

GLM(M)s for binary data

Bayesian statistics 7 – generalized linear models for binary data

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Some things that we learned the last time

- Poisson models good for rare events generating small counts
- Classical *link function* is the log-link so that
$$Y_i | \epsilon_i \sim \mathcal{P}(\exp(a + bx_i + [\text{stuff}] + \epsilon_i)) \text{ (with extra dispersion)}$$
- Posterior predictive checks

The Binomial distribution: reminders and GLM

Let $U_i \sim \text{Bernoulli}(p)$ a coin toss with probability p .

- Then $Y = \sum_{i=1}^n U_i \sim \mathcal{B}(n, p)$
- Converges to normal distribution for large np (≥ 10) as n grows
- $\mathbb{E}(Y) = np$ and $\mathbb{E}(Y) = np(1 - p)$
- Conjugate prior for p = Beta distribution.

Two ways to specify a GLM:

- $Y_i \sim \mathcal{B}(n, p)$ with $\text{logit}(p) = a + bx_i + [\text{stuff}]$ in which case the data resembles $c(31, 14, 5, 0, 19)$
- $U_i \sim \mathcal{B}(p)$ with $\text{logit}(p) = a + bx_i + [\text{stuff}]$ in which case the data resembles $c(0, 0, 1, 0, 1, 1, 0)$

Environnementally-driven turtle sex determination



Figure 1: Green turtle, Malaysia. Bernard Dupont. Licence: [CC BY SA 2.0](#)

First described in 1966 by [Madeleine Charnier](#) in a lizard, subsequent work on turtles (Chelonia) and Crocodylia, see [Janzen & Paukstis QRB 1991](#).

$$\mathbb{P}(\text{female hatchling}) = f(\text{temperature})$$



Figure 2: Stephan Hunt. Hatching green turtle, Ascension Island. [CC BY 3.0](#)

Empirical target

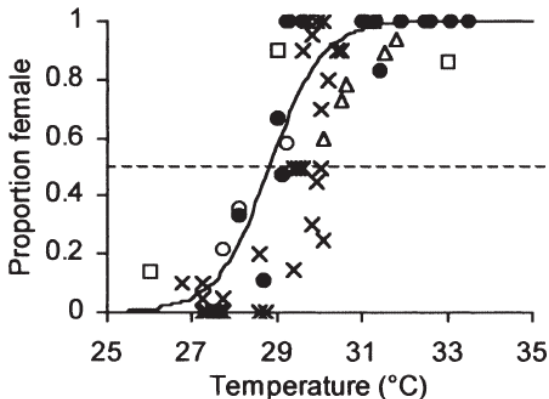


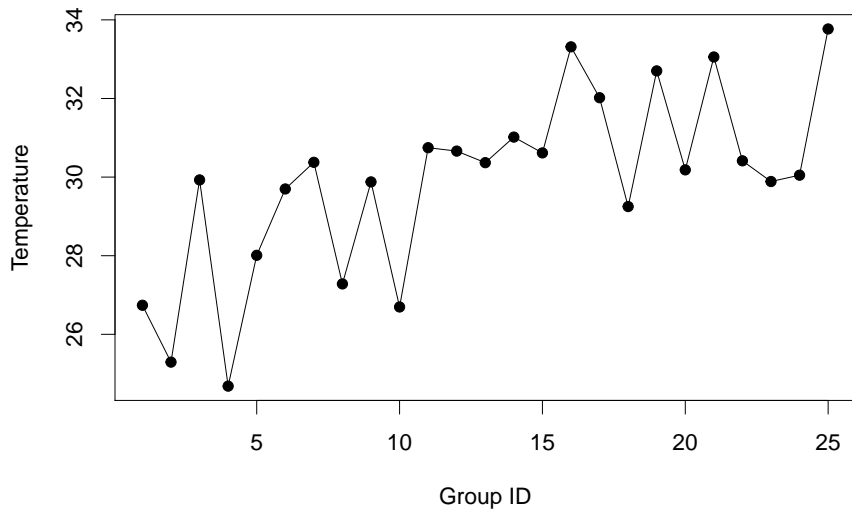
Figure 3: Temperature-dependent sex determination of Ascension Island green turtles, by Goldley et al. Marine Ecology Progress Series Vol. 226: 115–124, 2002.

