Macroeconomic Dynamics, **6**, 2002, 19–39. Printed in the United States of America.
DOI: 10.1017.S1365100501010197

SURVEY OF LINEAR QUADRATIC ROBUST CONTROL

Pierre Bernhard
University of Nice-Sophia Antipolis
and
CNRS

We review several control problems, all related to robust control in some way, that lead to a minimax linear quadratic problem. We stress the fact that although an augmented performance index appears, containing an L^2 norm of a disturbance signal, only the *nonaugmented* quadratic performance index is of interest *per se* in each case.

Keyword: Robust Control

1. INTRODUCTION

Robust control has been one of the most active areas of control research for the past 20 years or so. By the phrase "robust control," one means those control mechanisms that explicitly take into account the fact that the system model (or the noise model) is imprecise. Of course, one has to know something about this model. A distinctive feature of most robust control theory is that the *a priori* information on the unknown model errors (or signals) is nonprobabilistic in nature, but rather is in terms of *sets of possible realizations*.[1] Typically, though not always, the errors are bounded in some way; however, we shall show a strange parallel with a stochastic problem formulation in the so-called *risk-averse* control problem. As a consequence, robust control aims at synthesizing control mechanisms that control in a satisfactory fashion (e.g., stabilize, or bound, an output) a *family* of models.

If so-called $\mathcal{H}_\infty$-optimal control has been the subject of the largest share of that research, it is by no means the only approach to robust control. Most prominent among other approaches are those "à la Kharitonov." Kharitonov's theorem states a sufficient condition for a particular family of polynomials to have all their roots in the left half complex plane. Applied to the characteristic polynomial of a family of linear time-invariant (LTI) systems, it yields interesting robust control results. The so-called "edge theorem" is an attempt at a similar result for more interesting families of polynomials. It is the starting point of some robust control results. We shall not review that line of thought here; for a nice review, see Barmish (1998).

Address correspondence to: Professor Pierre Bernhard, Laboratoire I3S, Bâtement Euclide, 2000 route des Lucioles, Sophia Antipolis, F-06410 BIOT, France; e-mail: Pierre.Bernhard@i3s.unice.fr.

 1365-1005/02 $9.50

$\mathcal{H}_\infty$-optimal control started with the work of G. Zames (1981), although papers such as that of Doyle and Stein (1981) can be seen as forerunners of the new theory. It was developed in the context of LTI systems and used a frequency-domain representation, where it won its name. A good review of that approach is given by Francis (1987). It was with the important paper of Doyle et al. (1989), first given at the 1988 Conference on Decision and Control (CDC), that a link with state space was first established, the role of a game-like Riccati equation shown, and an observer-like form, reminiscent of the linear quadratic Gaussian (LQG) theory, exhibited. The link with games was elucidated the next year independently by Başar (1989), Papavassilopoulos and Safonov (1989), and Tadmor (1990). At that time, the available game theory did not allow one to explain the full results of Doyle et al. (1989) and its observer-like structure. This was to be explained by the minimax certainty equivalence theorem of Bernhard (1990, 1991) and Bernhard and Rapaport (1995) and exploited to its full strength by Başar and Bernhard (1991).

This allowed us to develop a theory entirely in state space, indeed in the realm of the classical linear quadratic theory, with the same type of tools. It also let us extend the theory to non-time-invariant systems and finite-horizon criteria, and successfully deal with such features as sampled data control, time lags, and $\mathcal{H}_\infty$-optimal estimation.

Other important work was conducted in parallel on the time-domain approach—see, for instance, Limebeer et al. (1989), Kwakernaak (1991), and Stoorvogel (1992)—most of them, however, restricted to LTI systems.

Consideration of L^2 norms in a linear system with state x and control u naturally leads to the consideration of a quadratic performance index, specifically, the integral of a quadratic form in $x(t)$ and $u(t)$, which we call J. In some sense, we wish to keep it small in spite of unknown factors, disturbances, and/or model errors. For technical reasons, we consider an augmented performance index $J_\gamma = J - \gamma^2 \|w(\cdot)\|^2$ and its minimax value. Our presentation is intended to stress the fact that this is for purely technical reasons, and that the real problems at hand are only concerned with J and keeping it small.

In Section 2, we first show a simple noise attenuation problem, which shows up as an alternative to the Gaussian noise model to deal with the control of a disturbed linear system. There is no uncertain model in that problem and no set of models *per se*, but it shares a feature of robust control inasmuch as the noise description is in terms of an admissible set of time functions: L^2, the set of finite "energy" signals (the set of finite "power" signals leads to the same theory).

In Section 3, we consider a slightly generalized version of Zames' original robust control formulation. Here lies the claim of $\mathcal{H}_\infty$-optimal control to solve robust control problems. The second edition of Başar and Bernhard (1995) serves as the basis of Sections 2 and 3 of the current review.

In Section 4, we show the link with risk-averse control. As a matter of fact, Whittle's separation theorem largely predates the certainty equivalence theorem of Bernhard (1990). It is a version of the latter restricted to the discrete-time,

time-invariant, linear quadratic case with a simple observation equation. We introduce the relationship between $\mathcal{H}_\infty$-optimal control and risk-averse control via what we nickname “Whittle’s magic formula.” This in a way makes that relationship look at best accidental, at worst magic. Some deeper reason is probably at work, but not completely clear to us at this time.

A final section gives a couple of elementary examples to show how the mathematics deals with model uncertainty. We emphasize there that *robust* is not synonymous with *cautious*. Robust control may be either more or less cautious than classical LQG, depending on the structure of the model uncertainties.

To keep the exposition as short and simple as possible, we restrict it to continuous-time models. There exist discrete-time parallels to (essentially) everything we discuss. In some instances the mathematics is even simpler, although the formulas are always more complicated.

2. ROBUST NOISE ATTENUATION

2.1. System and Notations

Consider a linear system with state $x \in \mathbf{R}^n$; two inputs, the control $u \in \mathbf{R}^m$ and a disturbance signal $w \in \mathbf{R}^\ell$; and two outputs, a measurement output $y \in \mathbf{R}^p$ and a controlled output $z \in \mathbf{R}^q$:

$$\dot{x} = Ax + Bu + Dw, \tag{1}$$

$$y = Cx + Ew, \tag{2}$$

$$z = Hx + Gu. \tag{3}$$

The matrices A, B, D, C, E, H, and G are of appropriate sizes. In infinite-horizon problems, they will be assumed constant. In finite-horizon problems, they may be time varying—say, piecewise continuous, right-continuous, and left-limited—but we do not consider that case. Notice that the following *system matrix* can be built from them:

$$\mathcal{S} = \begin{pmatrix} A & B & D \\ C & 0 & E \\ H & G & 0 \end{pmatrix}. \tag{4}$$

The 0 matrix in the definition of y is of no consequence. As a matter of fact, y will be the measured output, i.e., the information available to the controller to choose u. If there were a term $+Ju$ in it, we, knowing $u(t)$, could instantly subtract out that term and recover our y. Therefore, there is no loss of generality here. This is not so for the 0 matrix in z. We shall keep it because the theory is simpler that way, but Başar has extended the theory to the case in which this extra term is present [see Başar and Bernhard (1991)]. See also Bernhard (2000) for a discussion of that question in the framework of minimax control.

