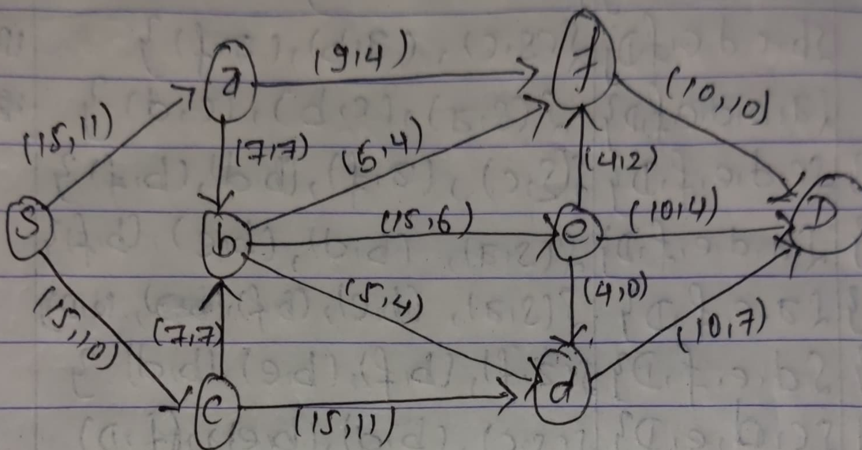
Ankit Pargew

Discrete Structures-2075

1. What is S-D cut? For the following flow, find the maximum flow from S to D



→) A S-D cut of transport network is a set of edge whose removal will divide the network into two halves X and $\bar{X}$ where,

Source vertex $S \in X$

Sink vertex $D \in \bar{X}$

It is denoted by ~~(X,X)~~ $(X, \bar{X})$

Let's find Maximum flow from S to D using S-D cut method.

X	Y	cuts (X,Y)	capacity c(X,Y)
{s}	{a,b,c,d,e,f,D}	{(s,a), (s,c)}	15+15=30
{s,a}	{b,c,d,e,f,D}	{(s,c), (a,b), (a,f)}	15+7+9=31
{s,c}	{a,b,d,e,f,D}	{(s,a), (c,b), (c,d)}	37
{s,a,b}	{c,d,e,f,D}	{(s,c), (a,f), (b,d), (b,f)}	34
{s,b,c}	{a,d,e,f,D}	{(s,a), (b,d), (b,e), (b,f)}	40
{s,b,c,d}	{a,e,f,D}	{(s,a), (b,e), (b,f), (d,D)}	45
{s,a,b,c}	{d,e,f,D}	{(a,f), (b,f), (b,e), (b,d)}	34
{s,a,b,f}	{c,d,e,D}	{(s,c), (b,d), (b,e), (f,D)}	45
{s,a,b,e,f}	{c,d,D}	{(s,c), (b,d), (e,d), (e,D), (f,D)}	44
{s,a,b,c,d,e,f}	{D}	{(d,D), (c,D), (f,D)}	35
{s,a,b,c,d}	{e,f,D}	{(a,f), (b,e), (b,f), (d,D)}	39
{s,b,c,d,e}	{a,f,D}	{(s,a), (b,f), (d,D), (e,f), (e,D)}	44

Here, the min cut = 30

So, by min cut, max flow

Maximum flow = 30.

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Group - A

Long answer questions.

2. Consider a set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What will be the computer representation for set containing the numbers which are multiple of 3 not exceeding 6? Describe injective, surjective and bijective function with example.

=) Given, set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

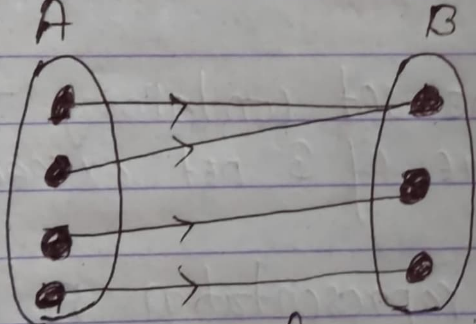
Here, we know the set containing the numbers which are multiple of 3 not exceeding is
$\{3, 6\}$
And its computer representation is its corresponding bit string i.e.
0 0 1 0 0 1 0 0 0 0

Let A and B be two non-empty sets. A function f from A to B is a set of ordered pairs with the property that for each element $x \in A$ there is an unique element $y \in B$.

