

## Homework 4 (HW4 Part B)

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### Step 1

1. The equality is trivial: we just have to set  $\vartheta = \theta$ .

To prove the equality

$$Q_\gamma(\vartheta, \theta) \geq (f + g)(\vartheta)$$

, we just need to prove that:

$$\forall \vartheta, \theta \in \Theta \quad f(\vartheta) \leq f(\theta) + \langle \nabla f(\theta), \vartheta - \theta \rangle + \frac{1}{2\gamma} \|\vartheta - \theta\|^2$$

Set  $\vartheta, \theta \in \Theta$ . Let's take  $t \in [0, 1]$ , we define:

$$F(t) \triangleq f(\vartheta_t) - f(\theta) - \langle \nabla f(\theta), \vartheta_t - \theta \rangle - \frac{1}{2\gamma} \|\vartheta_t - \theta\|^2 \quad (1)$$

where  $\vartheta_t = t\vartheta + (1-t)\theta$ . Since,  $\frac{d}{dt}\vartheta_t = \vartheta - \theta$ ,  $F$  is differentiable and:

$$\begin{aligned} \frac{d}{dt}F(t) &= \langle \nabla f(\vartheta), \vartheta - \theta \rangle - \langle \nabla f(\theta), \vartheta - \theta \rangle - \frac{1}{\gamma} \|\vartheta - \theta\|^2 \\ &= \langle \nabla f(\vartheta) - \nabla f(\theta), \vartheta - \theta \rangle - \frac{1}{\gamma} \|\vartheta - \theta\|^2 \\ &\leq \|\nabla f(\vartheta) - \nabla f(\theta)\| \cdot \|\vartheta - \theta\| - \frac{1}{\gamma} \|\vartheta - \theta\|^2 \\ &\leq (L - \frac{1}{\gamma}) \|\vartheta - \theta\|^2 \\ &\leq 0 \end{aligned}$$

The first inequality is the result of Cauchy-Schwartz inequality while the second one is due to the Lipschitz nature of the gradient. The third inequality comes from the fact that  $0 < \gamma \leq \frac{1}{L}$ .

This means that  $F$  is decreasing in  $[0, 1]$ , so:  $F(0) \geq F(1)$ :

$$F(0) = 0$$

and :

$$F(1) = f(\vartheta) - f(\theta) - \langle \nabla f(\theta), \vartheta - \theta \rangle - \frac{1}{2\gamma} \|\vartheta - \theta\|^2$$

2. Let  $n \in \mathbb{N}$ :

$$\begin{aligned}
 (f + g)(\theta_n) &= Q_{\gamma_{n+1}}(\theta_n, \theta_n) \\
 &\geq Q_{\gamma_{n+1}}(\theta_{n+1}, \theta_n) \\
 &\geq (f + g)(\theta_{n+1})
 \end{aligned}$$

The first inequality is the result of the definition of  $\theta_{n+1}$ . The second one comes from the first question.

3. Set  $\vartheta, \theta \in \Theta$ . we develop:

$$\frac{1}{2\gamma} \|\vartheta - \theta - \gamma \nabla f(\theta)\|^2 = \frac{1}{2\gamma} \|\vartheta - \theta\|^2 - \langle \vartheta - \theta, \nabla f(\theta) \rangle + \frac{\gamma}{2} \|\nabla f(\theta)\|^2$$

This means that :

$$Q_\gamma(\vartheta, \theta) = g(\vartheta) + f(\theta) + \frac{1}{2\gamma} \|\vartheta - \theta - \gamma \nabla f(\theta)\|^2 - \frac{\gamma}{2} \|\nabla f(\theta)\|^2$$