Also to keep things simple, we shall always make the following assumptions:

Assumption A.

(1) The matrix G is injective, and has its mth (smallest) singular value bounded away from 0 (hence $m < q$):
(2) The matrix E is surjective, and has its pth (smallest) singular value bounded away from zero (hence $p < \ell$).

We also use the following definitions:

$$\begin{pmatrix} H^tH & H^tG \\ G^tH & G^tG \end{pmatrix} =: \begin{pmatrix} Q & S \\ S^t & R \end{pmatrix}, \tag{5}$$

so that hypothesis A(1) translates into $R > 0$ and R^{-1} bounded, and

$$\begin{pmatrix} DD^t & DE^t \\ ED^t & EE^t \end{pmatrix} =: \begin{pmatrix} M & L^t \\ L & N \end{pmatrix}, \tag{6}$$

so that hypothesis A(2) translates into $N > 0$ and N^{-1} bounded.

In many applications, one has $S = 0$ (no cross terms in xu in J below) and $L = 0$ (the dynamics noise Dw and measurement noise Ew are unrelated). This simplifies somewhat the various equations below, but not by much.

We write $\|u\|_R^2 = (u, Ru) = u^tRu$, $\|w\|_N^2 = (w, Nw) = w^tNw$ and likewise for other quadratic forms, even when the weighting matrix is not positive definite.

We consider infinite-horizon problems where $t \in (-\infty, +\infty)$, where implicitly what is meant is that the state at time $-\infty$ was zero:

$$x(t) = \exp(At)\int_{-\infty}^{t} \exp(-As)[Bu(s) + Dw(s)]\,ds.$$

(This is well defined provided that the system is stable, or adequately stabilized by feedback.) Therefore, $x(\cdot)$ is then a function of $u(\cdot)$ and $w(\cdot)$ alone, and there is no consideration of the final state either. We shall always require that the system be stabilized: $x(t) \to 0$ as $t \to \infty$. We then have $z(\cdot) \in L^2(-\infty, +\infty)$ and use the notation

$$J(u(\cdot), w(\cdot)) = \|z(\cdot)\|^2 = \int_{-\infty}^{\infty} \left[\|x(t)\|_Q^2 + 2x(t)^tSu(t) + \|u(t)\|_R^2\right] dt, \tag{7}$$

and for any positive number γ,

$$\begin{aligned} J_\gamma &= J - \gamma^2\|w(\cdot)\|^2 \\ &= \int_{-\infty}^{\infty} \left[\|x(t)\|_Q^2 + 2x(t)^tSu(t) + \|u(t)\|_R^2 - \gamma^2\|w(t)\|^2\right] dt. \end{aligned} \tag{8}$$

For the finite-horizon case, where $t \in [0, T]$, we use the following notation:

$$\zeta = \begin{pmatrix} z(\cdot) \\ x(T) \end{pmatrix} \in L^2([0, T] \to \mathbf{R}^n) \times \mathbf{R}^n_X, \tag{9}$$

for a given nonnegative definite matrix X, so that

$$\|\zeta\|^2 = \|x(T)\|^2_X + \int_0^T \left[\|x(t)\|^2_Q + 2x(t)^t Su(t) + \|u(t)\|^2_R\right] dt, \tag{10}$$

and

$$\omega = \begin{pmatrix} x_0 \\ \omega(\cdot) \end{pmatrix} \in \mathbf{R}^n_Y \times L^2([0, T] \to \mathbf{R}^\ell) =: \Omega, \tag{11}$$

where $x_0 = x(0)$ and Y is a given positive definite matrix, so that

$$\|\omega\|^2 = \int_0^T \|w(t)\|^2 \, dt + \|x_0\|^2_Y. \tag{12}$$

And we write

$$J(x_0, u(\cdot), w(\cdot)) = J(u(\cdot), \omega) = \|\zeta\|^2, \tag{13}$$

and

$$J_\gamma = J - \gamma^2 \|\omega\|^2; \tag{14}$$

hence,

$$J_\gamma = \|x(T)\|^2_X + \int_{-\infty}^{\infty} \left[\|x(t)\|^2_Q + 2x(t)^t Su(t) + \|u(t)\|^2_R - \gamma^2 \|w(t)\|^2\right] dt - \gamma^2 \|x_0\|^2_Y.$$

2.2. The Problem

In an imprecise statement, the aim is to "choose $u(t)$, knowing only the past $y(s)$, $s < t$," in such a way as to "keep $z(\cdot)$ small in spite of the unpredictable disturbances." All we shall assume concerning these disturbances is that the time function $w(\cdot)$ is square integrable over the time interval considered, either finite or infinite.

The aim of the mathematical models is to propose mathematical metaphors of that problem, more or less well suited to various experimental or logical contexts. One very famous metaphor has been to construct a probabilistic model for the disturbances, and accordingly for the state trajectory, with the necessary apparatus to account for the causality of the admissible control laws. One then strives to minimize the *expected value* of J. This leads to the famous LQG theory.

This is known to be a very useful piece of theory, and a very brilliant one, but the point here is that it is only *one* possible way of making a mathematical metaphor of the basic problem. It is well suited if, on the one hand, one has reason to believe that the disturbances qualitatively resemble a random walk, and on the other hand, the average value of J over several experiments is of interest. However, if we assume that it is known, for instance, that the disturbance is a constant over time, there is no way by which this can resemble a random walk, or be represented as the output of a linear system driven by such a process, because it is not ergodic (the time average differs from the ensemble average). This is one situation, and others may arise, in which the foregoing approach might be better suited.

We stick with the decision that "keeping $z(\cdot)$ small" will be judged by looking at the L^2 norm of that output function, either of $z(\cdot)$ in the infinite-horizon case, or, to be slightly more general, ζ in the finite-horizon case. Thus our aim is, as previously, to keep J as given by (7) or (13) small.

Admissible control laws will be *causal* functions of the measured output, that is, of the form[2]

$$u(t) = \mu(t, y(s); s < t), \tag{15}$$

and such that, when substituted for u in (1) and (2), it yields for all $\omega \in \Omega$ a unique solution $x(\cdot)$. Let $\mathcal{M}$ be the set of all such admissible controls.

