SCEE09004: Partial Differential Equations 3 Analytical methods

Elliptic PDE Part 1: Introduction

Daniel Friedrich
d.friedrich@ed.ac.uk

School of Engineering

THE UNIVERSITY of EDINBURGH

Learning outcomes and prerequisites

Learning outcomes

After this part, you should be able to:

- Derive the Laplace equation in 2D

Prerequisites

Before this part, you should be able to:

- Distinguish ordinary and partial differential equations and their properties

Laplace Equation

The Laplace equation is defined by

$$\nabla^2 u(x, y) = 0$$

- It was first studied by Pierre-Simon Laplace (1749–1827), of whom you have already heard!
- It is an elliptic equation which models the steady state of a diffusion like process.
  - We will discuss the formal classification of PDEs later in this course
- We can use it to describe the pressure in an incompressible flow, steady state diffusion and heat conduction, stress and strain, etc.
- In 2D the Laplace operator Δ is given by

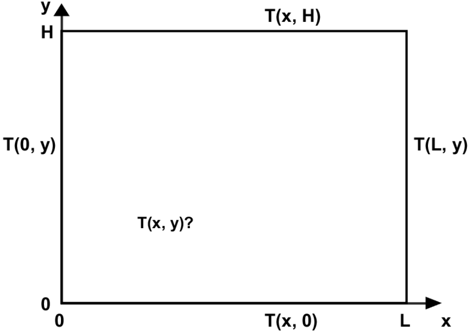
$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- It is closely related to the Poisson equation

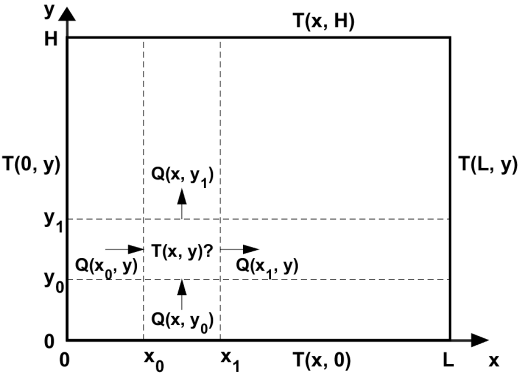
$$\nabla^2 u = f(x, y)$$

Problem statement

- Consider a flat, rectangular sheet of thermally conductive material
- The left side is heated while the other sides are keep cold
- There is no heat transfer in the z direction
- There is no source or sink inside the sheet
- Find the steady state temperature distribution in the sheet



Temperature in a small element



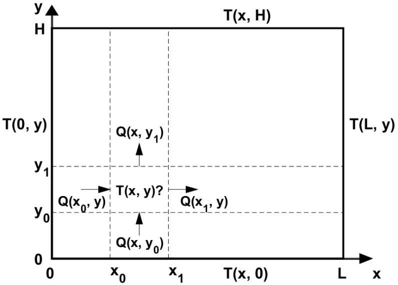
- Consider the small element between x_0, x_1, y_0 and y_1
- There are four heat flow terms at its boundaries
- The temperature inside is not changing over time, i.e. $\frac{\partial T}{\partial t} = 0$

Assumptions

- Heat flows from hot regions to cold regions
- The rate at which heat flows through a plane section drawn in a body is proportional to its area and to the temperature gradient normal to the section
- The quantity of heat in a body is proportional to its mass, specific heat capacity and to its temperature
- Fourier's law for the conduction of heat is

$$Q = -k\frac{\partial T}{\partial x}$$

Derivation 1/2



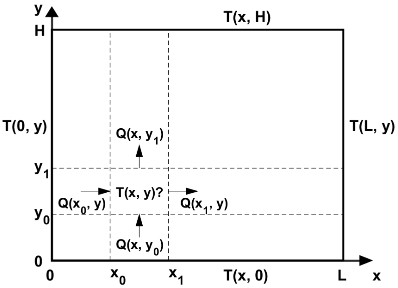
- The heat flows across the boundaries need to balance because T doesn't change with time

$$0 = [Q(x_0, y) - Q(x_1, y)](y_1 - y_0) + [Q(x, y_0) - Q(x, y_1)](x_1 - x_0)$$

- Divide by $(x_1 - x_0)(y_1 - y_0)$

$$0 = -\frac{Q(x_1, y) - Q(x_0, y)}{x_1 - x_0} - \frac{Q(x, y_1) - Q(x, y_0)}{y_1 - y_0}$$

