is a Hurwitz polynomial by (i) solving the equation $f(s) = 0$, (ii) applying Theorem 6.1, (iii) applying Theorem 6.2, and (iv) applying Theorem 6.4.

6.9. Let $A \in C[R^+, R^{n\times n}]$ and $x \in R^n$ and consider

$$\dot{x} = A(t)x. \tag{LH}$$

Show that the equilibrium $x_e = 0$ of (LH) is *uniformly stable* if there exists a $Q \in C^1[R^+, R^{n\times n}]$ such that $Q(t) = [Q(t)]^T$ for all t and if there exist constants $c_2 \geq c_1 > 0$ such that

$$c_1 I \leq Q(t) \leq c_2 I, \qquad t \in R, \tag{14.4}$$

and such that

$$[A(t)]^T Q(t) + Q(t)A(t) + \dot{Q}(t) \leq 0, \qquad t \in R, \tag{14.5}$$

where I is the $n \times n$ identity matrix. *Hint:* The proof of this assertion is similar to the proof of Theorem 7.1.

6.10. Show that the equilibrium $x_e = 0$ of (LH) is *exponentially stable* if there exists a $Q \in C^1[R^+, R^{n\times n}]$ such that $Q(t) = [Q(t)]^T$ for all t and if there exist constants $c_2 \geq c_1 > 0$ and $c_3 > 0$ such that (14.4) holds and such that

$$[A(t)]^T Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -c_3 I, \qquad t \in R. \tag{14.6}$$

Hint: The proof of this assertion is similar to the proof of Theorem 7.2.

6.11. Assume that the equilibrium $x_e = 0$ of (LH) is *exponentially stable* and that there exists a constant $a > 0$ such that $\|A(t)\| \leq a$ for all $t \in R$. Show that the matrix given by

$$Q(t) = \int_t^\infty [\Phi(\tau, t)]^T \Phi(\tau, t)\, d\tau \tag{14.7}$$

satisfies the hypotheses of the result given in Exercise 10. *Hint:* The proof of this assertion is similar to the proof of Theorem 7.5.

6.12. For (LH) let $\lambda_m(t)$ and $\lambda_M(t)$ denote the smallest and largest eigenvalues of $A(t) + [A(t)]^T$ at $t \in R$, respectively. Let $\phi(t, t_0, x_0)$ denote the unique solution of (LH) for the initial data $x(t_0) = x_0 = \phi(t_0, t_0, x)$. Show that for any $x_0 \in R^n$ and any $t_0 \in R$, the unique solution of (LH) satisfies the estimate,

$$\|x_0\| e^{(1/2)\int_{t_0}^t \lambda_m(\tau)d\tau} \leq \|\phi(t, t_0, x_0)\| \leq \|x_0\| e^{(1/2)\int_{t_0}^t \lambda_M(\tau)d\tau}, \qquad t \geq t_0. \tag{14.8}$$

Hint: Let $v(t, t_0, x_0) = [\phi(t, t_0, x_0)]^T[\phi(t, t_0, x_0)] = \|\phi(t, t_0, x_0)\|^2$, evaluate $\dot{v}(t, t_0, x_0)$, and then establish (14.8).

6.13. Use Exercise 6.12 to show that the equilibrium $x_e = 0$ of (LH) is *uniformly stable* if there exists a constant c such that

$$\int_\sigma^t \lambda_M(\tau)\, d\tau \leq c \tag{14.9}$$

for all t, σ such that $t \geq \sigma$, where $\lambda_M(t)$ denotes the largest eigenvalue of $A(t) + [A(t)]^T$, $t \in R$. *Hint:* Use (14.8) and the definition of uniform stability.

6.14. Use Exercise 6.12 to show that the equilibrium $x_e = 0$ of (LH) is *exponentially stable* if there exist constants $\epsilon > 0$, $\alpha > 0$ such that

$$\int_\sigma^t \lambda_M(\tau)\, d\tau \leq -\alpha(t - \sigma) + \epsilon \tag{14.10}$$

for all t, σ such that $t \geq \sigma$. *Hint.* Use (14.8) and the definition of exponential stability.