We assume for the present that we are restricted to *linear* control laws μ. Then, once μ is substituted into the dynamics, ζ becomes a linear function of ω alone, say $\zeta = T_\mu \omega$. Hence, there is no way to avoid $J = \|\zeta\|^2$ growing as $\|\omega\|^2$. A reasonable mathematical problem, which indeed is a valid metaphor of the original problem, is to try to keep the ratio $J/\|\omega\|^2$ as small as possible. Equivalently, since the norm of a linear operator T_μ is defined as the smallest number $\|T_\mu\|$ such that

$$\forall\, \omega \in \Omega, \quad \|\zeta\| \le \|T_\mu\| \|\omega\|,$$

the problem at hand is to *find an admissible control law μ that makes $\|T_\mu\|$ small.*

Thus, it would be nice to be able to solve the problem $\min_{\mu\in\mathcal{M}} \|T_\mu\|$. Unfortunately, this problem is not well behaved and does not admit a simple solution. In particular, the discussion of whether the *min* is reached is difficult.

It turns out that it is useful to rephrase that problem in the following way:

Problem P_γ. Given a positive number γ, does there exist an admissible control law μ that will ensure that $\|T_\mu\| \le \gamma$ or, equivalently, (16) below, and, if yes, find one.

Equivalently, this reads (remember that here $\zeta = T_\mu \omega$)

$$\forall\, \omega \in \Omega, \quad \|\zeta\| \le \gamma \|\omega\|. \tag{16}$$

Now, the above property is equivalent to

$$\forall\, \omega \in \Omega, \quad \|\zeta\|^2 - \gamma^2 \|\omega\|^2 = J_\gamma(\mu, \omega) \le 0,$$

and thus also to

$$\sup_{\omega\in\Omega} J_\gamma(\mu, \omega) \le 0.$$

Finally, existence of an admissible control law that achieves (16) is equivalent to (if the *min* exists)

$$\min_{\mu\in\mathcal{M}}\sup_{\omega\in\Omega} J_\gamma(\mu,\omega) \le 0 \tag{17}$$

(and this does not depend on μ being linear).

Hence, we end up solving a differential game, or minimax control problem, for the cost function J_γ, but only because checking whether (17) holds is a means of answering the question posed by Problem $\mathcal{P}_\gamma$ or, equivalently, to attempt to ensure (16), and because if the answer is yes, then the minimizing μ in (17) solves it.

2.3. Solution

For the sake of completeness, we recall here the solution of that game problem.

2.3.1. Finite horizon. Let us first consider the finite-horizon case. The solution of the problem involves two matrix Riccati equations, for symmetric matrices $P(t)$ and $\Sigma(t)$:

$$\dot P + PA + A^tP - (PB+S)R^{-1}(B^tP+S^t) + \gamma^{-2}PMP + Q = 0, \qquad P(T)=X, \tag{18}$$

and

$$\dot\Sigma = A\Sigma + \Sigma A^t - (\Sigma C^t + L^t)N^{-1}(C\Sigma + L) + \gamma^{-2}\Sigma Q\Sigma + M, \qquad \Sigma(0) = Z, \tag{19}$$

where we have set $Z := Y^{-1}$.

The main theorem of $\mathcal{H}_\infty$-optimal control is as follows. (For any square matrix K, $\rho(K)$ stands for *spectral radius* of K.)

THEOREM 1. *If equations* (18) *and* (19) *have solutions* $P(\cdot)$ *and* $\Sigma(\cdot)$ *over* $[0, T]$, *and if furthermore these solutions satisfy the inequality*

$$\forall t\in[0,T] \qquad \rho(\Sigma(t)P(t)) < \gamma^2,$$

then the answer to Problem $\mathcal{P}_\gamma$ *is positive, and a control law that achieves the desired disturbance attenuation level* γ *is given by equations* (20) *and* (21) *hereafter.*

Conversely, if one of the above conditions fails, then for any $\tilde\gamma < \gamma$ *the problem* $\mathcal{P}_{\tilde\gamma}$ *has no solution.*

The proposed control law is obtained as a "certainty equivalent" feedback on a "worst possible state" $\hat x(t)$:

$$u(t) = -R^{-1}(B^tP(t) + S^t(t))\hat x(t), \tag{20}$$

where $\hat x$ is the solution of the following differential equation:

$$\dot{\hat{x}} = (A - BR^{-1}(B^tP + S^t) + \gamma^{-2}MP)\hat{x} + (I - \gamma^{-2}\Sigma P)^{-1}(\Sigma C^t + L^t)N^{-1} \times [y - (C + \gamma^{-2}LP)\hat{x}], \qquad \hat{x}(0) = 0. \tag{21}$$

Notice the similarity with the optimal LQG control. Indeed, the feedback law (20) has exactly the same form (though with a different $\hat{x}$, of course), and equation (21) has the same structure as a Kalman filter. The differences with the latter case are in the presence of a "worst" disturbance $w = \gamma^{-2}D^tP\hat{x}$ in the dynamics and in the corrective term (it disappears from the correcting term if $L = 0$), and in the fact that the gain matrix of the corrective term is premultiplied by the coefficient $(I - \gamma^{-2}\Sigma P)^{-1}$. Notice that the spectral radius condition of the theorem precisely guarantees the required invertibility.

As a matter of fact, one way of showing this theorem is through a certainty equivalence theorem that states that under some conditions satisfied here, a minimax control in imperfect information is obtained by substituting in the minimax state feedback (i.e., the optimal control law in the case of perfect state information) a "worst current state compatible with the past measurements," which (21) provides.

Theorem 2 states a worthwhile fact (and *not* a corollary of the above).

THEOREM 2. *If the available measurement is $x(t)$ (exact state measurement), then Theorem 1 holds without the condition on equation (19) (existence of Σ), with the spectral radius condition restricted to initial time: $\rho(ZP(0)) < \gamma^2$, and with $x(t)$ instead of $\hat{x}(t)$ in (20). [Hence equation (21) is not required either.]*

2.3.2. Infinite horizon. The stationary theory, which predated the finite-horizon one, can be obtained as a limiting case of the above, with some care though. We assume now that S is constant and we need an extra set of assumptions.

Assumption B.

(1) The pair (A, D) is stabilizable.[3]
(2) The pair (H, A) is reconstructible.[4]

Then, we have the following theorem.

THEOREM 3. *Under Assumptions A and B, if the following three conditions hold,*

(i) *The Riccati equation (18) integrated from $P(0) = 0$ has a solution that converges to some P^* as $t \to -\infty$,*
(ii) *The Riccati equation (19) integrated from $\Sigma(0) = 0$ has a solution that converges to some Σ^* as $t \to \infty$, and*
(iii) $\rho(\Sigma^* P^*) < \gamma^2$.

then the answer to the Problem $\mathcal{P}_\gamma$ is positive, and an admissible controller is given by equations (20) and (21) with $P(t)$ and $\Sigma(t)$ replaced by P^* and Σ^*, respectively. In that case, P^* and Σ^* are the least positive definite solutions *of the* algebraic Riccati equations *obtained by placing $\dot{P} = 0$ and $\dot{\Sigma} = 0$ in (18) and (19), respectively.*

Conversely, if one of the above three conditions fails, for any $\tilde{\gamma} < \gamma$, Problem $\mathcal{P}_{\tilde{\gamma}}$ has no solution.

