ALGEBRAIC MODELS FOR CLASSIFYING SPACES OF FIBRATIONS

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Abstract. We prove new structural results for the rational homotopy type of the classifying space $B\operatorname{aut}(X)$ of fibrations with fiber a simply connected finite CW-complex X.

We first study nilpotent covers of $B\operatorname{aut}(X)$ and show that their rational cohomology groups are algebraic representations of the associated transformation groups. For the universal cover, this yields an extension of the Sullivan–Wilkerson theorem to higher homotopy and cohomology groups. For the cover corresponding to the kernel of the homology representation, this proves algebraicity of the cohomology of the homotopy Torelli space.

For the cover that classifies what we call *normal unipotent fibrations*, we then prove the stronger result that there exists a nilpotent dg Lie algebra $\mathfrak{g}(X)$ in algebraic representations that models its equivariant rational homotopy type. This leads to an algebraic model for the space $B\operatorname{aut}(X)$ and to a description of its rational cohomology ring as the cohomology of a certain arithmetic group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cohomology of $\mathfrak{g}(X)$. This has strong structural consequences for the cohomology ring and, in certain cases, allows it to be completely determined using invariant theory and calculations with modular forms. We illustrate these points with concrete examples.

As another application, we significantly improve on certain results on self-homotopy equivalences of highly connected even-dimensional manifolds due to Berglund–Madsen, and we prove parallel new results in odd dimensions.

1. Introduction

In this paper we prove new structural results about the classifying space $B\operatorname{aut}(X)$ of the topological monoid of self-homotopy equivalences of a simply connected finite CW-complex X from the point of view of rational homotopy theory.

Rational models are known for the universal cover of $B\operatorname{aut}(X)$ [66, 67] and for certain nilpotent covers [32]. However, there are no general results that incorporate the action of the associated transformation groups into such models, let alone integrate such information to structural results about $B\operatorname{aut}(X)$. Our aim is to establish results of this kind.

The fundamental group of $B\operatorname{aut}(X)$ is the group $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences of X. By a fundamental result due to Sullivan [66] and Wilkerson [71], the self-homotopy equivalences of the rationalization of X form a linear algebraic group $\mathcal{E}(X_{\mathbb{Q}})$ and the homomorphism $\mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ has finite kernel and image an arithmetic subgroup. Our first result could be viewed as an extension of the Sullivan–Wilkerson theorem to higher homotopy and (co)homology groups.

Theorem 1.1. *The representations of $\mathcal{E}(X)$ in the rational homotopy and (co)homology groups of the universal cover of $B\operatorname{aut}(X)$ are restrictions of algebraic representations of the linear algebraic group $\mathcal{E}(X_{\mathbb{Q}})$.*

In fact, this is a special case of an algebraicity result for a wide class of nilpotent covers of $B\operatorname{aut}(X)$ (Theorem 3.34), including the *homotopy Torelli space*

$B\operatorname{tor}(X)$—the classifying space of the monoid of self-homotopy equivalences of X that act trivially on $H_*(X;\mathbb{Z})$.

Determining the rational homotopy type of $B\operatorname{aut}(X)$ does however require more than knowledge of the action of $\mathcal{E}(X)$ on the rational cohomology groups of the universal cover. One approach is to study the fiberwise rationalization along the map $B\operatorname{aut}(X) \to B\,\mathcal{E}(X)$ using local systems of commutative differential graded algebras. The problem of finding a model for $B\operatorname{aut}(X)$ of this sort was raised in [38, §7.3]. By modifying the approach slightly, we propose a solution to this problem. Instead of working over $B\,\mathcal{E}(X)$, we study the fiberwise rationalization of the homotopy fiber sequence

$$\Gamma(X)/\operatorname{aut}(X) \to B\operatorname{aut}(X) \to B\Gamma(X) \tag{1}$$

for a certain arithmetic group $\Gamma(X)$, namely the image of $\mathcal{E}(X)$ in the maximal reductive quotient $R(X)$ of the algebraic group $\mathcal{E}(X_{\mathbb{Q}})$. The following is our main structural result.

Theorem 1.2. *The space $\Gamma(X)/\operatorname{aut}(X)$ is virtually nilpotent and admits an* algebraic Lie model*, meaning a nilpotent differential graded Lie algebra $\mathfrak{g}(X)$ of algebraic representations of $R(X)$, whose nerve $\langle\mathfrak{g}(X)\rangle$ is rationally equivalent to $\Gamma(X)/\operatorname{aut}(X)$ as a $\Gamma(X)$-space.*

In particular, $B\operatorname{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle\mathfrak{g}(X)\rangle_{h\Gamma(X)}$ and there is a quasi-isomorphism of commutative cochain algebras

$$\Omega^*\big(B\operatorname{aut}(X)\big) \sim \Omega^*\big(\Gamma(X), C^*_{CE}(\mathfrak{g}(X))\big).$$

Here $\Omega^*(B)$ stands for polynomial differential forms on B. It is a commutative dg algebra model for the singular cochains $C^*(B;\mathbb{Q})$ and it represents the rational homotopy type of B. Similarly, the right-hand side is a commutative dg algebra model for the cochains on the group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cochains of $\mathfrak{g}(X)$, see §2.4. In §3.5.2 we discuss how Theorem 1.2 gives a possible solution to [38, §7.3, Problem 3].

Let us point out that the existence of an algebraic Lie model immediately implies that the representations of the group $\Gamma(X)$ in the rational (co)homology groups of $\Gamma(X)/\operatorname{aut}(X)$ are restrictions of algebraic representations of $R(X)$. The existence of an algebraic Lie model is however stronger than algebraicity of (co)homology. Together with semisimplicity of algebraic representations of reductive groups, it implies the following result, which reduces the computation of the rational cohomology ring of $B\operatorname{aut}(X)$ to the computation of Chevalley–Eilenberg cohomology and cohomology of arithmetic groups with coefficients in algebraic representations.

Corollary 1.3. *There is an isomorphism of graded algebras*

$$H^*\big(B\operatorname{aut}(X);\mathbb{Q}\big) \cong H^*\big(\Gamma(X), H^*_{CE}(\mathfrak{g}(X))\big). \tag{2}$$ □

This implies, but is stronger than, collapse of the rational Serre spectral sequence of the fibration (1). Collapse of the spectral sequence only implies an isomorphism of algebras after passing to the associated graded algebra of $H^*(B\operatorname{aut}(X);\mathbb{Q})$ with respect to a filtration. In effect, Corollary 1.3 says that $H^*(B\operatorname{aut}(X);\mathbb{Q})$ is isomorphic to the E_2-page of this spectral sequence as a graded algebra.

When combined with finiteness of the virtual cohomological dimension of arithmetic groups [17], it also implies that the computation of the ring $H^*(B\operatorname{aut}(X);\mathbb{Q})$ modulo nilpotent elements reduces further to invariant theory.

Corollary 1.4. *The ring homomorphism*

$$H^*(B\operatorname{aut}(X);\mathbb{Q}) \to H^*\big(\Gamma(X)/\operatorname{aut}(X);\mathbb{Q}\big)^{\Gamma(X)} \tag{3}$$

is split surjective and its kernel I is a nilpotent ideal with $I^n = 0$ for $n > vcd(\Gamma(X))$.

In particular, (3) *induces an isomorphism modulo nilradicals. In other words, the algebraic variety of the ring* $H^*(B\operatorname{aut}(X);\mathbb{Q})$ *is isomorphic to that of the invariant ring*

$$H^*\big(\Gamma(X)/\operatorname{aut}(X);\mathbb{Q}\big)^{\Gamma(X)} \cong H^*_{CE}(\mathfrak{g}(X))^{\Gamma(X)}. \qquad \square$$

Let us underscore another immediate consequence of Corollary 1.3.

Corollary 1.5. *The ring homomorphism*

$$H^*(\Gamma(X),\mathbb{Q}) \to H^*(B\operatorname{aut}(X);\mathbb{Q})$$

is split injective. $\square$

This means in particular that all cohomology classes of the arithmetic group $\Gamma(X)$ are faithfully represented as characteristic classes of fibrations with fiber X. This is especially striking in view of the fact that many arithmetic groups can be realized as $\Gamma(X)$ for some X, *cf.* [66, Theorem 10.3(iv)]. See Remark 1.16 for further related remarks.

We have chosen to state our main result as an existence theorem in order to highlight the strong consequences of the mere existence of an algebraic Lie model, but the ingredients $R(X)$, $\Gamma(X)$, $\mathfrak{g}(X)$ can be described in more concrete terms. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = H_*(X;\mathbb{Q}) \tag{4}$$

be a composition series for $H_*(X;\mathbb{Q})$ as a representation of $\mathcal{E}(X_{\mathbb{Q}})$, meaning the filtration quotients V_i/V_{i-1} are simple $\mathcal{E}(X_{\mathbb{Q}})$-modules, and consider the associated graded representation

$$\operatorname{gr} H_*(X;\mathbb{Q}) = \bigoplus_{i=1}^{n} V_i/V_{i-1}.$$

The isomorphism type of the semisimple $\mathcal{E}(X_{\mathbb{Q}})$-module $\operatorname{gr} H_*(X;\mathbb{Q})$ is independent of the choice of composition series by the Jordan–Hölder theorem, so it is an invariant of the rational homotopy type of X. For a group G and a representation V, let $\mathsf{GL}^G(V)$ denote the image of the homomorphism $G \to \mathsf{GL}(V)$.

Theorem 1.6. *There are group isomorphisms*

$$R(X) \cong \mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}\big(\operatorname{gr} H_*(X;\mathbb{Q})\big),$$

$$\Gamma(X) \cong \mathsf{GL}^{\mathcal{E}(X)}\big(\operatorname{gr} H_*(X;\mathbb{Q})\big).$$

In other words, $R(X)$*, respectively* $\Gamma(X)$*, may be identified with the group of automorphisms of* $\operatorname{gr} H_*(X;\mathbb{Q})$ *that are induced by a self-homotopy equivalence of* $X_{\mathbb{Q}}$*, respectively* X*.*

In many cases of interest, in fact in all examples we consider except the last, the equivalent conditions in the corollary below are satisfied, which simplifies the descriptions.

Corollary 1.7. *The following are equivalent:*

(i) $\operatorname{gr} H_*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q})$ *as* $\mathcal{E}(X_{\mathbb{Q}})$*-modules.*
(ii) The $\mathcal{E}(X_{\mathbb{Q}})$*-module* $H_*(X;\mathbb{Q})$ *is semisimple.*
(iii) The group $\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ *is reductive.*

In this situation $R(X)$*, respectively* $\Gamma(X)$*, may be identified with the group of automorphisms of* $H_*(X;\mathbb{Q})$ *that are induced by a self-homotopy equivalence of* $X_{\mathbb{Q}}$*, respectively* X*.* $\square$

Remark 1.8. There is an inclusion of algebraic groups

$$\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q})) \leq \mathrm{Aut}_{coalg}(H_*(X;\mathbb{Q})),$$

where the latter is the group of automorphisms of the homology coalgebra. This inclusion is an equality if and only if X is formal (this follows from *e.g.* [66, Theorem 12.7]). Thus, the descriptions of $R(X)$ and $\Gamma(X)$ in Corollary 1.7 simplify even further for formal spaces.

Remark 1.9. If the equivalent conditions in Corollary 1.7 hold and in addition $H_*(X;\mathbb{Z})$ is torsion-free, then $\Gamma(X)$ may be identified with the group of automorphisms of $H_*(X;\mathbb{Z})$ that are induced by a self-homotopy equivalence. In this case, $\Gamma(X)/\operatorname{aut}(X)$ is weakly equivalent to the homotopy Torelli space $B\operatorname{tor}(X)$ so, in particular, the latter admits an algebraic Lie model. The existence of an algebraic Lie model for $B\operatorname{tor}(X)$ remains open in general. See Remark 3.42 for a further discussion.

In general, $\Gamma(X)/\operatorname{aut}(X)$ can be characterized as the classifying space for *normal unipotent fibrations*; we say that a fibration of connected spaces $X \to E \to B$ is normal unipotent if $\pi_1(B)$ acts trivially on the associated graded module of a filtration of $H_*(X;\mathbb{Q})$ by $\mathcal{E}(X_{\mathbb{Q}})$-submodules.[1] See §3.1 for a further discussion.

We will now give more concrete models for $\mathfrak{g}(X)$. Let Λ be the minimal Sullivan model for X and let $\mathbb{L}$ be the minimal Quillen model. Let $\operatorname{Der}\Lambda$ denote the dg Lie algebra of derivations of Λ and let $\operatorname{Der}^c\mathbb{L}$ denote the dg Lie algebra of curved derivations of $\mathbb{L}$ (see §2.5). We let $\operatorname{nil}\mathfrak{g}$ denote the nilradical of a dg Lie algebra $\mathfrak{g}$ (see §2.2).

The following refines the existence statement in Theorem 1.2, giving concrete descriptions of algebraic Lie models for $\Gamma(X)/\operatorname{aut}(X)$.

Theorem 1.10. *The algebraic group $R(X)$ admits algebraic actions on the Sullivan model Λ and the Quillen model $\mathbb{L}$, compatible with the action on* $\operatorname{gr} H_*(X;\mathbb{Q})$.

- *(i) If $H_*(X;\mathbb{Q})$ is finite-dimensional, then* $\operatorname{nil}\operatorname{Der}^c\mathbb{L}$*, with the induced* $\Gamma(X)$*-action, is an algebraic Lie model for* $\Gamma(X)/\operatorname{aut}(X)$.
- *(ii) If $\pi_*(X)\otimes\mathbb{Q}$ is finite-dimensional, then* $\operatorname{nil}\operatorname{Der}\Lambda$*, with the induced* $\Gamma(X)$*-action, is an algebraic Lie model for* $\Gamma(X)/\operatorname{aut}(X)$.

Remark 1.11. The minimal Sullivan model Λ is finitely generated if and only if $\pi_*(X)\otimes\mathbb{Q}$ is finite-dimensional. The minimal Quillen model $\mathbb{L}$ is finitely generated if and only if $H_*(X;\mathbb{Q})$ is finite-dimensional. If Λ is not finitely generated, then the graded components of $\operatorname{Der}\Lambda$ may not be finite-dimensional, but they are linear representations of $R(X)$ in the sense of [53, §4a]. In particular, they are filtered unions of finite-dimensional algebraic representations [53, Corollary 4.8]. Similar remarks apply to $\operatorname{nil}\operatorname{Der}^c\mathbb{L}$.

Remark 1.12. We have stated our main results for simply connected finite CW-complexes, but parallel results hold for simply connected finite Postnikov sections (*i.e.* spaces X with $\pi_n X = 0$ for all $n \gg 0$) with finite-dimensional rational homotopy groups. If the homology $H_*(X;\mathbb{Q})$ is not finite-dimensional, one can replace it by the spherical homology $SH_*(X;\mathbb{Q})$ in the descriptions of $R(X)$ and $\Gamma(X)$.

We should also add that many of our results go through for nilpotent spaces but we have chosen to state them for simply connected spaces for simplicity, see Remark 3.28.

[1]This notion is slightly stronger than what is usually meant by the term 'unipotent fibration' in that we require the filtration to be $\mathcal{E}(X_{\mathbb{Q}})$-stable and not only $\pi_1(B)$-stable.

Remark 1.13. By forgetting the action of $R(X)$ and taking the simply connected truncations of the Lie models in Theorem 1.10, we recover the known Lie models for the universal cover of $B\operatorname{aut}(X)$.

We will now discuss applications to specific spaces X. We highlight a few select results and refer the reader to §4 for further results and details.

Consider first the n-fold cartesian product $S^{d\times n} = S^d \times \ldots \times S^d$ of d-dimensional spheres. This is a good example for showcasing our results because all the ingredients can be computed explicitly. Corollary 1.3 assumes the following form.