4. Taking off the constants w.r.t to  $\vartheta$ , we get:

$$\begin{aligned}
 \theta_{n+1} &= \arg \min_{\vartheta \in \Theta} Q_{\gamma_{n+1}}(\vartheta, \theta_n) \\
 &= \arg \min_{\vartheta \in \Theta} g(\vartheta) + \frac{1}{2\gamma_{n+1}} \|\vartheta - \theta_n - \gamma_{n+1} \nabla f(\theta_n)\|^2 \\
 &= \arg \min_{\vartheta \in \Theta} \sum_{i=1, \dots, d} \lambda |\vartheta_i| + \frac{1}{2\gamma_{n+1}} (\vartheta_i - (\theta_n)_i - \gamma_{n+1} (\nabla f(\theta_n))_i)^2
 \end{aligned}$$

So we can just minimize on each direction independently:

$$(\theta_{n+1})_i = \arg \min_{\vartheta_i} \lambda |\vartheta_i| + \frac{1}{2\gamma_{n+1}} (\vartheta_i - (\theta_n)_i - \gamma_{n+1} (\nabla f(\theta_n))_i)^2$$

We set :  $u = \theta_n - \gamma_{n+1} \nabla f(\theta_n)$ . We rewrite the last equation as:

$$(\theta_{n+1})_i = \arg \min_{\vartheta_i} \lambda |\vartheta_i| + \frac{1}{2\gamma_{n+1}} (\vartheta_i - u_i)^2$$

So we consider the function:

$$L_u^\gamma(x) \triangleq \lambda |x| + \frac{1}{2\gamma} (x - u)^2$$

It is obvious that :

$$L_u^\gamma(x) = L_{-u}^\gamma(x)$$

So  $L$  is symmetric w.r.t.  $u$ . It means that we can just focus on the case where:  $u \in \mathbb{R}_+$ :

- $L_u^\gamma$  is decreasing in  $\mathbb{R}_-$  from  $+\infty$  to  $\frac{1}{2\gamma} u^2$ , because:

$$\forall x \in \mathbb{R}_- \quad L_u^\gamma(x) = -\lambda x + \frac{1}{2\gamma} (x - u)^2$$

- Since:

$$\forall x \in \mathbb{R}_+ \quad \frac{d}{dx} L_u^\gamma(x) = \lambda + \frac{1}{\gamma}(x - u)$$

Two cases present themselves:

- (i). Either  $u - \gamma\lambda \geq 0$  and then  $L_u^\gamma$  decreases in  $[0, u - \gamma\lambda]$  from  $\frac{1}{2\gamma}u^2$  to  $\lambda u - \frac{\lambda^2\gamma}{2}$  and then starts increasing in  $[u - \gamma\lambda, +\infty)$  from  $\lambda u - \frac{\lambda^2\gamma}{2}$  to  $+\infty$ .
- (ii). Or  $u - \gamma\lambda < 0$ , which means that  $L_u^\gamma$  increases in  $\mathbb{R}_+$  from  $\frac{1}{2\gamma}u^2$  to  $+\infty$

We make use of the symmetry of  $L$  w.r.t.  $u$  and we summarise the previous discussion:

$$\arg \min L_u^\gamma(x) = \begin{cases} u - \gamma\lambda & , \text{ if } u \geq \gamma\lambda \\ 0 & , \text{ if } u \in (-\lambda\gamma, \gamma\lambda) \\ u + \gamma\lambda & , \text{ if } u \leq -\gamma\lambda \end{cases}$$

Now we can conclude that :

$$\theta_{n+1} = P_{\gamma_{n+1}}(\theta_n - \gamma_{n+1} \nabla f(\theta_n))$$

## Step 2

1. For  $i = 1, \dots, N$ , we introduce:  $\tilde{u}_i \triangleq x_i' \beta + \sigma z_i' U$

$$\begin{aligned} \mathbb{P}(Y_i = y_i | \mathbf{U}) &= s(\tilde{u}_i)^{y_i} (1 - s(\tilde{u}_i))^{1-y_i} \\ &= \frac{e^{\tilde{u}_i y_i}}{(1 + e^{\tilde{u}_i})^{y_i}} \times \frac{1}{(1 + e^{\tilde{u}_i})^{1-y_i}} \\ &= \frac{e^{\tilde{u}_i y_i}}{(1 + e^{\tilde{u}_i})} \end{aligned}$$