Derivation 2/2



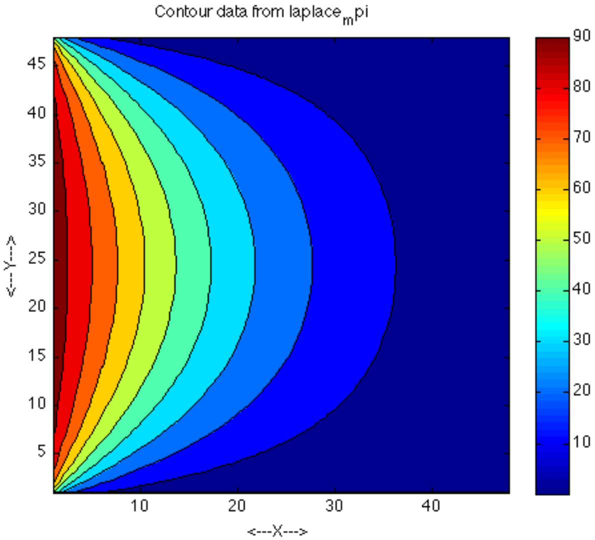
- Reduce the square size so that $x_1 - x_0 \to 0$ and $y_1 - y_0 \to 0$

$$0 = -\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}$$

- Apply Fourier's law and divide by k to get

$$0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \nabla^2 T$$

Contour plot of the solutionLaplace equation



- Contour plot for the case with heated left side while the other sides are keep cold

ConcepTest Elliptic-AP1: Boundary conditions

Which boundary condition does the solution

$$u(x, y) = e^{-y}\cos(x)$$

of the Laplace equation on $0 \leq x \leq 1$ and $0 \leq y \leq 1$ fulfil?

- Answer and discuss on the discussion board

ConcepTest: possible solutions

1. $u(1, y) = e^{-y}\cos(x)$ on $0 \leq y \leq 1$
2. $u(1, y) = e^{-y}\cos(1)$ on $0 \leq y \leq 1$
3. $u(1, y) = e^{-1}\cos(x)$ on $0 \leq y \leq 1$
4. $u(1, y) = e^{-1}\cos(1)$ on $0 \leq y \leq 1$

Laplace Equation

The Laplace equation is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

- It is the most fundamental elliptic PDE which models the steady state of a diffusion like process
- Used to describe the pressure in an incompressible flow, steady state diffusion and heat conduction, stress and strain, hydrostatics, electrostatics, ground water flow, etc.
- Unlike the other PDEs the solution only involves spatial derivatives – there is no time dependency
- It is easy to find **a** solution – finding **the** solution matching the boundary conditions is the hard part

Any questions?

- Go to the Discussion Board
  - Ask questions about this part or any other part on the forum
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Next part

- Topic: **Method of separation of variables**

SCEE09004: Partial Differential Equations 3
Analytical methods

Elliptic PDE Part 2: Separated solution

Daniel Friedrich
d.friedrich@ed.ac.uk

School of Engineering

THE UNIVERSITY of EDINBURGH

Learning outcomes and prerequisites

Learning outcomes

After this part, you should be able to:

- Find the three basic solutions for the method of separation of variabes for the Laplace equation

Prerequisites

Before this part, you should be able to:

- Derive the Laplace equation
- Show that a given equation is a solution to the Laplace equation

Laplace Equation

The Laplace equation is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

- It is the most fundamental elliptic PDE which models the steady state of a diffusion like process
- Used to describe the pressure in an incompressible flow, steady state diffusion and heat conduction, stress and strain, hydrostatics, electrostatics, ground water flow, etc.
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Separation of Variables

- Remember the following solution of the Laplace equation from the example video

$$u(x, y) = e^{-y} \cos(x)$$

- The two independent variables x and y are separated into individual parts
- Can we generalise this to find solutions of the Laplace equation of the form

$$u(x, y) = X(x)Y(y)$$

where $X(x)$ and $Y(y)$ are single functions of single variables?

  - We can sometimes find X and Y as solutions of ordinary differential equations
  - These are often much easier to solve than PDEs and it may be possible to build up full solutions of the PDE in terms of X and Y
  - This can be applied to many types of PDE

Separated solutions

- We seek solutions to the 2D Laplace equation of the form

$$u(x, y) = X(x)Y(y)$$

- Substituting this into the Laplace equation and applying the product rule gives

$$Y\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + X\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = 0$$

- Or by rearranging we get

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = \lambda$$

- Here λ is independent of both x and y
- We will later pick a λ so that the boundary conditions are fulfilled

Two simple harmonic ODEs

- Since the two equations are independent, we can split them into two ODEs of the simple harmonic type

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = \lambda X$$
$$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -\lambda Y$$

- The type of the solution depends on the sign of λ for which there are three possibilities
- Let us first look at the solutions for X:

$$\begin{aligned} X(x) =& A\cos(\mu x) + B\sin(\mu x) & \lambda = -\mu^2 < 0, \\ X(x) =& Ae^{\mu x} + Be^{-\mu x} & \lambda = \mu^2 > 0, \\ X(x) =& (Ax + B) & \lambda = 0 \end{aligned}$$

- The order of the first and second solution is swapped for Y due to the minus sign

