Therefore C will be a 2×1 matrix.

To find the element c_{11} we pair the elements in the first row of A with those in the first column of B, multiply and then add these together. Thus

$$c_{11} = 1 \times 5 + 2 \times -3 = 5 - 6 = -1$$

Similarly, to find the element c_{21} we pair the elements in the second row of A with those in the first column of B, multiply and then add these together. Thus

$$c_{21} = 4 \times 5 + 3 \times -3 = 20 - 9 = 11$$

The complete calculation is written as

$$\begin{aligned} AB = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \end{pmatrix} &= \begin{pmatrix} 1 \times 5 + 2 \times -3 \\ 4 \times 5 + 3 \times -3 \end{pmatrix} \\ &= \begin{pmatrix} 5 - 6 \\ 20 - 9 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 11 \end{pmatrix} \end{aligned}$$

If A is a $p \times q$ matrix and B is a $q \times s$ matrix, then the product $C = AB$ will be a $p \times s$ matrix. To find c_{ij} we take the ith row of A and pair its elements with the jth column of B. The paired elements are multiplied together and added to form c_{ij}.

Example 8

If $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ find BC.

Solution B has order 2×3 and C has order 3×1 so clearly the product BC exists and will have order 2×1. BC is formed as follows:

$$BC = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 2 + 3 \times 4 \\ 4 \times 1 + 5 \times 2 + 6 \times 4 \end{pmatrix} = \begin{pmatrix} 17 \\ 38 \end{pmatrix}$$

Note that the order of the product, 2×1, can be determined at the start by considering the orders of B and C.

Example 9

Find AB where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Solution A and B have orders 3×2 and 3×1, respectively, and consequently the product, AB, cannot be formed.

Example 10

Given

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

find, if possible, AB and BA, and comment upon the result.

Solution A and B both have order 3×3 and the products AB and BA can both be formed. Both will have order 3×3.

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 2 \\ 4 & 5 & 2 \\ 0 & 14 & 5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 2 & 2 \\ 2 & 3 & 4 \\ 9 & 3 & 4 \end{pmatrix}$$

Clearly AB and BA are not the same. Matrix multiplication is not usually commutative and we must pay particular attention to this detail when we are working with matrices.

In general $AB \neq BA$ and so matrix multiplication is not commutative.

In the product AB we say that B has been **premultiplied** by A, or alternatively A has been **postmultiplied** by B.

Example 11

Given

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

find BC, $A(BC)$, AB and $(AB)C$, commenting upon the result.

Solution

$$BC = \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 14 & 42 \\ 4 & 1 & 5 \\ 8 & -2 & 18 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0 & 14 & 42 \\ 4 & 1 & 5 \\ 8 & -2 & 18 \end{pmatrix} = \begin{pmatrix} 12 & 9 & 73 \\ 16 & 38 & 162 \\ 52 & -21 & 63 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 11 \\ 3 & 2 & 29 \\ 19 & -12 & -4 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 4 & -2 & 11 \\ 3 & 2 & 29 \\ 19 & -12 & -4 \end{pmatrix} \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 9 & 73 \\ 16 & 38 & 162 \\ 52 & -21 & 63 \end{pmatrix}$$

We note that $A(BC) = (AB)C$ so that in this case matrix multiplication is associative.

The result obtained in Example 11 is also true in general:

Matrix multiplication is associative:

$$(AB)C = A(BC)$$

Exercises

1 Evaluate

(a) $\begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix}$

(b) $\begin{pmatrix}3 & 7\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix}$

(c) $\begin{pmatrix}5 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}9\\1\end{pmatrix}$

(d) $\begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}\begin{pmatrix}5 & 2\\-3 & 4\end{pmatrix}$

(e) $\begin{pmatrix}5 & 1\\29 & 6\end{pmatrix}\begin{pmatrix}6 & -1\\-29 & 5\end{pmatrix}$

(f) $\begin{pmatrix}1 & 2 & 4\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix}$

(g) $\begin{pmatrix}5 & -1 & 3\end{pmatrix}\begin{pmatrix}3 & 3\\2 & 6\\4 & 1\end{pmatrix}$

(h) $\begin{pmatrix}1 & 9\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\2 & 2 & 2\end{pmatrix}$

(i) $\begin{pmatrix}1 & 3\\-1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\5 & 0\end{pmatrix}$

(j) $\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 3\\9 & 0\end{pmatrix}$

2 If $A = \begin{pmatrix}1 & 1\\3 & 4\end{pmatrix}$, $B = \begin{pmatrix}2\\1\end{pmatrix}$, $C = \begin{pmatrix}-7 & 1\\0 & 4\end{pmatrix}$,
$D = \begin{pmatrix}3 & 2 & 1\end{pmatrix}$ and $E = \begin{pmatrix}2 & 3 & 4\\1 & 2 & -1\end{pmatrix}$ find, if possible,

(a) $A + D$, $C - A$ and $D - E$

(b) AB, BA, CA, AC, DA, DB, BD, EB, BE and AE

(c) $7C$, $-3D$ and kE, where k is a scalar.

3 Plot the points A, B, C with position vectors given by

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix}2\\0\end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix}2\\3\end{pmatrix}$$

respectively. Treating these vectors as matrices of order 2×1 find the products $M\mathbf{v}_1$, $M\mathbf{v}_2$, $M\mathbf{v}_3$ when

(a) $M = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

(b) $M = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

(c) $M = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

In each case draw a diagram to illustrate the effect upon the vectors of multiplication by the matrix.

$$(A + B)^2 = A^2 + AB + BA + B^2$$

Why is $(A + B)^2$ not equal to $A^2 + 2AB + B^2$?

7 Find, if possible,

(a) $\begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\2\\1\end{pmatrix}$

(b) $\begin{pmatrix}0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\5\\5\\1\end{pmatrix}$

8 Find $\begin{pmatrix}1 & 3 & 6\\2 & -5 & 7\end{pmatrix}\begin{pmatrix}1 & 2\\3 & -5\\6 & 7\end{pmatrix}$.

4 Find AB and BA where

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \\ 5 & 1 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 3 & 4 \\ 1 & 3 & 5 \end{pmatrix}$$

5 Given that A^2 means the product of a matrix A with itself, find A^2 when $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. Find A^3.

6 If $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -4 & 5 \end{pmatrix}$ find AB, BA, $A + B$ and $(A + B)^2$. Show that

9 Given the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ calculate the vectors obtained when $\mathbf{v}$ is premultiplied by the following matrices:

(a) $\begin{pmatrix} 6 & 2 & 9 \\ 1 & 3 & 2 \\ -1 & 2 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 & 3 \\ 7 & 1 & 9 \\ 1 & 3 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 & 1 \\ 9 & 2 & 6 \\ 2 & 8 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 1 & 2 \\ 6 & 5 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 6 & 8 & 3 \\ 9 & 6 & 4 \\ 5 & 3 & 9 \\ 2 & 5 & 2 \end{pmatrix}$

SOME SPECIAL MATRICES

Square matrices

A matrix which has the same number of rows as columns is called a **square** matrix. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} \text{ is a square matrix, while } \begin{pmatrix} -1 & 3 & 0 \\ 2 & 4 & 1 \end{pmatrix} \text{ is not}$$

Diagonal matrices

Some square matrices have elements which are zero everywhere except on the leading diagonal (top-left to bottom-right). Such matrices are said to be **diagonal**. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are all diagonal matrices, whereas

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

is not.

Identity matrices

Diagonal matrices which have only ones on their leading diagonals, for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are called **identity** matrices and are denoted by the letter I.

Example 12

Find IA where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix}$ and comment upon the result.

Solution

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix}$$

The effect of premultiplying A by I has been to leave A unaltered. The product is identical to the original matrix A, and this is why I is called an identity matrix.

In general, if A is an arbitrary matrix and I is an identity matrix of the appropriate size, then

$$IA = A$$

> If A is a square matrix then $IA = AI = A$.

The transpose of a matrix

If A is an arbitrary $m \times n$ matrix, a related matrix is the **transpose** of A, written A^T, found by interchanging the rows and columns of A. Thus the first row of A becomes the first column of A^T and so on. A^T is an $n \times m$ matrix.

Example 13

If $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ find A^T.

Solution

$$A^T = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

Example 14

If $A = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ find A^T and evaluate AA^T.

Solution $A^T = \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix}$ $AA^T = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 56 & 62 \\ 62 & 114 \end{pmatrix}$

Symmetric matrices

If a square matrix A and its transpose A^T are identical, then A is said to be a **symmetric** matrix.

Example 15

If $A = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix}$ find A^T.

Solution $A^T = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix}$

which is clearly equal to A. Hence A is a symmetric matrix. Note that a symmetric matrix is symmetrical about its leading diagonal.

Skew symmetric matrices

If a square matrix A is such that $A^T = -A$ then A is said to be **skew symmetric**.

Example 16

If $A = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$, find A^T and deduce that A is skew symmetric.

Solution We have $A^T = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$ which is clearly equal to $-A$. Hence A is skew symmetric.

EXERCISES

1 If $A = \begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 \\ 3 & 8 \end{pmatrix}$

(a) find A^T,
(b) find B^T,
(c) find AB,
(d) find $(AB)^T$,
(e) deduce that $(AB)^T = B^T A^T$.

2 Treating the column vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ as a 3×1 matrix, find $I\mathbf{x}$ where I is the 3×3 identity matrix.

3 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ show that AA^T is a symmetric matrix.

4 If $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ -1 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -7 & 0 \\ 0 & 2 & 5 \\ 3 & 4 & 5 \end{pmatrix}$

find A^T, B^T, AB and $(AB)^T$.
Deduce that $(AB)^T = B^T A^T$.

5 Determine the type of matrix obtained when two diagonal matrices are multiplied together.

6 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is skew symmetric, show that $a = d = 0$, that is the diagonal elements are zero.

7 If $A = \begin{pmatrix} 1 & 13 \\ 15 & 7 \end{pmatrix}$

(a) find A^T,
(b) find $(A^T)^T$,
(c) deduce that $(A^T)^T$ is equal to A.

8 If $A = \begin{pmatrix} 9 & 4 \\ 3 & 2 \end{pmatrix}$

(a) find $A + A^T$ and show that this is a symmetric matrix,
(b) find $A - A^T$ and show that this is a skew symmetric matrix.

9 The sum of the elements on the leading diagonal of a square matrix is known as its **trace**. Find the trace of

(a) $\begin{pmatrix} 7 & 2 \\ -1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 2 & 1 \\ 8 & 2 & 3 \\ 9 & -1 & -4 \end{pmatrix}$

THE INVERSE OF A 2 × 2 MATRIX

When we are dealing with ordinary numbers it is often necessary to carry out the operation of division. Thus, for example, if we know that $3x = 4$, then clearly $x = 4/3$. If we are given matrices A and C and know that

$$AB = C$$

how do we find B? It might be tempting to write

$$B = \frac{C}{A}$$

Unfortunately, this would be entirely wrong since division of matrices is not defined. However, given expressions like $AB = C$ it is often necessary to be able to find the appropriate expression for B. This is where we need to introduce the concept of an inverse matrix.

If A is a square matrix and we can find another matrix B with the property that

$$AB = BA = I$$

then B is said to be the **inverse** of A and is written A^{-1}, that is

$$AA^{-1} = A^{-1}A = I$$

If B is the inverse of A, then A is also the inverse of B. Note that A^{-1} does not mean a reciprocal; there is no such thing as matrix division. A^{-1} is the notation we use for the inverse of A.

Multiplying a matrix by its inverse yields the identity matrix I, that is

$$AA^{-1} = A^{-1}A = I$$

Since A is a square matrix, A^{-1} is also square and of the same order, so that the products AA^{-1} and $A^{-1}A$ can be formed. The term 'inverse' cannot be applied to a matrix which is not square.

Example 17

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ show that the matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the inverse of A.

Solution Forming the products

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the inverse of A.

Finding the inverse of a matrix

For 2×2 matrices a simple formula exists to find the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This formula states

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 18

If $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ find A^{-1}.

Solution Clearly $ad - bc = 6 - 5 = 1$, so that

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

The solution should always be checked by forming AA^{-1}.

Example 19

If $A = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$ find A^{-1}.

Solution Here we have $ad - bc = 4 - 10 = -6$. Therefore

$$A^{-1} = \frac{1}{-6}\begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{5}{6} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

Example 20

If $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ find A^{-1}.

Solution This time, $ad - bc = 4 - 4 = 0$, so when we come to form $\dfrac{1}{ad - bc}$ we find 1/0 which is not defined. We cannot form the inverse of A in this case; it does not exist.

Clearly not all square matrices have inverses. The quantity $ad - bc$ is obviously the important determining factor since only if $ad - bc \neq 0$ can we find A^{-1}. This quantity is therefore given a special name: the **determinant** of A, denoted by $|A|$, or $\det A$. Given any 2×2 matrix A, its determinant, $|A|$, is the scalar $ad - bc$. This is easily remembered as

[product of $\searrow$ diagonal] − [product of $\swarrow$ diagonal]

Clearly not all square matrices have inverses. The quantity $ad - bc$ is obviously the important determining factor since only if $ad - bc \neq 0$ can we find A^{-1}. This quantity is therefore given a special name: the **determinant** of A, denoted by $|A|$, or $\det A$. Given any 2×2 matrix A, its determinant, $|A|$, is the scalar $ad - bc$. This is easily remembered as

[product of $\searrow$ diagonal] − [product of $\swarrow$ diagonal]

If A is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we write its determinant as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Note that the straight lines || indicate that we are discussing the determinant, which is a scalar, rather than the matrix itself. If the matrix A is such that $|A| = 0$, then it has no inverse and is said to be **singular**. If $|A| \neq 0$ then A^{-1} exists and A is said to be **non-singular**.

> A singular matrix A has $|A| = 0$.
>
> A non-singular matrix A has $|A| \neq 0$.

Example 21

If $A = \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$ find $|A|$, $|B|$ and $|AB|$.

Solution

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = (1)(0) - (2)(5) = -10 \\
|B| &= \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1)(1) - (2)(-3) = 5 \\
AB &= \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -5 & 10 \end{pmatrix} \\
|AB| &= (-7)(10) - (4)(-5) = -50
\end{aligned}$$

We note that $|A||B| = |AB|$.

The result obtained in Example 21 is true more generally:

If A and B are square matrices of the same order, $|A||B| = |AB|$.

Orthogonal matrices

A non-singular square matrix A such that $A^T = A^{-1}$ is said to be **orthogonal**. Consequently, if A is orthogonal $AA^T = A^T A = I$.

Example 22

Find the inverse of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Deduce that A is an orthogonal matrix.

Solution From the formula for the inverse of a 2×2 matrix we find

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This is clearly equal to the transpose of A. Hence A is an orthogonal matrix.

EXERCISES

1 If $A = \begin{pmatrix} 5 & 6 \\ -4 & 8 \end{pmatrix}$ find A^{-1}.

2 Find the inverse, if it exists, of each of the following matrices:

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 0 \\ -1 & 7 \end{pmatrix}$ (e) $\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} -6 & 2 \\ 9 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

3 If $A = \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 4 & 3 \end{pmatrix}$

find $|AB|$, $|BA|$.

4 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

find AB, $|A|$, $|B|$, $|AB|$.

Verify that $|AB| = |A||B|$.

5 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ find A^{-1}.

Find values of the constants a and b such that $A + aA^{-1} = bI$.

6 If $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

find AB, $(AB)^{-1}$, B^{-1}, A^{-1} and $B^{-1}A^{-1}$.

Deduce that $(AB)^{-1} = B^{-1}A^{-1}$.

7 Given that the matrix

$$M = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is orthogonal, find M^{-1}.

8 (a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and k is a scalar constant, show that the inverse of the matrix kA is $\frac{1}{k}A^{-1}$.

(b) Find the inverse of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and hence write down the inverse of $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$.

DETERMINANTS

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the value of its determinant, $|A|$, is given by

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If we choose an element of A, a_{ij} say, and cross out its row and column and form the determinant of the four remaining elements, this determinant is known as the **minor** of the element a_{ij}.

A moment's study will therefore reveal that the determinant of A is given by

$$|A| = (a_{11} \times \text{its minor}) - (a_{12} \times \text{its minor}) + (a_{13} \times \text{its minor})$$

This method of evaluating a determinant is known as **expansion along the first row**.

Example 23

Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$$

Solution The determinant of A, written as

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{vmatrix}$$

is found by expanding along its first row:

$$\begin{aligned} |A| &= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} \\ &= 1(2) - 2(-22) + 1(-16) \\ &= 2 + 44 - 16 \\ &= 30 \end{aligned}$$

Example 24

Find the minors of the elements 1 and 4 in the matrix

$$B = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 4 & 2 \end{pmatrix}$$

Solution To find the minor of 1 delete its row and column to form the determinant $\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$. The required minor is therefore $4 - 12 = -8$.

Similarly, the minor of 4 is $\begin{vmatrix} 7 & 3 \\ 1 & 3 \end{vmatrix} = 21 - 3 = 18$.

In addition to finding the minor of each element in a matrix, it is often useful to find a related quantity – the **cofactor** of each element. The cofactor is found by imposing on the minor a positive or negative sign depending upon its position, that is a **place sign**, according to the following rule:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Example 25

If

$$A = \begin{pmatrix} 3 & 2 & 7 \\ 9 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

find the cofactors of 9 and 7.

Solution The minor of 9 is $\begin{vmatrix} 2 & 7 \\ -1 & 2 \end{vmatrix} = 4 - (-7) = 11$, but since its place sign is negative, the required cofactor is -11.

The minor of 7 is $\begin{vmatrix} 9 & 1 \\ 3 & -1 \end{vmatrix} = -9 - 3 = -12$. Its place sign is positive, so that the required cofactor is simply -12.

Using determinants to find vector products

Determinants can also be used to evaluate the vector product of two vectors. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, we showed in Section 7.6 that $\mathbf{a} \times \mathbf{b}$ is the vector defined by

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

If we consider the expansion of the determinant given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

we find the same result. This definition is therefore a convenient mechanism for evaluating a vector product.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 25

If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$ find $\mathbf{a} \times \mathbf{b}$.

Solution We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 4 & 0 & 5 \end{vmatrix}$$
$$= 5\mathbf{i} - 23\mathbf{j} - 4\mathbf{k}$$

Cramer's rule

A useful application of determinants is to the solution of simultaneous equations. Consider the case of three simultaneous equations in three unknowns:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Cramer's rule states that x, y and z are given by the following ratios of determinants.

Cramer's rule:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Note that in all cases the determinant in the denominator is identical and its elements are the coefficients on the l.h.s. of the simultaneous equations. When this determinant is zero, Cramer's method will clearly fail.

Example 27

Solve

$$\begin{aligned} 3x + 2y - z &= 4 \\ 2x - y + 2z &= 10 \\ x - 3y - 4z &= 5 \end{aligned}$$

Solution We find

$$x = \frac{\begin{vmatrix} 4 & 2 & -1 \\ 10 & -1 & 2 \\ 5 & -3 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix}} = \frac{165}{55} = 3$$

Verify for yourself that $y = -2$ and $z = 1$.

EXERCISES

1 Find $\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 0 \\ 3 & 5 & -1 \end{vmatrix}$ and $\begin{vmatrix} 6 & 7 & 2 \\ 1 & 4 & 3 \\ -1 & 1 & 4 \end{vmatrix}$.

2 Find $\begin{vmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{vmatrix}$.

3 Evaluate $\begin{vmatrix} 5 & 0 & 0 \\ 6 & 3 & 2 \\ 4 & 5 & 7 \end{vmatrix}$ and $\begin{vmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{vmatrix}$.

4 If $A = \begin{pmatrix} 2 & -1 & 7 \\ 0 & 8 & 4 \\ 3 & 6 & 4 \end{pmatrix}$, find $|A|$ and $|A^T|$.

Comment upon your result.

5 Use Cramer's rule to solve

(a) $2x - 3y + z = 0$
$5x + 4y + z = 10$
$2x - 2y - z = -1$

(b) $3x + y = -1$
$2x - y + z = -1$
$5x + 5y - 7z = -16$

(c) $4x + y + z = 13$
$2x - y = 4$
$x + y - z = -3$

(d) $3x + 2y = 1$
$x + y - z = 1$
$2x + 3z = -1$

6 Given

$$A = \begin{pmatrix} 3 & 7 & 6 \\ -2 & 1 & 0 \\ 4 & 2 & -5 \end{pmatrix}$$

(a) find $|A|$

(b) find the cofactors of the elements of row 2, that is $-2, 1, 0$

(c) calculate

$-2 \times$ (cofactor of -2)
$+1 \times$ (cofactor of 1)
$+0 \times$ (cofactor of 0).

What do you deduce?

7 If $\mathbf{a} = 7\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ find $\mathbf{a} \times \mathbf{b}$.

8 Find $\mathbf{a} \times \mathbf{b}$ when

(a) $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 7\mathbf{k}$

(c) $\mathbf{a} = -7\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$

THE INVERSE OF A 3 × 3 MATRIX

Given a 3×3 matrix, A, its inverse is found as follows:

(1) Find the transpose of A, by interchanging the rows and columns of A.

(2) Replace each element of A^T by its cofactor; by its minor together with its associated place sign. The resulting matrix is known as the **adjoint** of A, denoted adj(A).

(3) Finally, the inverse of A is given by

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Example 28

Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$$

Solution $$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

Replacing each element of A^T by its cofactor, we find

$$\text{adj}(A) = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

The determinant of A is given by

$$\begin{aligned}|A| &= 1\begin{vmatrix}1 & 5\\ 2 & 3\end{vmatrix} - (-2)\begin{vmatrix}3 & 5\\ -1 & 3\end{vmatrix} + 0\begin{vmatrix}3 & 1\\ -1 & 2\end{vmatrix}\\ &= (1)(-7) + (2)(14)\\ &= 21\end{aligned}$$

Therefore,

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{1}{21}\begin{pmatrix}-7 & 6 & -10\\ -14 & 3 & -5\\ 7 & 0 & 7\end{pmatrix}$$

Note that this solution should be checked by forming AA^{-1} to give I.

It is clear that should $|A| = 0$ then no inverse will exist since then the quantity $1/|A|$ is undefined. Recall that such a matrix is said to be singular.

For any square matrix A, the following statements are equivalent:

$|A| = 0$

A is singular

A has no inverse

EXERCISES

1 Find $\operatorname{adj}(A)$, $|A|$ and, if it exists, A^{-1}, if

(a) $A = \begin{pmatrix}2 & -3 & 1\\ 5 & 4 & 1\\ 2 & -2 & -1\end{pmatrix}$

(b) $A = \begin{pmatrix}3 & 1 & 0\\ 2 & -1 & 1\\ 5 & 5 & -7\end{pmatrix}$

(c) $A = \begin{pmatrix}2 & -1 & 4\\ 5 & -2 & 9\\ 3 & 2 & -1\end{pmatrix}$

2 If $P = \begin{pmatrix}10 & -5 & -4\\ -5 & 10 & -3\\ -4 & -3 & 8\end{pmatrix}$, find $\operatorname{adj}(P)$ and $|P|$.

Deduce P^{-1}.

APPLICATION TO THE SOLUTION OF SIMULTANEOUS EQUATIONS

The matrix techniques we have developed allow the solution of simultaneous equations to be found in a systematic way.

Example 29

Use a matrix method to solve the simultaneous equations

$$2x + 4y = 14$$
$$x - 3y = -8$$

Solution We first note that the system of equations can be written in matrix form as follows:

$$\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \end{pmatrix}$$

To understand this expression it is necessary that matrix multiplication has been fully mastered, for, by multiplying out the l.h.s., we find

$$\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 1x - 3y \end{pmatrix}$$

We can write it as

$$AX = B$$

where A is the matrix $\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}$, X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ and B is the matrix $\begin{pmatrix} 14 \\ -8 \end{pmatrix}$.

In order to find $X = \begin{pmatrix} x \\ y \end{pmatrix}$ it is now necessary to make X the subject of the equation

AX=B . We can premultiply it by A−1, the inverse of A, provided such an inverse exists, to give

$$A^{-1}AX = A^{-1}B$$

Then, noting that $A^{-1}A = I$, we find

$$IX = A^{-1}B$$

that is

$$X = A^{-1}B$$

In this case

$$A^{-1} = \frac{1}{-10}\begin{pmatrix} -3 & -4 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3/10 & 2/5 \\ 1/10 & -1/5 \end{pmatrix}$$

and

$$A^{-1}B = \begin{pmatrix} 3/10 & 2/5 \\ 1/10 & -1/5 \end{pmatrix}\begin{pmatrix} 14 \\ -8 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

that is, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, so that $x = 1$ and $y = 3$ is the required solution.

If $AX = B$ then $X = A^{-1}B$ provided A^{-1} exists.

This technique can be applied to three equations in three unknowns in an analogous way.

Example 30

Express the following equations in the form $AX = B$ and hence solve them:

$$\begin{aligned} 3x + 2y - z &= 4 \\ 2x - y + 2z &= 10 \\ x - 3y - 4z &= 5 \end{aligned}$$

Solution Using the rules of matrix multiplication, we find

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$$

which is in the form $AX = B$. The matrix A is called the **coefficient matrix** and is simply the coefficients of x, y and z in the equations. As before,

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= X = A^{-1}B \end{aligned}$$

We must therefore find the inverse of A in order to solve the equations.

To invert A we use the adjoint. If

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 3 & 2 & 1 \\ 2 & -1 & -3 \\ -1 & 2 & -4 \end{pmatrix}$$

and you should verify that adj(A) is given by

$$\text{adj}(A) = \begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

The determinant of A is found by expanding along the first row:

$$\begin{aligned} |A| &= 3\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} \\ &= (3)(10) - (2)(-10) - (1)(-5) \\ &= 30 + 20 + 5 \\ &= 55 \end{aligned}$$

Therefore,

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

Finally, the solution X is given by

$$\begin{aligned} X &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

that is, the solution is $x = 3$, $y = -2$ and $z = 1$.

EXERCISES

1 By expressing the following equations in matrix form and finding an inverse matrix, solve

(a) $4x - 2y = 14$
$2x + y = 5$

(b) $2x - 2y = 0$
$x + 3y = -8$

(c) $8x + 3y = 59$
$-2x + y = -13$

2 Solve the following equations $AX = B$ by finding A^{-1}, if it exists.

(a) $\begin{pmatrix} 6 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 11 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \\ 20 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 17 \end{pmatrix}$

(f) $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Elementary row operations

Suppose we have a general linear system of m equations with n unknowns labelled $x_1, x_2, x_3, \ldots$ and x_n given by:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where the coefficients a_{ij}, b_i are real numbers and $x_1, x_2, x_3, \ldots$ and x_n are placeholders for real numbers that satisfy the equations. This general system can be stored in matrix form as

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

This is an **augmented matrix**, which is a matrix containing the coefficients of the unknowns x_1, x_2, $x_3, \ldots, x_n$ and the constant values on the right hand side of the equations. In everyday English language, augmented means ‘to increase’. Augmenting a matrix means adding one or more columns to the original matrix. In this case, we have added the b’s column to the matrix. These are divided by a vertical line as shown above.

