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An efficient and flexible mechanism for constructing membership functions

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Abstract

This paper introduces a Bézier curve-based mechanism for constructing membership functions of convex normal fuzzy sets. The mechanism can fit any given data set with a minimum level of discrepancy. In the absence of data, the mechanism can be intuitively manipulated by the user to construct membership functions with the desired shape. Some numerical experiments are included to compare the performance of the proposed mechanism with conventional methods. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Fuzzy set theory was first introduced in the 1960s by Zadeh [15] as a way to capture uncertainty and vagueness often overlooked in complex systems. It was pointed out that fuzzy set theory is a generalization of the classical set theory.

A fuzzy set $\tilde{A}$ is characterized by its *membership function* $\mu_{\tilde{A}}$, which maps each element of the universe X to the interval $[0,1]$. This function indicates the degree of belonging to $\tilde{A}$ for each element of X. One of the most important concepts of fuzzy sets is the concept of an α-cut. Given a fuzzy set $\tilde{A}$ defined on X and $\alpha \in (0,1]$, the α-cut is defined as ${}^{\alpha}\tilde{A} = \{x \in X : \mu_{\tilde{A}}(x) \geqslant \alpha\}$. For continuity purposes, we take ${}^{0}\tilde{A} = \lim_{\alpha\to 0} {}^{\alpha}\tilde{A}$. A fuzzy set $\tilde{A}$ is *convex* if and only if each of its α-cuts is a convex set. A fuzzy set $\tilde{A}$ is *normal* if ${}^{1}\tilde{A} \neq \emptyset$.

Even though there is no universal agreement on the question, Dombi [4] reported that there are some characteristics shared by the majority of continuous membership functions found in the literature. Among others, there is an apparent demand for membership functions with the following properties: they should be piecewise monotone nonincreasing or nondecreasing; they

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should achieve null and full membership for at least two different elements in the universal set; and they should be able to represent fuzzy convex sets. Commonly seen examples are the simple triangular, trapezoidal, and bell-shaped membership functions.

Problem formulations based on fuzzy sets can have greater expressive power than their counterparts based on crisp sets, but the applicability of fuzzy technology depends on the ability to construct membership functions that appropriately represent various concepts in different contexts [8]. To fully exploit the benefits provided by fuzzy technology, we need an efficient membership function generating mechanism with the following desirable characteristics:

1. *Accurate*. In the presence of data, the resulting membership functions should reflect the knowledge contained in the data in the most accurate way possible. Data in the form of membership values for points in the universe is usually obtained from experts.
2. *Flexible*. The methodology should provide a broad family of membership functions.
3. *Computationally affordable*. The method should be computationally tractable in order to be of any practical use. Medasani [10] has highlighted the importance of having membership functions that can be easily tuned and adjusted. Other authors have expressed the need for methods in which computer graphics can facilitate the process of constructing membership functions by allowing the user an easy and direct manipulation of different shapes [1].
4. *Easy to use*. Once a membership function has been generated, it should be easy to find $\mu_{\tilde{A}}(x)$ for a given x; and it should be easy to find ${}^{\alpha}\tilde{A}$ for a given α.

In this paper we propose a mechanism that exploits the properties of Bézier curves to address these issues and to provide the user with a flexible and efficient way of generating membership functions.

The paper is organized as follows. In Section 2, we review the basic techniques used for generating membership functions. Section 3 describes the proposed mechanism and some fundamental definitions and properties of Bézier curves. In Section 4, test problems found in the literature are used to illustrate the proposed mechanism and compare its performance with that of two methods which appear in the literature. Finally, conclusions and current research directions are given in Section 5.

2. Membership function generation

2.1. Overview

Membership functions can be constructed from data when it is available. This data can be elicited by interacting with experts using a *direct approach* (or *direct rating*) [8,11,13]. The direct approach requires the degree of membership of a collection of points in the universal set. A membership function that describes the underlying concept is fitted to the collected data points. This is known as *data-driven membership function estimation*. Sometimes this approach can be overly precise in capturing subjective judgment. By formulating easier and simpler questions, knowledge can also be acquired through an indirect approach. We will not deal with the indirect approach in this paper, but the reader is referred to the paper by Chameau and Santamarina [1] and the book by Klir and Yuan [8].

When data are not available in the form of value-membership pairs, a membership function has to be constructed subjectively. In this case, the conventional approach is to first pick the shape of the membership function from a list of families, and then to fine-tune the values of the parameters of that function. It is always desirable to have a parsimonious, meaningful parameterization of membership functions [4].

2.2. Current methods

In the literature, fuzzy sets are commonly modeled by triangular, trapezoidal, and bell-shaped membership functions. However, other parameterized functional shapes are useful in particular situations. More details can be found in Dombi [4] and Medasani et al. [10].

An effort to create a broad class of functions was made by Zysno [17] and Zimmermann and Zysno [16]. In their model, the membership function for a fuzzy set $\tilde{A}$ is given by

$$\mu_{\tilde{A}}(x) = \mathrm{mid}\left(0, \left(\frac{1}{1+\mathrm{e}^{-a(x+b)}} - c\right)\frac{1}{d} + \frac{1}{2}, 1\right) \quad \forall x \in X \subseteq R, \tag{1}$$

where

$$a, b \in R, \quad 0 \leqslant c \leqslant 1, \text{ and } 0 \leqslant d \leqslant 2\min(1-c, c).$$

The function $\mathrm{mid}(0, f(x), 1)$ is defined such that

$$\mathrm{mid}(0, f(x), 1) = f(x), \text{ if } 0 \leqslant f(x) \leqslant 1; \mathrm{mid}(0, f(x), 1) = 0, \text{ if } f(x) < 0; \text{ and } \mathrm{mid}(0, f(x), 1) = 1, \text{ if } f(x) > 1.$$

Even though the model provides the user with a commonly used family of S-shapes, the determination of the parameters from empirical data poses some problems and there is no direct numerical method for optimal parameter estimation [16,17]. The model may be used for estimating membership functions subjectively, with the parameters a, b, c, and d being fixed by the expert.

