1.23. For the magnetic ball suspension system given in Exercise 1.21, we view v and s as the system input and output, respectively.

(a) Determine a state-space representation for this system.

(b) Using the linearized equation obtained in part (c) of Exercise 1.21, obtain the input-output description of this system.

1.24. In Example 4.3, we view e_a and θ as the system input and output, respectively.

(a) Determine a state-space representation for this system.

(b) Determine the input-output description of this system.

1.25. For the second-order section digital filter in direct form, given in Fig. 1.13, determine the input-output description, where $x_1(k)$ and $u(k)$ denote the output and input, respectively.

1.26. In the circuit of Fig. 1.27, $V_i(t)$ and $V_0(t)$ are voltages (at time t) and R_1 and R_2 are resistors. There is also an ideal diode that acts as a short circuit when V_i is positive and as an open circuit when v_i is negative. We view V_i and V_0 as the system input and output, respectively.

(a) Determine an input-output description of this system.

(b) Is this system linear? Is it time-varying or time-invariant? Is it causal? Explain your answers.

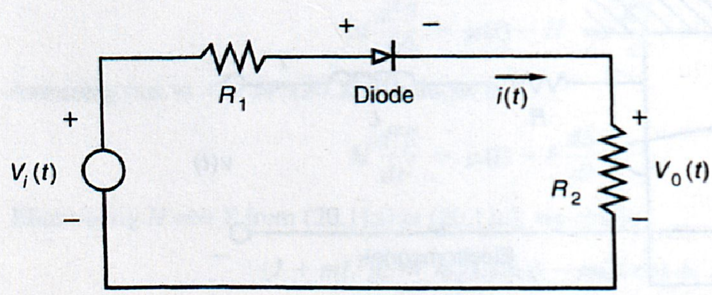

FIGURE 1.27
Diode circuit

1.27. We consider the *truncation operator* given by

$$y(t) = T_\tau(u(t))$$

as a system, where $\tau \in R$ is fixed, u and y denote system input and output, respectively, t denotes time, and $T_\tau(\cdot)$ is specified by

$$T_\tau(u(t)) = \begin{cases} u(t), & t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

Is this system causal? Is it linear? Is it time-invariant? What is its impulse response?

1.28. We consider the *shift operator* given by

$$y(t) = Q_\tau(u(t)) = u(t - \tau)$$

as a system, where $\tau \in R$ is fixed, u and y denote system input and system output, respectively, and t denotes time. Is this system causal? Is it linear? Is it time-invariant? What is its impulse response?

1.29. Consider the system whose input-output description is given by

$$y(t) = \min\{u_1(t), u_2(t)\},$$

where $u(t) = [u_1(t), u_2(t)]^T$ denotes the system input and $y(t)$ is the system output. Is this system linear?

1.30. Suppose it is known that a linear system has impulse response given by $h(t, \tau) = \exp(-|t - \tau|)$. Is this system causal? Is it time-invariant?

1.31. Consider a system with input-output description given by

$$y(k) = 3u(k+1) + 1, \qquad k \in Z,$$

where y and u denote the output and input, respectively (recall that Z denotes the integers). Is this system causal? Is it linear?

1.32. Use expression (16.8),

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k),$$

and $\delta(n) = p(n) - p(n-1)$ to express the system response $y(n)$ due to any input $u(k)$, as a function of the unit step response of the system [i.e., due to $u(k) = p(k)$].

1.33. (Simple pendulum) A system of first-order ordinary differential equations that characterize the simple pendulum considered in Exercise 1.1b is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix},$$

where $x_1 \triangleq \theta$ and $x_2 \triangleq \dot{\theta}$ with $x_1(0) = \theta(0)$ and $x_2(0) = \dot{\theta}(0)$ specified. A linearized model of this system about the solution $x = [0, 0]^T$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Let $g = 10$ (m/sec^2) and $l = 1$ (m).

(a) For the case when $x(0) = [\theta_0, 0]^T$ with $\theta_0 = \pi/18, \pi/12, \pi/6$, and $\pi/3$, plot the states for $t \geq 0$, for the nonlinear model.

(b) Repeat (a) for the linear model.

(c) Compare the results in (a) and (b).

recent texts on this subject, refer to Strang [16] and Michel and Herget [10]. Our presentation in Section 2.2 is in the spirit of the coverage given in [10].