Injective function

A function from A to B is one-to-one (injective) if $\forall x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. We can express that f is one to one (injective) using quantifiers as $\forall x_1 \forall x_2 (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$ where universe of discourse is the domain of the function.

Ex: Let A and B are two set

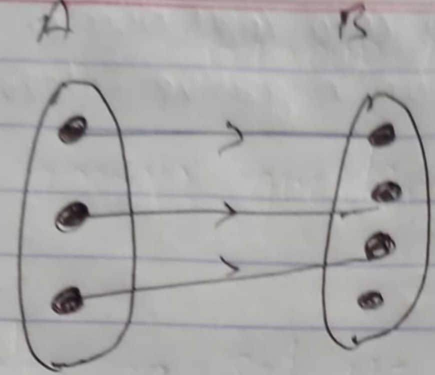


Injective.

surjective ~~Bijective~~ function

A function from A to B is an onto (surjective) ~~(bijective)~~ if every element of B is the image of some element in A. We can express that f is surjective using quantifiers as $\forall y \exists x (f(x) = y)$

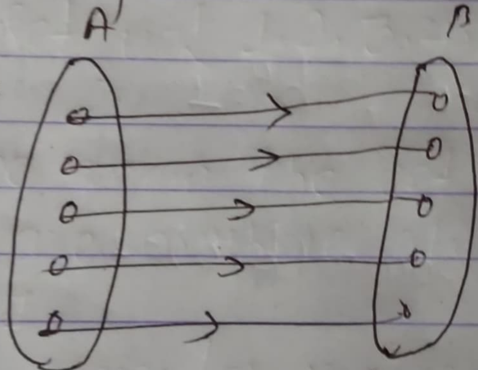
Ex:



Bijective (one-to-one correspondence)

A function f from A to B is said to be bijective if it is both injective and surjective.

Ex:



If a function is bijective, its inverse exists

Note: (you can also give algebraic function examples)

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3. Compute the following values

a) 3 mod 4
= 3 (mod 4)
= 3

b) 7 mod 5
= 7 (mod 5)
= 2

c) -5 mod 3
= -5 (mod 3)
= 1

d) 11 mod 5
= 11 (mod 5)
= 1

e) -8 mod 6
= -8 (mod 6)
= 4

Note: An algorithm is called recursive if it solves a problem by reducing it to instance of same problem with smaller output.

* Write down the recursive algorithm to find the value of b^n and prove its correctness using induction.

⇒ A recursive algorithm for b^n

Here, the power function can be defined recursively as:-

a) Base case: $b^0 = 1$

b) Recursive definition: $b^n = b \cdot b^{n-1}$ for $b \geq 1$

power(b, n) b = non zero real number
n = non negative integer

1. Start
2. if(n==0)
   return 1;
   else
   return b*power(b,n-1)
3. End.

Proof by induction

Base case: If $n=0$, $b^0=1$

Here, power(b,0)=1

Inductive Step:

Algorithm computes b^{k+1}

Suppose, power(b,k) = b^k $\forall b \neq 0$

and k is +ve integer

Now,

To show, power(b, k+1) = b^{k+1}

By inductive hypothesis,

power(b,k+1) = $b \times b^k$

$= b^{k+1}$

Proved

* Some extra recursive algorithms to practise:-

1) Recursive algorithm for computing gcd(a,b)

gcd(a,b: +ve integers a<b)

1. Start
2. if (a=0), then
   gcd(a,b) = b

   else
   gcd(a,b) = gcd(b mod a, a)

3. return gcd(a,b)
4. End.

2) Recursive algorithm for computing n!

factorial(n : nonnegative integer)

1. Start
2. if (n=0)
   return 1;

   else
   return n factorial(n-1)

   [output is n!]
3. End.

Group B.