Dombi [4] proposed a model with properties similar to the one presented by Zysno and Zimmermann. In his model a membership function for fuzzy set $\tilde{A}$ is constructed using the S-shaped monotonically increasing function

$$\mu_{\tilde{A}}(x) = \frac{(1-v)^{\lambda-1}(x-a)^{\lambda}}{(1-v)^{\lambda-1}(x-a)^{\lambda} + v^{\lambda-1}(b-x)^{\lambda}} \tag{2}$$

and/or the S-shaped monotonically decreasing function

$$\mu_{\tilde{A}}(x) = \frac{(1-v)^{\lambda-1}(b-x)^{\lambda}}{(1-v)^{\lambda-1}(b-x)^{\lambda} + v^{\lambda-1}(x-a)^{\lambda}}, \tag{3}$$

where $x \in [a, b]$; $a, b \in R$; the steepness is given by $\lambda \geqslant 1$; and the inflection point is determined by $0 < v < 1$. When data are available, Dombi proposed a method for estimating the parameters based on linearized forms of (2) and (3).

Both of these models provide similar membership functions because they use the same underlying form, i.e., $\mu_{\tilde{A}}(x) = 1/(1 + d(x))$, where $d(x)$ is a measure of distance. Even though these models provide flexibility for estimating S-shaped functions, they fail to provide more general monotonic curves.

Chen and Otto [2] present a novel method for constructing membership functions using interpolation and measurement theory. Following a systematic approach, their method is able to construct general monotonic functions from data. However, their methodology does not provide a mechanism for adjusting or building a membership function in the absence of data.

In the area of fuzzy system identification, sophisticated methods based on neural networks and evolutionary algorithms have been proposed to generate and tune both fuzzy rules and membership functions. However, they are basically case by case approaches [7,9].

In the following section we shall introduce an interactive and efficient approach for both data-driven and subjective estimation of membership functions. Based on Bézier curves, the method is able to generate a broad family of functions.

3. Proposed mechanism

3.1. Bézier curves

One of the major breakthroughs in computer aided design (CAD) is the theory of Bézier curves and surfaces, independently developed by P. de Casteljau and P. Bézier while working for the French automakers Citroën and Renault, respectively [6].

The theory of Bézier curves provides a mathematical foundation for representing a smooth curve that passes through the vicinity of a set of *control points*. Definition 1 gives a formal expression of a Bézier curve in terms of Bernstein polynomials.

Definition 1. A Bézier curve with $n+1$ control points $\mathbf{p} \triangleq (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ is given by

$$f(t, n, \mathbf{p}) \triangleq \sum_{k=0}^{n} B_{n,k}(t)\mathbf{p}_k,$$

where $t \in [0,1]$, $\mathbf{p}_k \triangleq (x_k, y_k)^{\mathrm{T}}$, and $B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ are the Bernstein polynomials. Since $f(t,n,\mathbf{p}) \in R^2$, we usually denote $f(t,n,\mathbf{p}) = [f_x(t,n,\mathbf{x}), f_y(t,n,\mathbf{y})]^{\mathrm{T}}$, where $\mathbf{x} \triangleq (x_0,\ldots,x_n)^{\mathrm{T}}$, $\mathbf{y} \triangleq (y_0,\ldots,y_n)^{\mathrm{T}}$.

Bézier curves have several properties that are particularly useful in the context of this paper [6].

Property 1. *The Bézier curve $f(t,n,\mathbf{p})$ defined over $t \in [0,1]$, lies in the convex hull of the polygon defined by the control points $\mathbf{p} \triangleq (\mathbf{p}_0,\ldots,\mathbf{p}_n)$.*

Property 2. *The Bernstein polynomial $B_{n,k}(t)$ achieves its unique maximum at $t = k/n$. If the control point $\mathbf{p}_k$ is moved, then the curve is mostly affected in the region around the parameter $t = k/n$.*

Property 3. *The Bézier curve interpolates its first $(\mathbf{p}_0)$ and last $(\mathbf{p}_n)$ control points. In other words, $f(0,n,\mathbf{p}) = \mathbf{p}_0$ and $f(1,n,\mathbf{p}) = \mathbf{p}_n$.*

These properties have practical effects in the curve design process. Property 1 guarantees that the curve will not fall outside the "control polygon". By using this property along with Property 2, a Bézier curve can be designed by exaggerating the target shape using the control polygon. Even though a single control point displacement will change the whole curve, this "pseudo-local control" property gives us the sense that the control points work locally as magnets on the curve. Property 3 is very useful for breaking the construction of a complex curve into simpler parts.

A complete discussion on Bézier curves and its properties can be found in the book by Farin [6].

3.2. Mathematical framework

In this section we give the mathematical framework of a broad family of membership shapes based on Bézier curves.

Let $\tilde{A}$ be a fuzzy set on the universal set X. The following conditions are commonly required for its membership function, $\mu_{\tilde{A}}(\cdot)$.

Condition 1. The membership function $\mu_{\tilde{A}}$ is a mapping from the universal set X to $[0,1]$, i.e., $\mu_{\tilde{A}} : X \to [0,1]$.

Condition 2. There exist $x_1, x_2 \in X$ such that $\mu_{\tilde{A}}(x_1) = 1$ and $\mu_{\tilde{A}}(x_2) = 0$. In other words, we say that $x_1 \in X$ fully belongs to the set $\tilde{A}$, while $x_2 \in X$ does not belong to $\tilde{A}$.

Condition 3. For $x_1, x_2 \in X$ and $\lambda \in [0,1]$, we have $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$.

Condition 1 is conventional in the fuzzy literature. The *normality* requirement implicit in Condition 2 (i.e., existence of $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$) can be easily relaxed, but we preserve it for the sake of clarity in our presentation. Condition 3 guarantees that the fuzzy set $\tilde{A}$ is *convex*.