Theorem 1.14. *For d odd, there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q}) \cong H^*\Big(\Gamma(S^{d\times n}), \operatorname{Sym}^\bullet\left(V_n[d+1]\right)\Big), \tag{5}$$

where $V_n[d+1]$ denotes the standard representation of $\mathsf{GL}_n(\mathbb{Q})$ concentrated in degree $d+1$,

$$\Gamma(S^{d\times n}) = \begin{cases} \mathsf{GL}_n(\mathbb{Z}), & d = 1,3,7, \\ \mathsf{GL}_n^{\Sigma}(\mathbb{Z}), & d \neq 1,3,7, \end{cases}$$

and $\mathsf{GL}_n^{\Sigma}(\mathbb{Z}) \leqslant \mathsf{GL}_n(\mathbb{Z})$ is the subgroup of matrices with exactly one odd entry in each row.

For $n = 2$ the right-hand side of (5) can be computed in terms of modular forms via the Eichler–Shimura isomorphism.

Theorem 1.15. *For d odd, there is an isomorphism of graded vector spaces*

$$\widetilde{H}^*\left(B\operatorname{aut}(S^d \times S^d);\mathbb{Q}\right) \cong \bigoplus_k S_k\left(\Gamma^+(S^{d\times 2})\right)[(k-2)(d+1)+1],$$

where $S_k(\Gamma)$ denotes the space of cuspidal modular forms of weight k for $\Gamma \leq \mathsf{SL}_2(\mathbb{Z})$, and where $\Gamma^+(S^{d\times 2}) = \Gamma(S^{d\times 2}) \cap \mathsf{SL}_2(\mathbb{Z})$. All cup products in reduced cohomology are trivial.

This in particular leads to an explicit formula for the Poincaré series of the cohomology. Similar results hold for $n = 3$. See §4.1 for more details.

Remark 1.16. The above example illustrates that even in cases where $\Gamma(X)$ and $H^*_{CE}(\mathfrak{g}(X))$ can be described explicitly, a complete computation of the right-hand side of (2) is in general out of reach due to the difficulty of calculating the cohomology of arithmetic groups. The cohomology of $\mathsf{GL}_n(\mathbb{Z})$ with coefficients in algebraic representations is not fully known.

This suggests a different perspective on the isomorphism (2). Rather than interpreting it as a computation of $H^*(B\operatorname{aut}(X);\mathbb{Q})$, it tells us that cohomology classes of arithmetic groups in the right-hand side, say classes constructed using automorphic forms, can be represented as characteristic classes of fibrations. Connections between characteristic classes and automorphic forms have been observed before in special cases (see *e.g.* [35]). Our results suggest that the connection between the cohomology of arithmetic groups and characteristic classes of fibrations may be deeper than expected.

The case d even is quite different. In this case, $\Gamma(S^{d\times n})$ is isomorphic to the hyperoctahedral group and the cohomology ring $H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q})$ turns out to be isomorphic to the invariant ring with respect to the action of $\Gamma(S^{d\times n})$ on a certain polynomial ring. By using standard methods of invariant theory, we show that it is a Cohen–Macaulay ring of Krull dimension n^2. In the case $n = 2$, we obtain an explicit presentation. See §4.1 for these and further results on $B\operatorname{aut}(S^{d\times n})$.

Another application of our results is that they lead to significant simplifications and improvements of certain key results of [15]. In fact, this was the original motivation for this work. Let $d > 1$ and consider the manifold

$$W_g = \#^g S^d \times S^d.$$

The group $\Gamma^+(W_g)$ agrees with the group Γ_g studied in [15]. It is equal to $\mathsf{Sp}_{2g}(\mathbb{Z})$ for $d = 1, 3, 7$, to a certain finite-index subgroup of $\mathsf{Sp}_{2g}(\mathbb{Z})$ for odd $d \neq 1, 3, 7$, and to $\mathsf{O}_{g,g}(\mathbb{Z})$ for even d. When adapted to the classifying space $B\operatorname{aut}_\partial(W_{g,1})$ of the monoid of self-homotopy equivalences of the manifold $W_{g,1} = W_g \setminus \operatorname{int} D^{2d}$ relative to the boundary, our methods yield the following result.

Theorem 1.17. *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*\big(\Gamma_g; H^*_{CE}(\operatorname{Der}^+_\omega \mathbb{L}_g)\big), \tag{6}$$

where $\mathbb{L}_g$ denotes the free graded Lie algebra on generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of degree $d-1$ and $\operatorname{Der}^+_\omega \mathbb{L}_g$ denotes the graded Lie algebra of positive degree derivations of $\mathbb{L}_g$ that annihilate the element

$$\omega = [\alpha_1, \beta_1] + \ldots + [\alpha_g, \beta_g].$$

We also obtain similar results for more general $(d-1)$-connected $2d$-dimensional manifolds. We stress that (6) is an *unstable* isomorphism of algebras. In [15], an isomorphism of this sort was only obtained stably (meaning for g large compared to the cohomological degree) using an argument that depends on several extraneous ingredients that are specific to W_g. The dependence on these extraneous ingredients can now be removed. See §4.2 for more details. Our methods also yield similar results for highly connected odd-dimensional manifolds without much extra effort. Such results were unattainable by the methods of [15], even stably. This is discussed in §4.3.

In the final section §4.4, we study a certain non-formal space X for which the equivalent conditions of Corollary 1.7 do not hold. This example also illustrates the necessity of working over $\Gamma(X)$ rather than $\mathcal{E}(X)$.

1.1. Some comments on related work.

As already mentioned, dg Lie algebra models for the universal cover of $B\operatorname{aut}(X)$ have been known since the early days of rational homotopy theory [66, 67] and it should come as no surprise that the algebraic Lie models in Theorem 1.10 are closely related to these. The first paragraph on p.314 in Sullivan's [66] contains, without proof, the idea of modeling $B\operatorname{aut}(|\Lambda|)$, where $|\Lambda|$ is the realization of a minimal Sullivan algebra Λ, by taking the nerve of the maximal nilpotent ideal of $\operatorname{Der}\Lambda$ modulo the action of the reductive part of $\operatorname{Aut}\Lambda$. This idea seems to have been largely overlooked in the subsequent rational homotopy theory literature; we are not aware of any source where this idea and its consequences have been properly developed (and in fact we only became aware of this paragraph in the final stages of writing this paper). Theorem 3.23 below could be viewed as giving a precise formulation and proof. The key points of the present paper—the treatment of $B\operatorname{aut}(X)$ for non-rational X, the algebraicity of the cohomology of nilpotent covers of $B\operatorname{aut}(X)$, the existence of algebraic Lie models and its strong consequences for the structure of the cohomology ring of $B\operatorname{aut}(X)$—are to the best of our knowledge new. Our results could be regarded as a strong vindication of Sullivan's idea.

We were inspired by Oprea [55] (via Burghelea [25]) for the idea of passing to the reductive quotient of $\mathcal{E}(X_\mathbb{Q})$ to rectify homotopy actions on the algebraic models. The idea of studying the fiberwise rationalization of $B\operatorname{aut}(X) \to B\Gamma(X)$ as we do here is similar in spirit to studying relative Malcev completions of mapping class groups as done by Hain [40]. The algebraicity result for the cohomology of the Torelli group of W_g of Kupers and Randal-Williams [49] inspired us to study similar questions for $B\operatorname{aut}(X)$.

Félix–Fuentes–Murillo [32] recently constructed Lie models for certain nilpotent covers of $B\operatorname{aut}(X)$ but they did not consider the action of the associated transformation groups. We recover these Lie models, see Theorem 3.34 and Remark 3.35. Our approach has the advantage that it lets us model the action of the relevant groups. This is what enables us to construct algebraic models for $B\operatorname{aut}(X)$.

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2. Background and preliminaries

2.1. **Affine algebraic groups and arithmetic groups.** In this section we will collect the basic facts about affine algebraic groups over $\mathbb{Q}$ that are relevant to us, following mainly [44, 53]. We are working over $\mathbb{Q}$: all algebras, vector spaces, undecorated tensor products, *etc.*, should be taken to be over $\mathbb{Q}$ unless explicitly specified otherwise.

Recall that an affine scheme X over $\mathbb{Q}$ is a corepresentable functor

$$X\colon \mathrm{Alg}_{\mathbb{Q}} \longrightarrow \mathrm{Set}$$

from the category $\mathrm{Alg}_{\mathbb{Q}}$ of finitely generated $\mathbb{Q}$-algebras to the category of sets. A morphism of affine schemes over $\mathbb{Q}$ is a natural transformation. An affine group scheme over $\mathbb{Q}$ (or an affine algebraic group over $\mathbb{Q}$) G is a functor

$$G\colon \mathrm{Alg}_{\mathbb{Q}} \longrightarrow \mathrm{Grp}$$

to the category of groups such that the underlying Set-valued functor is an affine scheme over $\mathbb{Q}$. If $A \in \mathrm{Alg}_{\mathbb{Q}}$ is a corepresenting object of G, then by the Yoneda lemma, the natural group structure on each $G(R)$ for $R \in \mathrm{Alg}_{\mathbb{Q}}$ induces a cogroup structure on A. In other words, A is a Hopf algebra. Homomorphisms of affine algebraic groups over $\mathbb{Q}$ are again natural transformations. We will refer to affine algebraic groups over $\mathbb{Q}$ simply as affine algebraic groups, as we will not have the occasion to consider other ground rings in this paper.

We fix the following notation for the sequel: given $n \geqslant 1$, we write GL_n for the affine algebraic group

$$\mathsf{GL}_n\colon R \longmapsto \mathsf{GL}_n(R)$$

of invertible $n \times n$ matrices. We also write $\mathbb{U}_n \leqslant \mathsf{GL}_n$ for the algebraic subgroup consisting of upper triangular matrices with 1s along the diagonal. More generally, if V is a finite-dimensional vector space, we write GL_V for the affine algebraic group

$$\mathsf{GL}_V\colon R \longmapsto \mathrm{Aut}_R(V \otimes R)$$

Remark 2.1. Note that while the definition of GL_V as a functor $\mathrm{Alg}_{\mathbb{Q}} \to \mathrm{Grp}$ makes sense even for infinite-dimensional vector spaces V, it fails to be an affine algebraic group in that case since it is not corepresentable by a finitely generated Hopf algebra.

An algebraic representation of an affine algebraic group G consists of a vector space V of finite dimension, and a homomorphism $G \to \mathsf{GL}_V$. Alternatively, writing

$$\begin{aligned} \widetilde{V}\colon \mathrm{Alg}_{\mathbb{Q}} &\longrightarrow \mathrm{Set}, \\ R &\longmapsto V \otimes R, \end{aligned}$$

the structure of a G-representation on V can equivalently be defined as a natural transformation of functors $\mathrm{Alg}_{\mathbb{Q}} \to \mathrm{Set}$

$$G \times \widetilde{V} \longrightarrow \widetilde{V}$$

which yields an R-linear action of the group $G(R)$ on the set $\widetilde{V}(R) = V \otimes R$ for each $R \in \mathrm{Alg}_{\mathbb{Q}}$. We will often abuse terminology and simply say that V is an (algebraic) representation of G, suppressing the rest of the data. We shall write $\mathrm{Rep}_{\mathbb{Q}}(G)$ for the category of finite-dimensional algebraic representations of the affine algebraic group G.

A linear algebraic group is an algebraic subgroup of GL_n for some n. It is a well-known fact that every affine algebraic group admits a faithful algebraic representation, *cf.* [53, Corollary 4.10], so the notions of affine algebraic group and linear algebraic group essentially coincide.

2.1.1. *Unipotent and reductive groups.* Suppose that V is an n-dimensional algebraic representation of an affine algebraic group G. It is is called unipotent if for some choice of basis of V, the action homomorphism $G \to \mathsf{GL}_V \cong \mathsf{GL}_n$ factors through the subgroup $\mathbb{U}_n \leqslant \mathsf{GL}_n$ of upper triangular matrices.

Unipotency and nilpotency are close cousins: if V is a unipotent representation of an affine algebraic group G, then the group $G(\mathbb{Q})$ of rational points of G acts on V and there exists a flag $0 = V_0 \subset \cdots \subset V_k = V$ of sub-$G(\mathbb{Q})$-modules of V such that each successive quotient V_i/V_{i-1} is a trivial $G(\mathbb{Q})$-module. Note that some authors (*e.g.* [31]) call such modules (and actions) nilpotent.

An affine algebraic group G is called unipotent if every algebraic representation of G is unipotent. Equivalently, G is unipotent if and only if it admits a faithful unipotent representation, *i.e.* if it is an algebraic subgroup of $\mathbb{U}_n$ for some n, *cf.* [53, Theorem 14.5]. Every unipotent group is nilpotent ([53, Proposition 14.21]).

Every affine algebraic group G admits a largest normal unipotent subgroup, called the *unipotent radical* G_u of G (*cf.* [44, Theorem 10.5]). An affine algebraic group G is called *reductive* if G_u is trivial. Every affine algebraic group G has a maximal reductive quotient, namely the quotient G/G_u by its unipotent radical.

Remark 2.2. Milne's definition [53, (6.46)] of the unipotent radical of a *connected* affine algebraic group G as the largest *connected* normal unipotent subgroup of G in fact agrees with the above since we are working in characteristic 0, *cf.* [44, Theorem 10.1]. See also [53, Corollary 14.15].

We do not require connectedness in our definition of reductive group, but an algebraic group G is reductive in the sense we consider here if and only if the identity component G° is reductive.

Recall that a group representation is called semisimple if it is a direct sum of simple (irreducible) subrepresentations.

Theorem 2.3 ([53, Corollary 22.43], see also [44, p.78])**.** *Every finite-dimensional algebraic representation of a reductive group is semisimple.* □

Thus, the category $\mathrm{Rep}_{\mathbb{Q}}(G)$ is a semisimple abelian category whenever G is reductive.

Next, we recall the Levi decomposition of an affine algebraic group G. An affine algebraic subgroup $L \leqslant G$ is called a Levi subgroup if the restriction to L of the projection

$$G \longrightarrow G/G_u$$

is an isomorphism. It is a classical fact that Levi subgroups always exist in characteristic 0, and moreover all Levi subgroups are conjugate:

Theorem 2.4 ([44, Theorem 14.2]). *Every affine algebraic group G affords a Levi subgroup. Any two Levi subgroups of G are conjugate in the action of G_u.* □

The choice of a Levi subgroup gives a splitting of the short exact sequence

$$1 \to U \to G \to R \to 1,$$

where $U = G_u$ is the unipotent radical and $R = G/G_u$ is the maximal reductive quotient. This means that G may be decomposed as a semi-direct product

$$G \cong R \ltimes U.$$

This is called the Levi decomposition of G.

The following lemma gives a concrete description of the unipotent radical and the maximal reductive quotient. Recall that a composition series of a representation V is a filtration

$$0 = V_0 \subset \cdots \subset V_r = V$$

by subrepresentations such that each V_i/V_{i-1} is a simple representation.

Lemma 2.5. *Let G be an affine algebraic group defined over $\mathbb{Q}$. If V is a representation of G with unipotent kernel, then the unipotent radical G_u is precisely the kernel of the associated graded representation,*

$$\operatorname{gr} V = \bigoplus_i V_i/V_{i-1},$$

associated to a composition series for V, and the maximal reductive quotient G/G_u may be identified with the image of $G \to \mathsf{GL}(\operatorname{gr} V)$.

Proof. Let K and N denote the kernel of the action of G on V and $\operatorname{gr} V$, respectively. Clearly, both K and N are normal in G and $K \leqslant N$. Furthermore, K is unipotent by hypothesis, and N/K is unipotent because V is a faithful unipotent representation of it. Since unipotent groups are closed under extensions (*cf.* [53, Corollary 14.7]), it follows that N is unipotent.