Our treatment of basic aspects of linear ordinary differential equations in Sections 2.3, 2.4, and 2.5 follows along lines similar to the development of this subject given in Miller and Michel [11].

State-space and input-output representations of continuous-time systems and discrete-time systems, addressed in Sections 2.6 and 2.7, respectively, are covered in a variety of textbooks, including Kailath [9], Chen [4], Brockett [3], DeCarlo [5], Rugh [14], and others. For further material on sampled-data systems, refer to Aström and Wittenmark [1] and to the early works on this subject that include Jury [8] and Ragazzini and Franklin [12].

Detailed treatments of the Laplace transform and the z-transform, discussed briefly in Sections 2.4 and 2.7, respectively, can be found in numerous texts on signals and linear systems, control systems, and signal processing.

The state representation of systems received wide acceptance in systems theory beginning in the late 1950s. This was primarily due to the work of R. E. Kalman and others in filtering theory and quadratic control theory and to the work of applied mathematicians concerned with the stability theory of dynamical systems. For comments and extensive references on some of the early contributions in these areas, refer to Kailath [9] and Sontag [15]. Of course, differential equations have been used to describe the dynamical behavior of artificial systems for many years. For example, in 1868 J. C. Maxwell presented a complete treatment of the behavior of devices that regulate the steam pressure in steam engines called flyball governors (Watt governors) to explain certain phenomena.

The use of state-space representations in the systems and control area opened the way for the systematic study of systems with multi-inputs and multi-outputs. Since the 1960s an alternative description is also being used to characterize time-invariant MIMO control systems that involves usage of polynomial matrices or differential operators. Some of the original references on this approach include Rosenbrock [13] and Wolovich [17]. This method, which corresponds to system descriptions by means of higher order ordinary differential equations (rather than systems of first-order ordinary differential equations, as is the case in the state-space description) is addressed in Chapter 7.

2.11 REFERENCES

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2.12 EXERCISES

2.1. (a) Let $(V, F) = (R^3, R)$. Determine the representation of $v = (1, 4, 0)^T$ with respect to the basis $v^1 = (1, -1, 0)^T$, $v^2 = (1, 0, -1)^T$, and $v^3 = (0, 1, 0)^T$.
(b) Let $V = F^3$ and let F be the field of rational functions. Determine the representation of $\tilde{v} = (s + 2, 1/s, -2)^T$ with respect to the basis $\{v^1, v^2, v^3\}$ given in (a).

2.2. Find the relationship between the two bases $\{v^1, v^2, v^3\}$ and $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ (i.e., find the matrix of $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ with respect to $\{v^1, v^2, v^3\}$), where $v^1 = (2, 1, 0)^T$, $v^2 = (1, 0, -1)^T$, $v^3 = (1, 0, 0)^T$, $\bar{v}^1 = (1, 0, 0)^T$, $\bar{v}^2 = (0, 1, -1)$, and $\bar{v}^3 = (0, 1, 1)$. Determine the representation of the vector $e_2 = (0, 1, 0)^T$ with respect to both of the above bases.

2.3. Let $\alpha \in R$ be fixed. Show that the set of all vectors $(x, \alpha x)^T$, $x \in R$, determines a vector space of dimension one over $F = R$, where vector addition and multiplication of vectors by scalars is defined in the usual manner. Determine a basis for this space.

2.4. Show that the set of all real $n \times n$ matrices with the usual operation of matrix addition and the usual operation of multiplication of matrices by scalars constitutes a vector space over the reals [denoted by $(R^{n\times n}, R)$]. Determine the dimension and a basis for this space. Is the above statement still true if $R^{n\times n}$ is replaced by $R^{m\times n}$, the set of real $m \times n$ matrices? Is the above statement still true if $R^{n\times n}$ is replaced by the set of nonsingular matrices? Justify your answers.

2.5. Let $v^1 = (s^2, s)^T$ and $v^2 = (1, 1/s)^T$. Is the set $\{v^1, v^2\}$ linearly independent over the field of rational functions? Is it linearly independent over the field of real numbers?

