Partial Differential Equations 3 – Static PDE Coursework

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1. The boundary condition on the inside and outside of the pipe are given by:

$$u(1, \theta) = 0, \quad 0 \leq \theta \leq 2\pi \tag{1}$$

$$\frac{\partial u}{\partial r}(3, \theta) = \begin{cases} 0, & 0 \leq \theta < \pi \\ 1, & \pi \leq \theta \leq 2\pi \end{cases} \tag{2}$$

The first boundary condition where $u = 0$, tells us that the temperature throughout the whole boundary of the inside pipe at radius = 1, is 0.

The second boundary conditions where $\frac{\partial u}{\partial r} = 0$, from 0 *to less than* π, informs us that the boundary of the outside wall of the upper part of the pipe is a perfectly insulated wall where there is no heat gained or loss throughout the whole boundary whereas when $\frac{\partial u}{\partial r} = 1$, from π *to* 2π tells us that the bottom part of the outside wall of the pipe is losing heat flux where the direction of the heat flux is normal to the domain.

2. We know that the basic solutions of cylindrical coordinate is given by:

$$\boldsymbol{\mu = 0} \qquad \boldsymbol{u(r, \theta) = (A\theta + B)(C \ln r + D)} \tag{3}$$

$$\boldsymbol{\mu^2 < 0} \qquad \boldsymbol{u(r, \theta) = (A \cosh(\mu\theta) + B \sinh(\mu\theta))(Cr^{-\mu} + Dr^{\mu})} \tag{4}$$

$$\boldsymbol{\mu^2 > 0} \qquad \boldsymbol{u(r, \theta) = (A \cos(\mu\theta) + B \cos(\mu\theta))(Cr^{-\mu} + Dr^{\mu})} \tag{5}$$

Since we are applying methods of seperation variable, we can generally write equation [3], [4] and [5] as

$$\boldsymbol{u(r, \theta) = \emptyset(\theta) R(r)} \tag{6}$$

Solving for equation [3] by applying the boundary conditions given from equation [1] and [2], we can see that;

$$\boldsymbol{\emptyset(\theta) = A\theta + B} \tag{7}$$

Since our solution needs to be periodic, and the relationship given from equation [7] will give a linear relationship for increasing θ, hence we have to let $A = 0$, hence we get $\emptyset(\theta) = B$.

$$\boldsymbol{R(1) = C \ln 1 + D = 0} \tag{8}$$

Applying the [1] boundary conditions, for equation [8], we get that the expression , $C \ln 1 = 0$, and this gives us that $D = 0$, however, we should not simply drop down the C term and make it equal to zero since coincidentally, $C \ln 1$ gives us zero. Therefore, we get $R(r) = C \ln r$. Substituting equation [7] and [8] into equation [6], we finally get that;

$$\mu = 0 \qquad u(r,\theta) = BC\ln r = B_n \ln r \tag{9}$$

Since our solution needs to be periodic, solving equation [4] is trivial. Here, we can see that it equation [4] consists of hyperbolic functions which cannot be used to model periodic behaviors, hence we can let $A = B = 0$, hence leaving us with $u_2(r,\theta) = 0$.

Last but not least, again, since our solution is periodic and equation [5] consists of sin and cosine functions, hence, it needs to be in our solution. We will first use equation [5] to solve for the first boundary condition where $u(1,\theta) = 0$. Substituting this boundary condition into equation [5], we get;

$$u(1,\theta) = (A\cos(\mu\theta) + B\cos(\mu\theta))(C + D) = 0$$

$$C = -D \tag{10}$$

Substituting equation [10] into equation [5], and expanding the expression, we can find that;

$$u(r,\theta) = (A_k\, cos\,(\mu\theta) + B_k\, cos\,(\mu\theta)\,)(r^{-n} - r^n) \tag{11}$$

where, $A_k = AC \; and \; B_k = BC$.

Superposing the solution we get from equation [9] and equation [11], we get the following equation;

$$u(r,\theta) = B_n \ln r + \sum_{n=1}^{\infty}[(A_k r^{-n} - A_k r^n)\cos(n\theta) + (B_k r^{-n} - B_k r^n)\sin(n\theta)] \tag{12}$$

Partially differentiating the following equation with respect to r, we obtain;

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{B_n}{r} + \sum_{n=1}^{\infty}[(-nA_k r^{-n-1} - nA_k r^{n-1})\cos(n\theta) + (-nB_k r^{-n-1} - nB_k r^{n-1})\sin(n\theta)] \tag{13}$$

From the second boundary condition given from equation [2], we obtain;

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{B_n}{3} + \sum_{n=1}^{\infty}[(-nA_k 3^{-n-1} - nA_k 3^{n-1})\cos(n\theta) + (-nB_k 3^{-n-1} - nB_k 3^{n-1})\sin(n\theta)]$$

Applying fourier series coefficient to the following above equation, we can determine the $a_o, a_n \; and \; b_n$ value which is given as below:

$$a_o = \frac{2}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\, d\theta \tag{14}$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\cos(n\theta)\, d\theta \tag{15}$$

$$\boldsymbol{b_n = \frac{2}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\sin(n\theta)\, d\theta} \tag{16}$$

Due to the fact that the boundary condition from $\mathbf{0}\ \boldsymbol{to}\ \boldsymbol{\pi}$ given from equation [2] is 0, we can futher simplify the above expression to

$$\boldsymbol{a_o = \frac{1}{\pi}\int_\pi^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\, d\theta} \tag{17}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_\pi^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\cos(n\theta)\, d\theta} \tag{18}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_\pi^{2\pi} \frac{\partial u}{\partial r}(r,\theta)\sin(n\theta)\, d\theta} \tag{19}$$

Solving the equation [17] should be trivial, and it is just a matter of integrating $\frac{\partial u}{\partial r}$ **from** $\boldsymbol{\pi\ to\ 2\pi}$, which we can simply obtain from the boundary conditions, hence;

$$\boldsymbol{\frac{B_n}{3} = \frac{1}{2}a_o}$$

Solving the above equation should also be trivial and we should obtain;

$$\boldsymbol{B_n = \frac{3}{2}} \tag{20}$$

Solving for equation [18];

$$\boldsymbol{(-nA_k 3^{-n-1} - nA_k 3^{n-1}) = \frac{\sin(2\pi n)}{\pi n} = 0 \quad for\ all\ n = 1,2,3\ \ldots.}$$

Therefore;

$$\boldsymbol{A_k = 0} \tag{21}$$

Solving for equation [19];

$$\boldsymbol{(-nB_k 3^{-n-1} - nB_k 3^{n-1}) = \begin{cases} \frac{-2}{\pi n}, & n\ odd \\ 0, & n\ even \end{cases}}$$

Therefore;

$$\boldsymbol{B_k = \frac{2}{\pi n^2(3^{-n-1} + 3^{n-1})}} \tag{22}$$

Substituting our coefficients from equation [20], [21] and [22] to our solution in equation [12], we should get;

$$\boldsymbol{u(r,\theta)=\frac{3}{2}\ln r+\sum_{k=1}^{\infty}\left[\frac{2}{\pi(2k-1)^2(3^{-2k}+3^{2k-2})}(r^{1-2k}-r^{2k-1})\sin((2k-1)\theta)\right]} \tag{23}$$