6.15. Let v be a quadratic function of the form

$$v(x,t) = x^T Q(t)x, \tag{14.11}$$

where $x \in R^n$, $Q \in C^1[R, R^{n\times n}]$, $Q(t) = [Q(t)]^T$, and $Q(t) \leq kI$, $k > 0$, for all $t \in R$. Evaluate the derivative of v with respect to t, along the solutions of (LH), to obtain

$$\dot{v}_{(LH)}(x,t) = x^T[[A(t)]^T Q(t) + Q(t)A(t) + \dot{Q}(t)]x. \tag{14.12}$$

Assume that there is a quadratic form $w(x) = x^T W x \leq 0$, where $W^T = W \in R^{n\times n}$, such that

$$\dot{v}_{(LH)}(x,t) \leq w(x) \tag{14.13}$$

for all $(x,t) \in G \times R$, where G is a closed and bounded subset of R^n. Let

$$E = \{x \in G : w(x) = 0\} \tag{14.14}$$

and assume that for (LH), $\|A(t)\|$ is bounded on R. Prove that any solution of (LH) that remains in G for all $t > t_0 \geq 0$ approaches E as $t \to \infty$.

6.16. Consider the system

$$\ddot{x} + a(t)\dot{x} + x = 0,$$

which by letting $x_1 = x$ and $x_2 = \dot{x}$ can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a(t)x_2 - x_1.$$

Assume that $a \in C[R, R^+]$ and that there are constants c_1, c_2 such that $0 < c_1 \leq a(t) \leq c_2$ for all $t \in R$. Let $v(x) = x_1^2 + x_2^2$. First, show that all solutions of this system are bounded. Next, use the results of Exercise 6.15 to show that $\phi_2(t, t_0, x_0) \to 0$ as $t \to \infty$.

6.17. Assume that for system (LH) there exists a quadratic function of the form $v(x,t) = x^T Q(t)x$, where $Q(t) = [Q(t)]^T \in C^1[R, R^{n\times n}]$ and $Q(t) \geq cI$ for some $c > 0$, such that $\dot{v}_{(LH)}(x,t) \leq x^T W x$, where $W = W^T \in R^{n\times n}$ is negative definite. Show that if v is negative for some (x,t), then the equilibrium $x_e = 0$ of system (LH) is *unstable*. *Hint.* The proof of this assertion follows along similar lines as the proof of Theorem 7.3.

6.18. It is shown that if the equilibrium $x_e = 0$ of system (LH) is exponentially stable, then there exists a function v that satisfies the requirements of the result given in Exercise 6.10, i.e., the present result is a *converse theorem* to the result given in Exercise 6.10.

In system (LH), let A be bounded for all $t \in R$, let $L = L^T \in C^1[R, R^{n\times n}]$ and assume that L is bounded for all $t \in R$. Show that the integral

$$Q(t) = \int_t^\infty [\Phi(\sigma,t)]^T L(\sigma)\Phi(\sigma,t)\,d\sigma$$

exists for all $t \in R$. Show that the derivative of the function

$$v(x,t) = x^T Q(t)x \tag{14.15}$$

with respect to t along the solutions of (LH) is given by

$$\dot{v}_{(LH)}(x,t) = -x^T L(t)x.$$

Next, show that if $L(t) \geq c_3 I$, $c_3 > 0$, for all $t \in R$, then there exist constants $c_2 \geq c_1 > 0$ such that for all $t \in R$,

$$c_1 I \leq Q(t) \leq c_2 I, \tag{14.16}$$

where $I \in R^{n\times n}$ denotes the identity matrix.

Note that the above result constitutes a generalization to Theorem 7.5 for time-invariant systems (L).

6.19. Apply Proposition 7.1 to determine the definiteness properties of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -1 & 10 \end{bmatrix}.$$

6.20. Use Theorem 7.3 to prove that the trivial solution of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is unstable.