Attempt any eight question

4. Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$

=) Given recurrance relation

$a_n = 5a_{n-1} - 6a_{n-2}$

Here, degree = 2

Now, the Characteristics Eqn is

$r^2 = 5r - 6$

Or, $r^2 - 5r + 6 = 0$

Or, $r^2 - 3r - 2r + 6 = 0$

Or, $r(r-3) - 2(r-3) = 0$

Or, $(r-2)(r-3) = 0$

We get, $r = 2, 3$

i.e. say, $r_1 = 2$, $r_2 = 3$

Now, the general solution of the homogeneous equation is

$a_n = A_1 r_1^n + A_2 r_2^n$

where, A_1 and A_2 are scalar

or, $a_n = A_1 2^n + A_2 3^n$

$\therefore$ $\boxed{a_n = A_1 2^n + A_2 3^n}$ — (i)

Given initial conditions.

$a_0 = 1$ $\therefore n=0$

$a_0 = A_1 2^0 + A_2 3^0$
$1 = A_1 + A_2$
$\therefore A_1 + A_2 = 1$ — (ii)

Also, $a_1 = 2$ $\therefore n=1$
$a_1 = A_1 2^1 + A_2 3^1$
$2A_1 + 3A_2 = 2$ — (iii)

Solving Eqns (ii) and (iii)

$\cancel{2A_1} + 3A_2 = 2$
$- \cancel{2A_1} - 2A_2 = -2$

$A_2 = 0$

$\therefore$ $A_1 + 0 = 1$
$A_1 = 1$

we get, $A_1 = 1$
$A_2 = 0$

∴ From eqn (i), the general solution is

$a_n = 2^n + 0 \times 3^n$

$$\boxed{a_n = 2^n}$$

5. Find the value of x such that $x \equiv 1 \pmod 5$ and $x \equiv 2 \pmod 7$ using Chinese remainder theorem.

=> Here $x \equiv 1 \pmod 5$

$x \equiv 2 \pmod 7$

We know, linear congruency is of form $x \equiv a_n \pmod{m_n}$ where n is an integer. So, let us suppose.

$a_1 = 1 \quad m_1 = 5$

$a_2 = 2 \quad m_2 = 7$

Now,

$m = m_1 \times m_2$

$m = 5 \times 7$

$\therefore m = 35$

Also,

$M_1 = m/m_1 = 35/5 = 7$

$M_2 = m/m_2 = 35/7 = 5$

Now, we need to find the inverse of $M_1 \bmod m_1$ & $M_2 \bmod m_2$.

Here, to find inverse of 7 mod 5

$7 \times 0 \equiv 0 \pmod 5$
$7 \times 1 \equiv 2 \pmod 5$
$7 \times 2 \equiv 4 \pmod 5$
$7 \times 3 \equiv 1 \pmod 5$

$\therefore$ 3 is the inverse of 7 mod 5

Let, $y_1 = 3$

Also, to find inverse of 5 mod 7

$5 \times 0 \equiv 0 \pmod 7$
$5 \times 1 \equiv 5 \pmod 7$
$5 \times 2 \equiv 3 \pmod 7$
$5 \times 3 \equiv 1 \pmod 7$

So, inverse is 3

Let, $y_2 = 3$

We get, $a_1 = 1 \quad M_1 = 7 \quad y_1 = 3$
$a_2 = 2 \quad M_2 = 5 \quad y_2 = 3$

The solutions to these systems are those x such that

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 \pmod{m}$$

$$x = 1 \times 7 \times 3 + 2 \times 5 \times 3 \pmod{35}$$

$$x = 21 + 30 \pmod{35}$$

$$x = 51 \pmod{35}$$

$$x = 16 \pmod{35}$$

Hence, we conclude that 16 is the smallest +ve integer that leaves remainders 1 when divided by 5 and 2 when divided by 7.

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6. Prove that $5^n - 1$ is divisible by 4 using mathematical induction

⇒ Let, $P(n) = 5^n - 1$

Basis step

we show $p(1)$ is true.
i.e. $p(1) = 5^1 - 1$
$= 4$ which is divisible by 4,

So $p(1)$ is true.

