

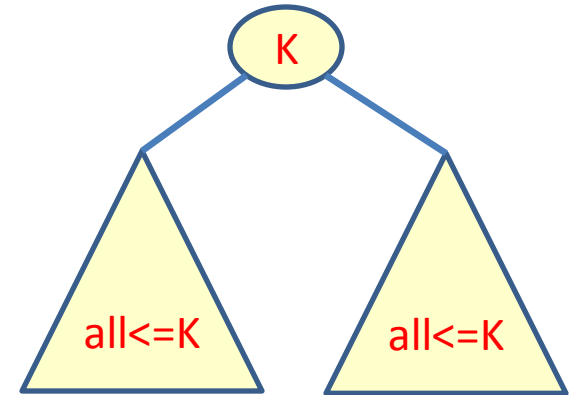
Chapter 8 - Heaps

- Binary Heap. Min-heap. Max-heap.
- Efficient implementation of heap ADT: **use of array**
- Basic heap algorithms: ReheapUp, ReheapDown, Insert Heap, Delete Heap, Built Heap
- d-heaps
- Heap Applications:
 - Select Algorithm
 - Priority Queues
 - Heap sort
- Advanced implementations of heaps: **use of pointers**
 - Leftist heap
 - Skew heap
 - Binomial queues

Binary Heaps

DEFINITION: A **max-heap** is a binary tree structure with the following properties:

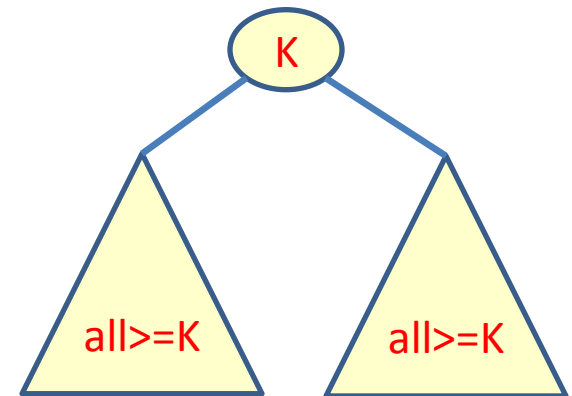
- The tree is complete or nearly complete.
- The key value of each node is **greater than or equal to** the key value



max-heap

DEFINITION: A **min-heap** is a binary tree structure with the following properties:

- The tree is complete or nearly complete.
- The key value of each node is **less than or equal to** the key value in each of its descendents.



min-heap

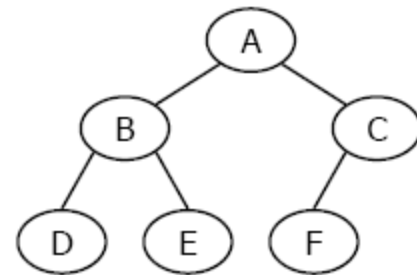
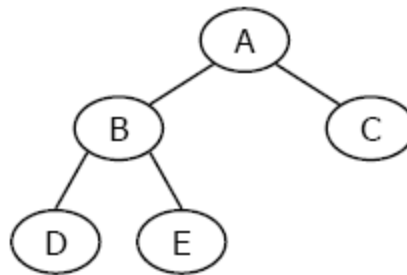
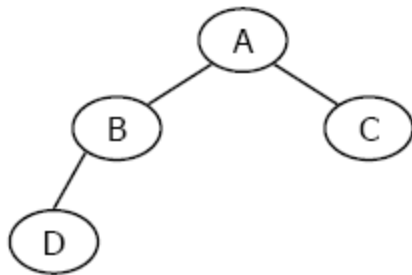
Properties of Binary Heaps

- Structure property of heaps
- Key value order of heaps

Properties of Binary Heaps

Structure property of heaps:

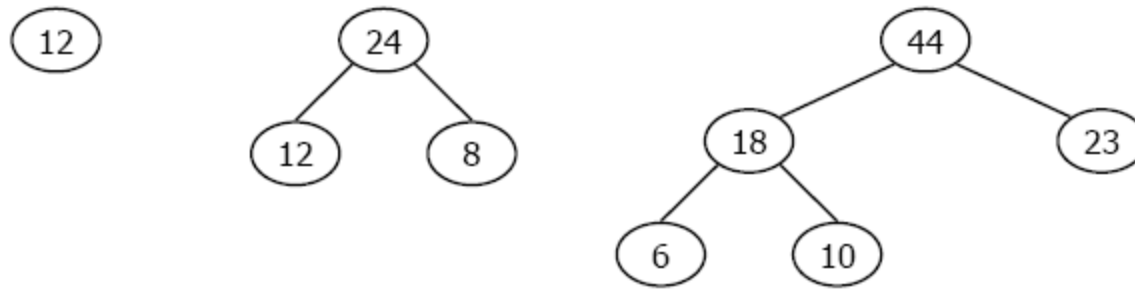
- A complete or nearly complete binary tree.
- If the height is h , the number of nodes n is between 2^{h-1} and $(2^h - 1)$
- **Complete tree**: $n = 2^h - 1$ when last level is full.
- **Nearly complete**: All nodes in the last level are on the left.