Example 1

Consider the following linear system

$$\begin{aligned} x + 2y + 4z &= 7 \\ 3x + 7y + 2z &= -11 \\ 2x + 3y + 3z &= 1 \end{aligned}$$

Write the augmented matrix of this linear system.

Solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 7 \\ 3 & 7 & 2 & -11 \\ 2 & 3 & 3 & 1 \end{array}\right)$$

Note that

Because each row of the augmented matrix corresponds to one equation of the linear system, we can carry out analogous operations such as:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row to another.
3. Interchange rows.

We refer to these operations as **elementary row operations**.

Gaussian elimination

Gauss (1777-1855) is widely regarded as one of the three greatest mathematicians of all time, the others being Archimedes and Newton. By the age of 11, Gauss could prove that $\sqrt{2}$ is irrational. At the age of 18, he constructed a regular 17-sided polygon with a compass and unmarked straight edge only. Gauss went as a student to the world-renowned centre for mathematics – Göttingen. Later in life, Gauss took up a post at Göttingen and published papers in number theory, infinite series, algebra, astronomy and optics. The unit of magnetic induction is named after Gauss.

A linear system of equations is solved by carrying out the above elementary row operations 1, 2 and 3 to find the values of the unknowns $x, y, z, w \ldots$ This method saves time because we do not need to write out the unknowns $x, y, z, w \ldots$ each time, and it is more methodical. In general, you will find there is less likelihood of making a mistake by using this Gaussian elimination process.

Example 2

Solve the following linear system by using the Gaussian elimination procedure:

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned}$$

Solution

What is the augmented matrix in this case?

Let R_1, R_2 and R_3 represent rows 1, 2 and 3 respectively. We have

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned} \quad \text{and} \quad \begin{array}{ll} \text{Row 1} & R_1 \\ \text{Row 2} & R_2 \\ \text{Row 3} & R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Note that each row represents an equation.

How can we find the unknowns x, y and z?

The columns in the matrix represent the x, y and z coefficients respectively. If we can transform this augmented matrix into

$$\begin{array}{rc} & \begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array} \\ \text{Final row} & \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$ where A, B and $*$ represents any real number

then we can find z.

How?

Look at the final row.

What does this represent?

$(0 \times x) + (0 \times y) + (A \times z) = B$

$$Az = B \text{ which gives } z = \frac{B}{A} \text{ provided } A \neq 0$$

Hence we have a value for $z = B/A$.

But how do we find the other two unknowns x and y?

Now we can use a method called **back substitution**. Examine the second row of the above matrix:

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \text{Second row} & \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$

By expanding the second row we get an equation in terms of y and z. From above we already know the value of $z = B/A$, so we can substitute $z = B/A$ and obtain y. Similarly from the first row we can find x by substituting the values of y and z.

We need to perform row operations on the augmented matrix to transform it from:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right) \text{ to } \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right)$$

We need to convert this augmented matrix to an equivalent matrix with zeros in the bottom left hand corner. That is 0 in place of 2, 3 and 1.

How do we get 0 in place of 2?

Remember, we can multiply an equation by a non-zero constant, and take one equation away from another. In terms of matrices, this means that we can multiply a row and take one row away from another because each row represents an equation.

To get 0 in place of 2 we multiple row 1, R_1, by 2 and subtract the result from row 2, R_2; that is, we carry out the row operation $\mathrm{R}_2 - 2\mathrm{R}_1$:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 - 2\,(1) & -1 - (2 \times (-3)) & -3 - (2 \times 5) & 19 - (2 \times (-9)) \\ 3 & 1 & 4 & -13 \end{array}\right)$$

We call the new middle row R_2^*. Completing the arithmetic, the middle row becomes

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Where else do we need a zero?
Need to get a 0 in place of 3 in the bottom row.
How?
We multiply the top row R_1 by 3 and subtract the result from the bottom row R_3; that is, we carry out the row operation, $R_3 - 3R_1$:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* = R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 - 3\,(1) & 1 - (3 \times (-3)) & 4 - (3 \times 5) & -13 - (3 \times (-9)) \end{array}\right)$$

We can call R_3^* the new bottom row of this matrix. Simplifying the arithmetic in the entries gives:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 10 & -11 & 14 \end{array}\right)$$

Note that we now only need to convert the 10 into zero in the bottom row.
How do we get a zero in place of 10?
We can only make use of the bottom two rows, R_2^* and R_3^*.
Why?
Looking at the first column, it is clear that taking any multiple of R_1 away from R_3 will interfere with the zero that we have just worked to establish.

We execute $R_3^* - 2R_2^*$ because

$$10 - (2 \times 5) = 0 \quad \text{(gives a 0 in place of 10)}$$

Therefore we have

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 - (2 \times 0) & 10 - (2 \times 5) & -11 - [2 \times (-13)] & 14 - (2 \times 37) \end{array}\right)$$

which simplifies to

$$
\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**} \end{array}
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 0 & 15 & -60 \end{array}\right)
\end{array}
\qquad (\dagger)
$$

From the bottom row R_3^{**} we have

$$15z = -60 \text{ which gives } z = -\frac{60}{15} = -4$$

How do we find the other two unknowns x and y?
By expanding the middle row R_2^* of (†) we have:

$$5y - 13z = 37$$

We can find y by substituting $z = -4$ into this

$$5y - 13(-4) = 37 \qquad [\text{Substituting } z = -4]$$

$$5y + 52 = 37 \ \text{ which implies } \ 5y = -15 \ \text{ therefore } \ y = -\frac{15}{5} = -3$$

How can we find the last unknown x?
By expanding the first row R_1 of (†) we have:

$$x - 3y + 5z = -9$$

Substituting $y = -3$ and $z = -4$ into this:

$$x - (3 \times (-3)) + (5 \times (-4)) = -9$$

$$x + 9 - 20 = -9 \text{ which gives } x = 2$$

Hence our solution to the linear system is $x = 2, y = -3$ and $z = -4$.

Remember, each of the given equations can also be graphically represented by planes in a 3d coordinate system.

In the above example we transformed the given system:

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x - 3y + 5z &= -9 \\ 5y - 13z &= 37 \\ 15z &= -60 \end{aligned}$$

The system on the right hand side is much easier to solve.

The above process is called **Gaussian elimination** with back substitution. The aim of Gaussian elimination is to produce a 'triangular' matrix with zeros in the bottom left corner of the matrix. This is achieved by the elementary row operations:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row from another.
3. Interchange rows.

Note that

1- We say two matrices are **row equivalent** if one matrix is derived from the other by using these three operations.

2- If augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

3- We can extend these row operations further and obtain the following matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right) \qquad (*)$$

This augmented matrix $(*)$ is said to be in reduced row echelon form.

A matrix is in **reduced row echelon form**, normally abbreviated to **rref**, if it satisfies all the following conditions:

1. If there are any rows containing only zero entries then they are located in the bottom part of the matrix.
2. If a row contains non-zero entries then the first non-zero entry is a 1. This 1 is called a **leading 1**.
3. The leading 1's of two consecutive non-zero rows go strictly from top left to bottom right of the matrix.
4. The only non-zero entry in a column containing a leading 1 is the leading 1.

If condition (4) is *not* satisfied then we say that the matrix is in **row echelon form** and drop the qualification 'reduced'. In some linear algebra literature the leading 1 condition is relaxed and it is enough to say that any non-zero number is the **leading coefficient**.

For example, the following are all in reduced row echelon form:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 8 & 0 \\ 0 & 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -6 \\ 0 & 0 & \boxed{1} & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following matrices are *not* in reduced row echelon form:

$$\mathbf{A} = \begin{pmatrix} 0 & \boxed{1} & 5 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 5 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Why not?

In matrix **A** the third column contains a leading one but has a non-zero entry, 5.
In matrix **B** the leading ones do not go from top left to bottom right.
In matrix **C** the top row of zeros should be relegated to the bottom of the matrix as stated in condition (1) above.

However, matrix **A** is in row echelon form but not in reduced row echelon form. Matrices **B** and **C** are not in row echelon form.

The procedure which places an augmented matrix into row echelon form is called Gaussian elimination and the algorithm which places an augmented matrix into a reduced row echelon form is called **Gauss–Jordan** elimination.

Example 3

Place the augmented matrix $\left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array}\right)$ into reduced row echelon form.

Solution

Why should we want to place this matrix into reduced row echelon form?

In a nutshell, it's to avoid back substitution. If we look at the bottom row of the given augmented matrix we have $5z = -15$.

We need to divide by 5 in order to find the z value.

The reduced row echelon form, rref, gives us the values of the unknowns directly, and we do not need to carry out further manipulation or elimination.

What does reduced row echelon form mean in this case?

It means convert the given augmented matrix

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array} \right) \text{ into something like } \left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right)$$

This means that we need to get 0 in place of the 5 in the second row, and 0's in place of the 5 and -3 in the first row. We also need a 1 in place of -13 in the middle row and 1 in place of the 5 in the bottom row.

How do we convert the 5 in the bottom row into 1?

Divide the last row by 5 (remember, this is the same as multiplying by $1/5$):

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3' = \mathrm{R}_3/5 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 1 & -3 \end{array} \right)$$

How do we get 0 in place of -3 in the first row and the 5 in the second row?

We execute the row operations $\mathrm{R}_1 + 3\mathrm{R}_3'$ and $\mathrm{R}_2 - 5\mathrm{R}_3'$:

$$\begin{array}{c} \mathrm{R}_1^* = \mathrm{R}_1 + 3\mathrm{R}_3' \\ \mathrm{R}_2^* = \mathrm{R}_2 - 5\mathrm{R}_3' \\ \mathrm{R}_3{}' \end{array} \left(\begin{array}{ccc|c} 1+3(0) & 5+3(0) & -3+3(1) & -9+3(-3) \\ 0-5(0) & -13-5(0) & 5-5(1) & 37-5(-3) \\ 0 & 0 & 1 & -3 \end{array} \right)$$

Simplifying the entries gives

$$\begin{array}{c} \mathrm{R}_1^* \\ \mathrm{R}_2^* \\ \mathrm{R}_3' \end{array} \left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & -13 & 0 & 52 \\ 0 & 0 & 1 & -3 \end{array} \right)$$

How do we get a 1 in place of -13?

Divide the middle row by -13 (remember, this is the same as multiplying by $-1/13$):

$$\begin{array}{r} R_1^* \\ R_2^{**} = R_2^*/(-13) \\ R_3' \end{array}\left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & -13/(-13) & 0 & 52/(-13) \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the second row gives

$$\begin{array}{r} R_1^* \\ R_2^{**} \\ R_3' \end{array}\left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

This matrix is now in row echelon form but not in reduced row echelon form. We need to convert the 5 in the top row into 0 to get it into reduced row echelon form.

How?

We carry out the row operation, $R_1^* - 5R_2^{**}$:

$$\begin{array}{r} R_1^{**} = R_1^* - 5R_2^{**} \\ R_2^{**} \\ R_3' \end{array}\left(\begin{array}{ccc|l} 1-5\,(0) & 5-5\,(1) & 0-5\,(0) & -18-5\,(-4) \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the top row entries gives

$$\begin{array}{r} \\ R_1^{**} \\ R_2^{**} \\ R_3' \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right) \end{array}$$

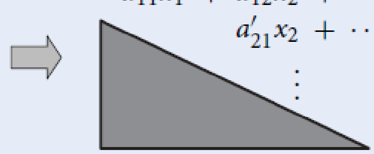
Hence we have placed the given augmented matrix into reduced row echelon form.

Now we can read off the x, y and z values, that is $x = 2$, $y = -4$ and $z = -3$.

Summary

To find the solution to a linear system of m equations by n unknowns we aim to produce:

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & = & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$



$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ & & a'_{21}x_2 & + & \cdots & + & a'_{2n}x_n & = & b'_2 \\ & & \vdots & & & & \vdots & & \vdots \\ & & & & & & a'_{mn}x_n & = & b'_m \end{array}$$

Exercises

1. Solve the following linear system by applying Gaussian elimination with back substitution:

(a) $\begin{aligned} x + y &= 7 \\ x - 2y &= 4 \end{aligned}$

(b) $\begin{aligned} x + 2y - 3z &= 3 \\ 2x - y - z &= 11 \\ 3x + 2y + z &= -5 \end{aligned}$

(c) $\begin{aligned} 2x + 2y + z &= 10 \\ x - 3y + 4z &= 0 \\ 3x - y + 6z &= 12 \end{aligned}$

(d) $\begin{aligned} x + 2y + z &= 1 \\ 2x + 2y + 3z &= 2 \\ 5x + 8y + 2z &= 4 \end{aligned}$

(e) $\begin{aligned} 10x + y - 5z &= 18 \\ -20x + 3y + 20z &= 14 \\ 5x + 3y + 5z &= 9 \end{aligned}$

2. Solve the following linear system by placing the augmented matrix in row echelon form.

(a) $\begin{aligned} x + 2y + 3z &= 12 \\ 2x - y + 5z &= 3 \\ 3x + 3y + 6z &= 21 \end{aligned}$

(b) $\begin{aligned} 2x - y - 4z &= 0 \\ 3x + 5y + 2z &= 5 \\ 4x - 3y + 6z &= -16 \end{aligned}$

(c) $\begin{aligned} 3x - y + 7z &= 9 \\ 5x + 3y + 2z &= 10 \\ 9x + 2y - 5z &= 6 \end{aligned}$

3. Solve the following linear system by placing the augmented matrix in reduced row echelon form.

(a) $\begin{aligned} x + y + 2z &= 9 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$

(b) $\begin{aligned} x + y + z &= -2 \\ 2x - y - z &= -4 \\ 4x + 2y - 3z &= -3 \end{aligned}$

(c) $\begin{aligned} 2x + y - z &= 2 \\ 4x + 3y + 2z &= -3 \\ 6x - 5y + 3z &= -14 \end{aligned}$

(d) $\begin{aligned} -2x + 3y - 2z &= 8 \\ -x + 2y - 10z &= 0 \\ 5x - 7y + 4z &= -20 \end{aligned}$

Notes

Physically, vectors express magnitude as well as direction. Scalars only define magnitude.

$\mathbb{R}^n$ is n-space where n is a natural number such as 1, 2, 3, 4, 5, . . . Vectors in $\mathbb{R}^n$ are denoted by

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

We can linearly combine vectors by applying the two fundamental operations of linear algebra – scalar multiplication and vector addition.

Generally, if $\mathbf{A}$ and $\mathbf{B}$ are matrices for which multiplication is a valid operation then

$$\mathbf{AB} \neq \mathbf{BA} \qquad [\text{not equal}]$$

We cannot blindly apply the properties of real number algebra to matrix algebra.
Let $\mathbf{A}$ be a square matrix then

$$\mathbf{A}^k = \underbrace{\mathbf{A} \times \mathbf{A} \times \mathbf{A} \times \cdots \times \mathbf{A}}_{k \text{ copies}}$$

The properties of the transpose matrices are given by:

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}; \quad (k\mathbf{A})^T = k\mathbf{A}^T; \quad (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T; \quad (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

A square matrix $\mathbf{A}$ is said to be invertible if there is a matrix $\mathbf{B}$ of the same size such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices then we have

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}; \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}; \quad \left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$$

Types of solutions

There are three types of solutions to a linear system:
1. No solution
2. Unique solution
3. Infinite number of solutions

If the system has no solution, we say it is **inconsistent**.

Example 4

Show that the following linear system is inconsistent:

$$\begin{aligned} x + y + 2z &= 3 \\ -x + 3y - 5z &= 7 \\ 2x - 2y + 7z &= 1 \end{aligned}$$

Solution

The augmented matrix with each row labelled is given by

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ -1 & 3 & -5 & 7 \\ 2 & -2 & 7 & 1 \end{array}\right)$$

We use Gaussian elimination to simplify the problem. To get a 0 in place of -1 in the second row and the 2 in the bottom row we execute operations, $\mathrm{R}_2 + \mathrm{R}_1$ and $\mathrm{R}_3 - 2\mathrm{R}_1$, respectively.

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 + \mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 - 2\mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ -1+1 & 3+1 & -5+2 & 7+3 \\ 2-(2\times 1) & -2-(2\times 1) & 7-(2\times 2) & 1-(2\times 3) \end{array}\right)$$

Simplifying the arithmetic in the entries gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^* \end{array} \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 4 & -3 & 10 \\ 0 & -4 & 3 & -5 \end{array}\right)$$

Adding the bottom two rows, $\mathrm{R}_3^* + \mathrm{R}_2^*$, and simplifying gives:

$$\begin{array}{rc} & \begin{array}{ccc} x & y & z \end{array} \\ \mathrm{R}_1 & \\ \mathrm{R}_2^* & \left(\begin{array}{ccr|r} 1 & 1 & 2 & 3 \\ 0 & 4 & -3 & 10 \\ 0 & 0 & 0 & 5 \end{array}\right) \\ \mathrm{R}_3^{**} = \mathrm{R}_3^* + \mathrm{R}_2^* & \end{array}$$

Expanding the bottom row, R_3^{**}, we have $0x + 0y + 0z = 5$ which means that $0 = 5$.
Clearly 0 cannot equal 5, therefore the given linear system is inconsistent.

Note

In general, if a linear system leads to

$$0x_1 + 0x_2 + 0x_3 + \cdots 0x_n = b$$

where $b \neq 0$ (b is not zero) then the system is inconsistent and does not have a solution.

A linear system $(\mathbf{A} \mid \mathbf{b})$ has no solution if and only if the row equivalent $\left(\mathbf{R} \mid \mathbf{b}'\right)$ has no solution.

What does the term homogeneous mean?

A linear system is called **homogeneous** if all the constant terms on the right hand side are zero; that is, $\mathbf{Ax} = \mathbf{O}$:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Non-homogeneous systems

A *non*-homogeneous linear system is defined as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{O}$:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Here, at least one of the b's must take a value other than zero.

- If the consistent linear system $\mathbf{Ax} = \mathbf{b}$ has more unknowns than non-zero equations in reduced row echelon form then this system has an infinite number of solutions.

Example 5

Solve the following non-homogeneous linear system using Gaussian elimination:

$$\begin{array}{rrrrrcr} x & -y & +2z & & +3u & = & 1 \\ -x & +y & & +2w & -5u & = & 5 \\ x & -y & +4z & +2w & +4u & = & 13 \\ -2x & +2y & -5z & -w & -3u & = & -1 \end{array}$$

Solution

The augmented matrix is

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{rrrrr|r} 1 & -1 & 2 & 0 & 3 & 1 \\ -1 & 1 & 0 & 2 & -5 & 5 \\ 1 & -1 & 4 & 2 & 4 & 13 \\ -2 & 2 & -5 & -1 & -3 & -1 \end{array}\right)$$

Note, that when a particular unknown does not exist in the equation, we place a 0 in the coefficient part. To transform this matrix into (reduced) row echelon form, we execute the following elementary row operations:

$$\begin{array}{r} R_1 \\ R_2^* = R_2 + R_1 \\ R_3^* = R_3 - R_1 \\ R_4^* = R_4 + 2R_1 \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ -1+1 & 1-1 & 0+2 & 2+0 & -5+3 & 5+1 \\ 1-1 & -1-(-1) & 4-2 & 2-0 & 4-3 & 13-1 \\ -2+2\,(1) & 2+2\,(-1) & -5+2\,(2) & -1+2\,(0) & -3+2\,(3) & -1+2\,(1) \end{array}\right)$$

Simplifying this gives

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^* \\ R_4^* \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & -2 & 6 \\ 0 & 0 & 2 & 2 & 1 & 12 \\ 0 & 0 & -1 & -1 & 3 & 1 \end{array}\right)$$

To obtain 0's in the bottom rows we have to carry out the following row operations:

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - R_2^* \\ R_4^{**} = 2R_4^* + R_2^* \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & -2 & 6 \\ 0-0 & 0-0 & 2-2 & 2-2 & 1-(-2) & 12-6 \\ 2\,(0)+0 & 2\,(0)+0 & 2\,(-1)+2 & 2\,(-1)+2 & 2\,(3)+(-2) & 2\,(1)+6 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & -2 & 6 \\ 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 4 & 8 \end{array}\right)$$

Dividing the third row by 3 and the fourth row by 4 (or multiplying by 1/3 and 1/4 respectively) gives

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^{\dagger} = R_3^{**}/3 \\ R_4^{\dagger} = R_4^{**}/4 \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & -2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Subtracting the last two rows from each other and dividing the second row R_2^* by 2 we have

$$\begin{array}{r} \\ R_1 \\ R_2^*/2 \\ R_3^{\dagger} \\ R_4^{\dagger} - R_3^{\dagger} \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & u \end{array} \\ \left(\begin{array}{ccccc|c} \boxed{1} & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \qquad (\dagger\dagger)$$

From the third row $R_3^{\dagger}$ we have $u = 2$. By expanding the second row $R_2^*/2$ we have

$$\begin{aligned} &z + w - u = 3 \\ &z + w - 2 = 3 \qquad [\text{substituting } u = 2] \\ &z + w = 5 \text{ which gives } z = 5 - w \end{aligned}$$

Let $w = t$ where t is an arbitrary real number then $z = 5 - t$.
From the first row R_1 we have

$$\begin{aligned} x - y + 2z + 3u &= 1 \\ x - y + 2(5 - t) + 3(2) &= 1 \qquad [\text{substituting } z = 5 - t \text{ and } u = 2] \\ x - y + 10 - 2t + 6 &= 1 \qquad [\text{simplifying}] \\ x &= y + 2t - 15 \end{aligned}$$

Let $y = s$, where s is an any real number, then

$$x = s + 2t - 15$$

Hence our solution to the given linear system is

$$x = s + 2t - 15, \quad y = s, \quad z = 5 - t, \quad w = t \text{ and } u = 2$$

This is the general solution and you can find particular values by substituting any real numbers for s and t. We have an infinite number of solutions.

Exercises

1. Determine whether the following linear systems have a unique, infinitely many or no solutions. Also determine the solutions in the case of the system consisting of infinitely many and unique solutions.

(a) $\begin{aligned} x + 3y + 2z &= 5 \\ 2x - y - z &= 1 \\ -x + 2y + z &= 3 \end{aligned}$

(b) $\begin{aligned} -x + y + z &= 0 \\ 3x - 2y + 5z &= 0 \\ 4x - y - 2z &= 0 \end{aligned}$

(c) $\begin{aligned} -x + y + z &= 2 \\ 2x + 2y + 3z &= 5 \\ 6x + 6y + 9z &= 7 \end{aligned}$

(d) $\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 4 \\ 3x + 3y - 3z &= 6 \end{aligned}$

(e) $\begin{aligned} 3x - 3y - z + 2w &= 0 \\ 6x - 7y + z + w &= 0 \\ x - y - 2z - w &= 0 \\ 2x - 2y + 6z + 8w &= 0 \end{aligned}$

(f) $\begin{aligned} 2x + 3y + 5z + 2w &= 6 \\ 2x + 3y + 2z + 2w &= 7 \\ 8x + 12y + 20z + 8w &= 24 \\ x + 2y + 4z + 5w &= 6 \end{aligned}$

The Inverse Matrix Method

You can find the inverse matrix of $\mathbf{A}$ by transforming the augmented matrix $(\mathbf{A} \mid \mathbf{I})$ into the augmented matrix $(\mathbf{I} \mid \mathbf{A}^{-1})$ by elementary row operations.

If in the process of this you get a row of zeros then the matrix $\mathbf{A}$ is non-invertible.

Note

$$(\mathbf{A} \mid \mathbf{I}) \times \mathbf{A}^{-1} = \left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$$

This means that we convert $(\mathbf{A} \mid \mathbf{I})$ to $\left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$. Hence the row operations that transform matrix $\mathbf{A}$ into $\mathbf{I}$, also transform $\mathbf{I}$ into $\mathbf{A}^{-1}$.

Example 6

Determine the inverse matrix $\mathbf{A}^{-1}$ given that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$.

Solution

Above we established that the row operations for transforming $\mathbf{A}$ into the identity matrix $\mathbf{I}$ are the same as for transforming the identity matrix $\mathbf{I}$ into the inverse matrix $\mathbf{A}^{-1}$, therefore we carry out the row operations simultaneously. This is achieved by transforming the augmented matrix $(\mathbf{A} \mid \mathbf{I})$ into the augmented matrix $\left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$. Labelling the rows:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) = (\mathbf{A} \mid \mathbf{I})$$

Remember that R_1, R_2 and R_3 represent the first, second and third rows respectively.

Our aim is to convert the given matrix $\mathbf{A}$ (left) into the identity matrix $\mathbf{I}$.

How?

Need to convert the 2 and 4 in the bottom two rows into zeros by executing the following row operations:

$$\begin{array}{r} R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 4R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2-2(1) & -1-2(0) & 3-2(2) & 0-2(1) & 1-2(0) & 0-2(0) \\ 4-4(1) & 1-4(0) & 8-4(2) & 0-4(1) & 0-4(0) & 1-4(0) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right)$$

We interchange the bottom two rows, R_2' and R_3':

$$\begin{array}{c} R_1 \\ R_2^* = R_3' \\ R_3^* = R_2' \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{array}\right)$$

Note that we have swapped the rows around so that the second row is in the correct format for the identity matrix on the *left hand side* of the vertical line.

What else do we need for the identity matrix?

Convert the first -1 in the bottom row into zero.

How?

Add the bottom two rows and simplify the entries:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^{**} = R_3^* + R_2^* \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right)$$

The -1 in the bottom row needs to be $+1$. Multiplying the bottom row by -1 yields

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^{\dagger} = -R_3^{**} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

We have nearly arrived at our destination, which is the identity matrix on the left.