Inductive step

Suppose, for any arbitary vawe k, $p(k)$ is true

i.e. $p(k)$: $5^k - 1 = 4a$, where, a is an integer.
$\therefore \boxed{5^k = 4a + 1}$

Now, we need to show $p(k+1)$ is divisible by 4.

$p(k+1)$: $5^{k+1} - 1$
$= 5^k \cdot 5 - 1$ [From assumption]
$= (4a+1) \cdot 5 - 1$
$= 20a + 5 - 1$
$= 20a + 4$
$= 4(5a+1)$
$= 4b$ where b is an integer.

Here, it is divisible by 4

So, by mathematical induction,
$5^n - 1$ is divisible by 4

7. Let, A = "Aido is Italian" and B = "Bob is English"
Formalize the following sentences in proposition.

a. Aido isn't Italian.
$\Rightarrow \neg A$

b. Aido is Italian while Bob is English.
$\Rightarrow A \wedge B$

c. If Aido is Italian then Bob is not English
$\Rightarrow A \rightarrow \neg B$

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7d. Aido is Italian or if Aido isn't Italian then Bob is English

=) $A \vee (\neg A \rightarrow B)$ logically equivalent to $A \vee B$

e. Either Aido is Italian and Bob is English, or neither Aido is Italian nor Bob is English

=) $(A \wedge B) \vee (\neg A \wedge \neg B)$ logically equivalent to $A \leftrightarrow B$

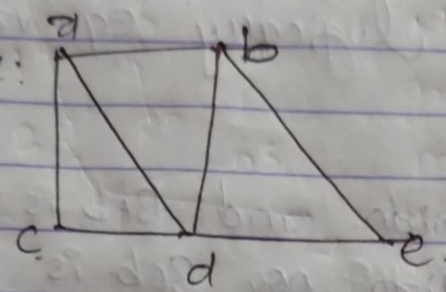
8. Define Euler path and Hamilton path with examples. Draw the Hasse diagram for the divisible relation on the set {1, 2, 5, 8, 16, 32} and find the maximal, minimal, greatest and least element if exist.

=) Euler path

A simple path in a graph G that passes through every edge once and only once is called euler path. An euler circuit is an eular path that returns to its starting vertex.

A connected multigraph has an euler path but not an euler circuit if and only if it has at most (max) two vertices of odd degree

Ex:



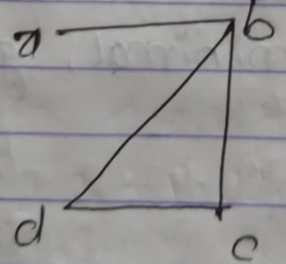
a → c → d → e → b → d → a → b

Here, it passes through all edges only once & no edge is repeated.

Hamilton path

A simple path in a graph G that passes through every vertex exactly once is called a hamilton path.

Ex:



a, b, c, d or a, b, d, c.

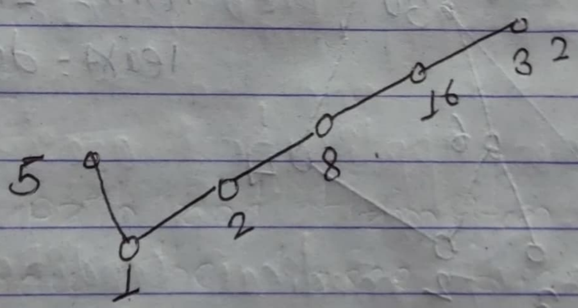
Here, it passes through every vertices & no vertices is repeated. So, it is hamilton path

Given set, $\{1, 2, 5, 8, 16, 32\}$

According to the given ~~set~~ question, we have to find the poset for the divisibility
Let the set is A

$$A = \{(1<2), (1<5), (1<8), (1<16), (1<32), (2<8), (2<16), (2<32), (8<16), (8<32), (16<32)\}$$

So, now the hasse diagram will be

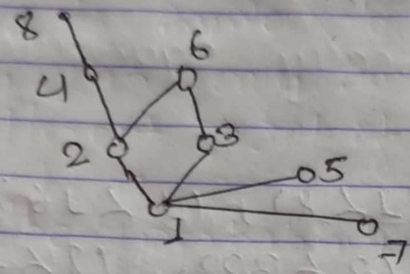


Here, Maximum = 5, 32 (which has no connect)
Greatest = Do not exist (which has connection with all)
Minimum = 1 (lowest none with a connection)
Least = 1 (lowest which has connection with all)

