The mean king's problem: Spin 1

Yakir Aharonov[a,b] and Berthold-Georg Englert[c,d]
[a]School of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel
[b]Physics Department, University of South Carolina, Columbia SC 29208, USA
[c]Atominstitut der Österreichischen Universitäten, Stadionallee 2, 1020 Wien, Austria
[d]Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany
(To appear in Zeitschrift für Naturforschung, received 26 November 2000)

We show how one can ascertain the values of four mutually complementary observables of a spin-1 degree of freedom.

PACS: 03.65.Bz

1. Introduction

About a dozen years ago, one of us (YA) co-authored a paper [1] with the somewhat provocative title "How to ascertain the values of σ_x, σ_y, and σ_z of a spin-$\frac{1}{2}$ particle". It reports the solution of what has later become known as *The King's Problem*:

> A ship-wrecked physicist gets stranded on a far-away island that is ruled by a mean king who loves cats and hates physicists since the day when he first heard what happened to Schrödinger's cat. A similar fate is awaiting the stranded physicist. Yet, mean as he is, the king enjoys defeating physicists on their own turf, and therefore he maliciously offers an apparently virtual chance of rescue.
>
> He takes the physicist to the royal laboratory, a splendid place where experiments of any kind can be performed perfectly. There the king invites the physicist to prepare a certain silver atom in any state she likes. The king's men will then measure one of the three cartesian spin components of this atom — they'll either measure σ_x, σ_y, or σ_z without, however, telling the physicist which one of these measurements is actually done. Then it is again the physicist's turn, and she can perform any experiment of her choosing. Only after she's finished with it, the king will tell her which spin component had been measured by his men. To save her neck, the physicist must then state correctly the measurement result that the king's men had obtained.
>
> Much to the king's frustration, the physicist rises to the challenge — and not just by sheer luck: She gets the right answer any time the whole procedure is repeated. How does she do it?

Readers who don't know the answer should try to figure it out themselves rather than consult the said reference. There is a lesson here about the wonderful things entanglement can do for you.

It is worth mentioning that this thought experiment of 1987 has not been realized as yet. Very recently, however, a quantum-optical analog has been formulated [2], and it is hoped that experimental data will be at hand shortly.

The present paper deals with a generalization of the king's problem. Instead of the traditional spin-$\frac{1}{2}$ atom, we consider the situation of a spin-1 atom. The two main questions are then: What are the appropriate spin-1 analogs of the spin-$\frac{1}{2}$ observables σ_x, σ_y, σ_z? And, how does the physicist save her neck now?

The first question is answered in Sect. 2 in terms of a complete set of mutually complementary observables. The answer to the second question is given in Sect. 3; it employs essentially the same strategy that works in the spin-$\frac{1}{2}$ case, so that we have a genuine generalization indeed. Further generalizations to even higher spins will be discussed elsewhere [3].

2. Mutually complementary observables

The three spin-$\frac{1}{2}$ observables σ_x, σ_y, σ_z are *complete* in the sense that the probabilities for finding their eigenvalues as the results of measurements specify uniquely the statistical operator that characterizes the spin-$\frac{1}{2}$ degree of freedom of the ensemble under consideration. They are not overcomplete because this unique specification is not ensured if one of the spin components is left out.

In addition to being complete, the observables σ_x, σ_y, σ_z are also pairwise *complementary*, which is to say that in a state where one of them has a definite value, all measurement results for the other ones are equally probable. For example, if $\sigma_x = 1$ specifies the ensemble, say, then the results of σ_y measurements are utterly unpredictable: $+1$ and -1 are found with equal frequency; and the same applies to σ_z measurements.

