Exercice 1 :

X : le poids de la tablette

$X \sim N(\mu = 100, \sigma = 1)$

$Z \sim N(0,1)$

$\mathbb{P}(a \leq Z \leq b) = \Phi(b) - \Phi(a)$

$X \sim N(m, \sigma)$

$\frac{X - m}{\sigma} \sim N(0,1)$

$\boxed{\mathbb{P}(-a \leq Z \leq a) = 2\Phi(a) - 1}$

$\Phi(a) - \Phi(-a) = \Phi(a) - (1 - \Phi(a))$
$= 2\Phi(a) - 1$

1) A : "La tablette est mise sur le marché"

$$\mathbb{P}(98 \leq X \leq 102) = \mathbb{P}\left(\underbrace{\frac{98 - 100}{1}}_{-2} \leq \underbrace{\frac{X - 100}{1}}_{Z \sim N(0,1)} \leq \underbrace{\frac{102 - 100}{1}}_{2}\right)$$

$$= 2 \times \Phi(\underbrace{2}_{2.00}) - 1$$

	0.00
2.0	x

2) Cherchons σ_0

tq

$$\mathbb{P}(98 \leq X \leq 102) = 0.97$$

avec $X \sim N(100, \sigma_0)$

$$\mathbb{P}(98 \leq X \leq 102) = 0.97$$

$$\Leftrightarrow \mathbb{P}\left(\underbrace{\frac{98-100}{\sigma_0}}_{\frac{-2}{\sigma_0}} \leq \underbrace{\frac{X-100}{\sigma_0}}_{Z \sim N(0,1)} \leq \underbrace{\frac{102-100}{\sigma_0}}_{\frac{2}{\sigma_0}}\right) = 0.97$$

$$\Leftrightarrow 2\Phi\left(\frac{2}{\sigma_0}\right) - 1 = 0.97$$

$$\Leftrightarrow \Phi\left(\frac{2}{\sigma_0}\right) = \frac{0.97+1}{2}$$

$$\Leftrightarrow \Phi\left(\frac{2}{\sigma_0}\right) = 0.985$$

$$\Leftrightarrow \frac{2}{\sigma_0} = \underbrace{\Phi^{-1}(0.985)}_{\text{lecture inverse du tableau}}$$

$$\Leftrightarrow \sigma_0 = \frac{2}{2.17}$$

Exercice 2 :

$$f_X(x) = \begin{cases} a\,x & \text{si } 0 \leq x \leq 2 \\ 0 & \text{sinon.} \end{cases}$$

$a = ?$

f_X vérifie $\begin{cases} f_X(x) \geq 0. \\ \int_{-\infty}^{+\infty} f_X(x)\,dx = 1. \end{cases}$

$\Leftrightarrow \begin{cases} a \geq 0 \\ \int_0^2 a\,x\,dx = 1 \end{cases}$

$\Leftrightarrow \left[a \frac{x^2}{2}\right]_0^2 = 1.$

$\Leftrightarrow a\left(\frac{2^2}{2} - 0\right) = 1$

$\Leftrightarrow 2a = 1 \Leftrightarrow a = \frac{1}{2}$

2) $E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx.$

$= \int_0^2 x \times \frac{1}{2} x\,dx.$

$= \int_0^2 \frac{1}{2} x^2\,dx = \left[\frac{1}{2}\frac{x^3}{3}\right]_0^2$

$= \frac{1}{2} \times \frac{2^3}{3} = \frac{4}{3}.$

$$\text{Var}(X) = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$$

$$= \int_0^2 x^2 \times \frac{1}{2} x \, dx$$

$$= \int_0^2 \frac{1}{2} x^3 \, dx = \left[\frac{1}{2} \frac{x^4}{4}\right]_0^2$$

$$= \frac{2^4}{4 \times 2}$$

D'où $\text{Var}(X) = 2^2 - \frac{4}{3} = 2$

3) $Y = X^2$

$\mathcal{L}(Y) = ?$

- $\boxed{Y(\Omega) = [0, 4]}$

Car : $0 \leq x \leq 2$

$0 \leq x^2 = y \leq 4$

- Si $y < 0$, $F_Y(y) = 0$
- Si $y > 4$, $F_Y(y) = 1$ (1)
- Si $0 \leq y \leq 4$,