To show that N is the unipotent radical, we need to show that every normal unipotent subgroup U of G acts trivially on $\operatorname{gr} V$. For this, it suffices to show that U acts trivially on every simple G-representation W. Since U is unipotent, there is a non-zero $w \in W$ that is fixed by U (*cf.* [53, Proposition 14.1]). Since U is normal in G, it also fixes gw for every $g \in G$. Indeed, for every $u \in U$ we have $g^{-1}ug \in U$, whence $(g^{-1}ug)w = w$ so that $ugw = gw$. Since W is simple, w generates W as a G-module, so U acts trivially on W. □

This has the following consequence:

Lemma 2.6. *Let G be an affine algebraic group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$. If V is a representation of G with unipotent kernel, then the Lie algebra of the unipotent radical,* $\operatorname{Lie} G_u$, *may be identified with the maximal ideal*

$$\operatorname{nil}_V \mathfrak{g} \subseteq \mathfrak{g},$$

consisting of elements which act nilpotently on V.

Proof. By [53, §10.14], the functor $G \mapsto \operatorname{Lie} G$ from affine algebraic groups to Lie algebras commutes with pullbacks, so in particular it preserves kernels of morphisms. Thus by the preceding lemma,

$$\operatorname{Lie} G_u = \ker\left(\mathfrak{g} \longrightarrow \mathfrak{gl}(\operatorname{gr} V)\right) \tag{7}$$

for any composition series $0 = V_0 \subset \cdots \subset V_r = V$ of V. Thus we have that $\operatorname{Lie} G_u \subseteq \operatorname{nil}_V \mathfrak{g}$.

For the reverse inclusion, it suffices to show that $\operatorname{nil}_V \mathfrak{g}$ is the Lie algebra of a normal unipotent subgroup H of G, as then $H \leqslant G_u$ and $\operatorname{nil}_V \mathfrak{g} \subseteq \operatorname{Lie} G_u$. Let K

be the kernel of the action of G on V, and let $\mathfrak{h}'$ be the image of $\mathrm{nil}_V\,\mathfrak{g}$ under the action morphism $\mathfrak{g} \to \mathfrak{gl}(V)$. Then $\mathfrak{h}'$ consists of nilpotent endomorphisms of V, so by Engel's theorem, $\mathfrak{h}'$ is a nilpotent subalgebra of $\mathrm{Lie}(G/K) \subseteq \mathfrak{gl}(V)$. Thus by [53, Theorem 14.37] there is a unipotent subgroup $H' \leqslant G/K \leqslant \mathsf{GL}(V)$ such that $\mathfrak{h}' = \mathrm{Lie}\,H'$. Let H be the preimage of H' in G. Then H is an extension of the unipotent group H' by the unipotent group K, hence unipotent, and $\mathrm{Lie}\,H$ is therefore nilpotent. Moreover, since the functor Lie commutes with pullbacks, we get that $\mathrm{nil}_V\,\mathfrak{g} = \mathrm{Lie}\,H$. Finally, since $\mathrm{nil}_V\,\mathfrak{g}$ is an ideal of $\mathfrak{g}$, the subgroup H of G is normal. □

The quotient G/H of an affine algebraic group G by a normal algebraic subgroup H always exists [53, Theorem 5.14], but the rational points of the quotient $(G/H)(\mathbb{Q})$ need not agree with $G(\mathbb{Q})/H(\mathbb{Q})$ in general. They do agree, however, if H is unipotent.

Lemma 2.7. *Let G be an affine algebraic group. If H is a normal unipotent algebraic subgroup of G, then the natural homomorphism*

$$G(\mathbb{Q})/H(\mathbb{Q}) \to (G/H)(\mathbb{Q})$$

is an isomorphism.

Proof. This is a consequence of the vanishing of Galois cohomology,

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H),$$

for unipotent groups H, see [71, Theorem 9.5]. □

2.1.2. *Arithmetic subgroups.* Given a linear algebraic group $G \leqslant \mathsf{GL}_n$, we write

$$G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathsf{GL}_n(\mathbb{Z})$$

for the integer matrices inside the rational points of G, and we call this group the integer points of G. By definition, an arithmetic subgroup Γ of G is a subgroup of $G(\mathbb{Q})$ which is commensurable with $G(\mathbb{Z})$, *i.e.* such that the intersection $\Gamma \cap G(\mathbb{Z})$ has finite index in both $G(\mathbb{Z})$ and Γ.

It is not possible to define $G(\mathbb{Z})$ unambiguously for an affine algebraic group G without specifying an embedding into some GL_n, since the intersection of $G(\mathbb{Q})$ with the integer points of the ambient general linear group is dependent on the choice of embedding. We can, however, speak of arithmetic subgroups of an affine algebraic group G, as the integer points of any two embeddings of G into a general linear group are commensurable.

Lemma 2.8 ([62, §1.1])**.** *Let G be an affine algebraic group over $\mathbb{Q}$ and let $H \leqslant G$ be an algebraic subgroup. If Γ is arithmetic in $G(\mathbb{Q})$, then $\Gamma \cap H(\mathbb{Q})$ is arithmetic in $H(\mathbb{Q})$.* □

Lemma 2.9 ([18, Theorem 6])**.** *Let $\varphi\colon G \to G'$ be a surjective morphism of affine algebraic groups defined over $\mathbb{Q}$. If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, then $\varphi(\Gamma)$ is an arithmetic subgroup of $G'(\mathbb{Q})$.* □

A group homomorphism $f\colon H \to G$ will be called $\mathbb{Q}$-injective if every element of $\ker(f)$ has finite order, and $\mathbb{Q}$-surjective if, for every $x \in G$, there is a positive integer k such that $x^k \in \mathrm{im}(f)$. We will say that f is a $\mathbb{Q}$-isomorphism if it is both $\mathbb{Q}$-injective and $\mathbb{Q}$-surjective, *cf.* [42, Definition I.1.3].

We will say that a map between topological spaces is a rational equivalence if it induces an isomorphism on all rational homology groups. For nilpotent spaces, this is equivalent to inducing an isomorphism on all rational homotopy groups.

By standard properties of localizations of nilpotent groups and spaces [42], the following are equivalent for a homomorphism $f\colon H \to G$ between nilpotent groups:

(i) The homomorphism f is a $\mathbb{Q}$-isomorphism.
(ii) The induced map in group cohomology $f^*\colon H^*(G,\mathbb{Q}) \to H^*(H,\mathbb{Q})$ is an isomorphism.
(iii) The map $Bf\colon BH \to BG$ is a rational equivalence.

For non-nilpotent groups, beware that (i) is not in general equivalent to (ii).

We record the following elementary lemma for later use. It is presumably well-known but we include a proof for completeness.

Lemma 2.10. *Let G be a unipotent algebraic group defined over $\mathbb{Q}$. The inclusion of any arithmetic subgroup into $G(\mathbb{Q})$ is a $\mathbb{Q}$-isomorphism.*

Proof. We may assume that G is a subgroup of $\mathbb{U}_n$ for some n. Inclusions of finite-index subgroups are clearly $\mathbb{Q}$-isomorphisms, so it suffices to show that the inclusion of $G(\mathbb{Z})$ into $G(\mathbb{Q})$ is $\mathbb{Q}$-surjective, *i.e.*, that for every $A \in G(\mathbb{Q})$, there is a positive integer k such that $A^k \in G(\mathbb{Z})$. Since A is a unipotent $n \times n$-matrix, the matrix $N = A - I$ satisfies $N^n = 0$. Pick a positive integer d such that dN^i has integer entries for all $i \geq 1$, and let $k = 1!2!\cdots n!d$. Then the matrix

$$A^k = \sum_{i=0}^{n-1} \binom{k}{i} N^i$$

has integer entries, because each coefficient $\binom{k}{i}$ is divisible by d. □

Remark 2.11. One can not relax unipotence to nilpotence in Lemma 2.10. The multiplicative group is abelian and in particular nilpotent, but the map $\mathbb{Z}^\times \to \mathbb{Q}^\times$ is not a $\mathbb{Q}$-isomorphism.

2.2. **Nilpotent radicals of differential graded Lie algebras.** In this section we introduce the notion of nilradical in the setting of differential graded Lie algebras. Let $(\mathfrak{g}, \delta)$ be a dg Lie algebra, possibly unbounded as a chain complex. We write $\Gamma^k\mathfrak{g}$ for the lower central series of $\mathfrak{g}$, so $\Gamma^1\mathfrak{g} = \mathfrak{g}$ and $\Gamma^{k+1}\mathfrak{g} = \left[\Gamma^k\mathfrak{g}, \mathfrak{g}\right]$. Given an integer k, we write $\mathfrak{g}\langle k\rangle$ for the truncation of $\mathfrak{g}$ given by

$$\mathfrak{g}\langle k\rangle_n = \begin{cases} 0 & \text{if } n < k \\ \ker(\delta : \mathfrak{g}_n \to \mathfrak{g}_{n-1}) & \text{if } n = k \\ \mathfrak{g}_n & \text{if } n > k \end{cases}$$

We call $\mathfrak{g}$ *connected* if $\mathfrak{g} = \mathfrak{g}\langle 0\rangle$ and we call $\mathfrak{g}$ *simply connected* if $\mathfrak{g} = \mathfrak{g}\langle 1\rangle$.

Definition 2.12. A connected dg Lie algebra $\mathfrak{g}$ is *nilpotent* if the following equivalent conditions are satisfied:

(i) For every n, we have $(\Gamma^k\mathfrak{g})_n = 0$ for $k \gg n$;
(ii) $\mathfrak{g}_0$ is a nilpotent Lie algebra which acts nilpotently on $\mathfrak{g}_n$ for every $n > 0$.

Definition 2.13. The *nilradical* of $\mathfrak{g}$, denoted $\operatorname{nil}\mathfrak{g}$, is the maximal nilpotent ideal of $\mathfrak{g}\langle 0\rangle$, provided such an ideal exists.

Remark 2.14. Ordinary Lie algebras may be identified with dg Lie algebras concentrated in degree 0. For these, the above definitions specialize to the usual definitions of nilpotence and nilradicals. In particular, the nilradical does not necessarily exist unless certain finiteness conditions are imposed. If the nilradical exists, however, then it is unique: if I, J are nilpotent ideals of $\mathfrak{g}$, then so is $I + J$. Thus if I is maximal, then $J \subseteq I + J = I$.

Lemma 2.15. *Suppose that $Z_0(\mathfrak{g})$ is finite-dimensional. Then* $\operatorname{nil}\mathfrak{g}$ *exists.*

Proof. We may assume that $\mathfrak{g}$ is connected. Note that the positively graded truncation $\mathfrak{g}\langle 1\rangle$ is always a nilpotent ideal of $\mathfrak{g}$, so the poset of all nilpotent ideals contains a maximal element if and only if the poset of nilpotent ideals above $\mathfrak{g}\langle 1\rangle$ does. But the latter poset injects into the poset of nilpotent (or indeed all) ideals of the quotient dg Lie algebra $\mathfrak{g}' = \mathfrak{g}/\mathfrak{g}\langle 1\rangle$. Clearly $\mathfrak{g}'_i = 0$ for $i > 1$, the differential $\mathfrak{g}'_1 \to \mathfrak{g}'_0$ is injective, and $\mathfrak{g}'_0 = \mathfrak{g}_0$ is finite-dimensional, so every chain of ideals of $\mathfrak{g}'$ is finite. □

2.3. **Geometric realizations of nilpotent dg Lie algebras.** To each nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$, one can associate a group $\exp(\mathfrak{g})$ with underlying set $\mathfrak{g}$ and with multiplication given by the Baker–Campbell–Hausdorff formula. The association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence of categories between nilpotent Lie algebras over $\mathbb{Q}$ and nilpotent uniquely divisible groups (*cf.* [57, Appendix A]). When $\mathfrak{g}$ is finite-dimensional, $\exp(\mathfrak{g})$ can be given the structure of an affine algebraic group, and the association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence between the category of finite-dimensional nilpotent Lie algebras over $\mathbb{Q}$ and the category of unipotent algebraic groups over $\mathbb{Q}$ (*cf.* [53, Theorem 14.37]).

Now let $\mathfrak{g}$ be a nilpotent dg Lie algebra. Following [12, §3.2], we consider the simplicial group

$$\exp_\bullet \mathfrak{g} = \exp Z_0(\mathfrak{g} \otimes \Omega_\bullet),$$

where $\Omega_\bullet$ is the simplicial commutative differential graded algebra over $\mathbb{Q}$ of polynomial differential forms on the standard simplices. The following is well-known, see *e.g.* the proof of Proposition 3.7 in [9].

Proposition 2.16. *For every nilpotent dg Lie algebra $\mathfrak{g}$, there is a natural isomorphism of groups,*

$$\exp H_0(\mathfrak{g}) \to \pi_0 \exp_\bullet(\mathfrak{g}),$$

and a natural isomorphism of abelian groups

$$H_k(\mathfrak{g}) \to \pi_k \exp_\bullet(\mathfrak{g}),$$

for every $k > 0$. □

A natural model for the classifying space $B\exp_\bullet(\mathfrak{g})$ is given by the nerve, or Maurer–Cartan space,

$$\langle \mathfrak{g} \rangle = \mathsf{MC}_\bullet(\mathfrak{g}).$$

It is the simplicial set defined by

$$\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g} \otimes \Omega_\bullet),$$

where MC denotes the set of Maurer–Cartan elements in a dg Lie algebra, i.e., the solutions to the equation

$$\delta(\tau) + \frac{1}{2}[\tau, \tau] = 0,$$

cf. [12, Corollary 3.10].

A *Lie model* for a nilpotent space B is by definition a nilpotent dg Lie algebra $\mathfrak{g}$ such that B is rationally equivalent to $\mathsf{MC}_\bullet(\mathfrak{g})$.

Given a commutative differential graded algebra (cdga) Λ, its spatial realization is the simplicial set

$$\langle \Lambda \rangle = \mathrm{Hom}_{\mathrm{cdga}}(\Lambda, \Omega_\bullet).$$

For $\mathfrak{g}$ nilpotent of finite type, there is an isomorphism of simplicial sets

$$\langle C^*(\mathfrak{g}) \rangle \cong \mathsf{MC}_\bullet(\mathfrak{g}),$$

where $C^*(\mathfrak{g})$ is the Chevalley–Eilenberg cochain algebra (see *e.g.* [12, Corollary 3.6])

2.4. **Commutative cochains with local coefficients.** In this section we will briefly review the definition of the commutative cochains of a simplicial set with local coefficients, following Halperin [41].

For a simplicial set K, we let $(\Delta \downarrow K)$ denote the simplex category. An object is a simplex $\sigma\colon \Delta^n \to K$ and a morphism from σ to τ is a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[d, "\varphi"'] \arrow[dr, "\sigma"] & \\ \Delta^m \arrow[r, "\tau"'] & K. \end{tikzcd}$$

Definition 2.17 ([41, pp. 150–151])**.** A local system on a simplicial set K with values in a category $\mathscr{C}$ is a functor

$$F\colon (\Delta \downarrow K)^{op} \to \mathscr{C}.$$

The global sections of a local system F on K is defined as the inverse limit

$$F(K) = \varprojlim_{\sigma \in (\Delta \downarrow K)} F_\sigma,$$

provided this limit exists in $\mathscr{C}$.

Remark 2.18. To compare this with the notion of a local system of coefficients defined in terms of representations of the fundamental groupoid, one should observe that the groupoidification of the category $(\Delta \downarrow K)$ is a model for the fundamental groupoid of K, *cf.* [37, §III.1]. This means that the category of local systems F such that $F(\varphi)$ is invertible for every morphism φ in $(\Delta \downarrow K)$ is equivalent to the category of representations of the fundamental groupoid. However, since we do not require invertibility of $F(\varphi)$ in general, the notion of a local system is more general.

Example 2.19.

(i) If Γ is a discrete group and M is an object of $\mathscr{C}$ with an action of Γ, then there is an associated local system on the nerve $B\Gamma$, also denoted M, where $M_\sigma = M$ for all $\sigma = (\gamma_1, \ldots, \gamma_n)\colon \Delta^n \to B\Gamma$ and $d_i\colon M_\sigma \to M_{d_i\sigma}$ is multiplication by γ_n for $i = n$ and the identity otherwise.