Simulating data I

We have got 25 groups i of size 50 or less (i.e., how many baby turtles are hatched at one beach location z_i), with a different temperature in each group.

```
sample_size_per_group = round(30*runif(25))+20
n_groups = length(sample_size_per_group)
temperature = 25 + (1:n_groups)*8/25 + rnorm(n_groups,0,2)
par(cex=1.5,pch=19)
plot(1:n_groups,temperature,type="o",
     xlab="Group ID",ylab="Temperature")
```

Simulating data II

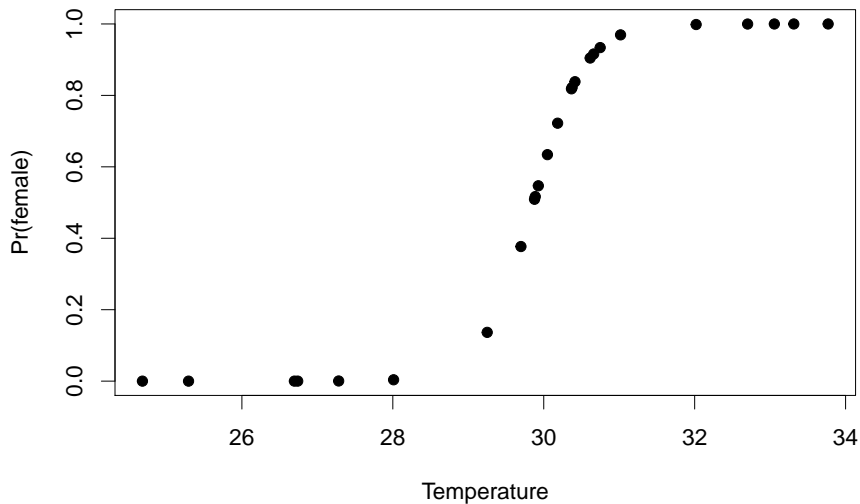


Binomial sampling I

Females are born with probability p_i in group i , which itself depends on temperature.

```
Y = p = temperature
for (i in 1:n_groups){
  p[i] = 1/(1+exp(-3*(temperature[i]-mean(temperature))) )
  Y[i] = rbinom(1,sample_size_per_group[i],p[i])
}
par(cex=1.5,pch=19)
plot(temperature,p,type="p",
     xlab="Temperature",ylab="Pr(female)")
```

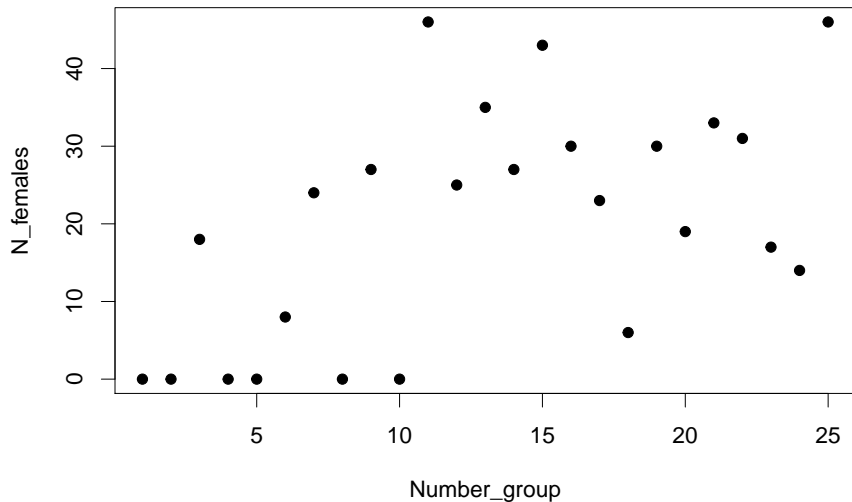
Binomial sampling II



The data I

```
par(cex=1.5,pch=19)
plot(1:n_groups,Y,type="p",
     xlab="Number_group",ylab="N_females")
```