Furthermore, if, in addition to Assumption B, (A, B) is stabilizable and (C, A) is reconstructible, then there will always be a positive γ^* such that the conditions of the theorem are satisfied for $\gamma > \gamma^*$ and violated for $\gamma < \gamma^*$.

A careful analysis of the problem shows that, usually, as γ is decreased from values larger than γ^*, the first condition of the theorem to be violated will be the third one. What happens for $\gamma = \gamma^*$ is more complicated (a reduced-order controller may exist), but is of little practical importance.

3. ROBUST STABILIZATION AND CONTROL

3.1. Model Uncertainty

We now turn to the original problem that brought $\mathcal{H}_\infty$-optimal control to life, a real robust control problem in that it deals with model uncertainty. To justify the description of plant uncertainty that we use, we begin with an example.

Let a linear system be of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned}$$

Assume that the matrices A, B, and C are not exactly known. All we know are approximate values A_0, B_0, and C_0 and bounds on how bad these approximations may be, in terms of norms of matrices: three positive numbers α, β, and γ are given, together with the information

$$\|A - A_0\| \le \alpha, \qquad \|B - B_0\| \le \beta, \qquad \|C - C_0\| \le \gamma.$$

We are concerned with the problem of stabilizing and controling that system, hence the family of all models thus described.

We rewrite the system's equations as

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u + [I \quad 0]w, \\ y &= C_0 x + [0 \quad I]w, \\ z &= \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u \end{aligned} \tag{22}$$

with the added relation

$$w = \begin{pmatrix} \Delta A & \Delta B \\ \Delta C & 0 \end{pmatrix} z, \tag{23}$$

where $\Delta A := A - A_0$, and likewise for ΔB and ΔC. This is indeed the same system.

System (22), called the *nominal* system, is of the form

$$\begin{pmatrix} y \\ z \end{pmatrix} = \mathcal{G}\begin{pmatrix} u \\ w \end{pmatrix},$$

where $\mathcal{G}$ is entirely known: There is no uncertain coefficient in it. Only an unknown disturbance input w. Furthermore, it is exactly of the form of our system of the preceding section (with $S = 0$ and $L = 0$). All uncertainty has been placed in the feedback term (23). We rewrite that last term as

$$w = \Delta\mathcal{G}z$$

and the available information on the uncertainties translates into (as a matter of fact, is degraded into)

$$\|\Delta\mathcal{G}\| \leq \delta \qquad \textbf{(24)}$$

for some number, δ, function of the given uncertainty bounds.[5]

The above example is meant to substantiate the claim that the following uncertainty description is indeed very general. For the sake of convenience, however, we rename r and s, respectively, as the input w and the output z above, which play a special role in the uncertainty description

We consider a linear system described as a linear operator $\mathcal{G}$ acting on inputs to deliver outputs. One may think of $\mathcal{G}$ as meaning an abstract operator from L^2 spaces into L^2 spaces, or, equivalently, as meaning the transfer function as a concrete representation of a linear operator (this latter interpretation, not the former one, being restricted to an infinite-horizon time-invariant problem). We use three (vector) inputs now: $\boldsymbol{w}$ renamed $\boldsymbol{r}$ and $\boldsymbol{u}$ as above, and an exogenous disturbance $\boldsymbol{v}$. Likewise, we may use three (vector) outputs: $\boldsymbol{z}$ renamed $\boldsymbol{s}$ and $\boldsymbol{y}$ as above, and a to-be-controlled output $\boldsymbol{e}$ (an "error" signal seen as the deviation of an actual output from a desired one, which should be kept small). By linearity, the system may be written as

$$s = \mathcal{G}_{sr}r + \mathcal{G}_{sv}v + \mathcal{G}_{su}u, \qquad \textbf{(25)}$$

$$e = \mathcal{G}_{er}r + \mathcal{G}_{ev}v + \mathcal{G}_{eu}u, \qquad \textbf{(26)}$$

$$y = \mathcal{G}_{yr}r + \mathcal{G}_{yv}v + \mathcal{G}_{yu}u. \qquad \textbf{(27)}$$

The uncertainty in the system resides in the fact that we know that

$$r = \Delta\mathcal{G}s \qquad \textbf{(28)}$$

for some linear operator $\Delta\mathcal{G}$ of which we only know a norm bound δ as in (24).

The output y is the measurement available to choose our control u, whose aim is to stabilize this family of models—this is the topic of the next subsection—and if possible, in doing so to attenuate as much as possible the effect of the exogenous disturbance v in the controlled output e, a topic dealt with in a later subsection.

3.2. Robust Stabilization

Because we want to deal with stability, we restrict our attention here to linear time-invariant systems and infinite-horizon controls.

Assume that we are constrained to linear operators (this restriction may be waived, but we do not consider that issue here) of the form (15), which we rewrite as

$$u = \mu y. \tag{29}$$

to stress the linearity. Then, substituting into (27), one may formally solve for y, and thus u, in terms of v and r, and substitute this form of u in (25), leading to a linear expression of the form

$$s = T_\mu r + S_\mu v. \tag{30}$$

This is indeed exactly the same argument as in Section 2.2. Thus the *controlled* system is now given by (30) (28).

How do we choose μ to ensure stability of this controlled system for any $\Delta\mathcal{G}$ within the norm bound? The fundamental remark is as follows: Under suitable assumptions, which are satisfied by a standard canonical state variable model, the system is stable if and only if the above equations have a solution in L^2 for any v in L^2. (Sufficiency stems from the fact that if all inputs are in L^2 so is $\dot{x}$; thus x is in H^1 and hence $x(t) \to 0$ as $t \to \infty$. Necessity requires some observability.)

Now, we substitute (28) into (30), giving

$$s = T_\mu \Delta\mathcal{G} s + S_\mu v. \tag{31}$$

This is a fixed-point equation for s. By Banach's theorem, a sufficient condition for the existence of a (unique) solution is that $\|T_\mu \Delta\mathcal{G}\| < 1$. Notice that we have

$$\|T_\mu \Delta\mathcal{G}\| \leq \|T_\mu\| \|\Delta\mathcal{G}\| \leq \|T_\mu\|\delta,$$

so that a sufficient condition of stability of all our models is that $\|T_\mu\| < \delta^{-1}$.

The so-called "small-gain theorem" states that if S_μ is onto, this condition is also necessary to ensure existence of a solution to (30) and (28) for all $\Delta\mathcal{G}$ of norm no more than δ.[6]

Hence, the search for a controller μ that stabilizes all the models in the family may, in practice, be replaced by the requirement that $\|T_\mu\| \leq \gamma$ for a well-chosen γ. Of course, the remarkable fact is that this is the problem considered for another reason in the preceding section.