2.6. Determine the rank of the following matrices, carefully specifying the field:

$$\text{(a)} \begin{bmatrix} j \\ 3j \\ -1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 4 & -5 \\ 7 & 0 & 2 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} s+4 & -2 \\ s^2-1 & 6 \\ 0 & 2s+3 \\ s & -s+4 \end{bmatrix}, \quad \text{(d)} \left(\frac{s+1}{s^2}\right),$$

where $j = \sqrt{-1}$.

2.7. Let V and W be vector spaces over the same field F and let $\mathscr{A}: V \to W$ be a linear transformation. Show that if $\{\mathscr{A}v^1,\ldots,\mathscr{A}v^n\}$ is a linearly independent set, then so is the set $\{v^1,\ldots,v^n\}$. Give an example to show that the converse of this statement is not true.

2.8. Let V and W be vector spaces over the same field F and let $\mathscr{A}: V \to W$ be a linear transformation. Show that $\mathscr{A}$ is a one-to-one mapping if and only if $\mathscr{N}(\mathscr{A}) = \{0\}$.

2.9. Let $\mathscr{C} \triangleq [B, AB, \ldots, A^{n-1}B]$ and

$$\mathscr{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$.

(a) Prove that if $\eta^1 \in \mathscr{N}(\mathscr{O})$, then $A\eta^1 \in \mathscr{N}(\mathscr{O})$. ($\eta^1$ denotes the coordinate representation of a vector $v^1 \in R^n$ with respect to the natural basis $\{e_1,\ldots,e_n\}$.)

(b) Prove that if $\eta^1 \in \mathscr{R}(\mathscr{C})$, then $A\eta^1 \in \mathscr{R}(\mathscr{C})$.

The above shows that $\mathscr{N}(\mathscr{O})$ and $\mathscr{R}(\mathscr{C})$ are invariant vector spaces under a transformation $\mathscr{A}$ that is represented by the matrix A.

2.10. Show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\Delta = ad - bc \neq 0$.

2.11. Determine the determinant, the (classical) adjoint, and the inverse of the matrix

$$A = \begin{bmatrix} \dfrac{s^2-3}{s} & 4s+3 \\ \dfrac{1}{s^2-2} & 3 \end{bmatrix}.$$

2.12. Determine the matrix X in $\begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & X \\ O & D^{-1} \end{bmatrix}$, where it is assumed that A and D are nonsingular. Also, determine the matrix $\begin{bmatrix} A & O \\ C & D \end{bmatrix}^{-1}$.

2.13. (a) Show that $det\begin{bmatrix} A & O \\ C & D \end{bmatrix} = (det\,A)(det\,D)$, where A and D are square matrices.

Hint: For D nonsingular, use the identity $\begin{bmatrix} A & O \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \\ O & D \end{bmatrix}\begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix}$.

(b) If A is nonsingular, show that

$$det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (det\,A)det\,(D - CA^{-1}B).$$

Hint: Note that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \\ O & I \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} I & O \\ -C & I \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ O & D - CA^{-1}B \end{bmatrix}$.

(c) In part (b), derive an expression for the case when it is known only that D is nonsingular.

2.14. Show that $e^{(A_1+A_2)t} = e^{A_1 t}e^{A_2 t}$ if $A_1A_2 = A_2A_1$.

2.15. Determine the characteristic and the minimal polynomials of the matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = I_4.$$

Hint: These matrices are in Jordan canonical form.

2.16. Determine the Jordan canonical form of the matrices

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

2.17. Show that there exists a similarity transformation matrix P such that

$$PAP^{-1} = A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}$$

if and only if there exists a vector $b \in R^n$ such that the rank of $[b, Ab, \ldots, A^{n-1}b]$ is n, i.e., $\rho[b, Ab, \ldots, A^{n-1}b] = n$.

2.18. Show that if λ_i is an eigenvalue of the companion matrix A_c given in Exercise 2.17, then a corresponding eigenvector is $v^i = (1, \lambda_i, \ldots, \lambda_i^{n-1})^T$.

2.19. Let λ_i be an eigenvalue of a matrix A and let v^i be a corresponding eigenvector. Let $f(\lambda) = \sum_{k=0}^{l} \alpha_k \lambda^k$ be a polynomial with real coefficients. Show that $f(\lambda_i)$ is an eigenvalue of the matrix function $f(A) = \sum_{k=0}^{l} \alpha_k A^k$. Determine an eigenvector corresponding to $f(\lambda_i)$.