(ii) A simplicial object $X_\bullet\colon \Delta^{op} \to \mathscr{C}$ determines a local system on every simplicial set K by restriction along $(\Delta \downarrow K)^{op} \to \Delta^{op}$. In particular, the simplicial cdga $\Omega_\bullet$ determines a local system Ω^* of cdgas on every simplicial set K. The global sections $\Omega^*(K)$ is the usual model for Sullivan's cdga of polynomial differential forms on K, also denoted $A^*_{PL}(K)$.

(iii) More generally, if F is a local system of cochain complexes on K, then $\Omega^*(K;F)$ may be defined as the global sections of the local system $\Omega^* \otimes F$, *cf.* [41, Definition 13.10]. If A is local system of cochain algebras, then $\Omega^*(K;A)$ is a cochain algebra, which is commutative if A is.

Proposition 2.20. *Let K be a simplicial set and let A be a local system of $\mathbb{Q}$-cochain complexes on K. The cochain complex $\Omega^*(K;A)$ is naturally quasi-isomorphic to $C^*(K;A)$. If A is a local system of dg algebras, then $\Omega^*(K;A)$ and $C^*(K;A)$ are quasi-isomorphic as dg algebras.*

Proof. Integration provides a natural quasi-isomorphism

$$\Omega^*(K;A) \to C^*(K;A)$$

cf. [41, Theorem 14.18]. The integration map is not multiplicative on the cochain level, but the argument in [33, §10(d)] goes through with coefficients A, providing a zig-zag of dg algebra quasi-isomorphisms. □

In particular, for a discrete group Γ and a cochain complex M of $\mathbb{Q}[\Gamma]$-modules, the cohomology of $\Omega^*(\Gamma, M) = \Omega^*(B\Gamma; M)$ agrees with group cohomology $H^*(\Gamma, M)$ (as defined in *e.g.* [24, p. VII.5]). If A is a commutative cochain algebra over $\mathbb{Q}$ with an action of Γ, then $\Omega^*(\Gamma, A)$ is a commutative cochain algebra model for $C^*(\Gamma, A)$.

We record the following proposition for later use.

Proposition 2.21. *Let X be a space with a Γ-action. For every cochain complex of Γ-modules M, we have that $\Omega^*(X_{h\Gamma}; M)$ is quasi-isomorphic to $\Omega^*(\Gamma, \Omega^*(X) \otimes M)$. If M is a commutative cochain algebra with a Γ-action, then they are quasi-isomorphic as commutative cochain algebras.*

Proof. The local system F of the fibration $X_{h\Gamma} \to B\Gamma$ in the sense of [41, p.248] may be identified up to quasi-isomorphism with the local system associated to the Γ-cdga $\Omega^*(X)$. Granted this, the combination of [41, Lemma 19.21] and [41, Theorem 13.12] applied to the local system F shows that

$$\Omega^*(X_{h\Gamma}) \cong F(B\Gamma) \sim \Omega^*(B\Gamma; F) \sim \Omega^*(B\Gamma; \Omega^*(X)).$$

One checks that the argument goes through with coefficients M as well. □

This is a commutative version, in characteristic zero, of well-known statements for singular cochains; *cf. e.g.* [24, §VII.7] or [52, §13].

2.5. **The dg Lie algebra of curved derivations.** By Quillen's theory [57], the rational homotopy category of simply connected pointed spaces is equivalent to the homotopy category of simply connected dg Lie algebras. If one wants to model unpointed spaces, one has to enlarge the set of morphisms of dg Lie algebras. A possible solution is to work with so called curved morphisms of dg Lie algebras (*cf.* [51]).

Let L be a finitely generated, positively graded dg Lie algebra and let $L_+ = (L * \mathbb{L}(\tau), d^\tau)$ be the dg Lie algebra obtained by freely adjoining a Maurer–Cartan element τ to L and twisting the differential by τ, so $d^\tau(x) = d(x) + [\tau, x]$ for $x \in L$. It is straightforward to check that morphisms from L_+ to a dg Lie algebra L' correspond to curved morphisms from L to L'. This is analogous to the fact that the space of free maps from a pointed space X to another pointed space Y can be recovered as the space of pointed maps from X_+ to Y, where X_+ is the space obtained from X by adding a disjoint basepoint.

The projection $p\colon L_+ \to L$ that restricts to the identity on L and sends τ to zero is a morphism of dg Lie algebras. Let $\mathrm{Der}^c(L)$ denote the chain complex of p-derivations from L_+ to L. Its elements are maps $\theta\colon L_+ \to L$ that satisfy

$$\theta[x, y] = [\theta(x), p(y)] + (-1)^{|\theta||x|}[p(x), \theta(y)],$$

for all $x, y \in L_+$.

As is well known, the mapping cone of the chain map $\mathrm{ad}\colon L \to \mathrm{Der}\, L$, denoted $\mathrm{Der}\, L /\!\!/ \mathrm{ad}\, L$ or $\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL$, admits a dg Lie algebra structure, see *e.g.* [67] or [12, p.252]. We now make the observation that this mapping cone may be identified with the chain complex of curved derivations.

Proposition 2.22. *The map $\varphi\colon \mathrm{Der}\, L /\!\!/ \mathrm{ad}\, L \to \mathrm{Der}^c(L)$, defined by*

$$\varphi(\theta, s\xi) = \theta \circ p + (-1)^{|\xi|}\xi \frac{\partial}{\partial \tau},$$

is an isomorphism of chain complexes, with inverse

$$\varphi^{-1}(\nu) = \big(\nu|_L, (-1)^{|\nu|+1} s\nu(\tau)\big).$$

Proof. Straightforward calculation. □

In particular, this implies that $\mathrm{Der}^c(L)$ admits a dg Lie algebra structure and that it acts on L by outer derivations in the sense of [12, §3.5]. We do not recall the full definition here, but we point out that the outer action of $\mathrm{Der}^c(L)$ on L gives rise to an (ordinary) action by coderivations on the Chevalley–Eilenberg chains

$$C_*(L) = (\Lambda \mathrm{s}L, d = d_0 + d_1)$$

of L, where d_0 and d_1 are characterized by

$$\begin{aligned} d_0(\mathrm{s}x) &= -\mathrm{s}(dx) \\ d_1(\mathrm{s}x_1 \wedge \mathrm{s}x_2) &= (-1)^{|x_1|}\mathrm{s}[x_1, x_2] \end{aligned}$$

This action may be constructed by noting that $\mathrm{s}L_+$ contains $\mathbb{Q}[0] \oplus \mathrm{s}L = \Lambda^{\leqslant 1}\mathrm{s}L$ as a graded subspace, where the copy of $\mathbb{Q}$ in degree 0 is generated by $\mathrm{s}\tau$. Hence every curved derivation $\phi \in \mathrm{Der}^c(L)$ determines a map $\Lambda^{\leqslant 1}\mathrm{s}L \to \mathrm{s}L$ by suspension and restriction, and thus it determines a unique coderivation of the cofree coalgebra $\Lambda \mathrm{s}L$. Explicitly,

$$\begin{aligned} \phi(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n) &= (-1)^{|\phi|}\mathrm{s}\phi(\tau) \wedge \mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n \\ &\quad + \sum_i \pm \mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}\phi(x_i) \wedge \cdots \wedge \mathrm{s}x_n \end{aligned} \tag{8}$$

Dually, $\mathrm{Der}^c(L)$ acts by derivations on the cdga $C^*(L) = C_*(L)^\vee$ of Chevalley–Eilenberg cochains on L.

2.6. **Algebraic groups of automorphisms.** The following goes back to Sullivan and Wilkerson, see [71, Theorem B], [66, Theorem 10.3]. We outline a proof for the reader's convenience.

Theorem 2.23. *Let X be simply connected, and either a finite CW-complex, or a finite Postnikov section.*

(i) The group $\mathcal{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$-points of an affine algebraic group.
(ii) The homomorphism $q_*\colon \mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ induced by rationalization has finite kernel and image an arithmetic subgroup.
(iii) The representation of $\mathcal{E}(X)$ in $H_*(X;\mathbb{Q})$ is the restriction of an algebraic representation of $\mathcal{E}(X_{\mathbb{Q}})$ with unipotent kernel.

Proof outline. Since $X_{\mathbb{Q}}$ is simply connected, we have an isomorphism $\mathcal{E}_*(X_{\mathbb{Q}}) \cong \mathcal{E}(X_{\mathbb{Q}})$. Quillen's equivalence [58] between the category of simply connected pointed spaces, localized at the rational homotopy equivalences, and the category of positively graded dg Lie algebras, localized at the quasi-isomorphisms, coupled with the fact that a quasi-isomorphism between minimal dg Lie algebras is an isomorphism, leads to an isomorphism

$$\mathcal{E}_*(X_{\mathbb{Q}}) \cong \mathrm{Aut}^h(L),$$

where the right-hand side denotes the quotient of the group $\mathrm{Aut}(L)$ of automorphisms of the minimal Quillen model L modulo the normal subgroup $\mathrm{Aut}_h L$ of automorphisms homotopic to the identity.

If X is a finite CW-complex, then L is finitely generated and the group $\mathrm{Aut}\, L$ forms the $\mathbb{Q}$-points of the affine algebraic group $\mathcal{A}\mathrm{ut}\, L$ with functor of points

$$R \longmapsto \mathrm{Aut}_{\mathrm{dgl}(R)}(L \otimes R),$$

sending a $\mathbb{Q}$-algebra R to the group of R-linear automorphisms of the dg Lie algebra $L \otimes R$. A faithful finite-dimensional representation is given by $L^{\leq N}$, where N is the maximal degree of a generator.

The key to realizing $\mathrm{Aut}^h(L)$, and hence $\mathcal{E}(X_{\mathbb{Q}})$, as the rational points of an algebraic group is to observe that $\mathrm{Aut}_h(L)$ may be identified with the unipotent group associated to the nilpotent Lie algebra $B_0 \operatorname{Der} L$ of derivations of L of the form $[d, \theta]$, for some derivation $\theta\colon L \to L$ of degree 1. See [66, §6] or [16, Theorem 3.4], or the proof of [32, Lemma 8.1]. In particular, this means that $\operatorname{Aut}_h L$ consists of the rational points of a unipotent subgroup $\mathcal{A}\mathrm{ut}_h L$ of $\mathcal{A}\mathrm{ut}\, L$. It follows from Lemma 2.7 that $\operatorname{Aut}^h(L)$ may be identified with the rational points of the quotient algebraic group $\mathcal{A}\mathrm{ut}^h(L) = \mathcal{A}\mathrm{ut}(L)/\mathcal{A}\mathrm{ut}_h(L)$.

The case of X a finite Postnikov section can be treated similarly: in this case, the minimal Sullivan model Λ is finitely generated, so $\operatorname{Aut}\Lambda$ forms the rational points of an affine algebraic group $\mathcal{A}\mathrm{ut}\,\Lambda$ with functor of points

$$R \longmapsto \operatorname{Aut}_{\mathrm{cdga}(R)}(\Lambda \otimes R),$$

and $\mathcal{E}(X_{\mathbb{Q}})$ can be identified with the $\mathbb{Q}$-points of the quotient of $\mathcal{A}\mathrm{ut}\,\Lambda$ by the unipotent subgroup $\mathcal{A}\mathrm{ut}_h\,\Lambda$ of automorphisms homotopic to the identity.

For a detailed proof of arithmeticity of the image of $\mathcal{E}(X)$ in $\mathcal{E}(X_{\mathbb{Q}})$, see *e.g.* [68].

For the third statement, one identifies

$$\widetilde{H}_*(X;\mathbb{Q}) = sL/[L,L],$$

and notes that the right-hand side is an algebraic representation of $\mathcal{A}\mathrm{ut}\, L$ on which $\mathcal{A}\mathrm{ut}_h L$ acts trivially. The associated graded $\operatorname{gr} L$ of L with respect to the lower central series is isomorphic to the free Lie algebra on $L/[L,L]$ as a representation of $\mathcal{A}\mathrm{ut}\, L$. It follows that the kernel of the representation $L/[L,L]$ acts trivially on $\operatorname{gr} L$, which implies that it acts unipotently on L in each degree. ☐

Remark 2.24. Block–Lazarev [16] observed that the Lie algebra of $\mathcal{E}(X_{\mathbb{Q}})$ and the higher homotopy groups of $B\operatorname{aut}(X_{\mathbb{Q}})$ can be expressed in terms of André–Quillen cohomology of $\Omega^*(X)$. We will not need this fact.

Sullivan also observes that the representation of $\operatorname{Aut}\Lambda$ in spherical homology has unipotent kernel [66, Proposition 6.4]. Sullivan's proof is a little brief, so we offer a proof below. In fact, the proof we give here can be adapted to yield an integral statement, strengthening a certain result of Dror–Zabrodsky, see Proposition 3.5 below. Recall that the spherical cohomology of a Sullivan algebra Λ is the quotient of the cocycles $Z^*(\Lambda)$ by the decomposable cocycles $Z^*(\Lambda) \cap (\Lambda^+)^2$.

Lemma 2.25. *Suppose that $\Lambda = (\Lambda V, d)$ is a finitely generated minimal Sullivan algebra such that $V = V^{\geqslant 2}$. If a group G acts on Λ in such a way that the induced action on the spherical cohomology $SH^*(\Lambda)$ is nilpotent, then G acts nilpotently on Λ. In particular, the kernel of the action of $\operatorname{Aut}\Lambda$ on $SH^*(\Lambda)$ is a unipotent algebraic subgroup.*

Proof. For $n \geqslant 2$, let $P_n\Lambda = \Lambda(V^{\leqslant n})$ be the subalgebra of Λ generated by $V^{\leqslant n}$. This is an $\operatorname{Aut}\Lambda$-subrepresentation (and hence by restriction a G-subrepresentation) of Λ, since it can equivalently be characterised as the subalgebra generated by the $\operatorname{Aut}\Lambda$-stable graded subspace $\Lambda^{\leqslant n} \subset \Lambda$. Since the differential d has no linear part and V is positively graded, $P_n\Lambda$ is closed under d.

We prove by induction on n that G acts nilpotently on each $P_n\Lambda$, noting that $\Lambda = P_n\Lambda$ for n sufficiently large. When $n = 2$, $P_2\Lambda$ is the tensor algebra generated by $V^{\leqslant 2} = V^2$, the differential is zero, and $\operatorname{Aut}(P_2\Lambda) \cong \mathsf{GL}\, V^2$ acts faithfully on $SH^*(P_2\Lambda) \cong V^2 \cong (P_2\Lambda)^2$.

Suppose that G acts nilpotently on $P_{n-1}\Lambda$ and consider the cofiber

$$C = P_n\Lambda \otimes_{P_{n-1}\Lambda} \mathbb{Q}$$

of the inclusion $P_{n-1}\Lambda \to P_n\Lambda$ in the category of cdgas. It can be seen that $C \cong \Lambda C^n$ with zero differential, and that the composite map $V^n \to (P_n\Lambda)^n \to C^n$

is an isomorphism of $\mathbb{Q}$-vector spaces. Consider the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & (P_{n-1}\Lambda)^n \arrow[r] \arrow[d, "d"'] & (P_n\Lambda)^n \arrow[r] \arrow[d, "d"'] & C^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^{n+1}(P_{n-1}\Lambda) \arrow[r] & Z^{n+1}(P_{n-1}\Lambda) \arrow[r] & H^{n+1}(P_{n-1}\Lambda) \arrow[r] & 0
\end{tikzcd} \tag{9}
$$

whose rows are short exact sequences. (Note that the middle vertical arrow does land inside $(P_{n-1}\Lambda)^{n+1} \subset (P_n\Lambda)^{n+1}$ since d has no linear part and Λ has no elements of degree 1.) The snake lemma applied to (9) shows that the kernel of the right-hand vertical arrow in (9) is the cokernel of the inclusion $Z^n(P_{n-1}\Lambda) \to Z^n(P_n\Lambda)$ since the left-hand vertical arrow is clearly surjective. But $Z^n(P_{n-1}\Lambda) \to Z^n(P_n\Lambda)$ is precisely the inclusion of the decomposable cocycles in the n-cocycles of $P_n\Lambda$, so the cokernel is precisely the spherical cohomology group $SH^n(P_n\Lambda)$. Thus we obtain the exact sequence

$$0 \to SH^n(P_n\Lambda) \to C^n \to H^{n+1}(P_{n-1}\Lambda)$$

where the left-hand term is a nilpotent G-module by assumption and the right-hand term is a nilpotent G-module by our inductive hypothesis. Hence C^n is a nilpotent G-module too.