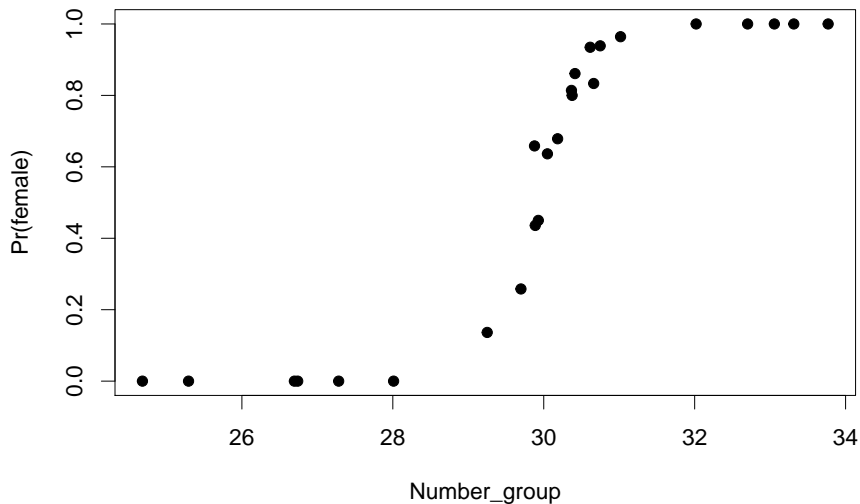
The data II



Empirical proportion estimates I

```
par(cex=1.5,pch=19)
plot(temperature,Y/sample_size_per_group,
     type="p",xlab="Number_group",
     ylab="Pr(female)")
```

Empirical proportion estimates II



Prior predictive checks

We have seen posterior predictive distribution = the distribution of imaginary data under the fitted model (given a posterior distribution).

Prior predictive distribution = the distribution of imaginary data under the priors.

Let's say we have proportion $\theta \sim \text{Beta}(\alpha, \beta)$ which is the prior for a very simple $Y_i \sim \mathcal{B}(n, \theta)$ model where n is known. Then simulating data under the prior predictive distribution can be done as

```
for (rep in 1:nrep){  
  theta[rep] = rbeta(1,alpha,beta)  
  Yrep[rep] = rbinom(1,size=n,prob=theta[rep])  
}
```

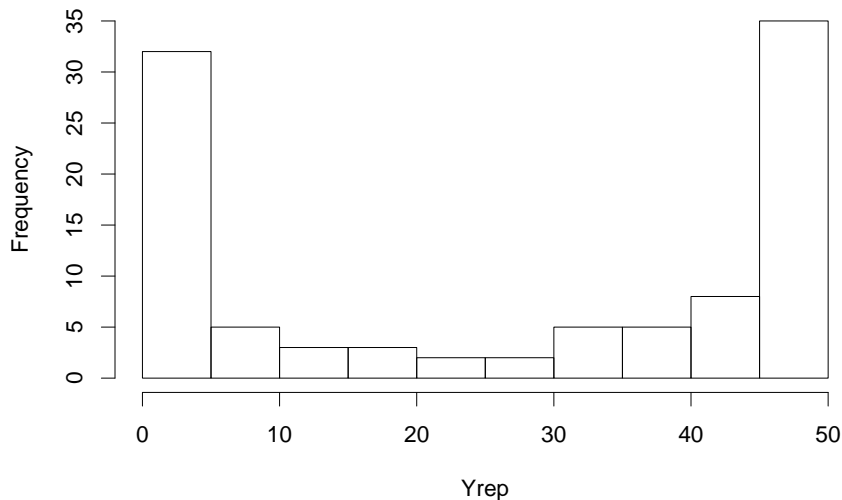
(it's a [Beta-Binomial distribution](#))