2.20. For the matrices

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

determine the matrices A_1^{100}, A_2^{100}, $e^{A_1 t}$, and $e^{A_2 t}$, $t \in R$.

2.21. Determine some bases for the range and null spaces of the matrices

$$A_1 = [1 \quad 0 \quad 1], \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

2.22. Determine all solutions of the equation $A\eta = v$, where

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 4 & -1 \\ 2 & 0 & 2 & 0 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

2.23. Let $\phi_1(t) = e^{-t}$ for $t \in [-1, 1]$ and let

$$\phi_2(t) = \begin{cases} e^t, & t \in [-1, 0], \\ e^{-t}, & t \in [0, 1]. \end{cases}$$

Show that ϕ_1 and ϕ_2 are linearly independent over the field of the real numbers on $[-1, 1]$, but not on $[0, 1]$.

Remark: This example illustrates the fact that linear independence of time functions over a time interval $[a, b]$ does not necessarily imply linear independence over a time subinterval $[a', b'] \subset [a, b]$.

2.24. Show that if two time functions $\phi_1(t)$, $\phi_2(t)$ are linearly independent over a field F on a time interval $[a, b]$, then they are linearly independent over F on any interval that contains $[a, b]$. Give a specific example.

2.25. Prove that for $A \in C[R, R^{n\times n}]$, (3.14) is true if and only if (3.21) is true for all $t, \tau \in R$.

2.26. Determine the state transition matrix $\Phi(t, t_0)$ for (LH) with

$$A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$$

by (a) directly solving differential equations, (b) using the Peano-Baker series, and (c) using (3.15).

2.27. Determine the state transition matrix $\Phi(t, t_0)$ for (LH) with $A(t) = \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix}$ and determine in this case the solution for (LH) when $x(1) = (1, 1)^T$.

2.28. Verify that $\phi_1(t) = (1/t^2, -1/t)^T$ and $\phi_2(t) = (2/t^3, -1/t^2)^T$ are two solutions of (LH) with

$$A(t) = \begin{bmatrix} -\frac{4}{t} & -\frac{2}{t^2} \\ 1 & 0 \end{bmatrix}.$$

(a) Determine the state transition matrix $\Phi(t, \tau)$ for this system.
(b) Determine a solution ϕ for this system that satisfies the initial conditions $x(1) = (1, 1)^T$.

2.29. Given is the system of first-order ordinary differential equations $\dot{x} = t^2 Ax$, where $A \in R^{n\times n}$ and $t \in R$. Determine the state transition matrix $\Phi(t, t_0)$. Apply your answer to the specific case when $t^2 A = \begin{bmatrix} t^2 & 0 \\ 2t^2 & -t^2 \end{bmatrix}$.

2.30. Show that the two linear systems

$$\dot{x}^{(1)} = \begin{bmatrix} 0 & 1 \\ 2 - t^2 & 2t \end{bmatrix} x^{(1)} \triangleq A_1(t)x^{(1)}$$

and

$$\dot{x}^{(2)} = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} x^{(2)} \triangleq A_2(t)x^{(2)}$$

are equivalent state-space representations of the differential equation

$$\ddot{y} - 2t\dot{y} - (2 - t^2)y = 0.$$

(a) For which choice is it easier to compute the state transition matrix $\Phi(t, t_0)$? For this case, compute $\Phi(t, 0)$.
(b) Determine the relation between $x^{(1)}$ and y and between $x^{(2)}$ and y.

2.31. Using the Peano-Baker series, show that when $A(t) = A$, then $\Phi(t, t_0) = e^{A(t-t_0)}$.

2.32. For (LH) with $A(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}$, determine $\lim_{t\to\infty} \phi(t, t_0, x_0)$ if $x(0) = (0, 1)^T$. This example shows that an attempt of trying to extend the concept of eigenvalue from a constant matrix A to a time-varying matrix $A(t)$, for the purpose of characterizing the asymptotic behavior of time-varying systems (LH), will in general not work.

2.33. For the system

$$\dot{x} = A(t)x + B(t)u, \tag{11.1}$$

where all symbols are as defined in (6.1a), derive the *variation of constants formula* (3.10), using the change of variables $z(t) = \Phi(t_0, t)x(t)$.

