flections of nearly identical sin θ value. Any indexing of powder photographs of soap crystals is consequently very questionable. It hardly need be said that no notion of symmetry can be obtained from powder photographs of crystals with such large cells as soaps.

There is also a prevailing opinion in some quarters that if the computed length of a soap molecule fits the spacing $d_{(001)}$, the crystal is orthorhombic, while if it must be tipped to fit the spacing, the crystal is monoclinic. Neither of these propositions is necessarily true.

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STATISTICAL METRICS

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We shall call *statistical metric* a set S such that with each two elements ("points") p and q of S a probability function $\Pi(x;\ p, q)$ is associated satisfying the following conditions:

1. $\Pi(0;\ p, p) = 1.$
2. If $p \neq q$, then $\Pi(0;\ p, q) < 1.$
3. $\Pi(x;\ p, q) = \Pi(x;\ q, p).$
4. $T[\Pi(x;\ p, q), \Pi(y;\ q, r)] \leq \Pi(x + y;\ p, r).$

where $T(\alpha, \beta)$ is a function defined for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ such that

(*a*) $0 \leq T(\alpha, \beta) \leq 1.$
(*b*) T is non-decreasing in either variable.
(*c*) $T(\alpha, \beta) = T(\beta, \alpha).$
(*d*) $T(1, 1) = 1.$
(*e*) If $\alpha > 0$, then $T(\alpha, 1) > 0.$

By a probability function we mean a non-decreasing function defined for all non-negative values of x, continuous to the right, with values between 0 and 1, and converging toward 1 as x increases beyond all bounds.

We call $\Pi(x;\ p, q)$ the *distance function* of p and q and interpret it as the probability that the points p and q have a distance $\leq x$. Condition 4, our "triangular inequality," implies that $\Pi(z;\ p, r) \geq \text{Max. } T[\Pi(x;\ p, q), \Pi(z - x;\ q, r)]$ for all points q and all numbers x between 0 and z. We shall call the function T the *triangular norm* of the statistical metric, and more specifically refer to the metric defined above as a T-metric. A triangular norm T will be called *simple* if

(*f*) $0 < T(\alpha, \beta) < 1$ for $0 < \alpha \cdot \beta < 1$.

An ordinary metric space is a statistical metric such that for each pair of points p, q there exists a number $d(p, q) \geq 0$ with the property that $\Pi(x;\ p, q)$ is $= 0$ if $x < d(p, q)$, and $= 1$ if $x \geq d(p, q)$.

On the basis of our postulates large parts of metric geometry can be developed, in particular, a theory of betweenness. We shall say that q lies *between* p and r (and we write pqr) if

$$T[1 - \Pi(x;\ p, q), 1 - \Pi(y;\ q, r)] \leq 1 - \Pi(x + y;\ p, r).$$

Equivalent is the assumption

$$\Pi(z;\ p, r) \leq 1 - \text{Max. } T[1 - \Pi(x;\ p, q), 1 - \Pi(z - x;\ q, r)] \text{ for } 0 \leq x \leq z.$$

Obviously, if pqr then rqp. In metric spaces if q and r are distinct, then pqr and prq are incompatible. In a statistical metric we can only prove: If q and r are apart, then pqr and prq are incompatible, where q and r are said to be apart if there exists a number $y > 0$ such that $\Pi(y;\ q, r) = 0$.

If for each two points of a statistical metric S the distance function $\Pi(x;\ p, q)$ belongs to a family $\mathfrak{P}$ of probability functions, we call S metrized by means of $\mathfrak{P}$. Let $\mathfrak{P}$ be a 2 parameter family of probability functions $\Pi(x;\ a', a'')$ defined for all real numbers a' and a'' such that $0 \leq a' \leq a''$ and satisfying the conditions

$$\begin{aligned} &\Pi(x;\ a', a'') = 0 \text{ if } x \leq a', \\ &\Pi(x;\ a', a'') = 1 \text{ if } x \geq a'', \\ &0 < \Pi(x;\ a', a'') < 1 \text{ for } a' < x < a''. \end{aligned}$$

If S is metrized by means of $\mathfrak{P}$ and T is simple, then for each three points p, q, r one of which lies between the two other ones, two of the distance functions $\Pi(x;\ p, q)$, $\Pi(y;\ q, r)$, $\Pi(z;\ p, r)$ determine the third. In particular, if pqr and $\Pi(x;\ p, q) = \Pi(x;\ a', a'')$ and $\Pi(y;\ q, r) = \Pi(y;\ b', b'')$, then $\Pi(z;\ p, r) = \Pi(z;\ a' + b', a'' + b'')$. From this theorem one readily derives the classical law: If pqr and prs, then pqs and qrs.

If

$$\begin{aligned} \Pi(x;\ p, q) &= \Pi(x;\ r, s) = \Pi(x;\ a', a'') \\ \Pi(x;\ q, r) &= \Pi(x;\ p, s) = \Pi(x;\ b', b'') \\ \Pi(x;\ p, r) &= \Pi(x;\ q, s) = \Pi(x;\ a' + b';\ a'' + b''), \end{aligned}$$

then p, q, r, s form what may be called a *pseudo-linear statistical quadruple*, i.e., a quadruple which cannot be ordered by means of the between-relation though for each three of the four points one lies between the other two.

If a statistical T-metric S metrized by means of $\mathfrak{P}$ contains more than four points, then by virtue of the properties of betweenness this relation can be used to order S. Moreover, the other ideas of metric geometry (convexity, geodesics, etc.) can be applied.

The three principal applications of statistical metrics are to macroscopic, microscopic and physiological spatial measurements. Statistical metrics are designed to provide us (1) with a method removing conceptual difficulties from microscopic physics and transferring them into the underlying geometry, (2) with a treatment of thresholds of spatial sensation eliminating the intrinsic paradoxes of the classical theory. For a given point p_0 the number $\Pi(0;\ p_0, q)$ considered as a function of the point q indicates the probability that q cannot be distinguished from p_0. The study of this function should replace the attempt to determine a definite set of points q which cannot be distinguished from p_0. This function could also be used advantageously instead of a relation of physical identity for which, as Poincaré emphasized on several occasions, we always have triples p, q, r for which

$$p = q,\ q = r, \text{ and } p \neq r.$$

Experiments indicate that q sometimes can and sometimes cannot be distinguished from p_0. Hence, the adequate description of the situation seems to arise from counting the relative frequency of these occurrences.

NATURAL ISOMORPHISMS IN GROUP THEORY

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1. Introduction.—Frequently in modern mathematics there occur phenomena of "naturality": a "natural" isomorphism between two groups or between two complexes, a "natural" homeomorphism of two spaces and the like. We here propose a precise definition of the "naturality" of such correspondences, as a basis for an appropriate general theory. In this preliminary report we restrict ourselves to the natural isomorphisms of group theory; with this limitation we can present the basic concepts of our theory without developing the axiomatic approach necessary for a general treatment applicable to various branches of mathematics.