We only need to convert the 2 in the top row into 0. Carry out the row operation $R_1 - 2R_3^{\dagger}$:

$$\begin{array}{c} R_1^* = R_1 - 2R_3^{\dagger} \\ R_2^* \\ R_3^{\dagger} \end{array} \left(\begin{array}{ccc|ccc} 1-2(0) & 0-2(0) & 2-2(1) & 1-2(6) & 0-2(-1) & 0-2(-1) \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

Simplifying the arithmetic in the top row gives the identity on the left:

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3^{\dagger} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right) = \left(\mathbf{I} \,\middle|\, \mathbf{A}^{-1}\right)$$

Hence the inverse matrix, $\mathbf{A}^{-1}$, is the matrix with the entries on the *right hand side* of the vertical line,

$$\mathbf{A}^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}.$$

Solving linear equations

Consider a general linear system which is written in matrix form as

$$\mathbf{Ax} = \mathbf{b} \qquad [\mathbf{A} \text{ is invertible}]$$

Multiplying this by the inverse matrix $\mathbf{A}^{-1}$ gives

$$\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{=\mathbf{I}}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \text{ implies } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad [\text{because } \mathbf{Ix} = \mathbf{x}]$$

Hence the solution of a linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is invertible is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Example 7

Solve both the linear systems:

(a) $\begin{aligned} x \qquad + 2z &= 5 \\ 2x - y + 3z &= 7 \\ 4x + y + 8z &= 10 \end{aligned}$

(b) $\begin{aligned} x \qquad + 2z &= 1 \\ 2x - y + 3z &= 2 \\ 4x + y + 8z &= 3 \end{aligned}$

Solution

(a) Writing this in matrix form $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$$

The matrix $\mathbf{A}$ is the same matrix as the previous example, therefore the inverse matrix is already evaluated:

$$\mathbf{A}^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

Hence

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix} \qquad \left[\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\right]$$

$$= \begin{pmatrix} -(11 \times 5) + (2 \times 7) + (2 \times 10) \\ -(4 \times 5) + (0 \times 7) + (1 \times 10) \\ (6 \times 5) - (1 \times 7) - (1 \times 10) \end{pmatrix} = \begin{pmatrix} -21 \\ -10 \\ 13 \end{pmatrix}$$

We have $x = -21$, $y = -10$ and $z = 13$. You can check this is the solution to the given linear system by substituting these values into the system.

(b) The coefficient matrix is the same and we have only changed the vector **b**. Hence

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \left[\text{because } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\right]$$

We have $x = -1$, $y = -1$ and $z = 1$.

In the above example, we have solved two linear systems in one go because we had $\mathbf{A}^{-1}$. Once we have the inverse matrix $\mathbf{A}^{-1}$, we can solve a whole sequence of systems such as

$$\mathbf{A}\mathbf{x}_1 = \mathbf{b}_1, \quad \mathbf{A}\mathbf{x}_2 = \mathbf{b}_2, \quad \mathbf{A}\mathbf{x}_3 = \mathbf{b}_3, \ldots, \quad \mathbf{A}\mathbf{x}_k = \mathbf{b}_k \text{ by using } \mathbf{x}_j = \mathbf{A}^{-1}\mathbf{b}_j.$$

Note

Let **A** be an *n* by *n* matrix, then the following five statements are equivalent:

(a) The matrix **A** is invertible (non-singular).

(b) The linear system **Ax** = **O** only has the trivial solution **x** = **O**.

(c) The reduced row echelon form of the matrix **A** is the identity matrix **I**.

(d) **Ax** = **b** has a unique solution.

- **Non-invertible (singular) matrices**

If the above approach of trying to convert *(***A**| **I** *)* into the augmented matrix $(I|A^{-1})$ cannot be achieved then the matrix A is non-invertible (singular). This means that the matrix A does not have an inverse.

We know that the matrix A is invertible (has an inverse)⇔the reduced row echelon form of A is the identity matrix I.

Remember, the reduced row echelon form of an n by n matrix can only: (1) be an identity matrix or (2) have a row of zeros. If we end up with a row of zeros then the given matrix is non-invertible.

<u>Example 8</u>

Show that the matrix $\mathbf{A}$ is non-invertible where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 2 & 5 & 6 & 9 \\ -3 & 1 & 2 & 3 \\ 1 & 13 & -30 & -49 \end{pmatrix}$$

Solution

Writing this in the augmented matrix form $(\mathbf{A} \mid \mathbf{I})$ we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{cccc|cccc} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 2 & 5 & 6 & 9 & 0 & 1 & 0 & 0 \\ -3 & 1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 13 & -30 & -49 & 0 & 0 & 0 & 1 \end{array}\right)$$

We need to convert the 2, -3 and 1 (bottom row) in the first column into 0's.

How?

By executing the following row operations:

$$\begin{array}{r} R_1 \\ R_2^* = R_2 - 2R_1 \\ R_3^* = R_3 + 3R_1 \\ R_4^* = R_4 - R_1 \end{array} \left(\begin{array}{cccc|cccc} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 2-2(1) & 5-2(-2) & 6-2(3) & 9-2(5) & -2 & 1 & 0 & 0 \\ -3+3(1) & 1+3(-2) & 2+3(3) & 3+3(5) & 3 & 0 & 1 & 0 \\ 1-1 & 13-(-2) & -30-3 & -49-5 & -1 & 0 & 0 & 1 \end{array}\right)$$

Simplifying the arithmetic gives

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* \\ R_4^* \end{array} \left(\begin{array}{cccc|cccc} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & -5 & 11 & 18 & 3 & 0 & 1 & 0 \\ 0 & 15 & -33 & -54 & -1 & 0 & 0 & 1 \end{array}\right)$$

Note, that the bottom row is -3 times the third row on the left. Executing $R_4^* + 3R_3^*$:

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^* \\ R_4^{**} = R_4^* + 3R_3^* \end{array} \left(\begin{array}{cccc|cccc} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & -5 & 11 & 18 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 3 & 1 \end{array}\right)$$

Since we have a zero row on the *left hand side* of the vertical line, therefore the reduced row echelon form will have a zero row. We conclude that

$rref$ (A) = R has at least one row of zeros ⇔ A is non-invertible.

The matrix **A** is non-invertible (singular). That is the matrix **A** does not have an inverse.

Note

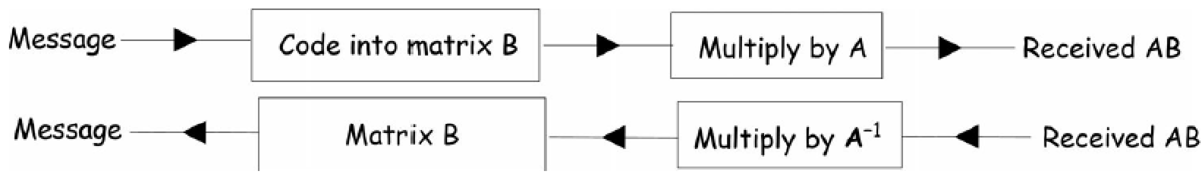
1. If the matrix **A** does not have an inverse then the linear system **Ax** = **b** cannot have a unique solution. We must have an infinite number or no solution.

2. We normally use the compact notation *rref* (**A**) = **R** to mean that the reduced row echelon form of **A** is **R**. This notation *rref (A)* is the command in MATLAB to find the rref of **A**.

Applications to cryptography

Cryptography is the study of communication by stealth. It involves the coding and decoding of messages. This is a growing area of linear algebra applications because agencies such as the CIA use cryptography to encode and decode information.

One way to code a message is to use matrices.



For example, let **A** be an invertible matrix. The message is encrypted into a matrix **B** such that the matrix multiplication **AB** is a valid operation. Send the message generated by the matrix multiplication **AB**. At the other end, they will need to know the inverse matrix $\mathbf{A}^{-1}$ in order to decode the message because $\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{B}$. Remember that the matrix **B** contains the message.

A simple way of encoding messages is to represent each letter of the alphabet by its position in the alphabet and then add 3 to this. For example, we can create the following Table

Alphabet	A	B	C	D	...	W	X	Y	Z	
Position	1	2	3	4	...	23	24	25	26	27
Position +3	4	5	6	7	...	26	27	28	29	30

The final column represents space and we nominate this by a value of 27 + 3 = 30

Example 9

Encode the message 'OPERATION BLUESTAR' by using matrix $\mathbf{A}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$$

Solution

Using the above table we have the numerical values as shown in Table 1.4.

Table 1.4

O	P	E	R	A	T	I	O	N		B	L	U	E	S	T	A	R
18	19	8	21	4	23	12	18	17	30	5	15	24	8	22	23	4	21

Since the matrix $\mathbf{A}$ is a 3 by 3 matrix, we write these numbers in 3 by 1 vectors so that matrix multiplication is a valid operation. Hence

$$\begin{pmatrix} 18 \\ 19 \\ 8 \end{pmatrix}, \begin{pmatrix} 21 \\ 4 \\ 23 \end{pmatrix}, \begin{pmatrix} 12 \\ 18 \\ 17 \end{pmatrix}, \begin{pmatrix} 30 \\ 5 \\ 15 \end{pmatrix}, \begin{pmatrix} 24 \\ 8 \\ 22 \end{pmatrix} \text{ and } \begin{pmatrix} 23 \\ 4 \\ 21 \end{pmatrix}$$

Putting these together as column vectors of matrix $\mathbf{B}$ so that we can multiply by $\mathbf{A}$ in one go:

$$\mathbf{B} = \begin{pmatrix} 18 & 21 & 12 & 30 & 24 & 23 \\ 19 & 4 & 18 & 5 & 8 & 4 \\ 8 & 23 & 17 & 15 & 22 & 21 \end{pmatrix}$$

Multiplying this matrix by $\mathbf{A}$ (use software to do this):

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 18 & 21 & 12 & 30 & 24 & 23 \\ 19 & 4 & 18 & 5 & 8 & 4 \\ 8 & 23 & 17 & 15 & 22 & 21 \end{pmatrix} = \begin{pmatrix} 80 & 98 & 99 & 85 & 106 & 94 \\ 71 & 92 & 76 & 95 & 100 & 92 \\ 124 & 163 & 140 & 160 & 176 & 161 \end{pmatrix}$$

These numbers in the columns of matrix $\mathbf{AB}$ are transmitted, that is the encoded message is:

$$80, 71, 124, 98, 92, 163, 99, 76, 140, 85, 95, 160, 106, 100, 176, 94, 92, 161$$

By examining these numbers you cannot say they are related to the alphabet.

How do we decode this message?

We need to multiply by the inverse matrix. By using software the inverse matrix $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 5 & -4 \\ -1 & -4 & 3 \end{pmatrix}$$

To decode the message we need to find $\mathbf{A}^{-1}(\mathbf{AB})$, which is evaluated by using software:

$$\mathbf{A}^{-1}(\mathbf{AB}) = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 5 & -4 \\ -1 & -4 & 3 \end{pmatrix} \begin{pmatrix} 80 & 98 & 99 & 85 & 106 & 94 \\ 71 & 92 & 76 & 95 & 100 & 92 \\ 124 & 163 & 140 & 160 & 176 & 161 \end{pmatrix} = \begin{pmatrix} 18 & 21 & 12 & 30 & 24 & 23 \\ 19 & 4 & 18 & 5 & 8 & 4 \\ 8 & 23 & 17 & 15 & 22 & 21 \end{pmatrix}$$

The columns of this matrix $\mathbf{A}^{-1}(\mathbf{AB})$ are the entries:

$$18, 19, 8, 21, 4, 23, 12, 18, 17, 30, 5, 15, 24, 8, 22, 23, 4, 21$$

Using the above Table backwards, we can read the message as ‘OPERATION BLUESTAR’.

Note: We use cryptography all the time. For example, emails, websites, ATM cards and digital passwords are all protected by encryption.

Exercises

1- Find the inverse matrix, if it exists, by using row operations:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & -5 \\ -6 & 1 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 3 & 6 & 0 \end{pmatrix}$

(d) $\mathbf{D} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ (e) $\mathbf{E} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

(f) $\mathbf{F} = \begin{pmatrix} 1 & 3 & 4 \\ -1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ (g) $\mathbf{G} = \begin{pmatrix} -2 & 5 & 3 & 1 \\ -9 & 2 & -5 & 6 \\ 2 & 4 & 8 & 16 \\ 4 & 8 & 16 & 32 \end{pmatrix}$

2- Determine whether the following linear systems have a unique solution by using the inverse matrix method. Also find the solutions in the unique case.

(a) $\begin{aligned} x + 2y &= 3 \\ -x + 4y &= 5 \end{aligned}$ (b) $\begin{aligned} 2x - 5y &= 3 \\ -6x + y &= -1 \end{aligned}$

$$\text{(c)}\quad \begin{aligned} x \qquad\quad + 2z &= -1 \\ 2x + 3y + z \quad &= 1 \\ 3x + 6y \qquad &= 9 \end{aligned} \qquad\qquad \text{(d)}\quad \begin{aligned} x - y + z &= 10 \\ x \qquad - z &= 3 \\ -z &= 5 \end{aligned}$$

$$\text{(e)}\quad \begin{aligned} 2x - y \qquad &= 5 \\ -x + 2y - z &= 7 \\ - y + 2z &= 3 \end{aligned} \qquad\qquad \text{(f)}\quad \begin{aligned} x + 3y + 4z &= -2 \\ -x + y + z &= 3 \\ 2x + y - 2z &= 6 \end{aligned}$$

3- By using the following Table

Alphabet	A	B	C	D	. . .	W	X	Y	Z	
Position	1	2	3	4	. . .	23	24	25	26	27
Position +3	4	5	6	7	. . .	26	27	28	29	30

encode the following messages using the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

(a) ATTACK (b) IS YOUR PARTNER HOME

4- Obtain a Gaussian array from the following set of equations and transform it to reduced row echelon form to obtain any solutions:

$$\begin{aligned} x + y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$$

5- Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}, \text{ and let } \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Reduce the augmented matrix for the system **Ax** = **b** to reduced echelon form.
(b) Write the solution set for the system **Ax** = **b** in parametric vector form.

6- Find the general solution in vector form to

$$\begin{aligned} x_1 - x_2 - 2x_3 - 8x_4 &= -3 \\ -2x_1 + x_2 + 2x_3 + 9x_4 &= 5 \\ 3x_1 - 2x_2 - 3x_3 - 15x_4 &= -9 \end{aligned}$$

using the Gaussian elimination method.

7- Show that the following system of linear equations is inconsistent.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 5x_4 &= -13 \\ 3x_1 + x_2 + 4x_3 - 4x_4 &= 5 \\ 2x_1 - 2x_2 + 3x_3 - 10x_4 &= 7 \\ x_1 + x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

8- Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 & 2 \\ -1 & -1 & 2 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

Find rref(A).

Eigenvalues

and Eigenvectors

Eigenvector/value problems crop up frequently in the physical sciences and engineering. They take the form Av =(scalar) x v where v is a non-zero vector and A is a square matrix. By knowing the eigenvalues and eigenvectors of a matrix we can easily find its determinant, decide whether the matrix has an inverse and determine the powers of the matrix. For an example of linear algebra at work, one needs to look no further than Google's search engine, which relies upon eigenvalues and eigenvectors to rank pages with respect to relevance.

1. Definition of eigenvalues and eigenvectors

Before we define what is meant by an eigenvalue and an eigenvector let's do an example which involves them.

Example 1

Let $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ then evaluate $\mathbf{Au}$.

Solution

Multiplying the matrix $\mathbf{A}$ and vector $\mathbf{u}$ we have

$$\mathbf{Au} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

What do you notice about the result?

We have $\mathbf{Au} = 3\mathbf{u}$. The matrix $\mathbf{A}$ scalar multiplies the vector $\mathbf{u}$ by 3, as shown in Fig. 1.
In general terms, this can be written as

$\mathbf{Au} = \lambda\mathbf{u}$ (matrix $\mathbf{A}$ scalar multiplies vector $\mathbf{u}$) (1)

where $\mathbf{A}$ is a square matrix, $\mathbf{u}$ is a vector and the Greek letter λ (lambda) is a scalar.

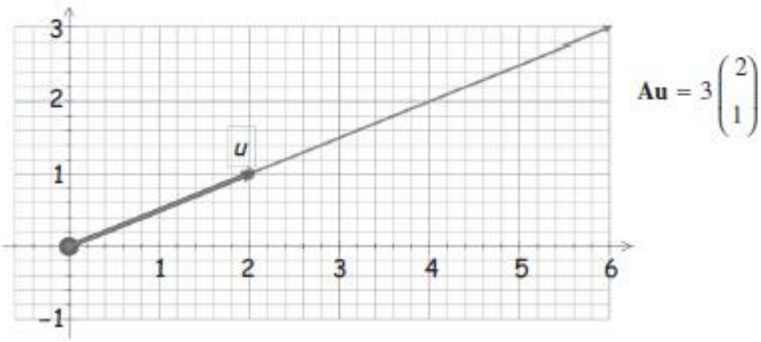


Fig. 1.

This is an important result which is used throughout this chapter and well worth becoming familiar with.

Why is formula (1) important?

Because the matrix A transforms the vector u by scalar multiplying it, which means that the transformation only changes the length of the vector u unless $\lambda = \pm 1$ (in which case the length remains unchanged). Note that (1) says that the matrix A applied to u gives a vector in the same or opposite (negative λ) direction of u.

Can you think of a vector, u, which satisfies equation (1)?

The zero vector u = O because AO = λO = O. In this case we say that we have the trivial solution u = O. In this chapter we consider the non-trivial solutions, u $\neq$ O (not zero), and these solutions are powerful tools in linear algebra.

For a non-zero vector u the scalar λ is called an eigenvalue of the matrix A and the vector u is called an eigenvector belonging to or corresponding to λ, which satisfies Au = λu.

In most linear algebra literature, the Greek letter lambda, λ, is used for eigenvalues. These terms eigenvalue and eigenvector are derived from the German word ‘Eigenwert’ which means ‘proper value’. The word eigen is pronounced ‘i-gun’.

Eigenvalues were initially developed in the field of differential equations by Jean d’Alembert.

Example 2

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$. Verify the following:

(a) $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of matrix $\mathbf{A}$ belonging to the eigenvalue $\lambda_1 = 2$.

(b) $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of matrix $\mathbf{A}$ belonging to the eigenvalue $\lambda_2 = 3$.

Solution

(a) Multiplying the given matrix $\mathbf{A}$ and vector $\mathbf{u}$ we have

$$\mathbf{Au} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus $\mathbf{u} = (1 \quad 1)^T$ is an eigenvector of the matrix $\mathbf{A}$ belonging to $\lambda_1 = 2$ because $\mathbf{Au} = 2\mathbf{u}$. Matrix $\mathbf{A}$ doubles the vector $\mathbf{u}$.

(b) Similarly we have

$$\mathbf{Av} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus $\mathbf{v} = (1 \quad 2)^T$ is an eigenvector of the matrix $\mathbf{A}$ belonging to $\lambda_2 = 3$ because $\mathbf{Av} = 3\mathbf{v}$. This $\mathbf{Av} = 3\mathbf{v}$ means matrix $\mathbf{A}$ triples the vector $\mathbf{v}$.

What do you notice about your results?
A 2 by 2 matrix can have more than one eigenvalue and eigenvector.

We have eigenvalues λ and eigenvectors u for any square matrix A such that Au = λ u.

<u>Example 3</u>

Let $\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Show that the matrix $\mathbf{A}$ scalar multiplies the vector $\mathbf{u}$ and find the value of this scalar, λ, the eigenvalue.

Solution

Applying the matrix $\mathbf{A}$ to the vector $\mathbf{u}$ we have

$$\mathbf{Au} = \begin{pmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

We have $\mathbf{Au} = 2\mathbf{u}$ so $\lambda = 2$. Hence $\lambda = 2$ is an eigenvalue of the matrix $\mathbf{A}$ with an eigenvector $\mathbf{u}$. Matrix $\mathbf{A}$ transforms the vector $\mathbf{u}$ by a scalar multiple of 2 because $\mathbf{Au} = 2\mathbf{u}$.

2. Characteristic equation

From the above formula (1) $\mathbf{Au} = \lambda\mathbf{u}$ we have

$$\mathbf{Au} = \lambda\mathbf{Iu}$$

[$\lambda\mathbf{Iu} = \lambda\mathbf{u}$ where multiplying by the identity keeps it the same]
where $\mathbf{I}$ is the identity matrix. We can rewrite this as

$$\mathbf{Au} - \lambda\mathbf{Iu} = \mathbf{O}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$$

Under what condition is the non-zero vector u a solution of this equation?

We know that

$$Ax = O \text{ has an infinite number of solutions} \Leftrightarrow \det(A) = 0.$$

Applying this result to $(A - \lambda I)u = O$ means that we must have a non-zero vector u (because there are an infinite number of solutions which satisfy this equation) $\Leftrightarrow$

$$\det(A - \lambda I) = 0$$

This is an important equation because we use this to find the eigenvalues and it is called the characteristic equation:

$$\det(A - \lambda I) = 0 \tag{2}$$

<u>The procedure for determining eigenvalues and eigenvectors is:</u>
1. Solve the characteristic equation (2) for the scalar λ.
2. For the eigenvalue λ determine the corresponding eigenvector u by solving the system $(A - \lambda I)u = O$.

Let's follow this procedure for the next example.

Note that eigenvalues and eigenvectors come in pairs. You cannot have one without the other. It is a relationship like mother and child because eigenvalues give birth to eigenvectors.

Example 4

Determine the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Also sketch the effect of multiplying the eigenvectors by matrix $\mathbf{A}$.

Solution

What do we find first, the eigenvalues or eigenvectors?

Eigenvalues, because they produce eigenvectors.

We carry out the above procedure:

Step 1.

We need to find the values of λ which satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. First we obtain $\mathbf{A} - \lambda\mathbf{I}$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix}$$

Substituting this into $\det(\mathbf{A} - \lambda\mathbf{I})$ gives

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix}$$

To find the determinant, we use formula (6.1), $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, thus:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 0$$

For eigenvalues we equate this determinant to zero:

$$(2-\lambda)(3-\lambda) = 0 \quad \text{implies } \lambda_1 = 2 \text{ or } \lambda_2 = 3$$

Step 2.

For each eigenvalue, λ determine the corresponding eigenvector $\mathbf{u}$ by solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$.

Let $\mathbf{u}$ be the eigenvector corresponding to $\lambda_1 = 2$. Substituting $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\lambda_1 = \lambda = 2$ into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\mathbf{u} = \mathbf{O}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{u} = \mathbf{O}$$

Remember, $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, so we have

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out gives

$$0 + 0 = 0$$
$$x + y = 0$$

Remember, the eigenvector *cannot* be the zero vector, therefore at least one of the values, x or y, must be non-zero. From the bottom equation we have $x = -y$.

The simplest solution is $x = 1$, $y = -1$ but we could have $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $\begin{pmatrix} \pi \\ -\pi \end{pmatrix}$, $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$, ...

Hence we have an infinite number of eigenvectors belonging to $\lambda = 2$. We can write down the general eigenvector $\mathbf{u}$.

How?

Let $x = s$ then $y = -s$ where $s \neq 0$ and is a real number. Thus the eigenvectors belonging to $\lambda = 2$ are

$$\mathbf{u} = \begin{pmatrix} s \\ -s \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ where } s \neq 0 \qquad \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is one of the simplest eigenvectors}\right]$$

Similarly, we find the general eigenvector $\mathbf{v}$ belonging to the other eigenvalue $\lambda_2 = 3$. Putting $\lambda_2 = \lambda = 3$ into $[\mathbf{A} - \lambda\mathbf{I}]\,\mathbf{v} = \mathbf{O}$ gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\mathbf{v} = \mathbf{O} \text{ simplifies to } \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\mathbf{v} = \mathbf{O}$$

By writing $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ [different x and y from those above] and $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we obtain

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out:

$$-x + 0 = 0, \quad x + 0 = 0$$

From these equations we must have $x = 0$.

What is y equal to?

We can choose y to be any real number apart from zero because the eigenvector cannot be zero. Thus

$$y = s \text{ where } s \neq 0$$

The general eigenvector belonging to $\lambda_2 = 3$ is

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ where } s \neq 0 \quad \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is one of the simplest eigenvectors}\right]$$

Summarizing the above we have:

Eigenvector $\mathbf{u} = s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ belonging to $\lambda_1 = 2$ and eigenvector $\mathbf{v} = s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ belonging to $\lambda_2 = 3$.

What does all this mean?

The given matrix $\mathbf{A}$ scalar multiplies the eigenvector $\mathbf{u}$ by 2 and $\mathbf{v}$ by 3 because

$$\mathbf{Au} = 2\mathbf{u} \text{ and } \mathbf{Av} = 3\mathbf{v}$$

Plotting these eigenvectors, the effect of multiplying by the matrix **A** is shown in Fig. 2.

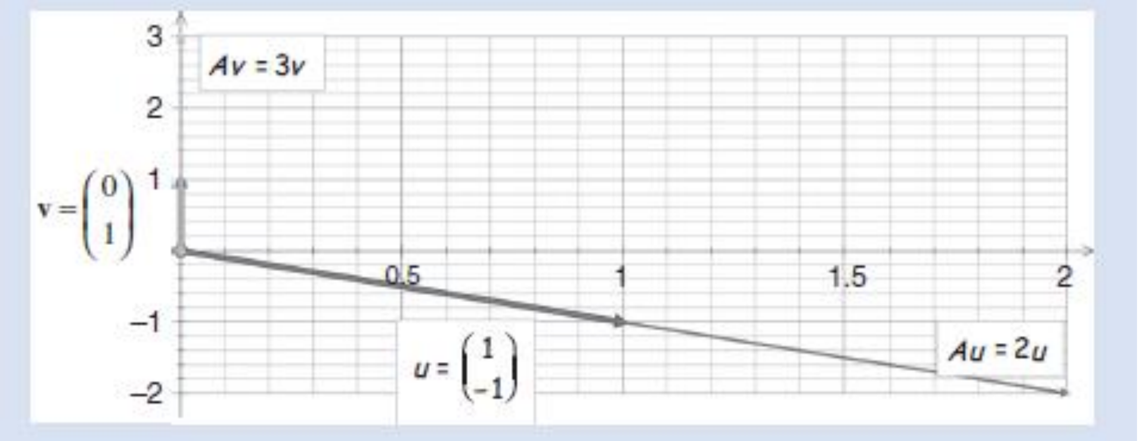


Fig. 2.

We note from Fig. 2. that matrix **A** doubles $(\lambda 1 = 2)$ the eigenvector **u** and triples $(\lambda 2 = 3)$ the eigenvector **v** as you can see in Fig. 2 Matrix **A** does *not* change the direction of the eigenvectors.

Eigenvectors are non-zero vectors which are transformed by the matrix A to a scalar multiple λ of itself.

Next, we find the eigenvalues and eigenvectors of a 3 by 3 matrix. Follow the algebra carefully because you will have to expand brackets like $(1 - \lambda)(-3 - \lambda)$.

To expand this, it is usually easier to take out two minus signs and then expand, that is:

$$(1 - \lambda)(-3 - \lambda) = --(-1 + \lambda)(3 + \lambda) = (\lambda - 1)(3 + \lambda) \text{ [Because } -- = +]$$

<u>Example 5</u>

Determine the eigenvalues of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix}$

Solution

We have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 4-\lambda & 0 \\ 3 & 5 & -3-\lambda \end{pmatrix}$$

It is easier to remember that $\mathbf{A} - \lambda\mathbf{I}$ is actually matrix $\mathbf{A}$ with $-\lambda$ along the leading diagonal (from top left to bottom right). We need to evaluate $\det(\mathbf{A} - \lambda\mathbf{I})$.

What is the simplest way to find $\det(\mathbf{A} - \lambda\mathbf{I})$*?*

From the properties of determinants of the last chapter, we know that it will be easier to evaluate the determinant along the middle row, containing the elements 0, $4 - \lambda$ and 0.

Why?