Three basic solutions

- By combining the solutions for X and Y we get

$$
\begin{aligned}
u(x,y) &= (A\sin\mu x + B\cos\mu x)\left(Ce^{\mu y} + De^{-\mu y}\right) && \lambda = -\mu^2 < 0,\\
&= (A\sin\mu x + B\cos\mu x)\left(\tilde{C}\cosh\mu y + \tilde{D}\sinh\mu y\right) \\
u(x,y) &= \left(Ae^{\mu x} + Be^{-\mu x}\right)(C\sin\mu y + D\cos\mu y) && \lambda = \mu^2 > 0,\\
&= \left(\tilde{A}\cosh\mu x + \tilde{B}\sinh\mu x\right)(C\sin\mu y + D\cos\mu y) \\
u(x,y) &= (Ax + B)(Cy + D) && \lambda = 0
\end{aligned}
$$

  where A, B, C, and D are arbitrary constants
- We will show in an example video how we can use these basic solutions to find the solution to the Laplace equation and its boundary conditions

Recipe for the method of separation of variables

Consider the three possible solutions

$$\begin{aligned}
u_1(x,y) =& (A\sin\mu x + B\cos\mu x)\left(Ce^{\mu y} + De^{-\mu y}\right) && \lambda = -\mu^2 < 0,\\
=& (A\sin\mu x + B\cos\mu x)\left(\tilde{C}\cosh\mu y + \tilde{D}\sinh\mu y\right) && \\
u_2(x,y) =& \left(Ae^{\mu x} + Be^{-\mu x}\right)(C\sin\mu y + D\cos\mu y) && \lambda = \mu^2 > 0,\\
=& \left(\tilde{A}\cosh\mu x + \tilde{B}\sinh\mu x\right)(C\sin\mu y + D\cos\mu y) && \\
u_3(x,y) =& (Ax+B)(Cy+D) && \lambda = 0
\end{aligned}$$

and go through the boundary conditions to

1. Eliminate basic solutions
2. Specify the permissible values of μ
3. Superpose solutions to fit the initial condition

Any questions?

- Go to the Discussion Board
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Next part

- Topic: **Boundary conditions for elliptic PDEs**

SCEE09004: Partial Differential Equations 3
Analytical methods

Elliptic PDE Part 3: Boundary conditions

Daniel Friedrich
d.friedrich@ed.ac.uk

School of Engineering

THE UNIVERSITY of EDINBURGH

Learning outcomes and prerequisites

Learning outcomes

After this part, you should be able to:

- Differentiate the different boundary conditions and explain their physical meaning

Prerequisites

Before this part, you should be able to:

- Solve the Laplace equation with the method of separation of variables

Recipe for separated solutions

Consider the three possible solutions

$$
\begin{aligned}
u_1(x,y) =&(A\sin\mu x + B\cos\mu x)\left(Ce^{\mu y} + De^{-\mu y}\right) && \lambda = -\mu^2 < 0,\\
=&(A\sin\mu x + B\cos\mu x)\left(\tilde{C}\cosh\mu y + \tilde{D}\sinh\mu y\right)\\
u_2(x,y) =& \left(Ae^{\mu x} + Be^{-\mu x}\right)(C\sin\mu y + D\cos\mu y) && \lambda = \mu^2 > 0,\\
=& \left(\tilde{A}\cosh\mu x + \tilde{B}\sinh\mu x\right)(C\sin\mu y + D\cos\mu y)\\
u_3(x,y) =&(Ax + B)(Cy + D) && \lambda = 0
\end{aligned}
$$

and go through the boundary conditions to

1. Eliminate basic solutions
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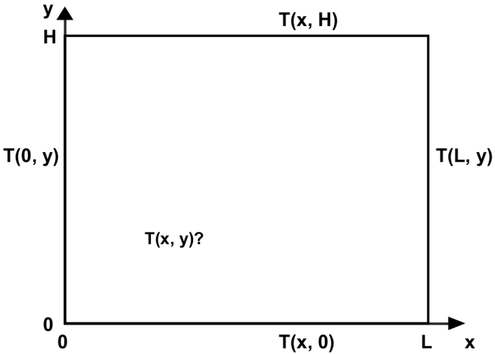
Types of boundary conditions

- So far we have only looked at the boundary conditions on the left
- What do these conditions signify?
- Can you think of any other boundary conditions?



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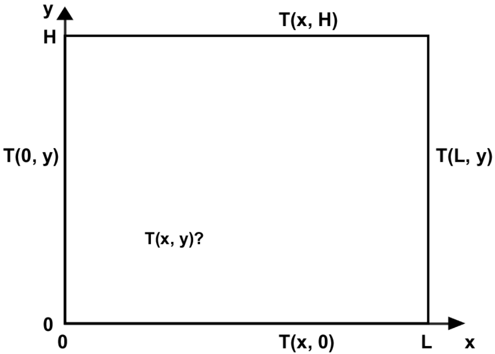
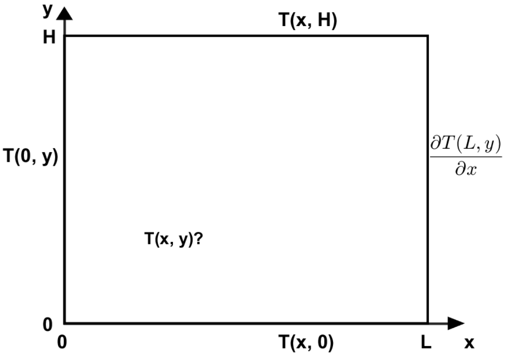




