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A survey of credibility theory

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Abstract This paper provides a survey of credibility theory that is a new branch of mathematics for studying the behavior of fuzzy phenomena. Some basic concepts and fundamental theorems are introduced, including credibility measure, fuzzy variable, membership function, credibility distribution, expected value, variance, critical value, entropy, distance, credibility subadditivity theorem, credibility extension theorem, credibility semicontinuity law, product credibility theorem, and credibility inversion theorem. Recent developments and applications of credibility theory are summarized. A new idea on chance space and hybrid variable is also documented.

Keywords Fuzzy variable · Credibility measure · Credibility theory · Fuzzy random variable · Random fuzzy variable · Conditional credibility

1 Introduction

The concept of fuzzy set was initiated by Zadeh (1965) via membership function. In order to measure a fuzzy event, Zadeh (1978) proposed the concept of possibility measure. Since then possibility theory has been studied by many researchers such as Nahmias (1978), Kaufman and Gupta (1985), Zimmermann (1985), Dubois and Prade (1988), Klir and Yuan (1995), De Cooman (1997) and Liu (2002a). Although possibility measure has been widely used, it has no

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self-duality property. However, a self-dual measure is absolutely needed in both theory and practice. In order to define a self-dual measure, Liu and Liu (2002) introduced the concept of credibility measure. In addition, a sufficient and necessary condition for credibility measure was given by Li and Liu (2006a).

Credibility theory was founded by Liu (2004) in 2004 as a branch of mathematics for studying the behavior of fuzzy phenomena. Credibility theory is based on five axioms from which a credibility measure is defined. Then we have some basic concepts such as fuzzy variable, membership function, credibility distribution, expected value, variance, critical value, entropy, distance, and fundamental theorems such as the credibility subadditivity theorem, the credibility extension theorem, the credibility semicontinuity law, the product credibility theorem, and the credibility inversion theorem.

The purpose of this paper is to provide a personal overview of credibility theory rather than a balanced survey of all activities in the area. The author hopes to provide his readers with an axiomatic approach of studying fuzzy phenomena.

The paper is organized as follows. Section 2 introduces the concepts of credibility measure and credibility space, including subadditivity theorem, extension theorem, semicontinuity law, and product credibility theorem. Section 3 defines the fuzzy variable as a function on a credibility space. Section 4 gives the concept of membership function, and provides a sufficient and necessary condition for membership functions as well as the credibility inversion theorem. Sections 5–14 introduce the main concepts and theorems of credibility theory.

As extensions of credibility theory, we introduce fuzzy random theory and random fuzzy theory. Roughly speaking, a fuzzy random variable is a function from a probability space to the set of fuzzy variables, and a random fuzzy variable is a function from a credibility space to the set of random variables. Both of them may be regarded as special cases of a hybrid variable which is a function from a chance space to the set of real numbers. This is discussed in Sections 15–17.

Many researchers have attempted to define conditional credibility (or possibility). However, the results in the literature are not convincing. In fact, within the framework of the fifth axiom (or equivalently the extension principle of Zadeh), it is impossible to define a reasonable and acceptable conditional credibility because the minimization operator loses partial information related to the credibilities of original fuzzy events. If we do want to have a concept of conditional credibility, we must replace the fifth axiom with a new one (or equivalently give a new extension principle). In order to do so, Liu (2004) presented a new mathematical system called *nonclassical credibility theory*. This is introduced in Section 18.

The last section provides some applications of credibility theory, including fuzzy optimization, scheduling, engineering design, portfolio selection, capital budgeting, vehicle routing, facility location, inventory control, production planning, reliability, and renewal process.

2 Credibility measure and credibility space

Let Θ be a nonempty set, and let $\mathcal{P}(\Theta)$ be the power set of Θ (i.e., all subsets of Θ). Each element in $\mathcal{P}(\Theta)$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $\text{Cr}\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $\text{Cr}\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following five axioms:

1. $\text{Cr}\{\Theta\} = 1$.
2. Cr *is increasing, i.e.,* $\text{Cr}\{A\} \le \text{Cr}\{B\}$ *whenever* $A \subset B$.
3. Cr *is self-dual, i.e.,* $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ *for any* $A \in \mathcal{P}(\Theta)$.
4. $\text{Cr}\{\cup_i A_i\} \wedge 0.5 = \sup_i \text{Cr}\{A_i\}$ *for any* $\{A_i\}$ *with* $\text{Cr}\{A_i\} \le 0.5$.
5. *Let* Θ_k *be nonempty sets on which* Cr_k *satisfy the first four axioms,* $k = 1, 2, \ldots, n$*, respectively, and let* $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$*. Then*

$$\text{Cr}\{(\theta_1, \theta_2, \ldots, \theta_n)\} = \text{Cr}_1\{\theta_1\} \wedge \text{Cr}_2\{\theta_2\} \wedge \cdots \wedge \text{Cr}_n\{\theta_n\} \tag{1}$$

for each $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

Definition 1 (*Liu & Liu, 2002*) The set function Cr is called a credibility measure if it satisfies the first four axioms.

It is easy to verify that $\text{Cr}\{\emptyset\} = 0$ and that the credibility measure takes values between 0 and 1. It follows from the axioms that the credibility measure is increasing and self-dual.

Theorem 1 (Liu, 2004, Credibility Subadditivity Theorem) *The credibility measure is subadditive. That is,* $\text{Cr}\{A \cup B\} \le \text{Cr}\{A\} + \text{Cr}\{B\}$ *for any* $A, B \in \mathcal{P}(\Theta)$.

A credibility measure on Θ is additive if and only if there are at most two elements in Θ taking nonzero credibility values. This means that a credibility measure is identical with probability measure if there are effectively two elements in the universal set.

Generally speaking, the credibility measure is neither lower semicontinuous nor upper semicontinuous. However, we have the following credibility semicontinuity law.