Prior predictive checks (practice) I

```
nrep = 100
n = 50
alpha=0.1
beta=0.1
Yrep = theta = rep(0,nrep)
for (rep in 1:nrep){
  theta[rep] = rbeta(1,alpha,beta)
  Yrep[rep] = rbinom(1,size=n,prob=theta[rep])
}
par(cex=1.5,pch=19)
hist(Yrep)
```


Prior predictive checks (practice) II

Histogram of Yrep

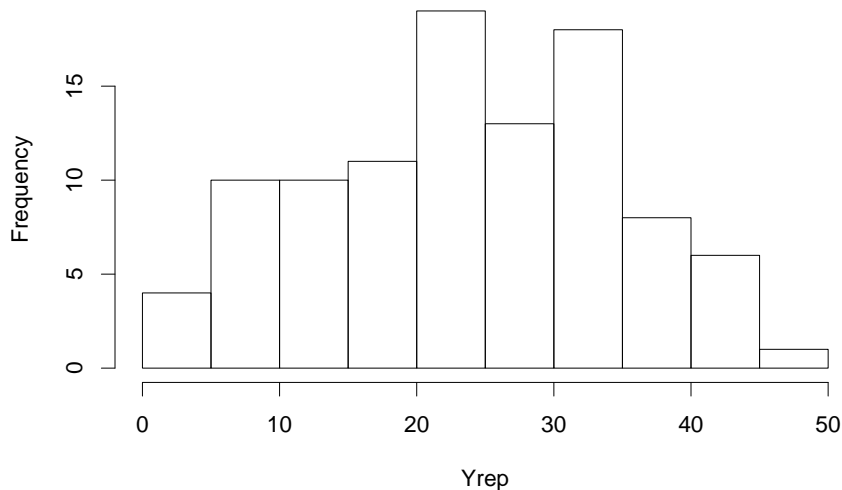


Prior predictive checks – better prior I

```
nrep = 100
n = 50
alpha=2
beta=2
Yrep = theta = rep(0,nrep)
for (rep in 1:nrep){
  theta[rep] = rbeta(1,alpha,beta)
  Yrep[rep] = rbinom(1,size=n,prob=theta[rep])
}
par(cex=1.5,pch=19)
hist(Yrep)
```

Prior predictive checks – better prior II

Histogram of Yrep



Fitting the binomial model I

Now we fit that model which writes mathematically like

$$y_i \sim \mathcal{B}(z_i, p(\text{temp}_i))$$

Fitting the binomial model II

```
m11.data <- list(N = n_groups, y = Y, temp = temperature,
                 z = sample_size_per_group)

cat(file="logistic.regression.txt", "
model {
  mu_temp ~ dnorm(2, 0.1) ## prior of the mean temp
  gamma ~ dnorm(1, 0.1)  ## prior of the slope

  for (k in 1:N){
    y[k] ~ dbin(p[k],z[k])      ## likelihood
    logit(p[k])<-gamma*(temp[k]-mu_temp)
  }

}
")
```

Running the model I

```
# Inits function
inits <- function(){list(gamma = rnorm(1, 0, 1),
                        mu_temp = rnorm(1,0,1))}

# Parameters to estimate
params <- c("gamma","mu_temp")

# MCMC settings
nc <- 3 ; ni <- 2000 ; nb <- 1000 ; nt <- 2

# Call JAGS, check convergence and summarize posteriors
out <- jags(m11.data, inits, params, "logistic.regression.txt", n.thin = nt,
           n.chains = nc, n.burnin = nb, n.iter = ni)
```

```
## Compiling model graph
##   Resolving undeclared variables
##   Allocating nodes
## Graph information:
##   Observed stochastic nodes: 25
##   Unobserved stochastic nodes: 2
##   Total graph size: 156
##
## Initializing model
```