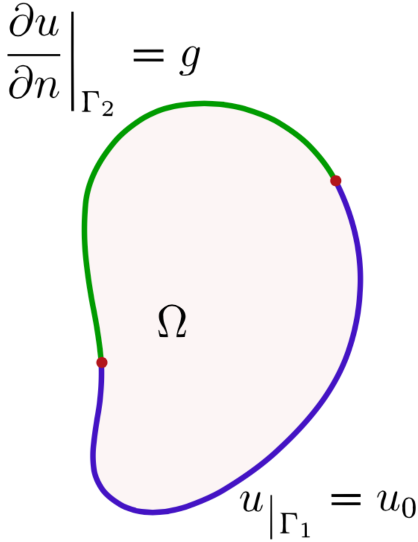
Types of boundary conditions

If Ω is the domain on which the given equation is to be solved and $\partial\Omega$ denotes its boundary, we can have

- Dirichlet
$$u \text{ is specified on } \partial\Omega.$$
- Neumann
$$\frac{\partial u}{\partial n} \text{ is specified on } \partial\Omega.$$
- Cauchy
$$\text{both } u \text{ and } \frac{\partial u}{\partial n} \text{ are specified on } \partial\Omega.$$
- Robin
$$au + B\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega.$$

Mixed boundary conditions

Two or more BC of different types are specified on $\partial\Omega$

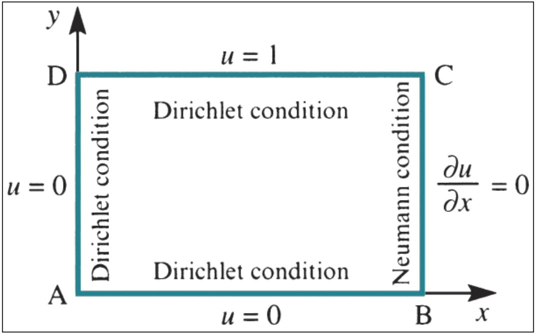


Boundary conditions and physics

Consider that the Laplace equation is used to model heat conduction in a sheet of metal,

- Dirichlet conditions – the edges of the sheet are held at a specified temperature
- Neumann conditions – the heat flux at the edges is given

The important question is: Can a solution be found?



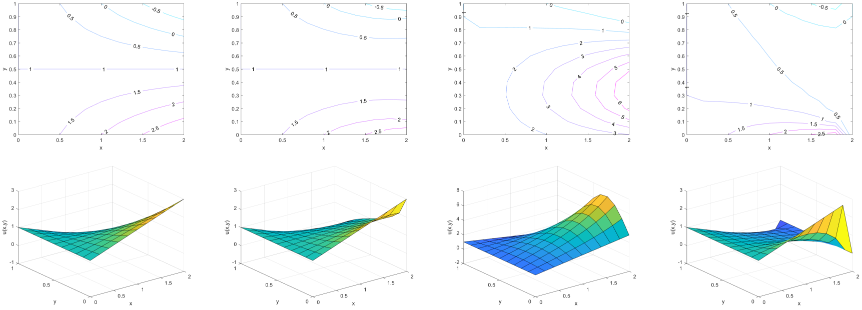
ConcepTest Elliptic-AP3a: Solution shape

Look at the possible solutions and find the one with the correct solution shape for the Laplace equation with following boundary conditions?

$$u(x,0) = 1 + x \qquad 0 \leq x \leq 2 \tag{1}$$
$$u(x,1) = 1 - x \qquad 0 \leq x \leq 2 \tag{2}$$
$$\frac{\partial u}{\partial x}(2,y) = 0, \qquad 0 \leq y \leq 1 \tag{3}$$
$$u(0,y) = 1 \qquad 0 \leq y \leq 1 \tag{4}$$

- Answer and discuss on the discussion board

ConcepTest: possible solutions



Is the PDE well-posed?

The behaviour of PDEs is defined by

- Initial and boundary conditions
- Coefficient functions
- Inhomogeneous terms
- These define the data on which the solution depends

Well-posed problem

1. A solution exists
2. The solution is unique
3. The solution depends continuously on the data: small changes in the data produce small changes in the solution

Ill-posed problem

- Any of the requirements of well-posed problems is not fulfilled

Appropriateness of Boundary Conditions

Data	Boundary	$\nabla^2 u = u_{tt}$ Hyperbolic	$\nabla^2 u = 0$ Elliptic	$\nabla^2 u = u_t$ Parabolic
Dirichlet or Neumann	Open	Insufficient data	Insufficient data	Unique, stable solution for $t > 0$
	Closed	Not unique	Unique, stable (to an arbitrary constant in the Neumann case)	Overspecified
Cauchy	Open	Unique, stable	Solution may exist, but is unstable	Overspecified
	Closed	Overspecified	Overspecified	Overspecified

ConcepTest Elliptic-AP3b: Extreme points of the solution

Where can the extreme points, e.g. maximum or minimum, of the solution to the Laplace equation be located for a non-constant solution?