Because it has two zeros we do *not* have to evaluate the 2 by 2 determinants associated with these zeros. [Spending a second or two in choosing an easy way forward can really help save on the arithmetic later on.] From above we have

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 4-\lambda & 0 \\ 3 & 5 & -3-\lambda \end{pmatrix} \longleftarrow \text{middle row}$$

$$= (4-\lambda)\left[\det\begin{pmatrix} 1-\lambda & 4 \\ 3 & -3-\lambda \end{pmatrix}\right] \qquad \left[\begin{array}{l}\text{expanding the} \\ \text{middle Row}\end{array}\right]$$

$$= (4-\lambda)\left[(1-\lambda)(-3-\lambda)-(3\times 4)\right] \qquad [\text{by determinant of 2 by 2}]$$

$$= (4-\lambda)\left[(\lambda-1)(3+\lambda)-12\right] \qquad [\text{taking out minus signs}]$$

$$= (4-\lambda)\left[3\lambda+\lambda^2-3-\lambda-12\right] \qquad [\text{opening brackets}]$$

$$= (4-\lambda)\left[\lambda^2+2\lambda-15\right] \qquad [\text{simplifying}]$$

$$= (4-\lambda)(\lambda+5)(\lambda-3) \qquad [\text{factorizing}]$$

By the characteristic equation (7.2), $\det(\mathbf{A}-\lambda\mathbf{I})=0$, we equate all the above to zero:

$$(4-\lambda)(\lambda+5)(\lambda-3)=0$$

Solving this equation gives the eigenvalues $\lambda_1 = 4$, $\lambda_2 = -5$ and $\lambda_3 = 3$.

Example 6

Determine the eigenvectors associated with $\lambda_3 = 3$ for the matrix **A** given in Example 5.

Solution

Substituting the eigenvalue $\lambda_3 = \lambda = 3$ and the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix}$ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ (subtract 3 from the leading diagonal) gives:

$$(\mathbf{A}-3\mathbf{I})\,\mathbf{u} = \begin{pmatrix} 1-3 & 0 & 4 \\ 0 & 4-3 & 0 \\ 3 & 5 & -3-3 \end{pmatrix}\mathbf{u} = \mathbf{O}$$

where $\mathbf{u}$ is the eigenvector corresponding to $\lambda_3 = 3$.

What is the zero vector, $\mathbf{O}$, equal to?

Remember, this zero vector is $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Let $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Substituting these into the above and simplifying gives

$$\begin{pmatrix} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 3 & 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding this yields the linear system

$$\begin{aligned} -2x + 0 + 4z &= 0 \qquad (1) \\ 0 + y + 0 &= 0 \qquad (2) \\ 3x + 5y - 6z &= 0 \qquad (3) \end{aligned}$$

From the middle equation (2) we have $y = 0$. From the top equation (1) we have

$$2x = 4z \text{ which gives } x = 2z$$

If $z = 1$ then $x = 2$; or more generally if $z = s$ then $x = 2s$ where $s \neq 0$ [not zero].

The general eigenvector $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 0 \\ s \end{pmatrix} = s \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ where $s \neq 0$ and corresponds to $\lambda_3 = 3$.

The given matrix $\mathbf{A}$ triples the eigenvector $\mathbf{u}$ because $\mathbf{Au} = 3\mathbf{u}$.

Homework: Find the eigenvectors belonging to $\lambda_1 = 4$ and $\lambda_2 = -5$.

<u>In MATLAB</u>, we enter the matrix **A** by typing:

```
A=[1  0  4 ; 0  4  0 ; 3  5  -3]
```

where the semicolon denotes the start of the new row. We'll let the matrix containing the eigenvectors be called V and the matrix containing the eigenvalues as d. We then use the following MATLAB command.

```
[V,d]=eig(A)
```

The result of this command is:

```
V =
1.0000     0.6667   -1.0000
     0          0   -0.4500
0.5000    -1.0000   -0.7500
```

Reading down each column gives the eigenvectors.

```
d =
3     0    0
0    -5    0
0     0    4
```

The leading diagonal entries give the eigenvalues.

By reading the above MATLAB output, the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9/20 \\ 3/4 \end{pmatrix}$$

So the general eigenvectors are

$$\mathbf{u} = r \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v} = s \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \mathbf{w} = t \begin{pmatrix} 20 \\ 9 \\ 15 \end{pmatrix}$$

where $r, s, t \neq 0$. Note that the eigenvector **u** is the eigenvector found in Example 6. You are asked to verify the other two by hand.

Summary
The eigenvector u belonging to eigenvalue λ satisfies:

$$Au = \lambda u \text{ (matrix A scalar multiplies eigenvector u by } \lambda)$$

The following equation is used to find the eigenvalues:

$$\det(A - \lambda I) = 0$$

Eigenvectors u are found using $(A - \lambda I)u = O$.

Exercises

In this exercise you may check your numerical answer using MATLAB

1. Find the eigenvalues and particular eigenvectors of the following matrices:

(a) $A = \begin{pmatrix} 7 & 3 \\ 0 & -4 \end{pmatrix}$ (b) $A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$ (c) $A = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$

2. Obtain the eigenvectors of $\lambda_1 = 4$ and $\lambda_2 = -5$ for the matrix in Example 5.

3. Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Plot the eigenspaces E_λ.

4. Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$. State the effect of multiplying the eigenvector by the matrix $\mathbf{A}$. Plot the eigenspaces E_λ and write down a basis vector for each of the eigenspaces.

5. Let $\mathbf{A} = \begin{pmatrix} -2 & 8 \\ 5 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 16 \\ 10 & 2 \end{pmatrix}$.

   (a) Determine the eigenvalues of $\mathbf{A}$.
   (b) Determine the eigenvalues of $\mathbf{B}$.
   (c) State a relationship between the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ and predict a general relationship.

6. Let $\mathbf{A}$ be a square matrix and $\mathbf{B} = r\mathbf{A}$, where r is a real number. Prove that if λ is the eigenvalue of matrix $\mathbf{A}$ then the eigenvalue of $\mathbf{B}$ is $r\lambda$.

7. Prove that the zero $n \times n$ matrix, $\mathbf{O} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, only has *zero* eigenvalues.

8. Determine the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. By using appropriate software plot the eigenspaces and write down a basis for each eigenspace.

Wronskian determinant

The Wronskian determinant is used in differential equations to test whether solutions to the differential equation are *linearly independent*. If the Wronskian determinant is *non-zero* then the solutions are *linearly independent*. Let $f(x)$ and $g(x)$ be two solutions to a differential equation then the Wronskian $W(f, g)$ is defined by:

$$W(f, g) = \det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}$$

where $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$ respectively. If $W(f, g) \neq 0$ then f and g are linearly independent.

For example, the Wronskian $W(\cos(x), \sin(x))$ is given by:

$$W(\cos(x), \sin(x)) = \det\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \qquad \left[\begin{array}{l} \text{because } (\cos(x))' = -\sin(x) \\ \text{and } (\sin(x))' = \cos(x) \end{array}\right]$$
$$= \cos^2(x) + \sin^2(x) = 1 \neq 0$$

Remember, the fundamental trigonometric identity $\cos^2(x) + \sin^2(x) = 1$. Hence sine and cosine are linearly independent solutions because $W(\cos(x), \sin(x)) \neq 0$ [Non-zero]

(1)

Show that the Wronskian $W(e^{-x}, e^{-3x}) \neq 0$.

(2)

The Wronskian $W(f, g, h) = \det\begin{pmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{pmatrix}$. Determine

$$W(1, \cos(x), \sin(x))$$

(3)

Find the values of x so that $\det\begin{pmatrix} 1 & 0 & -3 \\ 5 & x & -7 \\ 3 & 9 & x-1 \end{pmatrix} = 0$.

(4)

Find $\det(\mathbf{A})$ where $\mathbf{A} = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$.

(5)

Find the values of k for which the following matrix is invertible:

$$\mathbf{A} = \begin{pmatrix} k & 1 & 2 \\ 0 & k & 2 \\ 5 & -5 & k \end{pmatrix}$$

(6)

Show that the 3 by 3 Vandermonde determinant is written as

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix} = (x-y)(y-z)(z-x)$$

(7)

The volume of a parallelepiped (three-dimensional parallelogram) which is spanned by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by $\left|\det\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}\right|$.

Find the volume of the parallelepiped generated by the vectors

$$\mathbf{u} = (1 \;\; 2 \;\; 1)^T, \quad \mathbf{v} = (2 \;\; 3 \;\; 5)^T \text{ and } \mathbf{w} = (7 \;\; 10 \;\; -1)^T$$

(8)

Show that the determinant of the following rotational matrix $\mathbf{R}$ is 1.

$$\mathbf{R} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General Vector Spaces

In this chapter we describe what is meant by a vector space and how it is mathematically defined.

1. Vector space

Let V be a non-empty set of elements called vectors. We define two operations on the set V which are **vector addition** and **scalar multiplication**. Scalars are real numbers.

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the set V. The set V is called a **vector space** if it satisfies the following 10 axioms.

1. The vector addition $\mathbf{u} + \mathbf{v}$ is also in the vector space V. Generally in mathematics we say that we have **closure** under vector addition if this property holds.
2. Commutative law: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. Associative law: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. Neutral element. There is a vector called the **zero** vector in V denoted by $\mathbf{O}$ which satisfies

$$\mathbf{u} + \mathbf{O} = \mathbf{u} \text{ for every vector } \mathbf{u} \text{ in } V$$

5. Additive inverse. For every vector $\mathbf{u}$ there is a vector $-\mathbf{u}$ (minus $\mathbf{u}$) which satisfies the following:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{O}$$

6. Let k be a real scalar then $k\mathbf{u}$ is also in V. We say that we have **closure** under scalar multiplication if this axiom is satisfied.
7. Associative law for scalar multiplication. Let k and c be real scalars then

$$k(c\mathbf{u}) = (kc)\mathbf{u}$$

8. Distributive law for vectors. Let k be a real scalar then

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\,\mathbf{v}$$

9. Distributive law for scalars. Let k and c be real scalars then

$$(k + c)\mathbf{u} = k\mathbf{u} + c\mathbf{u}$$

10. Identity element. For every vector $\mathbf{u}$ in V we have

$$1\mathbf{u} = \mathbf{u}$$

We say that if the elements of the set V satisfy the above 10 axioms then V is called a vector space and the elements are known as vectors. We will use these axioms frequently in the next few sections, and you will soon become familiar with them.

Examples 1
Show that R^2 is a vector space.

Solution We need to check all the above 10 axioms. Let u, v and w are vectors in R^2 defined as follow:

$$\mathbf{u}=\begin{pmatrix}a\\b\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}c\\d\end{pmatrix}\ \text{and}\ \mathbf{w}=\begin{pmatrix}e\\f\end{pmatrix}$$

Axiom 1. We first check closure under vector addition. We have

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix}\ \text{and}\ \begin{pmatrix}a+c\\b+d\end{pmatrix}\ \text{is in}\ \mathbb{R}^2 \quad \surd$$

Hence we have closure under vector addition.

Axiom 2. Commutative.

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix}=\begin{pmatrix}c+a\\d+b\end{pmatrix}=\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} \quad \surd$$

Hence $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

Axiom 3. Associative law:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w}&=\left[\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right]+\begin{pmatrix}e\\f\end{pmatrix}\\&=\begin{pmatrix}a+c\\b+d\end{pmatrix}+\begin{pmatrix}e\\f\end{pmatrix} \qquad \surd\\&=\begin{pmatrix}a+c+e\\b+d+f\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c+e\\d+f\end{pmatrix}=\mathbf{u}+(\mathbf{v}+\mathbf{w})\end{aligned}$$

We have $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Axiom 4. Neutral element is $\mathbf{O}=\begin{pmatrix}0\\0\end{pmatrix}$ and for any $\mathbf{u}$ we have

$$\mathbf{u}+\mathbf{O}=\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}a+0\\b+0\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}=\mathbf{u} \quad \surd$$

Axiom 5. Additive inverse, we have

$$\mathbf{u}+(-\mathbf{u})=\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}-a\\-b\end{pmatrix}=\begin{pmatrix}a-a\\b-b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{O} \quad \surd$$

Axiom 6. Let k be scalar then

$$k\mathbf{u}=k\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}ka\\kb\end{pmatrix} \quad \surd$$

Hence $k\mathbf{u}$ is also in $\mathbb{R}^2$, therefore we have closure under scalar multiplication.

Axiom 7. Associative Law for scalar multiplication. Let k and c be real numbers then

$$k(c\mathbf{u})=k\left(c\begin{pmatrix}a\\b\end{pmatrix}\right)=k\begin{pmatrix}ca\\cb\end{pmatrix}=\begin{pmatrix}kca\\kcb\end{pmatrix}=(kc)\begin{pmatrix}a\\b\end{pmatrix}=(kc)\mathbf{u} \quad \surd$$

Axiom 8. Distributive Law for vectors. Let k be a real number then

$$k(\mathbf{u}+\mathbf{v}) = k\left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right]$$

$$= k\begin{pmatrix} a+c \\ b+d \end{pmatrix} \quad \surd$$

$$= \begin{pmatrix} ka+kc \\ kb+kd \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} + \begin{pmatrix} kc \\ kd \end{pmatrix} = k\begin{pmatrix} a \\ b \end{pmatrix} + k\begin{pmatrix} c \\ d \end{pmatrix} = k\mathbf{u} + k\mathbf{v}$$

Axiom 9. Distributive Law for scalars. Let k and c be real numbers then

$$(k+c)\mathbf{u} = (k+c)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} (k+c)a \\ (k+c)b \end{pmatrix} \quad \surd$$

$$= \begin{pmatrix} ka+ca \\ kb+cb \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} + \begin{pmatrix} ca \\ cb \end{pmatrix} = k\begin{pmatrix} a \\ b \end{pmatrix} + c\begin{pmatrix} a \\ b \end{pmatrix} = k\mathbf{u} + c\mathbf{u}$$

Axiom 10. Identity Element. For every vector $\mathbf{u}$ in V we have

$$1\mathbf{u} = 1\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1\times a \\ 1\times b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{u} \quad \surd$$

Since all 10 axioms are satisfied, we conclude that $\mathbb{R}^2$ is a vector space.

Examples 2

Show that the set M_{22} of 2 by 2 matrices is a vector space.

Solution We need to check all the above 10 axioms. Let u, v and w are matrices in M_{22} defined as follow:

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

Axiom 1. We check $\mathbf{u}+\mathbf{v}$ is also in the set M_{22}:

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \quad \surd$$

Since $\mathbf{u}+\mathbf{v}$ is a 2 by 2 matrix therefore it is in M_{22}.

Axiom 2. Commutative, $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. We know from Chapter 1 that matrix addition is commutative. You may like to check this if you want.

Axiom 3. Similarly we have the associative law, $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for matrix addition.

Axiom 4. Neutral element $\mathbf{O}$ which satisfies

$$\mathbf{u}+\mathbf{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{u} \quad \surd$$

Axiom 5. Additive inverse.

$$\mathbf{u}+(-\mathbf{u})=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$=\begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\mathbf{O} \qquad \surd$$

Axiom 6. Let k be scalar then

$$k\mathbf{u}=k\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \quad \surd$$

Since $\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ is a 2 by 2 matrix which means it is in M_{22} therefore we have closure under scalar multiplication.

Axiom 7. Associative Law for scalar multiplication. Let k_1 and k_2 be real numbers then

$$k_1(k_2\mathbf{u})=k_1\begin{pmatrix} k_2a & k_2b \\ k_2c & k_2d \end{pmatrix}=\begin{pmatrix} k_1k_2a & k_1k_2b \\ k_1k_2c & k_1k_2d \end{pmatrix}=(k_1k_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}=(k_1k_2)\mathbf{u} \qquad \surd$$

Axiom 8. Distributive Law for vectors. Let k be a real number then

$$k(\mathbf{u}+\mathbf{v})=k\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]$$

$$=k\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$=\begin{pmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{pmatrix} \qquad \surd$$

$$=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}+\begin{pmatrix} ke & kf \\ kg & kh \end{pmatrix}=k\begin{pmatrix} a & b \\ c & d \end{pmatrix}+k\begin{pmatrix} e & f \\ g & h \end{pmatrix}=k\mathbf{u}+k\mathbf{v}$$

Axiom 9. Distributive Law for scalars. Let k_1 and k_2 be scalars then

$$(k_1+k_2)\mathbf{u}=(k_1+k_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$=\begin{pmatrix} (k_1+k_2)a & (k_1+k_2)b \\ (k_1+k_2)c & (k_1+k_2)d \end{pmatrix}$$

$$=\begin{pmatrix} k_1a+k_2a & k_1b+k_2b \\ k_1c+k_2c & k_1d+k_2d \end{pmatrix} \qquad \surd$$

$$=\begin{pmatrix} k_1a & k_1b \\ k_1c & k_1d \end{pmatrix}+\begin{pmatrix} k_2a & k_2b \\ k_2c & k_2d \end{pmatrix}=k_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}+k_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}=k_1\mathbf{u}+k_2\mathbf{u}$$

Axiom 10. Identity element, 1. We have

$$1\mathbf{u}=1\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\mathbf{u} \quad \surd$$

Since **all** 10 axioms are satisfied therefore, we conclude that the set of 2 by 2 matrices M_{22} is a vector space.

Examples 3

Show that the following set of matrices does not form a vector space:

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

Solution Matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ do **not** form a vector space because there is **no neutral** element or there is **no zero** vector. We cannot have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in this set therefore it does **not** form a vector space.

Examples 4

Let V be the set of integers $\mathbb{Z}$. Let vector addition be defined as the normal addition of integers, and scalar multiplication by the usual multiplication of integers by a real scalar, which is any real number.

Show that this set is not a vector space with respect to this definition of vector addition and scalar multiplication.

Solution

The set of integers $\mathbb{Z}$ is not a vector space because when we multiply an integer by a real scalar, which is any real number, then the result may not be an integer.

For example, if we consider the integer 2 and multiply this by the scalar 1/3 then the result is $\frac{1}{3}(2) = \frac{2}{3}$ which is not an integer. Our result after scalar multiplication is not in the set of integers $\mathbb{Z}$. Hence the set of integers fails axiom 6.

We do not have closure under scalar multiplication, therefore the set of integers $\mathbb{Z}$ is not a vector space with respect to vector addition and scalar multiplication as defined above.

2. Basic properties of general vector spaces

Let V be a vector space and k be a real scalar. Then we have:

(a) For any real scalar k we have $k\mathbf{O} = \mathbf{O}$ where $\mathbf{O}$ is the zero vector.
(b) For the real number 0 and any vector $\mathbf{u}$ in V we have $0\mathbf{u} = \mathbf{O}$.
(c) For the real number -1 and any vector $\mathbf{u}$ in V we have $(-1)\mathbf{u} = -\mathbf{u}$.
(d) If $k\mathbf{u} = \mathbf{O}$, where k is a real scalar and $\mathbf{u}$ in V, then $k = 0$ or $\mathbf{u} = \mathbf{O}$.

Summary

A vector space is formed by a non-empty set within which the operation of vector addition and scalar multiplication satisfies the above 10 axioms.

Examples of vector spaces include the sets of R^2, matrices, polynomials and functions.

Some properties of vector spaces are:

(a) $k\mathbf{O} = \mathbf{O}$
(b) $0\mathbf{u} = \mathbf{O}$
(c) $(-1)\mathbf{u} = -\mathbf{u}$
(d) If $k\mathbf{u} = \mathbf{O}$ then $k = 0$ or $\mathbf{u} = \mathbf{O}$

Exercises

1. Show that R3 is a vector space.
2. Show that the set M_{23} of 2 by 3 matrices is a vector space.
3. Show that matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ form a vector space.
4. Show that the set of 2 by 2 non-invertible matrices do not form a vector space.
5. Let P2 be the set of polynomials of degree 2 or less and

$$p(x) = ax^2 + bx + c \text{ and } q(x) = dx^2 + ex + f$$

be members of P_2.

Show that the set P_2 is a vector space with respect to the usual vector addition and scalar multiplication.

6. Consider the set R^2 with the normal vector addition, but scalar multiplication defined by

$$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ kb \end{pmatrix}$$

Show that the set R^2 with this scalar multiplication definition is not a vector space.

7. Show that the set F[a, b] which is the set of all functions defined on the interval [a, b] is a vector space.
8. Let V be the set $\begin{pmatrix} a \\ b \end{pmatrix}$ in R2 where $a \geq 0$ and $b \geq 0$. Show that with respect to normal vector addition and scalar multiplication, the given set is not a vector space.
9. Let V be the set of rationals (fractions) Q. Let vector addition be defined as the normal addition of rationals and scalar multiplication by the usual multiplication of rationals by a scalar, which is any real number.

Show that this set is not a vector space with respect to this definition of vector addition and scalar multiplication.

10. Let V be the set of real numbers R. Show that this set is a vector space with respect to the usual definition of vector addition and scalar multiplication.
11. Let V be the set of positive real numbers R^+. Show that this set is not a vector space with respect to the usual definition of vector addition and scalar multiplication.

3. Subspace of a Vector Space

In section 1 we discussed the whole vector space *V*. In this section we show that parts of the whole vector space also form a vector space in their own right. We will show that a non-empty set *within* *V*, which is *closed* under the basic operations of vector addition and scalar multiplication, is also a legitimate vector space.

Examples of vector subspaces

Let *V* be a vector space and *S* be a non-empty subset of *V*. If the set *S* satisfies all 10 axioms of a vector space with respect to the same vector addition and scalar multiplication as *V* then *S* is also a vector space. We say *S* is a subspace of *V*.

Definition
A non-empty subset *S* of a vector space *V* is called a **subspace** of *V* if it is also a vector space with respect to the same vector addition and scalar multiplication as *V*.

Note the difference between subspace and subset. A subset is merely a specific set of elements chosen from V. A subset must also satisfy the 10 axioms of vector space to be called a subspace.

More generally, we will use the following proposition to check if a given subset qualifies as a subspace.

Proposition (1)
Let S be a non-empty subset of a vector space V. Then S is subspace of V:$\Leftrightarrow$

(a) If u and v are vectors in the set S then the vector addition u + v is also in S.
(b) If u is a vector in S then for every scalar k we have, ku is also in S.

Note that this proposition means that we must have closure under both vector addition and scalar multiplication. This means that S is a subspace of V $\Leftrightarrow$ both the following are satisfied, as shown in Fig. 1.

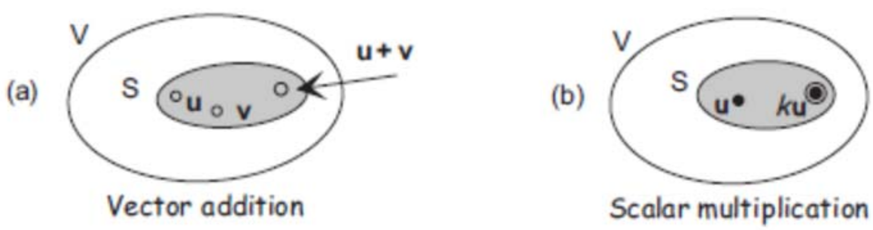


Figure 1.

What does Proposition (1) mean?
Proposition (1) makes life a lot easier because we only need to check closure under vector addition and scalar multiplication to show that a given set is a subspace. No need to check all 10 axioms.

This mean that if vectors u and v are in subspace S then u + v and ku cannot escape from S. As shown in fig.2.

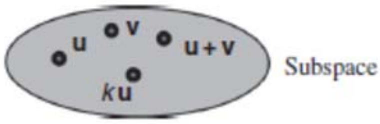


Figure 2.

Example 1

Let V be the set $\mathbb{R}^2$ and vector addition and scalar multiplication be defined as normal. Let S be the set of vectors of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$. Show that S is a subspace of V.

Solution

We only need to check conditions (a) and (b) of Proposition (1). These are the closure conditions under vector addition and scalar multiplication. We need to show:
(a) If **u** and **v** are vectors in the set S then the vector addition **u** + **v** is also in S.
(b) If **u** is a vector in S and k is any real scalar then k**u** is also in S.

Let $\mathbf{u} = \begin{pmatrix} 0 \\ a \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ be vectors in S. Then

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} 0 \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a+b \end{pmatrix} \qquad \begin{bmatrix} \text{closure under} \\ \text{vector addition} \end{bmatrix}$$

which is in the set S and

$$k\mathbf{u} = k\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ ka \end{pmatrix} \qquad \begin{bmatrix} \text{closure under} \\ \text{scalar multiplication} \end{bmatrix}$$

which is in S as well.

Conditions (a) and (b) are satisfied, therefore the given set S is a subspace of the vector space R2.
Note that the set S is the y axis in the xy plane as shown in Fig. 3.

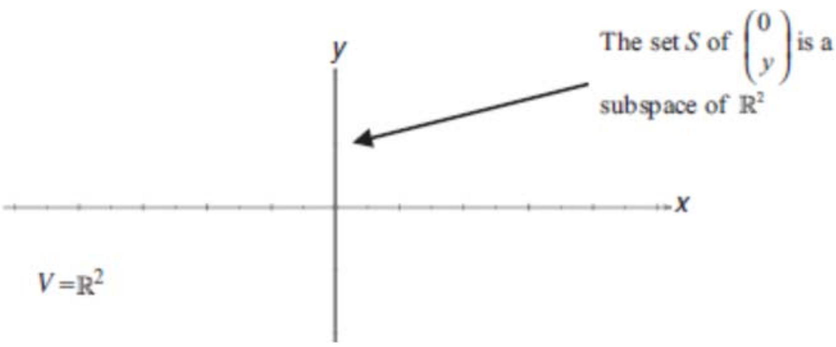


Figure 3.

Example 2

Let S be the subset of vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x \geq 0$ in the vector space $\mathbb{R}^2$. Show that S is not a subspace of $\mathbb{R}^2$.

Solution

How do we show that S is not a subspace of $\mathbb{R}^2$?

If we can show that we do not have closure under vector addition or scalar multiplication of vectors in S, then we can conclude that S is not a subspace. Consider the scalar $k = -1$ and the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; then clearly $\mathbf{u}$ is in the set S, but the scalar multiplication

$$k\mathbf{u} = (-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

is not in S.

Why not?

Because the first entry in the vector $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ is -1 which is less than 0 and the set S only contains vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where the first entry $x \geq 0$ (greater than or equal to zero).

Hence S is *not* a subspace of R^2

Example 3

Let M_{22} be the set of 2 by 2 matrices. From the last section we know that this set is a vector space. Let S be a subset of M_{22}, containing matrices of the form $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$.

Show that S is not a subspace of V.

Solution

We need to show that one of closure conditions (a) or (b) of Proposition (3.5) fails.

Let $\mathbf{u} = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be vectors in the subset S. Then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} 2 & b \\ c & d \end{pmatrix}$$

Hence $\mathbf{u} + \mathbf{v}$ is not a member of the set S because the elements in S are of the form $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ but $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 & b \\ c & d \end{pmatrix}$. The first entry in the matrix $\mathbf{u} + \mathbf{v}$ needs to be 1, not 2, to qualify as an element of the subset S. Therefore S is not a subspace of the vector space M_{22}.