Running the model II

```
print(out, dig = 3)      # Bayesian analysis
```

```
## Inference for Bugs model at "logistic.regression.txt", fit using jags,
## 3 chains, each with 2000 iterations (first 1000 discarded), n.thin = 2
## n.sims = 1500 iterations saved
##
```

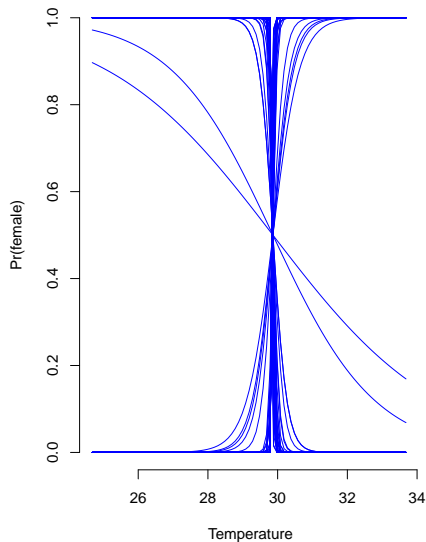
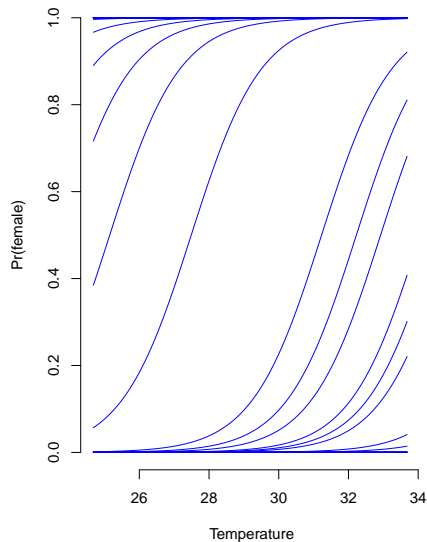
	mu.vect	sd.vect	2.5%	25%	50%	75%	97.5%	Rhat	n.eff
## gamma	3.048	0.284	2.499	2.857	3.037	3.246	3.624	1.001	1500
## mu_temp	29.889	0.041	29.806	29.861	29.891	29.917	29.968	1.000	1500
## deviance	61.511	2.060	59.526	60.051	60.858	62.245	67.298	1.006	660

```
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 2.1 and DIC = 63.6
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Priors on γ and μ_{temp} I

```
mu_temp = rnorm(100,2, 100) # prior of the mean temp
gamma = rnorm(100,1, 100)   # prior on the slope
x=seq(min(temperature),max(temperature),by=0.1)
par(mfrow=c(1,2))
plot(0, bty = 'n', pch = '', ylab = "Pr(female)",
     xlab = "Temperature",ylim=c(0,1),
     xlim=c(min(temperature),max(temperature)))
for (kprior in 1:100) {
  prob = 1/(1+exp(-1*(x-mu_temp[kprior]))) )
  lines(x,prob,type="l",col="blue")}
plot(0, bty = 'n', pch = '', ylab = "Pr(female)",
     xlab = "Temperature",ylim=c(0,1),
     xlim=c(min(temperature),max(temperature)))
for (kprior in 1:100) {
  prob = 1/(1+exp(-gamma[kprior]*(x-mean(temperature)))) )
  lines(x,prob,type="l",col="blue")}
```

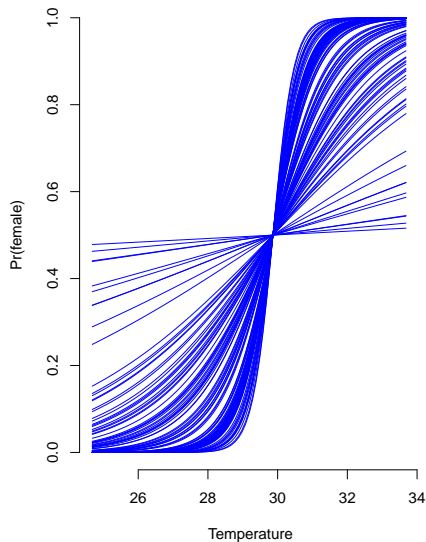
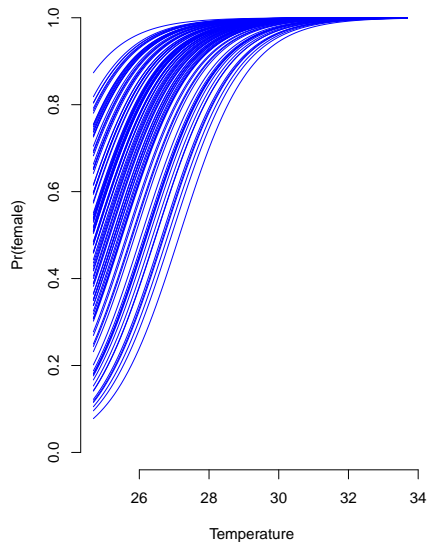

Priors on γ and μ_{temp} II



Better priors I