A convenient, parametric form for expressing our membership function model is

$$\mu_{\tilde{A}}(x(t)) = \begin{cases} 0 & \text{if } x(t) < m_{\mathrm{L}} - \gamma, \\ \mu_{\tilde{A}_{\mathrm{L}}}(x(t)) & \text{if } m_{\mathrm{L}} - \gamma \leqslant x(t) \leqslant m_{\mathrm{L}}, \\ 1 & \text{if } m_{\mathrm{L}} < x(t) < m_{\mathrm{R}}, \\ \mu_{\tilde{A}_{\mathrm{R}}}(x(t)) & \text{if } m_{\mathrm{R}} \leqslant x(t) \leqslant m_{\mathrm{R}} + \beta, \\ 0 & \text{if } x(t) > m_{\mathrm{R}} + \beta, \end{cases} \tag{4}$$

where γ and β are the left and right spreads, respectively; $m_{\mathrm{L}}, m_{\mathrm{R}} \in X$ are the lowest and highest values with full membership, respectively; and $\mu_{\tilde{A}_{\mathrm{L}}}(x(t))$ and $\mu_{\tilde{A}_{\mathrm{R}}}(x(t))$ are the left and right membership values. Assume that $\mathbf{p}_{\mathrm{L}} = (\mathbf{p}_{\mathrm{L},0},\ldots,\mathbf{p}_{\mathrm{L},n_{\mathrm{L}}})^{\mathrm{T}}$ and $\mathbf{p}_{\mathrm{R}} = (\mathbf{p}_{\mathrm{R},0},\ldots,\mathbf{p}_{\mathrm{R},n_{\mathrm{R}}})^{\mathrm{T}}$ are $n_{\mathrm{L}}+1$ and $n_{\mathrm{R}}+1$ control points for generating the left and right membership functions, respectively. The left and right membership functions are part of the following parametric expressions:

$$[x(t), \mu_{\tilde{A}_{\mathrm{L}}}(x(t))]^{\mathrm{T}} = \vec{\mu}_{\tilde{A}_{\mathrm{L}}}(t, n_{\mathrm{L}}, \mathbf{p}_{\mathrm{L}}) \triangleq \sum_{k=0}^{n_{\mathrm{L}}} B_{n_{\mathrm{L}},k}(t)\mathbf{p}_{\mathrm{L},k}, \tag{5}$$

$$[x(t), \mu_{\tilde{A}_{\mathrm{R}}}(x(t))]^{\mathrm{T}} = \vec{\mu}_{\tilde{A}_{\mathrm{R}}}(t, n_{\mathrm{R}}, \mathbf{p}_{\mathrm{R}}) \triangleq \sum_{k=0}^{n_{\mathrm{R}}} B_{n_{\mathrm{R}},k}(t)\mathbf{p}_{\mathrm{R},k}, \tag{6}$$

where $\vec{\mu}_{\tilde{A}_L}(\cdot)$ and $\vec{\mu}_{\tilde{A}_R}(\cdot)$ are the Bézier curves for the left and right membership functions, respectively; $t \in [0,1]$; $\mathbf{p}_{L,k} \triangleq (x_{L,k}, y_{L,k})^T$ is the kth Bézier control point for the left membership function (for $k = 0, \dots, n_L$); $\mathbf{p}_{R,k} \triangleq (x_{R,k}, y_{R,k})^T$ is the kth Bézier control point for the right membership function (for $k = 0, \dots, n_R$); and $B_{n_L,k}(t)$ and $B_{n_R,k}(t)$ are Bernstein polynomials. As before, in two-dimensional space, we denote $\vec{\mu}_{\tilde{A}_L}(t, n_L, \mathbf{p}_L) = [f_x(t, n_L, \mathbf{x}_L), f_y(t, n_L, \mathbf{y}_L)]^T$ and $\vec{\mu}_{\tilde{A}_R}(t, n_R, \mathbf{p}_R) = [f_x(t, n_R, \mathbf{x}_R), f_y(t, n_R, \mathbf{y}_R)]^T$, where $\mathbf{x}_L \triangleq (x_{L,0}, \dots, x_{L,n_L})^T$, $\mathbf{y}_L \triangleq (y_{L,0}, \dots, y_{L,n_L})^T$, $\mathbf{x}_R \triangleq (x_{R,0}, \dots, x_{R,n_R})^T$ and $\mathbf{y}_R \triangleq (y_{R,0}, \dots, y_{R,n_R})^T$.

The type of shapes that can be obtained using the family of membership functions described by (4) are presented in Fig. 1.

In order to satisfy Conditions 1–3 we need to impose some restrictions on the parametric form expressed by (5) and (6).

For Conditions 1 *and* 2,

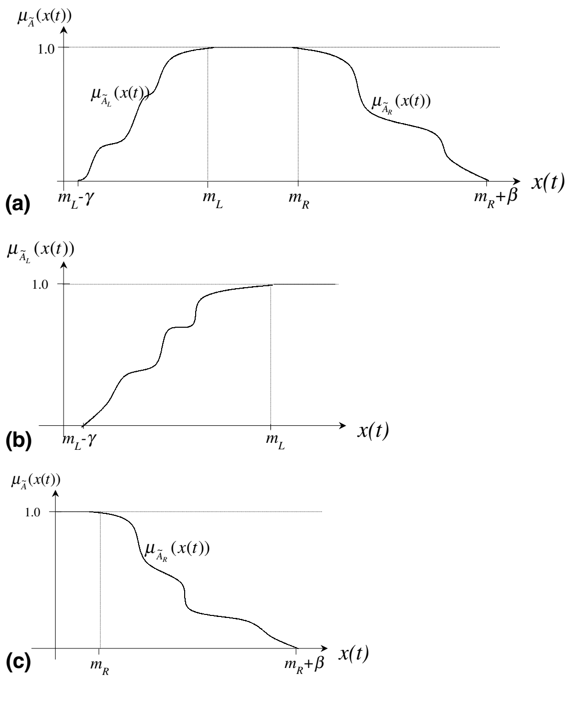

Fig. 1. Types of membership functions. (a) Monotonic nondecreasing and nonincreasing. (b) Monotonic nondecreasing (left). (c) Monotonic nonincreasing (right).

Proposition 1. *The first and last control points of* $\vec{\mu}_{\tilde{A}_L}(\cdot)$ *are* $\mathbf{p}_{L,0} = (m_L - \gamma, 0)^T$ *and* $\mathbf{p}_{L,n_L} = (m_L, 1)^T$.

Proof. It follows from Property 3 of the Bézier curves. □

Proposition 2. *The first and last control points of* $\vec{\mu}_{\tilde{A}_R}(\cdot)$ *are* $\mathbf{p}_{R,0} = (m_R, 1)^T$ *and* $\mathbf{p}_{R,n_R} = (m_R + \beta, 0)^T$.