But now by inspecting the first row of (9), we see that $(P_n\Lambda)^n$ is also a nilpotent G-module. Since $P_n\Lambda$ is generated in degrees $\leqslant n$, it follows that G acts nilpotently on $P_n\Lambda$ in each degree. □

Remark 2.26. If Λ is the minimal Sullivan model of a simply connected space X, then $P_n\Lambda$ is the minimal Sullivan model of the n-th Postnikov section P_nX of X.

3. Proofs of the main results

This section is the core of the paper. We begin in §3.1 by studying the spaces $B\operatorname{aut}_U(X)$ for subgroups $U \leqslant \mathcal{E}(X)$. We observe that the space $\Gamma(X)/\operatorname{aut}(X)$ is weakly equivalent to $B\operatorname{aut}_U(X)$ for a suitably chosen U, and we use this to identify it as the classifying space for normal unipotent fibrations. In §3.2 we discuss (virtual) nilpotence results for the spaces $B\operatorname{aut}_U(X)$. In §3.3 we prove a lemma on bar constructions that will be the key for showing that the algebraically defined $\Gamma(X)$-action on the model for $\Gamma(X)/\operatorname{aut}(X)$ corresponds to the holonomy action. In §3.4 we establish algebraic models for $B\operatorname{aut}(X_{\mathbb{Q}})$ as a preparation for §3.5 where we prove the main results about $B\operatorname{aut}(X)$.

3.1. Normal unipotent fibrations.

In this section, we will show that the space $\Gamma(X)/\operatorname{aut}(X)$ can be characterized as the classifying space for normal unipotent fibrations. We start by proving Theorem 1.6.

Proof of Theorem 1.6. By definition, the group $R(X)$ is the maximal reductive quotient of $\mathcal{E}(X_{\mathbb{Q}})$. By Theorem 2.23(iii), the rational homology $H_*(X;\mathbb{Q})$ is an algebraic representation of $\mathcal{E}(X_{\mathbb{Q}})$ with unipotent kernel. Hence by Lemma 2.5, the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$ is precisely the image of $\mathcal{E}(X_{\mathbb{Q}})$ in $\mathsf{GL}(\operatorname{gr} H_*(X;\mathbb{Q}))$.

By definition, the group $\Gamma(X)$ is the image of $\mathcal{E}(X)$ in $R(X)$. In view of the above, this agrees with $\mathsf{GL}^{\mathcal{E}(X)}(\operatorname{gr} H_*(X;\mathbb{Q}))$. □

Definition 3.1. For a subgroup $U \leqslant \mathcal{E}(X)$, let $\operatorname{aut}_U(X)$ denote the union of the components of $\operatorname{aut}(X)$ that belong to U. Thus, there is a pullback square

$$
\begin{tikzcd}
\operatorname{aut}_U(X) \arrow[r] \arrow[d] & U \arrow[d] \\
\operatorname{aut}(X) \arrow[r] & \mathcal{E}(X).
\end{tikzcd}
$$

If $U \leqslant \mathcal{E}(X)$ is a normal subgroup, then we denote the quotient group by

$$\mathcal{E}^U(X) = \mathcal{E}(X)/U.$$

In this case, the outer rectangle in the diagram of homotopy pullback squares

$$\begin{tikzcd} B\operatorname{aut}_U(X) \arrow[r] \arrow[d] & BU \arrow[r] \arrow[d] & * \arrow[d] \\ B\operatorname{aut}(X) \arrow[r] & B\,\mathcal{E}(X) \arrow[r] & B\,\mathcal{E}^U(X). \end{tikzcd}$$

shows that there is a homotopy fiber sequence

$$B\operatorname{aut}_U(X) \to B\operatorname{aut}(X) \to B\,\mathcal{E}^U(X).$$

Definition 3.2. Let $\mathcal{E}_u(X)$ denote the preimage of the unipotent radical of the affine algebraic group $\mathcal{E}(X_{\mathbb{Q}})$ under the homomorphism $\mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ induced by rationalization.

When $U = \mathcal{E}_u(X)$, we will use the notation $\operatorname{aut}_u(X) = \operatorname{aut}_U(X)$. In this case, the group $\mathcal{E}^U(X)$ is isomorphic to $\Gamma(X)$ and there is a homotopy fiber sequence

$$B\operatorname{aut}_u(X) \to B\operatorname{aut}(X) \to B\Gamma(X).$$

In particular, $\Gamma(X)/\operatorname{aut}(X)$ is weakly equivalent to $B\operatorname{aut}_u(X)$. The next result characterizes the latter as the classifying space for normal unipotent fibrations with fiber X.

Proposition 3.3. *The following conditions are equivalent for a fibration*

$$X \to E \to B$$

of connected spaces:

(i) The image of the holonomy action $\pi_1(B) \to \mathcal{E}(X_{\mathbb{Q}})$ is contained in a normal unipotent subgroup.

(ii) The classifying map for the fibration admits a lift up to homotopy

$$\begin{tikzcd} & B\operatorname{aut}_u(X) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & B\operatorname{aut}(X). \end{tikzcd}$$

(iii) The holonomy representation of $\pi_1(B)$ in $H_*(X;\mathbb{Q})$ is $\mathcal{E}(X_{\mathbb{Q}})$-unipotent, in the sense that $\pi_1(B)$ acts trivially on the associated graded module of a filtration of $H_*(X;\mathbb{Q})$ by $\mathcal{E}(X_{\mathbb{Q}})$-submodules.

Proof. The first two conditions are equivalent because a lift exists if and only if the image of $\pi_1(B) \to \pi_0 \operatorname{aut}(X) = \mathcal{E}(X)$ is contained in $\pi_1\big(B\operatorname{aut}_u(X)\big)$; the latter group is equal to the preimage of the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$ under $\mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ by definition of $B\operatorname{aut}_u(X)$.

To show that the first and the last conditions are equivalent, observe that Theorem 1.6 implies that the unipotent radical $\mathcal{E}_u(X_{\mathbb{Q}})$ may be described as the kernel of the action of $\mathcal{E}(X_{\mathbb{Q}})$ on the associated graded $\operatorname{gr} H_*(X;\mathbb{Q})$ of a composition series for the $\mathcal{E}(X_{\mathbb{Q}})$-module $H_*(X;\mathbb{Q})$. Hence, if $\pi_1(B) \to \mathcal{E}(X_{\mathbb{Q}})$ has image in the unipotent radical, then $\pi_1(X)$ acts trivially on $\operatorname{gr} H_*(X;\mathbb{Q})$ as well. Conversely, if $\pi_1(B)$ acts trivially on the associated graded module of a filtration of $H_*(X;\mathbb{Q})$ by $\mathcal{E}(X_{\mathbb{Q}})$-submodules, then the image of $\pi_1(B) \to \mathcal{E}(X_{\mathbb{Q}})$ is contained in the unipotent radical, because any such filtration can be refined to a composition series. $\square$

Remark 3.4. The first characterization justifies calling fibrations satisfying the conditions *normal unipotent fibrations*. The second characterization shows that $B\operatorname{aut}_u(X)$ may be interpreted as the classifying space for normal unipotent fibrations. The third condition is useful for comparing with existing notions of unipotent fibrations:

A fibration $X \to E \to B$ of connected spaces is often called *unipotent* (or nilpotent or quasi-nilpotent) if the holonomy representation of $\pi_1(B)$ in $H_*(X;\mathbb{Q})$ is unipotent, in the sense that $\pi_1(B)$ acts trivially on the associated graded module of a filtration of $H_*(X;\mathbb{Q})$ by $\pi_1(B)$-submodules. Since a subgroup of $\mathcal{E}(X_\mathbb{Q})$ is contained in a unipotent subgroup if and only if it acts unipotently on $H_*(X;\mathbb{Q})$ (see Lemma 3.33 below), we note that a fibration is unipotent in this sense if and only if the image of $\pi_1(B) \to \mathcal{E}(X_\mathbb{Q})$ is contained in a unipotent subgroup.

3.2. **Virtual nilpotence.** The main purpose of this section is to show that the space $B\operatorname{aut}_u(X)$ is virtually nilpotent. Along the way, we establish a slight strengthening of certain results of Dror and Zabrodsky [31] on the nilpotency of the spaces $B\operatorname{aut}_U(X)$.

Recall that the spherical homology of a space X, which we denote by $SH_*(X)$, is the image of the Hurewicz homomorphism $\mathrm{hur}\colon \pi_*X \to H_*(X;\mathbb{Z})$. If X is simply connected, then $\mathcal{E}(X)$ and the group $\mathcal{E}_*(X)$ of pointed homotopy classes of pointed self-equivalences are isomorphic. Thus $\mathcal{E}(X)$ acts not only on $H_*(X;\mathbb{Z})$, but also on π_*X in such a way that the Hurewicz homomorphism is equivariant, and thus we obtain an action of $\mathcal{E}(X)$ on $SH_*(X)$.

Proposition 3.5. *Let X be a simply connected space which is either a finite-dimensional CW-complex, or a finite Postnikov section. Let $G \leqslant \mathcal{E}(X)$ be a group which acts nilpotently on spherical homology $SH_*(X)$. Then G is a nilpotent group, and $B\operatorname{aut}_G(X)$ is a nilpotent space.*

Proof. We explain how this statement may be derived from the results of [31]. We write $\dim X$ for the dimension of X if X is a finite-dimensional complex, and for the largest n such that $\pi_n X \neq 0$ if X is a finite Postnikov section.

Firstly, note that the proof of [31, Theorem 3.3] readily generalizes to show that G acts nilpotently on $\pi_n X$ for all n by induction. Indeed, for every $n > 1$, the Serre spectral sequence of the homotopy fiber sequence

$$K(\pi_n X, n) \longrightarrow P_n X \longrightarrow P_{n-1}X$$

gives rise to the exact sequence

$$H_{n+1}(P_{n-1}X;\mathbb{Z}) \longrightarrow H_n(K(\pi_n X, n);\mathbb{Z}) \longrightarrow H_n(P_n X;\mathbb{Z})$$

where the first arrow is the only possible differential landing in bigrading $(0,n)$. The second arrow is the edge homomorphism induced by the map $K(\pi_n X, n) \to P_n X$; under the usual identification of the middle term with $\pi_n X$ and of the right-hand term with $H_n(X;\mathbb{Z})$, the second arrow is precisely the Hurewicz homomorphism. Hence the exact sequence can be rewritten as

$$H_{n+1}(P_{n-1}X;\mathbb{Z}) \longrightarrow \pi_n X \xrightarrow{\mathrm{hur}} SH_n(X) \longrightarrow 0.$$

Now if G acts nilpotently on $\pi_j X$ for $j < n$, then it acts nilpotently on all homology groups of $P_{n-1}X$. By assumption, it also acts nilpotently on $SH_n(X)$. Thus, since nilpotent modules are closed under extensions (*cf.* [42, Proposition I.4.3]), it acts nilpotently on $\pi_n X$. It follows from [31, Theorems B, C] that G is a nilpotent group, and its corresponding cover $B\operatorname{aut}_{G,*}(X)$ of the classifying space $B\operatorname{aut}_*(X)$ of the monoid of *pointed* self-equivalences is a nilpotent space.

As pointed out before the statement of the proposition, G acts on $\pi_* X$ since X is simply connected. Therefore the evaluation fiber sequence

$$\operatorname{aut}_{G,*}(X) \longrightarrow \operatorname{aut}_G(X) \longrightarrow X$$

gives rise to G-equivariant exact sequences

$$\pi_n \operatorname{aut}_{G,*}(X) \longrightarrow \pi_n \operatorname{aut}_G(X) \longrightarrow \pi_n X$$

for all $n \geqslant 1$. Since the outer terms are nilpotent G-modules by the above, so is the middle term $\pi_n \operatorname{aut}_G(X) \cong \pi_{n+1} B \operatorname{aut}_G(X)$. □

Remark 3.6. Since the $\mathcal{E}(X)$-module $SH_*(X;\mathbb{Z})$ is a quotient module of $\pi_*(X)$ and a submodule of $H_*(X;\mathbb{Z})$, we have that $\mathcal{E}(X)$ acts nilpotently on $SH_*(X;\mathbb{Z})$ whenever it acts nilpotently on either $\pi_*(X)$ or $H_*(X;\mathbb{Z})$. Therefore, Proposition 3.5 is a common generalization of Theorem C and Theorem D in [31], in the case when X is simply connected.

Recall [29, 30] that a connected space B is called *virtually nilpotent* if $\pi_1(B)$ has a nilpotent subgroup of finite index and if, for each $n > 1$, $\pi_1(B)$ has a subgroup of finite index that acts nilpotently on $\pi_n(B)$.

Remark 3.7. For many purposes in rational homotopy theory, virtually nilpotent spaces are just as good as nilpotent spaces. For instance, by [30, Proposition 3.4], virtually nilpotent spaces are $\mathbb{Q}$-good in the sense of Bousfield–Kan [23]. This means in particular that the spatial realization of the minimal Sullivan model for $\Omega^*(B)$ is a $\mathbb{Q}$-localization of B provided B has finite-dimensional rational cohomology groups (this follows from [22, Theorem 12.2]).

Next, we will prove a variant of Proposition 3.5 where we consider the action on $SH_*(X;\mathbb{Q})$ instead of $SH_*(X;\mathbb{Z})$.

Proposition 3.8. *Let X be simply connected, and either a finite CW-complex, or a finite Postnikov section with finitely generated homotopy groups. Let $G \leq \mathcal{E}(X)$ be a subgroup which acts nilpotently on rational spherical homology $SH_*(X;\mathbb{Q})$. Then the space $B \operatorname{aut}_G(X)$ is virtually nilpotent. It is nilpotent if G acts nilpotently on $SH_*(X;\mathbb{Z})$, which is automatic if $SH_*(X;\mathbb{Z})$ is torsion-free.*

Proof. We will in fact prove the slightly stronger statement that $B \operatorname{aut}_G(X)$ admits a finite cover which is nilpotent. Let $\mathcal{T} \leqslant \mathcal{E}(X)$ be the subgroup of self-equivalences that act trivially on the torsion subgroup $SH_*(X;\mathbb{Z})_{tor}$. As the torsion subgroup is finite, so is its automorphism group, so the exact sequence

$$1 \to \mathcal{T} \to \mathcal{E}(X) \to \operatorname{Aut}\left(SH_*(X;\mathbb{Z})_{tor}\right)$$

shows that $\mathcal{T}$ has finite index in $\mathcal{E}(X)$. Hence, $G \cap \mathcal{T}$ has finite index in G. The space $B \operatorname{aut}_{G\cap\mathcal{T}}(X)$ is weakly homotopy equivalent to the finite cover of $B \operatorname{aut}_G(X)$ that corresponds to the finite-index subgroup $G \cap \mathcal{T} \leq G$, so we are done if we can show that $B \operatorname{aut}_{G\cap\mathcal{T}}(X)$ is nilpotent. This will follow from Proposition 3.5 once we prove that $G \cap \mathcal{T}$ acts nilpotently on $SH_*(X;\mathbb{Z})$. By hypothesis, G acts nilpotently on $SH_*(X;\mathbb{Q})$, so there exists a filtration of G-modules

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = SH_*(X;\mathbb{Q})$$

such that G acts trivially on V_i/V_{i-1} for each i. Let $W_i \subseteq SH_*(X;\mathbb{Z})$ be the preimage of V_i under the change of coefficients homomorphism $SH_*(X;\mathbb{Z}) \to SH_*(X;\mathbb{Q})$. Then we get a filtration of G-modules

$$W_0 \subseteq W_1 \subseteq \ldots \subseteq W_n = SH_*(X;\mathbb{Z}),$$

and it follows that W_i/W_{i-1} has trivial G-action for $1 \leq i \leq n$. If $SH_*(X;\mathbb{Z})$ is torsion-free, then $W_0 = 0$ and the above shows that G acts nilpotently on

$SH_*(X;\mathbb{Z})$, whence $B\,\mathrm{aut}_G(X)$ is nilpotent. In general, the G-module $W_0 = SH_*(X;\mathbb{Z})_{\mathrm{tor}}$ need not be trivial, but the action of $G\cap\mathcal{T}$ on it is trivial by definition of $\mathcal{T}$. Thus, extending the filtration by $W_{-1}=0$ yields a filtration witnessing that $G\cap\mathcal{T}$ acts nilpotently on $SH_*(X;\mathbb{Z})$. □

Specializing the above to $G=\mathcal{E}_u(X)$, the preimage of the unipotent radical under the homomorphism $\mathcal{E}(X)\to\mathcal{E}(X_{\mathbb{Q}})$, we obtain the following result.