Theorem 2 (Liu, 2004, Credibility Semicontinuity Law). *For* $A_1, A_2, \ldots \in \mathcal{P}(\Theta)$, *we have*

$$\lim_{i\to\infty} \text{Cr}\{A_i\} = \text{Cr}\left\{\lim_{i\to\infty} A_i\right\} \tag{2}$$

if one of the following conditions is satisfied:
(a) $\text{Cr}\{A\} \le 0.5$ *and* $A_i \uparrow A$; (b) $\lim_{i\to\infty} \text{Cr}\{A_i\} < 0.5$ *and* $A_i \uparrow A$;
(c) $\text{Cr}\{A\} \ge 0.5$ *and* $A_i \downarrow A$; (d) $\lim_{i\to\infty} \text{Cr}\{A_i\} > 0.5$ *and* $A_i \downarrow A$.

Suppose that the credibility of each singleton set is given. Is the credibility measure fully and uniquely determined? The credibility extension theorem will answer this question.

Theorem 3 (Li & Liu, 2006a, Credibility Extension Theorem) *Suppose that* Θ *is a nonempty set, and* $\mathrm{Cr}\{\theta\}$ *is a nonnegative function on* Θ *satisfying the credibility extension condition*

$$\begin{aligned} &\sup_{\theta\in\Theta} \mathrm{Cr}\{\theta\} \geq 0.5, \\ &\mathrm{Cr}\{\theta^*\} + \sup_{\theta\neq\theta^*} \mathrm{Cr}\{\theta\} = 1 \quad \textit{if}\ \mathrm{Cr}\{\theta^*\} \geq 0.5. \end{aligned} \tag{3}$$

Then $\mathrm{Cr}\{\theta\}$ *has a unique extension to a credibility measure on* $\mathcal{P}(\Theta)$ *as follows,*

$$\mathrm{Cr}\{A\} = \begin{cases} \sup\limits_{\theta\in A} \mathrm{Cr}\{\theta\}, & \text{if } \sup\limits_{\theta\in A} \mathrm{Cr}\{\theta\} < 0.5 \\ 1 - \sup\limits_{\theta\in A^c} \mathrm{Cr}\{\theta\}, & \text{if } \sup\limits_{\theta\in A} \mathrm{Cr}\{\theta\} \geq 0.5. \end{cases} \tag{4}$$

The credibility extension theorem enables us to give numerical credibility measures. In other words, in order to give a credibility measure, it suffices to give the credibility value of each singleton set. Without the credibility extension theorem, we cannot determine a nontrivial credibility measure because it is impossible for us to list the credibility values of all subsets of Θ.

Definition 2 Let Θ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ, and Cr a credibility measure. Then the triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called a credibility space.

Theorem 4 (Product Credibility Theorem) *Suppose that* Θ_k *are nonempty sets,* Cr_k *the credibility measures on* $\mathcal{P}(\Theta_k)$*,* $k = 1, 2, \ldots, n$*, respectively. Let* $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$*. Then* $\mathrm{Cr} = \mathrm{Cr}_1 \wedge \mathrm{Cr}_2 \wedge \cdots \wedge \mathrm{Cr}_n$ *defined by Axiom* 5 *has a unique extension to a credibility measure on* $\mathcal{P}(\Theta)$ *as follows,*

$$\mathrm{Cr}\{A\} = \begin{cases} \sup\limits_{(\theta_1,\theta_2,\ldots,\theta_n)\in A} \min\limits_{1\leq k\leq n} \mathrm{Cr}_k\{\theta_k\}, \\ \qquad \textit{if} \sup\limits_{(\theta_1,\theta_2,\ldots,\theta_n)\in A} \min\limits_{1\leq k\leq n} \mathrm{Cr}_k\{\theta_k\} < 0.5 \\ 1 - \sup\limits_{(\theta_1,\theta_2,\ldots,\theta_n)\in A^c} \min\limits_{1\leq k\leq n} \mathrm{Cr}_k\{\theta_k\}, \\ \qquad \textit{if} \sup\limits_{(\theta_1,\theta_2,\ldots,\theta_n)\in A} \min\limits_{1\leq k\leq n} \mathrm{Cr}_k\{\theta_k\} \geq 0.5 \end{cases} \tag{5}$$

for each $A \in \mathcal{P}(\Theta)$*. The triplet* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ *will be called the product credibility space.*

The product credibility theorem will play an important role in credibility theory. Without it, it is impossible to define the arithmetic of fuzzy variables.

3 Fuzzy variable

Traditionally, a fuzzy variable is defined by a membership function (Zadeh, 1965). Now we define it as a function on a credibility space just as a random variable is defined as a measurable function on a probability space.

Definition 3 A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to the set of real numbers.

An n-dimensional fuzzy vector is defined as a function from a credibility space to the set of n-dimensional real vectors. It is easy to prove that $(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy vector if and only if $\xi_1, \xi_2, \ldots, \xi_n$ are fuzzy variables.

Definition 4 (*Fuzzy Arithmetic*) Let $f : \Re^n \to \Re$ be a function, and let $\xi_1, \xi_2, \ldots, \xi_n$ be fuzzy variables on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy variable defined as $\xi(\theta) = f(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta))$ for any $\theta \in \Theta$.

If the fuzzy variables are defined on different credibility spaces, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy variable defined on the product credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ as $\xi(\theta_1, \theta_2, \ldots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), \ldots, \xi_n(\theta_n))$ for any $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$. Essentially, fuzzy arithmetic is just the classical operation of functions on a credibility space.

4 Membership function

A fuzzy variable has been defined as a function from a credibility space to the set of real numbers rather than as a membership function. Now we have to introduce a membership function for a fuzzy variable.

Definition 5 Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then its membership function is derived from the credibility measure by

$$\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re. \tag{6}$$

It is clear that a fuzzy variable has a unique membership function. However, a membership function may produce multiple fuzzy variables.

Theorem 5 (Sufficient and Necessary Condition for Membership Function) *A function* $\mu : \Re \to [0, 1]$ *is a membership function if and only if* $\sup \mu(x) = 1$.

In practice, a fuzzy variable may be specified by a membership function. In this case, we need a formula to calculate the credibility value of some fuzzy event. The credibility inversion theorem provides this.