Proof. It follows from Property 3 of the Bézier curves. □

For Condition 3,

Proposition 3. *If the control points* $\mathbf{p}_L$ *of* $\vec{\mu}_{\tilde{A}_L}(\cdot)$ *are chosen such that* $x_{L,0} \leqslant \cdots \leqslant x_{L,n_L}$ *and* $y_{L,0} \leqslant \cdots \leqslant y_{L,n_L}$, *then* $\mu_{\tilde{A}_L}(x(t))$ *is monotonically nondecreasing for* $m_L - \gamma \leqslant x(t) \leqslant m_L$ *and* $x(t)$ *is monotonically nondecreasing for* $0 \leqslant t \leqslant 1$.

Proof.

$$
\begin{aligned}
f'_y(t, n_L, \mathbf{p}_L) &= \mu'_{\tilde{A}_L}(x(t)) \\
&= \lim_{\delta \to 0} \left[\left\{ \frac{\mu_{\tilde{A}_L}(x(t+\delta)) - \mu_{\tilde{A}_L}(x(t))}{\delta} \right\} \Bigg/ \left\{ \frac{x(t+\delta) - x(t)}{\delta} \right\} \right] \\
&= \frac{f'_y(t, n_L, \mathbf{y}_L)}{f'_x(t, n_L, \mathbf{x}_L)} \qquad (7) \\
&= \frac{\sum_{k=0}^{n_L} n_L [B_{n_L-1,k-1}(t) - B_{n_L-1,k}(t)] y_k}{\sum_{k=0}^{n_L} n_L [B_{n_L-1,k-1}(t) - B_{n_L-1,k}(t)] x_k} \\
&= \frac{\sum_{k=0}^{n_L-1} B_{n_L-1,k}(t) \Delta y_k}{\sum_{k=0}^{n_L-1} B_{n_L-1,k}(t) \Delta x_k},
\end{aligned}
$$

where $t \in [0,1]$, $\Delta y_k \triangleq y_{k+1} - y_k$, and $\Delta x_k \triangleq x_{k+1} - x_k$, for $k = 0, \dots, n_L - 1$. From the result in Eq. (7), if $\Delta y_k \geqslant 0$ and $\Delta x_k \geqslant 0$, then $\mu'_{\tilde{A}_L}(x(t)) \geqslant 0$. Thus we conclude that $\mu_{\tilde{A}_L}(x(t))$ is monotonically nondecreasing. □

The basic results used in the proof of Proposition 3 can be found in Farin [6] and Wagner and Wilson [14].

Similar to Proposition 3, the following result applies for the monotonically nonincreasing membership function, $\vec{\mu}_{\tilde{A}_R}(\cdot)$.

Proposition 4. *If the control points* $\mathbf{p}_R$ *of* $\vec{\mu}_{\tilde{A}_R}(\cdot)$ *are chosen such that* $x_{R,0} \leqslant \cdots \leqslant x_{R,n_R}$ *and* $y_{R,0} \geqslant \cdots \geqslant y_{R,n_R}$, *then* $\mu_{\tilde{A}_R}(x(t))$ *is monotonically nonincreasing for* $m_R \leqslant x(t) \leqslant m_R + \beta$ *and* $x(t)$ *is monotonically nondecreasing for* $0 \leqslant t \leqslant 1$.

The next result follows from Propositions 3 and 4.

Proposition 5. *If the control points* $\mathbf{p}_L$ *of* $\vec{\mu}_{\tilde{A}_L}(\cdot)$ *are chosen such that* $x_{L,0} \leqslant \cdots \leqslant x_{L,n_L}$ *and* $y_{L,0} \leqslant \cdots \leqslant y_{L,n_L}$; *and the control points* $\mathbf{p}_R$ *of* $\vec{\mu}_{\tilde{A}_R}(\cdot)$ *are chosen such that* $x_{R,0} \leqslant \cdots \leqslant x_{R,n_R}$ *and* $y_{R,0} \geqslant \cdots \geqslant y_{R,n_R}$, *then the fuzzy set* $\tilde{A}$ *is convex and satisfies Condition 3.*

3.3. Methodology

3.3.1. Basic operations

In the previous section we imposed conditions on the placement of the control points to guarantee the generation of membership functions that satisfy Conditions 1–3. It remains to discuss how to calculate $\mu_{\tilde{A}}(x)$ given x and ${}^{\alpha}\tilde{A}$ given α. Assuming the location of the control points $\mathbf{p}_L$ and $\mathbf{p}_R$ are known, the following algorithms may be used.

Algorithm 1 (*Finding* $\mu_{\tilde{A}}(x)$ *given* x).

if $x \leqslant m_L - \gamma$ **or** $x \geqslant m_R + \beta$
  then $\mu_{\tilde{A}}(x) = 0$.
if $m_L \leqslant x \leqslant m_R$
  then $\mu_{\tilde{A}}(x) = 1$.
if $m_L - \gamma < x < m_L$
  then
    Find $t \in [0, 1]$ such that

$$\sum_{k=0}^{n_L} \binom{n_L}{k} t^k (1-t)^{n_L-k} x_{L,k} = x$$

    and compute

$$\mu_{\tilde{A}}(x) = \sum_{k=0}^{n_L} \binom{n_L}{k} t^k (1-t)^{n_L-k} y_{L,k}.$$

if $m_R < x < m_R + \beta$
  then
    Find t such that

$$\sum_{k=0}^{n_R} \binom{n_R}{k} t^k (1-t)^{n_R-k} x_{R,k} = x$$

    and compute

$$\mu_{\tilde{A}}(x) = \sum_{k=0}^{n_R} \binom{n_R}{k} t^k (1-t)^{n_R-k} y_{R,k}.$$

return $\mu_{\tilde{A}}(x)$.

The computational burden of Algorithm 1 is the solution of a root finding problem on a polynomial of degree n_L or n_R. This problem can be solved efficiently using the bisection method [3] or the methods proposed by Müller or Laguerre [12].