```
### Better priors
mu_temp = rnorm(100,25, 1) # prior of the mean temp
gamma = rnorm(100,1, 1)    # prior on the slope
par(mfrow=c(1,2))
plot(0, bty = 'n', pch = '', ylab = "Pr(female)",
     xlab = "Temperature",ylim=c(0,1),
     xlim=c(min(temperature),max(temperature)))
for (kprior in 1:100) {
  prob = 1/(1+exp(-1*(x-mu_temp[kprior])))
  lines(x,prob,type="l",col="blue")}
plot(0, bty = 'n', pch = '', ylab = "Pr(female)",
     xlab = "Temperature",ylim=c(0,1),
     xlim=c(min(temperature),max(temperature)))
for (kprior in 1:100) {
  prob = 1/(1+exp(-abs(gamma[kprior])*(x-mean(temperature))))
  lines(x,prob,type="l",col="blue")}
```

Better priors II



Weakly informative priors I

What are they?

- Priors that provide *regularization* or *shrinkage*
- In practice, often $\mathcal{N}(0, [\text{small}])$ instead of $\mathcal{N}(0, [\text{huge}])$,
e.g. `dnorm(0, 1)` or `dnorm(0, 0.1)` instead of `dnorm(0, 0.0001)` in JAGS.

A more detailed explanation

A detour on identifiability, convergence, and priors

Borrowed from Mc Elreath's Statistical rethinking

We consider the (obviously wrong) model

$$Y_i \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = \alpha_1 + \alpha_2$$

$$\alpha_j \sim \text{Unif}(-10000, 10000) \text{ i.i.d}$$

$$\sigma \sim \text{Exp}(1/10)$$

for 100 data points simulated as $\mathcal{N}(0, 1)$.

Coding the stupid model

```
gaussian.data = list(y=rnorm(100,0,1),N=100)

cat(file="stupid.model.txt", "
model {
  # Priors
  alpha[1] ~ dunif(-10000,10000) #dnorm(0,0.00001)
  alpha[2] ~ dunif(-10000,10000) #dnorm(0,0.00001)
  sigma ~ dunif(0,1000) #dexp(0.01) #dexp(1)
  tau<-pow(sigma,-2)

  # Likelihood
  mu<-alpha[1]+alpha[2]
  for (i in 1:N){
    y[i] ~ dnorm(mu,tau)
  }

}
")
```

Fitting the stupid model I

```
# Initial values
inits <- function(){list(alpha=rnorm(2,0,1000))}
# Parameters to estimate
params <- c("alpha","sigma","mu")
# MCMC settings
nc <- 3 ; ni <- 2000 ; nb <- 1000 ; nt <- 2
# Call JAGS, check convergence and summarize posteriors
out <- jags(gaussian.data, inits, params, "stupid.model.txt", n.thin = nt,
            n.chains = nc, n.burnin = nb, n.iter = ni)