Corollary 3.9. *The space $\Gamma(X)/\operatorname{aut}(X)\sim B\,\mathrm{aut}_u(X)$ is virtually nilpotent. It is nilpotent if $SH_*(X;\mathbb{Z})$ is torsion-free.* □

The following example shows that the space $B\,\mathrm{aut}_u(X)$ is not nilpotent in general.

Example 3.10. Consider the Moore space $X = M(\mathbb{Z}/3\mathbb{Z},2)$, i.e., the homotopy cofiber of a degree 3 self-map of S^2. This space is rationally equivalent to point, so $\Gamma(X)=\mathcal{E}(X_{\mathbb{Q}})=1$, whence $B\,\mathrm{aut}_u(X)=B\,\mathrm{aut}(X)$. The group of self-equivalences is not difficult to compute (see *e.g.* [63, Theorem 2]); there is an isomorphism

$$\mathcal{E}(X)\cong(\mathbb{Z}/3\mathbb{Z})^\times\ltimes\mathbb{Z}/3\mathbb{Z}\cong\Sigma_3,$$

showing $\pi_1(B\,\mathrm{aut}_u(X))=\mathcal{E}(X)$ is not nilpotent in this case. The hypotheses in Proposition 3.5 are not satisfied because the action on $SH_2(X;\mathbb{Z})=\mathbb{Z}/3\mathbb{Z}$ is through the projection onto $(\mathbb{Z}/3\mathbb{Z})^\times$ and this action is not nilpotent.

3.3. **A lemma on bar constructions.** Let $f\colon G\to H$ be a map of topological monoids. If G and H are grouplike, then by Proposition 8.8 and Remark 8.9 of [52], there is a homotopy fiber sequence

$$H/G\to BG\xrightarrow{Bf}BH$$

where by definition H/G is the geometric bar construction $B(H,G,*)$. Moreover, the evident left action of H on H/G is compatible with the holonomy action of $\pi_1BH=\pi_0H$ on the homotopy fiber.

Lemma 3.11. *Let $f\colon G\to H$ be a map of topological (or simplicial) monoids and let $K\to G$ be a homomorphism from a group K. Then there is a zig-zag of K-equivariant weak equivalences between H/G where K acts on the left of H, and the same space where K acts by conjugation on both H and G.*

Remark 3.12. Suppose that

$$1\to G'\to G\xrightarrow{f}G''\to 1$$

is a split short exact sequence of groups, giving rise to a homotopy fiber sequence

$$BG'\to BG\xrightarrow{Bf}BG''.$$

It is well known that the conjugation action of G'' on BG' models the holonomy action of G'' on the homotopy fiber of Bf. On the other hand, as recalled before the statement of the lemma, the left action of G'' on G''/G also models the holonomy action. The evident map $BG'\to G''/G$ is, however, not equivariant; it becomes equivariant when the target is replaced with $(G''/G)_{\mathrm{conj}}$, *i.e.* the same space equipped with the conjugation action of G''. The lemma offers a resolution of this seeming incongruity by showing that G''/G and $(G''/G)_{\mathrm{conj}}$ are weakly equivalent as G''-spaces.

Before we prove this result, we recall the existence of a certain model category structure. Given a simplicial monoid G, we write Mod_G for the category of G-modules in simplicial sets, or G-modules for short.

Proposition 3.13. *Let G be a simplicial monoid. Then the category* Mod_G *admits a combinatorial model category structure where a morphism is a weak equivalence or a fibration if the underlying map of simplicial sets is a weak equivalence or a fibration.*

Proof. The existence of the model structure is well known and goes all the way back to Quillen [58, § II.4] — or see [60, Theorem 4.1] for a more modern formulation. By [1, § 2.78], Mod_G is locally presentable, hence combinatorial. $\square$

Lemma 3.14. *Let G be a simplicial monoid. Then $B(G, G, *)$ is a cofibrant object of* Mod_G.

Proof. Let sMod_G be the category of simplicial objects in Mod_G. By Corollary 18.4.12(1) of [43], the geometric realization functor $\mathrm{sMod}_G \to \mathrm{Mod}_G$ carries Reedy cofibrant objects to cofibrant objects. Hence it suffices to prove that the simplicial G-module $X = B_\bullet(G, G, *) = G \times G^\bullet$ is Reedy cofibrant. The n-th latching object $L_n X$ of X is identified with the union of the images of the degeneracy maps in $X_n = G \times G^n$. The map $L_n X \to X_n$ is thus a free map of G-modules generated by a cofibration of simplicial sets, and hence it is a cofibration of G-modules. $\square$

Proof of Lemma 3.11. We only prove the simplicial version of the statement; the topological version can be deduced from it by considering the singular simplicial sets of the topological monoids G, H.

The induction–restriction adjunction

$$\mathrm{Mod}_G \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{Mod}_H$$

is a Quillen adjunction with respect to the model structures of Proposition 3.13. Moreover, since these model structures are combinatorial, the category Mod_G^K of K-objects in Mod_G admits the injective model structure, where a morphism is a (trivial) cofibration if and only if the underlying G-module morphism is a (trivial) cofibration. Thus f induces a Quillen adjunction

$$\mathrm{Mod}_G^K \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{Mod}_H^K$$

between the injective model structures.

Let $B(G, G, *)_{\mathrm{triv}} \in \mathrm{Mod}_G^K$ be the bar construction $B(G, G, *)$ equipped with the trivial action of K. Let $B(G, G, *)_{\mathrm{conj}} \in \mathrm{Mod}_G^K$ be $B(G, G, *)$ equipped with the following action of K:

$$k \cdot (g; g_1, \dots, g_n) = (gk^{-1}; kg_1k^{-1}, \dots, kg_nk^{-1})$$

for $k \in K$ and $(g, g_1, \dots, g_n) \in B_n(G, G, *)$. Both $B(G, G, *)_{\mathrm{triv}}$ and $B(G, G, *)_{\mathrm{conj}}$ are weakly equivalent to the terminal object $*$ in Mod_G^K with the injective model structure. Thus their images under the left derived functor $\mathbb{L}f_!$ of $f_!$ are weakly equivalent in Mod_H^K. But by the preceding lemma, they are both cofibrant objects of Mod_G^K, so

$$\mathbb{L}f_! B(G, G, *)_{\mathrm{triv}} \simeq f_! B(G, G, *)_{\mathrm{triv}} = B(H, G, *)_{\mathrm{triv}}$$

and similarly for $B(G, G, *)_{\mathrm{conj}}$.

Finally, note that objects of Mod_H^K can equivalently be thought of as $K \times H$-modules.[2] Restricting the $K \times H$-module structures on both $B(H, G, *)_{\mathrm{triv}}$ and

[2] We chose the notation Mod_H^K over $\mathrm{Mod}_{K\times H}$ to emphasize that we are using a different model structure.

$B(H, G, *)_{\mathrm{conj}}$ along the diagonal homomorphism $K \to K \times H$ yields precisely the two K-spaces from the statement of the lemma. □

3.4. **A model for $B\operatorname{aut}(X_{\mathbb{Q}})$.** Before proving our main theorem about $B\operatorname{aut}(X)$, we treat the somewhat simpler case of $B\operatorname{aut}(X_{\mathbb{Q}})$. In this section, we let Y denote the nerve of a finitely generated simply connected minimal dg Lie algebra L. If L is the minimal Quillen model of a simply connected finite complex X, then Y is a model for $X_{\mathbb{Q}}$. The following definition should be viewed as a dg Lie algebra companion to Definition 3.1.

Definition 3.15. For a Lie subalgebra $\mathfrak{u} \subseteq H_0(\operatorname{Der}^c L)$, let $\operatorname{Der}^c_{\mathfrak{u}} L$ denote the dg Lie subalgebra of $\operatorname{Der}^c L\langle 0\rangle$ defined by the pullback

$$\begin{tikzcd} \operatorname{Der}^c_{\mathfrak{u}} L \arrow[r] \arrow[d] & \mathfrak{u} \arrow[d] \\ \operatorname{Der}^c L\langle 0\rangle \arrow[r] & H_0(\operatorname{Der}^c L\langle 0\rangle). \end{tikzcd}$$

When $\mathfrak{u} = \operatorname{Lie}\mathcal{E}_u(Y) \subseteq \operatorname{Lie}\mathcal{E}(Y) = H_0(\operatorname{Der}^c L)$ is the Lie algebra of the unipotent radical of $\mathcal{E}(Y)$, we will use the notation $\operatorname{Der}^c_u L = \operatorname{Der}^c_{\mathfrak{u}} L$.

Lemma 3.16. *Let L be a positively graded, finitely generated, minimal dg Lie algebra with nerve Y. For every unipotent algebraic subgroup $U \leqslant \mathcal{E}(Y)$, with Lie algebra $\mathfrak{u}$, the dg Lie algebra $\operatorname{Der}^c_{\mathfrak{u}} L$ is nilpotent and there is a weak equivalence of grouplike monoids,*

$$\exp_\bullet\left(\operatorname{Der}^c_{\mathfrak{u}} L\right) \to \operatorname{aut}_U(Y). \tag{10}$$

If U is normal in $\mathcal{E}(Y)$, then this weak equivalence is equivariant with respect to the conjugation action of $\operatorname{Aut} L$ *on the domain and codomain.*

Proof. We first verify that the dg Lie algebra $\mathfrak{g} = \operatorname{Der}^c_{\mathfrak{u}} L$ is nilpotent. In degree zero, $\mathfrak{g}_0$ may be identified with the Lie algebra of the preimage $\widetilde{U} = p^{-1}(U)$ under the projection $p\colon \mathcal{A}\mathrm{ut}\, L \to \mathcal{A}\mathrm{ut}^h L$. By the proof of Theorem 2.23, the homomorphism p has unipotent kernel. It follows that $\widetilde{U}$ is an extension of unipotent groups and is hence unipotent (see *e.g.* [53, §6.45]). Since $(\operatorname{Der}^c L)_k$ is an algebraic representation of $\widetilde{U}$ for each $k > 0$, it follows that $\widetilde{U}$ acts unipotently on it (see [53, Proposition 14.3]). Hence, $\mathfrak{g}_0 = \operatorname{Lie}\widetilde{U}$ is nilpotent and acts nilpotently on $\mathfrak{g}_k = (\operatorname{Der}^c L)_k$ for each $k > 0$.

The action of the nilpotent dg Lie algebra $\mathfrak{g}$ on L by outer derivations induces an action of $\exp_\bullet(\mathfrak{g})$ on $\mathsf{MC}_\bullet(L)$ (*cf.* [9, §3.5]), giving rise to a morphism of grouplike monoids

$$\alpha\colon \exp_\bullet(\mathfrak{g}) \to \operatorname{aut}(\mathsf{MC}_\bullet(L)). \tag{11}$$

The map (11) is known to induce an isomorphism on $\pi_k(-)$ for $k > 0$. To see this, one can apply [9, Proposition 3.7] to the Sullivan model $\Lambda = C^*(L)$. Let us add that this is essentially equivalent to the statement, going back to Tanré [67, VII.4.(4)], that $\operatorname{Der}^c L\langle 1\rangle$ is a Lie model for the simply connected cover of $B\operatorname{aut}(\mathsf{MC}_\bullet(L))$.

It is clear from the construction that $\pi_0(\operatorname{aut}_U(Y)) = U$ and that $H_0(\operatorname{Der}^c_{\mathfrak{u}} L) = \mathfrak{u}$. By Proposition 2.16, there is an isomorphism

$$\pi_0(\exp_\bullet \operatorname{Der}^c_{\mathfrak{u}} L) \cong \exp(H_0(\operatorname{Der}^c_{\mathfrak{u}} L)) = \exp(\mathfrak{u}) = U.$$

One checks that the map induced by (11) on π_0 may be identified with the inclusion of U into $\mathcal{E}(Y)$. Hence, (11) corestricts to a weak equivalence $\exp_\bullet(\mathfrak{g}) \to \operatorname{aut}_U(Y)$. The statement about $\operatorname{Aut} L$-equivariance is quickly verified by inspection. □

Corollary 3.17. *For every unipotent algebraic subgroup $U \leqslant \mathcal{E}(Y)$, the space $B\operatorname{aut}_U(Y)$ is nilpotent and rational with Lie model $\operatorname{Der}^c_{\mathfrak{u}}(L)$.*

Proof. Apply the classifying space functor to the monoid map (10) and note that the space $\mathsf{MC}_\bullet(\mathfrak{g}) \sim B\exp_\bullet(\mathfrak{g})$ is nilpotent and rational whenever $\mathfrak{g}$ is a nilpotent dg Lie algebra. □

Remark 3.18. The full classifying space $B\operatorname{aut}(Y)$ is not nilpotent in general. For instance, if $X = S^2 \vee S^2$ then $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathsf{GL}_2(\mathbb{Q})$. As pointed out in [32, Proposition 6.1], $B\operatorname{aut}(X_{\mathbb{Q}})$ does not even have the homotopy type of the nerve of a connected complete dg Lie algebra in general.

An important special case of the above lemma is when $U = \mathcal{E}_u(Y)$, the unipotent radical of $\mathcal{E}(Y)$. In this case, $\operatorname{Der}^c_{\mathrm{u}} L$ may be identified with the nilradical $\operatorname{nil}\operatorname{Der}^c L$ (as defined in §2.2) by the following lemma.

Lemma 3.19. *The dg Lie algebra* $\operatorname{Der}^c_u L$ *agrees with the nilradical* $\operatorname{nil}\operatorname{Der}^c L$*. Furthermore, its degree* 0 *component is the Lie algebra of the unipotent radical* $\mathcal{A}\mathrm{ut}_u L$ *of* $\mathcal{A}\mathrm{ut}\, L$*,*

$$(\operatorname{nil}\operatorname{Der}^c L)_0 = (\operatorname{Der}^c_u L)_0 = \operatorname{Lie}(\mathcal{A}\mathrm{ut}_u L).$$

This Lie algebra may also be identified with the maximal ideal $\operatorname{nil}_H Z_0(\operatorname{Der}^c L) \subseteq Z_0(\operatorname{Der}^c L)$ *of derivations that act nilpotently on* $H_*(X;\mathbb{Q})$*.*

Proof. By Lemma 3.16 and its proof, the dg Lie algebra $\operatorname{Der}^c_u L$ is nilpotent and its degree zero component may be identified with the Lie algebra of the preimage $p^{-1}(U)$ of $U = \mathcal{E}_u(Y)$. Since the homomorphism $p\colon \mathcal{A}\mathrm{ut}\, L \to \mathcal{A}\mathrm{ut}^h L$ has unipotent kernel, it follows that $p^{-1}(U) = \mathcal{A}\mathrm{ut}_u L$. This shows that $(\operatorname{Der}^c_u L)_0$ is the Lie algebra of the unipotent radical of $\mathcal{A}\mathrm{ut}\, L$. In particular, since this is an ideal in $\operatorname{Lie}(\mathcal{A}\mathrm{ut}\, L) = Z_0(\operatorname{Der} L)$, it follows that $\operatorname{Der}^c_u L$ is an ideal in $\operatorname{Der}^c L\langle 0\rangle$, and it is nilpotent by Lemma 3.16. By definition, $\operatorname{nil}\operatorname{Der}^c L$ is the maximal nilpotent ideal. Hence, $\operatorname{Der}^c_u L \subseteq \operatorname{nil}\operatorname{Der}^c L$.