Algorithm 2 (*Finding* ${}^{\alpha}\tilde{A}$ *given* α).

if $\alpha = 0$
  then $l \leftarrow m_L - \gamma$, $u \leftarrow m_R + \beta$.
else
  if $\alpha = 1$
    then $l \leftarrow m_L$, $u \leftarrow m_R$.
  else
    if $\gamma \neq 0$
      then
        Find $t \in [0, 1]$ such that

$$\sum_{k=0}^{n_L} \binom{n_L}{k} t^k (1-t)^{n_L-k} y_{L,k} = \alpha$$

        and compute

$$x = \sum_{k=0}^{n_L} \binom{n_L}{k} t^k (1-t)^{n_L-k} x_{L,k}.$$

        set $l \leftarrow x$.
    else $l \leftarrow m_L$
    if $\beta \neq 0$
      then
        Find t such that

$$\sum_{k=0}^{n_R} \binom{n_R}{k} t^k (1-t)^{n_R-k} y_{R,k} = \alpha$$

        and compute

$$x = \sum_{k=0}^{n_R} \binom{n_R}{k} t^k (1-t)^{n_R-k} x_{R,k}.$$

set $u \leftarrow x$.
else $u \leftarrow m_{\mathrm{R}}$
set ${}^{\alpha}\tilde{A} \leftarrow [l, u]$.
return ${}^{\alpha}\tilde{A}$.

Again the computational bottleneck of Algorithm 2 is a root finding problem on a polynomial of degree n_{L} or n_{R}.

3.3.2. Data-driven estimation

In a direct approach to knowledge acquisition, experts are required to provide the degree of membership for each of a collection of points in the universal set [8]. The resulting set of value-membership pairs is used to construct the membership function of the underlying concept. This section provides a mechanism for constructing membership functions from data by determining the number of control points and their locations in the $(x, \mu(x))$ space.

The left side of the membership function can be estimated independently from the right side. We formulate a mathematical model and propose an algorithm for estimating the monotonically nonincreasing portion (right side) of a membership function. A similar approach can be used for estimating the nondecreasing (left side) portion.

Let the given data points be $\mathbf{d}_{\mathrm{R},i} = (\check{x}_{\mathrm{R},i}, \check{y}_{\mathrm{R},i})^{\mathrm{T}}$ for $i = 1, \ldots, M_{\mathrm{R}}$, where M_{R} is the total number of data points and $\check{y}_{\mathrm{R},i}$ is the membership given by the expert through the direct approach to the ith value $\check{x}_{\mathrm{R},i} \in X$. Without loss of generality, assume there are at least three data points (i.e., $M_{\mathrm{R}} \geqslant 3$) which are sorted in ascending order by their first component. Also let the $n_{\mathrm{R}} + 1$ control points be $\mathbf{p}_{\mathrm{R}} = ((x_{\mathrm{R},0}, y_{\mathrm{R},0}) \cdots (x_{\mathrm{R},n_{\mathrm{R}}}, y_{\mathrm{R},n_{\mathrm{R}}}))^{\mathrm{T}}$.

Let the decision variables be $x_{\mathrm{R},k}$ and $y_{\mathrm{R},k}$, the first and second coordinates of the kth control point ($k = 0, \ldots, n_{\mathrm{R}}$); t_i, the parameter value of the Bézier curve for the ith data point ($i = 1, \ldots, M_{\mathrm{R}}$); and n_{R}, the maximum value of the index associated with the control points to be placed. By Proposition 2, the first and last control points are fixed in $\mathbf{p}_{\mathrm{R},0} = (\check{x}_1, 1)^{\mathrm{T}}$ and $\mathbf{p}_{\mathrm{R},n_{\mathrm{R}}} = (\check{x}_{M_{\mathrm{R}}}, 0)^{\mathrm{T}}$. Thus the final value of some variables is known before performing any optimization, namely, $x_{\mathrm{R},0} = \check{x}_{\mathrm{R},1}$, $y_{\mathrm{R},0} = 1$, $x_{\mathrm{R},n_{\mathrm{R}}} = \check{x}_{\mathrm{R},M_{\mathrm{R}}}$, $y_{\mathrm{R},n_{\mathrm{R}}} = 0$, $t_1 = 0$, and $t_{M_{\mathrm{R}}} = 1$.

The following mathematical program minimizes the sum of the squared errors (SSE) between the fitted membership function and the empirical data.

$$\min \sum_{i=2}^{M_{\mathrm{R}}-1} \left(\check{y}_{\mathrm{R},i} - \sum_{k=0}^{n_{\mathrm{R}}} \binom{n_{\mathrm{R}}}{k} t_i^k (1 - t_i)^{n_{\mathrm{R}}-k} y_{\mathrm{R},k} \right)^2 \tag{8}$$

subject to:

$$\begin{aligned}
&\sum_{k=0}^{n_{\mathrm{R}}} \binom{n_{\mathrm{R}}}{k} t_i^k (1 - t_i)^{n_{\mathrm{R}}-k} x_{\mathrm{R},k} = \check{x}_{\mathrm{R},i} \\
&\quad \text{for } i = 2, \ldots, M_{\mathrm{R}} - 1, \\
&t_i \leqslant t_{i+1} \quad \text{for } i = 1, \ldots, M_{\mathrm{R}} - 1, \\
&x_{\mathrm{R},k} \leqslant x_{\mathrm{R},k+1} \quad \text{for } k = 0, \ldots, n_{\mathrm{R}} - 1, \\
&y_{\mathrm{R},k} \geqslant y_{\mathrm{R},k+1} \quad \text{for } k = 0, \ldots, n_{\mathrm{R}} - 1, \\
&x_{\mathrm{R},k} \geqslant \check{x}_{\mathrm{R},1} \quad \text{for } k = 1, \ldots, n_{\mathrm{R}} - 1, \\
&x_{\mathrm{R},k} \leqslant \check{x}_{\mathrm{R},M_{\mathrm{R}}} \quad \text{for } k = 1, \ldots, n_{\mathrm{R}} - 1, \\
&y_{\mathrm{R},k} \geqslant 0 \quad \text{for } k = 1, \ldots, n_{\mathrm{R}} - 1, \\
&y_{\mathrm{R},k} \leqslant 1 \quad \text{for } k = 1, \ldots, n_{\mathrm{R}} - 1, \\
&t_i \geqslant 0 \quad \text{for } i = 2, \ldots, M_{\mathrm{R}} - 1, \\
&t_i \leqslant 1 \quad \text{for } i = 2, \ldots, M_{\mathrm{R}} - 1,
\end{aligned} \tag{9}$$