## Compiling model graph
##   Resolving undeclared variables
##   Allocating nodes
## Graph information:
##   Observed stochastic nodes: 100
##   Unobserved stochastic nodes: 3
##   Total graph size: 112
##
## Initializing model

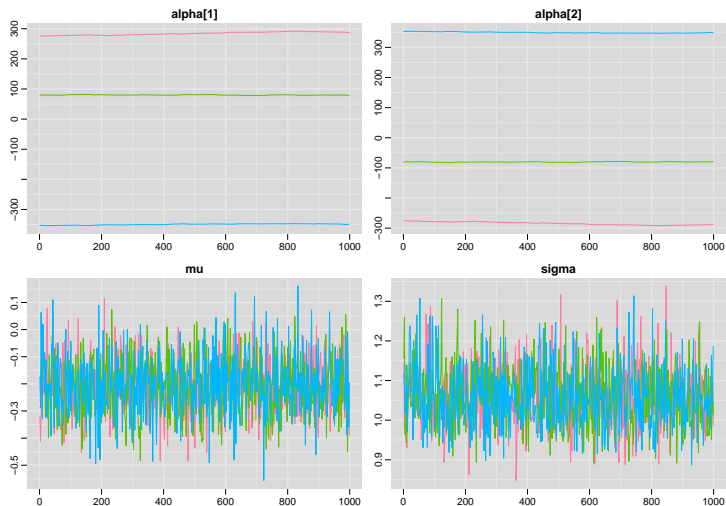
print(out, dig = 3)      # Bayesian analysis
```

Fitting the stupid model II

```
## Inference for Bugs model at "stupid.model.txt", fit using jags,
## 3 chains, each with 2000 iterations (first 1000 discarded), n.thin = 2
## n.sims = 1500 iterations saved
##           mu.vect sd.vect      2.5%      25%      50%      75%      97.5%      Rhat
## alpha[1]   4.890 264.180 -352.779 -347.788  80.031 278.998 290.612 153.944
## alpha[2]  -5.098 264.181 -290.799 -279.295 -80.247 347.594 352.606 154.065
## mu         -0.208   0.106  -0.407  -0.279  -0.207  -0.139   0.009   1.001
## sigma       1.060   0.076   0.922   1.007   1.057   1.109   1.227   1.001
## deviance 293.290   2.017  291.294  291.827 292.716 294.006 298.886   1.006
##           n.eff
## alpha[1]      3
## alpha[2]      3
## mu           1500
## sigma         1500
## deviance      1400
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule,  $pD = \text{var}(\text{deviance})/2$ )
##  $pD = 2.0$  and  $DIC = 295.3$ 
## DIC is an estimate of expected predictive error (lower deviance is better).
```


Traceplots

```
library(mcmcplots)
traplot(as.mcmc(out),parms=c("alpha", "mu", "sigma"))
```



Less stupid model

```
cat(file="less.stupid.model.txt", "  
model {  
  # Priors  
  alpha[1] ~ dnorm(0,0.1)  
  alpha[2] ~ dnorm(0,0.1)  
  sigma ~ dexp(1)  
  tau<-pow(sigma,-2)  
  
  # Likelihood  
  mu<-alpha[1]+alpha[2]  
  for (i in 1:N){  
    y[i] ~ dnorm(mu,tau)  
  }  
}  
")
```

Actually even a much smaller precision on α_j 's prior would work

Fitting the less stupid model I

```
# Initial values
inits <- function(){list(alpha=rnorm(2,0,1))}
# Parameters to estimate
params <- c("alpha","sigma","mu")
# MCMC settings
nc <- 3 ; ni <- 2000 ; nb <- 1000 ; nt <- 2
# Call JAGS, check convergence and summarize posteriors
out <- jags(gaussian.data, inits, params, "less.stupid.model.txt",
            n.thin = nt, n.chains = nc, n.burnin = nb, n.iter = ni)