2.34. For (11.1) with $x(t_0) = x_0$, show under what conditions it is possible to determine $u(t)$ so that $\phi(t, t_0, x_0) = x_0$ for all $t \geq t_0$. Use your result to find such $u(t)$ for the particular case $\dot{x} = x + e^{-t}u$.

2.35. Show that $(\partial/\partial\tau)\Phi(t, \tau) = -\Phi(t, \tau)A(\tau)$ for all $t, \tau \in R$.

2.36. Determine the state transition matrix $\Phi(t, t_0)$ for the system of equations $\dot{x} = e^{-At}Be^{At}x$, where $A \in R^{n\times n}$ and $B \in R^{n\times n}$. Investigate the case when in particular $AB = BA$.

2.37. The *adjoint equation* of (LH) is given by

$$\dot{z} = -A(t)^T z. \tag{11.2}$$

Let $\Phi(t, t_0)$ and $\Phi_a(t, t_0)$ denote the state transition matrices of (LH) and its adjoint equation, respectively. Show that $\Phi_a(t, t_0) = [\Phi(t_0, t)]^T$.

2.38. Consider the system described by

$$\dot{x} = A(t)x + B(t)u \tag{11.3a}$$

$$y = C(t)x, \tag{11.3b}$$

where all symbols are as in (6.1a), (6.1b) with $D(t) \equiv 0$, and consider the *adjoint equation* of (11.3a), (11.3b), given by

$$\dot{z} = -A(t)^T z + C(t)^T v \tag{11.4a}$$

$$w = B(t)^T z. \tag{11.4b}$$

(a) Let $H(t, \tau)$ and $H_a(t, \tau)$ denote the impulse response matrices of (11.3a), (11.3b) and (11.4a), (11.4b), respectively. Show that at the times when the impulse responses are nonzero, they satisfy $H(t, \tau) = H_a(\tau, t)^T$.

(b) If $A(t) \equiv A$, $B(t) \equiv B$, and $C(t) \equiv C$, show that $H(s) = -H_a(-s)^T$, where $H(s)$ and $H_a(s)$ are the transfer matrices of (11.3a), (11.3b) and (11.4a), (11.4b), respectively.

2.39. Show that if for (LH),

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix},$$

where $A_{11}(t)$, $A_{12}(t)$, and $A_{22}(t)$ are submatrices of appropriate dimensions, then

$$\Phi(t, t_0) = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ 0 & \Phi_{22}(t, t_0) \end{bmatrix},$$

where $\Phi_{ii}(t)$ satisfies the matrix equation $(\partial/\partial t)\Phi_{ii}(t,t_0) = A_{ii}(t)\Phi_{ii}(t,t_0)$ and where the matrix $\Phi_{12}(t,t_0)$ satisfies the equation $(\partial/\partial t)\Phi_{12}(t,t_0) = A_{11}(t)\Phi_{12}(t,t_0) + A_{12}(t)\Phi_{22}(t,t_0)$ with $\Phi_{12}(t_0,t_0) = O$.

Use the above result to determine the state transition matrix $\Phi(t,0)$ for

$$A(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}.$$

2.40. Compute e^{At} for

$$A = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

2.41. Given is the matrix

$$A = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(a) Determine e^{At}, using the different methods covered in this text. Discuss the advantages and disadvantages of these methods.

(b) For system (L) let A be as given. Plot the components of the solution $\phi(t,t_0,x_0)$ when $x_0 = x(0) = (1,1,1)^T$ and $x_0 = x(0) = (\frac{2}{3},1,0)^T$. Discuss the differences in these plots, if any.

2.42. Show that for $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, we have $e^{At} = e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$.

2.43. Given is the system of equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

with $x(0) = (1,0)^T$ and

$$u(t) = p(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Plot the components of the solution of ϕ. For different initial conditions $x(0) = (a,b)^T$, investigate the changes in the asymptotic behavior of the solutions.