- $h = \lfloor \log_2 n \rfloor + 1$
- Can be represented in an array and no pointers are necessary.

Properties of Binary Heaps

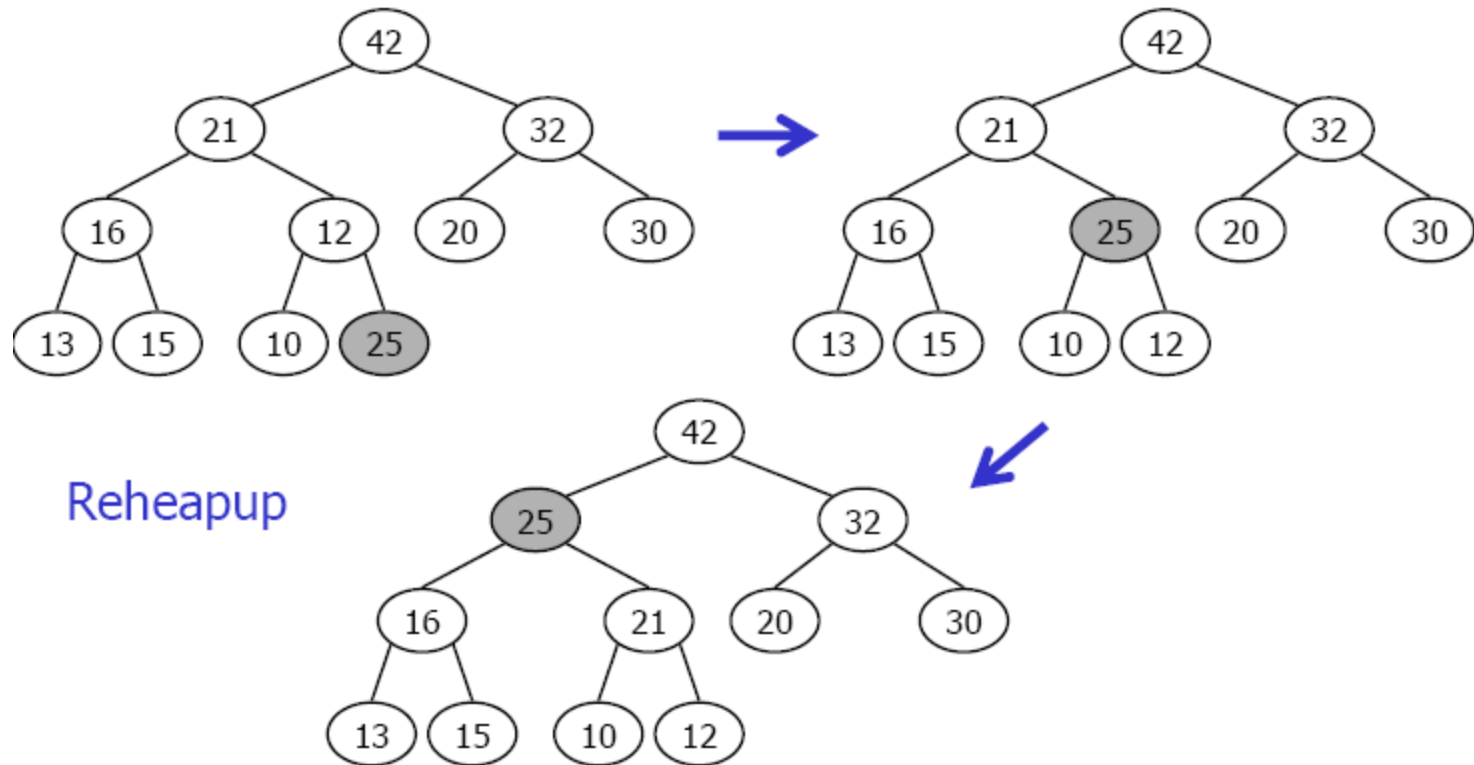
Key value order of max-heap:



(max-heap is often called as *heap*)