Theorem 6 (Credibility Inversion Theorem) *Let ξ be a fuzzy variable with membership function μ. Then for any set B of real numbers, we have*

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right). \tag{7}$$

If a fuzzy variable is defined as a function on a credibility space, then we may get its membership function via (6). Conversely, if a fuzzy variable is given by a membership function, then we may get the credibility value via (7).

5 Credibility distribution

Following the idea of a probability distribution of a random variable, here we introduce the credibility distribution for a fuzzy variable.

Definition 6 (*Liu, 2002b*) The credibility distribution $\Phi : \Re \to [0,1]$ of a fuzzy variable ξ is defined by $\Phi(x) = \mathrm{Cr}\{\theta \in \Theta | \xi(\theta) \le x\}$.

That is, $\Phi(x)$ is the credibility that the fuzzy variable ξ takes a value less than or equal to x. If the fuzzy variable ξ is given by a membership function μ, then its credibility distribution is determined by

$$\Phi(x) = \frac{1}{2}\left(\sup_{y \le x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right), \quad \forall x \in \Re. \tag{8}$$

Generally speaking, the credibility distribution is neither left-continuous nor right-continuous. However, we have the following sufficient and necessary condition for credibility distribution.

Theorem 7 (Liu, 2004) *A function* $\Phi : \Re \to [0,1]$ *is a credibility distribution if and only if it is an increasing function with*

$$\lim_{x \to -\infty} \Phi(x) \le 0.5 \le \lim_{x \to \infty} \Phi(x), \tag{9}$$

$$\lim_{y \downarrow x} \Phi(y) = \Phi(x) \text{ if } \lim_{y \downarrow x} \Phi(y) > 0.5 \text{ or } \Phi(x) \ge 0.5. \tag{10}$$

Definition 7 (*Liu, 2004*) The credibility density function ϕ: $\Re \to [0, +\infty)$ of a fuzzy variable ξ is a function such that

$$\int_{-\infty}^{+\infty} \phi(y)\mathrm{d}y = 1, \quad \Phi(x) = \int_{-\infty}^{x} \phi(y)\mathrm{d}y, \quad \forall x \in \Re \tag{11}$$

where Φ is the credibility distribution of the fuzzy variable ξ.

Different from the random case, generally speaking, $\mathrm{Cr}\{a \le \xi \le b\} \ne \int_a^b \phi(y)\mathrm{d}y$. However, Liu (2004) proved that

$$\mathrm{Cr}\{\xi \le x\} = \int_{-\infty}^{x} \phi(y)\mathrm{d}y, \quad \mathrm{Cr}\{\xi \ge x\} = \int_{x}^{+\infty} \phi(y)\mathrm{d}y. \tag{12}$$

Some mathematical properties of credibility distributions were investigated by Yang and Liu (2006) and continuity theorems of characteristic functions were also proved.

Peng, Mok, and Tse (2005a) introduced a new concept of fuzzy dominance based on credibility distributions, and provided some basic properties of fuzzy dominance.

6 Independent and identical distribution

The independence of fuzzy variables has been discussed by many authors from different angles, for example, Zadeh (1978), Nahmias (1978), Yager (1992), Liu (2004), and Liu and Gao (2005a). A lot of equivalence conditions for independence have been presented. Here we use the condition given by Liu and Gao (2005a).

Definition 8 The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le m} \mathrm{Cr}\{\xi_i \in B_i\} \tag{13}$$

for any sets $B_1, B_2, \ldots, B_m$ of $\Re$.

Liu and Gao (2005a) also proved that the fuzzy variables $\xi_1, \xi_2, \ldots, \xi_m$ are independent if and only if

$$\mathrm{Cr}\left\{\bigcup_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \max_{1\le i\le m} \mathrm{Cr}\{\xi_i \in B_i\} \tag{14}$$

for any sets $B_1, B_2, \ldots, B_m$ of $\Re$.

Theorem 8 (Extension Principle of Zadeh) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and f: $\Re^n \to \Re$ a function. Then the membership function μ of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is derived from the membership functions $\mu_1, \mu_2, \ldots, \mu_n$ by*

$$\mu(x) = \sup_{x=f(x_1,x_2,\ldots,x_n)} \min_{1\le i\le n} \mu_i(x_i), \quad x \in \Re. \tag{15}$$

The extension principle of Zadeh is only applicable to operations on independent fuzzy variables. Previously, in the literature, the extension principle has been treated as a postulate. However, in credibility theory it is treated as a theorem.

Theorem 9 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and $f : \Re^n \to \Re^m$ a function. Then for any set B of $\Re^m$, the credibility* $\mathrm{Cr}\{f(\xi_1, \xi_2, \ldots, x_n) \in B\}$ *is*

$$\frac{1}{2}\left(\sup_{f(x_1,x_2,\ldots,x_n)\in B}\min_{1\le i\le n}\mu_i(x_i)+1-\sup_{f(x_1,x_2,\ldots,x_n)\in B^c}\min_{1\le i\le n}\mu_i(x_i)\right).$$

Definition 9 (*Liu, 2004*) The fuzzy variables ξ and η are said to be identically distributed if $\text{Cr}\{\xi\in B\}=\text{Cr}\{\eta\in B\}$ for any set B of real numbers.

It is easy to prove that the fuzzy variables are identically distributed if and only if they have the same membership function. Furthermore, identically distributed variables have the same credibility distribution and the same credibility density function.

7 Expected value

The expected value operator of random variable plays an extremely important role in probability theory. For fuzzy variables, there are many ways to define an expected value operator. See, for example, Dubois and Prade (1987), Heilpern (1992), Campos and González (1989), González (1990) and Yager (1981). The most general definition of expected value operator for fuzzy variables was given by Liu and Liu (2002). This definition is applicable to both continuous and discrete fuzzy variables.

Definition 10 (*Liu and Liu, 2002*) Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi]=\int_0^{+\infty}\text{Cr}\{\xi\ge r\}\text{d}r-\int_{-\infty}^0\text{Cr}\{\xi\le r\}\text{d}r \tag{16}$$

provided that at least one of the two integrals is finite.