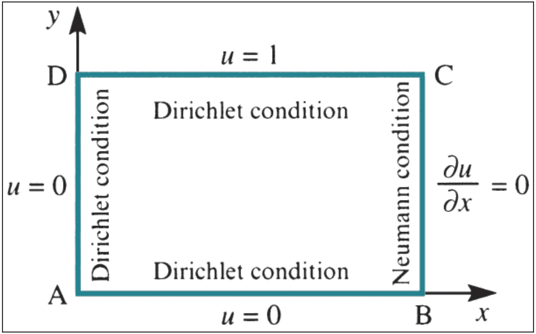
- Answer and discuss on the discussion board

ConcepTest: possible solutions

1. On the boundary
2. Inside the solution domain
3. Anywhere

Well-posed elliptic PDE

- Boundary conditions are given on all sides of the domain



- The boundary conditions are either
  - Dirichlet conditions – the edges of the sheet are held at a specified temperature
  - Neumann conditions – the heat flux at the edges is given

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Next part

- Topic: **Nondimensionalisation**

SCEE09004: Partial Differential Equations 3
Analytical methods

Elliptic PDE Example 1: Laplace equation solutions

Daniel Friedrich
d.friedrich@ed.ac.uk

School of Engineering

THE UNIVERSITY of EDINBURGH

Laplace Equation

The Laplace equation is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

- It is the most fundamental elliptic PDE which models the steady state of a diffusion like process
- Used to describe the pressure in an incompressible flow, steady state diffusion and heat conduction, stress and strain, hydrostatics, electrostatics, ground water flow, etc.
- It is easy to find **a** solution – finding **the** solution matching the boundary conditions is the hard part

Inviscid, irrotational flow

- Used in subsonic aerodynamics
- Can be used to calculate lift on aerofoils
- One solution to the Laplace equation is the stream function

$$\psi(x, y) = Uy \left(1 - \frac{a^2}{x^2 + y^2}\right)$$

- This functions satisfies the Laplace equation
    1. Calculate the second derivatives with respect to x and y
    2. Insert these into the Laplace equation
    3. Check that they add up to zero

Show the stream function satisfies the Laplace equation

- Calculate the first and second partial derivatives with respect to both dimensions of

$$\psi(x, y) = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right)$$

- The derivatives are given by

Show the stream function satisfies the Laplace equation

- Calculate the first and second partial derivatives with respect to both dimensions of

$$\psi(x, y) = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right)$$

- The derivatives are given by

$$\psi_x(x, y) = \frac{2xyUa^2}{(x^2 + y^2)^2}$$

$$\psi_y(x, y) = U - \frac{Ua^2}{x^2 + y^2} + \frac{2y^2 Ua^2}{(x^2 + y^2)^2}$$

$$\psi_{xx}(x, y) = \frac{2yUa^2}{(x^2 + y^2)^2} - \frac{8x^2 yUa^2}{(x^2 + y^2)^3}$$

$$\psi_{yy}(x, y) = \frac{6yUa^2}{(x^2 + y^2)^2} - \frac{8y^3 Ua^2}{(x^2 + y^2)^3}$$

Show the stream function satisfies the Laplace equation

- Take the second partial derivatives

$$\psi_{xx}(x,y) = \frac{2yUa^2}{(x^2+y^2)^2} - \frac{8x^2yUa^2}{(x^2+y^2)^3}$$
$$\psi_{yy}(x,y) = \frac{6yUa^2}{(x^2+y^2)^2} - \frac{8y^3Ua^2}{(x^2+y^2)^3}$$

- Insert them into the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

- To get

Show the stream function satisfies the Laplace equation

- Take the second partial derivatives

$$\psi_{xx}(x,y) = \frac{2yUa^2}{(x^2+y^2)^2} - \frac{8x^2yUa^2}{(x^2+y^2)^3}$$
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- Insert them into the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

- To get

$$\begin{aligned}\nabla^2\psi(x,y) &= \frac{2yUa^2}{(x^2+y^2)^2} - \frac{8x^2yUa^2}{(x^2+y^2)^3} + \frac{6yUa^2}{(x^2+y^2)^2} - \frac{8y^3Ua^2}{(x^2+y^2)^3} \\ &= \frac{8yUa^2}{(x^2+y^2)^2} - \frac{8y(x^2+y^2)Ua^2}{(x^2+y^2)^3} \\ &= \frac{8yUa^2}{(x^2+y^2)^2} - \frac{8yUa^2}{(x^2+y^2)^2} = 0\end{aligned}$$

Multivariate polynomial

- Consider the function

$$u(x, y) = a(x^2 - y^2) + bxy + c$$

  with constants a, b, c
- The partial derivatives are given by

Multivariate polynomial

- Consider the function

$$u(x, y) = a(x^2 - y^2) + bxy + c$$

  with constants a, b, c
- The partial derivatives are given by

$$\frac{\partial u}{\partial x} = 2ax + by \qquad \frac{\partial u}{\partial y} = -2ay + bx$$

$$\frac{\partial^2 u}{\partial x^2} = 2a \qquad \frac{\partial^2 u}{\partial y^2} = -2a$$

$$\frac{\partial^2 u}{\partial x \partial y} = b \qquad \frac{\partial^2 u}{\partial y \partial x} = b$$

