Laplace's Equation

A. The Laplacian

For a single-variable function $u = u(x)$, $u'(x)$ measures *slope* and $u''(x)$ measures *concavity* or *curvature*. When $u = u(x, y)$ depends on two variables, the *gradient* (a vector) and the *Laplacian* (a scalar) record the corresponding quantities:

$$\begin{aligned} \nabla u(x,y) &= \left(u_x(x,y),\, u_y(x,y)\right), && \text{(the gradient)} \\ \Delta u(x,y) &= u_{xx}(x,y) + u_{yy}(x,y). && \text{(the Laplacian).} \end{aligned}$$

The Laplacian operator Δ is important enough to deserve an intuitive understanding on its own. However, we'll still think of it as being somehow related to concavity/curvature. It certainly does the same job in problems of wave motion or heat flow in 2D domains, and similar extensions hold in 3 or more dimensions.

Wave Equation. For a 2D membrane stretched across a wire frame around a region Ω in the (x, y)-plane (like a drum head), the lateral displacement at point (x, y) and time t is a function $u = u(x, y, t)$ that obeys

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) = c^2 \Delta u, \qquad (x,y) \in \Omega,\ t > 0.$$

("Curvature drives acceleration.") [*Sketch something*.]

Heat Equation. For a 2D metal plate occupying the plane region Ω, sandwiched between insulation slabs on its flat sides so heat can only flow in the (x, y)-plane, the temperature at point (x, y) and time t is a function $u = u(x, y, t)$ that obeys

$$u_t = \alpha^2\left(u_{xx} + u_{yy}\right) = \alpha^2 \Delta u, \qquad (x,y) \in \Omega,\ t > 0$$

("Curvature drives flow rate.")

Laplace's Equation. Given an open set Ω in the (x, y)-plane, Laplace's Equation for $u = u(x, y)$ is

$$0 = \Delta u \stackrel{\text{def}}{=} u_{xx} + u_{yy}, \qquad (x,y) \in \Omega. \tag{$*$}$$

Applications:

(1) Steady-state temperature in a 2D region with fixed boundary temperatures. (The case of a 1D region is easy: $u''(x) = 0$ implies $u(x) = mx + c$. For a higher-dimensional region, however, a much greater variety of solutions is possible.)

(2) Potential fields (electrostatic, gravitational, etc.) in regions free from potential sources (charges, masses, etc.) obey Laplace's equation.

(3) Minimal surfaces (e.g., soap films in equilibrium) obey the nonlinear PDE

$$(1 + u_y^2)u_{xx} + 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \qquad (x,y) \in \Omega.$$

When ∇u is so small that terms of second and higher order in its components are negligible, this is *approximated by* Laplace's equation.

Boundary Conditions. Write Γ for the boundary curve of Ω.

(i) Dirichlet (prescribed function values): a function g is given, and we seek a function u obeying $(*)$ and

$$u(x,y) = g(x,y) \qquad \text{for } (x,y) \in \Gamma.$$

(ii) Neumann (prescribed directional derivatives normal to boundary): a function h is given and we seek u satisfying $(*)$ and

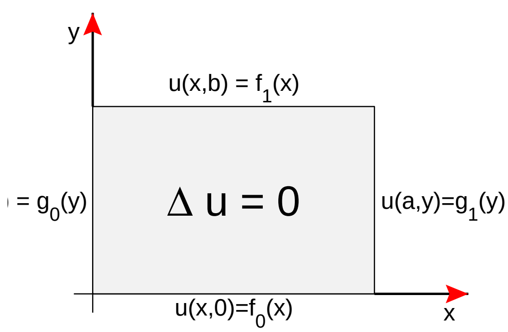
$$\nabla u(x,y) \bullet \widehat{\mathbf{N}}(x,y) = h(x,y) \quad \text{for } (x,y) \in \Gamma.$$

Here $\widehat{\mathbf{N}}(x,y)$ is the outward unit normal to the curve Γ at point (x,y).

(iii) Mixed (Dirichlet on some segments of Γ, Neumann on the rest).