Let ξ be a fuzzy variable whose credibility density function ϕ exists. Liu (2002b) proved that

$$E[\xi]=\int_{-\infty}^{+\infty}x\phi(x)\text{d}x \tag{17}$$

provided that the Lebesgue integral is finite. If $\lim_{x\to-\infty}\Phi(x)=0$ and $\lim_{x\to\infty}\Phi(x)=1$, then

$$E[\xi]=\int_{-\infty}^{+\infty}x\text{d}\Phi(x) \tag{18}$$

provided that the Lebesgue-Stieltjes integral is finite. Zhu and Ji (2006) presented some methods to calculate the expected value of a function of fuzzy variable.

Let ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have $E[a\xi+b\eta]=aE[\xi]+bE[\eta]$. This is called the *linearity of expected value operator* (Liu & Liu, 2003b).

8 Variance

The variance of a fuzzy variable provides a measure of the spread of the distribution around its expected value. A small value of variance indicates that the fuzzy variable is tightly concentrated around its expected value; and a large value of variance indicates that the fuzzy variable has a wide spread around its expected value. The variance can be used to represent the level of risk in a fuzzy system.

Definition 11 (*Liu and Liu, 2002*) Let ξ be a fuzzy variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E[(\xi - e)^2]. \tag{19}$$

If ξ is a fuzzy variable whose variance exists, a and b are real numbers, then $V[a\xi + b] = a^2 V[\xi]$. It is also easy to prove that $V[\xi] = 0$ if and only if ξ is essentially a constant.

Let ξ be a fuzzy variable that takes values in $[a, b]$, but whose membership function is otherwise arbitrary. If its expected value is given, say e, what is the possible maximum variance? It was proved by Li and Liu (2006b) that the maximum variance is $(e - a)(b - e)$. This conclusion is called the *maximum variance theorem*. It may play an important role in treating games against nature.

9 Critical values

In order to rank fuzzy variables, we may use two critical values: the optimistic value and the pessimistic value.

Definition 12 (*Liu, 2002b*) Let ξ be a fuzzy variable, and $\alpha \in (0, 1]$. Then

$$\xi_{\sup}(\alpha) = \sup\left\{r \;\middle|\; \mathrm{Cr}\{\xi \ge r\} \ge \alpha\right\} \tag{20}$$

is called the α-optimistic value to ξ, and

$$\xi_{\inf}(\alpha) = \inf\left\{r \;\middle|\; \mathrm{Cr}\{\xi \le r\} \ge \alpha\right\} \tag{21}$$

is called the α-pessimistic value to ξ.

This means that the fuzzy variable ξ will exceed the α-optimistic value $\xi_{\sup}(\alpha)$ with credibility α, and will be below the α-pessimistic value $\xi_{\inf}(\alpha)$ with credibility α. In other words, the α-optimistic value $\xi_{\sup}(\alpha)$ is the supremum value that ξ achieves with credibility α, while the α-pessimistic value $\xi_{\inf}(\alpha)$ is the infimum value that ξ achieves with credibility α.

Some properties of optimistic and pessimistic values of fuzzy variables were investigated by Peng and Liu (2004b). It has been proven that (a) $\xi_{\inf}(\alpha)$ is an increasing and left-continuous function of α; (b) $\xi_{\sup}(\alpha)$ is a decreasing and left-continuous function of α; and (c) if $\alpha > 0.5$, then $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} \ge \alpha$ and $\mathrm{Cr}\{\xi \ge \xi_{\sup}(\alpha)\} \ge \alpha$.

10 Entropy

Fuzzy entropy is a measure of uncertainty. It has been studied by many researchers such as Knopfmacher (1975), Loo (1977), Yager (1979, 1980), Higashi and Klir (1982), Dubois and Prade (1985), and Pal and Pal (1992). For the case in which ξ is a fuzzy set taking values x_i with membership degrees μ_i, $i = 1, 2, \ldots, n$, respectively, De Luca and Termini (1972) defined its entropy as $S(\mu_1) + S(\mu_2) + \cdots + S(\mu_n)$, where $S(t) = -t \ln t - (1-t)\ln(1-t)$.

The entropy of De Luca and Termini (1972) characterizes uncertainty resulting primarily from the linguistic vagueness rather than resulting from information deficiency, and vanishes when the fuzzy variable takes all the values with membership degree 1. However, we hope that the degree of uncertainty is 0 when the fuzzy variable degenerates to a crisp number, and is maximum when the fuzzy variable is an equipossible one, i.e., all values have the same possibility. In order to meet such a requirement, Li and Liu (2005) provided a new definition based on credibility measure.

Definition 13 (*Li and Liu, 2005*) Let ξ be a discrete fuzzy variable taking values in $\{x_1, x_2, \ldots, x_n\}$. Then its entropy is defined by

$$H[\xi] = \sum_{i=1}^{n} S(\mathrm{Cr}\{\xi = x_i\}) \tag{22}$$

where $S(t) = -t \ln t - (1-t)\ln(1-t)$.

It is clear that the entropy depends only on the number of values and their credibilities and does not depend on the actual values that the fuzzy variable takes. Furthermore, we have $0 \le H[\xi] \le n \ln 2$. The entropy of a fuzzy variable reaches its minimum 0 when the fuzzy variable degenerates to a crisp number. In this case, there is no uncertainty. The entropy reaches its maximum $n \ln 2$ when the fuzzy variable is an equipossible one. In this case, there is no preference among all the values that the fuzzy variable will take.

Definition 14 (*Li and Liu, 2005*) Let ξ be a continuous fuzzy variable. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mathrm{Cr}\{\xi = x\})\mathrm{d}x. \tag{23}$$

11 Distance

The distance between fuzzy variables is also an important concept in credibility theory. It may be employed in image processing and pattern recognition.

Definition 15 (*Liu, 2004*) The distance between fuzzy variables ξ and η is defined as $d(\xi, \eta) = E[|\xi - \eta|]$.

Li and Liu (2006c) proved that (a) $d(\xi,\eta) \geq 0$ and $d(\xi,\eta) = 0$ if and only if $\xi = \eta$; (b) $d(\xi,\eta) = d(\eta,\xi)$; and (c) $d(\xi,\eta) \leq 2d(\xi,\tau) + 2d(\eta,\tau)$.