This means that:

$$\begin{aligned} \mathbb{P}((Y_1, \dots, Y_N) = (y_1, \dots, y_N) | \mathbf{U}) &= \prod_{i=1, \dots, N} \frac{e^{\tilde{u}_i y_i}}{(1 + e^{\tilde{u}_i})} \\ &= \frac{\exp(\sum_{i=1, \dots, N} \tilde{u}_i y_i)}{\prod_{i=1, \dots, N} (1 + e^{\tilde{u}_i})} \end{aligned}$$

$$\begin{aligned} \mathbb{P}((Y_1, \dots, Y_N) = (y_1, \dots, y_N)) &= \int_{\mathbb{R}} \mathbb{P}((Y_1, \dots, Y_N) = (y_1, \dots, y_N) | \mathbf{U} = u) \phi(u) du \\ &= \int_{\mathbb{R}} \frac{\exp(\sum_{i=1, \dots, N} \tilde{u}_i y_i)}{\prod_{i=1, \dots, N} (1 + e^{\tilde{u}_i})} \phi(u) du \end{aligned}$$

Now we can write that:

$$l(\theta|Y_1, \dots, Y_N) = \log\left(\int_{\mathbb{R}} \frac{\exp(\sum_{i=1, \dots, N} (x'_i \beta + \sigma z'_i u) Y_i)}{\prod_{i=1, \dots, N} (1 + \exp(x'_i \beta + \sigma z'_i u))} \phi(u) du\right)$$

2. We can derive inside the integral using the bounded convergence since the function is continuous:

$$\begin{aligned} \nabla l(\theta) &= \frac{1}{\int_{\mathbb{R}} \frac{\exp(\sum_{i=1, \dots, N} (x'_i \beta + \sigma z'_i u) Y_i)}{\prod_{i=1, \dots, N} (1 + \exp(x'_i \beta + \sigma z'_i u))} \phi(u) du} \times \nabla \int_{\mathbb{R}} \frac{\exp(\sum_{i=1, \dots, N} (x'_i \beta + \sigma z'_i u) Y_i)}{\prod_{i=1, \dots, N} (1 + \exp(x'_i \beta + \sigma z'_i u))} \phi(u) du \\ &= \frac{1}{\exp(l(\theta))} \times \nabla \int_{\mathbb{R}} \exp\left(\sum_{i=1, \dots, N} (x'_i \beta + \sigma z'_i u) Y_i - \log(1 + \exp(x'_i \beta + \sigma z'_i u))\right) \phi(u) du \\ &= \exp(-l(\theta)) \int_{\mathbb{R}} \nabla \exp(l_c(\theta|\mathbf{u})) \phi(u) du \\ &= \int_{\mathbb{R}} \nabla l_c(\theta|\mathbf{u}) \exp(l_c(\theta|\mathbf{u}) - l(\theta)) \phi(u) du \\ &= \int_{\mathbb{R}} \sum_{i=1}^N (Y_i \begin{bmatrix} x_i \\ z'_i u \end{bmatrix} - \frac{\exp(x'_i \beta + \sigma z'_i u)}{1 + \exp(x'_i \beta + \sigma z'_i u)} \begin{bmatrix} x_i \\ z'_i u \end{bmatrix}) \exp(l_c(\theta|\mathbf{u}) - l(\theta)) \phi(u) du \end{aligned}$$

## Step 3

1.

$$\begin{aligned}
\tilde{\pi}_\theta(u, w) &= \pi_\theta(u) \left( \prod_{i=1}^N \bar{\pi}(w_i; x'_i \beta + \sigma z'_i u) \right) \\
&= \frac{1}{\exp(l(\theta))} \phi(u) \exp(l_c(\theta|u)) \left( \prod_{i=1}^N \bar{\pi}(w_i; x'_i \beta + \sigma z'_i u) \right) \\
&= \frac{1}{\exp(l(\theta))} \phi(u) \prod_{i=1}^N Z \cosh\left(\frac{x'_i \beta + \sigma z'_i u}{2}\right) \rho(w_i) \mathbf{1}_{\mathbb{R}^+}(w_i) \exp(-w(x'_i \beta + \sigma z'_i u)^2/2) \\
&\quad \frac{\exp(Y_i(x'_i \beta + \sigma z'_i u))}{1 + \exp(x'_i \beta + \sigma z'_i u)}
\end{aligned}$$

