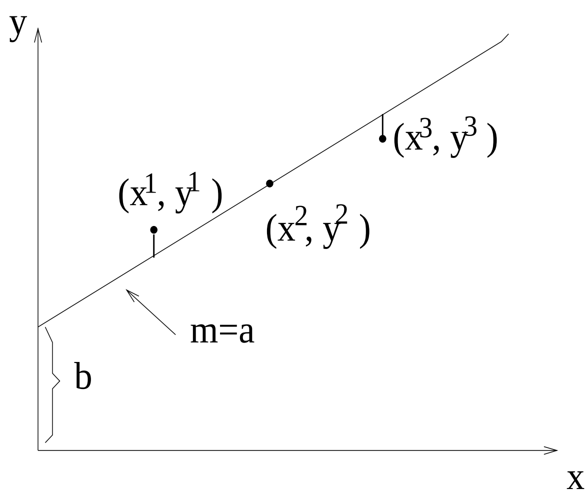
§5 Fuzzy Regression Models

Linear Regression Model

One factor

$$y = f(x) = ax + b$$



$$\underbrace{\varepsilon_j}_{\text{observation error}} \triangleq \underbrace{y^j}_{\text{data value}} - \underbrace{(ax^j + b)}_{\text{inferred value}}$$

Task: Find a, b such that a criterion

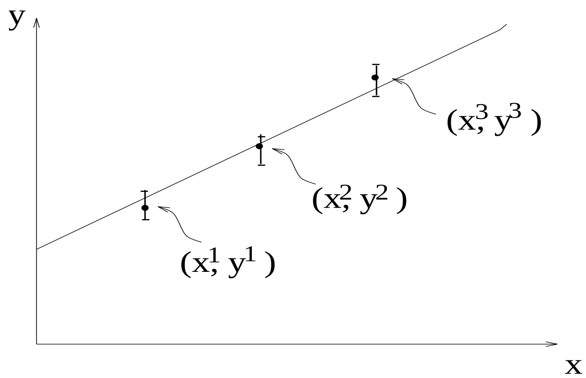
based on total observation errors is minimized.

Underlying Assumption:

The system is well-defined with no " vagueness ". Therefore, its output is crisply determined by a linear function and any deviation is caused by the " observation error" in data collection.

Fuzzy Regression Model:

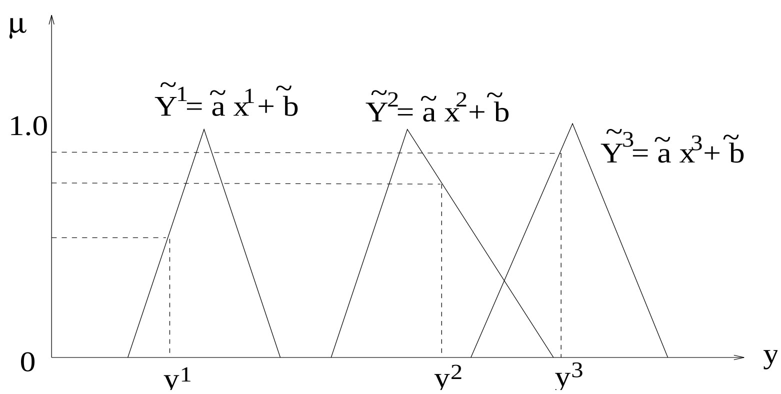
The underlying system has inherent vagueness which can cause many possible outputs to be observed. Our objective is to characterize this (linear) possibility system.