Let us examine the triangle inequality. In fact, it has been proved that $d(\xi,\eta) \leq \lambda[d(\xi,\tau) + d(\eta,\tau)]$, where λ is a constant between 1.5 and 2. An open problem is to determine the smallest value of λ for which the above inequality holds.

12 Inequalities

There are several well-known inequalities in probability theory, such as Markov's inequality, Chebyshev's inequality, Minkowski's inequality, Hölder's inequality, and Jensen's inequality. They play an important role in both theory and applications. Liu (2003) discussed the analogous inequalities for fuzzy variables.

Let ξ be a fuzzy variable, and f a nonnegative function. If f is even and increasing on $[0,\infty)$, then for any given number $t > 0$, Liu (2003) proved that

$$\mathrm{Cr}\{|\xi| \geq t\} \leq \frac{E[f(\xi)]}{f(t)}. \tag{24}$$

Based on such an inequality, Liu (2003) also proved the Markov inequality

$$\mathrm{Cr}\{|\xi| \geq t\} \leq \frac{E[|\xi|^p]}{t^p}, \quad t > 0,\ p > 0 \tag{25}$$

and the Chebyshev inequality

$$\mathrm{Cr}\{|\xi - E[\xi]| \geq t\} \leq \frac{V[\xi]}{t^2}, \quad t > 0. \tag{26}$$

Let p and q be two positive real numbers with $1/p + 1/q = 1$, and let ξ and η be independent fuzzy variables. Liu (2003) proved Hölder's inequality, i.e.,

$$E[|\xi\eta|] \leq \sqrt[p]{E[|\xi|^p]}\sqrt[q]{E[|\eta|^q]}, \quad p > 0,\ q > 0,\ \frac{1}{p} + \frac{1}{q} = 1 \tag{27}$$

and Minkowski's inequality, i.e.,

$$\sqrt[p]{E[|\xi+\eta|^p]} \leq \sqrt[p]{E[|\xi|^p]} + \sqrt[p]{E[|\eta|^p]}, \quad p \geq 1. \tag{28}$$

Let f be a convex function. Liu (2004) proved Jensen's inequality, $f(E[\xi]) \leq E[f(\xi)]$. Also, $|E[\xi]|^p \leq E[|\xi|^p]$ for $p \geq 1$.

13 Convergence concepts

Liu (2003) presented some convergence concepts of fuzzy sequence, including convergence almost surely (a.s.), convergence in credibility, convergence in mean, and convergence in distribution.

Suppose that $\xi, \xi_1, \xi_2, \ldots$ are fuzzy variables defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. The sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if there exists a set $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{i\to\infty} |\xi_i(\theta) - \xi(\theta)| = 0 \tag{29}$$

for every $\theta \in A$. In this case we write $\xi_i \to \xi$, a.s. We say that the sequence $\{\xi_i\}$ converges in credibility to ξ if

$$\lim_{i\to\infty} \mathrm{Cr}\{|\xi_i - \xi| \geq \varepsilon\} = 0 \tag{30}$$

for every $\varepsilon > 0$. We say that the sequence $\{\xi_i\}$ converges in mean to ξ if

$$\lim_{i\to\infty} E[|\xi_i - \xi|] = 0. \tag{31}$$

We say that $\{\xi_i\}$ converges in distribution to ξ if $\Phi_i(x) \to \Phi(x)$ for all continuity points x of Φ. The relationship among the convergence concepts was proved by Wang and Liu (2003).

14 Fuzzy simulations

Fuzzy simulation, developed by Liu and Iwamura (1998), Liu (1998, 1999) and Liu and Liu (2002), was defined as a technique for conducting sampling experiments on models of fuzzy systems. Liu (2006) also discussed convergence issues in fuzzy simulation. Numerous numerical experiments have shown that fuzzy simulation indeed works very well for handling fuzzy systems. In this section, we will introduce the technique of fuzzy simulation for computing credibility, finding critical values, and calculating expected value.

Case I Suppose that f is a function and that $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy vector with membership function μ. We design a fuzzy simulation to compute the credibility $L = \mathrm{Cr}\{f(\boldsymbol{\xi}) \leq 0\}$. We randomly generate $\boldsymbol{u}_k$ from the ε-level set of $\boldsymbol{\xi}$ for $k = 1, 2, \ldots, N$, where ε is a small number. It follows from the credibility inversion theorem that

$$L = \frac{1}{2}\left(\max_{1\leq k\leq N}\left\{\mu(\boldsymbol{u}_k) \,\middle|\, f(\boldsymbol{u}_k) \leq 0\right\} + \min_{1\leq k\leq N}\left\{1 - \mu(\boldsymbol{u}_k) \,\middle|\, f(\boldsymbol{u}_k) > 0\right\}\right).$$

Case II We design a fuzzy simulation to find the maximal $\overline{f}$ such that the inequality $\mathrm{Cr}\left\{f(\boldsymbol{\xi}) \geq \overline{f}\right\} \geq \alpha$ holds. We randomly generate $\boldsymbol{u}_k$ from the ε-level set of $\boldsymbol{\xi}$ for $k = 1, 2, \ldots, N$. For any number r, we set

$$L(r) = \frac{1}{2}\left(\max_{1\le k\le N}\{\mu(\boldsymbol{u}_k)|f(\boldsymbol{u}_k) \ge r\} + \min_{1\le k\le N}\{1-\mu(\boldsymbol{u}_k)|f(\boldsymbol{u}_k) < r\}\right).$$

It follows from monotonicity that we may employ bisection search to find the maximal value r such that $L(r) \ge \alpha$. This value is an estimation of $\overline{f}$.

