# Optimization with Max-Min Fuzzy Relational Equations

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# Problem Facing

- Problem (P)

Minimize $f(x)$

s.t. $A \circ x = b$

$x \in [0,1]^n$

where $f$: $R^n \to R$ is a function,

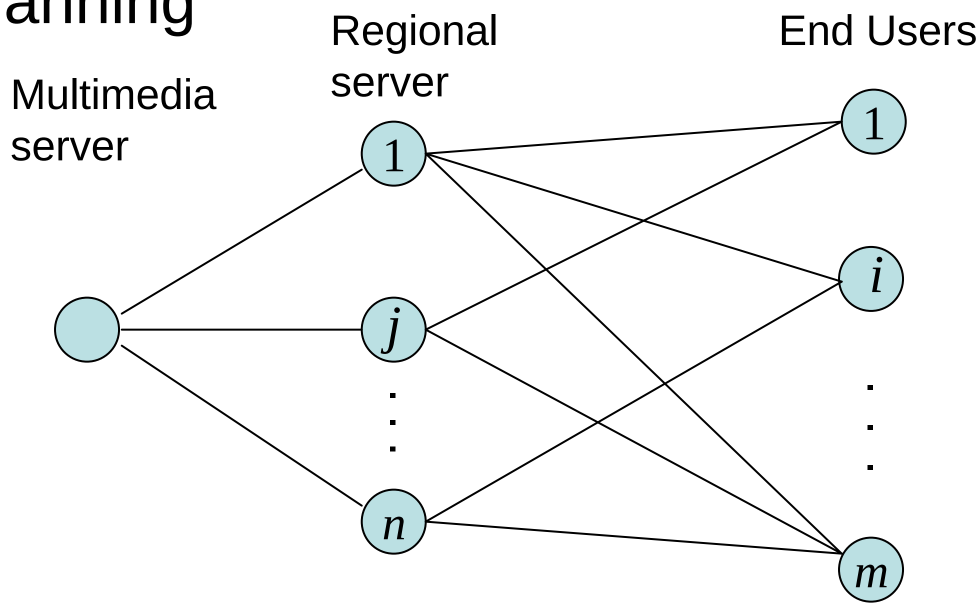
$$A = (a_{ij})_{m \times n} \in [0,1]^{mn}, \qquad b = (b_i)_{m \times 1} \in [0,1]^m,$$

"$\circ$ " is a matrix operation replacing "product" by "minimum" and "addition" by "maximum", i.e.,

$$\sum_{1 \le j \le n} a_{ij} x_j = b_i \Rightarrow \max_{1 \le j \le n} \min(a_{ij}, x_j) = b_i, \text{ for } i = 1, \ldots, m.$$

# Examples

1. Capacity Planning




$a_{ij}$ : bandwidth in field from server $j$ to user $i$

$b_i$ : bandwidth required by user $i$

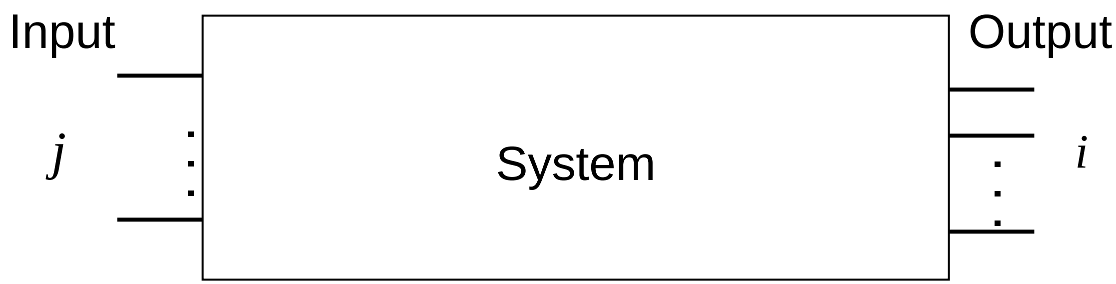
$x_j$ : capacity of server $j$

Consider

$$\max_{1 \le j \le n} \min \ (a_{ij}, x_j) = b_i \ , \quad \text{for } i = 1, \ldots, m.$$

# Examples

2. Fuzzy control / diagnosis / knowledge system




$a_{ij}$ : degree of input $j$ relating to output $i$

$b_i$ : degree of output at state $i$ (symptom)

$x_j$ : degree of input at state $j$ (cause)

A fuzzy system is usually characterized by

$$\max_{1\le j\le n} t(a_{ij}, x_j) = b_i, \quad \forall i,$$

or

$$\min_{1\le j\le n} s(a_{ij}, x_j) = b_i, \quad \forall i,$$

where "$t$" is a triangular norm and "$s$" is a triangular co-norm.