6.21. Determine the equilibrium points of a system described by the differential equation

$$\dot{x} = -x + x^2$$

and determine the stability properties of the equilibrium points, if applicable, by using Theorem 8.1 or 8.2.

6.22. The system described by the differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2) \end{aligned} \tag{14.17}$$

has an equilibrium at the origin $x^T = (x_1, x_2) = (0, 0)$. Show that the trivial solution of the *linearization* of system (14.17) is stable. Prove that the equilibrium $x = 0$ of system (14.17) is unstable. (This example shows that the assumptions on the matrix A in Theorems 8.1 and 8.2 are absolutely essential.)

6.23. Prove that the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -t & 0 \\ (2t - t) & -2t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} u(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is BIBO stable.

6.24. Use Theorem 9.3 to analyze the stability properties of the system given by

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad C = [0, 1].$$

6.25. Determine all equilibrium points for the discrete-time systems given by

(a)
$$\begin{aligned} x_1(k+1) &= x_2(k) + |x_1(k)| \\ x_2(k+1) &= -x_1(k) + |x_2(k)| \end{aligned}$$

(b)
$$x_1(k+1) = x_1(k)x_2(k) - 1$$
$$x_2(k+1) = 2x_1(k)x_2(k) + 1.$$

6.26. Consider the discrete-time system given by

$$x(k+1) = sat[Ax(k)] \tag{14.18}$$

where for $\theta = (\theta_1, \ldots, \theta_n)^T \in R^n$, $sat\,\theta = [sat\,\theta_1, \ldots, sat\,\theta_n]^T$, and

$$sat\,\theta_i = \begin{cases} 1, & \theta_i > 1, \\ \theta_i, & |\theta_i| \le 1, \\ -1, & \theta_i < 1. \end{cases}$$

(a) For $A \in R^{n\times n}$ arbitrary, use Theorem 10.1 to analyze system (14.18).
(b) Imposing various restrictions on the locations of the eigenvalues of A in the complex plane, use as many results of this chapter as you can to analyze the stability properties of the trivial solution of system (14.18).

6.27. Determine the stability properties of the trivial solution of the discrete-time system given by the equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

with θ fixed.

6.28. Analyze the stability of the equilibrium $x = 0$ of the system described by the scalar-valued difference equation

$$x(k+1) = \sin[x(k)].$$

6.29 Analyze the stability of the equilibrium $x = 0$ of the system described by the difference equations

$$x_1(k+1) = x_1(k) + x_2(k)[x_1(k)^2 + x_2(k)^2]$$
$$x_2(k+1) = x_2(k) - x_1(k)[x_1(k)^2 + x_2(k)^2].$$

6.30. Determine a basis of the solution space of the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Use your answer in analyzing the stability of the trivial solution of this system.

6.31. Let $A \in R^{n\times n}$. Prove that part (iii) of Theorem 10.8 is equivalent to the statement that all eigenvalues of A have modulus less than 1, i.e.,

$$\lim_{k\to\infty} \|A^k\| = 0$$

if and only if for any eigenvalue λ of A, it is true that $|\lambda| < 1$.

6.32. Use Theorem 10.7 to show that the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} x(k)$$

is unstable.

6.33. (a) Use Theorem 10.9 to determine the stability of the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 9 & -1 \end{bmatrix} x(k).$$

(b) Use Theorem 10.9 to determine the stability of the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 9 & -1 \end{bmatrix} x(k).$$

6.34. Apply the Schur-Cohn criterion (Theorem 10.10) in analyzing the stability of the trivial solution of the system given by the equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}.$$

6.35. Apply Theorems 7.2 and 10.11 to show that if the equilibrium $x = 0$ $(x \in R^n)$ of the system

$$x(k+1) = e^A x(k)$$

is asymptotically stable, then the equilibrium $x = 0$ of the system

$$\dot{x} = Ax$$

is also asymptotically stable.

6.36. Apply Theorem 10.11 to show that the trivial solution of the system given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

is unstable.