Case III Now we compute the expected value $E[f(\boldsymbol{\xi})]$. We randomly generate $\boldsymbol{u}_k$ from the ε-level set of $\boldsymbol{\xi}$ for $k = 1, 2, \ldots, N$. Then for any number $r \ge 0$, the credibility values $\mathrm{Cr}\{f(\boldsymbol{\xi}) \ge r\}$ and $\mathrm{Cr}\{f(\boldsymbol{\xi}) \le r\}$ may be estimated by using the samples. After that, we may employ simulation to calculate the integral,

$$E[f(\boldsymbol{\xi})] = \int_0^{+\infty} \mathrm{Cr}\{f(\boldsymbol{\xi}) \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{f(\boldsymbol{\xi}) \le r\}\mathrm{d}r. \tag{32}$$

15 Fuzzy random theory

Sometimes, fuzziness and randomness simultaneously appear in a system. Fuzzy random theory is a branch of mathematics that studies the behavior of fuzzy random phenomena. Roughly speaking, a fuzzy random variable is a measurable function from a probability space to the set of fuzzy variables. In other words, a fuzzy random variable is a random variable taking fuzzy values. Kwakernaak (1978, 1979) first introduced the notion of fuzzy random variable. This concept was then developed by several researchers such as Puri and Ralescu (1986), Kruse and Meyer (1987), and Liu and Liu (2003a) according to different requirements of measurability. For our purpose, we use the following mathematical definition of fuzzy random variable.

Definition 16 (*Liu and Liu, 2003a*) A fuzzy random variable is a function ξ from a probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$ to the set of fuzzy variables such that $\mathrm{Cr}\{\xi(\omega) \in B\}$ is a measurable function of ω for any Borel set B of $\Re$.

Now let us consider the chance of fuzzy random event. Recall that the probability of random event and the credibility of fuzzy event are each defined as a real number. However, the chance a fuzzy random event is defined as a function rather than a number.

Definition 17 (*Liu, 2001a, b; Gao & Liu, 2001*) Let ξ be a fuzzy random variable, and B a Borel set of $\Re$. Then the chance of fuzzy random event $\xi \in B$ is a function from $(0, 1]$ to $[0, 1]$, defined as

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \sup_{\mathrm{Pr}\{A\}\ge\alpha} \inf_{\omega\in A} \mathrm{Cr}\{\xi(\omega) \in B\}. \tag{33}$$

It is easy to prove that (a) $\mathrm{Ch}\{\xi \in \emptyset\}(\alpha) = 0$; (b) $\mathrm{Ch}\{\xi \in \Re\}(\alpha) = 1$; (c) $0 \le \mathrm{Ch}\{\xi \in B\}(\alpha) \le 1$ for each Borel set B; (d) $\mathrm{Ch}\{\xi \in B\}(\alpha)$ is increasing with respect to B for each α; and (e) $\mathrm{Ch}\{\xi \in B\}(\alpha)$ is decreasing with respect to α for each Borel set B.

Liu and Liu (2003a) defined the expected value and variance of fuzzy random variables.

Liu and Gao (2005b) presented several new convergence concepts for sequences of fuzzy random variables, established their convergence criteria and discussed their convergence relation.

Yang and Liu (2005a, b) proved some useful inequalities and conceptualized critical values of fuzzy random variables.

Feng and Liu (2006) discussed measurability criteria for fuzzy random vectors, and showed that measurability criteria for upper semicontinuous fuzzy random vectors can be expressed in several different but equivalent formulations.

16 Random fuzzy theory

Random fuzzy theory is a branch of mathematics that studies the behavior of random fuzzy phenomena. Random fuzzy variable was defined by Liu (2002b) as a function from a credibility space to the set of random variables. In other words, a random fuzzy variable is a fuzzy variable taking "random variable" values. For example, in many statistics problems, the probability distribution is completely known except for the values of one or more parameters. It might be known that the lifetime ξ of a modern engine is an exponentially distributed variable with an unknown expected value β. If the value of β is provided as a fuzzy variable, then ξ is a random fuzzy variable. Generally, we have the following definition of a random fuzzy variable.

Definition 18 (*Liu, 2002b*) A random fuzzy variable is a function from the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to the set of random variables.

Definition 19 (*Liu, 2002b*) Let ξ be a random fuzzy variable, and B a Borel set of real numbers. Then the chance of random fuzzy event $\xi \in B$ is a function from $[0, 1]$ to $[0, 1]$, defined as

$$\mathrm{Ch}\left\{\xi \in B\right\}(\alpha) = \sup_{\mathrm{Cr}\{A\}\geq\alpha} \inf_{\theta\in A} \Pr\left\{\xi(\theta) \in B\right\}. \tag{34}$$

Zhu and Liu (2004) proved that, for each α, we have (a) $\mathrm{Ch}\{\xi \in \emptyset\}(\alpha) = 0$; (b) $\mathrm{Ch}\{\xi \in \Re\}(\alpha) = 1$; (c) $0 \leq \mathrm{Ch}\{\xi \in B\}(\alpha) \leq 1$ for each Borel set B; (d) $\mathrm{Ch}\{\xi \in B\}(\alpha)$ is an increasing function of B, and (e) $\mathrm{Ch}\{\xi \in B\}(\alpha)$ is a decreasing function of α.

Liu (2001b) conceptualized critical values of random fuzzy variables. Liu and Liu (2003b) presented the expected value and variance. Zhu and Liu (2005) gave some useful inequalities in random fuzzy theory.

17 Chance space and hybrid variable

In order to clearly understand the relation between fuzzy random variable and random fuzzy variable, we introduce the concepts of chance space and hybrid variable.

Suppose that $(\Omega, \mathcal{A}, \Pr)$ is a probability space, and $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ a credibility space. By a *chance space* we mean the product $(\Omega, \mathcal{A}, \Pr) \times (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ of the probability and credibility spaces. More generally, let Λ be a nonempty set, $\mathcal{A}$ a σ-algebra over Λ, Pr a measure on $\mathcal{A}$, $\mathcal{P}(\Lambda)$ the power set over Λ, and Cr a credibility measure on $\mathcal{P}(\Lambda)$. Then $(\Lambda, \mathcal{A}, \Pr, \mathcal{P}(\Lambda), \mathrm{Cr})$ is called a chance space.

Roughly speaking, a *hybrid variable* is a function ξ from a chance space $(\Omega, \mathcal{A}, \Pr) \times (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to the set of real numbers.