# Triangular Norms

## t-norm:

$$t:[0,1]\times[0,1]\rightarrow[0,1] \quad \text{such that}$$

1) $t(x,y)=t(y,x)$ (commutative)

2) $t(x,t(y,z))=t(t(x,y),z)$ (associative)

3) $t(x,y)\leq t(x,z)$, if $y\leq z$ (monotonically nondecreasing)

4) $t(x,0)=0$ and $t(x,1)=x$ (boundary condition).

## s-norm (t co-norm):

$$s:[0,1]\times[0,1]\rightarrow[0,1] \quad \text{such that}$$

$$s(x,y)=1-t(1-x,\ 1-y) \quad \forall\ x,y\in[0,1]$$

[ Menger K. (1942), "Statistical metrics", Proceedings of the National Academy of Sciences of the United States of America, 28, 535-537.]

[ Schweizer B. and Sklar A. (1961), "Associative functions and statistical triangle inequalities", Mathematical Debrecen 8, 169-186.]

# Triangular Norms

$$t_w(x,y)=\begin{cases}\min\{x,y\} & \text{if } \max\{x,y\}=1\\ 0, & \text{otherwise}\end{cases}$$ drastic product

$$s_w(x,y)=\begin{cases}\max\{x,y\} & \text{if } \min\{x,y\}=0\\ 1, & \text{otherwise}\end{cases}$$ drastic sum

$$t_1(x,y)=\max\{0,x+y-1\}$$ bounded difference

$$s_1(x,y)=\min\{1,x+y\}$$ bounded sum

$$t_{1.5}(x,y)=\frac{x\cdot y}{2-[x+y-x\cdot y]}$$ Einstein product

$$s_{1.5}(x,y)=\frac{x+y}{1+x\cdot y}$$ Einstein sum

# Triangular Norms

$$t_2(x, y) = x \cdot y$$ algebraic product

$$s_2(x, y) = x + y - x \cdot y$$ algebraic sum

$$t_{2.5}(x, y) = \frac{x \cdot y}{x + y - x \cdot y}$$ Hamacher product

$$s_{2.5}(x, y) = \frac{x + y - 2x \cdot y}{1 - x \cdot y}$$ Hamacher sum

$$t_3(x, y) = \min\{x, y\}$$ minimum

$$s_3(x, y) = \max\{x, y\}$$ maximum

$$t_w \le \cdots \le t_1 \cdots \le t_2 \cdots \le t_3 = \min \quad \le s_3 = \max \cdots \le s_2 \cdots \le s_1 \cdots \le s_w$$

[ Klement E.P. et al. (2000), “Triangular norms”, Kluwer, Dordrecht, 2000.]

# Fuzzy Relational Equations

Given

$$A = (a_{ij}) \in [0,1]^{m\times n},$$

$$b = (b_1, \ldots, b_m) \in [0,1]^m,$$

find

$$x = (x_1, \ldots, x_n) \in [0,1]^n \quad \text{such that}$$

(max-t-norm composition $A \circ x = b$)

$$\max_{1\le j\le n} t(a_{ij}, x_j) = b_i,\ \forall i.$$

(min-s-norm composition $A \circ x = b$)

$$\min_{1\le j\le n} s(a_{ij}, x_j) = b_i,\ \forall i.$$

The solution set is denoted by $\sum(A,b)$.