B. Dirichlet Problem in a Rectangular Box

Consider $\Omega = \{(x,y) \,:\, 0 < x < a,\ 0 < y < b\}$. Here the boundary curve Γ consists of the four segments on the sides of the rectangle, and BC's of Dirichlet type can be drawn right onto the picture:



In detail, functions f_0, f_1, g_0, and g_1 are given, and we seek a function u obeying both $\Delta u = 0$ in Ω and

$$\begin{aligned}
u(x,0) &= f_0(x), && 0 < x < a && \text{(bottom)},\\
u(x,b) &= f_1(x), && 0 < x < a && \text{(top)},\\
u(0,y) &= g_0(y), && 0 < y < b && \text{(left)},\\
u(a,y) &= g_1(y), && 0 < y < b && \text{(right)}.
\end{aligned}$$

Trivial Case. If $f_0 \equiv f_1 \equiv g_0 \equiv g_1 \equiv 0$, what's u? Physical intuition and mathematical outcome agree: $u \equiv 0$.

Simplest Nontrivial Case. All but one boundary function are zero. Suppose $f_1(x) \not\equiv 0$, whereas $f_0 \equiv g_0 \equiv g_1 \equiv 0$. Follow our usual 6-step process.

1: Splitting. Not required here.

2: Eigenfunctions. Look for simple nontrivial solutions in product for $u(x,y) = X(x)Y(y)$. Substitute into PDE/BC, remembering that separation of variables is worse than futile (it's misleading) on nonhomogeneous conditions. So only three BC's give useful info, namely,

$Y(0) = 0$ (from the bottom), $\quad X(0) = 0$ (from the left), $\quad X(a) = 0$ (from the right).

In PDE, substitution leads to

$$0 = X''(x)Y''(y) + X(x)Y''(y) \iff \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

for some separation constant λ. Since we have a pair of BC's for X and only one for Y, it is the X-component that provides a well-formed eigenvalue problem:

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < a; \qquad X(0) = 0 = X(a).$$

The corresponding eigenfunctions are well known (FSS):

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \; n = 1, 2, \ldots.$$

3: Superposition. Separation has done all it can for us. The solution we expect will not have simple separated form, but rather combine all possible product-form solutions found above like this:

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y) \sin\left(\frac{n\pi x}{a}\right). \qquad (**)$$

Finding those coefficient functions $Y_n(y)$ will complete the solution.

4: Auxiliary Information (formerly "Initialization"). Any series of form $(**)$ will satisfy the BC's on the left and right sides. On the bottom, we need

$$0 = u(x,0) = \sum_{n=1}^{\infty} Y_n(0) \sin\left(\frac{n\pi x}{a}\right).$$

Here is a FSS expansion for the zero function, giving

$$Y_n(0) = 0, \qquad n = 1, 2, \ldots. \qquad (\dagger)$$

On the top, we need

$$f_1(x) = u(x,b) = \sum_{n=1}^{\infty} Y_n(b) \sin\left(\frac{n\pi x}{a}\right).$$

This is a FSS expansion for the function f_1. Standard coefficient formulas give

$$Y_n(b) = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right)\, dx. \qquad (\ddagger)$$

5: Propagation. Plug series into PDE to get

$$0 = u_{xx} + u_{yy} = \sum_{n=1}^{\infty} \left[-\left(\frac{n\pi}{a}\right)^2 Y_n(y) + Y_n''(y) \right] \sin\left(\frac{n\pi x}{a}\right).$$

This is a FSS expansion on $0 < x < a$, with coefficients independent of x, for the zero function. Again it requires that all coefficients must vanish, i.e.,

$$Y_n''(y) - \left(\frac{n\pi}{a}\right)^2 Y_n(y) = 0, \qquad n = 1, 2, \ldots.$$

Guess $Y_n = e^{sy}$ and plug in to get $s = \pm\dfrac{n\pi}{a}$, and hence the general solution

$$Y_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

Now use (†) to get $0 = Y_n(0) = A_n + B_n$, so $B_n = -A_n$,

$$Y_n(y) = A_n \left[e^{n\pi y/a} - e^{-n\pi y/a} \right].$$

Plug in $y = b$ and appeal to (‡) to get

$$A_n = \frac{Y_n(b)}{e^{n\pi b/a} - e^{-n\pi b/a}} = \frac{2}{a\left[e^{n\pi b/a} - e^{-n\pi b/a}\right]} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

6: Conclusion. The series solution we want is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \left[e^{n\pi y/a} - e^{-n\pi y/a} \right] \sin\left(\frac{n\pi x}{a}\right),$$

with constants A_n given in terms of f_1 by the integral formula above. This appearance is typical: a sum of products, with one factor being an eigenfunction and the other some kind of exponential. ////

Eigenfunction Setup. Notice that the key to Step 2 above was the presence of homogeneous BC's on the opposite faces (left and right) of the domain Ω. These gave us the FSS eigenproblem encoded in the series solution; the homogeneous BC on the bottom edge helped us only in Step 4, and a nonhomogeneous one could have been handled in Step 4 with very little extra work.