4. linear combination

Linear combination combines the two fundamental operations of linear algebra – vector addition and scalar multiplication.

<u>Definition</u> Let v_1, v_2, . . . and v_n be vectors in a vector space. If a vector x can be expressed as

$$x = k_1v_1 + k_2v_2 + k_3v_3 + \cdot\cdot\cdot + k_nv_n \text{ (where k's are scalars)}$$

then we say x is a linear combination of the vectors v_1, v_2, v_3, . . . and v_n.

<u>Example 1</u>

Let P_2 be the set of all polynomials of degree less than or equal to 2.

Let $\mathbf{v}_1 = t^2 - 1$, $\mathbf{v}_2 = t^2 + 3t - 5$ and $\mathbf{v}_3 = t$ be vectors in P_2.

Show that the quadratic polynomial

$$\mathbf{x} = 7t^2 - 15$$

is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution

How do we show $\mathbf{x}$ is a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$?

We need to find the values of the scalars k_1, k_2 and k_3 which satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x} \quad (*)$$

How can we determine these scalars?

By substituting $\mathbf{v}_1 = t^2 - 1$, $\mathbf{v}_2 = t^2 + 3t - 5$, $\mathbf{v}_3 = t$ and $\mathbf{x} = 7t^2 - 15$ into (*):

$$\begin{aligned} k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= k_1(t^2 - 1) + k_2(t^2 + 3t - 5) + k_3t \\ &= k_1t^2 - k_1 + k_2t^2 + 3k_2t - 5k_2 + k_3t && \text{[expanding]} \\ &= (k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) && \text{[factorizing]} \\ &= 7t^2 - 15 && \text{[remember } \mathbf{x} = 7t^2 - 15\text{]} \end{aligned}$$

By equating coefficients of the last two lines

$$(k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) = 7t^2 - 15$$

gives

$$\begin{aligned} k_1 + k_2 &= 7 && \text{[equating } t^2\text{]} \\ 3k_2 + k_3 &= 0 && \text{[equating } t\text{]} \\ k_1 + 5k_2 &= 15 && \text{[equating constants]} \end{aligned}$$

Solving these equations gives the values of the scalars: $k_1 = 5$, $k_2 = 2$ and $k_3 = -6$.
Substituting these into $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x}$:

$$5\mathbf{v}_1 + 2\mathbf{v}_2 - 6\mathbf{v}_3 = \mathbf{x}$$

This means that adding 5 lots of $\mathbf{v}_1$, 2 lots of $\mathbf{v}_2$ and -6 lots of $\mathbf{v}_3$ gives the vector $\mathbf{x}$:

$$5(t^2 - 1) + 2(t^2 + 3t - 5) - 6t = 7t^2 - 15$$

You may like to check the algebra.
We conclude that $\mathbf{x}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The next proposition allows us to check that *S* is a subspace, by carrying out the test for scalar multiplication and vector addition in a single calculation.

Proposition (2)
A non-empty subset S containing vectors u and v is a subspace of a vector space V ⇔ any linear combination ku + cv is also in S (k and c are scalars).

Example 2

Let S be the subset of vectors of the form $(x \quad y \quad 0)^T$ in the vector space $\mathbb{R}^3$. Show that S is a subspace of $\mathbb{R}^3$.

Solution
How do we show that S is a subspace of R^3?
We can use the above Proposition (2), which means we need to show that any linear combination ku + cv is in S for any vectors u and v in S.

Let $\mathbf{u} = (a \quad b \quad 0)^T$ and $\mathbf{v} = (c \quad d \quad 0)^T$ be in S. Then for real scalars k_1 and k_2 we have

$$\begin{aligned} k_1\mathbf{u} + k_2\mathbf{v} &= k_1\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + k_2\begin{pmatrix} c \\ d \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} k_1a \\ k_1b \\ 0 \end{pmatrix} + \begin{pmatrix} k_2c \\ k_2d \\ 0 \end{pmatrix} = \begin{pmatrix} k_1a + k_2c \\ k_1b + k_2d \\ 0 \end{pmatrix} \end{aligned}$$

Hence $k_1u + k_2v$ is also in S.

By the above Proposition (2)
S is subspace of V ⇔ any linear combination ku + cv is also in S.
We conclude that the given set S is a subspace of the vector space R^3.

You might find it easier to use this test given by Proposition (2) rather than Proposition (1) because you only need to recall that the linear combination is closed in S.

Note that the given set S in the above example describes the xy plane in three dimensional space R^3 as shown in Fig. 4:

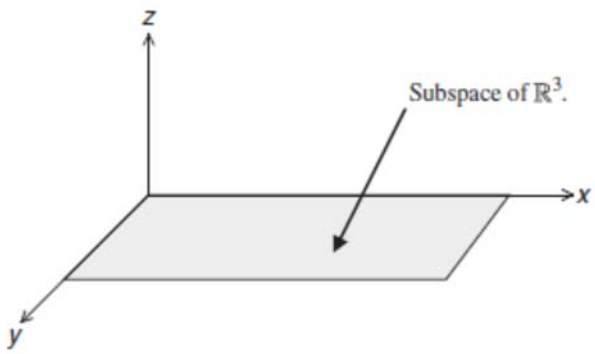


Figure 4.

Sometimes we write $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ as the set $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; z = 0 \right\}$ where the vertical line in the set means 'such that'. That is, S contains the set of vectors in $\mathbb{R}^3$ such that the last entry is zero.

5. spanning sets

Definition

If *every* vector in V can be produced by a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, . . . and $\mathbf{v}_n$ then these vectors **span** or **generate** the vector space V. We write this as

$$span\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}.$$

Example 1

Let P_3 be the vector space containing the set of polynomials of degree 3 or less. Show that the set $\{1, t, t^2, t^3\}$ spans $\{P_3\}$.

Solution

Let $at^3 + bt^2 + ct + d$ be a general polynomial in the space P_3. Then we need to show that there are scalars k_1, k_2, k_3 and k_4 which satisfy the following:

$$k_1t^3 + k_2t^2 + k_3t + k_4 = at^3 + bt^2 + ct + d$$

Equating coefficients of t^3, t^2, t and constant gives $k_1 = a$, $k_2 = b$, $k_3 = c$ and $k_4 = d$ respectively. We have found scalars, $k_1 = a$, $k_2 = b$, $k_3 = c$ and $k_4 = d$, therefore the linear combination of $\{1, t, t^2, t^3\}$ can generate any vector in P_3 which means $\{1, t, t^2, t^3\}$ spans P_3.

Example 2

Let P_2 be the set of polynomials of degree 2 or less. Let the following vectors be in P_2:

$$\mathbf{v}_1 = t^2 - t - 1, \quad \mathbf{v}_2 = 6t^2 + 3t - 3 \text{ and } \mathbf{v}_3 = t^2 + 5t + 1$$

Determine whether the vector $\mathbf{u} = 3t^2 + 2t + 1$ is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution

How do we check that the vector $\mathbf{u}$ *is spanned by* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*?*

We need to see if scalars, k_1, k_2 and k_3 exist which satisfy the following linear combination:

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{u}$$

Substituting $\mathbf{v}_1 = t^2 - t - 1$, $\mathbf{v}_2 = 6t^2 + 3t - 3$, $\mathbf{v}_3 = t^2 + 5t + 1$ and $\mathbf{u} = 3t^2 + 2t + 1$ into this:

$$\begin{aligned} k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= k_1(t^2 - t - 1) + k_2(6t^2 + 3t - 3) + k_3(t^2 + 5t + 1) \\ &= (k_1 + 6k_2 + k_3)t^2 + (-k_1 + 3k_2 + 5k_3)t + (-k_1 - 3k_2 + k_3) \qquad [\text{rearranging}] \\ &= 3t^2 + 2t + 1 \end{aligned}$$

Equating the coefficients of the last two lines gives:

$$\begin{aligned} k_1 + 6k_2 + k_3 &= 3 \qquad &(\text{equating } t^2) \\ -k_1 + 3k_2 + 5k_3 &= 2 \qquad &(\text{equating } t) \\ -k_1 - 3k_2 + k_3 &= 1 \qquad &(\text{equating constants}) \end{aligned}$$

What are we trying to find?

We need to find values of the scalars k_1, k_2 and k_3 which satisfy the above linear system. Writing the above system as an augmented matrix gives:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 6 & 1 & 3 \\ -1 & 3 & 5 & 2 \\ -1 & -3 & 1 & 1 \end{array}\right)$$

Executing row operations, $\mathrm{R}_2 + \mathrm{R}_1$ and $\mathrm{R}_3 + \mathrm{R}_1$, gives:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 + \mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 + \mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 6 & 1 & 3 \\ 0 & 9 & 6 & 5 \\ 0 & 3 & 2 & 4 \end{array}\right)$$

Carrying out the row operation $3\mathrm{R}_3^* - \mathrm{R}_2^*$:

$$\begin{array}{cc} & \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ 3\mathrm{R}_3^* - \mathrm{R}_2^* \end{array} & \left(\begin{array}{ccc|c} 1 & 6 & 1 & 3 \\ 0 & 9 & 6 & 5 \\ 0 & 0 & 0 & 7 \end{array}\right) \end{array}$$

From the bottom row we have $0k_1 + 0k_2 + 0k_3 = 0 = 7$, which means that the linear system is inconsistent, therefore we have no solution. There are no scalars (k's) which satisfy the above linear system. Hence the vector $\mathbf{u}$ does not exist in the *span*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. This means that we cannot make $\mathbf{u}$ out of a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Example 3

Let $F[-\pi, \pi]$ be the vector space of continuous functions defined in the interval $[-\pi, \pi]$. Let the functions $\mathbf{f} = \cos^2(x)$ and $\mathbf{g} = \sin^2(x)$. Determine whether 2 is in *span*$\{\mathbf{f}, \mathbf{g}\}$.

Solution

We need to find scalars k and c such that

$$k\mathbf{f} + c\mathbf{g} = k\cos^2(x) + c\sin^2(x) = 2$$

From trigonometry we have the fundamental identity $\cos^2(x) + \sin^2(x) = 1$. Multiplying this by 2 gives us our result $2\cos^2(x) + 2\sin^2(x) = 2$.

Hence with $k = c = 2$ we have $k\mathbf{f} + c\mathbf{g} = 2$, therefore 2 is in the *span*$\{\mathbf{f}, \mathbf{g}\}$.

Example 4

Let M_{22} be the vector space of 2 by 2 matrices. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix}$$

Determine whether the matrix $\mathbf{D} = \begin{pmatrix} 1 & 2 \\ -4 & -2 \end{pmatrix}$ is within the *span*$\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$.

Solution

We need to solve the linear combination

$$k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C} = \mathbf{D}$$

for scalars k_1, k_2 and k_3:

$$\begin{aligned} k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C} &= k_1\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} + k_2\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} k_1 & -k_1 \\ k_1 & 2k_1 \end{pmatrix} + \begin{pmatrix} 0 & 2k_2 \\ 0 & -k_2 \end{pmatrix} + \begin{pmatrix} 0 & k_3 \\ 5k_3 & 2k_3 \end{pmatrix} \\ &= \begin{pmatrix} k_1 & -k_1 + 2k_2 + k_3 \\ k_1 + 5k_3 & 2k_1 - k_2 + 2k_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -4 & -2 \end{pmatrix} \quad [\text{matrix } \mathbf{D}] \end{aligned}$$

Equating entries of the matrix gives

$$k_1 = 1, \quad -k_1 + 2k_2 + k_3 = 2, \quad k_1 + 5k_3 = -4 \quad \text{and} \quad 2k_1 - k_2 + 2k_3 = -2$$

We have $k_1 = 1$, but how do we find k_2 and k_3?

Substituting $k_1 = 1$ into the third equation $k_1 + 5k_3 = -4$ yields

$$1 + 5k_3 = -4 \text{ gives } k_3 = -1$$

Substituting $k_1 = 1$ and $k_3 = -1$ into the last equation $2k_1 - k_2 + 2k_3 = -2$:

$$\begin{aligned} 2(1) - k_2 + 2(-1) &= -2 \\ -k_2 &= -2 \text{ which gives } k_2 = 2 \end{aligned}$$

Hence we have found scalars, $k_1 = 1$, $k_2 = 2$ and $k_3 = -1$, which satisfy

$$k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C} = \mathbf{A} + 2\mathbf{B} - \mathbf{C} = \mathbf{D}$$

Therefore $\mathbf{D}$ belongs to *span*$\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ because $\mathbf{D}$ can be made by a linear combination of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

Summary

(1) The set S is a subspace of a vector space $\Leftrightarrow$:

(a) S is closed under vector addition.

(b) S is closed under scalar multiplication.

(2) If a vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \text{ (where } k\text{'s are scalars)}$$

then we say $\mathbf{x}$ is a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$.

(3) If *every* vector in V can be produced by a linear combination of vectors $\mathbf{v}1, \mathbf{v}2, \mathbf{v}3, \ldots$ and $\mathbf{v}n$ then we say these vectors *span* or *generate* the vector space V.

Exercises

2. Let S be the set of vectors $\begin{pmatrix} a \\ 0 \end{pmatrix}$ in the vector space $\mathbb{R}^2$. Show that S is a subspace of $\mathbb{R}^2$.
3. Let S be the set of vectors $\begin{pmatrix} a \\ a \end{pmatrix}$ in the vector space $\mathbb{R}^2$. Show that S is a subspace of $\mathbb{R}^2$.
4. Let S be the set of vectors $(0 \quad 0 \quad c)^T$ in the vector space $\mathbb{R}^3$. Show that S is a subspace of $\mathbb{R}^3$.
5. Let S be the set of vectors $(\,1 \quad b \quad c \quad d\,)^T$ in the vector space $\mathbb{R}^4$. Show that S is not a subspace of $\mathbb{R}^4$.
6. Let S be the set of vectors $(\,a \quad b \quad c\,)^T$ where $a + b + c = 0$ in the vector space $\mathbb{R}^3$. Determine whether S is a subspace of $\mathbb{R}^3$.
7. Let $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, a \text{ and } b \text{ are integers} \right\}$ be a subset of $\mathbb{R}^2$. Show that S is not a subspace of $\mathbb{R}^2$.
8. Let $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \,\middle|\, a,\ b \text{ and } c \text{ are rational numbers} \right\}$ be a subset of $\mathbb{R}^3$. Show that S is not a subspace of $\mathbb{R}^3$.
9. Let M_{22} be the set of matrices of size 2 by 2. Let S be the subset of matrices of the form $\begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$. Show that S is not a subspace of M_{22}.
10. Let M_{22} be the set of matrices of size 2 by 2. Let

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,\ b,\ c,\ d \text{ are all integers} \right\}$$

Show that S is not a subspace of M_{22}.

11. Let S be the set of symmetric matrices (these are matrices $\mathbf{A}$ such that $\mathbf{A}^T = \mathbf{A}$, that is $\mathbf{A}$ transposed is equal to $\mathbf{A}$). Let V be the set of all matrices. Show that S is a subspace of V.

12. Let $\mathbf{v}_1 = t^2 - 1$, $\mathbf{v}_2 = t + 1$ and $\mathbf{v}_3 = 2t^2 + t - 1$ be vectors in P_2 where P_2 is the set of polynomials of degree 2 or less. Show that $\mathbf{x} = 7t^2 + 8t + 1$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

13. Let $\mathbf{p}_1 = t^2 + 2t - 1$, $\mathbf{p}_2 = 2t + 1$ and $\mathbf{p}_3 = 5t^2 + 2t - 3$ be vectors in P_2. Show that the following are linear combinations of these vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$:
(a) $\mathbf{x} = 4t^2 - 2t - 3$ (b) $\mathbf{x} = -2t^2 - 2$ (c) $\mathbf{x} = 6$

14. Let V be the set of real valued continuous functions. Then V is a vector space. Let $\mathbf{v}_1 = \sin^2(t)$ and $\mathbf{v}_2 = \cos^2(t)$. Show that the following are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
(a) $\mathbf{x} = 1$ (b) $\mathbf{x} = \pi$ (c) $\mathbf{x} = \cos(2t)$

15. Let S be a subspace of $\mathbb{R}^3$ given by $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$. Show that the vectors $\mathbf{u}$ and $\mathbf{v}$ span S.

16. Let P_2 be the set of polynomials of degree 2 or less. Let

$$\mathbf{p}_1 = t^2 + 3,\ \mathbf{p}_2 = 2t^2 + 5t + 6 \text{ and } \mathbf{p}_3 = 5t$$

Determine whether the vector $\mathbf{x} = t^2 + 3$ is in the span $\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\}$.

17. Let M_{22} be the set of 2 by 2 matrices. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} -1 & 2 \\ 5 & 7 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 2 & 6 \\ 8 & 0 \end{pmatrix}$$

Determine whether the matrix $\mathbf{D} = \begin{pmatrix} 7 & -3 \\ -14 & -26 \end{pmatrix}$ belongs to span $\{\mathbf{A},\ \mathbf{B},\ \mathbf{C}\}$.

18. Let $F[0,\ 2\pi]$ be the vector space of continuous functions. Let the functions $\mathbf{f} = \cos(2x)$ and $\mathbf{g} = \sin(2x)$. Determine whether the following are in span $\{\mathbf{f},\ \mathbf{g}\}$:
(a) 0 (b) $\sin(2x)$ (c) $\cos^2(x) - \sin^2(x)$ (d) 1

19. Determine whether $x + 1$ and $(x + 1)^2$ span P_2.

المعادلات التفاضلية

(1–1) مقدمة

نعلم من دراستنا السابقة أنه إذا كانت $x = f\,(t)$ دالة تفاضلية (أي قابلة للتفاضل) حيث t المتغير المستقل و x المتغير التابع فإنه يمكن حساب $\frac{dx}{dt}$ وهي هندسياً تمثل ميل المماس للمنحنى $x = f\,(t)$ وجبرياً معدل تغير الدالة x بالنسبة للمتغير t. أيضاً يمكن حساب $\ddot{x} = \frac{d^2x}{dt^2}$ وكذلك يمكن حساب المشتقات الأعلى إذا كانت الدالة قابلة لذلك.

فإذا كانت x هي المسافة و t هو الزمن فإن $\dot{x} = \frac{dx}{dt}$ تمثل سرعة جسيم يتحرك على المحور x عند أي لحظة t. وتكون عجلة الجسم عند أي لحظة هي $\frac{d^2x}{dt^2}$ فإذا كان هناك جسم يتحرك في خط مستقيم هو المحور x بحيث تكون سرعته عند أي لحظة t تساوي مقدار ثابت α فإن:

$$\frac{dx}{dt} = \alpha \qquad (1)$$

كذلك إذا كان هناك جسيم يتحرك في خط مستقيم تحت تأثير قوة مقدارها $\alpha\, x$ حيث x بعد الجسيم عن نقطة ثابتة فإن معادلة حركته طبقاً لقانون نيوتن هي:

$$m\frac{d^2x}{dt^2} = \alpha\, x$$

$$\frac{d^2x}{dt^2} = \beta x \qquad (2)$$

حيث $\beta = \frac{\alpha}{m} =$ ثابت، m كتلة الجسيم.

يلاحظ أن المعادلتان $(1), (2)$ هما ببساطة علاقتين بين المتغير التابع x ومشتقة ذلك المتغير بالنسبة للمتغير التابع t.

تعريف (1):

المعادلة التفاضلية هي علاقة بين المتغير التابع x ومشتقاته $\dot{x}, \ddot{x}, \ldots$ والمتغير المستقل t.

مثال (1):

اكتب المعادلة التفاضلية التي تمثل منحنى الدالة $y = f(x)$ والتي يكون فيها ميل المماس للمنحنى عند أي نقطة (x, y) مساوياً لمجموع بعدي النقطة عن المحورين x, y.

الحــل

ميل المماس عند أي نقطة هو $\frac{dy}{dx}$

ومجموع بعدي النقطة عن المحاور هو $(x + y)$ وعلى ذلك يكون:

$$\frac{dy}{dx} = x + y \qquad (3)$$

مثال (2):

اكتب المعادلة التفاضلية التي تمثل حركة جسيم يتحرك في خط مستقيم تحت تأثير عجلة تتناسب عند أي لحظة مع سرعة الجسيم وثابت التناسب هو λ.

الحـــل

نفرض أن الجسيم يتحرك على المحور x وأن t تمثل الزمن. سرعة الجسيم هي $\frac{dx}{dt}$ وعجلته $\frac{d^2x}{dt^2}$ والمعادلة التفاضلية للحركة هي:

$$\frac{d^2x}{dt^2} = \lambda \frac{dx}{dt}$$

مثال (3):

إذا كان لتر الكحول يتبخر بمعدل يتناسب مع كمية الكحول الموجودة حيث ثابت التناسب α اكتب المعادلة التفاضلية التي تمثل كمية الكحول الموجودة عند أي لحظة t.

الحـــل

بفرض أن كمية الكحول الموجود عند أي لحظة هي Q فإن معدل تناقص الكحول هو $\frac{dQ}{dt}$ علماً بأن:

$$\frac{dQ}{dt} = -\alpha Q$$

تعريف (2):

حل المعادلة (بصفة عامة) هو إيجاد المتغير التابع كدالة في المتغير المستقل بحيث يكون ومشتقته تحقق المعادلة التفاضلية.

أمثلة:

(1) حل المعادلة التفاضلية (1) هو

$$x = \alpha t + c$$

وذلك لأي ثابت c لأن x و $\frac{dx}{dt}$ تحقق المعادلة (1).

(2) حل المعادلة التفاضلية (2) هو

$$x = c_1 e^{\sqrt{\beta}\, t} + c_2 e^{-\sqrt{\beta}\, t}$$

حيث c_1, c_2 مقادير ثابتة وذلك لأي x و $\frac{d^2x}{dt^2}$ تحقق المعادلة (2)

(3) حل المعادلة التفاضلية (3) بالصورة

$$y = ce^x - (1 + x)$$

وذلك لأي مقدار ثابت c.

(تحقق من أن $\frac{dy}{dx}$ y, تحقق المعادلة (3)).

(2-1) تصنيف المعادلات التفاضلية:

لنأخذ في الاعتبار المعادلات التفاضلية الآتية:

$$\frac{dy}{dx} = x + y \qquad (1)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0 \qquad (2)$$

$$xy' + y^2 = x^2 + 1 \qquad (3)$$

$$y''' + 2(y')^4 + yy' = \ln x \qquad (4)$$

$$(y'')^2 + x^3 yy'' + y \cos x = 1 \qquad (5)$$

$$(x y')' + e^x y = 0 \qquad (6)$$

$$y^3 (y')^3 + xy^6 = y \qquad (7)$$

حيث $\frac{dy}{dx} = y'$ ، $\frac{d^2 y}{dx^2} = y''$ وهكذا.

تعريف (1):

رتبة المعادلة التفاضلية هي رتبة أعلى مشتقة فيها.

وعلى ذلك تكون المعادلات (1)، (3)، (7) من الرتبة الأولى، والمعادلات (2)، (5)، (6) من الرتبة الثانية والمعادلة (4) من الرتبة الثالثة.

تعريف (2):

درجة المعادلة التفاضلية هي درجة (أس) أعلى مشتقة فيها.

وعلى ذلك فإن المعادلات (1)، (2)، (3)، (4)، (6) من الدرجة الأولى و المعادلة (5) من الدرجة الثانية والمعادلة (7) من الدرجة الثالثة.

تعريف (3):

يقال أن المعادلة التفاضلية خطية إذا كان المتغير التابع ومشتقاته كمقادير من الدرجة الأولى أي لا يوجد فيها حدود بالصورة ... y^2, y^3, أو yy' أو $(y')^2, (y'')^2, ...$ أو $(y''y')$ وهكذا.

وعلى ذلك فإن المعادلات (1)، (2)، (6) معادلا خطية بينما المعادلة (3) غير خطية

لاحتوائها على y^2 والمعادلة (4) غير خطية لوجود $\left(y'\right)^4$ والمعادلة (5) غير خطية لوجود الحد yy'' والمعادلة (7) غير خطية لأكثر من سبب.

وعموماً فإن المعادلة الخطية التي من الرتبة النونية (والدرجة الأولى) لها الصورة العامة الآتية:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x) \quad (8)$$

تعريف (4):

يقال أن المعادلة التتفاضلية متجانسة إذا كان لا يوجد بها حد (أو أكثر) يحتوي على المتغير المستقل فقط. أما إذا كان بها حد أو أكثر يحتوي على المتغير المستقل فقط فإن المعادلة تكون غير متجانسة.

وعلى ذلك المعادلة (1) غير متجانسة نظراً لوجود x والمعادلة (3) غير متجانسة لاحتوائها على $(x^2 + 1)$ والمعادلة (4) غير متجانسة لاحتوائها $\ln x$ والمعادلة (5) غير متجانسة لاحتوائها على $\cos x$ أما المعادلات (2)، (6)، (7) فهي معادلات متجانسة.

أما المعادلة (8) فتكون متجانسة إذا كانت $f(x) = 0$ وغير متجانسة إذا كانت $f(x) \neq 0$

(3-1) الحل العام للمعادلة التفاضلية:

نعود الآن إلى المعادلة التفاضلية الآتية:

$$\frac{dx}{dt} = 2 \quad (1)$$

نلاحظ أن الدالة $x_1 = 2t$ ومشتقاتها الأولى تحقق المعادلة (1) حسب تعريف الحل. كذلك فإن المعادلة $x_2 = 2t + 1$ ومشتقاتها تحقق المعادلة (1) وعلى ذلك فهي أيضاً حل (مجرد حل) يحقق المعادلة 10) وكذلك $x_3 = 2t - 5$ مجرد حل. وهكذا يوجد عدد لا نهائي من

الدوال كل منها مجرد حل. وعموماً فإن:

$$x = 2t + c \qquad (2)$$

(حيث c ثابت اختياري) هي أيضاً حل ولكنه يختلف عن x_1, x_2, x_3 لأنه يمكن الحصول على x_1, x_2, x_3 بوضع $c = 0, c = 1, c = -5$ على الترتيب بينما لا يمكن الحصول على x_1 من x_2 أو x_2 من x_3 وهكذا. لذلك فإن x تسمى بالحل العام للمعادلة (1) وذلك لأنه يمكن الحصول على جميع الحلول الأخرى وذلك باختيار مناسب للثابت c.

تعريف (1):

الحل العام للمعادلة التفاضلية هو الحل الذي يحتوي على عدد من الثوابت مساوي لرتبة المعادلة.

ويلاحظ أن المعادلة (1) من الرتبة الأولى لذلك فإن (2) هو حلها العام لاحتوائه على ثابت واحد.

ولفهم العلاقة بين الحل العام وبقية الحلول نرسم معادلات الخطوط المستقيمة (المنحنيات) $x, x_1, x_2, x_3, \ldots$ فإن المعادلة (2) هي مجموعة من الخطوط المستقيمة (وذلك بإعطاء قيم مختلفة للثابت c) أما $x_1, x_2, x_3, \ldots$ فإن أي منها يمثل أحد أعضاء تلك المجموعة.