$$n_{\mathrm{R}} \in \{2, 3, \ldots\}. \tag{10}$$

The fact that the number of control points is unknown and integer increases dramatically the complexity of the problem described by (8)–(10). Fortunately, in most practical applications the number of control points required is small. By treating this number as a parameter, we can solve a series of nonlinear programs, instead of dealing directly with a more difficult mixed integer nonlinear program. For a given n_{R}, the nonlinear program has $2n_{\mathrm{R}} + M_{\mathrm{R}} - 4$ continuous variables, $M_{\mathrm{R}} - 2$ nonlinear constraints, $M_{\mathrm{R}} + 2n_{\mathrm{R}} - 1$ linear constraints, and $2(M_{\mathrm{R}} + 2n_{\mathrm{R}} - 4)$ lower and upper bounds.

Given n_{R}, let $e(n_{\mathrm{R}})$ be the sum of the square errors between the fitted membership function and the empirical data when $n_{\mathrm{R}} + 1$ control points are used. Let $\mathrm{NLP}(\mathbf{d}_{\mathrm{R}}, n_{\mathrm{R}})$ be a function that solves the nonlinear program described by (8) and (9). $\mathrm{NLP}(\mathbf{d}_{\mathrm{R}}, n_{\mathrm{R}})$ takes the empirical data $\mathbf{d}_{\mathrm{R}}$ and a specified value of n_R as its arguments and returns the optimal value of the objective function described in (8), $e(n_R)$, and the optimal locations of the control points, $\mathbf{p}_R(n_R)$. Then Algorithm 3 can be used to solve the data-driven estimation for the right membership functions. The algorithm stops when the improvement in SSE is less than a given small quantity ϵ_0 (say, $\epsilon_0 = 0.0010$) or when the maximum number of control points to be placed is reached.

Algorithm 3 (*Data-driven estimation of the right membership function*).

set $\epsilon \leftarrow \epsilon_0$, $n_R \leftarrow 1$, $e(1) \leftarrow +\infty$.
do
  $n_R \leftarrow n_R + 1$
  $(e(n_R), \mathbf{p}_R(n_R)) \leftarrow \mathrm{NLP}(\mathbf{d}_R, n_R)$
  if $e(n_R - 1) - e(n_R) \leqslant \epsilon$ **or** $n_R = M_R - 1$
    if $e(n_R - 1) - e(n_R) < 0$
      then return $\mathbf{p}_R(n_R - 1)$, $e(n_R - 1)$, $n_R - 1$.
    else
      return $\mathbf{p}_R(n_R)$, $e(n_R)$, n_R.
end

Note that the algorithm may terminate with an increase in the SSE. In this case a local minimum has been obtained.

4. Performance

4.1. Flexibility

In current practice, users choose the shape of the membership functions from a pool of commonly used parameterized families. After the shape is selected, the parameters are manipulated to tune the shape. As discussed in Section 2.1 the pool of parameterized families of membership functions include triangular, trapezoidal, Gaussian, generalized bell curve, sigmoid, and S-shaped. In contrast, our approach can be used to produce the membership function of almost any imprecise concept. Basically, our approach can be viewed as a generalized free form generator of membership functions that satisfy the basic requirements presented in Section 3.2.

The example in Table 1 and Fig. 2 illustrates the ease with which a membership function can be

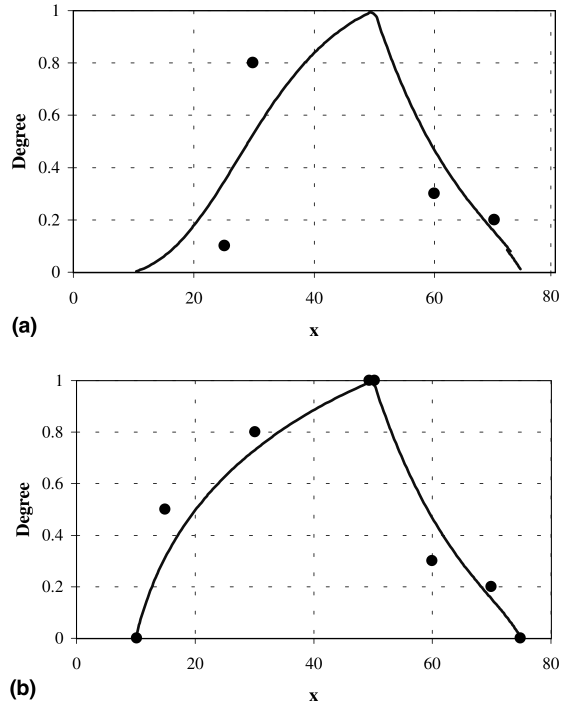

Fig. 2. Effect on the change of a single control point. (a) before; (b) after.

Table 1
Control points (before change)

k	$\mathbf{p}_{L,k}$		$\mathbf{p}_{R,k}$	
	$x_{L,k}$	$y_{L,k}$	$x_{R,k}$	$y_{R,k}$
0	10	0.0	50	1.0
1	25	0.1	60	0.3
2	30	0.8	70	0.2
3	50	1.0	75	0.0

constructed and tuned interactively using our approach. By placing the control points in the locations shown in Table 1, the membership function depicted in Fig. 2(a), with the control points being represented by black dots, can be obtained. By changing the location of the second control point on the left side ($k = 1$) from $(25, 0.1)$ to $(15, 0.5)$, the curve bends toward the new point as if there were some magnetic attraction between the control point and the membership function (left portion). This is shown in Fig. 2(b). Moreover, due to the Property 2 of the Bézier curves presented in Section 3.1, we observe that, even though this change affects the whole left membership function, the change is more noticeable in the vicinity of the control point.

This new flexible and interactive way of building and tuning a membership function can be leveraged by using a graphical user interface (GUI). Currently, we are developing a GUI that helps the user add, move, and delete control points to obtain the desired free-form membership function.