Writing  $\cosh(x) = \exp(-x/2)(1 + \exp(x))^{1/2}$ , we obtain :

$$\begin{aligned}
\tilde{\pi}_\theta(u, w) &= \frac{1}{\exp(l(\theta))} \phi(u) \prod_{i=1}^N Z \rho(w_i) \mathbf{1}_{\mathbb{R}^+}(w_i) \exp(Y_i(x'_i \beta + \sigma z'_i u) - (x'_i \beta + \sigma z'_i u)/2 - w_i(x'_i \beta + \sigma z'_i u)^2/2) \\
&= \frac{1}{\exp(l(\theta))} \phi(u) Z^N \prod_{i=1}^N \exp(Y_i x'_i \beta - x'_i \beta/2) \\
&\quad \prod_{i=1}^N \rho(w_i) \mathbf{1}_{\mathbb{R}^+}(w_i) \exp(\sigma(Y_i - 1/2) z'_i u - w_i(x'_i \beta + \sigma z'_i u)^2/2) \\
&= C(\theta) \phi(u) \prod_{i=1}^N \rho(w_i) \mathbf{1}_{\mathbb{R}^+}(w_i) \exp(\sigma(Y_i - 1/2) z'_i u - w_i(x'_i \beta + \sigma z'_i u)^2/2)
\end{aligned}$$

With  $C(\theta) = \frac{1}{\exp(l(\theta))} \phi(u) Z^N \prod_{i=1}^N \exp(Y_i x'_i \beta - x'_i \beta/2),$

$$\tilde{\pi}_\theta(u, w) = C(\theta) \phi(u) \prod_{i=1}^N \rho(w_i) \mathbf{1}_{\mathbb{R}^+}(w_i) \exp(\sigma(Y_i - 1/2) z'_i u - w_i(x'_i \beta + \sigma z'_i u)^2/2)$$

2. Let us factorize the expression of  $\tilde{\pi}_\theta(u, w)$  under the form :

$$\begin{aligned}
\tilde{\pi}_\theta(u, w) &= h(\theta, w, \beta, \sigma, \dots) \exp\left(-\frac{1}{2} u^t u - \sum_{i=1}^N \frac{w_i}{2} \sigma^2 (z'_i u)^2 + \sum_{i=1}^N \sigma(Y_i - \frac{1}{2}) z'_i u - w_i x'_i \beta \sigma z'_i u\right) \\
&= h(\theta, w, \beta, \sigma, \dots) \exp\left(-\frac{1}{2} u^t (I + \sigma^2 \sum_{i=1}^n \frac{w_i}{2} z_i z'_i) u + \sigma \langle u, \sum_{i=1}^N ((Y_i - \frac{1}{2}) - w_i x'_i \beta) z_i \rangle\right)
\end{aligned}$$

Where  $h$  does not depend on  $u$ . We can directly identify the density of a Gaussian :

$$\tilde{\pi}_\theta(u|w) \propto \exp\left(-\frac{1}{2}u^t(I + \sigma^2 \sum_{i=1}^n \frac{w_i}{2} z_i z_i')u + \sigma < u, \sum_{i=1}^N ((Y_i - \frac{1}{2}) - w_i x_i' \beta) z_i >\right)$$

i.e (we can identify since the Gaussian is normalized)

$$\boxed{\tilde{\pi}_\theta(u|w) = \mathcal{N}(\mu_\theta, \Gamma_\theta(w))}$$

With  $\boxed{\Gamma_\theta(w) = (I + \sigma^2 \sum_{i=1}^n \frac{w_i}{2} z_i z_i')^{-1}}$ , and  $\boxed{\mu_\theta(w) = \sigma \Gamma_\theta(w) \sum_{i=1}^N ((Y_i - \frac{1}{2}) - w_i x_i' \beta) z_i}$ .