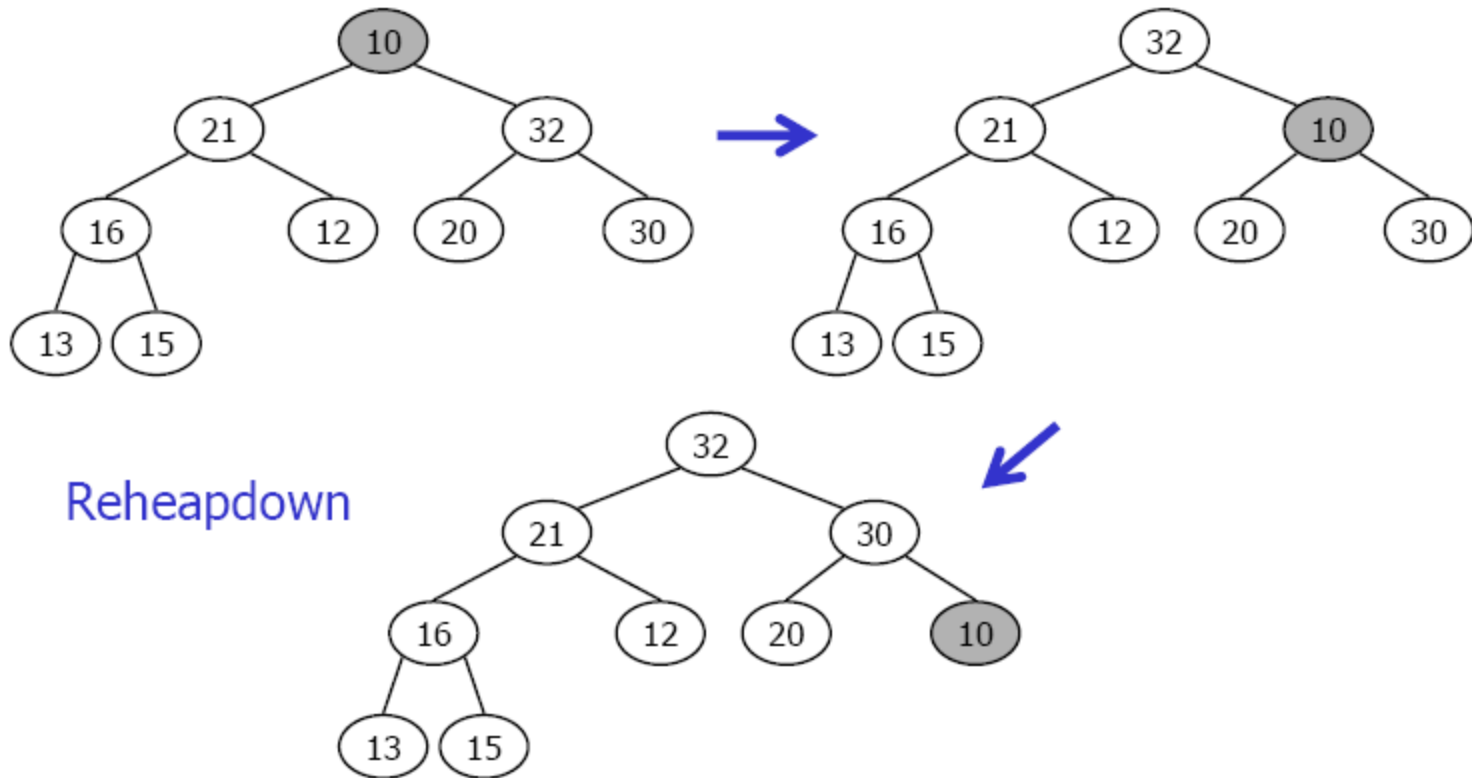
Basic heap algorithms

ReheapUp: repairs a "broken" heap by floating the last element up the tree until it is in its correct location.



Basic heap algorithms

ReheapDown: repairs a "broken" heap by pushing the root of the subtree down until it is in its correct location.



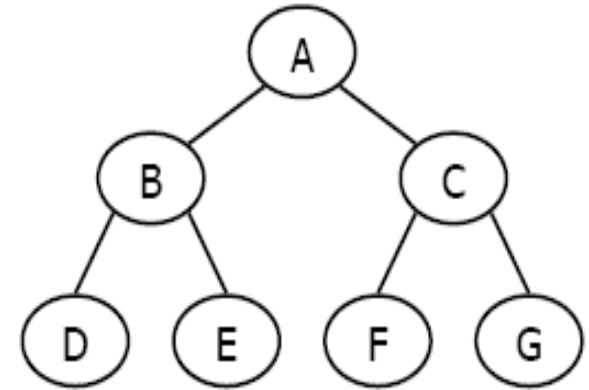
Contiguous Implementation of Heaps

Heap

data <Array of <DataType> >

count <int> *//number of elements in heap*

End Heap



0 1 2 3 4 5 6



Physical

Conceptual

$\lfloor (i-1)/2 \rfloor$

$2i+1$

$2i+2$

ReheapUp

Algorithm **ReheapUp** (val **position** <int>)

Reestablishes heap by moving data in **position** up to its correct location.

Pre All data in the heap above this **position** satisfy key value order of a heap, except the data in **position**.

Post Data in **position** has been moved up to its correct location.