What is essential here are not the eigenvalues of σ_x, σ_y, σ_z, but their sets of eigenstates. It is familiar that they are related to each other by

$$\begin{aligned} \left|\sigma_x = \pm 1\right\rangle &= 2^{-\frac{1}{2}} \left(\left|\sigma_z = +1\right\rangle \pm \left|\sigma_z = -1\right\rangle\right) , \\ \left|\sigma_y = \pm 1\right\rangle &= 2^{-\frac{1}{2}} \left(\left|\sigma_z = +1\right\rangle \pm \mathrm{i}\left|\sigma_z = -1\right\rangle\right) , \end{aligned} \tag{1}$$

if the usual phase conventions are adopted. The fact that the transition probabilities

$$\begin{aligned}\left|\langle\sigma_x=\pm1|\sigma_y=\pm1\rangle\right|^2&=\frac{1}{2}\,,\\ \left|\langle\sigma_y=\pm1|\sigma_z=\pm1\rangle\right|^2&=\frac{1}{2}\,,\\ \left|\langle\sigma_z=\pm1|\sigma_x=\pm1\rangle\right|^2&=\frac{1}{2}\,,\end{aligned}\tag{2}$$

do not depend on the quantum numbers ±1, is the statement of the pairwise complementary nature of σ_x, σ_y, and σ_z. Their algebraic completeness is then an immediate consequence of the insight that a spin-$\frac{1}{2}$ degree of freedom can have at most three mutually complementary observables [4].

Analogously, there can be no more than four such observables for a spin-1 degree of freedom. Let's call them A_0, A_1, A_2, and A_3, and to be specific, we take their eigenvalues to be 0, 1, and 2. We denote by $|m_k\rangle$ the eigenstate of A_m to eigenvalue k, and we express the eigenstates of A_1, A_2, A_3 in terms of those of A_0. With

$$x=\mathrm{e}^{\mathrm{i}2\pi/3}\,,\tag{3}$$

the basic cubic root of unity, it is a matter of inspection to verify that the choice

$$\begin{aligned}\left(|1_0\rangle\,,|1_1\rangle\,,|1_2\rangle\right)&=\left(|0_0\rangle\,,|0_1\rangle\,,|0_2\rangle\right)\frac{1}{\sqrt{3}}\begin{pmatrix}x&1&1\\1&x&1\\1&1&x\end{pmatrix}\,,\\ \left(|2_0\rangle\,,|2_1\rangle\,,|2_2\rangle\right)&=\left(|0_0\rangle\,,|0_1\rangle\,,|0_2\rangle\right)\frac{1}{\sqrt{3}}\begin{pmatrix}x^2&1&1\\1&x^2&1\\1&1&x^2\end{pmatrix}\,,\\ \left(|3_0\rangle\,,|3_1\rangle\,,|3_2\rangle\right)&=\left(|0_0\rangle\,,|0_1\rangle\,,|0_2\rangle\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&x&x^2\\1&x^2&x\end{pmatrix}\end{aligned}\tag{4}$$

is indeed such that

$$\left|\langle m_k|m'_{k'}\rangle\right|^2=\begin{cases}\delta_{kk'} & \text{if}\quad m=m'\,,\\ \frac{1}{3} & \text{if}\quad m\neq m'\,,\end{cases}\tag{5}$$

so that each set consists of 3 orthonormal states, as it should, and any two different sets are complementary.

Repeated measurements of the observables A_m (on identically prepared spin-1 systems) eventually determine the probabilities $p_k^{(m)}$ for finding their eigenstates $|m_k\rangle$. As a consequence of their mutual complementarity, knowledge of the probabilities for one A_m contains no information whatsoever about the probabilities for any other one. These 12 probabilities represent 8 parameters in total, since $p_0^{(m)}+p_1^{(m)}+p_2^{(m)}=1$ for each of the four A_ms. The statistical operator that characterizes the ensemble of identically prepared spin-1 systems,

$$\rho=\sum_{m=0}^{3}\sum_{k=0}^{2}|m_k\rangle\left(p_k^{(m)}-\frac{1}{4}\right)\langle m_k|\,,\tag{6}$$

is therefore uniquely determined by the probabilities $p_k^{(m)}=\langle m_k|\,\rho\,|m_k\rangle$. Indeed, the A_ms constitute a complete set of mutually complementary observables for the spin-1 degree of freedom.