3. We use the same reasoning as the previous question.

$$\begin{aligned} \tilde{\pi}_\theta(w|u) &\propto_w \prod_{i=1}^N \rho(w_i) \mathbf{1}_{R^+}(w_i) \exp\left(-\frac{w_i}{2} (x_i' \beta + \sigma z_i' u)^2\right) \\ &\propto_w \prod_{i=1}^N Z \cosh\left(\frac{x_i' \beta + \sigma z_i' u}{2}\right) \mathbf{1}_{R^+}(w_i) \exp\left(-\frac{w_i}{2} (x_i' \beta + \sigma z_i' u)^2\right) \end{aligned}$$

Since  $\cosh(x) = \cosh(|x|)$ ,

$$\begin{aligned} \tilde{\pi}_\theta(w|u) &\propto_w \prod_{i=1}^N Z \cosh\left(\frac{|x_i' \beta + \sigma z_i' u|}{2}\right) \mathbf{1}_{R^+}(w_i) \exp\left(-\frac{w_i}{2} (x_i' \beta + \sigma z_i' u)^2\right) \\ &\propto_w \prod_{i=1}^N \bar{\pi}(w_i; |x_i' \beta + \sigma z_i' u|) \end{aligned}$$

Since the last line a function normalized in  $w$ , i.e

$$\begin{aligned} \int_{\Omega} \prod_{i=1}^N \bar{\pi}(w_i; |x_i' \beta + \sigma z_i' u|) dw &= \prod_{i=1}^N \int_{\Omega_i} \bar{\pi}(w_i; |x_i' \beta + \sigma z_i' u|) dw_i \\ &= 1 \end{aligned}$$

We have

$$\boxed{\tilde{\pi}_\theta(w|u) = \prod_{i=1}^N \bar{\pi}(w_i; |x_i' \beta + \sigma z_i' u|)}$$

4.

5. To sample  $(w_1, \dots, w_N, u)$  from  $\tilde{\pi}_\theta(u, w)$ , we will use a Gibbs Sampler

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**Algorithm 1** Gibbs Sampler to sample from  $\tilde{\pi}_\theta$ 

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Given,  $w_1^{(0)}, \dots, w_N^{(0)}$ 

Loop on k

Draw  $u^{(k+1)}$  from  $\pi_\theta(\cdot|w^{(k)}) = \mathcal{N}(\mu_\theta, \Gamma_\theta(\omega))$ 

-for i=1:N

 $w_i^{(k+1)}$  drawn thanks to HW1Sampler( $\cdot, |x'_i\beta + \sigma z'_i u^{(k+1)}|/2$ )

-end for i

end k

return  $(u^{(k)}, w_{1 \leq k \leq K})$ 

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Cf code for implementation.

6. One can use a sample generated by the previous algorithm to approximate the integral of  $\nabla l(\theta)$ . Let us denote  $H_\theta(u) = \sum_{i=1}^N (Y_i - s(x'_i\beta + \sigma z'_i u)) \begin{pmatrix} x_i \\ z'_i u \end{pmatrix}$ . Then,

$$\int H_\theta(u) \pi_\theta(u) du \sim \frac{1}{K} \sum_{k=1}^K H_\theta(u^k)$$

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**Algorithm 2** Algorithm to approximate  $\nabla l(\theta)$ 

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Sample a chain  $(u^{(k)}, w^{(k)})_{1 \leq k \leq K}$  with algorithm 1.return  $\frac{1}{K} \sum_{k=1}^K H_\theta(u^k)$ 

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Cf code for implementation.

## Step 4

1. c.f. the code directory  $\rightarrow$  'dataset\_generator.m'.