$Y = f(X)$

$\mathcal{L}(X)$ donnée

- $X(\Omega)$
- $f_X(x)$
- $F_X(x)$

$\xrightarrow{?} \mathcal{L}(Y)$

(1) $Y(\Omega)$

(2) $F_Y(y) = \ldots$

$F_Y'(y) = f_Y(y)$

$$F_Y(y) = \mathbb{P}(Y \leq y)$$
$$= \mathbb{P}(X^2 \leq y)$$
$$= \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$\mathbb{P}(a \leq X \leq b)$
$= F_X(b) - F_X(a)$

Calculons F_X

$$f_X(x) = \begin{cases} \frac{1}{2}x & \text{si } 0 \leq x < 2 \\ 0 & \text{sinon} \end{cases}$$

- si $x < 0$.

$F_X(x) = 0$

- si $x > 2$.

$F_X(x) = 1$

- si $x \in [0, 2]$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{1}{2}t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^x$$
$$= \frac{1}{2} \times \frac{x^2}{2} = \frac{x^2}{4}$$

$X(\Omega) = [a, b]$

$$f_X(x) = \begin{cases} * & \text{si } x \in [a, b] \\ 0 & \text{sinon} \end{cases} \rightarrow F_X(x) = \begin{cases} 0 & \text{si } x < a \\ * & \text{si } a \leq x \leq b \\ 1 & \text{si } x \geq b \end{cases}$$

2.

$X(\Omega) = [a, +\infty[$

$$f_X(x) = \begin{cases} * & \text{si } x \geq a \\ 0 & \text{sinon} \end{cases} \rightarrow F_X(x) = \begin{cases} * & \text{si } x \geq a \\ 0 & \text{si } x < a \end{cases}$$

Finalement

$$F_X(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 2 \\ \frac{x^2}{4} & \text{si } 0 \leq x \leq 2. \end{cases}$$

Revenons à F_Y : On a, $0 \leq y \leq 4$ d'où $0 \leq \sqrt{y} \leq 2$
$-\sqrt{y} < 0$; ainsi

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$= \frac{(\sqrt{y})^2}{4} - 0 = \frac{y}{4}$$

Par la suite $F_Y(y) = \begin{cases} \frac{y}{4} & \text{si } 0 \leq y \leq 4 \\ 0 & \text{si } y < 0 \\ 1 & \text{si } y > 4 \end{cases}$

D'où

$$f_Y(y) = \begin{cases} \frac{1}{4} & \text{si } 0 \leq y \leq 4 \\ 0 & \text{si } y < 0 \\ 0 & \text{si } y > 4 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & \text{si } 0 \leq y \leq 4 \\ 0 & \text{sinon} \end{cases}$$

4/ $\mathbb{E}(Y) = \mathbb{E}(X^2) = 2$

$\text{var}(Y) = \mathbb{E}(Y^2) - \{\mathbb{E}(Y)\}^2$

avec

$\mathbb{E}(Y^2)$
$= \int_0^4 y^2 \frac{1}{4} \, dy$
$= \left[\frac{1}{4} \frac{y^3}{3}\right]_0^4$
$= \frac{1}{4}\left(\frac{4^3}{3}\right)$

Ex3

$U \sim \mathcal{U}([0,1])$

$X = \dfrac{-\log U}{\lambda}$

$\lambda > 0$

① $Z \sim \mathcal{E}(\lambda)$

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z} & \text{si } z \geq 0 \\ 0 & \text{sinon.} \end{cases}$$

$Z(\Omega) = [0, +\infty[$.

2) si $z < 0$; $F_Z(z) = 0$

si $z \geq 0$

$\int u' e^u = e^u$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)\,dt$$

$$= \int_0^z \lambda e^{-\lambda t}\,dt$$

$$= \int_0^z -(-\lambda)\,e^{-\lambda t}\,dt$$

$$= \left[-e^{-\lambda t}\right]_0^z$$

$$= -e^{-\lambda z} - \left(-e^{-0}\right)$$

$= 1 - e^{-\lambda z}.$

D'où

$$F_Z(z) = \begin{cases} 1 - e^{-\lambda z} & \text{si } z \geq 0. \\ 0 & \text{si } z < 0. \end{cases}$$