3.3. Robust Stabilizing Control

We now want to simultaneously stabilize and control our family of models. Consider now the combined input

$$w = \begin{pmatrix} r \\ v \end{pmatrix}$$

and the combined output

$$z = \begin{pmatrix} \delta s \\ \beta e \end{pmatrix}$$

for some positive β (and the same δ as above). Assume a control law μ is chosen, and let $\tilde{T}_\mu$ be the ensuing linear operator from w to z. Choose $\gamma < 1$ (but very close to 1). If it is possible to choose μ such that $\|\tilde{T}_\mu\| \leq \gamma$, then this in particular implies that, on the one hand, the operator from r to δs has norm less than 1; hence the operator T_μ from r to s has norm less than δ^{-1}, ensuring robust stability, and on the other hand, that the operator from v to βe also has norm less than 1, ensuring a disturbance rejection ratio of at least β^{-1}. Thus, the larger the β for which this is possible, the better the control law.

Again, we are back to a problem of the form treated to begin with in Section 2.

Notice, however, that up to this point, the control problem addressed by this approach is not completely satisfactory because the system norm that we have strived to control to ensure noise attenuation is that of the *nominal* system. It would be interesting to be able to say something of the operator from v to e in the *perturbed* system, where indeed $r = \delta\mathcal{G}s$. It is a recent and surprising theorem [see Chilali (1996)] that, indeed, in that case we have *also* ensured that the perturbed system admits the same norm bound. Thus, this does provide simultaneous robust stabilization and control.

This is still an elementary stage of the theory, however. Two important extensions have been developed. On the one hand, it is possible to exploit a more refined knowledge on the disturbance system than just a norm bound, typically in terms of a frequency-dependent bound. This is done using shaping filters, very much as is done with the classical LQG theory to deal with serially correlated noise. On the other hand, we have stressed in the example that, by reducing our knowledge about the disturbance to a single operator norm, we degrade our information. Thus, means have been developed to distinguish several channels in both r and s, with a diagonal structure on $\Delta\mathcal{G}$, and separate norm bounds on each block of that structure. This is the aim of "μ-synthesis," after the name of the "structured singular value" often called μ.

4. RISK-AVERSE CONTROL

We now outline a seemingly completely different problem that leads to the consideration of the same minimax problem as in (17).

Consider a linear model as in (1), (2), and (3), but for the time being, and following Whittle (1981), in discrete time:

$$x(t+1) = Ax(t) + Bu(t) + Dw(t), \tag{32}$$

$$y(t) = Cx(t) + Ew(t), \tag{33}$$

$$z(t) = Hx(t) + Gu(t). \tag{34}$$

This will make things simpler on technical grounds, but the theory has since been extended to the continuous-time problem, though with much technicality [Bensoussan and Van Schuppen (1985)]. We also restrict our attention to a finite-horizon problem. Thus, $w(\cdot)$ is now a finite sequence, and thus a finite dimensional variable, which we still write as w when no confusion is possible. Its l^2 norm is exactly the Euclidean norm of the composite vector of dimension $T\ell$ made of all the $w(t)$'s.

As in classical LQG theory, we model the disturbances $w(\cdot)$ as a normalized white noise, that is, a sequence of independent normal Gaussian random variables.

We want to model a risk-averse controller. One way of doing so is to assume that the controller seeks to minimize the expected value of the exponential of the classical quadratic performance index. Because the exponential function is convex, this penalizes upward deviations from the mean more than it saves on downward deviations, making it important to reduce the variance of the quadratic performance index.

More precisely, we take as the performance index

$$G_\gamma(x_0, u) = \mathbf{E}\exp\left(\frac{1}{2\gamma^2} J(x_0, u, w)\right). \tag{35}$$

(For obvious reasons, it is customary to consider more precisely $\tilde{G}_\gamma := 2\gamma^2 \ln G_\gamma$, but it is clearly equivalent to minimize $\tilde{G}_\gamma$ or G_γ.)

Expanding the expectation operator, this leads to

$$G_\gamma(x_0, u) = (2\pi)^{-\frac{T\ell}{2}} \int \exp\left(\frac{1}{2\gamma^2}[J(u, w) - \gamma^2\|w\|^2]\right) dw,$$

The exponent involves the familiar $J_\gamma = J - \gamma^2\|w\|^2$. It is a nonhomogeneous quadratic form in u and w, and can be written as

$$J_\gamma = (u, \mathcal{R}u) + 2(w, \mathcal{S}u) - \gamma^2(w, \mathcal{T}w) + 2(a, u) + 2(b, w) + c$$

with

$$\mathcal{T} = I - \frac{1}{2\gamma^2} J_{ww},$$

and for some linear operators $\mathcal{R}$, $\mathcal{S}$, some time functions a and b, and a number c. We have used a minus sign in front of the quadratic term in w to stress the fact that the expectation is defined (finite) if and only if the operator $\mathcal{T}$ is positive definite. Otherwise, the integral in w diverges.

It is a classical fact that one may "complete the square," i.e., rewrite the above quadratic form in terms of the linear operators $\mathcal{N} = \gamma^{-2}\mathcal{T}^{-1}\mathcal{S}$ and $v = \gamma^{-2}\mathcal{T}^{-1}b$ as

$$J_\gamma = -\gamma^2(w - \mathcal{N}u - v, \mathcal{T}(w - \mathcal{N}u - v)) + K_\gamma(u). \tag{36}$$

The remainder K_γ is easily computed. The important fact is that it does not depend on w. Because we need $\mathcal{T}$ to be positive definite, the form (36) immediately shows that

$$K_\gamma(u) = \max_w J_\gamma(u, w). \tag{37}$$

However, we also have

$$G_\gamma = \exp\left(\frac{1}{2\gamma^2}K_\gamma(u)\right)(2\pi)^{-\frac{T\ell}{2}} \int \exp\left(-\frac{1}{2}\|w - \mathcal{N}u - v\|_{\mathcal{T}}^2\right) dw,$$

and a simple change of variable shows that the last integral does not depend on u, yielding

$$G_\gamma = \frac{1}{\sqrt{\det \mathcal{T}}} \exp\left(\frac{1}{2\gamma^2}K_\gamma(u)\right).$$

Therefore, the problem of minimizing G_γ is equivalent to that of minimizing K_γ, which in view of (37) is indeed equivalent to problem (17).

The above assumes an open-loop control u (a prior commitment), but we have a similar situation if we want to accept a control law of the form (15). Let us again restrict the control law to be linear. It amounts to an affine map $u = \mathcal{F}w + f$ in an admissible family of such maps (in particular, the matrix of $\mathcal{F}$ will be triangular to ensure causality). Substituting this in J_γ, we again obtain a nonhomogeneous quadratic form in w. The same technique of completing the square will lead to the same conclusion, that G_γ is proportional to the exponential of $(1/2\gamma^2)\max_w J_\gamma (\mu(y), w)$.

The above result is what we like to call Whittle's magic formula. Whittle was able to go further, proving, in a slightly simpler case, a separation theorem that implies our certainty equivalence theorem in that case. This approach was used by James et al. (1993, 1994) to derive a solution to the partial-information minimax control problem when the certainty equivalence theorem does not hold. Their derivation was contemporaneous to, and independent from, our own derivation of the same result using tools introduced by Başar and Bernhard (1991), and since generalized in the framework of the (max, +) algebra, see Bernhard (2000).