RTFT. In the textbook by Trench, please read Section 12.3 and try problems 5, 9, 19, 31, 34.

Symmetry Argument #1. Suppose $u = u(x, y)$ obeys Laplace's Equation in the given rectangle. Define $v = u(x, b - y)$. Notice that

$$v_y(x, y) = -u_y(x, b - y), \qquad v_{yy}(x, y) = (-1)^2 u_{yy}(x, b - y) = u_{yy} x, b - y,$$

while $v_{xx}(x, y) = u_{xx}(x, b - y)$. Now when $0 < y < b$, of course $0 < b - y < b$ also, so we get

$$v_{xx}(x, y) + v_{yy}(x, y) = u_{xx}(x, b - y) + u_{yy}(x, b - y) = 0.$$

Meanwhile,

$$\begin{aligned} v(x,0) &= u(x,b) = f_1(x),\\ v(x,b) &= u(x,0) = f_0(x) = 0,\\ v(0,y) &= u(0,b-y) = g_0(b-y) = 0,\\ v(a,y) &= u(a,b-y) = g_1(b-y) = 0. \end{aligned}$$

Now the function u is known, so v is too, and v solves a problem very similar to the u-problem except that it has a nontrivial temperature on the bottom edge of Ω. [*Sketch a pictorial representation for the v-problem.*] Putting this back into the notation of the original u-problem lets us cover the case where $g_0 \equiv f_1 \equiv g_1 \equiv 0$ but $f_0 \not\equiv 0$:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \left[e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right] \sin\left(\frac{n\pi x}{a}\right),$$

$$\text{where} \qquad A_n = \frac{2}{a\left[e^{n\pi b/a} - e^{-n\pi b/a}\right]} \int_0^a f_0(x) \sin\left(\frac{n\pi x}{a}\right)\, dx.$$

Symmetry Argument #2. Switching letters $x \leftrightarrow y$, $a \leftrightarrow b$, $f \leftrightarrow g$ changes the appearance but not the validity of the solution. (Physically it makes sense too: we're just flipping our metal plate with its steady temperature distribution along the axis $y = x$.) This swap leaves the PDE unchanged, but gives a BC where it's $g_1(y)$ that is nontrivial. So for the case $f_0 \equiv g_0 \equiv f_1 \equiv 0$, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \left[e^{n\pi x/b} - e^{-n\pi x/b}\right] \sin\left(\frac{n\pi y}{b}\right), \tag{5}$$

$$B_n = \frac{2}{b\left[e^{n\pi a/b} - e^{-n\pi a/b}\right]} \int_0^a g_1(y) \sin\left(\frac{n\pi y}{b}\right)\, dy.$$

Approaching this problem directly, we would separate variables as shown above in the PDE, but the separated BC's would lead to the homogeneous conditions $Y(0) = 0 = Y(b)$. So in this situation, the eigenvalue problem of interest would involve the unknowns Y:

$$Y''(y) - \lambda Y(y) = 0,\ 0 < y < b; \qquad Y(0) = 0 = Y(b).$$

The natural series form to postulate would then be $u(x,y) = \sum_{n=1}^{\infty} X_n(x) \sin\left(\frac{n\pi y}{b}\right)$, and this is precisely what we see in line (5) above.

Practice. Without lengthy calculation, find a series solution formula for the case where $f_1 \equiv g_1 \equiv f_0 \equiv 0$ but $g_0 \not\equiv 0$.

Splitting and Superposition. To handle arbitrary f_0, f_1, g_0, and g_1, consider four subproblems of the form above, each with one nonzero boundary function. Write a series solution for each subproblem, then add them up.