Some extra questions

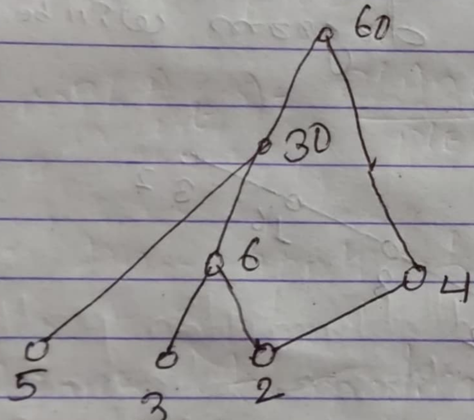
Draw a Hasse diagram for (A, |) (divisibility relation)

i) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$



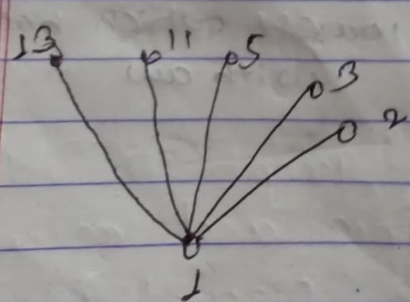
Max = 5, 6, 7, 8
greatest = do not exist
min = 1
least = 1

ii) $A = \{2, 3, 4, 5, 6, 30, 60\}$

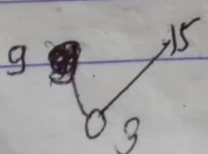


max = 60
greatest = 60
min = 2, 3, 5
least = do not exist

iii) $A = \{1, 2, 3, 5, 11, 13\}$



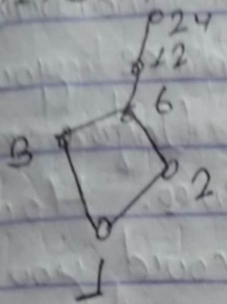
max = 2, 3, 5, 11, 13
greatest = do not exist
min = 1
least = 1



max = 9, 15
great = D.N.E
min = 3, least = 3

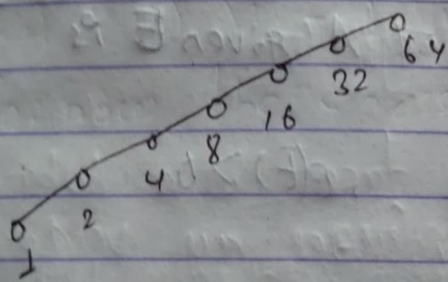
Note: for greatest, the number should have connection with all the number
same with least

iv) $A = \{1, 2, 3, 6, 12, 24\}$



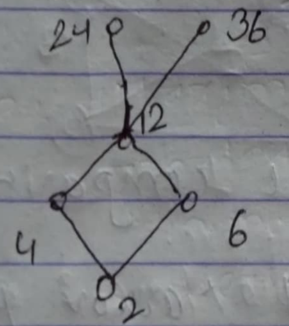
max = 24
greatest = 24
min = 1
least = 1

v) $A = \{1, 2, 4, 8, 16, 32, 64\}$



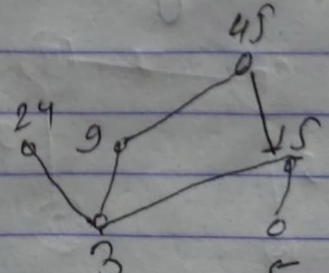
max = 64
greatest = 64
min = 1
least = 1

vi) $A = \{2, 4, 6, 12, 24, 36\}$



max = 24, 36
greatest = do not exist
min = 2
least = 2

vii) $A = \{3, 5, 9, 15, 24, 45\}$



max = 24, 45
greatest = Do not exist
min = 3, 5
least = D.N.E

10. List any two applications of conditional probability. You have 9 families you would like to invite to a wedding. Unfortunately, you can only invite 6 families. How many different set of invitations could you write?