Uses Recursive function **ReheapUp**.

```
1. if (position <> 0)           // the parent of position exists.
    1. parent = (position-1)/2
    2. if (data[position].key > data[parent].key)
        1. swap(position, parent) // swap data at position with data at parent.
        2. ReheapUp(parent)
2. return
End ReheapUp
```

ReheapDown

Algorithm **ReheapDown** (val **position** <int>, val **lastPosition** <int>)

Reestablishes heap by moving data in **position** down to its correct location.

Pre All data in the subtree of **position** satisfy key value order of a heap, except the data in **position**.

Post Data in **position** has been moved down to its correct location.

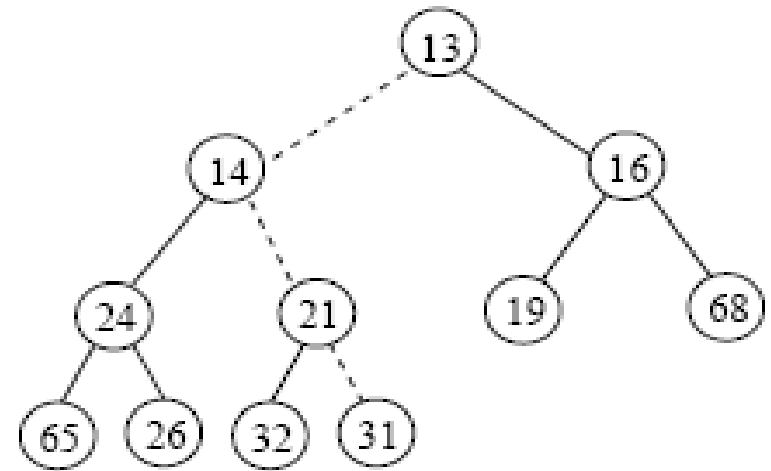
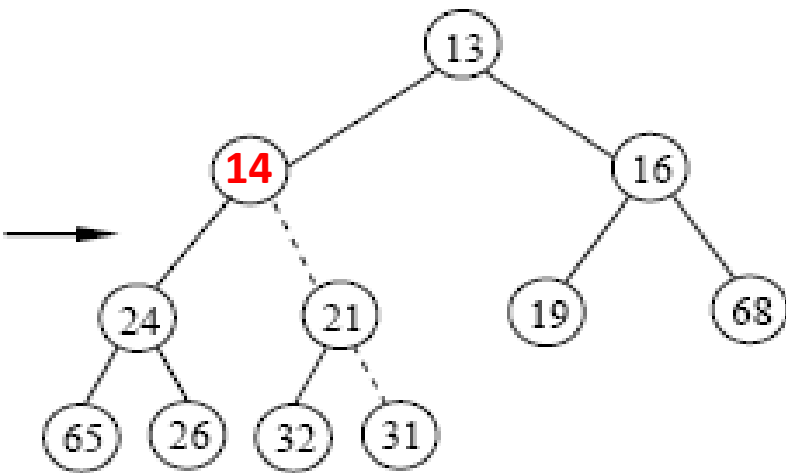
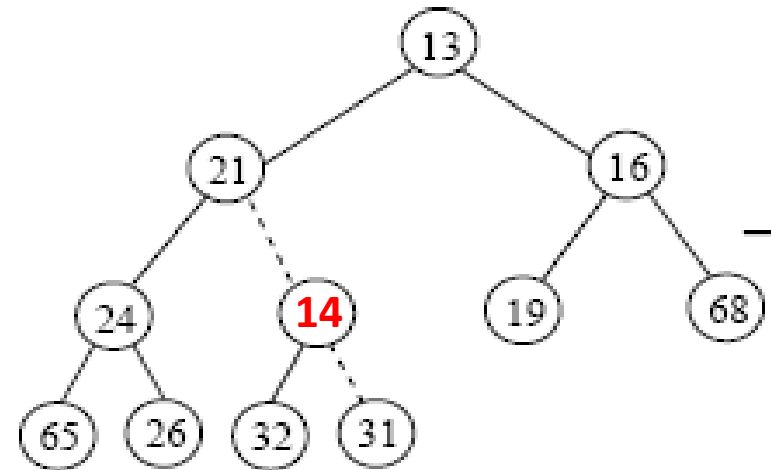
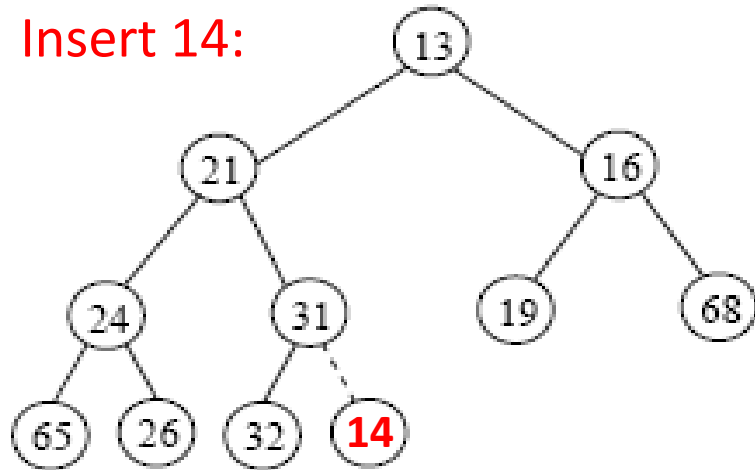
Uses Recursive function **ReheapDown**.

