

Homework 3 : Part A

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Exercise 1

1. Let $\gamma \in \mathbb{N}^*$ and $X = \{1, \dots, 6\}$. We consider the Hastings-Metropolis algorithm with target π associated to the proposal distribution:

$$q(x, \cdot) = \mathcal{U}(\{X_n - \gamma, X_n - \gamma + 1, \dots, X_n - 1, X_n + 1, \dots, X_n + \gamma\})$$

Algorithm 1 Metropolis Hastings algorithm

Given one initial point in $x_0 \in X$

for $n=1:N$

-Generate $Y_n \sim q(x_n, \cdot)$

- $X_{n+1} = Y_n$ with probability $\alpha(X_n, Y_n)$

and $X_{n+1} = X_n$ with probability $1 - \alpha(X_n, Y_n)$.

end

Since $\forall x \in X, \forall y \in X, q(x, y) = \frac{1}{2\gamma}$, the acceptance ratio is equal to

$$\alpha(x, y) = \begin{cases} 1 \wedge \frac{\pi(y)}{\pi(x)} & \text{if } y \in X \\ 0 & \text{otherwise} \end{cases}$$

To avoid technical discussions, we will consider the extended distribution π on \mathbb{Z} , with $\pi(k) = 0$ if $k \notin X$ so that :

$$\alpha(x, y) = 1 \wedge \frac{\pi(y)}{\pi(x)}$$

And the transition kernel is equal, for $y \neq x$ to:

$$\begin{aligned} p(x, y) &= q(x, y)\alpha(x, y) \\ &= \frac{1}{2\gamma} \left(1 \wedge \frac{\pi(y)}{\pi(x)}\right) \end{aligned}$$

and :

$$p(x, x) = 1 - \sum_{y \neq x} q(x, y)\alpha(x, y)$$

First :

$$\begin{aligned}
 p(x, y \neq x) &= \frac{1}{2\gamma} (1 \wedge \frac{\pi(y)}{\pi(x)}) \\
 &\geq \frac{1}{2\gamma} (1 \wedge \frac{\pi(y)}{\max_x \pi(x)}) \\
 &= \frac{\pi(y)}{2\gamma \max_x \pi(x)} \\
 &\geq \frac{\pi(y)}{2\gamma}
 \end{aligned}$$

The third equality is due to $\frac{\pi(y)}{\max_x \pi(x)} \leq 1$. The fourth inequality is due to $\max_x \pi(x) \leq 1$.

Second,

$$\begin{aligned}
 p(x, x) &= 1 - \sum_{y \neq x} q(x, y) \alpha(x, y) \\
 &= q(x, x) + \sum_{y \neq x} q(x, y) - \sum_{y \neq x} q(x, y) \alpha(x, y) \\
 &= q(x, x) + \sum_{x \neq y} q(x, y) (1 - \alpha(x, y)) \\
 &\geq q(x, x) \\
 &= \frac{1}{2\gamma} \\
 &\geq \frac{\pi(x)}{2\gamma}
 \end{aligned}$$

The second inequality is obtained by upper bounding the sum terms by 1, and using $|X| = 6$.

Let $\epsilon = \frac{1}{2\gamma}$, then

$$\forall x, \forall y \quad p(x, y) \geq \epsilon \pi(y)$$

which can be written in terms of subsets B of X :

$$p(x, B) \geq \epsilon \pi(B)$$

2. Given the inequality written above, the algorithm satisfies the Doeblin condition. The Doeblin Lemma gives : $\Delta(P) \leq (1 - \epsilon)$

Given one initial distribution ξ , we have in the general case:

$$\|\xi P^n - \pi\|_{TV} \leq (\Delta(P))^n \|\xi - \pi\|_{TV}$$

which yields here :

$$\|\xi P^n - \pi\|_{TV} \leq (1 - \frac{1}{2\gamma})^n \|\xi - \pi\|_{TV}$$

Where ξP^n is actually the distribution of X_n . We have geometric ergodicity

3. Cf code for implementation.

The following error is computed with comparison of the numerical mean and the theoretical mean : $\pi(1) = \frac{1}{2}$, $\pi(2) = \frac{2}{6}$, $\pi(3) = \frac{1}{24}$, $\pi(4) = \frac{1}{24}$, $\pi(5) = \frac{1}{24}$, $\pi(6) = \frac{1}{24}$, using many simulations (Monte Carlo Simulation).

$\gamma = 1$, *error* = 0.1440.

$\gamma = 4$ *error* = 0.0641

$\gamma = 50$ *error* = 0.1818

Small values of γ provide a better theoretical upperbound (cf 2). Here, we see that there is a trade off with regards of the value of γ . Small values will not promote the exploration of the set X , and this explains the bad convergence of the algorithm.

Exercise 2

1. Let a family of transition matrices

$$P_t = \begin{bmatrix} t & (1-t) \\ (1-t) & t \end{bmatrix}$$

The obvious invariant distribution is $\pi = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

Since $t \in]0, 1[$, the Markov Chain is obviously irreducible.

-> récurrente positive -> apériodique

2. Given the algorithm written X_n depends only on X_{n-1} and,

$$\mathbb{P}(X_{n+1} = x | X_n = y) = P_{t_0}(y, x) \mathbf{1}_{y=0} + P_{t_1}(y, x) \mathbf{1}_{y=1}$$

, then

$$P_* = \begin{bmatrix} t_0 & (1-t_1) \\ (1-t_0) & t_1 \end{bmatrix}$$

Solving

$$\pi P = \pi$$

gives $\pi_1^* = \frac{\pi_2(t_1-1)}{t_0-1}$, and $\pi_1 + \pi_2 = 1$ yields :

$$\pi_1^* = \frac{t_1 - 1}{t_0 + t_1 - 2}$$

and

$$\pi_2^* = \frac{t_0 - 1}{t_0 + t_1 - 2}$$

The chain is obviously irreducible (each state is available from another since $t_0 \in]0, 1[$ and $t_1 \in]0, 1[$).

Exercise 3