$$\tilde{Y} = \tilde{a}x + \tilde{b} \qquad\qquad y = ax + b$$

$$\downarrow \quad \downarrow$$

fuzzy fuzzy



Task: Find $\tilde{a}, \tilde{b}$ with minimal spreads in $\tilde{Y}^j$

such that

$$\mu_{\tilde{Y}^j}(y^j) \geq h \qquad j = 1, 2, 3$$

$$\uparrow$$

preassigned possibility

n-factor Linear Possibility System

$$\tilde{Y} = \tilde{A}_0 + \tilde{A}_1 x_1 + \cdots + \tilde{A}_n x_n$$

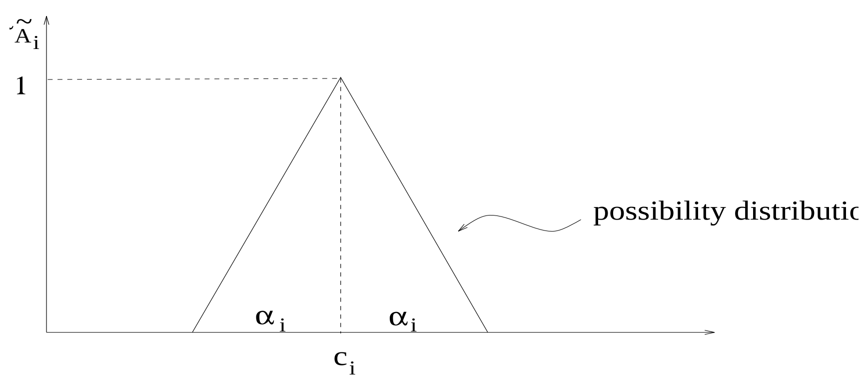
$\tilde{A}_i$: fuzzy number

$(m, \alpha, \beta)_{LR}$

For simplicity, we take

$$\left. \begin{array}{l} m = c_i \\ \alpha = \beta = \alpha_i > 0 \\ L = R \end{array} \right\} \mu_{\tilde{A}_i}(x) = L((x - c_i)/\alpha_i)$$

Example



$$\mu_{\tilde{A}_i}(x) = 1 - \frac{|\ x - c_i\ |}{\alpha_i}$$

In general:

(1) $L(x) = L(-x)$

(2) $L(0) = 1,\ L(1) = 0$

(3) $L(x)$ strictly decreasing

Example:

$$L_1(x) = \max\{0, 1- \mid x \mid^p\}, \quad \text{for } p > 0$$

$$L_2(x) = e^{-|x|p}, \quad \text{for } p > 0$$

$$L_3(x) = \frac{1}{1+ \mid x \mid^p}, \quad \text{for } p > 0$$

Properties:

(1) $(c_1, \alpha_1)_L + (c_2, \alpha_2)_L$

$$= (c_1 + c_2, \alpha_1 + \alpha_2)_L$$

(2) $(c_1, \alpha_1)_L - (c_2, \alpha_2)_L$

$$= (c_1 - c_2, \alpha_1 + \alpha_2)_L$$

(3) $\lambda \cdot (c_1, \alpha_1)_L = (\lambda c_1, \mid \lambda \mid \alpha_1)_L$

Theorem:

The membership function for the output of a linear possibility system

$$\tilde{Y} = \tilde{A}_0 + \tilde{A}_1 x_1 + \cdots + \tilde{A}_n x_n$$

with $\tilde{A}_i = (c_i, \alpha_i)_L$

is given by

$$\mu_{\tilde{Y}}(y) = L((y - \Sigma_{i=0}^n x_i c_i) / \Sigma_{i=0}^n \mid x_i \mid \alpha_i)$$

where $x_0 = 1$.

Observation:

· Possibility distribution

$$(c_1, \alpha_1)_L x_1 + \cdots + (c_n, \alpha_n)_L x_n = (c^t x, \alpha^t \mid x \mid)_L$$

· Probability distribution

$$N(e_1, \sigma_1^2) x_1 + \cdots + N(e_n, \sigma_n^2) x_n = N(e^t x, (\sigma^2)^t x^2)$$

where $\mid x \mid = (\mid x_1 \mid, \cdots, \mid x_n \mid)^t,\ x^2 = (x_1^2, \cdots, x_n^2)^t$

Linear Possibility Regression Model

Regular Data Set (n-factors, N points)

$$(x_1^1, x_2^1, \cdots, x_n^1) \longrightarrow y^1$$

$$(x_1^2, x_2^2, \cdots, x_n^2) \longrightarrow y^2$$

$$\vdots \qquad\qquad \vdots$$

$$(x_1^N, x_2^N, \cdots, x_n^N) \longrightarrow y^N$$

· Linear Possibility system

$$\tilde{Y}^j = \tilde{A}_0 + \tilde{A}_1 x_1^j + \cdots + \tilde{A}_n x_n^j, \; j = 1, \cdots, N$$

with $\tilde{A}_i = (c_i, \alpha_i)_L$, for $i = 1, 2, \cdots, n$.