# Difficulties in Solving Problem (P)

1. Algebraically, neither "maximum" nor "minimum" operations has an inverse operation.

$$0.6x + 0.3 = 0.6 \quad \Rightarrow \quad x = \frac{0.6 - 0.3}{0.6} = 0.5$$

$$\max\left(0.3, \min\left(0.6, x\right)\right) = 0.6 \;\Rightarrow\; x = ?$$

2. Geometrically, the solution set $\sum(A, b)$ is a "combinatorially" generated "non-convex" set.

# Solution Set of Max-t Equations

1. Definition: $\hat{x} \in \Sigma(A,b)$ is a maximum solution

   if $x \leq \hat{x}, \ \forall x \in \Sigma(A,b).$

2. Definition: $\check{x} \in \Sigma(A,b)$ is a minimum solution

   if $x \geq \check{x}, \ \forall x \in \Sigma(A,b).$

3. Definition: $\hat{x} \in \Sigma(A,b)$ is a maximal solution

   if $x \geq \hat{x}$ implies $x = \hat{x}, \ \forall x \in \Sigma(A,b).$

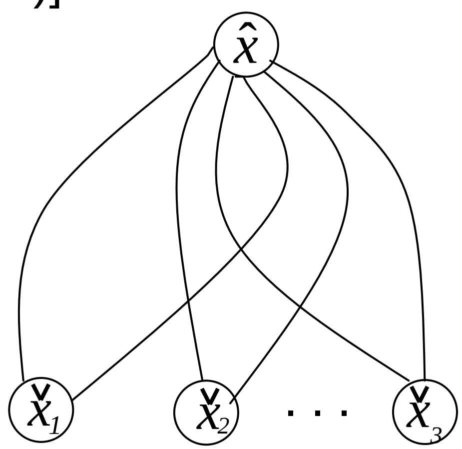
4. Definition: $\check{x} \in \Sigma(A,b)$ is a minimal solution

   if $x \leq \check{x}$ implies $x = \check{x}, \ \forall x \in \Sigma(A,b).$

# Solution Set of Max-t Equations

- Theorem: For a continuous t-norm, if $\Sigma(A, b)$ is nonempty, then $\Sigma(A, b)$ can be completely determined by one maximum and a finite number of minimal solutions.

[Sanchez (1976, 1977), Czogala / Drewniak / Pedrycz (1982), Higashi / Klir (1984), di Nola (1985)]




[Root System]

# Characteristics of Solution Sets

- Existence

[di Nola / Sessa / Pedrycz / Sanchez (1989)]

Theorem : For a continuous t - norm, $\Sigma(A,b) \neq \phi$ if and only if

it has a maximum solution $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_j)$

with $\hat{x}_j = \min_{1 \le i \le m} (a_{ij} \ \varphi \ b_i)$ where

$$a \ \varphi \ b \equiv \sup \left\{ u \in [0,1] \ \middle| \ t(a,u) \le b \right\}.$$

**Remark:** The maximum solution $\hat{x}$ can be obtained in ***O(mn)***.

# Characteristics of Solution Sets

- Uniqueness

Theorem: For a continuous t-norm, $\Sigma(A,b)$ has its minimum solution if and only if the kernel vector $\breve{x}^b$ is feasible. $\Sigma(A,b)$ has a unique solution if and only if $\breve{x}^b = \hat{x}$.

**Remark:** The kernel vector $\breve{x}^b$ can be obtained in $O(mn)$.

[ Gavalec M. (2001), "Solvability and unique solvability of max-min fuzzy equations", Fuzzy Sets and Systems, 124, 385-393.]

[ Li P. and Fang S.-C. (2009), "On the unique solvability of fuzzy relational equations", Soft Computing, submitted.]

# Characteristics of Solution Sets

- Complexity

The upper bound of the number of minimal solutions is $n^m$.

[ Wang P.Z. et al. (1984), “How many lower solutions does a fuzzy relation have?”, BUSEFAL, 18, 67-74.]

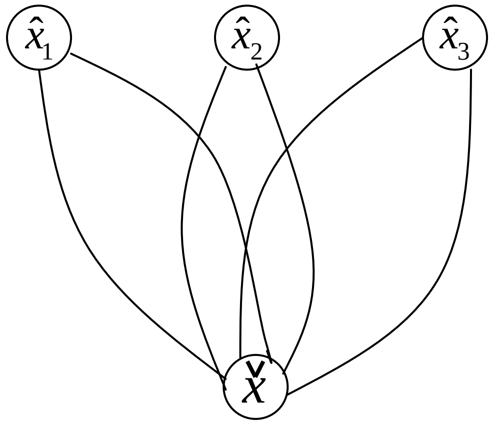
The set of all minimal solutions can be computed in incremental quasi-polynomial time.

[ Fredman  M. and Khachiyan L. (1996), “On the complexity of dualization of monotone disjunctive normal forms”, Journal of Algorithms, 21, 618-628.]

[ Elbassioni K.M. (2008), “A note on systems with max-min and max-product constraints”, Fuzzy Sets and Systems, 159, 2272-2277.]

# Characteristics of Solution Sets

- Theorem: For a continuous s-norm, if $\sum(A, b)$ is nonempty, then $\sum(A, b)$ is completely determined by one minimum and a finite number of maximal solutions.