We now show the reverse inclusion $\operatorname{nil}\operatorname{Der}^c L \subseteq \operatorname{Der}^c_u L$. The components in degrees > 0 are clearly equal, so we only need to consider the degree 0 component. Note that since L is finitely generated as a graded Lie algebra and of finite type, there is some n such that the finite-dimensional graded vector space $L_{\leqslant n} = \bigoplus_{i=1}^n L_i$ is a faithful algebraic representation of $\mathcal{A}\mathrm{ut}\, L$. Thus, by Lemma 2.6, $\operatorname{Lie}(\mathcal{A}\mathrm{ut}_u L)$ is precisely the maximal ideal of $\operatorname{Lie}(\mathcal{A}\mathrm{ut}\, L) = Z_0(\operatorname{Der} L)$ consisting of derivations of L which are nilpotent when restricted to $L_{\leqslant n}$. Since $(\operatorname{nil}\operatorname{Der}^c L)_0$ is an ideal in $Z_0(\operatorname{Der} L)$ which acts nilpotently on $(\operatorname{Der}^c L)_k \cong (\operatorname{Der} L)_k \oplus L_{k-1}$ for each $k > 0$, it in particular acts nilpotently on $L_{\leq n}$. Hence, $(\operatorname{nil}\operatorname{Der}^c(L))_0 \subseteq \operatorname{Lie}(\mathcal{A}\mathrm{ut}_u L)$.

The last statement follows from Lemma 2.6. □

Thus, for the unipotent radical $U = \mathcal{E}_u(Y)$, we can restate Lemma 3.16 as follows.

Lemma 3.20. *Let* L *be a positively graded, finitely generated, minimal dg Lie algebra with nerve* $Y = \mathsf{MC}_\bullet(L)$*. There is an* $\operatorname{Aut} L$*-equivariant weak equivalence of grouplike monoids,*

$$\exp_\bullet\big(\operatorname{nil}\operatorname{Der}^c(L)\big) \to \operatorname{aut}_u(Y)$$

where $\operatorname{Aut} L$ *acts on the domain and codomain by conjugation.* □

We record the following descriptions of the Lie algebra of the unipotent radical of $\mathcal{E}(Y)$.

Corollary 3.21. *The following three Lie algebras agree:*

(i) The Lie algebra of the unipotent radical of $\mathcal{E}(Y)$*;*
(ii) the degree zero homology $H_0(\operatorname{nil}\operatorname{Der}^c L)$*;*
(iii) the maximal ideal $\operatorname{nil}_H H_0(\operatorname{Der}^c L) \subseteq H_0(\operatorname{Der}^c L)$ *of classes of derivations that act nilpotently on* $H_*(X;\mathbb{Q})$*.* □

For a normal subgroup $U \leqslant \mathcal{E}(Y)$, recall that $\mathcal{E}^U(Y)$ denotes $\mathcal{E}(Y)/U$.

Lemma 3.22. *Let $U \leqslant \mathcal{E}(Y)$ be a normal unipotent algebraic subgroup. The* $\operatorname{Aut} L$*-space $B\operatorname{aut}_U(Y)$, equipped with the conjugation action, is weakly equivalent to the* $\operatorname{Aut} L$*-space $\mathcal{E}^U(Y)/\operatorname{aut}(Y)$, where* $\operatorname{Aut} L$ *acts on the left via the homomorphism* $\operatorname{Aut} L \to \mathcal{E}^U(Y)$.

Proof. The map

$$B\operatorname{aut}_U(Y) = B\big(*, \operatorname{aut}_U(Y), *\big) \to B\big(\mathcal{E}^U(Y), \operatorname{aut}(Y), *\big) = \mathcal{E}^U(Y)/\operatorname{aut}(Y)$$

induced by the inclusion of $\operatorname{aut}_U(Y)$ into $\operatorname{aut}(Y)$ is clearly equivariant with respect to the conjugation action of $\operatorname{Aut} L$, and it is a weak equivalence because

$$B\operatorname{aut}_U(Y) \to B\operatorname{aut}(Y) \to B\,\mathcal{E}^U(Y)$$

is a homotopy fiber sequence. Lemma 3.11, applied to the map of monoids $\operatorname{aut}(Y) \to \mathcal{E}^U(Y)$ with $K = \operatorname{Aut} L$, shows that $\mathcal{E}^U(Y)/\operatorname{aut}(Y)$ with the conjugation action of $\operatorname{Aut} L$ is weakly equivalent to the same space but where $\operatorname{Aut} L$ acts on the left via $\operatorname{Aut} L \to \mathcal{E}^U(Y)$. □

The above lemma gives a model for the homotopy type of the $\operatorname{Aut} L$-space $\mathcal{E}^U(Y)/\operatorname{aut}(Y)$, but this is in general not enough to recover the homotopy type of $B\operatorname{aut}(Y)$, because it does not give a model for $\mathcal{E}^U(Y)/\operatorname{aut}(Y)$ as an $\mathcal{E}^U(Y)$-space. However, when U is equal to the unipotent radical $\mathcal{E}_u(Y)$, the group $\mathcal{E}^U(Y)$ is the maximal reductive quotient $R(Y)$, and the fact that $\operatorname{Aut} L$ and $\mathcal{E}(Y)$ have the same maximal reductive quotient allows us to use the existence of Levi subgroups to find a splitting of the composite homomorphism

$$\operatorname{Aut} L \to \mathcal{E}(Y) \to R(Y)$$

in the category of algebraic groups. This solves the problem.

Theorem 3.23. *The left $R(Y)$-space $R(Y)/\operatorname{aut}(Y)$ is weakly equivalent to the $R(Y)$-space $B\exp_\bullet(\operatorname{nil}\operatorname{Der}^c(L))$, where $R(Y)$ acts by conjugation via any choice of splitting of* $\operatorname{Aut} L \to R(Y)$*. Consequently,*

$$B\operatorname{aut}(Y) \sim \langle \operatorname{nil}\operatorname{Der}^c(L)\rangle_{hR(Y)}$$

Proof. By Lemma 3.20 and Lemma 3.22 applied to the unipotent radical $U = \mathcal{E}_u(Y)$, the $\operatorname{Aut} L$-space $R(Y)/\operatorname{aut}(Y)$, where $\operatorname{Aut} L$ acts on the left via the homomorphism $\operatorname{Aut} L \to R(Y)$, is weakly equivalent to the $\operatorname{Aut} L$-space $\langle \operatorname{nil}\operatorname{Der}^c(L)\rangle$ where $\operatorname{Aut} L$ acts by conjugation. This implies that the $R(Y)$-space $R(Y)/\operatorname{aut}(Y)$ is weakly equivalent to the $R(Y)$-space $\langle \operatorname{nil}\operatorname{Der}^c(L)\rangle$, where $R(Y)$ acts through any choice of splitting of $\operatorname{Aut} L \to R(Y)$. Thus, there are weak equivalences

$$B\operatorname{aut}(Y) \sim \big(R(Y)/\operatorname{aut}(Y)\big)_{hR(Y)} \sim \langle \operatorname{nil}\operatorname{Der}^c(L)\rangle_{hR(Y)}.$$

□

3.4.1. *Finitely generated Sullivan models.* The results in the previous section have analogs for spaces X that have a finitely generated minimal Sullivan model, or equivalently, $\dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} < \infty$. We briefly remark on the necessary changes here.

Let Λ be the minimal Sullivan model of X. We write $Y = \langle \Lambda \rangle \simeq X_{\mathbb{Q}}$. Analogously to Definition 3.15, if $\mathfrak{u} \subset H_0(\operatorname{Der}\Lambda)$ is a Lie subalgebra, we write $\operatorname{Der}_{\mathfrak{u}} \Lambda$ for the subalgebra of $\operatorname{Der}\Lambda\langle 0\rangle$ defined by the pullback

$$\begin{array}{ccc} \operatorname{Der}_{\mathfrak{u}} \Lambda & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow \\ \operatorname{Der}\Lambda\langle 0\rangle & \longrightarrow & H_0(\operatorname{Der}\Lambda\langle 0\rangle) \end{array}$$

When $\mathfrak{u} = \operatorname{Lie}\mathcal{E}_u(Y)$ is the Lie algebra of the unipotent radical of $\mathcal{E}(Y)$, we write $\operatorname{Der}_u \Lambda = \operatorname{Der}_{\mathfrak{u}} \Lambda$. We remark that we view Λ as cohomologically graded whereas $\operatorname{Der}\Lambda$ is graded homologically. Thus the degree k piece $(\operatorname{Der}\Lambda)_k$ consists of derivations of Λ that *lower* the cohomological degree of Λ by k.

Lemma 3.16 then admits the following analog:

Lemma 3.24. *Let Λ be a finitely generated minimal Sullivan algebra generated in positive degrees, and let $Y = \langle\Lambda\rangle$ be its spatial realization. For every unipotent algebraic subgroup $U \leqslant \mathcal{E}(Y)$ with Lie algebra $\mathfrak{u}$, the dg Lie algebra $\operatorname{Der}_{\mathfrak{u}}\Lambda$ is nilpotent, and there is a weak equivalence*

$$\exp_\bullet(\operatorname{Der}_{\mathfrak{u}}\Lambda) \longrightarrow \operatorname{aut}_U(Y)$$

of grouplike monoids. If U is normal in $\mathcal{E}(Y)$, then this weak equivalence is equivariant with respect to the conjugation action of $\operatorname{Aut}\Lambda$ on the domain and codomain.

Proof. The map is defined as in [9, Proposition 3.7]: the action of the nilpotent dg Lie algebra $\mathfrak{g} = \operatorname{Der}_{\mathfrak{u}}\Lambda$ on Λ by derivations induces an action of the simplicial nilpotent group

$$\exp_\bullet(\mathfrak{g}) = \exp(Z_0(\Omega_\bullet \otimes \mathfrak{g}))$$

on $\Omega_\bullet \otimes \Lambda$ by $\Omega_\bullet$-linear automorphisms, which in turn induces an action of $\exp_\bullet(\mathfrak{g})$ on

$$\langle\Lambda\rangle = \operatorname{Hom}_{cdga(\Omega_\bullet)}(\Omega_\bullet \otimes \Lambda, \Omega_\bullet).$$

The rest of the proof is entirely analogous to Lemma 3.16. We omit the details. $\square$

Next comes the cdga analog of Lemma 3.19. Recall that we write $\operatorname{nil}_H Z_0(\operatorname{Der}\Lambda)$ and $\operatorname{nil}_S Z_0(\operatorname{Der}\Lambda)$ for the maximal ideals of $Z_0(\operatorname{Der}\Lambda)$ consisting of derivations that act nilpotently on $H^*(X;\mathbb{Q})$ and on $SH^*(X;\mathbb{Q})$, respectively.

Lemma 3.25. *The following subalgebras of $Z_0(\operatorname{Der}\Lambda)$ are equal:*

(i) $(\operatorname{nil}\operatorname{Der}\Lambda)_0$
(ii) $(\operatorname{Der}_u\Lambda)_0$,
(iii) $\operatorname{Lie}\left(\mathcal{A}\mathrm{ut}_u\Lambda\right)$,
(iv) $\operatorname{nil}_H Z_0(\operatorname{Der}\Lambda)$,
(v) $\operatorname{nil}_S Z_0(\operatorname{Der}\Lambda)$.

Proof. We show only that (i) and (iii) are equal; the rest is an easy modification of the proof of Lemma 3.19.

First, observe that the graded ideal $I \leqslant \operatorname{Der}\Lambda\langle 0\rangle$ given by

$$I_n = \begin{cases} B_0(\operatorname{Der}\Lambda) & \text{if } n = 0 \\ (\operatorname{Der}\Lambda)_n & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

is nilpotent—the degree 0 part is the Lie algebra of the unipotent group $\mathcal{A}\mathrm{ut}_h\Lambda$, and the higher parts are algebraic representations of this group. Thus $I \leqslant \operatorname{nil}\operatorname{Der}\Lambda$, and (i) is the largest nilpotent ideal of $Z_0(\operatorname{Der}\Lambda)$ consisting of derivations which act nilpotently on $(\operatorname{Der}\Lambda)_k$ for each $k > 0$. Therefore the equality between (i) and (iii) will follow immediately from Lemma 2.6 as soon as we prove that for every sufficiently large n,

$$\bigoplus_{i=1}^{n} (\operatorname{Der}\Lambda)_i$$

is a faithful representation of $\operatorname{Aut}\Lambda$.

We have an isomorphism

$$\operatorname{Der}\Lambda \cong V^\vee \otimes \Lambda V \cong \bigoplus_{i\geqslant 0} V^\vee \otimes \Lambda^i V$$

of graded vector spaces, where $V^\vee$ is the linear dual of V, and $V^\vee \otimes \Lambda^i V$ is precisely the space of derivations of Λ that increase weight by $i-1$. We claim that the $\operatorname{Aut}\Lambda$-subrepresentation of $\operatorname{Der}\Lambda$ generated by $V^\vee \otimes \Lambda^0 V$ is faithful. Indeed, once we fix a homogenenous basis $x_1, \dots, x_k$ of V, the derivations in $V^\vee \otimes \Lambda^0 V$ admit the dual basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$. Now suppose that $\phi \in \operatorname{Aut}\Lambda$ acts trivially on $V^\vee \otimes \Lambda^0 V$, *i.e.*

$$\phi \frac{\partial}{\partial x_i} \phi^{-1}(x_j) = \delta_{ij} \tag{12}$$

for all i, j. Every cdga automorphism $\psi \colon \Lambda \to \Lambda$ can be uniquely expressed as $\psi = \psi_0 + \psi_1$ where ψ_0 preserves the weight of its argument and ψ_1 raises it strictly. Thus we have

$$\phi \frac{\partial}{\partial x_i} \phi^{-1}(x_j) = \phi \frac{\partial}{\partial x_i} (\phi^{-1})_0(x_j) + \phi \frac{\partial}{\partial x_i} (\phi^{-1})_1(x_j)$$

where the first term is a constant and the second term has no constant term (as $(\phi^{-1})_1(x_j) \in \Lambda^{\geqslant 2} V$, the derivation $\frac{\partial}{\partial x_i}$ lowers weight by 1, and the cdga automorphism ϕ does not lower weight). Also, the first term is equal to $\frac{\partial}{\partial x_i}(\phi^{-1})_0(x_j)$ since ϕ restricts to the identity on $\Lambda^0 V$, and the second term vanishes if and only if $\frac{\partial}{\partial x_i}(\phi^{-1})_1(x_j)$ does since ϕ is invertible. Thus condition (12) is equivalent to requiring that

$$\begin{aligned} \frac{\partial}{\partial x_i}(\phi^{-1})_0(x_j) &= \delta_{ij} \\ \frac{\partial}{\partial x_i}(\phi^{-1})_1(x_j) &= 0 \end{aligned}$$

for all i, j. But if the first line holds for a fixed j and all i, then clearly $(\phi^{-1})_0(x_j) = x_j$. Likewise, if the second line holds for a fixed j and all i, then $(\phi^{-1})_1(x_j) = 0$. Thus $\phi^{-1} = \mathrm{id}$, which is what we wanted to prove.