وعموماً فإن الحل العام يمثل عائلة (مجموعة) المنحنيات التي يكون أي من أفرادها محققاً للمعادلة التفاضلية وعند اختيار قيمة عددية للثابت (أو للثوابت) فإننا نقوم باختيار أحد أعضاء تلك العائلة من المنحنيات.

مثال (1):

المعادلة

$$\frac{d^2x}{dt^2} = x \qquad (3)$$

فإن أي من الدوال

$$x_1 = e^t, \qquad x_2 = -5e^t, \qquad x_3 = e^{-t},$$

$$x_4 = 2e^{-t}, \qquad x_5 = c_1e^t, \qquad x_6 = c_2e^{-t}$$

هو مجرد حل للمعادلة (3) بمعنى أن أي منها يحققها مع ملاحظة أن أي منها لا يمثل الحل العام للمعادلة (3)، وذلك لأن أي منها لا يحتوي على ثابتين (حيث أن المعادلة من الرتبة الثانية).

أما الدالة

$$x = c_1e^t + c_2e^{-t} \qquad (4)$$

حيث c_1, c_2 ثوابت.

فهي حل بالإضافة لاحتوائه على ثابتين وعلى ذلك فإن x هي الحل العام للمعادلة (3). بمعنى أن (4) هي مجموعة المنحنيات التي تمثل الحل العام، أما $x_1, x_2, \ldots, x_6$ فإن أي منها أحد أفراد تلك العائلة ويلاحظ أنه يمكن الحصول على

x_1 من x باختيار $c_1 = 1, c_2 = 0$ ، x_2 من x باختيار $c_1 = -5, c_2 = 0$

x_3 من x باختيار $c_1 = 0, c_2 = 1$ ، x_5 من x باختيار $c_2 = 0$ وهكذا.

(4-1) الحل الوحيد للمعادلة التفاضلية:

إذا أضفنا إلى المعادلة التفاضلية عدد من الشروط (تسمى الشروط الابتدائية أو الشروط الحدية) مساوي لرتبة المعادلة التفاضلية فإنه يمكن تعيين الثوابت الاختيارية في الحل العام للمعادلة. هذا الحل يسمى بالحل الوحيد للمعادلة لأنه يحقق الشروط الابتدائية (الحدية) المعطاة.

تعريف (1):

الحل الوحيد هو الحل المستنتج من الحل العام بعد تعيين الثوابت فيه (وذلك باستخدام الشروط الحدية أو الابتدائية).

مثال (1):

الحل العام للمعادلة $\frac{dx}{dt} = 2$ هو

$$x = 2t + c$$

فإذا $t = 0$ عندما $x = 0$ ويكتب

$$x(0) = 0$$

فإنه بالتعويض عن $x = 0, t = 0$ فإن:

$$0 = 0 + c$$

ومنها

$$c = 0$$

أي أن الحل الوحيد الذي يحقق الشرط المطلوب هو:

$$x = 2t$$

كذلك فإذا أردنا إيجاد الحل الوحيد الذي يحقق

$$x(1) = -7$$

فإنه بالتعويض في الحل العام فإن $c = -5$ وعلى ذلك يكون الحل الوحيد هو:

$$x = 2t - 5.$$

مثال (2):

أوجد الحل الوحيد للمعادلة

$$\frac{d^2x}{dt^2} = x$$

حيث

$$x(0) = 0, \qquad x(1) = 1$$

الحــل

نعلم أن الحل العام هو (مثال ص8)

$$x = c_1 e^t + c_2 e^{-t}$$

وباستخدام الشروط المعطاة فإن:

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e + c_2 e^{-1} &= 1 \end{aligned}\right\}$$

وهما معادلتان في المجهولين c_1, c_2 بحلهما يكون:

$$c_1 = -c_2 = \frac{e}{e^2 - 1}$$

والحل الوحيد الذي يحقق الشروط المعطاة هو

$$x = \frac{e}{e^2 - 1}\left(e^t - e^{-t}\right)$$

يلاحظ أن المعادلة التفاضلية من الرتبة الثانية يلزم شرطان لإيجاد الحل الوحيد.

مثال (3):

أوجد الحل الوحيد في المثال السابق الذي يحقق الشروط

$$x(0) = 0, \qquad \dot{x}(0) = 1$$

الحـــل

بالتعويض عن $x(0) = 0$ في الحل العام نحصل على

$$c_1 + c_2 = 0 \qquad (*)$$

وبالتعويض عن الشرط $\dot{x}(0) = 1$ فإننا نحسب أولاً $\dot{x}$ من الحل العام

$$\dot{x} = c_1 e^t - c_2 e^{-t}$$

ومنها

$$1 = c_1 - c_2 \qquad (**)$$

بحل المعادلتين $(*), (**)$ في المجهولين c_1, c_2 نحصل على:

$$c_1 = -c_2 = \frac{1}{2}$$

وعلى ذلك فإن الحل الوحيد الذي يحقق الشروط هو:

$$x = \frac{1}{2}\left(e^t - e^{-t}\right).$$

ملاحظة هامة:

عرفنا الحل أو الحلول للمعادلة التفاضلية وذلك على فرض أن للمعادلة التفاضلية حل موجود. وفي الواقع هناك أكثر من نظرية وكل منها بشروط معينة تضمن لنا وجود حل للمعادلة التفاضلية وسوف نتعرض لبعض من هذه النظريات لاحقاً. أما الآن فسوف نقوم بتقسيم المعادلات التفاضلية تبعاً لرتبتها إلى مجموعات وسوف نعطي طريقة أو أكثر لكيفية إيجاد حل أو حلول كل منها.

الباب الثاني

معادلات الرتبة الأولى

هذه المعادلات لها الصورة العامة الآتية:

$$y' = f(x, y)$$

(2–1) المعادلات القابلة للفصل Separable equations

إذا كانت المعاملات على الصورة:

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$

التي يمكن كتابتها على الصورة:

$$\boxed{N(y)dy = M(x)dx}$$

وبذلك أمكن كتابة x, dx في طرف و y, dy في الطرف الآخر وهذه المعادلات يمكن حلها وذلك بتكامل طرفي المعادلة. والأمثلة سوف توضح ذلك.

مثال (1):

حل المعادلة التفاضلية الآتية:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

حيث $y(0) = -1$.

الحــل

حيث أنه أضيف الشرط $y(0) = -1$ لذلك فإن المطلوب هو الحل الوحيد الذي يحقق بالشرط وسوف نحصل أولاً على الحل العام ومنه نوجد الحل الوحيد لذلك نكتب المعادلة كالآتي:

$$2(y - 1)\,dy = (3x^2 + 4x + 2)\,dx$$

وبإجراء تكامل الطرفين فإن:

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

حيث c ثابت اختياري وهذا هو الحل العام للمعادلة التفاضلية. ولإيجاد الحل الوحيد نعوض بالشرط $y(0) = -1$ في الحل العام فإن:

$$1 + 2 = c = 3$$

ويكون الحل الوحيد هو:

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

مثال (2):

أوجد الحل الوحيد للمعادلة

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}, \qquad y(0) = 1.$$

الحــل

بفصل المتغيرات في المعادلة المعطاة يكون:

$$\left(\frac{1+2y^2}{y}\right)dy = \cos x\,dx$$

وبإجراء تكامل الطرفين يكون:

$$\ln|y| + y^2 = \sin x + c$$

وهذا هو الحل العام للمعادلة وبالتعويض عن الشرط المطلوب في الحل العام يكون

$$0 + 1 = \sin(0) + c \quad \Rightarrow \quad c = 1$$

وعلى ذلك فإن الحل الوحيد هو:

$$\ln|y| + y^2 = \sin x + 1.$$

<u>ملاحظة:</u>

ليس صحيحاً أن كل معادلة قابلة للفصل (يمكن فصلها) والدليل لنعتبر المثال الآتي:

$$y' = \sin(x\,y).$$

تمارين

أوجد الحل العام لكل من المعادلات الآتية:

(1) $y' = \dfrac{x^2}{y}$

(2) $y' + y^2 \sin x = 0$

(3) $y' = \dfrac{x^2}{y\left(1+x^3\right)}$

(4) $y' = 1 + x + y^2 + xy^2$

(5) $y' = \left(\cos^2 x\right)\left(\cos^2 2y\right)$

(6) $xy' = \sqrt{1+y^2}$

(7) $y' = \dfrac{x - e^{-x}}{y + e^{y}}$

(8) $y' = \dfrac{x^2}{1+y^2}$

أوجد الحل الوحيد لكل من المعادلات الآتية:

(9) $\sin(2x)dx + \cos(3y)dy = 0$; $\quad y\left(\dfrac{\pi}{2}\right) = \dfrac{1}{3}$

(10) $xdx + ye^{-x}dy = 0$; $\quad y(0) = 1$

(11) $\dfrac{dr}{d\theta} = r$; $\quad r(0) = 2$

(12) $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$; $y(0) = -2$

(13) $\dfrac{dy}{dx} = xy^3\left(1 + x^2\right)^{-\frac{1}{2}}$; $y(0) = 1$

(14) $y' = \dfrac{2x}{1 + 2y}$; $y(2) = 0$

أوجد الحل العام لكل مما يأتي:

(15) $y'\sqrt{1 - x^2}\,dy = \left(\sin^{-1} x\right)dx$; $|x| < 1$

(16) $y' = \dfrac{ax + b}{cx + d}$

حيث a, b, c, d ثوابت، $ad - bc \neq 0$

(17) $\dfrac{dy}{dx} = \dfrac{ay + b}{cy + d}$

حيث a, b, c, d ثوابت، $ad - bc \neq 0$

(18) $1 = y x^2 y' e^{\left(y + \frac{1}{x}\right)}$; $x \neq 0$

(2-2) معادلات على الصورة

$$\boxed{y' + p(x)y(x) = g(x)} \qquad (1)$$

لإيجاد حل هذه المعادلة (1) سوف نبحث عن دالة ما $\mu(x) > 0$ بحيث إذا ضربنا (1) في $\mu(x)$ يكون الطرف الأيسر من (1) بالصورة $[\mu(x)y(x)]'$ أي أن:

$$\mu(y' + py) = (\mu y)'$$

$$= \mu y' + \mu' y$$

$$\Rightarrow \qquad \mu(x)\, p(x) = \mu'(x)$$

$$\Rightarrow \qquad \frac{d\mu(x)}{\mu(x)} = p(x)dx$$

وبإجراء تكامل الطرفين نحصل على:

$$\ln \mu(x) = \int^{x} p(t)dt$$

ومنها

$$\mu(x) = \exp\int^{x} p(t)dt \qquad (2)$$

بضرب طرفي المعادلة (1) في $\mu(x)$ حيث نختار $\mu(x)$ كما في (2) فإن:

$$\mu(y' + py) = (\mu y)' = \mu g \quad \Rightarrow \quad \frac{d}{dx}(\mu y) = \mu g$$

بضرب الطرفين في dx وبإجراء التكامل يكون:

$$y(x)=\frac{1}{\mu(x)}\left[c+\int^{x}\mu(t)g(t)dt\right] \qquad (3)$$

هو الحل العام للمعادلة التفضلية (1) حيث $\mu(x)$ هي كما في (2).

$\mu(x)$ تسمى بمعامل التكامل.

مثال (1):

أوجد الحل العام للمعادلة التفاضلية

$$y' - 2xy = x$$

الحــل

في هذه المسألة نجد أن:

$$p(x) = -2x, \quad g(x) = x$$

وعلى ذلك يكون

$$\mu(x)=\exp\int^{x}p(t)dt=\exp\int^{x}-2tdt=\exp\left(-x^{2}\right)=e^{-x^{2}}$$

وعليه يكون

$$y(x)=\frac{1}{\mu(x)}\left[c+\int^{x}te^{-t^{2}}dt\right]$$

$$=e^{x^{2}}\left(c-\frac{1}{2}e^{-x^{2}}\right)$$

وهذا هو الحل العام للمعادلة.

مثال (2):

حل المعادلة التفاضلية:

$$(x-2)\frac{dy}{dx}=y+2(x-2)^3; \qquad x\neq 2$$

الحــل

بكتابة المعادلة المعطاة على صورة المعادلة (1) يكون

$$\frac{dy}{dx}-\left(\frac{1}{x-2}\right)y=2(x-2)^2$$

وعلى ذلك يكون:

$$p(x)=\frac{1}{2-x}, \qquad g(x)=2(x-2)^2$$

باستخدام (2) فإن:

$$\mu(x)=\exp\int^x p(t)dt=\exp\int^x\frac{dt}{2-t}=\exp\left(\ln\left(\frac{1}{x-2}\right)\right)=\frac{1}{x-2}$$

من (3) فإن:

$$y(x)=(x-2)\left[c+\int^x\frac{2(t-2)^2}{(t-2)}dt\right]$$

$$=(x-2)\left[c+(x-2)^2\right]$$

$$=(x-2)^3+c(x-2).$$

مثال (3):

أوجد الحل العام للمعادلة:

$$y' + y\cot x = 5e^{\cos x}; \qquad x \in (0, \pi)$$

الحــل

$$p(x) = \cot x, \qquad g(x) = 5e^{\cos x}$$

من العلاقة (2)

$$\mu(x) = \exp\int^{x} p(t)dt = \exp\int^{x} \cot t\, dt = \exp(\ln(\sin x)) = \sin x$$

ومن العلاقة (3) فإن الحل العام $y(x)$ يأخذ الصورة:

$$y(x) = \frac{1}{\sin x}\left[c + 5\int^{x} \sin t\, e^{\cos t} dt\right]$$

$$= \frac{1}{\sin x}\left[c - 5e^{\cos x}\right].$$

مثال (4):

أوجد الحل العام للمعادلة:

$$x^3 y' + \left(2 - 3x^2\right)y = x^3; \qquad x \neq 0$$

الحــل

بإعادة كتابة المعادلة فإنها تأخذ الصورة:

$$y' + \left(\frac{2-3x^2}{x^3}\right)y = 1$$

وعلى ذلك

$$p(x) = \frac{2-3x^2}{x^3}, \qquad\qquad g(x) = 1$$

$$\mu(x) = \exp\int^x p(t)dt = \exp\int^x \left(\frac{2-3t^2}{t^3}\right)dt$$

$$= \exp\int^x \left(2\frac{dt}{t^3} - 3\frac{dt}{t}\right)$$

$$= \exp\left[-\frac{1}{x^2} + \ln\left(\frac{1}{x^3}\right)\right]$$

$$= \frac{1}{x^3}e^{-\frac{1}{x^2}}$$

وعلى ذلك فإن الحل العام

$$y(x) = x^3 e^{\frac{1}{x^2}}\left[c + \int^x \frac{1}{t^3}e^{-\frac{1}{t^2}}dt\right]$$

$$= x^3 e^{\frac{1}{x^2}}\left[c + \frac{1}{2}e^{-\frac{1}{x^2}}\right]$$

$$= \frac{1}{2}x^3 + cx^3 e^{-\frac{1}{x^2}}.$$

مثال (5):

حل المعادلة

$$y' - 2y\cot 2x = 1 - 2x\cot 2x - 2\operatorname{cosec} 2x\ ; \quad x \in (0,\ \pi)$$

الحــل

$$\mu(x) = \exp\int^{x} p(t)dt = \exp\int^{x} -2\cot(2t)dt$$

$$= \exp(-\ln\sin 2x)$$

$$= \operatorname{cosec} 2x\ .$$

وعلى ذلك

$$y(x) = \sin 2x\left\{c + \int^{x}\operatorname{cosec}(2t)[1 - 2t\cot 2t - 2\operatorname{cosec} 2t]dt\right\}$$

$$= \sin(2x)\{c + x\operatorname{cosec}(2x) + \cot(2x)\}$$

$$= c\sin 2x + x + \cos 2x.$$

وهو الحل العام للمعادلة.

(3–2) معادلة برنولي Bernoulli Equation

هي معادلة تفاضلية من الرتبة الأولى وهي غير خطية وصورتها كالآتي:

$$\boxed{y' + p(x)y = y^{n}g(x)}$$

حيث n ثابت لا يساوي الواحد الصحيح.

وبإجراء التحويل

$$v(x) = y^{1-n}$$

$$\therefore \ \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y^n}{1-n}\frac{dv}{dx}$$

بالتعويض عن y' في معادلة برنولي نحصل على:

$$\frac{y^n}{1-n}\frac{dv}{dx} + yp(x) = y^n g(x)$$

$$\Rightarrow \qquad \frac{dv}{dx} + (1-n)v(x)p(x) = (1-n)g(x)$$

وهي معادلة تفاضلية على شكل المعادلة (1) ص18 المجهول فيها هو $v(x)$... وبحلها يمكن إيجاد قيمة $v(x)$ وبالتالي $y(x)$.

مثال (1):

حل المعادلة

$$y' - y = xy^5$$

الحــل

هذه المعادلة هي معادلة برنولي حيث $n = 5$ لذلك نفرض أن:

$$v(x) = y^{-4} \qquad \Rightarrow \qquad y' = -\frac{1}{4}y^5\frac{dv}{dx}$$

بالتعويض في المعادلة الأصلية يكون

$$-\frac{1}{4}y^{5}\frac{dv}{dx}-y=xy^{5}$$

$$\Rightarrow \qquad \frac{dv}{dx}+4v=-4x$$

$$\therefore\ \mu(x)=\exp\int^{x}4dx=e^{4x}$$

$$\therefore\ v(x)=e^{-4x}\left[c-4\int 4e^{4x}dx\right]$$

$$=e^{-4x}\left[c-xe^{4x}+\frac{1}{4}e^{4x}\right]$$

$$\therefore\ y(x)=e^{x}\left[c-xe^{4x}+\frac{1}{4}e^{4x}\right]^{-\frac{1}{4}}$$

وهو الحل العام للمعادلة المطلوبة.

مثال (2):

حل المعادلة

$$y'+2xy=-xy^{4}$$

الحـــل

هي معادلة برنولي حيث $n = 4$

نضع

$$v(x) = y^{-3} \qquad \Rightarrow \qquad y' = -\frac{1}{3} y^4 \frac{dv}{dx}$$

بالتعويض في المعادلة المعطاة فإن:

$$-\frac{1}{3} y^4 \frac{dv}{dx} + 2xy = -xy^4$$

$$\therefore \; \frac{dv}{dx} - 6xv = 3x$$

$$\therefore \; \mu(x) = \exp \int^x -6x\,dx = \exp\left(-3x^2\right) = e^{-3x^2}$$

$$\therefore \; v(x) = e^{3x^2}\left[c + 3\int^x xe^{-3x^2}\,dx\right]$$

$$= e^{3x^2}\left[c - \frac{1}{2}e^{-3x^2}\right]$$

$$= ce^{3x^2} - \frac{1}{2}$$

$$\therefore \; y(x) = \left[ce^{3x^2} - \frac{1}{2}\right]^{-\frac{1}{3}}.$$

مثال (3):

حل المعادلة

$$y' + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$$

الحــل

هي معادلة برنولي حيث $n = 4$ لذلك نضع

$$v = y^{-3} \qquad \Rightarrow \qquad y' = -\frac{1}{3} y^{4} \frac{dv}{dx}$$

بالتعويض في المعادلة الأصلية

$$\therefore \; -\frac{1}{3} y^{4} \frac{dv}{dx} + \frac{1}{3} y = \frac{1}{3}(1-2x) y^{4}$$

$$\Rightarrow \qquad \frac{dv}{dx} - v = 2x - 1$$

$$\therefore \; \mu(x) = \exp \int^{x} - dt = e^{-x}$$

$$\therefore \; v(x) = e^{x}\left[c + \int^{x} (2t-1) e^{-t} dt \right]$$

$$= ce^{x} - 1 - 2x$$

$$\therefore \; y(x) = \left[ce^{x} - 1 - 2x \right]^{-\frac{1}{3}}.$$

مثال (4):

حل المعادلة:

$$y' - \frac{y}{x} = (1 + \ln x) y^{3}; \qquad x > 0$$

الحــل

هي معادلة برنولي حيث $n = 3$. بوضع

$$v = y^{-2} \qquad \Rightarrow \qquad y' = -\frac{1}{2} y^3 \frac{dv}{dx}$$

$$\therefore \; -\frac{1}{2} y^3 \frac{dv}{dx} - \frac{1}{x} y = (1 + \ln x) y^3$$

$$\therefore \; \frac{dv}{dx} + \frac{2}{x} v = 1 + \ln x$$

$$\therefore \; \mu(x) = \exp\left[2\int \frac{dx}{x}\right] = x^2$$

$$\therefore \; v(x) = \frac{1}{x^2}\left[c + \int x^2 (1 + \ln x) dx\right]$$

$$= \frac{1}{x^2}\left[c - \frac{4}{9} x^3 - \frac{2}{3} x^3 \ln x\right]$$

$$\therefore \; y(x) = \left[\frac{c}{x^2} - \frac{2}{3} x \left(\frac{2}{3} + \ln x\right)\right]^{-\frac{1}{2}}.$$

مثال (5):

أوجد الحل العام للمعادلة:

$$y' + \frac{2}{x} y = \frac{1}{x^2 y}; \qquad x > 0$$

الحــل

هي معادلة برنولي حيث $n = -1$ لذلك نفرض أن

$$v = y^2 \qquad \Rightarrow \qquad \frac{dv}{dx} = 2y\frac{dy}{dx}$$

بالتعويض عن y' في المعادلة الأصلية

$$\frac{1}{2y}v' + \frac{2}{x}y = \frac{1}{x^2 y}$$

$$\Rightarrow \qquad v' + \frac{4}{x}v = \frac{2}{x^2}$$

$$\therefore\ \mu(x) = \exp\int\frac{4dx}{x} = x^4$$

$$\therefore\ v(x) = \frac{1}{x^4}\left[c + 2\int x^2 dx\right]$$

$$= \frac{c}{x^4} + \frac{2}{3x}$$

$$\therefore\ y = \sqrt{\frac{c}{x^4} + \frac{2}{3x}}\ .$$

تمارين

أوجد الحل العام لكل مما يأتـي:

(1) $y' + 3y = x + e^{-2x}$

(2) $y' - 2y = x^2 e^{2x}$

(3) $y' + y = xe^{-x} + 1$

(4) $y' + \frac{1}{x} y = 3\cos 2x\,;$ $\quad x > 0$

(5) $y' - y = 2e^{x}$

(6) $xy' + 2y = \sin x\,;$ $\quad x > 0$

(7) $y' + \frac{1}{x} y = \sin x\,;$ $\quad x > 0$

(8) $x^2 y' + 3xy = \frac{\sin x}{x}\,;$ $\quad x > 0$

(9) $y' + y\tan x = x\sin 2x\,;$ $\quad x \in \left(0, \frac{\pi}{2}\right)$

(10) $xy' + 2y = e^{x}\,;$ $\quad x < 0$

أوجد الحل الوحيد لكل مما يأتـي:

(11) $y' - y = 2xe^{2x}\,;$ $\quad y(0) = 1$

(12) $y' + y = \frac{1}{1 + x^2}\,;$ $\quad y(0) = 0$

(13) $y' + \frac{2}{x} y = \frac{\cos x}{x^2}\,;$ $\quad x > 0,\; y(\pi) = 0$

(14) $y' - 2y = e^{2x};$ $\qquad y(0) = 2$

(15) $xy' + 2y = \sin x;$ $\qquad x > 0,\ y\left(\frac{\pi}{2}\right) = 1$

(16) $xy' + y = e^{x};$ $\qquad x > 0, y(1) = 1$

(17) $xy' + 2y = x^2 - x + 1;$ $\qquad x > 0,\ y(1) = 0$

(18) $y' + y\cot x = 2\operatorname{cosec} x,$ $\qquad x \in \left(0, \frac{\pi}{2}\right)$

(19) $y' + 2y = xe^{-2x};$ $\qquad y(1) = 0$

أوجد الحل العام لكل مما يأتي:

(20) $x^2 y' + 2xy - y^3 = 0;$ $\qquad x > 0$

(21) $y' - y = xy^2;$

(22) $y' + y = y^2 e^x$

(23) $xy' + y = x^3 y^6;$ $\qquad x > 0$

(24) $yy' - xy^2 + x = 0$

أوجد الحل العام للمعادلة التفاضلية الآتية:

(25) $\dfrac{dy}{dx} = \dfrac{1}{e^y - x}$

(26) نفرض أن $y_1(x)$ هو حل للمعادلة:

$$y' + p(x)\, y = 0 \qquad \text{(i)}$$

ونفرض أن $y_2(x)$ هو حل للمعادلة

$$y' + p(x)\, y = g(x) \qquad \text{(ii)}$$

أثبت أن $y(x) = y_1(x) + y_2(x)$ هي أيضاً حل للمعادلة التفاضلية (ii).

تحويـــــــلات لابـــلاس
Laplace Transforms

(1) مقدمة:

من دراستنا السابقة للتكاملات المعتلة نجد أن

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

مثال (1):

احسب قيمة

$$\int_0^{\infty} e^{sx}\,dx$$

حيث s بارامتر

الحــل

$$\int_0^{\infty} e^{sx}\,dx = \lim_{b\to\infty}\int_0^b e^{sx}\,dx$$

$$= \lim_{b\to\infty}\frac{1}{s}\left[e^{sx}\right]_0^b$$

$$= \frac{1}{s}\lim_{b\to\infty}\left(e^{sb}-1\right)$$

وعلى ذلك يكون التكامل تقاربي إذا كانت $s < 0$ وتباعدي إذا كانت $s > 0$.

(2) تحويلات لابلاس:

تعريف:

يعرف تحويل لابلاس للدالة $f(x)$ بأنه

$$L\{f(x)\}=\int_0^\infty e^{-sx}f(x)dx=F(s)$$

حيث s بارامتر، $L\{.\}$ هو مؤثر لابلاس

واضح أن نتيجة التكامل النهائية تعتمد على s فقط وهذا إذا كان التكامل تقاربي.

والآن سوف نعطي الشرط الكافي لكي يكون للدالة تحويل لابلاس.

الشرط الكافي لوجود تحويل لابلاس:

نفرض أن $f(x)$ هي دالة piece wise continuous وذلك في الفترة $[0,\infty)$ كذلك نفرض أن $f(x)\le k\,e^{ax}$ حيث k ثابت موجب، a ثابت حقيقي. فإن الدالة $f(x)$ قابلة لتحويلات لابلاس، أي أنه يمكن حساب تحويل لابلاس لهذه الدالة (أي أن التكامل تقاربي) ويعطى من العلاقة

$$L\{f(x)\}=F(s)=\int_0^\infty e^{-sx}f(x)dx$$

البرهان:

$$\int_0^\infty e^{-sx}f(x)dx\le k\int_0^\infty e^{-sx}e^{ax}dx$$

$$=k\int_0^\infty e^{(a-s)x}dx$$

$$=\frac{k}{a-s}e^{(a-s)x}\Big|_0^\infty$$

ويكون التكامل تقاربي إذا كانت $s > a$.

