Homework 3 : Part A

Oussama Ennafii & Sammy Khalife

25/02/2015

Exercise 1

1. Let $\gamma \in \mathbb{N}^*$ and $X = \{1, .., 6\}$. We consider the Hastings-Metropolis algorithm with target π associated to the proposal distribution:

$$q(x, .) = \mathcal{U}(\{X_n - \gamma, X_n - \gamma + 1, ..., X_n - 1, X_n + 1, ..., X_n + \gamma\})$$

Algorithm 1 Metropolis Hastings algorithm

Given one initial point in $x_0 \in X$
for n=1:N
-Generate $Y_n \sim q(x_n, .)$
-$X_{n+1} = Y_n$ with probability $\alpha(X_n, Y_n)$
and $X_{n+1} = X_n$ with probability $1 - \alpha(X_n, Y_n)$.
end

Since $\forall x \in X, \forall y \in X, q(x, y) = \frac{1}{\gamma}$, the acceptance ratio is equal to

$$\alpha(x, y) = \begin{cases} 1 \wedge \frac{\pi(y)}{\pi(x)} & \text{if} \quad y \in X \\ 0 & \text{otherwise} \end{cases}$$

To avoid technical discussions, we will consider the extended distribution π on $\mathbb{N}$, with $\pi(k) = 0$ if $k \notin X$ so that :

$$\alpha(x, y) = 1 \wedge \frac{\pi(y)}{\pi(x)}$$

And the transition kernel is equal, for $y \neq x$ to:

$$\begin{aligned} p(x, y) &= q(x, y)\alpha(x, y) \\ &= \frac{1}{\gamma}(1 \wedge \frac{\pi(y)}{\pi(x)}) \end{aligned}$$

and :

$$p(x, x) = 1 - \sum_{y \neq x} q(x, y)\alpha(x, y)$$

First :

$$p(x, y \neq x) = \frac{1}{\gamma}(1 \wedge \frac{\pi(y)}{\pi(x)})$$
$$\geq \frac{1}{\gamma}(1 \wedge \frac{\pi(y)}{\max_x \pi(x)})$$
$$= \frac{\pi(y)}{\gamma \max_x \pi(x)}$$
$$\geq \frac{\pi(y)}{\gamma}$$

The third equality is due to $\frac{\pi(y)}{\max_x \pi(x)} \leq 1$. The fourth inequality is due to $max_x \pi(x) \leq 1$.

Second,

$$p(x, x) = 1 - \sum_{y \neq x} q(x, y)\alpha(x, y)$$
$$= q(x, x) + \sum_{y \neq x} q(x, y) - \sum_{y \neq x} q(x, y)\alpha(x, y)$$
$$= q(x, x) + \sum_{x \neq y} q(x, y)(1 - \alpha(x, y))$$
$$\geq q(x, x)$$
$$= \frac{1}{\gamma}$$
$$\geq \frac{\pi(x)}{\gamma}$$

The second inequality is obtained by upper bounding the sum terms by 1, and using $|X| = 6$.
Let $\boxed{\epsilon = \frac{1}{\gamma}}$, then

$$\forall x, \forall y \quad p(x, y) \geq \epsilon\pi(y)$$

which can be written in terms of subsets B of X :

$$\boxed{p(x, B) \geq \epsilon\pi(B)}$$

2. Given the inequality written above, the algorithm satisfies the Doeblin condition. The Doeblin Lemma gives : $\boxed{\Delta(P) \leq (1 - \epsilon)}$

Given one initial distribution ξ, we have in the general case:

$$||\xi P^n - \pi||_{TV} \leq (\Delta(P))^n ||\xi - \pi||_{TV}$$

which yields here :

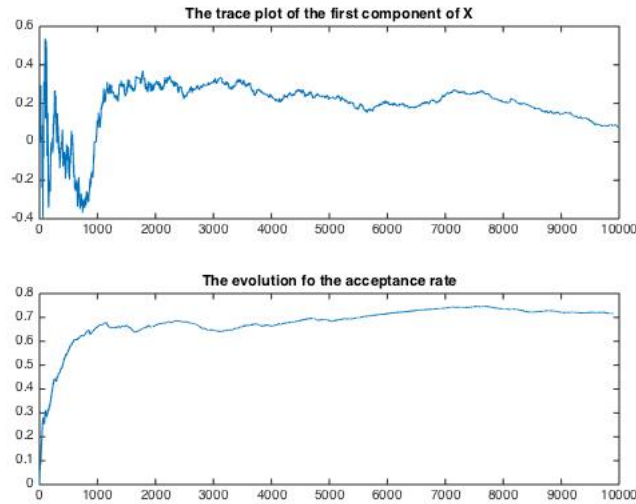$$\boxed{||\xi P^n - \pi||_{TV} \leq (1 - \frac{1}{\gamma})^n ||\xi - \pi||_{TV}}$$

Figure 1: The naive symmetric random walk Hastings-Metropolis chain

Where ξP^n is actually the distribution of X_n. We have a geometric ergodicity

3.

Exercise 3

1.

2.

Exercise 3

We choose for the numerical simulation the parameters:

$$
\begin{aligned}
N : &= 1000 \\
c : &= 2 \\
n_0 : &= 100
\end{aligned}
$$

1. In figure 1, we see how the sequence X_1^n varies too much and takes a long time to converge to the mean which is 0 in this case. The second plot shows that the acceptance ratio also converges , slowly though, up to around 70%. So there is room to do better.

2. In figure 2, we see that the sequence X_n converges abruptly to the mean after $n0$. The second plot shows that the acceptance ratio also converges rapidly up to around 100%.
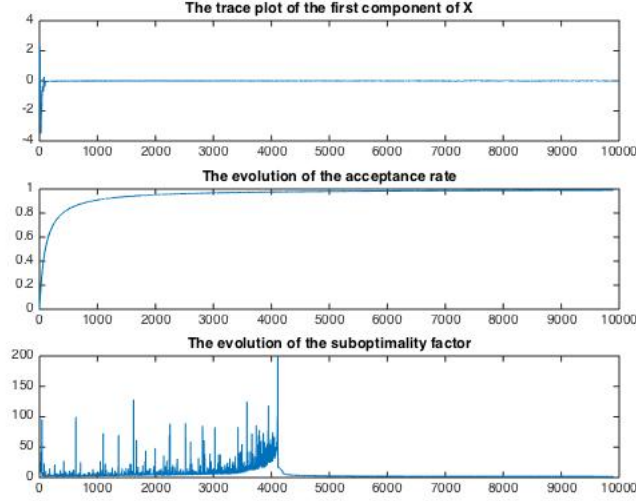
Figure 2: The adaptive symmetric random walk Hastings-Metropolis chain

Further, we can see that the suboptimality factor depend on n in 2 ways. First, it varies a lot and stays strictly over 1 (the optimal case). In the second phase, it starts decreasing very slowly to 1.

3. The suboptimality factor of the first algorithm is constant independent from c:

$$n \mapsto d \frac{trace(\Gamma_\pi^2)}{trace(\Gamma_\pi)^2} > 1$$

The adaptive algorithm yields a sequence that adapts and thus convergences, slowly, to the optimal case. In conclusion, the adaptive random walk is much better than the naive one.