1. leftChild = **position** * 2 + 1
2. rightChild = **position** * 2 + 2
3. **if** (leftChild <= **lastPosition**) *// the left child of position exists.*
 1. **if** (rightChild <= **lastPosition**) AND (data[rightChild].key > data[leftChild].key)
 1. child = rightChild
 2. **else**
 1. child = leftChild *// choose larger child to compare with data in position*
3. **if** (data[child].key > data[**position**].key)
 1. swap(child, **position**) *// swap data at position with data at child.*
 2. **ReheapDown**(child, **lastPosition**)
4. return

End ReheapDown

Insert new element into min-heap

Insert 14:



The new element is put to the last position, and **ReheapUp** is called for that position.

<ErrorCode> **InsertHeap** (val **DataIn** <DataType>) // *Recursive version.*

Inserts new data into the min-heap.

Post **DataIn** has been inserted into the heap and the heap order property is maintained.

Return *overflow* or *success*

Uses recursive function **ReheapUp**.

1. **if** (heap is full)
 1. return *overflow*
2. **else**
 1. **data**[**count**] = **DataIn**
 2. **ReheapUp**(**count**)
 3. **count** = **count** + 1
 4. return *success*

End InsertHeap

<ErrorCode> **InsertHeap** (val **DataIn** <DataType>) *// Iterative version*

Inserts new data into the min-heap.

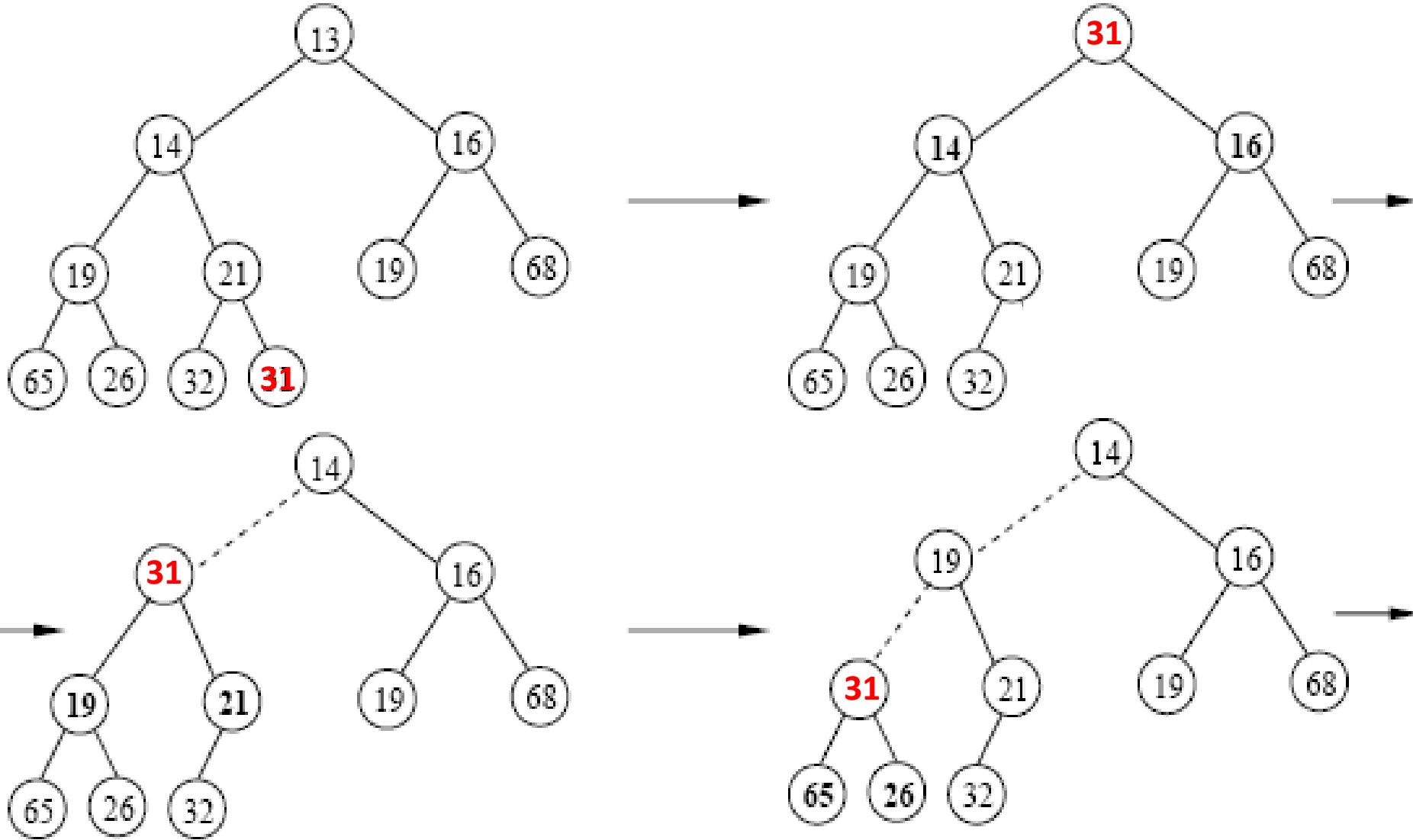
Post **DataIn** has been inserted into the heap and the heap order property is maintained.