## Compiling model graph
##   Resolving undeclared variables
##   Allocating nodes
## Graph information:
##   Observed stochastic nodes: 100
##   Unobserved stochastic nodes: 3
##   Total graph size: 111
##
## Initializing model

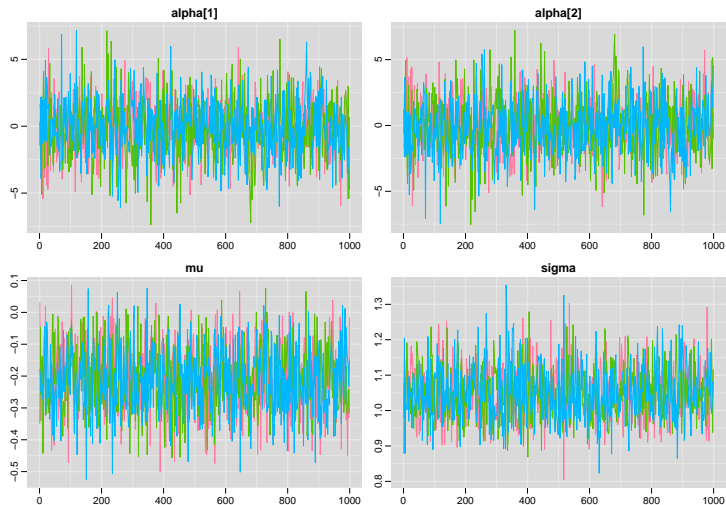
print(out, dig = 3)      # Bayesian analysis
```

Fitting the less stupid model II

```
## Inference for Bugs model at "less.stupid.model.txt", fit using jags,
## 3 chains, each with 2000 iterations (first 1000 discarded), n.thin = 2
## n.sims = 1500 iterations saved
##           mu.vect sd.vect   2.5%   25%   50%   75%   97.5% Rhat n.eff
## alpha[1] -0.118   2.156 -4.460 -1.491 -0.108   1.352   4.055 1.001 1500
## alpha[2] -0.092   2.156 -4.196 -1.562 -0.100   1.292   4.219 1.001 1500
## mu        -0.209   0.105 -0.415 -0.281 -0.210  -0.138  -0.008 1.000 1500
## sigma      1.049   0.075  0.915  0.995  1.043   1.097   1.206 1.002 1500
## deviance 293.249   1.940 291.303 291.866 292.676 294.112 298.291 1.009 450
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule,  $pD = \text{var}(\text{deviance})/2$ )
##  $pD = 1.9$  and  $DIC = 295.1$ 
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Traceplots again

```
library(mcmcplots)
traplot(as.mcmc(out),parms=c("alpha", "mu", "sigma"))
```



Now a real binary data example: bald eagles I



Figure 4: Bald eagle with salmon, Alaska. KJ Gill aka Gillfoto. [CC BY 2.0](#)

Now a real binary data example: bald eagles II

Records of (160!) salmon-pirating attempts by one Bald eagle on another Bald eagle (not always the same!). Also borrowed from McElreath's Statistical Rethinking p. 330.

```
library(MASS)
data(eagles)
head(eagles)
```

```
##      y   n P A V
## 1 17 24 L A L
## 2 29 29 L A S
## 3 17 27 L I L
## 4 20 20 L I S
## 5  1 12 S A L
## 6 15 16 S A S
```

Now a real binary data example: bald eagles III

```
#P // Size of pirating eagle (L = large, S = small).  
#A // Age of pirating eagle (I = immature, A = adult).  
#V // Size of victim eagle (L = large, S = small).  
  
m2.data = list(N=nrow(eagles),y=eagles$y,z=eagles$n,  
              P=as.numeric(eagles$P)-1,A=as.numeric(eagles$A)-1,  
              V=as.numeric(eagles$V)-1)  
# m2.data = list(N=nrow(eagles),y=eagles$y,z=eagles$n,P=eagles$P,A=
```