أمثلة:

يلاحظ أن الدوال الآتية هي من درجة الدالة ke^{ax}.

$$x^n, \sin x, xe^{bx}$$

الحــل

$$\text{(i)} \quad \lim_{x\to\infty} \frac{x^n}{e^{mx}} = \lim_{x\to\infty} \frac{n!}{m^n e^{mx}} = 0$$

$$\text{(ii)} \quad \lim_{x\to\infty} \frac{\sin x}{e^{mx}} = 0$$

$$\text{(iii)} \quad \lim_{x\to\infty} \frac{xe^{bx}}{e^{mx}} = \lim_{x\to\infty} \frac{x}{e^{(m-b)x}} \to 0 \; if \;\; m > b$$

في جميع الأمثلة السابقة يمكن إيجاد k, a بحيث تكون $f(x) \le k\, e^{ax}$.

مثال (2):

الدالة e^{x^2} ليست من درجة e^{mx}

الحــل

$$\lim_{x\to\infty} \frac{e^{x^2}}{e^{mx}} = \lim_{x\to\infty} e^{(x-m)x}$$

يلاحظ أنه لعدد اختيارى m فإن x تكون أكبر منها وعلى ذلك فإن النهاية تؤول إلى ∞.

الآن سوف نبدأ بإيجاد تحويلات لابلاس لبعض الدوال المشهورة.

مثال (3):

إذا كانت

$$f(x) = 1$$

فإن

$$L\{1\} = \int_0^{\infty} e^{-sx}(1)\,dx = \lim_{b\to\infty}\int_0^{b} e^{-sx}\,dx$$

$$= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}; \qquad s > 0$$

لأحظ أن

$$e^{-sb} \to 0 \ as\ b \to \infty \ \text{ for s > 0.}$$

مثال (4):

إذا كانت

$$f(x) = x$$

فإن

$$F(s) = L\{x\} = \int_0^{\infty} xe^{-sx}\,dx$$

$$= -\frac{1}{s}\int_0^{\infty} x\,d\left(e^{-sx}\right)$$

بإستخدام التكامل بالتجزىء

$$= -\frac{1}{s} x e^{-sx} \Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-sx}\,dx$$

$$= zero - \frac{1}{s^2} e^{-sx} \Big|_0^{\infty}$$

$$= \frac{1}{s^2}$$

مثال (5):

باستخدام مبدأ الاستنتاج الرياضي أثبت أن:

$$F(s) = L\{x^n\} = \frac{n!}{s^{n+1}}$$

حيث n **صحيح موجب.** (يترك للطالب)

مثال (6):

إذا كانت

$$f(x) = e^{ax}$$

حيث a ثابت

$$F(s) = L\{e^{ax}\} = \int_0^{\infty} e^{ax} e^{-sx}\,dx = \int_0^{\infty} e^{-(s-a)x}\,dx$$

$$= -\frac{1}{s-a} e^{-(s-a)x} \Big|_0^{\infty}$$

$$= \frac{1}{s-a} \qquad ; s > a$$

مثال (7):

أثبت أن $L\{.\}$ مؤثر خطي أي أن:

$$L\{c_1 f_1 + c_2 f_2\} = c_1 L\{f_1\} + c_2 L\{f_2\}$$

الحـل

$$\begin{aligned} L\{c_1 f_1 + c_2 f_2\} &= \int_0^\infty (c_1 f_1 + c_2 f_2) e^{-sx} dx \\ &= c_1 \int_0^\infty f_1(x) e^{-sx} dx + c_2 \int_0^\infty f_2(x) e^{-sx} dx \\ &= c_1 L\{f_1\} + c_2 L\{f_2\} \end{aligned}$$

وهو المطلوب

مثال (8):

إذا كانت

$$f(x) = \sinh bx$$

فإن

$$\begin{aligned} L\{\sinh bx\} &= L\left\{\frac{1}{2}\left(e^{bx} - e^{-bx}\right)\right\} \\ &= \frac{1}{2}\left(L\{e^{bx}\} - L\{e^{-bx}\}\right) \\ &= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) \end{aligned}$$

$$= \frac{b}{s^2 - b^2}$$

مثال (9):

إذا كانت

$$f(x) = \cos bx$$

فإن

$$F(s) = L\{\cos bx\} = \int_0^\infty \cos bx \; e^{-sx} dx$$

$$= -\frac{1}{s}\int_0^\infty \cos bx \; d\left(e^{-sx}\right)$$

بإستخدام التكامل يالتجزىء

$$= -\frac{1}{s}\cos bx \; e^{-sx}\Big|_0^\infty - \frac{b}{s}\int_0^\infty \sin bx \; e^{-sx} dx$$

$$= \frac{1}{s} + \frac{b}{s^2}\int \sin bx \; d\left(e^{-sx}\right)$$

بإستخدام التكامل بالتجزىء مرة أخرى للتكامل أعلاه

$$= \frac{1}{s} + \frac{b}{s^2}\sin bx \; e^{-sx}\Big|_0^\infty - \frac{b^2}{s^2}\int_0^\infty \cos bx \; e^{-sx} dx$$

$$F(s) = \frac{1}{s} + 0 - \frac{b^2}{s^2}F(s)$$

$$\left(1+\frac{b^2}{s^2}\right)F(s)=\frac{1}{s}$$

$$\therefore F(s)=\frac{s}{b^2+s^2}$$

ملاحظة:

عرفنا سابقاً الشرط الكافي لوجود تحويل لابلاس.

مثال (10):

الدالة $f(x)=\frac{1}{\sqrt{x}}$ هي ليست piece wise continuous في الفترة $[0,\infty]$

ولكن يوجد لها تحول لابلاس. وذلك لأن

$$F(s)=\int_0^\infty e^{-sx}x^{\frac{-1}{2}}dx=L\left\{\frac{1}{\sqrt{x}}\right\}$$

نفرض أن

$$\sqrt{sx}=t \Rightarrow sx=t^2 \Rightarrow \qquad dx=\frac{2}{s}t\,dt$$

$$F(s)=\frac{2}{\sqrt{s}}\int_0^\infty e^{-t^2}dt=\frac{2}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}\right)=\sqrt{\frac{\pi}{s}}$$

مثال (11):

الدالة $f(x)=x^p$ حيث $-1<p<0$ ليست P-C ولكن لها تحويل لابلاس لأن

$$F(s)=L\{x^p\}=\int_0^\infty e^{-sx}x^p dx$$

نفرض أن

$$s\,x=t \qquad \Rightarrow \qquad dx=\frac{dt}{s}$$

$$F(s)=\int_0^\infty e^{-t}\left(\frac{t}{s}\right)^p\frac{dt}{s}=\frac{1}{s^{p+1}}\int_0^\infty t^p e^{-t}dt$$

ومن تعريف دالة جاما نعلم أن

$$\Gamma(p+1)=\int_0^\infty t^p e^{-t}dt$$

$$\therefore\ F(s)=\frac{\Gamma(p+1)}{s^{p+1}}$$

مثال (12):

إذا كانت $f(x)=x^p$ حيث p عدد قياسي موجب

$$F(s)=L\{x^p\}=\int_0^\infty x^p e^{-sx}dx$$

بوضع

$$s\,x=t \qquad \Rightarrow \qquad dx=\frac{dt}{s}$$

$$F(s)=\frac{1}{s^{p+1}}\int_0^\infty t^p e^{-t}dt=\frac{\Gamma(p+1)}{s^{p+1}}$$

من مثال (10)، (11) يكون

$$L\left\{x^{p}\right\}=\frac{\Gamma\left(p+1\right)}{s^{p+1}} \qquad ; s>0$$

حيث p عدد قياسي موجب أكبر من 1- .

مثال (13):

احسب تحويلات لابلاس للدالة $f'(x)$

الحـل

$$F(s)=L\{f'(x)\}=\int_{0}^{\infty}e^{-sx}f'(x)dx$$

$$=\int_{0}^{\infty}e^{-sx}df(x)$$

$$=\left.\frac{f(x)}{e^{sx}}\right|_{0}^{\infty}+s\int_{0}^{\infty}e^{-sx}f(x)dx$$

$$=0-f(0)+sF(s)$$

$$=sF(s)-f(0)$$

مثال (14):

احسب تحويلات لابلاس للدالة $f''(x)$

الحـل

$$L\{f''(x)\}=\int_0^\infty e^{-sx}f''(x)\,dx$$

$$=\int_0^\infty e^{-sx}d\left(f'(x)\right)$$

$$=e^{-sx}f'(x)\Big|_0^\infty+s\int_0^\infty e^{-sx}f'(x)dx$$

$$=0-f'(0)+sL\{f'(x)\}$$

$$=-f'(0)+s\,[s\,F(s)-f(0)]$$

$$=s^2F(s)-sf(0)-f'(0)$$

ويمكن باستخدام الاستنتاج الرياضي إثبات أن:

$$L\{f^{(n)}(x)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0).$$

مثال (15):

احسب تحويل لابلاس للتكامل

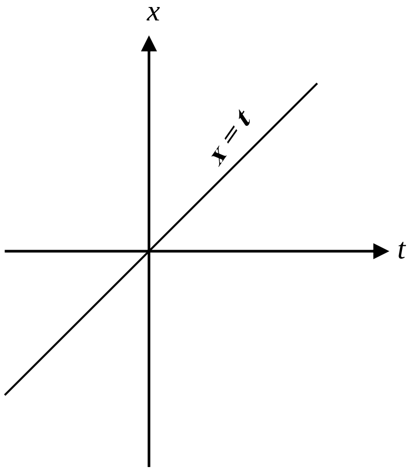
$$\int_0^x f(t)g(x-t)dt$$

الحـل

$$L\left\{\int_0^x f(t)g(x-t)dt\right\} = \int_0^\infty e^{-sx}dx\int_0^x f(t)g(x-t)dt$$

$$= \int_0^\infty\int_0^x e^{-sx} f(t)g(x-t)dt\,dx$$

وبعكس ترتيب التكامل يكون



$$= \int_{t=0}^{\infty} f(t)dt \int_{x=t}^{x=\infty} g(x-t)e^{-sx}dx$$

نفرض أن:

$u = x - t$ (في التكامل الداخلي)

$\therefore\ du = dx$

$$= \int_{t=0}^{\infty} f(t)dt \int_{u=0}^{u=\infty} g(u)e^{-s(u+t)}du$$

$$= \int_{t=0}^{\infty} f(t)e^{-st}dt \int_{u=0}^{u=\infty} g(u)e^{-su}du$$

$$= \int_{t=0}^{\infty} f(t) e^{-st} G(s) dt$$

$$= G(s) \int_{t=0}^{\infty} e^{-st} f(t) dt$$

$$= G(s)\, F(s)$$

حيث

$$L\{g(x)\} = G(s), \qquad L\{f(x)\} = F(s)$$

مثال (16):

أثبت أن

$$L\left\{\int_0^x f(t) dt\right\} = \frac{F(s)}{s}$$

لحــل

نفرض أن:

$$G(x) = \int_0^x f(t) dt$$

ومنها

$$G'(x) = \frac{dG}{dx} = f(x), \qquad G(0) = 0$$

وبأخذ مؤثر لابلاس للطرفين يكون

$$\therefore\ L\{G'(x)\} = L\{f(x)\}$$

$$\therefore\ sL\{G(x)\}-G(0)=F(s)$$

$$\therefore\ sL\left\{\int_0^x f(t)dt\right\}-0=F(s)$$

$$\therefore\ L\left\{\int_0^x f(t)dt\right\}=\frac{F(s)}{s}$$

وهو المطلوب

مثال (17):

أثبت أن

$$l\left\{\int_0^x\int_0^t f(u)du\,dt\right\}=\frac{F(s)}{s^2}$$

الحــل

نفرض أن

$$G(x)=\int_0^x\int_0^t f(u)du\,dt$$

$$\therefore\ G'(x)=\frac{dG}{dx}=\int_0^x f(u)du$$

$$G''(x)=f(x),\qquad G(0)=G'(0)=0$$

وبأخذ مؤثر لابلاس للطرفين يكون

$$L\{G''(x)\}=L\{f(x)\}$$

$$\therefore\ s^2L\left\{G\left(x\right)\right\}-sG\left(0\right)-G'\left(0\right)=F\left(s\right)$$

$$\therefore\ s^2L\left\{G\left(x\right)\right\}-0-0=F\left(s\right)$$

$$\therefore\ L\left\{\int_0^x\int_0^t f\left(u\right)du\,dt\right\}=\frac{F\left(s\right)}{s^2}$$

مثال (18):

أثبت أنه لأي عدد صحيح غير سالب n يكون:

$$L\left\{x^n f\left(x\right)\right\}=\left(-1\right)^n\frac{d^nF\left(s\right)}{ds^n}$$

حيث

$$L\{f(x)\}=F(s)$$

الحـل

$$\frac{d^nF\left(s\right)}{ds^n}=\frac{d^n}{ds^n}\left[\int_0^\infty e^{-sx}f\left(x\right)dx\right]$$

$$=\int_0^\infty\frac{\partial^n}{\partial s^n}\left(e^{-sx}f\left(x\right)dx\right)$$

$$=\left(-1\right)^n\int_0^\infty e^{-sx}\,x^nf\left(x\right)dx$$

$$=\left(-1\right)^nL\left\{x^nf\left(x\right)\right\}$$

وهو المطلوب.

مثال (19):

باستخدام القاعدة في المثال السابق حيث $n = 2$ يكون

$$L\left\{x^2 \sin bx\right\} = (-1)^\wedge 2 \frac{d^2}{ds^2} L\{\sin bx\} = \frac{d^2}{ds^2}\left(\frac{b}{s^2+a^2}\right)$$

$$= \frac{a^2 - 3s^2}{\left(s^2+a^2\right)^3}$$

No	$f(x)=L^{-1}\{F(s)\}$	$F(s)=L\{f(x)\}$
1	1	$\frac{1}{s}$; $s>0$
2	e^{ax}	$\frac{1}{s-a}$; $s>a$
3	x^n, صحيح موجب n	$\frac{n!}{s^{n+1}}$; $s>a$
4	x^p , عدد قياسي أكبر من سالب واحد p	$\frac{\Gamma(p+1)}{s^{p+1}}$; $s>0$
5	$\sin bx$	$\frac{b}{s^2+b^2}$; $s>0$
6	$\cos bx$	$\frac{s}{s^2+b^2}$; $s>0$
7	$\sinh bx$	$\frac{b}{s^2-b^2}$; $s>0$
8	$\cosh bx$	$\frac{s}{s^2-b^2}$; $s>0$
9	$e^{ax}\sin bx$	$\frac{b}{(s-a)^2+b^2}$; $s>a$
10	$e^{ax}\cos bx$	$\frac{s-a}{(s-a)^2+b^2}$; $s>a$
11	$e^{ax}\sinh bx$	$\frac{b}{(s-a)^2-b^2}$; $s>a$
12	$e^{ax}\cosh bx$	$\frac{s-a}{(s-a)^2-b^2}$; $s>a$

13	$x^n e^{ax}$ صحيح موجب n	$\frac{n!}{(s-a)^{n+1}}$; $s>a$
14	$e^{ax} f(x)$	$F(s-a)$
15	$f(ax)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$; $a>0$
16	$\int_0^x f(x-t)g(t)dt$ or $\int_0^x f(t)g(x-t)dt$	$F(s)G(s)$
17	$(-x)^n f(x)$, صحيح موجب n	$F^{(n)}(s)$
18	$f^{(n)}(x)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19	$\int_0^x f(t)dt$	$\frac{F(s)}{s}$; $s>0$
20	$\int_0^x \int_0^t f(u)dudt$	$\frac{F(s)}{s^2}$; $s>0$

تمارين

1- في الجدول السابق أثبت صحة العلاقات 18 ,17 ,15 ,14 ,13 ,12

2- أوجد $f(x)$ إذا كانت:

(i) $$F(s)=\frac{2s}{s^2+2s-1}$$

(ii) $$F(s)=\frac{2s^2+s+1}{(s^2+s+1)(s^2-2s+3)}$$

(iii) $$F(s)=\frac{s+1}{(s\ \ -2s-4)(s+1)^2}$$

(iv) $$F(s)=\frac{1}{(s^2+2)^2}$$

(v) $$F(s)=\frac{1}{(s^2+2)(s+1)(s-3)}$$

Inverse transforms

Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs.

For example, we know that $\dfrac{a}{s^2+a^2}$ is the Laplace transform of $\sin at$,

so we can now write $L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$, the symbol L^{-1} indicating the inverse transform and **not** a reciprocal.

$\therefore$ (a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = \dots\dots\dots\dots$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = \dots\dots\dots\dots$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\}\dots\dots\dots\dots$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = \dots\dots\dots\dots$

(a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = 4$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\} = \cos 5t$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = 4\sinh 3t$

Therefore, given a transform, we can write down the corresponding expression in t, provided we can recognise it from our table of transforms.

▶

But what about $L^{-1}\left\{\dfrac{3s+1}{s^2-s-6}\right\}$? This certainly did not appear in our list of standard transforms.

In considering $L^{-1}\left\{\dfrac{3s+1}{s^2-s-6}\right\}$, it happens that we can write $\dfrac{3s+1}{s^2-s-6}$ as the sum of two simpler functions $\dfrac{1}{s+2}+\dfrac{2}{s-3}$ which, of course, makes all the difference, since we can now proceed

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}$$

which we immediately recognise as ....

$$\boxed{e^{-2t}+2e^{3t}}$$

The two simpler expressions $\dfrac{1}{s+2}$ and $\dfrac{2}{s-3}$ are called the *partial fractions* of $\dfrac{3s+1}{s^2-s-6}$, and the ability to represent a complicated algebraic fraction in terms of its partial fractions is the key to much of this work. Let us take a closer look at the rules.

Rules of partial fractions

1. The numerator must be of lower degree than the denominator. This is usually the case in Laplace transforms. If it is not, then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fractions.
3. A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\dfrac{A}{(s+a)}+\dfrac{B}{(s+a)^2}$.
5. Similarly $(s+a)^3$ gives $\dfrac{A}{(s+a)}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$.
6. A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$.
7. Repeated quadratic factors $(s^2+ps+q)^2$ give

   $$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

So $\dfrac{s-19}{(s+2)(s-5)}$ has partial fractions of the form ....

$$\boxed{\frac{A}{s+2}+\frac{B}{s-5}}$$

and $\dfrac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fractions of the form ....

Be careful of the repeated factor.

$$\boxed{\frac{A}{s+3}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}}$$

Let us work through the various steps with an example.

Example 1

To determine $L^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$.

(a) First we check that the numerator is of lower degree than the denominator. In fact, this is so.

(b) Factorise the denominator $\dfrac{5s+1}{s^2-s-12}=\dfrac{5s+1}{(s-4)(s+3)}$.

(c) Then the partial fractions are of the form ....

$$\boxed{\frac{A}{s-4}+\frac{B}{s+3}}$$

We therefore have the identity

$$\frac{5s+1}{s^2-s-12}\equiv\frac{A}{s-4}+\frac{B}{s+3}$$

If we multiply through both sides by the denominator $s^2-s-12\equiv(s-4)(s+3)$ we have

$$5s+1\equiv A(s+3)+B(s-4)$$

This is also an identity and true for any value of s we care to substitute – our job is now to find the values of A and B.

We now substitute convenient values for s

(a) Let $(s-4)=0$, i.e. $s=4$ $\quad\therefore\;21=A(7)+B(0)\quad\therefore\;A=3$

(b) Let $(s+3)=0$, i.e. $s=-3$ and we get ....

$$\boxed{B = 2}$$

$$\therefore \quad \frac{5s+1}{s^2 - s - 12} \equiv \frac{3}{s-4} + \frac{2}{s+3}$$

$$\therefore \quad L^{-1}\left\{\frac{5s+1}{s^2 - s - 12}\right\} = \ldots\ldots\ldots\ldots$$

$$\boxed{3e^{4t} + 2e^{-3t}}$$

Example 2

Determine $L^{-1}\left\{\dfrac{9s-8}{s^2-2s}\right\}$.

Working as before, $f(t) = \ldots\ldots\ldots\ldots$

$$\boxed{4 + 5e^{2t}}$$

Because

$$L\{f(t)\} = \frac{9s-8}{s^2-2s}.$$

(a) Numerator of first degree; denonominator of second degree. Therefore rule satisfied.

(b) $\dfrac{9s-8}{s(s-2)} \equiv \dfrac{A}{s} + \dfrac{B}{s-2}$.

(c) Multiply by $s(s-2)$. $\therefore$ $9s - 8 = A(s-2) + Bs$.

(d) Put $s = 0$. $\quad -8 = A(-2) + B(0) \quad \therefore \ A = 4$.

(e) Put $s - 2 = 0$, i.e. $s = 2$. $\quad 10 = A(0) + B(2) \quad \therefore \ B = 5$.

$$\therefore \quad f(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Example 3

Express $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

$\dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ has partial fractions of the form $\ldots\ldots\ldots\ldots$

$$\boxed{\frac{A}{s+2}+\frac{B}{s-3}+\frac{C}{(s-3)^2}}$$

Now we multiply throughout by $(s+2)(s-3)^2$ and get

$$s^2-15s+41 \equiv A(s-3)^2+B(s+2)(s-3)+C(s+2)$$

Putting $(s-3)=0$ and then $(s+2)=0$ we obtain ....

$$\boxed{A=3 \text{ and } C=1}$$

Now that we have run out of 'crafty' substitutions, we equate coefficients of the highest power of s on each side, i.e. the coefficients of s^2. This gives ....

$$\boxed{1=A+B \quad \therefore\ 1=3+B \quad \therefore\ B=-2}$$

So $\dfrac{s^2-15s+41}{(s+2)(s-3)^2}=\dfrac{3}{s+2}-\dfrac{2}{s-3}+\dfrac{1}{(s-3)^2}$

Now $L^{-1}\left\{\dfrac{3}{s+2}\right\}=$ .... and $L^{-1}\left\{\dfrac{2}{s-3}\right\}=$ ....

$$\boxed{3e^{-2t} \text{ and } 2e^{3t}}$$

But what about $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$?

We remember that $L^{-1}\left\{\dfrac{1}{s^2}\right\}=$ ....

$$\boxed{t}$$

and that by Theorem 1, if $L\{f(t)\}=F(s)$ then $L\{e^{-at}f(t)\}=F(s+a)$.

$\therefore\ \dfrac{1}{(s-3)^2}$ is like $\dfrac{1}{s^2}$ with s replaced by $(s-3)$ i.e. $a=-3$.

$$\therefore\ L^{-1}\left\{\frac{1}{(s-3)^2}\right\}=te^{3t}$$

$$\therefore\ L^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\}=3e^{-2t}+2e^{3t}+te^{3t}$$

▶

Example 4

Determine $L^{-1}\left\{\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$.

Notice that this time we have a quadratic factor in the denominator

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)} \equiv \frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$
$$\therefore\ 4s^2-5s+6 \equiv A(s^2+4)+(Bs+C)(s+1).$$

(a) Putting $(s+1)=0$, i.e. $s=-1$, $\quad 15=5A \quad \therefore\ A=3$

(b) Equate coefficients of highest power, i.e. s^2

$4=A+B \quad \therefore\ 4=3+B \quad \therefore\ B=1$

(c) We now equate the lowest power on each side, i.e. the constant term

$6=4A+C \quad \therefore\ 6=12+C \quad \therefore\ C=-6$

Now you can finish it off. $f(t)=\ldots\ldots\ldots\ldots$

$$\boxed{f(t)=3e^{-t}+\cos 2t-3\sin 2t}$$

Because

$$L\{f(t)\}=\frac{3}{s+1}+\frac{s}{s^2+4}-\frac{6}{s^2+4}$$
$$\therefore\ f(t)=3e^{-t}+\cos 2t-3\sin 2t$$

The 'cover up' rule

While we can always find A, B, C, etc., there are many cases where we can use the 'cover up' methods and write down the values of the constant coefficients almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors. The following examples illustrate the method.

Example 1

We know that $F(s) = \dfrac{9s-8}{s(s-2)}$ has partial fractions of the form $\dfrac{A}{s} + \dfrac{B}{s-2}$.

By the 'cover up' rule, the constant A, that is the coefficient of $\dfrac{1}{s}$, is found by temporarily covering up the factor s in the denominator of $F(s)$ and finding the limiting value of what remains when s (the factor covered up) tends to zero.

Therefore A = coefficient of $\dfrac{1}{s} = \underset{s\to 0}{Lim}\left\{\dfrac{9s-8}{s-2}\right\} = 4$. That is $A = 4$.

Similarly, B, the coefficient of $\dfrac{1}{s-2}$, is obtained by covering up the factor $(s-2)$ in the denominator of $F(s)$ and finding the limiting value of what remains when $(s-2) \to 0$, that is $s \to 2$.

Therefore B = coefficient of $\dfrac{1}{s-2} = \underset{s\to 2}{Lim}\left\{\dfrac{9s-8}{s}\right\} = 5$. That is $B = 5$.

So that

$$\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{s-2}$$

Another example

Example 2

$$F(s) = \frac{s+17}{(s-1)(s+2)(s-3)} \equiv \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

A: cover up $(s-1)$ in $F(s)$ and find

$$\underset{s\to 1}{Lim}\left\{\frac{s+17}{(s+2)(s-3)}\right\} = \frac{18}{-6} \quad \therefore\ A = -3$$

Similarly

B: .... $\therefore\ B =$ ....

C: .... $\therefore\ C =$ ....

$$B = \underset{s\to -2}{Lim}\left\{\frac{s+17}{(s-1)(s-3)}\right\} = \frac{15}{(-3)(-5)} = 1 \qquad \therefore\ B = 1$$

$$C = \underset{s\to 3}{Lim}\left\{\frac{s+17}{(s-1)(s+2)}\right\} = \frac{20}{(2)(5)} = 2 \qquad \therefore\ C = 2$$

$$\therefore\ F(s) = \frac{1}{s+2} + \frac{2}{s-3} - \frac{3}{s-1}$$

So $f(t) = e^{-2t} + 2e^{3t} - 3e^{t}$

Every entry in our table of standard transforms gives rise to a corresponding entry in a similar table of inverse transforms. Let us tabulate such a list.

Table of inverse transforms

$F(s)$	$f(t)$	
$\dfrac{a}{s}$	a	
$\dfrac{1}{s+a}$	e^{-at}	
$\dfrac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\dfrac{a}{s^2+a^2}$	$\sin at$	
$\dfrac{s}{s^2+a^2}$	$\cos at$	
$\dfrac{a}{s^2-a^2}$	$\sinh at$	
$\dfrac{s}{s^2-a^2}$	$\cosh at$	

Theorem 1

The first shift theorem can be stated as follows.

If $F(s)$ is the Laplace transform of $f(t)$ then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

Here is a short revision exercise.