If $\xi(\omega, \theta)$ is a measurable function of ω for each $\theta \in \Theta$, then $\xi(\cdot, \theta)$ is random variable for each $\theta \in \Theta$. Thus the hybrid variable is a random fuzzy variable because it is a function from a credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to the set $\{\xi(\cdot, \theta) : \theta \in \Theta\}$ of random variables.

If $\mathrm{Cr}\{\theta \in \Theta | \xi(\omega, \theta) \in B\}$ is a measurable function of ω for any set B of real numbers, then the hybrid variable is a fuzzy random variable because it is a measurable function from a probability space $(\Omega, \mathcal{A}, \Pr)$ to the set $\{\xi(\omega, \cdot) : \omega \in \Omega\}$ of fuzzy variables.

It follows that both fuzzy random variable and random fuzzy variable are special cases of hybrid variable. In other words, they can be regarded as a hybrid variable on a chance space except that some measurability conditions are needed for each case.

18 Nonclassical credibility theory

Credibility theory is based on the five axioms given in section 2. The first four axioms are all fairly straightforward and easy to accept. The fifth one however causes problems. In fact, Liu (2004) replaced Axiom 5 with a new one, thus producing a new branch of mathematics called *nonclassical credibility theory*. The new fifth axiom is shown below:

5′. *Let* Θ_k *be nonempty sets on which* $\tilde{\mathrm{Cr}}_k$ *satisfy the first four axioms,* $k = 1, 2, \ldots, n$ *and let* $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$*. Then*

$$\tilde{\mathrm{Cr}}\{(\theta_1, \theta_2, \ldots, \theta_n)\} = \begin{cases} \dfrac{1}{2} \displaystyle\prod_{i=1}^{n} \left((2\tilde{\mathrm{Cr}}_i\{\theta_i\}) \wedge 1 \right), & \text{if } \displaystyle\min_{1 \le k \le n} \tilde{\mathrm{Cr}}_k\{\theta_k\} < 0.5 \\ \displaystyle\min_{1 \le k \le n} \tilde{\mathrm{Cr}}_k\{\theta_k\}, & \text{if } \displaystyle\min_{1 \le k \le n} \tilde{\mathrm{Cr}}_k\{\theta_k\} \ge 0.5 \end{cases}$$

for each $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$*. In that case we write* $\tilde{\mathrm{Cr}} = \tilde{\mathrm{Cr}}_1 \times \tilde{\mathrm{Cr}}_2 \times \cdots \times \tilde{\mathrm{Cr}}_n$

Note that in this section we use $\tilde{\mathrm{Cr}}$ to represent the credibility measure in order to differentiate the new credibility measure from the old one.

Theorem 10 (Nonclassical Product Credibility Theorem) *Suppose that* Θ_k *are nonempty sets,* $\tilde{\mathrm{Cr}}_k$ *the credibility measures on* $\mathcal{P}(\Theta_k)$*,* $k = 1, 2, \ldots, n$*, respectively.*

Let $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. *Then* $\tilde{\mathrm{Cr}} = \tilde{\mathrm{Cr}}_1 \times \tilde{\mathrm{Cr}}_2 \times \cdots \times \tilde{\mathrm{Cr}}_n$ *defined by Axiom 5′ has a unique extension to a credibility measure on* $\mathcal{P}(\Theta)$.

The triplet $(\Theta, \mathcal{P}(\Theta), \tilde{\mathrm{Cr}})$ is called the nonclassical product credibility space.

Definition 20 (*Liu, 2004, Nonclassical Independence*) The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\left(2\tilde{\mathrm{Cr}}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\}\right) \wedge 1 = \prod_{i=1}^{m}\left(\left(2\tilde{\mathrm{Cr}}\{\xi_i \in B_i\}\right) \wedge 1\right) \tag{35}$$

for any sets $B_1, B_2, \ldots, B_m$ of real numbers.

Based on the new fifth axiom, we obtain a new fuzzy arithmetic that is different from the extension principle of Zadeh.

Theorem 11 (Liu, 2004, Nonclassical Extension Principle) *Suppose that* $\xi_1, \xi_2, \ldots, \xi_n$ *are independent fuzzy variables with membership functions* $\mu_1, \mu_2, \ldots, \mu_n$*, respectively, and* $f : \Re^n \to \Re$ *a function. Then the membership function* μ *of* $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ *is derived from the membership functions* $\mu_1, \mu_2, \ldots, \mu_n$ *by*

$$\mu(x) = \sup_{x=f(x_1,x_2,\ldots,x_n)} \prod_{1\le i\le n} \mu_i(x_i), \quad x \in \Re. \tag{36}$$

Theorem 12 (Nonclassical Credibility Inversion Theorem) *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be independent fuzzy variables with membership functions* $\mu_1, \mu_2, \ldots, \mu_n$*, respectively, and* $f : \Re^n \to \Re^m$ *a function. Then for any set* B *of* $\Re^m$*, the credibility* $\tilde{\mathrm{Cr}}\{f(\xi_1, \xi_2, \ldots, x_n) \in B\}$ *is*

$$\frac{1}{2}\left(\sup_{f(x_1,x_2,\ldots,x_n)\in B} \prod_{1\le i\le n} \mu_i(x_i) + 1 - \sup_{f(x_1,x_2,\ldots,x_n)\in B^c} \prod_{1\le i\le n} \mu_i(x_i)\right).$$

We now consider the credibility of an event A after it has been learned that some other event B has occurred. This new credibility of A is called the conditional credibility of the event A given B.

Definition 21 (*Liu, 2004*) Let $(\Theta, \mathcal{P}(\Theta), \tilde{\mathrm{Cr}})$ be a credibility space, and $A, B \in \mathcal{P}(\Theta)$. Then the conditional credibility of A given B is

$$\tilde{\mathrm{Cr}}\{A|B\} = \frac{1}{2}\left(\frac{\left(2\tilde{\mathrm{Cr}}\{A \cap B\}\right) \wedge 1}{\left(2\tilde{\mathrm{Cr}}\{B\}\right) \wedge 1} + 1 - \frac{\left(2\tilde{\mathrm{Cr}}\{A^c \cap B\}\right) \wedge 1}{\left(2\tilde{\mathrm{Cr}}\{B\}\right) \wedge 1}\right) \tag{37}$$

provided that $\tilde{\mathrm{Cr}}\{B\} > 0$.