- $u(x, y)$ is a solution to the Laplace equation
- The order of integration makes no difference if the function is smooth enough
- True for most functions in engineering

Trigonometric-exponential solution

- Consider the function
$$v(x, y) = e^{-y} \cos(x)$$
- The partial derivatives are

Trigonometric-exponential solution

- Consider the function

$$v(x, y) = e^{-y}\cos(x)$$

- The partial derivatives are

$$\frac{\partial v}{\partial x} = -e^{-y}\sin(x) \qquad \frac{\partial v}{\partial y} = -e^{-y}\cos(x)$$

$$\frac{\partial^2 v}{\partial x^2} = -e^{-y}\cos(x) \qquad \frac{\partial^2 v}{\partial y^2} = e^{-y}\cos(x)$$

- $v(x, y)$ is also a solution of the Laplace equation
- The superposition of u and v is also a solution

$$w(x, y) = u(x, y) + v(x, y) = a(x^2 - y^2) + bxy + c + e^{-y}\cos(x)$$

- The particular solution for a given problem is determined by the boundary conditions

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SCEE09004: Partial Differential Equations 3
Analytical methods

Elliptic PDE Example 2: Separation of variables

Daniel Friedrich
d.friedrich@ed.ac.uk

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Example 9.29

Solve the Laplace equation $\nabla^2 u = 0$ subject to the following boundary conditions

$$u(x,0) = 0 \qquad 0 \le x \le 2 \tag{1}$$
$$u(x,1) = 0 \qquad 0 \le x \le 2 \tag{2}$$
$$u(0,y) = 0 \qquad 0 \le y \le 1 \tag{3}$$
$$u(2,y) = a\sin 2\pi y \qquad 0 \le y \le 1 \tag{4}$$

- The last boundary condition 4 is zero at the corners
- The boundary conditions are consistent $\rightarrow$ what does this mean?

Recipe for separated solutions

Consider the three possible solutions

$$
\begin{aligned}
u_1(x,y) =& (A\sin\mu x + B\cos\mu x)\left(Ce^{\mu y} + De^{-\mu y}\right) && \lambda = -\mu^2 < 0,\\
=& (A\sin\mu x + B\cos\mu x)\left(\tilde{C}\cosh\mu y + \tilde{D}\sinh\mu y\right) &&\\
u_2(x,y) =& \left(Ae^{\mu x} + Be^{-\mu x}\right)(C\sin\mu y + D\cos\mu y) && \lambda = \mu^2 > 0,\\
=& \left(\tilde{A}\cosh\mu x + \tilde{B}\sinh\mu x\right)(C\sin\mu y + D\cos\mu y) &&\\
u_3(x,y) =& (Ax+B)(Cy+D) && \lambda = 0
\end{aligned}
$$

and go through the boundary conditions to

1. Eliminate basic solutions
2. Specify the permissible values of μ
3. Superpose solutions to fit the initial condition

Solution of example 9.29

- To satisfy the first two conditions, i.e. $u(x, 0) = u(x, 1) = 0$, we need solutions which are zero for varying x and for two values of y
- Thus, we need solutions of the form

Solution of example 9.29

- To satisfy the first two conditions, i.e. $u(x, 0) = u(x, 1) = 0$, we need solutions which are zero for varying x and for two values of y
- Thus, we need solutions of the form

$$u_2(x, y) = \left(Ae^{\mu x} + Be^{-\mu x}\right)\left(C \sin \mu y + D \cos \mu y\right)$$

- The first boundary condition, $u(x, 0) = 0$, gives

Solution of example 9.29

- To satisfy the first two conditions, i.e. $u(x, 0) = u(x, 1) = 0$, we need solutions which are zero for varying x and for two values of y
- Thus, we need solutions of the form

$$u_2(x, y) = \left(Ae^{\mu x} + Be^{-\mu x}\right)(C \sin \mu y + D \cos \mu y)$$

- The first boundary condition, $u(x, 0) = 0$, gives

$$0 = \left(Ae^{\mu x} + Be^{-\mu x}\right) D, \quad \text{for } 0 \leq x \leq 2$$

- Thus, $D = 0$ and we have

$$u(x, y) = \left(A^* e^{\mu x} + B^* e^{-\mu x}\right) \sin \mu y$$

where $A^* = AC$ and $B^* = BC$

Solution of example 9.29, continued

- Applying the second boundary condition, $u(x, 1) = 0$ we get

Solution of example 9.29, continued

- Applying the second boundary condition, $u(x, 1) = 0$ we get

$$0 = \left(A^* e^{\mu x} + B^* e^{-\mu x}\right) \sin \mu, \quad \text{for } 0 \leq x \leq 2$$

  so $\sin \mu = 0$ and we need $\mu = n\pi$ where n is an integer
- The third boundary condition, $u(0, y) = 0$, gives