5. "ROBUST" IS NOT NECESSARILY "CAUTIOUS"

Among the misconceptions concerning robust control, one is that it is by nature cautious because it does not rely on an uncertain model. This is not necessarily so, and the following examples are meant to illustrate that point. All along, we assume perfect state information and concentrate on the state feedback gain.

5.1. Disturbance Attenuation

Let us consider the simple system where variables x, u, and w are scalar:

$$\dot{x} = -x + u + w,$$
$$z = \begin{pmatrix} x \\ u \end{pmatrix}.$$

As a reference, consider LQG control theory, with $w(\cdot)$ taken as a normalized white noise. The corresponding algebraic Riccati equation is

$$-2P - P^2 + 1 = 0.$$

The positive root is $P = \sqrt{2} - 1 \simeq 0.414$. This is also the optimal feedback gain $F = P$ as well as the expectation of the (limit of the) integrand in the quadratic performance index; that is,

$$\lim_{t\to\infty} \mathbf{E}\|z(t)\|^2 = \mathbf{E} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|z(t)\|^2 \,\mathrm{d}t = \sqrt{2} - 1 \simeq 0.414.$$

Using essentially the same theory (or that of $\mathcal{H}_\infty$ norms), one can also check that, with that control, the noise attenuation from w to z is $\gamma = \sqrt{2 - \sqrt{2}} \simeq 0.765$.

Let us now use the theory of robust noise attenuation. The Riccati equation is

$$-2P - (1 - \gamma^{-2})P^2 + 1 = 0.$$

It has a positive real root down to $\gamma = 1/\sqrt{2} = 0.707$, for which the positive real root is unique and is $P = 1$. Again, since the feedback gain is $F = P$, we see that it is *larger* than in the previous case. Of course, the noise attenuation, as measured by L^2 norms, is better (it is optimized here): 0.707 versus 0.765, while correlatively, if we assume that w is a normalized white noise as previously, this leads to a worse output covariance (it was minimized in the previous case); specifically, $\mathbf{E}x^2 = 1/4$, and thus (since $u = -x$) $\mathbf{E}\|z\|^2 = 0.5$ versus 0.414.

To summarize, the robust noise attenuation control leads to more control effort, for a better L^2-norm noise attenuation, at the expense of a larger control power that degrades the output covariance in the case in which the disturbance is (looks like) a normalized white noise.

5.2. Robust Stabilization

Let us now examine how the theory of robust stabilization (still with perfect state measurement) works on such simple examples. We consider two situations, depending on whether the plant uncertainty resides with the free dynamics or the control channel efficiency.

5.2.1. Uncertain free dynamics. Let us consider the system

$$\dot{x} = -x + \delta a\, x + u, \qquad |\delta a| < \alpha,$$

to be stabilized with the exact value of δa unknown.

Application of the above theory leads to consideration of the system

$$\dot{x} = -x + u + r,$$
$$s = \begin{pmatrix} \alpha x \\ \varepsilon u \end{pmatrix}$$

with the uncertain feedback

$$r = \Delta G s, \qquad \Delta G = \begin{bmatrix} \dfrac{\delta a}{\alpha} & 0 \end{bmatrix}.$$

The bound on the uncertainty is now

$$\|\Delta G\| < 1. \tag{38}$$

We were obliged to introduce the extra εu component in s to ensure that the equivalent of the G matrix of equation (3) is injective. We use it as a tuning parameter of the design method.

The Riccati equation is now

$$-2P - (\varepsilon^{-2} - 1)P^2 + \alpha^2 = 0.$$

Its smallest positive root is, for small ε's,

$$P = \frac{1}{\varepsilon^{-2} - 1}(\sqrt{1 + \alpha^2(\varepsilon^{-2} - 1)} - 1).$$

The corresponding feedback gain is $F = \varepsilon^{-2}P$. We see that, for small ε's, it is close to α/ε. Hence, the theory says: If you want to stabilize the above uncertain system, just use a large negative feedback gain. The larger the uncertainty (α), the larger the feedback gain. However, we may make it arbitrarily large since we did not attempt to simultaneously control an output containing u.

If we are interested in limiting the feedback gain, we may look at the same design procedure for large ε's. We see that there exists a positive root to the Riccati equation if and only if $1 - (1 - \varepsilon^{-2})\alpha^2 > 0$. Thus, if $\alpha < 1$, we may take ε arbitrarily large, and, correlatively, F arbitrarily small. We do not have to control at all; the system is spontaneously stable whatever δa within its bounds. If $\alpha > 1$, however, the limiting ε is $\alpha/\sqrt{\alpha^2 - 1}$, and the corresponding feedback gain is $F = \alpha^2 - 1$.

As a matter of fact, this leads to the closed-loop system

$$\dot{x} = -(\alpha^2 - \delta a)x$$

which is stable for every $\delta a < \alpha^2$, and a fortiori for the bound (38).

That we do not find $F = \alpha - 1$ is only a reflection of the fact that our design procedure is conservative. As a matter of fact, with (38), we have allowed any $\Delta G = [p \;\; q]$ with $\sqrt{p^2 + q^2} < 1$. Thus we have controlled the family of systems

$$\dot{x} = -(1 - p\alpha)x + (1 + q\varepsilon)u, \qquad p^2 + q^2 < 1.$$

It is a simple exercise to place $u = -Fx$ in that system and investigate for which ε there is an F that ensures $-1 + p\alpha - (1 + q\varepsilon)F < 0$. One indeed finds that ε should not be larger than $\alpha/\sqrt{\alpha^2 - 1}$, and that, at this limiting value, the only satisfactory F is $\alpha^2 - 1$. (Avoiding that degree of conservatism is the aim of μ-synthesis.)

5.2.2. Uncertain control channel efficiency. We use a similar approach to stabilize the unstable system

$$\dot{x} = x + (1 + \delta b)u, \qquad |\delta b| < \beta.$$

(Applying the design procedure to a stable system would lead to no control.) We proceed in the same fashion, using the system

$$\dot{x} = x + u + r,$$
$$s = \begin{pmatrix} \varepsilon x \\ \beta u \end{pmatrix},$$

with

$$r = \Delta G s, \qquad \Delta G = \begin{bmatrix} 0 & \dfrac{\delta b}{\beta} \end{bmatrix}.$$

and again the bound (38). The term εx in s is now needed to ensure the observability condition of the theory.