· Given a required possibility level h,

we want to have

$$\mu_{\tilde{Y}^j}(y^j) \geq h, \quad \text{for } j = 1, \cdots, N$$

(The possibility of having observation y^j from the inferred system is higher than h.)

- The total spread of the system outputs $\{\tilde{Y}^j\}$

  is

$$J(\alpha) = \Sigma_{j=1}^{N}(\Sigma_{i=0}^{n} \mid x_i^j \mid \alpha_i)$$

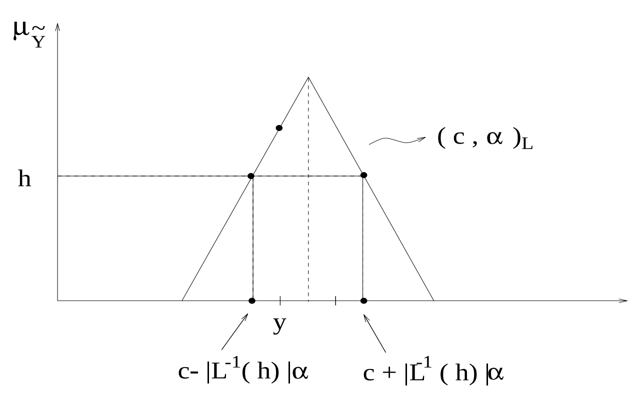
- Minimize$_{\alpha,c}$ $J(\alpha) = \Sigma_{j=1}^{N}(\Sigma_{i=0}^{n} \mid x_i^j \mid \alpha_i)$

$$\text{s.t.} \quad \mu_{\tilde{Y}^j}(y^j) \geq h, \quad j = 1, 2, \cdots, N$$

$$\alpha_i \geq 0, \quad i = 0, 1, \cdots, n$$

$$c_i \in \mathbf{R}, \quad i = 0, 1, \cdots, n$$

- Note that



· Linear Fuzzy Regression

$$\min \;\; J(\alpha) = \Sigma_{j=1}^{N}(\Sigma_{i=0}^{n} \mid x_i^j \mid \alpha_i)$$

s.t.

$$y^j \leq \Sigma_{i=0}^{n} x_i^j c_i + \mid L^{-1}(h) \mid (\Sigma_{i=0}^{n} \mid x_i^j \mid \alpha_i)$$
$$j = 1, 2, \cdots, N$$

$$y^j \geq \Sigma_{i=0}^{n} x_i^j c_i - \mid L^{-1}(h) \mid (\Sigma_{i=0}^{n} \mid x_i^j \mid \alpha_i)$$
$$j = 1, 2, \cdots, N$$

$$\alpha_i \geq 0, \quad i = 0, 1, \cdots, n$$

$$c_i \in \mathbf{R}, \quad i = 0, 1, \cdots, n$$

· LP with 2(n+1) variables ((n+1) free variables and (n+1) non-negative variables) and 2N explicit constraints.

Observations:

(1) For $h = 1.0$, the LP problem requires that

$$y^j = \Sigma_{i=0}^{n} x_j^i c_i, \quad j = 1, 2, \cdots, N$$

When $N > n + 1$, LP may have no solution.

(2) However, for any $h < 1.0$, if α_i is large enough, the LP problem is always feasible.

Theorem: Given data (y^j, x^j), for $0 \leq h < 1$, the LP problem has an optimal solution with

$$\tilde{A}_i^h = (c_i^h, \alpha_i^h)_L$$

Example: p.77, 78, 79, 80.

Theorem : Given $\tilde{A}_i^h = (c_i^h, \alpha_i^h)_L$ is known,

for $1 > h' \neq h$, we have

$$\tilde{A}_i^{h'} = (c_i^h, \frac{L^{-1}(h)}{L^{-1}(h')}\alpha_i^h)_L$$