2.44. The system (L) with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is called the *harmonic oscillator* (refer to Chapter 1) because it has periodic solutions $\phi(t) = (\phi_1(t), \phi_2(t))^T$. Simultaneously, for the same values of t, plot $\phi_1(t)$ along the horizontal axis and $\phi_2(t)$ along the vertical axis in the x_1-x_2 plane to obtain a *trajectory* for this system for the specific initial condition $x(0) = x_0 = (x_1(0), x_2(0))^T = (1,1)^T$. In plotting such trajectories, time t is viewed as a parameter, and arrows are used to indicate increasing time. When the horizontal axis corresponds to position and the vertical axis corresponds to velocity, the x_1-x_2 plane is called the *phase plane* and ϕ_1, ϕ_2 (resp. x_1, x_2) are called *phase variables*.

2.45. There are various ways of obtaining the coefficients $\alpha_i(t)$ given in (2.101). One of these was described in Subsection 2.2J. In the following, we present another method.

We consider the relation $(d/dt)e^{At} = Ae^{At}$ and we use (2.101) to obtain

$$\frac{d}{dt}\sum_{j=0}^{n-1}\alpha_j(t)A^j = A\sum_{j=0}^{n-2}\alpha_j(t)A^j + \alpha_{n-1}(t)[-(a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)], \tag{11.5}$$

where the Cayley-Hamilton Theorem was used. The coefficients $\alpha_i(t)$ that satisfy this relation generate a matrix $\Phi = \Sigma\alpha_j(t)A^j$ that satisfies the equation $\dot{\Phi} = A\Phi$. For Φ to equal e^{At}, we also require that $\Phi(0) = \Sigma\alpha_j(0)A^j = I$ (why?).

(a) Show that the $\alpha_j(t)$ can be generated as solutions of the system of equations

$$\begin{bmatrix} \dot{\alpha}_0(t) \\ \dot{\alpha}_1(t) \\ \vdots \\ \dot{\alpha}_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \vdots \\ \alpha_{n-1}(t) \end{bmatrix} \tag{11.6}$$

with $\alpha_0(0) = 1, \alpha_j(0) = 0, j \geq 1$. Also, show that the $\alpha_j(t)$ generated via (11.6) are linearly independent.

(b) Express the solution of the equation

$$\dot{x} = Ax + Bu, \tag{11.7}$$

where all symbols are as defined in (6.8a) and $x(0) = x_0$, in terms of $\alpha_j(t)$. Also, show that for $x(0) = x_0 = 0, \phi(t,0,0) = \phi(t) = \Sigma_{j=0}^{n-1} A^j B w_j(t)$, where $w_j(t) = \int_0^t \alpha_j(t-\tau)u(\tau)d\tau$.

2.46. First, determine the solution ϕ of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x(0) = (1,1)^T$. Next, determine the solution ϕ of the above system for $x(0) = \alpha(1,-1)^T, \alpha \in R, \alpha \neq 0$, and discuss the properties of the two solutions.

2.47. In Subsection 2.4C it is shown that when the n eigenvalues λ_i of a real $n \times n$ matrix A are distinct, then $e^{At} = \Sigma_{i=1}^n A_i e^{\lambda_i t}$, where $A_i = \lim_{s \to \lambda_i}[(s-\lambda_i)(sI-A)^{-1}] = v_i\tilde{v}_i$ [refer to (4.39), (4.40), and (4.43)], where $v_i, \tilde{v}_i$ are the right and left eigenvectors of A, respectively, corresponding to the eigenvalue λ_i. Show that (a) $\Sigma_{i=1}^n A_i = I$, where I denotes the $n \times n$ identity matrix, (b) $AA_i = \lambda_i A_i$, (c) $A_i A = \lambda_i A_i$, (d) $A_i A_j = \delta_{ij} A_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

2.48. Show that two state-space representations $\{A,B,C,D\}$ and $\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$ are zero-state equivalent if and only if $CA^kB = \tilde{C}\tilde{A}^k\tilde{B}, k = 0,1,2,\ldots$, and $D = \tilde{D}$.

2.49. Find an equivalent time-invariant representation for the system described by the scalar differential equation $\dot{x} = \sin 2tx$.