3. Spin-1 version of the mean king's problem

Accordingly, in the spin-1 version of *The King's Problem* either one of A_0, A_1, A_2, or A_3 is measured by the mean king's men, on a spin-1 atom suitably prepared by the physicist. Without knowing which measurement was done actually, the physicist performs a subsequent measurement of her own, and — after then being told which A_m was measured by the king's men — she has to state correctly what they found: 0 or 1 or 2.

The physicist solves the problem by first preparing a state $|\Psi_0\rangle$ in which the given spin-1 atom is entangled with another, auxiliary, spin-1 atom. Two-atom states in which the given atom is in state $|m_k\rangle$ and the auxiliary atom in $|m'_{k'}\rangle$ are denoted by $|m_k m'_{k'}\rangle$. Then

$$\begin{aligned}|\Psi_0\rangle&=3^{-\frac{1}{2}}\left(|0_00_0\rangle+|0_10_1\rangle+|0_20_2\rangle\right)\\ &=3^{-\frac{1}{2}}\left(|1_02_0\rangle+|1_12_1\rangle+|1_22_2\rangle\right)\\ &=3^{-\frac{1}{2}}\left(|2_01_0\rangle+|2_11_1\rangle+|2_21_2\rangle\right)\\ &=3^{-\frac{1}{2}}\left(|3_03_0\rangle+|3_13_2\rangle+|3_23_1\rangle\right)\end{aligned}\tag{7}$$

are alternative ways of writing the state she prepares. Their equivalence is easily verified with the aid of the transformation laws (4).

If the king's men then measure A_m on the given atom and find the value k, the resulting two-atom state is the respective $|m_k m'_{k'}\rangle$ component of $|\Psi_0\rangle$. After their measurement, there are thus all together 4 trios of possible two-atom states. We write them compactly as

$$\begin{aligned}\left(|0_00_0\rangle\,,|0_10_1\rangle\,,|0_20_2\rangle\right)&=\left(|\Psi_0\rangle\,,|\Psi_1\rangle\,,|\Psi_2\rangle\right)\mathcal{U}\,,\\ \left(|1_02_0\rangle\,,|1_12_1\rangle\,,|1_22_2\rangle\right)&=\left(|\Psi_0\rangle\,,|\Psi_3\rangle\,,|\Psi_4\rangle\right)\mathcal{U}\,,\\ \left(|2_01_0\rangle\,,|2_11_1\rangle\,,|2_21_2\rangle\right)&=\left(|\Psi_0\rangle\,,|\Psi_5\rangle\,,|\Psi_6\rangle\right)\mathcal{U}\,,\\ \left(|3_03_0\rangle\,,|3_13_2\rangle\,,|3_23_1\rangle\right)&=\left(|\Psi_0\rangle\,,|\Psi_7\rangle\,,|\Psi_8\rangle\right)\mathcal{U}\,,\end{aligned}\tag{8}$$

where the 3-rows on the right are multiplied by the unitary 3×3 matrix

$$\mathcal{U}=\frac{1}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&x&x^2\\1&x^2&x\end{pmatrix}\tag{9}$$

which we met in (4) as well. Since the members of each trio are orthogonal to each other, the 8 two-atom states $|\Psi_1\rangle$, …, $|\Psi_8\rangle$ introduced here are orthogonal to $|\Psi_0\rangle$ by construction. It is equally immediate that the paired states $|\Psi_{2m+1}\rangle\,,|\Psi_{2m+2}\rangle$ are orthogonal to each other for $m=0,1,2,3$. That, more generally, the orthonormality relation

$$\langle\Psi_j|\Psi_k\rangle = \delta_{jk} \quad \text{for } j,k = 0,\ldots,8 \tag{10}$$

holds also for states from different trios can be checked explicitly (or one recognizes a special case of a more general statement [3]).