C. Laplace's Equation in Polar Coordinates (Pizza Problems)

In standard polar coordinates, where

$$x = r\cos\theta, \quad y = r\sin\theta,$$

the Laplacian of a given function $u = u(r,\theta)$ is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

[Proof idea: Define $U(x,y) = u(\sqrt{x^2+y^2}, \tan^{-1}(y/x))$, compute $\Delta U = U_{xx} + U_{yy}$ with chain rule, express result in terms of r and θ.] Hence Laplace's equation is equivalent to

$$0 = r^2\Delta u = r^2 u_{rr} + r u_r + u_{\theta\theta}.$$

Separation of this PDE with $u(r,\theta) = R(r)\Theta(\theta)$ leads to

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \sigma$$

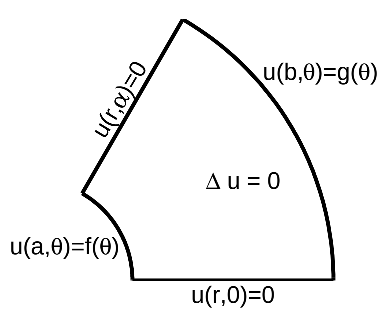
for some separation constant σ, and this produces two linked ODE problems:

$$\begin{aligned} &(1) \qquad r^2R''(r) + rR'(r) - \sigma R(r) = 0, \\ &(2) \qquad \Theta''(\theta) + \sigma\Theta(\theta) = 0. \end{aligned}$$

Depending on what sort of BC's are present, either of these could be the ODE of choice in an eigenvalue problem.

Example (A). Pizza slice with first bite gone: polar region

$$\Omega = \{(r,\theta) \,:\, a < r < b,\ 0 < \theta < \alpha\}.$$



Here $\alpha \in (0, 2\pi)$ and $a > 0$, $b > a$ are some preassigned constants. If the BC's on the flat sides are homogeneous, i.e.,

$$\begin{aligned} &u(r,0) = 0, && u(r,\alpha) = 0, && a < r < b, \\ &u(a,\theta) = f(\theta), && u(b,\theta) = g(\theta), && 0 < \theta < \alpha, \end{aligned}$$

then separation of variables in the BC's leads to

$$\Theta(0) = 0 = \Theta(\alpha),$$

so equation (2) participates in the eigenvalue problem

$$\Theta''(\theta) + \sigma\Theta(\theta) = 0,\ 0 < \theta < \alpha, \qquad \Theta(0) = 0 = \Theta(\alpha).$$

Thus we get FSS eigenfunctions $\Theta_n(\theta) = \sin\left(\frac{n\pi\theta}{\alpha}\right)$, and a series-form solution

$$u(r,\theta) = \sum_{n=1}^{\infty} R_n(r) \sin\left(\frac{n\pi\theta}{\alpha}\right).$$

The BC's give

$$f(\theta) = u(a,\theta) = \sum_{n=1}^{\infty} R_n(a) \sin\left(\frac{n\pi\theta}{\alpha}\right), \text{ so } R_n(a) = \frac{2}{\alpha}\int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)\, d\theta,$$

$$g(\theta) = u(b,\theta) = \sum_{n=1}^{\infty} R_n(b) \sin\left(\frac{n\pi\theta}{\alpha}\right), \text{ so } R_n(b) = \frac{2}{\alpha}\int_0^{\alpha} g(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)\, d\theta.$$

Plugging the series into the PDE leads to

$$0 = r^2 u_{rr} + r u_r + u_{\theta\theta} = \sum_{n=1}^{\infty} \left[r^2 R_n'' + r R_n' - \left(\frac{n\pi}{\alpha}\right)^2 R_n \right] \sin\left(\frac{n\pi\theta}{\alpha}\right),$$

whence

$$0 = r^2 R_n'' + r R_n' - \left(\frac{n\pi}{\alpha}\right)^2 R_n.$$

This equation has Euler type: a function $R_n(r) = r^s$ gives a solution iff

$$s(s-1) + s - \left(\frac{n\pi}{\alpha}\right)^2 = 0, \quad \text{i.e.,} \quad s = \pm\frac{n\pi}{\alpha}.$$

So the general solution is

$$R_n(r) = A_n r^{n\pi/\alpha} + B_n r^{-n\pi/\alpha}, \qquad A_n, B_n \in \mathbb{R}.$$

When functions f, g are given in detail, the right-hand sides in the system below are known constants, and it is possible to solve for A_n, B_n:

$$A_n a^{n\pi/\alpha} + B_n a^{-n\pi/\alpha} = R_n(a) = \frac{2}{\alpha}\int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)\, d\theta,$$

$$A_n b^{n\pi/\alpha} + B_n b^{-n\pi/\alpha} = R_n(b) = \frac{2}{\alpha}\int_0^{\alpha} g(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)\, d\theta.$$

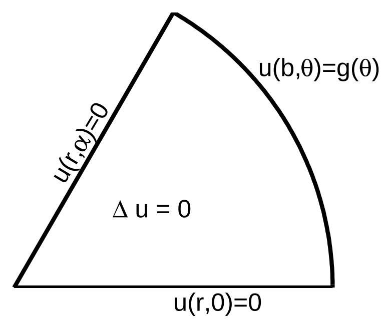
The series solution is then

$$u(r,\theta) = \sum_{n=1}^{\infty} \left[A_n r^{n\pi/\alpha} + B_n r^{-n\pi/\alpha} \right] \sin\left(\frac{n\pi\theta}{\alpha}\right).$$

Consider same region with other BC's later (see (D) below).