⇒ The probability that an event A occurs given that event E has already occurred written as $P(A|E)$ and read as the conditional probability of A given E is

$$P(A/E) = \frac{P(A \cap E)}{P(E)} \quad , P(E) > 0$$

Its two applications are:

i) Diagnoses of medical condition (Sensitivity / Specificity)

ii) Data analyses and model comparision

iii) In Baye's theorem and Markov processes

Here, given,
total families = 9
families I can invite = 6

$\therefore$ Total set of invitations I could write = $^9C_6$

$$= \frac{9!}{(9-6)! \times 6!}$$

$$= 84$$

So, 84 different set of invitations can be written by me.

Note: $^nC_r = \frac{n!}{(n-r)!\,r!}$

$^nP_r = \frac{n!}{(n-r)!}$

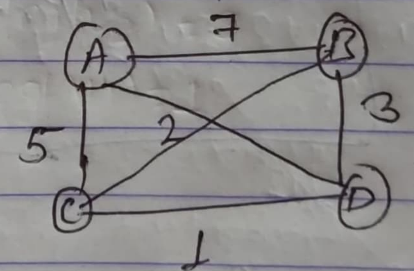
11. Define spanning tree and minimum spanning tree. Mention the conditions for two graphs for being isomorphic with an example.

⇒ Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing all vertex of G.
Spanning tree has n vertices and n-1 edges as compared to graph G.

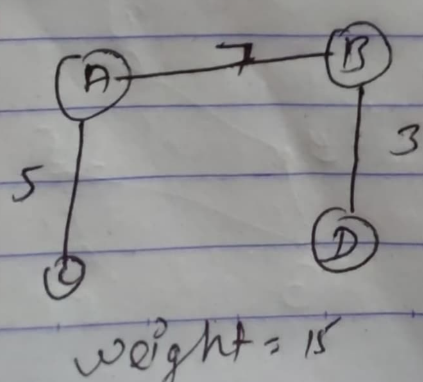
A minimum spanning tree is a subgraph of a weighted graph G that includes all vertices of graph G and has minimum weight than all other spanning trees of the same graph.

~~extra~~

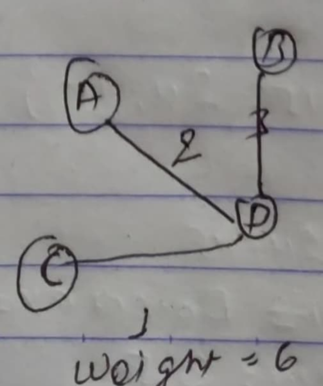
Ex: Given graph,



no. of vertices (n) = 4.
So, all spanning trees should have 3 edges

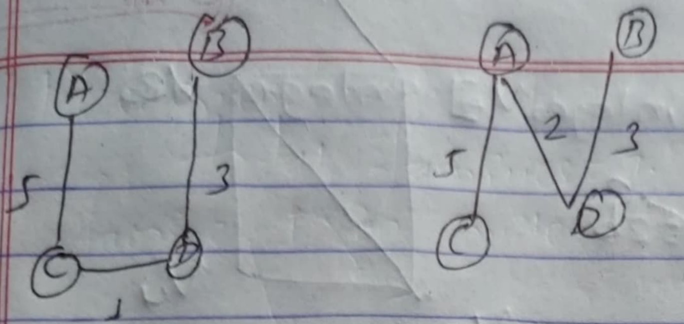


weight = 15



~~minimum spanning tree~~

weight = 6



weight = 9 weight = 10

Here, the one with the minimum weight among all the spanning trees above is called minimum spanning tree.

* Two graphs are said to be isomorphic if they are perhaps the same graphs just drawn differently with different names ie they have identical behaviour for any graph-theoretic property.