When A and B are independent events, and B has occurred, it is reasonable that the the credibility of the event A remains unchanged. The following formula shows the fact,

$$\tilde{\mathrm{Cr}}\{A|B\} = \frac{1}{2}\left((2\tilde{\mathrm{Cr}}\{A\}) \wedge 1 + 1 - (2\tilde{\mathrm{Cr}}\{A^c\}) \wedge 1\right) = \tilde{\mathrm{Cr}}\{A\}.$$

Let (ξ, η) be a fuzzy vector with a joint membership function μ. If $\sup_r \mu(r, y) \neq 0$ for some y, it follows from the definition of conditional credibility that the conditional membership function of ξ given $\eta = y$ is

$$\nu(x|\eta = y) = \frac{\mu(x, y)}{\sup\limits_r \mu(r, y)}. \tag{38}$$

Based on the conditional credibility, Liu (2004) defined conditional credibility distribution, conditional credibility density function, conditional expected value, etc.

19 Applications

Based on credibility theory, Liu (2002b) suggested chance-constrained programming and dependent-chance programming, and Liu and Liu (2002) provided a fuzzy expected value model. In addition, Liu (2005) also applied credibility measure to fuzzy programming with recourse.

Liu (2001a, b) and Liu and Liu (2002b, 2003c, 2005a, b) suggested some fuzzy random programming models, and Liu (2002b), Liu (2002c) and Liu and Liu (2002a, 2003b) proposed some random fuzzy programming models. Zhou and Liu (2003, 2004) employed credibility measures to construct bifuzzy programming models.

In order to model generally uncertain decentralized decision systems, Liu (2002b) proposed a general framework of uncertain multilevel programming and introduced the concept of Stackelberg-Nash equilibrium to uncertain multilevel programming. After that, Gao and Liu (2005) investigated fuzzy multilevel programming, and designed a series of hybrid intelligent algorithms for finding the Stackelberg-Nash equilibrium. Xu, Zhao, and Shu (2005a) discussed equilibrium strategies for two-person zero-sum game with fuzzy payoffs.

Liu (2002b) suggested some fuzzy dynamic programming models based on credibility measures and designed a hybrid intelligent algorithm for solving general models.

Credibility theory was also applied to fuzzy scheduling problems. Peng and Liu (2004a) discussed parallel machine scheduling problems with fuzzy processing times. Li and Liu (2003) discussed critical path problems with fuzzy activities. Ke and Liu (2004, 2005) constructed some optimization models for fuzzy project scheduling problems. Ji, Iwamura, and Shao (2006) considered the shortest path problem with fuzzy arc lengths.

Yang and Liu (2005b) presented dependent-chance goal programming for assignment problems in which the elements of the profit matrix and the task time matrix are fuzzy variables. Feng and Yang (2006) investigated an assignment problem and constructed a chance-constrained goal programming model by using credibility measures. Liu (2006b) gave some new models for the fuzzy quadratic assignment problem with penalties.

Credibility theory has also been applied to engineering design. Yang and Sun (2004) presented an expected value model for a fuzzy random warehouse layout problem. Chen, Fung, and Yang (2005) employed a fuzzy expected value model for determining target values of engineering characteristics. Gao and Lu (2005) discussed the fuzzy quadratic minimum spanning tree problem. Gao (2005) constructed a fuzzy multi-criteria model for the minimum spanning tree problem. Yang and Wen (2005) presented a chance-constrained programming model for transmission system expansion planning. Liu (2006a) discussed maximum fuzzy weighted matching models.

Lu and Gao (2001) proposed fuzzy expected value integer programming models for capital budgeting problems. Gao, Zhao, and Ji (2005) constructed a fuzzy chance-constrained programming model for capital budgeting problems with fuzzy decisions. Huang (2006a) proposed a credibility-based chance-programming model for capital budgeting.

Peng, Mok, and Tse (2005b) treated portfolio selection problems in fuzzy environments by fuzzy programming. Huang (2006b) presented a chance-constrained programming model for portfolio selection in which the returns are assumed to be fuzzy variables.

Fung, Chen, and Chen (2005) provided a fuzzy expected value goal programming model for production planning. Yan, Zhao, and Cao (2005) suggested some fuzzy programming models for lot-sizing production planning problem. Shao and Ji (2006) modelled a multi-product newsboy problem with fuzzy demands subject to the budget constraint. Ji and Shao (2006) applied multilevel programming to a newsboy problem with fuzzy demand and price discount policy.

In order to model uncertain vehicle routing problem, Zheng and Liu (2004) gave a fuzzy optimization model based on credibility measures. He and Xu (2005) presented a random fuzzy programming model for vehicle routing problem with random fuzzy demands.

Zhou (2002) constructed a fuzzy programming model for a minimax facility location problem. Wen and Iwamura (2004) discussed a fuzzy facility location–allocation problem under the Hurwicz criterion. Zhou and Liu (2006) modeled a capacitated location–allocation problem with fuzzy demands.

Zhao and Liu (2005) constructed some optimization models for system reliability design with fuzzy lifetimes. Furthermore, Zhao and Liu (2003) modeled a redundancy optimization problem in which uncertainty involves a combination of randomness and fuzziness.

Zhao and Liu (2003) considered a renewal process in which the interarrival times and costs are characterized as fuzzy variables. Zhao, Tang, and Yun (2004) extended this work to a more general case. Zhao, Tang, and Li (2005) and Zhao and Tang (2006) addressed some properties of fuzzy random renewal processes generated by a sequence of iid fuzzy random interarrival times. In addition, Zhao, Tang, and Yun (2006) and Xu, Zhang, and Zhao (2005b) discussed a random fuzzy renewal process.

Wu, Tang, and Zhao (2004, 2005) studied web mining of preferred traversal patterns in a fuzzy environment by using credibility theory.

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