Example (B). Pizza slice before first bite taken: polar region

$$\Omega = \{(r, \theta) \,:\, 0 < r < b,\ 0 < \theta < \alpha\}\,.$$



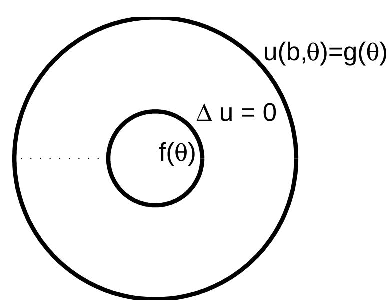
This is the limiting case $a \to 0^+$ of (1); we use the same boundary conditions. Mathematically, there is no place for the function $f(\theta)$ describing u-values on the curved inner boundary shown in problem (A), so we don't have enough information to determine both constants A_n and B_n above. However, problems of physical interest typically have **bounded solutions.** The requirement that $u(r,\theta)$ behave well near the boundary (which includes the origin, where $r = 0$) forces us to choose all $B_n = 0$, so the solution simplifies to

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right), \qquad A_n = \frac{2}{b^n \alpha} \int_0^{\alpha} g(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)\, d\theta.$$

Example (B′). Infinite pizza slice missing one bite. Practice: how should the solution shown in part (A) be modified in the limiting case $b \to +\infty$?

Example (C). Annulus. Solve $\Delta u = 0$ in the polar region $\Omega = \{(r,\theta) \,:\, a < r < b\}$, where $a > 0$ and $b > a$ are given constants and

$$u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta), \qquad -\pi < \theta < \pi.$$



Main Idea: Periodicity implicit in the polar coordinate representation provides boundary conditions not written explicitly in the problem statement, namely,

$$u(r,-\pi) = u(r,\pi), \quad u_\theta(r,-\pi) = u_\theta(r,\pi), \qquad a < r < b.$$

Separating $u(r,\theta) = R(r)\Theta(\theta)$ provides two pieces of boundary information about Θ, so we focus the ODE for Θ in line (2) above. This produces an eigenvalue problem for Θ:

$$\begin{aligned}&\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad -\pi < \theta < \pi;\\ &\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).\end{aligned}$$

This is a standard problem: we know that its eigenfunctions are precisely the set of all basis functions associated with the Full Fourier Series on $[-\pi, \pi]$. Hence we postulate a solution of the form

$$u(r,\theta) = \tfrac{1}{2}A_0(r) + \sum_{n=1}^{\infty} \left[A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)\right],$$

for some coefficient functions A_n and B_n to be determined.

To see how the coefficients evolve with r, plug that series into Laplace's Equation:

$$\begin{aligned}0 &= r^2 u_{rr} + r u_r + u_{\theta\theta}\\ &= \tfrac{1}{2}\left[r^2 A_0''(r) + rA_0'(r)\right] + \sum_{n=1}^{\infty}\left[r^2 A_n''(r) + rA_n'(r) - n^2 A_n(r)\right]\cos(n\theta)\\ &\qquad + \sum_{n=1}^{\infty}\left[r^2 B_n''(r) + rB_n'(r) - n^2 B_n(r)\right]\sin(n\theta)\end{aligned}$$

In this full Fourier series for the zero function, all coefficients must be 0. For case $n \geq 1$, this gives a pair of identical Euler-type equations. Exactly as in problem (A) above, we find the general solutions

$$A_n(r) = a_n r^n + b_n r^{-n}, \quad B_n(r) = c_n r^n + d_n r^{-n}, \qquad a_n, b_n, c_n, d_n \in \mathbb{R}.$$

For case $n = 0$, a shrewd observation gives

$$0 = r^2 A_0''(r) + rA_0'(r) \stackrel{!}{=} r\frac{d}{dr}\left(rA_0'(r)\right).$$