Exercise

1 Find the inverse transforms of

(a) $\dfrac{1}{2s-3}$; (b) $\dfrac{5}{(s-4)^3}$; (c) $\dfrac{3s+4}{s^2+9}$.

2 Express in partial fractions

(a) $\dfrac{22s+16}{(s+1)(s-2)(s+3)}$; (b) $\dfrac{s^2-11s+6}{(s+1)(s-2)^2}$.

3 Determine

(a) $L^{-1}\left\{\dfrac{4s^2-17s-24}{s(s+3)(s-4)}\right\}$; (b) $L^{-1}\left\{\dfrac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$.

▶

(3) استخدام تحويلات لابلاس لحل المعادلات التفاضلية:

مثال (1):

أوجد الحل الوحيد الآتي:

$$y'' + y = \sin(2x); \qquad y(0) = 0, y'(0) = 1$$

الحــل

بأخذ تحويل لابلاس لطرفي المعادلة يكون

$$L\{y''\} + L\{y\} = L\{\sin 2x\}$$

وباستخدام الجداول نجد أن: (حيث $L\{y(x)\} = Y(s)$)

$$s^2Y - sy(0) - y'(0) + Y = \frac{2}{s^2+4}$$

ومنها

$$Y = \frac{s^2+6}{(s^2+1)(s^2+4)}$$

وباستخدام الكسور الجزيئية يكون

$$Y(s) = \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{2}{3}\left(\frac{1}{s^2+4}\right)$$

وبأخذ مؤثر لابلاس العكسي وباستخدام الجداول نجد أن:

$$y(x) = \frac{5}{3}\sin x - \frac{1}{3}\sin 2x$$

مثال (2):

باستخدام لابلاس أوجد الحل الوحيد

$$y'' + 2y' - y = e^{x}\sin x\ ; \quad y(0) = 0,\ y'(0) = 1$$

الحـل

بأخذ مؤثر لابلاس للطرفين يكون

$$s^2Y - sy(0) - y'(0) + 2\left[sY - y(0)\right] - Y = \frac{1}{(s-1)^2+1}$$

$$\therefore\ \left(s^2 + 2s - 1\right)Y = \frac{1}{(s-1)^2+1} + 1$$

$$Y = \frac{s^2 - 2s + 3}{\left(s^2 - 2s + 2\right)\left(s^2 + 2s - 1\right)}$$

$$= \frac{1}{17}\left[\frac{-16s + 37}{s^2 - 2s + 2} + \frac{16s + 44}{s^2 + 2s - 1}\right]$$

$$= \frac{1}{17}\left[\frac{-16(s-1) + 21}{(s-1)^2+1} + \frac{16(s+1) + 28}{(s+1)^2 - 2}\right]$$

$$= \frac{1}{17}\left[-16\frac{(s-1)}{(s-1)^2+1} + 21\frac{1}{(s-1)^2+1}\right.$$

$$\left. + 16\frac{(s+1)}{(s+1)^2-2} + \frac{28}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2-2}\right]$$

وبأخذ مؤثر لابلاس العكسي للطرفين، يكون

$$y(x) = \frac{1}{17}\left[-16e^{x}\cos x + 21e^{x}\sin x + 16e^{-x}\cosh\sqrt{2}x + 14\sqrt{2}\ e^{x}\sinh\sqrt{2}x\right]$$

تمارين

باستخدام تحويلات لابلاس أوجد الحل الوحيد لكل مما يأتي:

(1) $y'' - y' - 6y = 0$; $y(0) = 1,\ y'(0) = -1$

(2) $y'' + 3y' + 2y = 1$; $y(0) = 1,\ y'(0) = 0$

(3) $y'' - 2y' + 2y = x$; $y(0) = 0,\ y'(0) = 1$

(4) $y'' - 4y' + 4y = e^x$; $y(0) = y'(0) = 0$

(5) $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$; $y(0) = 0,\ y'(0) = 1,$ $y''(0) = 0,\ y'''(0) = 1$

(6) $y^{(4)} - y = 0$; $y(0) = 1,\ y'(0) = 0,$ $y''(0) = 1,\ y'''(0) = 0$

(7) $y'' - 2y' + 2y = \cos x$; $y(0) = 1,\ y'(0) = 0$

(8) $y'' + 2y = \sinh\sqrt{2}x$; $y(0) = y'(0) = 0$

(9) $y'' + 3y' - y = e^x \sin x$; $y(0) = y'(0) = 0$

(10) $y'' - y' - y = e^x \cosh x$; $y(0) = 0,\ y'(0) = 1$

(11) $y''' - 2y'' + y = 1$; $y(0) = y'(0) = y''(0) = 0$

(12) $y''' - y = e^x$; $y(0) = y'(0) = 0,\ y''(0) = 1$

(4) استخدام تحويلات لابلاس لحل مجموعة من المعادلات التفاضلية:

مثال (1):

أوجد الحل الوحيد للمجموعة

$$\frac{dy_1}{dx} + 2y_2 = 1 \qquad ; y_1(0) = 0$$

$$\frac{dy_2}{dx} - y_1 = x \qquad ; y_2(0) = 0$$

الحـل

بأخذ مؤثر لابلاس لطرفي كل من المعادلتين حيث

$$L\{y_1(x)\} = Y_1(s),$$

$$L\{y_2(x)\} = Y_2(s)$$

$$sY_1 + 2Y_2 = \frac{1}{s}$$

$$sY_2 - Y_1 = \frac{1}{s^2}$$

وبحل هاتين المعادلتين يكون

$$Y_1 = \frac{s^2 - 2}{s^2\left(s^2 + 2\right)},$$

$$Y_2 = \frac{2}{s\left(s^2 + 2\right)}$$

وباستخدام الكسور الجزيئية يكون

$$Y_1 = \frac{-1}{s^2} + \frac{2}{s^2+2}$$

$$Y_2 = \frac{1}{s} - \frac{s}{s^2+2}$$

وبأخذ مؤثر لابلاس العكسي يكون

$$y_1(x) = -x + \sqrt{2}\sin\sqrt{2}x$$

$$y_2(x) = 1 - \cos\sqrt{2}x$$

تمارين

أوجد الحل الوحيد لكل من المجموعات الآتية:

$$(1)\quad \begin{cases} y_1' - 2y_2 = x^2 & , y_1(0) = 0 \\ y_2' + 3y_1 = e^x & , y_2(0) = 0 \end{cases}$$

$$(2)\quad \begin{cases} y_1' + 2y_2 + 3y_1 = x & , y_1(0) = 1 \\ y_2' - y_1 = 1 & , y_2(0) = 0 \end{cases}$$

$$(3)\quad \begin{cases} 2y_1' + 4y_2 = 1 & , y_1(0) = 0 \\ y_2' + 3y_1 - y_1' = x & , y_2(0) = 1 \end{cases}$$

$$(4)\quad \begin{cases} y_1' - 2y_2 = \cos x & , y_1(0) = 0 \\ y_2' - y_1 = \sin x & , y_2(0) = 0 \end{cases}$$

$$(5)\quad \begin{cases} y_1' - 2y_2 + y_1 = 1 & , y_1(0) = 0 \\ y_2' - 2y_1 + 3y_2 = x & , y_2(0) = 0 \end{cases}$$

$$(6)\quad \begin{cases} y_1' - y_2 - y_3 = 1 & , y_1(0) = 0 \\ y_2' + y_2 - y_1 = x & , y_2(0) = 0 \\ y_3' - y_1 + 2y_2 = e^x & , y_3(0) = 0 \end{cases}$$

$$(7)\quad \begin{cases} y_1' - 2y_2 = \sin x & , y_1(0) = 0 \\ y_2' + y_1 = 1 + x & , y_2(0) = 0 \\ y_3' - y_2 = \cos x & , y_3(0) = 0 \end{cases}$$

Introduction

The Laplace transform deals with continuous functions and can be used to solve many differential equations that arise in science and engineering. There are occasions, however, when we have to deal with discrete functions – *sequences* – and their associated **difference equations**. For example, the central processing unit of your computer can only handle information in the form of pulses of electricity. This information transmission is called **digital** transmission. There are, however, times when information is fed into the computer in the form of a continuously varying signal called an **analogue** signal. For instance, a mouse can be moved about the flat surface of your desk in a continuous manner but the central processing unit will only recognise position on the screen to the nearest pixel. The analogue signal coming from the mouse needs to be converted into a digital signal for recognition by the computer's central processing unit. This conversion of a signal from analogue to digital is achieved by a device called a **demodulator** that *samples* the analogue signal at regular intervals of time and outputs the sampled values as the digital signal – as a sequence of values. The *Z* transform, which is allied to the Laplace transform, deals with such sequences and the recurrence relations – or difference equations – that arise. 1

Sequences

The sequence $\dots, 3^{-2}, 3^{-1}, 3^0, 3, 3^2, 3^3, \dots$ has a general term of the form 3^k and as a shorthand notation we use $\{3^k\}_{-\infty}^{\infty}$ to represent this sequence and to indicate that the powers range from $-\infty$ to ∞. The sum 2

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \dots + \left(\frac{3}{z}\right)^{-1} + \left(\frac{3}{z}\right)^0 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \dots$$

is called the ***Z* transform** of the sequence, $Z\{3^k\}_{-\infty}^{\infty}$, and is denoted by $F(z)$, where the complex number z is chosen to ensure that the sum is finite. We say that

$\{3^k\}_{-\infty}^{\infty}$ and $Z\{3^k\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k$ form a *Z transform pair*.

For our purposes we shall consider only *causal sequences* of the form $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$ which for brevity we shall denote by $\{x_k\}$ with corresponding Z transform

$$Z\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}.$$

Notice that this is the *definition* of the Z transform of the sequence $\{x_k\}$. For example, the *unit impulse* sequence $\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$ has the Z transform

$Z\{\delta_k\} = \ldots\ldots\ldots\ldots$ valid for $\ldots\ldots\ldots\ldots$ values of z

---

3

$$\boxed{Z\{\delta_k\} = 1 \text{ valid for all values of } z}$$

Because

$$Z\{\delta_k\} = \sum_{k=0}^{\infty} \frac{\delta_k}{z_k}$$
$$= 1 + \frac{0}{z} + \frac{0}{z^2} + \ldots = 1$$

Try another.

The sequence $\{u_k\} = \{1, 1, 1, \ldots\} = \{1\}$ is called the *unit step* sequence and has the Z transform

$\ldots\ldots\ldots\ldots$ provided $|z| \ldots\ldots\ldots\ldots$

Next frame

4

$$\boxed{\frac{z}{z-1} \text{ provided } |z| > 1}$$

Because

$$Z\{u_k\} = F(z)$$
$$= \sum_{k=0}^{\infty} \frac{u_k}{z^k} = \sum_{k=0}^{\infty} \frac{1}{z^k}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$F(z) = \frac{1}{1 - \frac{1}{z}} \text{ provided } \left|\frac{1}{z}\right| < 1$$
$$= \frac{z}{z-1} \text{ provided } |z| > 1$$

And another.

Given the causal sequence $\{x_k\} = \{1, a, a^2, a^3, a^4, \ldots\} = \{a^k\}$

the Z transform is $\ldots\ldots\ldots\ldots$

Next frame

5

$$\boxed{\frac{z}{z-a} \text{ provided } |z| > a}$$

Because

$$\begin{aligned} Z\{a^k\} &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \\ &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots \end{aligned}$$

Comparing this to the series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots \\ &= \frac{1}{1 - \dfrac{a}{z}} \text{ provided } \left|\frac{a}{z}\right| < 1. \end{aligned}$$

That is, multiplying numerator and denominator by z

$$F(z) = \frac{z}{z-a} \text{ provided } |z| > |a|$$

Therefore $\{a^k\}$ and $F(z) = \dfrac{z}{z-a}$, $(|z| > |a|)$ form a Z transform pair.

Let's try another. The sequence $\{x_k\} = \{0, 1, 2, 3, 4, \ldots\} = \{k\}$ has the Z transform

$$Z\{k\} = F(z) = \ldots\ldots\ldots\ldots$$

Answer in the next frame

6

$$\boxed{F(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ldots}$$

Because

$$\begin{aligned} Z\{k\} &= F(z) \\ &= \sum_{k=0}^{\infty} \frac{x_k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\ &= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ldots \end{aligned}$$

By comparing this sequence with the derivative of $(1-x)^{-1}$ and its series representation, this sequence can be written as a rational expression in z as $F(z) = \ldots\ldots\ldots\ldots$

7

$$\boxed{F(z) = \frac{z}{(z-1)^2}}$$

Because

$$F(z) = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ldots$$

Comparing this with the series expansion

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) \\ &= \frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2} \end{aligned}$$

then we can see that by multiplying $F(z)$ by z

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots = \frac{1}{(1-1/z)^2}$$

so, dividing both sides by z gives

$$F(z) = \frac{1}{z(1-1/z)^2} = \frac{z}{(z-1)^2}$$

Next frame

Table of *Z* transforms

8

We list the results that we have obtained so far as well as some additional ones for future reference.

Sequence	**Transform $F(z)$**	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \ldots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, \ldots\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\frac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

Next frame

Properties of Z transforms

1 *Linearity*

9

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\}+b\{y_k\})=aZ\{x_k\}+bZ\{y_k\}$$

For example, the Z transform of the sequence $\{k\}$ is $Z\{k\}=$ .... and the Z transform of the sequence $\{e^{-2k}\}$ is $Z\{e^{-2k}\}=$ ....

10

$$\boxed{Z\{k\}=\frac{z}{(z-1)^2} \text{ and } Z\{e^{-2k}\}=\frac{z}{z-e^{-2}}}$$

Because

$Z\{k\}=\dfrac{z}{(z-1)^2}$ from the table and, also from the table,

$Z\{a^k\}=\dfrac{z}{z-a}$ so when $a=e^{-2}$,

$$Z\{e^{-2k}\}=\frac{z}{z-e^{-2}}$$

Consequently, the Z transform of $3\{k\}-5\{e^{-2k}\}$ is ....

11

$$\boxed{\frac{-5z^3+13z^2-z(3e^{-2}+5)}{(z-1)^2(z-e^{-2})}}$$

Because

$$\begin{aligned}Z(3\{k\}-5\{e^{-2k}\}) &= 3Z\{k\}-5Z\{e^{-2k}\}\\ &= \frac{3z}{(z-1)^2}-\frac{5z}{(z-e^{-2})}\\ &= \frac{3z(z-e^{-2})-5z(z-1)^2}{(z-1)^2(z-e^{-2})}\\ &\frac{3z^2-3ze^{-2}-5z^3+10z^2-5z}{(z-1)^2(z-e^{-2})}\\ &= \frac{-5z^3+13z^2-z(3e^{-2}+5)}{(z-1)^2(z-e^{-2})}\end{aligned}$$

▶

2 ***First shift theorem*** (shifting to the left)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

is the Z transform of the sequence that has been shifted by m places to the left. For example

$$Z\{x_{k+1}\} = zF(z) - zx_0$$
$$Z\{x_{k+2}\} = z^2F(z) - z^2x_0 - zx_1$$

These will be used later when solving difference equations. Note the similarity between these results and the Laplace transforms for the first and second derivatives for continuous functions.

For example, given that $Z\{4^k\} = \dfrac{z}{z-4}$ then

$$Z\{4^{k+3}\} = \ldots\ldots\ldots\ldots$$

12

$$\boxed{\dfrac{64z}{z-4}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

so

$$\begin{aligned}
Z\{4^{k+3}\} &= z^3 Z\{4^k\} - \left[z^3 4^0 + z^2 4^1 + z4^2\right] \text{ where } Z\{4^k\} = \frac{z}{z-4}\\
&= z^3\frac{z}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4 - (z^3 + 4z^2 + 16z)(z-4)}{z-4}\\
&= \frac{z^4 - (z^4 - 64z)}{z-4}\\
&= \frac{64z}{z-4}
\end{aligned}$$

In this way we have derived the Z transform of the sequence $\{64, 256, 1024, \ldots\}$ by shifting the sequence $\{1, 4, 16, 64, 256, \ldots\}$ three places to the left and losing the first three terms.

Try another. Given that $Z\{k\} = \dfrac{z}{(z-1)^2}$ then

$$Z\{(k+1)\} = \ldots\ldots\ldots\ldots$$

13

$$\boxed{\dfrac{z^2}{(z-1)^2}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

so

$$\begin{aligned} Z\{k+1\} &= z\frac{z}{(z-1)^2} - [z \times 0] \\ &= \frac{z^2}{(z-1)^2} \end{aligned}$$

3 ***Second shift theorem*** (shifting to the right)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

For example, given that $Z\{x_k\} = \dfrac{z}{z-1}$ then

$$Z\{x_{k-3}\} = \ldots\ldots\ldots\ldots$$

14

$$\boxed{\dfrac{1}{z^2(z-1)}}$$

Because

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

so

$$\begin{aligned} Z\{x_{k-3}\} &= z^{-3}\frac{z}{z-1} \\ &= \frac{1}{z^2(z-1)} \end{aligned}$$

In this way we have derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, \ldots\}$ by shifting the sequence $\{1, 1, 1, 1, \ldots\}$ three places to the right and defining the first three terms as zeros.

Try this one. The sequence $\{x_k\}$ with Z transform

$$Z\{x_k\} = \frac{1}{(z-a)}, \text{ where } a \text{ is a constant, is } \{\ldots\ldots\ldots\ldots\}$$

15

$$\boxed{\{a^{k-1}\}}$$

Because

From the table of transforms the nearest transform to the one in question is $\dfrac{z}{(z-a)}$ which is the Z transform of $\{a^k\}$. Now

$$\begin{aligned}\frac{1}{(z-a)} &= \frac{1}{z}\times\frac{z}{(z-a)}\\ &= z^{-1}F(z) \quad \text{where } F(z) = Z\{a^k\}\end{aligned}$$

and so

$$\frac{1}{(z-a)} = Z\{a^{k-1}\}$$

which is the Z transform of $\{a^k\}$, shifted one place to the right.

4 *Translation*

If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

For example, $Z\{k\} = \dfrac{z}{(z-1)^2}$ so that $Z\{2^k k\} = \ldots\ldots\ldots\ldots$

16

$$\boxed{\frac{2z}{(z-2)^2}}$$

Because

Since $Z\{k\} = \dfrac{z}{(z-1)^2} = F(z)$ then by the translation property

$$\begin{aligned}Z\{2^k k\} &= F(2^{-1}z)\\ &= \frac{2^{-1}z}{(2^{-1}z-1)^2}\\ &= \frac{2z}{(z-2)^2}\end{aligned}$$

▶

5 *Final value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

For example, the sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}.$$

Now

$$\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \underset{z\to 1}{Lim}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

and

$$\underset{k\to\infty}{Lim}\left\{\left(\frac{1}{2}\right)^k\right\} = 0 \text{ which confirms the final value theorem.}$$

Using the final value theorem the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2+2z}{(z-1)(5z-1)^2} \text{ is } \ldots\ldots\ldots\ldots$$

17

$\boxed{0{\cdot}75}$

Because

$$\begin{aligned}\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} &= \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\}\\ &= \underset{z\to 1}{Lim}\left\{\frac{10z+2}{(5z-1)^2}\right\}\\ &= \frac{12}{16}\\ &= 0{\cdot}75\end{aligned}$$

6 *The initial value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\{F(z)\}$$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and $\underset{z\to\infty}{Lim}\, F(z) = \underset{z\to\infty}{Lim}\dfrac{z}{z-a} = \underset{z\to\infty}{Lim}\dfrac{1}{1} = 1$ by L'Hôpital's rule. Furthermore $x_0 = a^0 = 1$ so demonstrating the validity of the theorem.

▶

7 *The derivative of the transform*

If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$

This is easily proved.

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k} \text{ and so } F'(z) = \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k}$$

$$= -\frac{1}{z}Z\{kx_k\}$$

and so $-zF'(z) = Z\{kx_k\}$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and so the sequence $\{ka^k\}$ has Z transform

$$Z\{kx_k\} = -zF'(z) = \ldots\ldots\ldots\ldots$$

18

$$\boxed{Z\{kx_k\} = \frac{az}{(z-a)^2}}$$

Because

$$-zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Notice that this is in agreement with the Table of transforms in Frame 8.

Next frame

Inverse transforms

19

If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}$$

There are many times when, given the Z transform of a sequence, it is not possible to immediately read off the sequence from the Table of transforms. Instead some manipulation may be required and, as with Laplace transforms, very often this involves using partial fractions.

Example

The sequence $\{x_k\}$ has Z transform $F(z) = \dfrac{z}{z^2 - 5z + 6}$. To find the inverse transform, and hence the sequence, we recognise that the denominator can be factorised and separated into partial fractions as

$$F(z) = \ldots\ldots\ldots\ldots$$

20

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

Because

$$\begin{aligned} F(z) &= \frac{z}{z^2 - 5z + 6} \\ &= \frac{z}{(z-2)(z-3)} \\ &= \frac{A}{z-2} + \frac{B}{z-3} \\ &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)} \end{aligned}$$

Equating numerators gives $z = A(z-3) + B(z-2)$, giving $A + B = 1$ and $-3A - 2B = 0$. From these two equations we find that $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $Z\{a^k\} = \frac{z}{z-a}$. Therefore if we write

$$\begin{aligned} F(z) &= \frac{3}{z-3} - \frac{2}{z-2} \\ &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \end{aligned}$$

so

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

21

$$Z^{-1}F(z) = \{3^k - 2^k\}$$

Because

$$\begin{aligned} F(z) &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \\ &= 3 \times z^{-1}Z\{3^k\} - 2 \times z^{-1}Z\{2^k\} \end{aligned}$$

and so

$$\begin{aligned} Z^{-1}F(z) &= 3 \times \{3^{k-1}\} - 2 \times \{2^{k-1}\} \text{ by the second shift theorem} \\ &= \{3^k\} - \{2^k\} \\ &= \{3^k - 2^k\} \text{ giving } x_k = 3^k - 2^k \end{aligned}$$

There is a simpler way of doing this without employing the second shift theorem. Recognising that z appears in the numerator of $F(z)$, we consider instead the partial fraction breakdown of $\frac{F(z)}{z}$

$$\frac{F(z)}{z} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{\frac{1}{z-3}-\frac{1}{z-2}}$$

Because

$$\begin{aligned}\frac{F(z)}{z}&=\frac{1}{z}\times\frac{z}{z^2-5z+6}\\&=\frac{1}{z^2-5z+6}\\&=\frac{1}{(z-2)(z-3)}\\&=\frac{A}{z-2}+\frac{B}{z-3}\\&=\frac{A(z-3)+B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators gives $1=A(z-3)+B(z-2)$, giving

[z]: $A+B=0$

[CT]: $-3A-2B=1$ with solution $A=-1$ and $B=1$. So that

$$\begin{aligned}\frac{F(z)}{z}&=\frac{1}{z-3}-\frac{1}{z-2}\text{ that is}\\F(z)&=\frac{z}{z-3}-\frac{z}{z-2}\\&=Z\{3^k\}-Z\{2^k\}\text{ and so}\\Z^{-1}F(z)&=\{3^k\}-\{2^k\}\\&=\{3^k-2^k\}\end{aligned}$$

Thus the use of the second shift theorem is avoided.

So try one yourself. The sequence $\{x_k\}$ has Z transform

$$F(z)=\frac{5z}{(z^2-4z+4)(z+2)}$$

therefore $\{x_k\}=$ ....

$$\boxed{\{x_k\} = \left\{\frac{5k}{4} - \frac{5}{16} \times \left(2^k + (-2)^k\right)\right\}}$$

Because

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z} \times \frac{5z}{(z^2 - 4z + 4)(z+2)} \\
&= \frac{5}{(z-2)^2(z+2)} \\
&= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z+2} \\
&= \frac{A(z+2) + B(z-2)(z+2) + C(z-2)^2}{(z-2)^2(z+2)}
\end{aligned}$$

Equating numerators gives $5 = A(z+2) + B(z^2 - 4) + C(z^2 - 4z + 4)$, giving

$[z^2]$: $\quad B + C = 0$

$[z]$: $\quad A - 4C = 0$

$[CT]$: $\quad 2A - 4B + 4C = 5$

with solution $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2} \text{ giving}$$

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2} \text{ and so}$$

$$\begin{aligned}
Z^{-1}F(z) &= \frac{5}{8} \times \{k2^k\} - \frac{5}{16} \times \{2^k\} + \frac{5}{16} \times \left\{(-2)^k\right\} \\
&= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\}
\end{aligned}$$

Revision summary

1 *Sequences*

The sequence $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ is represented by the notation $\{x_k\}_{-\infty}^{\infty}$. The sequence $\{x_k\}_0^{\infty}$ is called a causal sequence and is denoted simply by $\{x_k\}$.

2 *Z transform*

The Z transform of the causal sequence $\{x_k\}$ is

$$Z\{x_k\} = \sum_{k=0}^{\infty}\left(\frac{x_k}{z^k}\right) = F(z)$$ where the value of z is chosen to ensure that the sum converges.

$\{x_k\}$ and $Z\{x_k\}$ form a Z transform pair.

3 *Table of Z transforms*

Sequence	**Transform $F(z)$**	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \ldots\}$	1	All values of z
$\{x_k\} = \{1, 1, 1, \ldots\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\frac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

4 *Linearity*

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\} + b\{y_k\}) = aZ\{x_k\} + bZ\{y_k\}.$$

5 *First shift theorem* (shifting to the left)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + z x_{m-1}\right]$$

the Z transform of the sequence that has been shifted by m places to the left.

6 *Second shift theorem* (shifting to the right)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

7 *Translation*
If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

8 *Final value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

9 *The initial value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\{F(z)\}.$$

10 *The derivative of the transform*
If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$.

11 *Inverse transformations*
If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}.$$

1 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-1)^k$.

2 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = 4k - 2a^k$.

3 Find the Z transform of the causal sequences:

(a) $\{k - 3\}$

(b) $\{5^{k+2}\}$

4 Find the inverse Z transformation of

$$F(z) = \frac{z^2(z-3)}{(z^2-2z+1)(z-2)}.$$

5 Find the Z transform of each of the following sequences.

(a) $\{1, 0, 1, 0, 1, 0, \ldots\}$

(b) $\{0, 1, 0, 1, 0, 1, \ldots\}$

(c) $\{1, 0, 1, 1, 0, 0, 0, 1\}$

(d) $\{1, 1, 1, 0, 0, 0, 1, 1\}$

(e) $\{0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1\}$

(f) $\{1, 1, 0, 0, 0, 1, 1\}$

Note that the last four of these are finite sequences.

6 Find the inverse transform of

(a) $F(z) = \dfrac{z}{(z+1)(z+2)(z+3)}$

(b) $F(z) = \dfrac{z^2}{(z+1)(z+2)(z+3)}$

(c) $F(z) = \dfrac{z(3z+1)}{(z-2)(z-3)}$

(d) $F(z) = \dfrac{z^2}{2-3z+z^2}.$