4.2. Numerical examples

For data-driven estimation, we tested our approach using data originally published by Zysno [17] and compared its performance to that of the methods reported in Zysno [17] and Dombi [4]. Sixty-four persons from 21 to 25 years of age were asked to rate 52 different statements related to age concepts. The group was divided into four subgroups of 16. The individuals within a subgroup were asked to rate one of the 4 concepts: *very young man*, *young man*, *old man*, and *very old man*. The subjects were asked to give the degree of membership in the designated fuzzy set of a man of x years of age on a 0% to 100% scale.

Fig. 3 shows the progress of our algorithm when applied to automatically estimate the membership function for the fuzzy set *old man* based on the data collected by interviewing subject 35 in Zysno [17]. In the figure, a black square represents a control point. A number beside a control point is used when more than one control point shares the same location. The number represents the total number of control points in the given location. Empty circles represent data points. Lines are used to display the estimated membership functions.

As is customary in the literature, to compare our method with those of Zysno and Dombi we used the sum of the squared errors (SSE) as the measure of goodness of fit between a membership function model and the empirical data. Dombi used data for subjects 9, 18, 35, 44 and 58 in Zysno [17] as his benchmark test cases and measured the corresponding SSEs. Zysno estimated the parameters of his model for all the data sets (64 subjects), but did not provide SSE as the measure of goodness of fit. In order to make valid comparisons, we calculated the SSE for Zysno's model for the benchmark test cases chosen by Dombi. Table 2 gives the SSE for the benchmark test cases for the three models, namely, Dombi, Zysno, and ours. The superior performance of our approach is clearly seen.

Table 3 shows the evolution of the SSE for each of the test benchmark cases when our data-driven estimation mechanism is used. The resulting estimated membership functions are shown in Figs. 3 and 4. Note that most of the intermediate solutions shown in Table 3 are better than the final solutions provided by Zysno and Dombi. By monitoring the progress of the SSE, the algorithm may be interrupted as soon as the user is satisfied with the current SSE. We have used a very small value for ϵ which could potentially cause overfitting. However, our method for membership function generation can get arbitrarily close to the empirical data.

A final remark should be made. After fitting a membership function to data, the user can still go back and tune the membership by moving the control points as described in Section 4.1. This high level of interaction and flexibility between the model and the user is a desirable feature when designing imprecise concepts.

4.3. Computational efficiency

In the absence of data, our approach requires from the user the number and location of the control points. We have shown in Section 4.1 how easy it is to change the shape of the membership

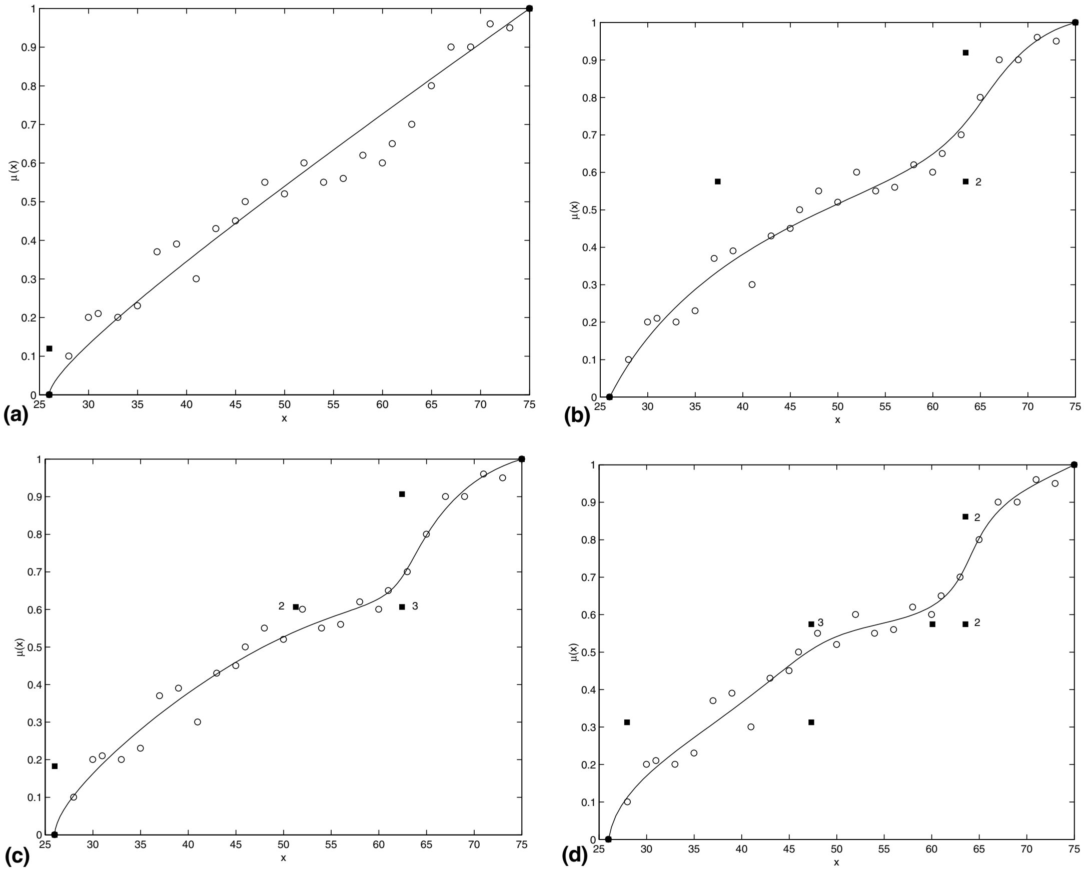

Fig. 3. Data-driven estimation. Data: subject 35 (*Old man*). (a) $n_{\mathrm{L}} = 2, \mathrm{SSE} = 0.08320$; (b) $n_{\mathrm{L}} = 5, \mathrm{SSE} = 0.03517$; (c) $n_{\mathrm{L}} = 8, \mathrm{SSE} = 0.02846$. (d) $n_{\mathrm{L}} = 11, \mathrm{SSE} = 0.02469$.