Hence $rA_0'(r) = b_0$ for some $b_0 \in \mathbb{R}$, and this implies

$$A_0'(r) = \frac{b_0}{r} \implies A_0(r) = a_0 + b_0\ln(r),\ a_0, b_0 \in \mathbb{R}.$$

Thus we may express our series solution as

$$\begin{aligned}u(r,\theta) = \tfrac{1}{2}\left[a_0 + b_0\ln(r)\right] &+ \sum_{n=1}^{\infty}\left[a_n r^n + b_n r^{-n}\right]\cos(n\theta)\\ &+ \sum_{n=1}^{\infty}\left[c_n r^n + d_n r^{-n}\right]\sin(n\theta).\end{aligned} \tag{1}$$

Now using standard coefficient formulas on the BC's

$$f(\theta) = u(a,\theta), \quad g(\theta) = u(b,\theta)$$

will give 2×2 systems of linear equations to solve for (a_n, b_n) and (c_n, d_n). [Try it!]

Example (C′). Steady temperature in a disk of radius $b > 0$, with given boundary temperature $g(\theta)$. This is the limiting case $a \to 0^+$ of (C) above. Again we must apply the *boundedness requirement* imposed by the physical interpretation (steady temperature). This takes the place of the prescribed temperature f on the inner edge; now boundedness requires $b_n = 0$, $d_n = 0$ for all n. (Please think about b_0 separately.) The result is

$$u(r,\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + c_n \sin(n\theta)\right).$$

Now the BC $u(b,\theta) = g(\theta)$ gives

$$b^n a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta, \qquad n = \mathbf{0},1,2,3,\ldots,$$
$$b^n c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta, \qquad n = 1,2,3,\ldots.$$

- **Averaging Property:** Midpoint temperature is average of boundary temperatures. (Proof. Plug in $r = 0$, recall definition of Fourier Coefficient a_0.)

  Important Consequence: For any solution u of Laplace's equation, in *any 2D domain* Ω*,* there can be no local min and no local max for u in interior of Ω.

  Reason: Suppose $P_0 \stackrel{\text{def}}{=} (x_0, y_0)$ is a point where a local min occurs. Choose a little disk with centre P_0 where all the boundary values are higher than $u(P_0)$. Then $\Delta u = 0$ in that disk, and the averaging property just proved is violated. This can't happen.

  Physics: Steady temperature in a 2D region can't have isolated local extrema. This makes sense—"hot spots" would be unstable. Likewise for a soap-film stretched between curved wires—simple bumps get pulled down by surface tension.

- **Poisson Kernel Formula (Optional):** Plug integral coefficients straight into the series and interchange sum and integral to get the following:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt\right)\cos(n\theta)\right.\\
&\qquad\qquad \left. + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt\right)\sin(n\theta)\right]\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \left[\cos(nt)\cos(n\theta) + \sin(nt)\sin(n\theta)\right]\right] dt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \cos(n(t-\theta))\right] dt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\left[-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{r}{b}\right)^n \cos(n(t-\theta))\right] dt
\end{aligned}$$

For real x, y with $|x| < 1$, geometric series calculation shows

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \cos(ny) &= \sum_{n=0}^{\infty} \Re\left(x^n e^{iny}\right) = \Re \sum_{n=0}^{\infty} \left(xe^{iy}\right)^n \\ &= \Re\left(\frac{1}{1-xe^{iy}} \cdot \frac{1+xe^{iy}}{1+xe^{iy}}\right) = \frac{1-x\cos(y)}{1-2x\cos(y)+x^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \cos(ny) - \frac{1}{2} &= \frac{1-x\cos y}{1-2x\cos y+x^2} - \frac{1}{2}\frac{1-2x\cos y+x^2}{1-2x\cos y+x^2} \\ &= \frac{1}{2}\frac{1-x^2}{1-2x\cos y+x^2}\end{aligned}$$

Use calc above with $x = r/b$ and $y = t - \theta$ to get a famous and useful formula:

$$u(r,\theta) = \int_{-\pi}^{\pi} \left[\frac{1}{2\pi}\frac{b^2-r^2}{b^2-2br\cos(t-\theta)+r^2}\right] g(t)\,dt, \qquad 0 < r < b,\ \theta \in \mathbb{R}.$$