The Riccati equation is now

$$(1 - \beta^{-2})P^2 + 2P + \varepsilon^2 = 0.$$

For $\beta < 1$, it always has a positive root,

$$P = \frac{1}{\beta^{-2} - 1}(1 + \sqrt{1 + \varepsilon^2(\beta^{-2} - 1)}),$$

leading to the feedback gain $F = P/\beta^2$, the limit of which as $\varepsilon \to 0$ is now $F = 2/(1 - \beta^2)$. The worst closed-loop system is then, for $\delta b = -\beta$,

$$\dot{x} = -\frac{1 - \beta}{1 + \beta}x,$$

which is indeed stable since here $\beta < 1$.

Again, for β close to 1, we need a large feedback gain to compensate for the fact that the control channel may be very inefficient.

For $\beta > 1$, the problem clearly has no solution: Our system is unstable, and we do not know the sign of the coefficient of u in the dynamics. The Riccati equation always has two negative roots.

5.2.3. Mixed case. We consider the uncertain system

$$\dot{x} = -(1 - \delta a)x + (1 + \delta b)u, \qquad (\delta a)^2 + (\delta b)^2 < \rho^2.$$

We may expect that, for $\rho \leq \sqrt{2}$, that family of systems can be stabilized because, if $1 - \delta a < 0$, making the free system unstable, then the sign of $1 + \delta b$ is known to be positive, so that it is possible to control the system.

In fact, the Riccati equation associated with that problem is

$$(1 - \rho^{-2})P^2 - 2P + \rho^2 = 0,$$

which has a positive root, provided that $\rho \leq \sqrt{2}$. For $\rho > 1$, that root leads to a feedback gain $F = (1 - \sqrt{2-\rho^2})/(\rho^2 - 1)$, equal to 1 if $\rho = \sqrt{2}$.

We do not have large gains anymore; this is indeed a cautious control because one has to balance the risk of not compensating unstable free dynamics and that of exerting a positive feedback.

Notice that, if the a priori bound on the uncertainty is of the form $|\delta a| < \alpha$ and $|\delta b| < \beta$, within the current simple $\mathcal{H}_\infty$-optimal control theory we cannot do better than the above, with $\rho^2 = \alpha^2 + \beta^2$.

5.3. Robust Stabilizing Control

At last, we consider the simultaneous stabilization and control of our simple system.

5.3.1. Uncertain free dynamics. We want to control z in the disturbed uncertain system

$$\begin{aligned} \dot{x} &= -x + (\delta a)x + u + w, \qquad |\delta a| < \alpha, \\ z &= \begin{pmatrix} x \\ u \end{pmatrix}. \end{aligned}$$

We may seek to minimize the system norm from $(r \;\; w)$ to z, and check whether this norm is less than $1/\alpha$, which, in view of the small-gain theorem, is sufficient to ensure stability. One finds that the limiting γ is 1. Thus, this procedure succeeds only if $\alpha < 1$.

For $\alpha > 1$, we can apply the standard procedure advocated above: Introduce both $s = \alpha x$ and $z' = (1/\gamma)z$, that is, an output in $\mathbf{R}^3$. Then, seek the γ that will ensure that the operator norm from $(r \;\; w)$ to $(s \;\; z')$ will be less than 1, guaranteeing both robust stability and a disturbance attenuation of γ from w to z. We propose a slightly different approach, which turns out to give much better results. (This also serves the purpose of showing that this whole theory is to be applied with cleverness.)

We write the system

$$\begin{aligned} \dot{x} &= -x + u + r + w, \\ s &= \begin{pmatrix} \alpha x \\ \dfrac{1}{\gamma} u \end{pmatrix}; \end{aligned}$$

again, $r = \Delta G s$ with $\|\Delta G\| < 1$. We attempt to ensure an operator norm from $(r \;\; w)$ to s of less than 1. This ensures that the system is stable and, if $\gamma > 1/\alpha$ (as will occur for $\alpha > 1$), *a fortiori* an operator norm from w to z of less than γ (since $\|z\| < \gamma\|s\|$).

The Riccati equation associated with that new problem is

$$(2 - \gamma^2)P^2 - 2P + \alpha^2 = 0,$$

which has a positive root, provided that $\gamma^2 \geq 2 - \alpha^{-2}$. (Notice that this gives back $\gamma \geq 1$ if $\alpha = 1$. We recover the same limiting case as above.) For the limiting γ, we have $P = \alpha^2$ and $F = \gamma^2 P = 2\alpha^2 - 1$. Thus, this solution rules out large gains. In that respect, it embodies a degree of caution. However, as compared to the smallest gain obtained in the robust stabilization section, with no regard for an output, we use a larger feedback gain: $2\alpha^2 - 1$ versus $\alpha^2 - 1$. In that respect, robust control is not cautious control.

5.3.2. Uncertain control channel. For the uncertain control channel case, a symmetric situation results in

$$\dot{x} = -x + (1 + \delta b)u + w, \qquad |\delta b| < \beta,$$

with the same controlled output, leading when $\beta > 1/\sqrt{2}$ to the limit $\gamma^2 \leq 2 - \beta^{-2}$ and a feedback gain $F = 2\beta^2 - 1$.

5.3.3. Mixed case. As a last example, we consider the system with uncertainties on both the free dynamics and the control channel, which we want to simultaneously stabilize and control:

$$\begin{aligned} \dot{x} &= -(1 - \delta a)x + (1 + \delta b)u + w, \qquad |\delta a| < \alpha, \qquad |\delta b| < \beta, \\ z &= \begin{pmatrix} x \\ u \end{pmatrix}. \end{aligned}$$

Again, the standard approach proposed in the general theory would have us introduce a two-dimensional output s, in addition to the controlled output z, and thus apply $\mathcal{H}_\infty$-optimal control theory with a four-dimensional output. We avoid that higher dimension via another trick. We need to have a way to tune the relative weight of the two objectives: stabilization and control of z in such a way as to achieve the best possible disturbance attenuation in z without sacrificing stability. A way to achieve those goals is to introduce an output s modeled as previously, embodying both x and u, and to introduce a scaling weight $1/\gamma$ on the disturbance's input channel; that is, let $v = (1/\sqrt{2}\gamma)w$. (The $\sqrt{2}$ is there for normalization purposes.) Then we apply the $\mathcal{H}_\infty$-control theory to the system from $(r \;\; v)$ to s.

We therefore work with the system

$$\begin{aligned} \dot{x} &= -x + u + r + \frac{1}{\gamma\sqrt{2}} v, \\ s &= \begin{pmatrix} \alpha x \\ \beta u \end{pmatrix}, \end{aligned}$$

with the uncertainty model

$$r = \begin{bmatrix} \dfrac{\delta a}{\alpha} & \dfrac{\delta b}{\beta} \end{bmatrix} s = \Delta G s,$$

and the bound $\|\Delta G\| < \sqrt{2}$. Stability is ensured if the norm of the operator from $(r \;\; v)$ to s is less than $1/\sqrt{2}$. In that case, the disturbance attenuation factor, in the L^2 norm, is better than $\gamma/\min\{\alpha, \beta\}$. (Of course, this is an efficient design procedure only if α and β are of the same order of magnitude.)