Solution of example 9.29, continued

- Applying the second boundary condition, $u(x,1) = 0$ we get

$$0 = \left(A^* e^{\mu x} + B^* e^{-\mu x}\right) \sin \mu, \quad \text{for } 0 \le x \le 2$$

  so $\sin \mu = 0$ and we need $\mu = n\pi$ where n is an integer
- The third boundary condition, $u(0,y) = 0$, gives

$$(A^* + B^*) \sin n\pi y = 0, \quad \text{for } 0 \le y \le 1$$

- Which implies $A^* = -B^*$ giving

$$\begin{aligned} u(x,y) &= A^* \left(e^{n\pi x} - e^{-n\pi x}\right) \sin n\pi y \\ &= 2A^* \sinh n\pi x \sin n\pi y \end{aligned}$$

Solution of example 9.29, continued

- The final boundary condition $u(2, y) = a \sin 2\pi y$ gives

$$u(2, y) = 2A^* \sinh 2n\pi \sin n\pi y = a \sin 2\pi y, \quad 0 \leq y \leq 1$$

Solution of example 9.29, continued

- The final boundary condition $u(2, y) = a \sin 2\pi y$ gives

$$u(2, y) = 2A^* \sinh 2n\pi \sin n\pi y = a \sin 2\pi y, \quad 0 \leq y \leq 1$$

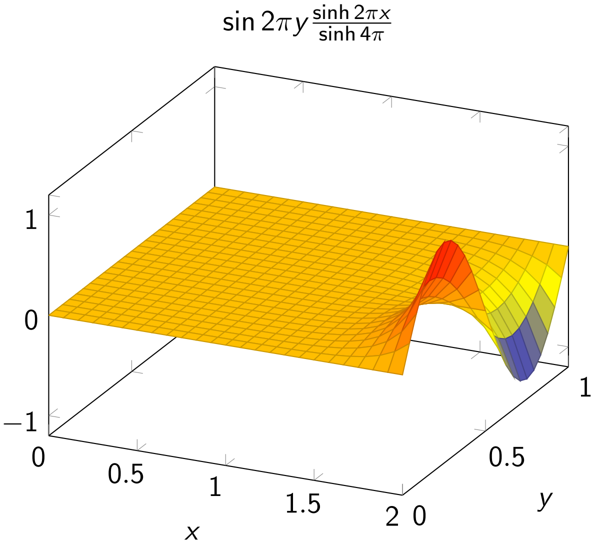
- So $n = 2$ and $a = 2A^* \sinh 4\pi$
- With this we can calculate

$$A^* = \frac{a}{2\sinh(4\pi)}$$

- So the solution is

$$u(x, y) = a \sin 2\pi y \frac{\sinh 2\pi x}{\sinh 4\pi}$$

Example 9.29



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SCEE09004: Partial Differential Equations 3
Analytical methods

Elliptic PDE Example 3: Separated solution with Fourier series

Daniel Friedrich
d.friedrich@ed.ac.uk

School of Engineering

THE UNIVERSITY of EDINBURGH

Recap of Fourier series

Was covered extensively in Engineering Mathematics 2A

1. For any periodic function $f(x)$ with period T we can find coefficients a_n and b_n so that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{T}\right)$$

2. The coefficients of the full-range extension can be calculated by

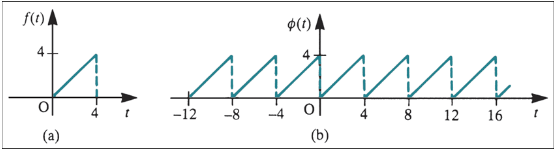
$$a_0 = \frac{2}{T}\int_0^T f(x)dx$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

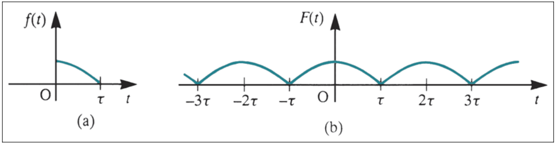
$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

Periodic extensions

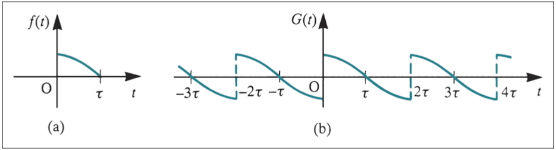
- Full-range periodic extension



- Even periodic extension



- Odd periodic extension



Half-range extensions

- The half-range extensions have half the frequency: the factor 2 is removed from the cosine/sine
- Even periodic extension: cosine series

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

- Odd periodic extension: sine series

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

Example 9.30

Solve the Laplace equation $\nabla^2 u = 0$ subject to the following boundary conditions

$$u(x,0) = 0 \qquad x \geq 0 \qquad (1)$$
$$u(x,1) = 0 \qquad x \geq 0 \qquad (2)$$
$$u(x,y) \to 0 \qquad x \to \infty, 0 \leq y \leq 1 \qquad (3)$$
$$u(0,y) = 1 \qquad 0 \leq y \leq 1 \qquad (4)$$