Return *overflow* or *success*

1. **if** (heap is full)
 1. return *overflow*
2. **else**
 1. current_position = **count** - 1
 2. **loop** (the parent of the element at the current_position is exists) AND (parent.key > **DataIn**.key)
 1. **data**[current_position] = parent
 2. current_position = position of parent
 3. **data**[current_position] = **DataIn**
 4. **count** = **count** + 1
 5. return *success*

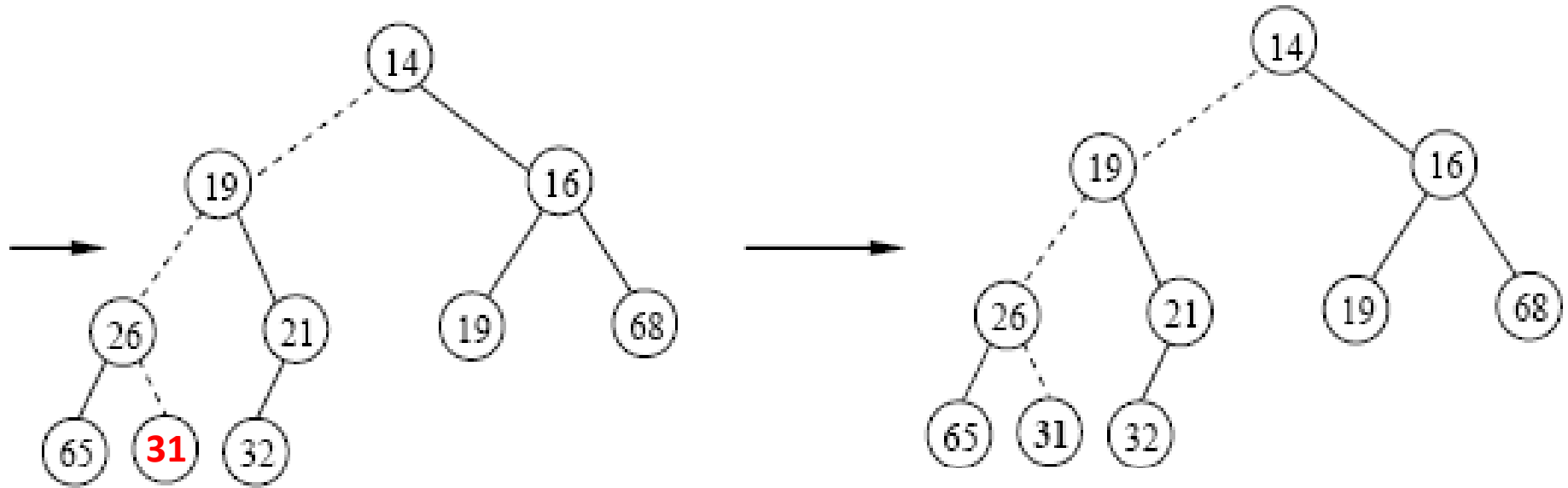
End InsertHeap

Delete minimum element from min-heap



The element in the last position is put to the position of the root, and **ReheapDown** is called for that position.

Delete minimum element from min-heap



The element in the last position is put to the position of the root, and **ReheapDown** is called for that position.

<ErrorCode> **DeleteHeap** (ref **MinData** <DataType>) *// Recursive version*

Removes the minimum element from the min-heap.

Post **MinData** receives the minimum data in the heap and this data has been removed. The heap has been rearranged.

Return *underflow* or *success*

Uses recursive function **ReheapDown**.