2.50. Consider the system

$$\dot{x} = Ax + Bu \tag{11.7a}$$

$$y = Cx, \tag{11.7b}$$

where all symbols are defined as in (6.8a), (6.8b) with $D = 0$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = [1,0,1,0]. \tag{11.8}$$

(a) Find equivalent representations for system (11.7a), (11.7b), (11.8), given by

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \tag{11.9a}$$

$$y = \tilde{C}\tilde{x}, \tag{11.9b}$$

where $\tilde{x} = Px$, when $\tilde{A}$ is in (i) the Jordan canonical (or diagonal) form, and (ii) the companion form.

(b) Determine the transfer function matrix for this system.

2.51. Consider the system (11.7a), (11.7b) with $B = 0$.

(a) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad C = [1,1,1].$$

If possible, select $x(0)$ in such a manner so that $y(t) = te^{-t}, t \geq 0$.

(b) Determine conditions under which it is possible to assign $y(t), t \geq 0$, using only the initial data $x(0)$.

2.52. Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} u, \qquad y = [1,0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Determine $x(0)$ so that for $u(t) = e^{-4t}, y(t) = ke^{-4t}$, where k is a real constant. Determine k for the present case. Notice that $y(t)$ does not have any transient components.

(b) Let $u(t) = e^{\alpha t}$. Determine $x(0)$ that will result in $y(t) = ke^{\alpha t}$. Determine the conditions on α for this to be true. What is k in this case?

2.53. Consider the system (11.7a), (11.7b) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & 0 & -3 & 1 \\ -1 & 1 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) For $x(0) = [1,1,1,1]^T$ and for $u(t) = [1,1]^T$, $t \geq 0$, determine the solution $\phi(t,0,x(0))$ and the output $y(t)$ for this system and plot the components $\phi_i(t,0,x(0)), i = 1,2,3,4$ and $y_i(t), i = 1,2$.

(b) Determine the transfer function matrix $H(s)$ for this system.

2.54. Consider the system

$$x(k+1) = Ax(k) + Bu(k) \tag{11.10a}$$

$$y(k) = Cx(k), \tag{11.10b}$$

where all symbols are defined as in (7.2a), (7.2b) with $D = 0$. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = [1,1],$$

and let $x(0) = 0$ and $u(k) = 1, k \geq 0$.

(a) Determine $\{y(k)\}, k \geq 0$, by working in the (i) time domain, and (ii) z-transform domain, using the transfer function $H(z)$.

(b) If it is known that when $u(k) = 0$, then $y(0) = y(1) = 1$, can $x(0)$ be uniquely determined? If your answer is affirmative, determine $x(0)$.

2.55. Consider $\hat{y}(z) = H(z)\hat{u}(z)$ with transfer function $H(z) = 1/(z + 0.5)$.

(a) Determine and plot the unit pulse response $\{h(k)\}$.

(b) Determine and plot the unit step response.

(c) If

$$u(k) = \begin{cases} 1, & k = 1,2, \\ 0, & \text{elsewhere,} \end{cases}$$

determine $\{y(k)\}$ for $k = 0,1,2,3,$ and 4 via (i) convolution, and (ii) the z-transform. Plot your answer.

(d) For $u(k)$ given in (c), determine $y(k)$ as $k \to \infty$.

2.56. Consider the system (11.10a) with $x(0) = x_0$ and $k \geq 0$. Determine conditions under which there exists a sequence of inputs so that the state remains at x_0, i.e., so that $x(k) = x_0$ for all $k \geq 0$. How is this input sequence determined? Apply your method to the specific case

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

2.57. For system (7.7) with $x(0) = x_0$ and $k \geq 0$, it is desired to have the state go to the zero state for any initial condition x_0 in at most n steps, i.e., we desire that $x(k) = 0$ for any $x_0 = x(0)$ and for all $k \geq n$.

(a) Derive conditions in terms of the eigenvalues of A under which the above is true. Determine the minimum number of steps under which the above behavior will be true.

(b) For part (a), consider the specific cases

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hint: Use the Jordan canonical form for A. Results of this type are important in *dead-beat control*, where it is desired that a system variable attain some desired value and settle at that value in a finite number of time steps.

2.58. Consider the system representations given by

$$x(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k),$$

$$y(k) = [1, 1]x(k) + u(k)$$

and

$$\tilde{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [1, 0]\tilde{x}(k).$$

Are these representations equivalent? Are they zero-input equivalent?

2.59. For the Jordan block given by

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_i \end{bmatrix},$$