3) $X(\Omega) = [0, +\infty[$

$0 \leq U \leq 1$

$\log U \leq 0$

$-\log U \geq 0$

$x = \frac{-\log u}{\lambda}$

si

• si $x < 0$, $F_X(x) = 0$

• si $x \geq 0$,

$F_X(x) = \mathbb{P}(X \leq x)$

$= \mathbb{P}\left(\frac{-\log U}{\lambda} \leq x\right)$

$= \mathbb{P}(-\log U \leq x\lambda)$

$= \mathbb{P}(\log U \geq -x\lambda)$

$= \mathbb{P}(U \geq e^{-x\lambda})$

$= 1 - \underbrace{\mathbb{P}(U \leq e^{-x\lambda})}_{F_U(e^{-\lambda x})}$

$F_U(u) = \mathbb{P}(U \leq u$

$U \overset{\text{ou}}{\sim} \mathcal{U}([0,1])$

$$F_U(u) = \begin{cases} 0 & \text{si } u < 0 \\ u = \dfrac{u-0}{1-0} & \text{si } 0 \leq u \leq 1 \\ 1 & \text{si } u \geq 1. \end{cases}$$

$x \geq 0$
$\Rightarrow -\lambda x \leq 0$
$\Rightarrow 0 \leq e^{-\lambda x} < 1.$

Ainsi $F_X(x) = 1 - e^{-\lambda x}$.

finalement $F_X(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - e^{-\lambda x} & \text{si } x \geq 0. \end{cases}$

4/ Puisque

$\left. \begin{array}{l} F_X = F_Z \\ Z \sim \mathcal{E}(\lambda) \end{array} \right\}$ d'où $X \sim \mathcal{E}(\lambda)$.

5) $X = \dfrac{-\log U}{\lambda} \sim \mathcal{E}(\lambda)$ avec $U \sim \mathcal{U}([0,1])$.

λ donné

$U = [U_1, U_2, \ldots, U_{1000}]$ réalisations selon $\mathcal{U}([0,1])$ (rand)

$Y = \left[\dfrac{-\log(U_1)}{\lambda}, \dfrac{-\log(U_2)}{\lambda}, \ldots, \dfrac{-\log(U_{1000})}{\lambda} \right]$ réalisations selon la loi exponentielle $\mathcal{E}(\lambda)$

6)

```
import matplotlib.pyplot as plt
plt.hist(V)
```

U = . . .
lamda = 1
V = -log(U)/lamda

7) $U \sim U(0,1)$

$Y = \lambda \tan\left(\pi\left(u - \frac{1}{2}\right)\right)$

$0 \leq u \leq 1$

$-\frac{1}{2} \leq u - \frac{1}{2} \leq \frac{1}{2}$

$-\frac{\pi}{2} \leq \pi\left(u - \frac{1}{2}\right) \leq \frac{\pi}{2}$

$\lambda \tan\left(\pi\left(u - \frac{1}{2}\right)\right) \in \mathbb{R}$

$y(u) = \mathbb{R}$

$\tan:]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$

$Y \sim C(\lambda)$

$f_Y(y) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + y^2}$

$y \in \mathbb{R}$

Soit $y \in \mathbb{R}$

$$F_Y(y) = \mathbb{P}(Y \leq y)$$
$$= \mathbb{P}(\lambda \tan(\pi(U - \tfrac{1}{2})) \leq y)$$
$$= \mathbb{P}(\tan(\pi(U - \tfrac{1}{2})) \leq \tfrac{y}{\lambda})$$
$$= \mathbb{P}(\pi(U - \tfrac{1}{2}) \leq \arctan(\tfrac{y}{\lambda}))$$

$(\arctan x)' = \frac{1}{1+x^2}$

$$= \mathbb{P}(U - \tfrac{1}{2} \leq \tfrac{1}{\pi} \arctan(\tfrac{y}{\lambda}))$$
$$= \mathbb{P}(U \leq \tfrac{1}{\pi} \arctan(\tfrac{y}{\lambda}) + \tfrac{1}{2})$$
$$= F_U(\tfrac{1}{\pi} \arctan(\tfrac{y}{\lambda}) + \tfrac{1}{2})$$

$$F_Y(y) = \frac{1}{\pi} \arctan\left(\frac{y}{\lambda}\right) + \frac{1}{2}$$
$$f_Y(y) = F_Y'(y) = \frac{1}{\pi} \cdot \frac{1}{\lambda} \cdot \frac{1}{1+\left(\frac{y}{\lambda}\right)^2}$$
$$(f \circ g)' = g' \times f'(g)$$