The Riccati equation of that problem is

$$(2 + \gamma^{-2} - \beta^{-2})P^2 - 2P + \alpha^2 = 0.$$

It has a positive solution, provided that

$$\alpha^{-2} + \beta^{-2} \geq 2 \tag{39}$$

and

$$\gamma^{-2} \leq \alpha^{-2} + \beta^{-2} - 2$$

and the feedback gain that yields the limiting attenuation factor is

$$F = \frac{\alpha^2}{\beta^2}. \tag{40}$$

That this gain is indeed stabilizing can be checked directly: It leads to

$$\dot{x} = -\alpha^2\left(\alpha^{-2} + \beta^{-2} - \frac{\delta a}{\alpha^2} - \frac{\delta b}{\beta^2}\right)x + w, \tag{41}$$

and it is easily seen that

$$\frac{\delta a}{\alpha^2} + \frac{\delta b}{\beta^2} < \frac{1}{\alpha} + \frac{1}{\beta} < \sqrt{2}\sqrt{\alpha^{-2} + \beta^{-2}},$$

so that condition (39) is precisely sufficient to ensure that (41) will be stable.

Formula (40) has an interesting interpretation: The feedback gain should be chosen large or small depending on whether the larger uncertainty is in the free dynamics or the control channel, respectively. If we trust the control channel, we may use it to correct uncertain free dynamics. If, on the contrary, we trust stable dynamics more than the control channel, we should exert control with care. Hence, $\mathcal{H}_\infty$-optimal control theory appears as more or less cautious depending on where the uncertainties in the system lie.

NOTES

1. One may argue *a posteriori* that this is because being concerned with disturbances in the coefficients of the model, a stochastic formulation would lead to a differential equation with stochastic coefficients, a technically complex object, difficult to use.

2. Roxin has proposed the following equivalent definition of causality: An application μ from $L^2([0, T] \to \mathbf{R}^p)$ to $L^2([0, T] \to \mathbf{R}^m)$ is causal if $\forall\, t \in [0, T]$, the equality $y(s) = y'(s)$ for almost all $s < t$ implies $\mu(y)(t) = \mu(y')(t)$.

3. Recall that (A, D) stabilizable means that there exists a feedback matrix F such that $A - DF$ is asymptotically stable and (A, D) controllable suffices.

4. (H, A) is reconstructible if (A^t, H^t) is stabilizable. Thus, (H, A) observable suffices.
5. It is an elementary matter to check that

$$\delta^2 = (\alpha^2 + \beta^2 + \gamma^2 + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\beta^2\gamma^2})/2.$$

6. Assume that $\|T_\mu\| = \gamma \geq \delta^{-1}$. There exists an $\hat{s}$ of norm 1 such that $T_\mu T_\mu^* \hat{s} = \gamma^2 \hat{s}$. Choose $\Delta\mathcal{G}$ defined by $\Delta\mathcal{G}u = \gamma^{-2} T_\mu^* \hat{s}(\hat{s}, u)$, and pick v such that $S_\mu v = \hat{s}$. Here $\|\Delta\mathcal{G}\| = \gamma^{-1} \leq \delta$. And it is readily seen that if s were the solution of (31), one would have $s = [(\hat{s}, s) + 1]\hat{s}$, and taking the scalar product with $\hat{s}$, $(\hat{s}, s) = (\hat{s}, s) + 1$, a contradiction.

REFERENCES

Barmish, B.R. (1988) New tools for robustness analysis. *27th IEEE CDC*, pp. 1–6. Austin, TX.

Başar, T. (1989) A dynamic games approach to controller design: Disturbance rejection in discrete time. *29th IEEE CDC*, pp. 407–414. Tampa, USA.

Başar, T. & P. Bernhard (1991) *$\mathcal{H}_\infty$-Optimal Control and Related Minimax Design Problems: A Differential Games Approach* (2nd ed. 1995). Birkhauser.

Bensoussan, A. & J.H. Van Schuppen (1985) Optimal control of partially observable stochastic systems with an exponential-of-integral performance index. *SIAM Journal on Control and Optimisation* 23, 599–613.

Bernhard, P. (1990) A Certainty Equivalence Principle and Its Application to Continuous Time, Sampled Data, and Discrete Time $\mathcal{H}_\infty$-Optimal Control. Report 1347, INRIA.

Bernhard, P. (1991) Application of the minimax certainty equivalence principle to sampled data output feedback $\mathcal{H}_\infty$-optimal control. *Systems & Control Letters* 16, 229–234.

Bernhard, P. (2001) Minimax—or Feared value—L^1/L^∞ control. *Theoretical Computer Science*. (To appear.)

Bernhard, P. & A. Rapaport (1995) Min-max certainty equivalence principle and differential games. *International Journal of Robust and Nonlinear Control* 6, 825–842.

Chilali, M. (1996) *Méthodes LMI pour l'Analyse et la Synthèse Multicritère*. Ph.D. Dissertation. Université de Paris-Dauphine.

Doyle, J.C. & G. Stein (1981) Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Transactions on Automatic Control* AC-26, 4–16.

Doyle, J.C., K. Glover, P. Khargonekar & B. Francis (1989) State space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions on Automatic Control* AC-34, 831–847.

Francis, B. (1987) *A Course in $\mathcal{H}_\infty$ Control Theory,* Vol. 88, *Lecture Notes in Control and Information Sciences*. New York: Springer-Verlag.

James, M.R., J.S. Barras & R.J. Elliott (1993) Output feedback risk sensitive control and differential games for continuous time nonlinear systems. *32nd IEEE CDC*, San Antonio, USA.

James, M.R., J.S. Barras & R.J. Elliott (1994) Risk sensitive control and dynamic games for partially observed discrete time nonlinear systems. *IEEE Transactions on Automatic Control* AC-39, 780–792.

Kwakernaak, H. (1991) *The Polynomial Approach to $\mathcal{H}_\infty$-Optimal Regulation*. CIME notes, Villa Olmo, Italy, 1990, and Springer-Verlag.

Limebeer, D., M. Green & D. Walker (1989) Discrete time $\mathcal{H}_\infty$ control. *29th IEEE CDC*, pp. 392–396. Tampa, USA.

Papavassilopoulos, G. & M.G. Safonov (1989) Robust control design via game theoretic methods. *29th IEEE CDC*, Tampa, USA.

Stoorvogel, A. (1992) *The $\mathcal{H}_\infty$ Control Problem: A State Space Approach*. Prentice-Hall.

Tadmor, G. (1990) Worst case design in the time domain: The Maximum Principle and the standard $\mathcal{H}_\infty$ problem. *Mathematics of Control, Signal and Systems* 3, 301–324.

Whittle, P. (1981) Risk sensitive linear quadratic Gaussian control. *Advances in Applied Probability* 13, 764–777.

Zames, G. (1981) Feedback and optimal sensitivity: Model reference transformation, multiplicative semi-norms and approximate inverses. *IEEE Transactions on Automatic Control* AC-26, 301–320.