1. **if** (heap is empty)
 1. return *underflow*
2. **else**
 1. **MinData** = **Data**[0]
 2. **Data**[0] = **Data**[**count** - 1]
 3. **count** = **count** - 1
 4. **ReheapDown**(0, **count** - 1)
 5. return *success*

End DeleteHeap

<ErrorCode> **DeleteHeap** (ref **MinData** <DataType>) *// Iterative version*

Removes the minimum element from the min-heap.

Post **MinData** receives the minimum data in the heap and this data has been removed. The heap has been rearranged.

Return *underflow* or *success*

1. **if** (heap is empty)
 1. return *underflow*
2. **else**
 1. **MinData** = **Data**[0]
 2. **lastElement** = **Data**[**count** - 1] *// The number of elements in the heap is decreased so the last element must be moved somewhere in the heap.*

// DeleteHeap(cont.)

// Iterative version

3. current_position = 0
4. continue = TRUE
5. **loop** (the element at the current_position has children) AND
(continue = TRUE)
 1. Let child is the smaller of two children
 2. **if** (lastElement.key > child.key)
 1. Data[current_position] = child
 2. current_position = current_position of child
 3. **else**
 1. continue = FALSE
6. Data[current_position] = lastElement
7. count = count - 1
8. return *success*

End DeleteHeap

Build heap

<ErrorCode> **BuildHeap** (val *listOfData* <List>)

Builds a heap from data from *listOfData*.

Pre *listOfData* contains data need to be inserted into an empty heap.

Post Heap has been built.

Return *overflow* or *success*

Uses Recursive function **ReheapUp**.

1. *count* = 0
2. **loop** (heap is not full) AND (more data in *listOfData*)
 1. *listOfData*.Retrieve(*count*, newData)
 2. *data*[*count*] = newData
 3. **ReheapUp**(*count*)
 4. *count* = *count* + 1
3. **if** (*count* < *listOfData*.Size())
 1. return *overflow*
4. **else**
 1. return *success*

End BuildHeap

Build heap

Algorithm **BuildHeap2** ()

Builds a heap from an array of random data.

Pre Array of **count** random data.

Post Array of data becomes a heap.

Uses Recursive function **ReheapDown**.

1. position = **count** / 2 - 1
2. **loop** (position >= 0)
 1. **ReheapDown**(position, **count**-1)
 2. position = position - 1
3. return

End BuildHeap2

Complexity of Binary Heap Operations

- ReheapUp: $O(\log_2 n)$
- ReheapDown: $O(\log_2 n)$
- BuildHeap: $O(n \log_2 n)$
- InsertHeap: $O(\log_2 n)$
- DeleteHeap: $O(\log_2 n)$

d-heaps

- d-heap is a simple generalization of a binary heap.
- In d-heap, all nodes have d children.
- d-heap improve the running time of `InsertElement` to $O(\log_d n)$.
- For large d , `DeleteMin` operation is more expensive: the minimum of d children must be found, which takes $d-1$ comparisons.
- The multiplications and divisions to find children and parents are now by d , which increases the running time. (If $d=2$, use of the bit shift is faster).
- d-heap is suitable for the applications where the number of Insertion is greater than the number of DeleteMin.

Heap Applications

- Select Algorithms.
- Priority Queues.
- Heap sort (*we will see in the Sorting Chapter*).

Select Algorithms

Determine the k^{th} largest element in an unsorted list

Algorithm 1a:

- Read the elements into an array, sort them.
- Return the appropriate element.

The running time of a simple sorting algorithm is $O(n^2)$

Select Algorithms

Determine the k^{th} largest element in an unsorted list

Algorithm 1b:

- Read k elements into an array, sort them.
- The smallest of these is in the k^{th} position.
- Process the remaining elements one by one.
- Compare the coming element with the k^{th} element in the array.
- If the coming element is large, the k^{th} element is removed, the new element is placed in the correct place.

The running time is $O(n^2)$

Select Algorithms

Determine the k^{th} largest element in an unsorted list

Algorithm 2a:

- Build a **max-heap**.
- Delete $k-1$ elements from the heap.
- The desired element will be at the top.

The running time is $O(n \log_2 n)$

Select Algorithms

Determine the k^{th} largest element in an unsorted list

Algorithm 2b:

- Build a **min-heap** of k elements.
- Process the remaining elements one by one.
- Compare the coming element with the minimum element in the heap (the element on the root of heap).
- If the coming element is large, the minimum element is removed, the new element is placed in the correct place (*reheapdown*).

The running time is $O(n \log_2 n)$

Priority Queue ADT

- Jobs are generally placed on a queue to wait for the services.
- In the multiuser environment, the operating system scheduler must decide which of several processes to run.
- Short jobs finish as fast as possible, so they should have precedence over other jobs.
- Otherwise, some jobs are still very important and should also have precedence.

These applications require a special kind of queue: [a priority queue](#).

Priority Queue ADT

- Each element has a priority to be dequeued.
- Minimum value of key has highest priority order.