6.37. Determine the stability of the equilibrium $x = 0$ of the scalar-valued system given by

$$x(k+1) = \tfrac{1}{2}x(k) + \tfrac{2}{3}\sin x(k).$$

6.38. Analyze the stability properties of the discrete-time system given by

$$x(k+1) = x(k) + \tfrac{1}{2}u(k)$$
$$y(k) = \tfrac{1}{2}x(k)$$

where x, y, and u are scalar-valued variables. Is this system BIBO stable? Can Theorem 10.16 be applied in the analysis of this system?

CHAPTER 7

Polynomial Matrix Descriptions and Matrix Fractional Descriptions of Systems

In this chapter, representations of linear time-invariant systems based on polynomial matrices, called *Polynomial Matrix Description* (*PMD*) or *Differential* (*Difference*) *Operator Representation* (*DOR*) are introduced. Such representations arise naturally when differential (or difference) equations of order higher than one are used to describe the behavior of systems, and the differential (or difference) operator is introduced to represent the operation of differentiation (or of time-shift). Polynomial matrices in place of polynomials are involved since this approach is typically used to describe MIMO systems. Note that state-space system descriptions involve only first-order differential (or difference) equations, and as such, PMDs include the state-space descriptions as special cases.

A rational function matrix can be written as a ratio or fraction of two polynomial matrices or of two rational matrices. If the transfer function matrix of a system is expressed as a fraction of two polynomial or rational matrices, this leads to a *Matrix Fraction*(*al*) *Description* (*MFD*) of the system. The MFDs that involve polynomial matrices, called polynomial MFDs, can be viewed as representations of internal realizations of the transfer function matrix, i.e., as system PMDs of special form. These polynomial fractional descriptions (PMFD) help establish the relationship between internal and external system representations in a clear and transparent manner. This can be used to advantage, for example, in the study of feedback control problems, leading to clearer understanding of the phenomena that occur when systems are interconnected in feedback configurations. The MFDs that involve ratios of rational matrices, in particular, ratios of proper and stable rational matrices, offer convenient characterizations of transfer functions in feedback control problems.

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6.14 EXERCISES

6.1. Determine the set of equilibrium points of a system described by the differential equations

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 + x_3\\ \dot{x}_2 &= 2x_1 + 3x_2 + x_3\\ \dot{x}_3 &= 3x_1 + 2x_2 + 2x_3.\end{aligned}$$

6.2. Determine the set of equilibria of a system described by the differential equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \begin{cases} x_1 \sin\left(\dfrac{1}{x_1}\right), & \text{when } x_1 \neq 0,\\ 0, & \text{when } x_1 = 0.\end{cases}$$

6.3. Determine the equilibrium points and their stability properties of a system described by the ordinary differential equation

$$\dot{x} = x(x-1) \tag{14.1}$$

by solving (14.1) and then applying the definitions of stability, uniform stability, asymptotic stability, etc.

6.4. Determine the set of equilibria and their stability properties of a system described by the ordinary differential equation

$$\dot{x} = (cost)x \tag{14.2}$$

by solving (14.2) and then applying the definitions of stability, uniform stability, asymptotic stability, etc.

6.5. Determine the set of equilibria and their stability properties of a system described by the ordinary differential equation

$$\dot{x} = (4t\sin t - 2t)x \tag{14.3}$$

by solving (14.3) and then applying the definitions of stability, uniform stability, asymptotic stability, etc.

6.6. Determine the state transition matrix $\Phi(t, t_0)$ of the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-t & 0\\ (2t-t) & -2t\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

Use Theorems 5.1 to 5.4 to determine the stability properties of the trivial solution of this system.

6.7. Show that the second-degree polynomial

$$f(s) = s^2 + 2as + b$$

is a Hurwitz polynomial if and only if $a > 0$ and $b > 0$ by (i) solving the equation $f(s) = 0$, and (ii) using the Routh-Hurwitz criterion (Theorem 6.4).

6.8. Determine whether the third-degree polynomial

$$f(s) = s^3 + 3s^2 + 3s + 2$$