Homework 4 (HW4 Part B)

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Step 1

Step 2

Step 3

1.

2. Let us factorize the expression of $\tilde{\pi}_\theta(u, w)$ under the form :

$$\begin{aligned}
\tilde{\pi}_\theta(u, w) &= h(\theta, w, \beta, \sigma, ...)exp(-\frac{1}{2}u^t u - \sum_{i=1}^{N} \frac{w_i}{2}\sigma^2 (z_i' u)^2 + \sum_{i=1}^{N} \sigma(Y_i - \frac{1}{2}) z_i' u - w_i x_i' \beta \sigma z_i' u) \\
&= h(\theta, w, \beta, \sigma, ...)exp(-\frac{1}{2}u^t (I + \sigma^2 \sum_{i=1}^{n} \frac{w_i}{2} z_i z_i') u + \sigma < u, \sum_{i=1}^{N} ((Y_i - \frac{1}{2}) - w_i x_i' \beta) z_i >)
\end{aligned}$$

Where h does not depend on u. We can directly identify the density of a Gaussian :

$$\tilde{\pi}_\theta(u|w) \quad \propto \quad exp(-\frac{1}{2}u^t (I + \sigma^2 \sum_{i=1}^{n} \frac{w_i}{2} z_i z_i') u + \sigma < u, \sum_{i=1}^{N} ((Y_i - \frac{1}{2}) - w_i x_i' \beta) z_i >)$$

i.e

$$\boxed{\tilde{\pi}_\theta(u|w) \quad \propto \mathcal{N}(\mu_\theta, \Gamma_\theta(w))}$$

With $\boxed{\Gamma_\theta(w) = (I + \sigma^2 \sum_{i=1}^{n} \frac{w_i}{2} z_i z_i')^{-1}}$, and $\boxed{\mu_\theta(w) = \sigma \Gamma_\theta(w) \sum_{i=1}^{N} ((Y_i - \frac{1}{2}) - w_i x_i' \beta) z_i}$.

3. We use the same reasoning as the previous question.

$$\begin{aligned}
\tilde{\pi}_\theta(w|u) &\underset{w}{\propto} \prod_{i=1}^{N} \rho(w_i) \mathbf{1}_{R^+}(w_i) exp(-\frac{w_i}{2}(x_i' \beta + \sigma z_i' u)^2) \\
&\underset{w}{\propto} \prod_{i=1}^{N} Z cosh(\frac{x_i' \beta + \sigma z_i' u}{2}) \mathbf{1}_{R^+}(w_i) exp(-\frac{w_i}{2}(x_i' \beta + \sigma z_i' u)^2)
\end{aligned}$$

Since $cosh(x) = cosh(|x|)$,

$$\begin{aligned}\tilde{\pi}_\theta(w|u) &\underset{w}{\propto} \prod_{i=1}^{N} Zcosh(\frac{|x_i'\beta + \sigma z_i' u|}{2})\mathbf{1}_{R^+}(w_i)exp(-\frac{w_i}{2}(x_i'\beta + \sigma z_i' u)^2) \\ &\underset{w}{\propto} \prod_{i=1}^{N} \bar{\pi}(w_i; |x_i'\beta + \sigma z_i' u|)\end{aligned}$$

Since the last line a function normalized in w, i.e

$$\begin{aligned}\int_\Omega \prod_{i=1}^{N} \bar{\pi}(w_i; |x_i'\beta + \sigma z_i' u|)dw &= \prod_{i=1}^{N} \int_{\Omega_i} \bar{\pi}(w_i; |x_i'\beta + \sigma z_i' u|)dw_i \\ &= 1\end{aligned}$$

We have

$$\boxed{\tilde{\pi}_\theta(w|u) = \prod_{i=1}^{N} \bar{\pi}(w_i; |x_i'\beta + \sigma z_i' u|)}$$

4.

5. To sample $(w_1, ..., w_N, u)$ from $\tilde{\pi}_\theta(u, w)$, we will use a Gibbs Sampler

Algorithm 1 Gibbs Sampler to sample from $\tilde{\pi}_\theta$

Given, $w_1^{(0)}, ..., w_N^{(0)}$

Loop on k
Draw $u^{(k+1)}$ from $\pi_\theta(.|w^{(k)}) = \mathcal{N}(\mu_\theta, \Gamma_\theta(\omega))$
-for i=1:N
$w_i^{(k+1)}$ drawn thanks to HW1Sampler(.,$|x_i'\beta + \sigma z_i' u^{(k+1)}|/2$)
-end for i
end k

6.