DEFINITION of Priority Queue ADT:

Elements are enqueued accordingly to their priorities.

Minimum element is dequeued first.

Basic Operations:

- *Create*
- *InsertElement*: Inserts new data to the position accordingly to its priority order in queue.
- *DeleteMin*: Removes the data with highest priority order.
- *RetrieveMin*: Retrieves the data with highest priority order.

Priority Queue ADT

Extended Operations:

- *Clear*
- *isEmpty*
- *isFull*
- *RetrieveMax*: Retrieves the data with lowest priority order.
- *IncreasePriority*
- *DecreasePriority* } Changes the priority of some data which has been inserted in queue.
- *DeleteElement*: Removes some data out of the queue.

Specifications for Priority Queue ADT

```
<ErrorCode> InsertElement (val DataIn <DataType>)
<ErrorCode> DeleteMin (ref MinData <DataType>)
<ErrorCode> RetrieveMin (ref MinData <DataType>)
<ErrorCode> RetrieveMax (ref MaxData <DataType>)
<ErrorCode> IncreasePriority (val position <int>,
                             val PriorityDelta <KeyType>)
<ErrorCode> DecreasePriority (val position <int>,
                             val PriorityDelta <KeyType>)
<ErrorCode> DeleteElement (val position <int>,
                           ref DataOut <DataType>)

<bool> isEmpty()
<bool> isFull()
<void> clear()
```

Implementations of Priority Queue

➤ Use linked list:

■ Simple linked list:

- Insertion performs at the front, requires $O(1)$.
- DeleteMin requires $O(n)$ for searching of the minimum data.

■ Sorted linked list:

- Insertion requires $O(n)$ for searching of the appropriate position.
- DeleteMin requires $O(1)$.

Implementations of Priority Queue

➤ Use BST:

- Insertion requires $O(\log_2 n)$.
- DeleteMin requires $O(\log_2 n)$.
- But DeleteMin , repeatedly removing node in the left subtree, seem to hurt **balance of the tree**.

Implementations of Priority Queue

➤ Use min-heap:

- Insertion requires $O(\log_2 n)$.
- DeleteMin requires $O(\log_2 n)$.

Insert and Remove element into/from priority queue

<ErrorCode> **InsertElement** (val DataIn <DataType>):

InsertHeap Algorithm

<ErrorCode> **DeleteMin** (ref MinData <DataType>):

DeleteHeap Algorithm

Retrieve minimum element in priority queue

<ErrorCode> **RetrieveMin** (ref **MinData** <DataType>)

Retrieves the minimum element in the heap.

Post **MinData** receives the minimum data in the heap and the heap remains unchanged.

Return *underflow* or *success*

1. **if** (heap is empty)
 1. return *underflow*
2. **else**
 1. **MinData** = **Data**[0]
 2. return *success*

End RetrieveMin

Retrieve maximum element in priority queue

<ErrorCode> **RetrieveMax** (ref **MaxData** <DataType>)

Retrieves the maximum element in the heap.

Post **MaxData** receives the maximum data in the heap and the heap remains unchanged.

Return *underflow* or *success*

1. if (heap is empty)

 1. return *underflow*

2. else

 1. Sequential search the maximum data in the right half elements of the heap (the leaves of the heap). The first leaf is at the position **count**/2.

 2. return *success*

End RetrieveMax

Change the priority of an element in priority queue

```
<ErrorCode> IncreasePriority (val position <int>,  
                                val PriorityDelta <KeyType>)
```

Increases priority of an element in the heap.

Post Element at `position` has its priority increased by `PriorityDelta` and has been moved to correct position.

Return *rangeError* or *success*

Uses `ReheapDown`.

Change the priority of an element in priority queue

```
<ErrorCode> DecreasePriority (val position <int>,  
                                val PriorityDelta <KeyType>)
```

Decreases priority of an element in the heap.

Post Element at `position` has its priority decreased by `PriorityDelta` and has been moved to correct position.

Return *rangeError* or *success*

Uses `ReheapUp`.

Remove an element out of priority queue

```
<ErrorCode> DeleteElement (val position <int>,  
                             ref DataOut <DataType>)
```

Removes an element out of the min-heap.

Post **DataOut** contains data in the element at **position**, this element has been removed. The heap has been rearranged.

Return *rangeError* or *success*

1. if (**position** ≥ **count**) OR (**position** < 0)
 1. return *rangeError*
2. else
 1. **DataOut** = **Data**[**position**]
 2. DecreasePriority(**position**, VERY_LARGE_VALUE),
 3. DeleteMin(MinData)
 4. return *success*

End DeleteElement

Advanced implementations of heaps

➤ Advanced implementations of heaps: **use of pointers**

- Leftist heap
- Skew heap
- Binomial queues

Use of pointers allows the **merge operations** (combine two heaps into one) to perform easily.