Defining $P(\vec{z},t) = \dfrac{1}{2\pi}\dfrac{b^2-r^2}{b^2-2br\cos(t-\theta)+r^2}$ makes this formula look like matrix-vector multiplication:

$$u(\vec{z}) = \int_{-\pi}^{\pi} P(\vec{z},t)g(t)\,dt. \tag{$*$}$$

Interpretation: Let Y be the set of all integrable 2π-periodic functions $g = g(\theta)$. Let X be the collection of all well-behaved functions $u = u(x,y)$ that satisfy Laplace's Equation on $r < b$. Define a linear operator $A\colon X \to Y$ like this:

$$A[u] = g \iff g(\theta) = \lim_{r\to b^-} u(r,\theta).$$

E.g., if $u(r,\theta) = r\cos\theta$ then $g = A[u]$ is the function

$$g(\theta) = b\cos\theta, \qquad -\pi < \theta < \pi.$$

It's easy to find g once u is given, but the really interesting problem is usually just the opposite, namely, solve for u in $A[u] = g$. Ideally, we would like to have an "inverse" for A, so we could just write

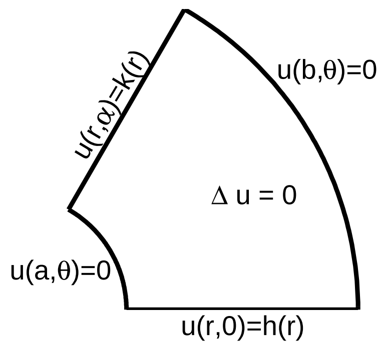
$$u = A^{-1}[g].$$

Formula $(*)$ does exactly this, with P replacing A^{-1}. (Analogy: For n-dimensional vectors, $u = Pg$ iff $u_k = \sum_{t=1}^{n} P_{kt}g_t$; for functions, $u(\vec{z}) = \int_{-\pi}^{\pi} P(\vec{z},t)g(t)\,dt$.) Froese's notes are a good source for this. (See link on course home page.)

Example (D). Standard bitten slice, different boundary conditions. Again we solve $\Delta u = 0$ in the polar region

$$\Omega = \{(r,\theta) \,:\, 1 < r < b,\ 0 < \theta < \alpha\}$$

where $a = 1$, $b > 1$ and $\alpha \in (0, 2\pi)$ are given.



This time, however, nonhomogeneous boundary data are given on the flat sides of the domain:

$$\begin{array}{llll} u(r,0) = h(r) & u(r,\alpha) = k(r) & 1 < r < b, & \\ u(1,\theta) = 0 & u(b,\theta) = 0 & & 0 < \theta < \alpha. \end{array}$$

[Shortcut: If both given functions h and k are constant, careful choices of the constants A and B together with the substitution $u(r,\theta) = A + B\theta + w(r,\theta)$ will reduce this problem to an instance of (A) above. But if one of these functions is nonconstant, the method shown below seems inevitable.]

A new eigenvalue problem. Separating $u(r,\theta) = R(r)\Theta(\theta)$ in the homogeneous BC gives

$$R(1)\Theta(\theta) = 0 = R(b)\Theta(\theta), \qquad \text{i.e.,} \qquad R(1) = 0 = R(b).$$

These homogeneous conditions on R force us to build our eigenvalue problem using the ODE in line (1) of the separation-of-variables result above. We arrive at this eigenvalue problem *for the function R*:

$$r^2 R''(r) + rR'(r) + \lambda R(r) = 0, \quad 1 < r < b; \qquad R(1) = 0 = R(b). \tag{11}$$

This is not a FSS problem, because the ODE has form different from the one familiar so far. To find eigenfunctions will take grinding case-by-case analysis. Since the ODE has Euler type, guess $R(r) = r^p$ and plug in:

$$r^2 \left[p(p-1)r^{p-2}\right] + r\left[pr^{p-1}\right] + \lambda r^p = 0 \iff p^2 + \lambda = 0.$$

- Case $\lambda < 0$: Write $\lambda = -s^2$ for some $s > 0$ and get $p = \pm s$, so the general solution is

$$R(r) = Ar^s + Br^{-s}, \qquad A, B \in \mathbb{R}.$$

Now $0 = R(1) = A + B$ gives $B = -A$, so $R(r) = A(r^s - r^{-s})$, and $0 = R(b) = A(b^s - b^{-s})$ forces $A = 0$ (since $b^s > 1 > b^{-s}$). Thus only trivial solutions appear when $\lambda < 0$.

- Case $\lambda = 0$: Repeated roots, so

$$R(r) = A + B\ln(r), \qquad A, B \in \mathbb{R}.$$

Now $0 = R(1) = A$ gives $R(r) = B\ln(r)$ and $0 = R(b) = B\ln(b)$ forces $B = 0$ (since $\ln(b) > 0$). Only trivial solutions here too.