To satisfy the third condition we need a solution which is exponential in x, so we need to use u_2

$$u_2(x,y) = \left(Ae^{\mu x} + Be^{-\mu x}\right)\left(C \sin \mu y + D \cos \mu y\right)$$

Solution of example 9.30

Remove the subscript and step through the boundary conditions to find the parameters.

$$u(x, y) = \left(Ae^{\mu x} + Be^{-\mu x}\right)\left(C \sin \mu y + D \cos \mu y\right)$$

1. The condition $\lim_{x \to \infty} u(x, y) = 0$ is only satisfied if $A = 0$ because the positive exponential is unbounded. Thus the solution simplifies to

$$u(x, y) = e^{-\mu x}(C^* \sin \mu y + D^* \cos \mu y)$$

   where $C^* = BC$ and $D^* = BD$.
2. Now, consider the first boundary condition, $u(x, 0) = 0$

Solution of example 9.30

Remove the subscript and step through the boundary conditions to find the parameters.

$$u(x,y) = \left(Ae^{\mu x} + Be^{-\mu x}\right)\left(C \sin \mu y + D \cos \mu y\right)$$

1. The condition $\lim_{x \to \infty} u(x,y) = 0$ is only satisfied if $A = 0$ because the positive exponential is unbounded. Thus the solution simplifies to

$$u(x,y) = e^{-\mu x}(C^* \sin \mu y + D^* \cos \mu y)$$

   where $C^* = BC$ and $D^* = BD$.
2. Now, consider the first boundary condition, $u(x,0) = 0$

$$\begin{aligned} u(x, y = 0) &= e^{-\mu x}(C^* \sin(\mu 0) + D^* \cos(\mu 0)) \\ &= e^{-\mu x} D^* \stackrel{!}{=} 0 \end{aligned}$$

   which implies that $D^* = 0$.

Solution of example 9.30, continued

3. We are left with the following solution

$$u(x,y) = C^* e^{-\mu x} \sin \mu y$$

4. The second boundary condition, $u(x,1) = 0$ gives

Solution of example 9.30, continued

3. We are left with the following solution

$$u(x,y) = C^* e^{-\mu x} \sin \mu y$$

4. The second boundary condition, $u(x,1) = 0$ gives

$$u(x, y = 1) = C^* e^{-\mu x} C^* \sin(\mu 1) \stackrel{!}{=} 0$$
$$\implies \quad \sin \mu = 0$$

   so $\mu = n\pi$, $n = 1, 2, \ldots$

5. Because the Laplace equation is linear we can superpose the solutions, to get

$$u(x,y) = \sum_{n=1}^{\infty} C_n^* e^{-n\pi x} \sin n\pi y, \quad 0 \le y \le 1.$$

Solution of example 9.30, continued

6. From the final boundary condition, $u(0, y) = 1,\ 0 \leq y \leq 1$ it follows that

Solution of example 9.30, continued

6. From the final boundary condition, $u(0, y) = 1,\ 0 \leq y \leq 1$ it follows that

$$1 \stackrel{!}{=} \sum_{n=1}^{\infty} C_n^* \sin n\pi y, \quad 0 \leq y \leq 1.$$

   which is a classical Fourier series problem. This is a half-range Fourier sine series with period $T = 1$, so that with the formula from earlier we get

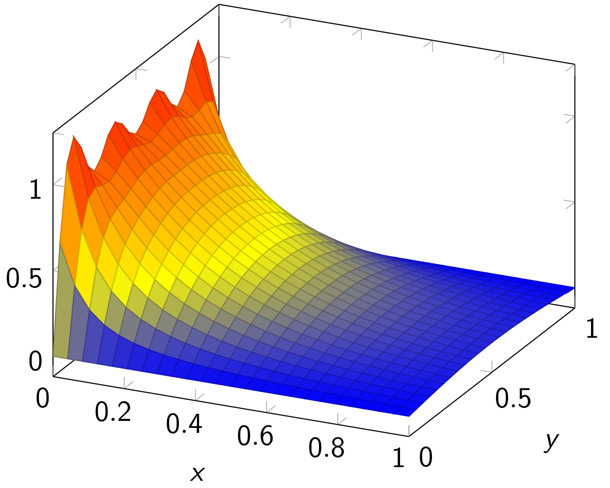
$$C_n^* = 2\int_0^1 \sin n\pi y \, \mathrm{d}y = \begin{cases} 4/n\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

The complete solution is therefore

$$u(x, y) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} e^{-(2k-1)\pi x} \sin\left((2k-1)\pi y\right)$$

Example 9.30 - plot the first 4 terms of the sum

$u(x,y) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} e^{-(2k-1)\pi x} \sin\left((2k-1)\pi y\right)$



- Evaluating this series at $x = 0$ requires more than 30 terms to be evaluated, while at $x = 1$ only one or two terms are needed.

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