The physicist will be able to state correctly the measurement result found by the king's men if she can find a two-atom observable P with a set of eigenstates $|P_0\rangle, \ldots, |P_8\rangle$ such that each $|P_k\rangle$ is orthogonal to two members each of the four trios on the left of (8). It is convenient to specify such states by indicating which members they are *not* orthogonal to, so that

$$\big|[k_0k_1k_2k_3]\big\rangle \tag{11}$$

has the defining property of being orthogonal to the two-atom states that result when measurements of A_m do *not* give the eigenvalue k_m.

In order to see how this enables her to infer the measured value, suppose the physicist finds the two-atom system in state $\big|[1012]\big\rangle$. She then knows that if the king's men had measured A_0, A_1, A_2, or A_3, the respective results must have been 1, 0, 1, and 2, because she would never find $\big|[1012]\big\rangle$ for other measurement results.

Accordingly, all that is needed to complete the solution of the spin-1 version of the mean king's problem is the demonstration that we can have a complete orthonormal set of two-atom states of the kind (11). First note that the expansion of $\big|[k_0k_1k_2k_3]\big\rangle$ in the $\big|\Psi_j\big\rangle$ basis is given by

$$\begin{aligned}\big|[k_0k_1k_2k_3]\big\rangle &= \frac{1}{3}\,|\Psi_0\rangle \\ &+\frac{1}{3}\sum_{m=0}^{3}\Big(|\Psi_{2m+1}\rangle\, x^{k_m} + |\Psi_{2m+2}\rangle\, x^{-k_m}\Big)\,.\end{aligned} \tag{12}$$

Then observe that

$$\big\langle[k_0k_1k_2k_3]\big|[k_0'k_1'k_2'k_3']\big\rangle = \frac{1}{3}\sum_{m=0}^{3}\delta_{k_m,k_m'} - \frac{1}{3}\,, \tag{13}$$

so that two such states are orthogonal if $k_m = k_m'$ for one and only one m value. Therefore, a possible choice of basis states for the physicist's final measurement is

$$\begin{aligned}|P_0\rangle &= \big|[0000]\big\rangle\,, & |P_1\rangle &= \big|[0111]\big\rangle\,, & |P_2\rangle &= \big|[0222]\big\rangle\,,\\ |P_3\rangle &= \big|[1012]\big\rangle\,, & |P_4\rangle &= \big|[1120]\big\rangle\,, & |P_5\rangle &= \big|[1201]\big\rangle\,,\\ |P_6\rangle &= \big|[2021]\big\rangle\,, & |P_7\rangle &= \big|[2102]\big\rangle\,, & |P_8\rangle &= \big|[2210]\big\rangle\,.\end{aligned} \tag{14}$$

After being told which measurement the king's men performed on the given atom, she can then infer their measurement result correctly, and with certainty, in the manner described above for $|P_3\rangle = \big|[1012]\big\rangle$.

Acknowledgments

YA gratefully acknowledges the kind hospitality extended to him by Herbert Walther during a visit at the MPI für Quantenoptik in Garching. BGE would like to thank Wolfgang Schleich and the University of Ulm for financial support while part of this work was done. YA's research is supported in part by Grant No. 471/98 of the Basic Research Foundation (administered by the Israel Academy of Sciences and Humanities) and NSF Grant No. PHY-9971005.

[1] L. Vaidman, Y. Aharonov, and D. Z. Albert, *How to ascertain the values of σ_x, σ_y, and σ_z of a spin-$\frac{1}{2}$ particle*, Phys. Rev. Lett. **58**, 1385–1387 (1987).

[2] B.-G. Englert, Ch. Kurtsiefer, and H. Weinfurter, *Universal unitary gate for single-photon 2-qubit states*, Phys. Rev. A (in print), LANL eprint quant-ph/0101064.

[3] Y. Aharonov and B.-G. Englert, *The mean king's problem: Prime degrees of freedom*, in preparation.

[4] W. K. Wootters and B. D. Fields, *Optimal State-Determination by Mutually Unbiased Measurements*, Ann. Phys. (NY) **191**, 363–381 (1989).