- Case $\lambda > 0$: Write $\lambda = \omega^2$ for some $\omega > 0$, so $p^2 = -\omega^2$ and $p = \pm i\omega$. Recall

$$r^{i\omega} = e^{i\omega\ln(r)} = \cos(\omega\ln(r)) + i\sin(\omega\ln(r)),$$

and that both real and imaginary parts give a solution. The general solution is

$$R(r) = A\cos(\omega\ln(r)) + B\sin(\omega\ln(r)), \qquad A, B \in \mathbb{R}.$$

Now $0 = R(1) = A$ gives $R(r) = B\sin(\omega\ln(r))$, so

$$0 = R(b) = B\sin(\omega\ln(b)).$$

This time nonzero values of B can occur, provided $\omega > 0$ satisfies

$$\sin(\omega\ln(b)) = 0, \quad \text{i.e.,} \quad \omega\ln(b) = n\pi,\ n = 1, 2, 3, \ldots.$$

Thus we have a sequence of eigenvalues

$$\lambda_n = \omega_n^2 = \left(\frac{n\pi}{\ln(b)}\right)^2, \qquad n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions are (multiples of)

$$R_n(r) = \sin\left(\frac{n\pi}{\ln(b)}\ln(r)\right), \qquad n = 1, 2, 3, \ldots.$$

Expansion Formulas. Dividing the ODE in (11) by r puts it into Sturm-Liouville form:

$$0 = rR'' + R' + \lambda\left(\frac{1}{r}\right)R = (rR'(r))' + \lambda\left(\frac{1}{r}\right)R, \qquad R(1) = 0 = R(b).$$

Hence for any reasonable $f = f(r)$ defined for $1 < r < b$, we have

$$\boxed{\begin{aligned} f(r) &= \sum_{n=1}^{\infty} b_n R_n(r) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ln(b)}\ln(r)\right), \qquad 1 < r < b, \\ \iff b_n &= \frac{2}{\ln(b)}\int_{r=1}^{b} f(r)\sin\left(\frac{n\pi}{\ln(b)}\ln(r)\right)\frac{dr}{r}, \qquad n = 1, 2, 3, \ldots. \end{aligned}} \tag{$*$}$$

Back in problem (D), we postulate an eigenfunction-series solution:

$$u(r,\theta) = \sum_{n=1}^{\infty} \Theta_n(\theta) \sin\left(\frac{n\pi}{\ln(b)} \ln(r)\right)$$

for some Θ_n to be determined. Plugging in $\theta = 0$ and $\theta = \alpha$ will give two opportunities to use $(*)$, one with h and the other with k, and these will reveal the values of $\Theta_n(0)$ and $\Theta_n(\alpha)$. The evolution of Θ_n will be governed by the PDE, i.e.,

$$0 = r^2 u_{rr} + r u_r + u_{\theta\theta} = \sum_{n=1}^{\infty} \left(\Theta_n(\theta) \left[r^2 R_n''(r) + r R_n'(r)\right] + R_n(r)\Theta_n''(\theta)\right)$$

Now remember the eigenvalue problem:

$$r^2 R_n''(r) + r R_n'(r) = -\left(\frac{n\pi}{\ln b}\right)^2 R_n(r).$$

Hence we have

$$0 = \sum_{n=1}^{\infty} \left[\Theta''(\theta) - \left(\frac{n\pi}{\ln b}\right)^2 \Theta_n(\theta)\right] \sin\left(\frac{n\pi}{\ln(b)} \ln(r)\right).$$

This is another chance to use $(*)$, this time with the zero function as the expansion result. The coefficient formulas give, for each n,

$$0 = \Theta_n''(\theta) - \left(\frac{n\pi}{\ln b}\right)^2 \Theta_n(\theta), \quad \text{i.e.,} \quad \Theta_n(\theta) = A_n e^{n\pi\theta/\ln b} + B_n e^{-n\pi\theta/\ln b}.$$

Answer:

$$u(r,\theta) = \sum_{n=1}^{\infty} \left[A_n e^{n\pi\theta/\ln b} + B_n e^{-n\pi\theta/\ln b}\right] \sin\left(\frac{n\pi}{\ln(b)} \ln(r)\right),$$

with A_n and B_n determined by using $(*)$ with h and k.