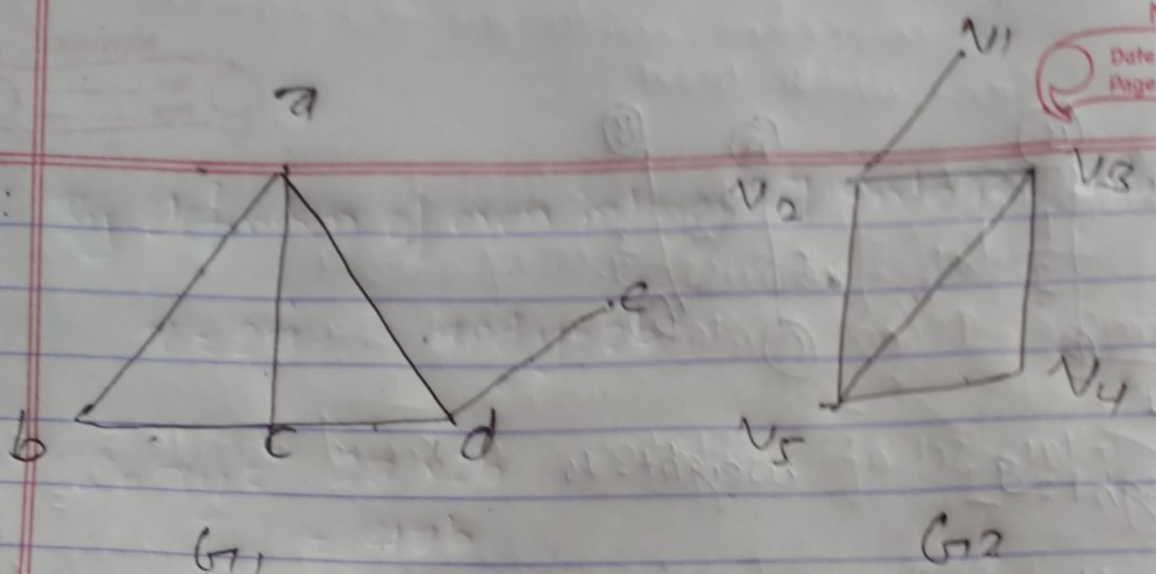
Some conditions for two graphs to be isomorphic are:

i) Both graphs should have same number of vertices ie. $|V_1| = |V_2|$

ii) No. of edges of both graphs should be equal ie $|E_1| = |E_2|$

iii) Both graphs should have same sequences and vertices of both graphs should have same degrees.

ex:



* Both have same no. of vertices
* Both have same no. of edges

degree

G_1	G_2
deg(a) = 3	deg(v_2) = 3
deg(b) = 2	deg(v_4) = 2
deg(c) = 3	deg(v_5) = 3
deg(d) = 3	deg(v_3) = 3
deg(e) = 1	deg(v_1) = 1

Hence, they are isomorphic.

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1) Prove that the product xy is odd if and only if x and y are odd integers.

→ Let us suppose, p = x and y are odd
$q = xy$ is odd
i.e. $p \to q$

1st case $p \to q$ (direct proof)
If x and y are odd then xy is odd.
By definition of odd integer; $\exists$ integers a, b.
$x = 2a+1$
and $y = 2b+1$

$\therefore xy = (2a+1)(2b+1)$
$= 4ab + 2a + 2b + 1$
$= 2(2ab + a + b) + 1$
$= 2c + 1$, for some integer c
Here, xy is also odd by direct proof.

2nd case $\neg q \to \neg p$. (indirect proof) (contraposition)
The product xy is odd if and only if x & y are odd

p = product xy is odd
q = x & y are odd
Let us suppose x and y are even.
Now we prove $\neg q \to \neg p$.

By definition of even integer, $\exists$ integer a, b

$x = 2a$ and $y = 2b$

$\therefore xy = 2a \times 2b$
$= 4ab$
$= 2(2ab)$
$= 2c$, for some integer

So, xy is ~~odd~~ even
Hence, by contraposition we can say xy is odd

$\therefore$ By direct proof and contraposition, the product xy is odd if and only if x & y is odd.