Table 2
Sum of square errors (SSE)

Model	Data set (subject)				
	9	18	35	44	58
Zysno	0.10074	0.05054	0.17808	0.07572	0.02641
Dombi	0.13204	0.05103	0.14841	0.05284	0.03027
Proposed approach ($\epsilon = 0.0010$)	0.07149	0.02390	0.02469	0.03610	0.01941

function by displacing the control points. It is important to note that the computational effort needed to redraw a membership function, whenever a control point is moved, is just the simple evaluation of (5) and (6).

Once a membership function has been generated (either with or without data), as shown in Section 3.3.1, the calculations required to find $\mu_{\tilde{A}}(x)$ for a given x and find ${}^{\alpha}\tilde{A}$ for a given α reduce to solving a computationally inexpensive root finding problem in a closed interval ($t \in [0, 1]$).

When our approach is used to fit membership functions to data, it was seen in Section 3.3.2 that the computational bottleneck is finding a solution to a nonlinear program with $2n_{\mathrm{R}} + M_{\mathrm{R}} - 4$ variables, $M_{\mathrm{R}} - 2$ nonlinear constraints, $M_{\mathrm{R}} + 2n_{\mathrm{R}} - 1$

Table 3
SSE progress for the test benchmark cases ($\epsilon = 0.0010$)[a]

Control points	Data set (subject)				
($n_R + 1$)	9	18	35	44	58
3	0.09231	0.07353	0.08320	0.12333	0.06127
4	0.09044	0.04557	0.05242	0.04838	0.02100
5	0.08822	0.03420	0.04498	0.04071	0.01981
6	0.08167	0.02406	0.03517	0.03800	0.01941[a]
7	0.07538	0.02390[a]	0.03133	0.03665	
8	0.07149[a]		0.03011	0.03610[a]	
9	0.07428		0.02846		
10			0.02715		
11			0.02549		
12			0.02469[a]		

[a] Final solution.

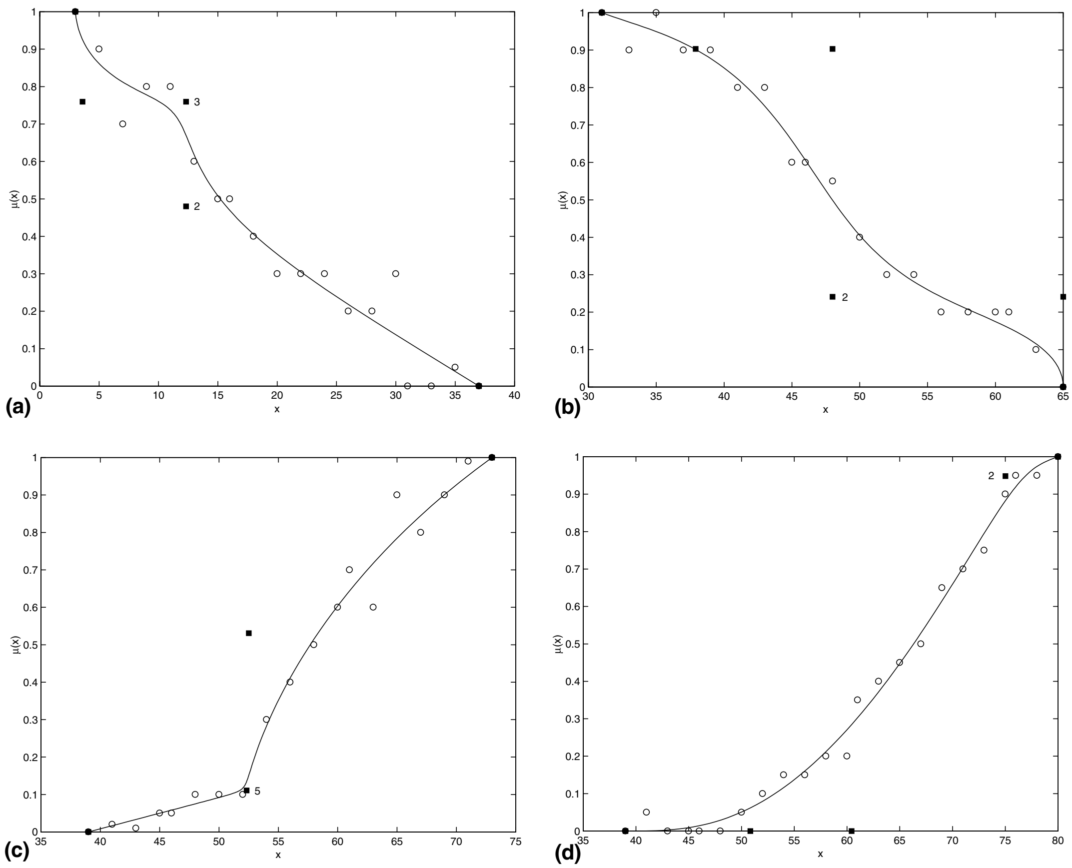

Fig. 4. Data-driven estimation ($\epsilon = 0.0010$). (a) Subject 9: $n_R = 7, \mathrm{SSE} = 0.07149$. (b) Subject 18: $n_R = 6, \mathrm{SSE} = 0.02390$. (c) Subject 44: $n_R = 7, \mathrm{SSE} = 0.03610$. (d) Subject 58: $n_R = 5, \mathrm{SSE} = 0.01941$.

linear constraints, and $2(M_R + 2n_R - 4)$ lower and upper bounds. For the numerical examples presented in Section 4.2 we used AMPL as the algebraic modeling language and MINOS 5.4 as the nonlinear optimizer. Using a computer with a 266 MHz Pentium II processor, all the nonlinear programs took less than 4 seconds to run.

5. Conclusion

We have proposed a new mechanism based on Bézier curves for generating membership functions well suited for a broad spectrum of fuzzy modeling. By placing control points in different locations, the shape of the membership functions can be altered in a very natural and intuitive way. Mechanisms for dealing with subjective and data-driven estimation of membership functions were discussed. Some advantages of this approach are its flexibility, ease of use, computational efficiency, and suitability for a graphical interactive implementation. The major advantage is its immense power of fitting data as close as possible without a priori assumption of the shape of the function.

Several aspects of this work are in progress. First, a tailored interior point algorithm that can exploit the structure of the nonlinear program presented in (8) and (9) is currently under investigation [5]. We are currently exploring applications of this methodology to the fields of fuzzy engineering design, fuzzy decision making, and fuzzy control.

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