[Crown System]

# Problem Facing

- Problem(P)

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{s.t.} \quad & A \circ x = b \\ & x \in [0,1]^n \end{aligned}$$

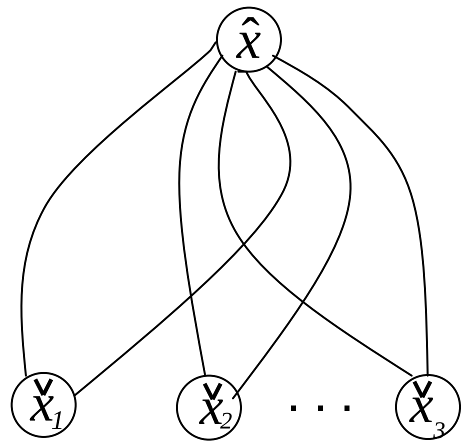
A nonconvex optimization problem over a region defined by a combinatorial number of vertices.

# Optimization with Fuzzy Relation Equations

- $f(x) = c^T x$ linear function

Lemma 1: If $c_j \le 0$ for all $j$, then $\hat{x}$ is an optimal solution.

Lemma 2: If $c_j \ge 0$ for all $j$, then one of the minimal solutions is an optimal solution.




[Root System]

[ Fang S.-C. and Li G. (1999), "Solving fuzzy relation equations with a linear objective function", Fuzzy Sets and Systems, 103, 107-113.]

# Optimization with Fuzzy Relation Equations

Theorem : Let

$$c_j' = \begin{cases} c_j & if \;\; c_j > 0 \\ 0 & if \;\; c_j \le 0 \end{cases} \quad \text{and} \quad x^* = \begin{cases} \check{x}_j^* & if \;\; c_j > 0 \\ \hat{x}_j & if \;\; c_j \le 0 \end{cases},$$

where $\check{x}^*$ solves the problem with $f(x) = (c')^T x$, then $x^*$ is an optimal solution.

0-1 integer programming with a branch-and-bound solution technique.

# Optimization with Fuzzy Relational Equations

- Extensions

1. Objective function $f(x)$
   - linear fractional
   - geometric
   - “max-t” or “min-s” operated
     (max-separable or min-separable)
   - general nonlinear
   - vector-valued

2. Constraints
   - interval-valued
   - “max-t” or “min-s” operated
   - “max-average” or “min-average” operated

# Results

## Li and Fang (2008, 2009)

**Theorem 1**: Let $A \circ x = b$ be a consistent system of max-min (max-t) equations with a maximum solution $\hat{x}$.

The Problem (P) can be reduced to

$$
\text{(MIP)} \qquad
\begin{aligned}
\text{Minimize} \quad & f(x) \\
\text{s. t.} \quad & Qu \geq e^m \\
& Gu \leq e^n \\
& Vu \leq x \leq \hat{x} \\
& u \in \{0,1\}^r
\end{aligned}
$$

where $Q$ is $m$-by-$r$, $G$ is $n$-by-$r$, $V$ is $n$-by-$r$, $e^m$, $e^n$ are vectors of all ones, and $r$ is an integer (up to $m \times n$).

# Results

**Theorem 2**: As in Theorem 1,

if $f(x)$ is linear, or generally, separable and monotone in each variable, then Problem (P) can be further reduced to **a set covering problem**;

if $f(x)$ is linear fractional, then Problem (P) can be further reduced to a 0-1 linear fractional integer programming problem;

if $f(x)$ is monotone in each variable, then Problem (P) can be further reduced to a 0-1 (nonlinear) integer programming problem.

# Results

**Corollary**: Problem (P) is in general NP-hard.

**Theorem 3**: In case $f(x) = \max_{1 \le j \le n} f_j(x_j)$ with $f_j(\cdot)$ being continuous and monotone for each $j$, then Problem (P) can be solved in polynomial time.

# Challenges Remain

- Efficient algorithms for optimization problems with fuzzy relational equation constraints.

- Efficient algorithms to generate an approximate solution of $\sum(A, b)$, i.e.,

$$\begin{aligned} \text{Minimize} \quad & dist\left(A \circ x,\ b\right) \\ \text{s. t.} \quad & x \in [0,1]^n. \end{aligned}$$

**Remark**: When $l^\infty$ norm is employed, this problem is polynomial time solvable.

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3. Li, P., Fang, S.-C., A survey on fuzzy relational equations, Part I: Classification and solvability, *Fuzzy Optimization and Decision Making*, 8 (2009) 179-229.
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*Thank you!*

*Question?*

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