It follows that $\bigoplus_{i=1}^n (\operatorname{Der}\Lambda)_i$ is a faithful $\operatorname{Aut}\Lambda$-representation as soon as V is concentrated below (cohomological) degree n. ☐

The rest of the argument carries through as in the previous section practically without change. In particular, we obtain

Theorem 3.26. *The left $R(Y)$-space $R(Y)/\operatorname{aut}(Y)$ is weakly equivalent to the $R(Y)$-space $B\exp_\bullet(\operatorname{Der}_u \Lambda)$, where the maximal reductive quotient $R(Y)$ of $\mathcal{E}(Y)$ acts via any choice of splitting of $\operatorname{Aut}\Lambda \to R(Y)$. Consequently,*

$$B\operatorname{aut}(Y) \sim \langle \operatorname{Der}_u \Lambda \rangle_{\mathrm{h}R(Y)}$$ ☐

Remark 3.27. As long as X has finite-dimensional rational spherical cohomology, the automorphism groups $\mathcal{A}\mathrm{ut}\,\Lambda$ and $\mathcal{A}\mathrm{ut}\,L$ of the minimal Sullivan model and Lie model of X, respectively, will be pro-affine algebraic groups even if they fail to be affine [59]. In fact, Levi decompositions exist for pro-affine algebraic groups in characteristic zero [45]. Moreover, in this case these groups will be semidirect products of the reductive group $R(X)$ (identified with the maximal reductive quotient of the affine algebraic group $\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(SH_*(X;\mathbb{Q}))$) with a pro-unipotent group.

Remark 3.28. The results of the preceding section can be modified to deal with nilpotent spaces of finite type as well. The minimal Sullivan algebra approach goes through with no modifications. The Quillen model approach goes through provided one is willing to work with *complete* dg Lie algebras, as in [32]. We opted to state our results for simply connected spaces in order to avoid the unedifying technical wrinkles that arise in the non-simply connected case.

3.5. **Algebraic models for $B\operatorname{aut}(X)$.** For a normal subgroup $G \leqslant \mathcal{E}(X)$, recall that we use the notation

$$\mathcal{E}^G(X) = \mathcal{E}(X)/G.$$

Let $q_*\colon \mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ be the homomorphism induced by rationalization.

Lemma 3.29. *Let $G \leqslant \mathcal{E}(X)$ be a normal subgroup and let $U \leqslant \mathcal{E}(X_{\mathbb{Q}})$ be a normal unipotent algebraic subgroup such that $q_*(G) \leqslant U$. If the inclusion $q_*(G) \leqslant U$ is $\mathbb{Q}$-surjective, then the $\mathcal{E}^G(X)$-spaces $\mathcal{E}^G(X)/\operatorname{aut}(X)$ and $\mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}})$ are rationally equivalent.*

Proof. Let $r\colon X \to X_{\mathbb{Q}}$ be a rationalization. We may assume that r is a cofibration. This holds in many models for rationalizations (*e.g.* cellular rationalization [33, Theorem 9.7] or the $H_*(-;\mathbb{Q})$-localization of Bousfield [21]), but if necessary, one can achieve this by abstract nonsense, *e.g.*, by factoring r as a cofibration followed by a weak homotopy equivalence. Now consider the pullback square

$$\begin{tikzcd} \operatorname{aut}(r) \arrow[r, "q"] \arrow[d] & \operatorname{aut}(X_{\mathbb{Q}}) \arrow[d, "r^*"] \\ \operatorname{aut}(X) \arrow[r, "r_*"] & \operatorname{map}(X, X_{\mathbb{Q}})_{re}, \end{tikzcd}$$

where $\operatorname{map}(X, X_{\mathbb{Q}})_{re}$ denotes the space of rational equivalences from X to $X_{\mathbb{Q}}$ and $\operatorname{aut}(r)$ denotes the space of self-equivalences of r viewed as an object in the category of maps.

The map r^* is a fibration since r is a cofibration, and it is a weak homotopy equivalence by standard properties of localizations. Hence, the left vertical map is a weak homotopy equivalence as well. The map r_* is in general not a bijection on π_0, but its restriction to each component is a rational equivalence to the component it hits by [42, Theorem II.3.11]. It follows that the top horizontal map q has the same property.

This yields a zig-zag of grouplike monoids

$$\operatorname{aut}(X) \xleftarrow{\sim} \operatorname{aut}(r) \xrightarrow{q} \operatorname{aut}(X_{\mathbb{Q}}), \tag{13}$$

where the left map is a weak equivalence and the right map q induces an isomorphism on $\pi_k(-)\otimes\mathbb{Q}$ for all $k > 0$ and may be identified with $q_*\colon \mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ on π_0. In particular, for our purposes, we may replace $\operatorname{aut}(X)$ by $\operatorname{aut}(r)$; we will blur the distinction in what follows and think about (13) as a map $\operatorname{aut}(X) \to \operatorname{aut}(X_{\mathbb{Q}})$.

Since $U \leqslant \mathcal{E}(X_{\mathbb{Q}})$ is unipotent, it acts unipotently on $H_*(X;\mathbb{Q})$. Hence so does G. As in the proof of Proposition 3.8, let $\mathcal{T} \leqslant \mathcal{E}(X)$ denote the subgroup of those automorphisms that act trivially on the torsion subgroup $SH_*(X)_{tor}$. Then $G' = G \cap \mathcal{T}$ is a finite-index subgroup of G. We have a map of $\mathcal{E}^{G'}(X)$-spaces

$$\mathcal{E}^{G'}(X)/\operatorname{aut}(X) \to \mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}}), \tag{14}$$

and we will now show that it is a rational equivalence between nilpotent spaces.

As in the proof of Proposition 3.8, the group G' acts nilpotently on $SH_*(X;\mathbb{Z})$, so the space $B\operatorname{aut}_{G'}(X) \sim \mathcal{E}^{G'}(X)/\operatorname{aut}(X)$ is nilpotent. Similarly, the space $B\operatorname{aut}_U(X_{\mathbb{Q}}) \sim \mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}})$ is nilpotent by Corollary 3.17. We have a commutative diagram

$$\begin{tikzcd} B\operatorname{aut}_{G'}(X) \arrow[r, "\sim"] \arrow[d, "f"] & \mathcal{E}^{G'}(X)/\operatorname{aut}(X) \arrow[d] \\ B\operatorname{aut}_U(X_{\mathbb{Q}}) \arrow[r, "\sim"] & \mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}}), \end{tikzcd}$$

where the right vertical map is (14), and the left vertical map f is induced by the restriction of (13) to the relevant components (strictly speaking, one should here either replace $B\operatorname{aut}(X)$ by $B\operatorname{aut}(r)$ or work in the homotopy category). Hence, f induces an isomorphism on $\pi_k(-)\otimes\mathbb{Q}$ for $k>1$. The map $\pi_1(f)$ may be identified with the homomorphism $G'\to U$, which factors as

$$G' \longrightarrow G \longrightarrow q_*(G) \longrightarrow U.$$

The first map is the inclusion of a finite-index subgroup, so it is a $\mathbb{Q}$-isomorphisms. The second map is surjective with finite kernel, whence a $\mathbb{Q}$-isomorphism. The third map is injective and $\mathbb{Q}$-surjective by hypothesis. Hence, the homomorphism $\pi_1(f)$ is a $\mathbb{Q}$-isomorphism between nilpotent groups (note: we do not claim that G is nilpotent in general). Hence, (14) is a rational equivalence between nilpotent spaces.

Finally, we can now show that the map of $\mathcal{E}^G(X)$-spaces

$$\mathcal{E}^G(X)/\operatorname{aut}(X)\to\mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}}) \tag{15}$$

is a rational homology isomorphism. We have just seen that the composite map

$$\mathcal{E}^{G'}(X)/\operatorname{aut}(X)\to\mathcal{E}^G(X)/\operatorname{aut}(X)\to\mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}})$$

is a rational equivalence between nilpotent spaces, whence a rational homology isomorphism, so we only need to show that the first map is a rational homology isomorphism. There is an exact sequence

$$1\to K\to\mathcal{E}^{G'}(X)\to\mathcal{E}^G(X)\to 1,$$

where $K=G/G'$, so the homotopy fiber $\mathcal{E}^G(X)/\mathcal{E}^{G'}(X)$ of $B\,\mathcal{E}^{G'}(X)\to B\,\mathcal{E}^G(X)$ is weakly equivalent to the classifying space of the finite group K. Hence, the rational Serre spectral sequence of the homotopy fiber sequence

$$\mathcal{E}^{G'}(X)/\operatorname{aut}(X)\to\mathcal{E}^G(X)/\operatorname{aut}(X)\to\mathcal{E}^G(X)/\mathcal{E}^{G'}(X)$$

reduces to an isomorphism

$$H_*\big(\mathcal{E}^{G'}(X)/\operatorname{aut}(X);\mathbb{Q}\big)_K\cong H_*\big(\mathcal{E}^G(X)/\operatorname{aut}(X);\mathbb{Q}\big),$$

where the left-hand side denotes the coinvariants with respect to the action of K. The action of K is the restriction of the evident action of $\mathcal{E}^{G'}(X)$. The rational equivalence of $\mathcal{E}^{G'}(X)$-spaces (14) shows that this action factors through $\mathcal{E}^G(X)$, meaning that the action of K is trivial. This completes the proof. □

Remark 3.30. If one ignores group actions, the assumption that U and G are normal can be dropped; a small modification of the preceding proof shows that the space $B\operatorname{aut}_G(X)$ is rationally equivalent to $B\operatorname{aut}_U(X_{\mathbb{Q}})$ for every unipotent algebraic subgroup $U\leqslant\mathcal{E}(X_{\mathbb{Q}})$ and every subgroup $G\leqslant\mathcal{E}(X)$ such that $q_*(G)$ includes into U by a $\mathbb{Q}$-isomorphism.

The following lemma will be useful for verifying the hypotheses in the previous lemma.

Lemma 3.31. *Let $G\leqslant\mathcal{E}(X)$ be a subgroup and let $U\leqslant\mathcal{E}(X_{\mathbb{Q}})$ be a unipotent algebraic subgroup such that $q_*(G)\leqslant U$. The following are equivalent:*

(i) The homomorphism $q_*\colon G\to U$ is a $\mathbb{Q}$-isomorphism.
(ii) The inclusion $q_*(G)\leqslant U$ is $\mathbb{Q}$-surjective.
(iii) The inclusion $G\leqslant q_*^{-1}(U)$ is $\mathbb{Q}$-surjective.

Proof. In the commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "q_*"'] & q_*^{-1}(U) \arrow[d, "q_*"] \\ q_*(G) \arrow[r] & U \end{tikzcd}$$

the left vertical map is surjective with finite kernel and hence a $\mathbb{Q}$-isomorphism. The right vertical map has finite kernel and image an arithmetic subgroup of U by Theorem 2.23(ii) and Lemma 2.8. Hence it is a $\mathbb{Q}$-isomorphism by Lemma 2.10. The claim follows. □

Remark 3.32. Unipotent algebraic groups are nilpotent and $\mathbb{Q}$-local (*i.e.* uniquely divisible), so U will be a model for the $\mathbb{Q}$-localization of the nilpotent group $q_*(G)$ whenever the equivalent conditions of the above lemma are fulfilled, and a model for the $\mathbb{Q}$-localization of G if, in addition, G is nilpotent.

Lemma 3.33. *Let $G \leqslant \mathcal{E}(X)$ be a subgroup. The following are equivalent:*

(i) G acts unipotently on $H_*(X;\mathbb{Q})$.
(ii) $q_*(G)$ is contained in a unipotent algebraic subgroup of $\mathcal{E}(X_\mathbb{Q})$.
(iii) $q_*(G)$ is contained in a unipotent algebraic subgroup U of $\mathcal{E}(X_\mathbb{Q})$ such that the inclusion $q_*(G) \leqslant U$ is $\mathbb{Q}$-surjective.

Proof. Evidently (iii) implies (ii) implies (i).

First we show that (i) implies (ii). If G acts unipotently on $H_*(X;\mathbb{Q})$, then the image of G in $\mathsf{GL}(H_*(X;\mathbb{Q}))$ lies in a unipotent algebraic subgroup U''. The preimage U' of U'' in $\mathcal{E}(X_\mathbb{Q})$ is a unipotent algebraic subgroup of $\mathcal{E}(X_\mathbb{Q})$, since it is an extension of U'' by the kernel of the $\mathcal{E}(X_\mathbb{Q})$-representation $H_*(X;\mathbb{Q})$, which is unipotent by Theorem 2.23(iii). Also, U' contains $q_*(G)$.

The implication (ii) $\Rightarrow$ (iii) is an easy consequence of [61, p. 104, Theorem 2]. Indeed, suppose that $q_*(G)$ is contained in a unipotent algebraic subgroup $U' = \exp(\mathfrak{u}')$ for some nilpotent Lie subalgebra $\mathfrak{u}'$ of $\operatorname{Lie}\mathcal{E}(X_\mathbb{Q})$. The underlying varieties of U' and $\mathfrak{u}'$ are equal, and the linear span L of $q_*(G)$ in $\mathfrak{u}'$ is a Lie subalgebra of $\mathfrak{u}'$. Then $U = \exp L$ (which is isomorphic to the Malcev completion of $q_*(G)$) is an affine algebraic subgroup of U' containing $q_*(G)$ such that the inclusion $q_*(G) \leqslant U$ is $\mathbb{Q}$-surjective [61, p. 104, Theorem 2(iv)]. □

3.5.1. *Algebraicity of homotopy and (co)homology.* For a normal subgroup $G \leqslant \mathcal{E}(X)$, recall that $\mathcal{E}^G(X)$ denotes $\mathcal{E}(X)/G$. In this situation, we have a homotopy fiber sequence

$$B\operatorname{aut}_G(X) \to B\operatorname{aut}(X) \to B\,\mathcal{E}^G(X).$$

Theorem 3.34. *Let X be a simply connected finite CW-complex, let L be its minimal Quillen model, let $U \leqslant \mathcal{E}(X_\mathbb{Q})$ be a normal unipotent algebraic subgroup with Lie algebra $\mathfrak{u}$, and let $G \leqslant \mathcal{E}(X)$ be a normal subgroup such that $q_*(G) \leqslant U$ and the inclusion is $\mathbb{Q}$-surjective.*

(i) The group $\mathcal{E}^U(X_\mathbb{Q})$ may be identified with the $\mathbb{Q}$-points of a linear algebraic group,
(ii) the map $\mathcal{E}^G(X) \to \mathcal{E}^U(X_\mathbb{Q})$ has finite kernel and image an arithmetic subgroup,
(iii) the holonomy representations of $\mathcal{E}^G(X)$ in the rational (co)homology groups of the space $B\operatorname{aut}_G(X)$ are restrictions of finite-dimensional algebraic representations of $\mathcal{E}^U(X_\mathbb{Q})$,
(iv) the space $B\operatorname{aut}_G(X)$ is virtually nilpotent and has Lie model $\operatorname{Der}^c_\mathfrak{u} L$.

Proof. The first claim follows from Lemma 2.7. The second claim follows from Theorem 2.23(ii) together with Lemma 2.9. The space $B\operatorname{aut}_G(X)$ is virtually nilpotent by Proposition 3.8.

The holonomy action of $\mathcal{E}^G(X)$ on the cohomology of $B\operatorname{aut}_G(X)$ may be computed via the left action on $\mathcal{E}^G(X)/\operatorname{aut}(X) \sim B\operatorname{aut}_G(X)$. By Lemma 3.29, the $\mathcal{E}^G(X)$-space $\mathcal{E}^G(X)/\operatorname{aut}(X)$ is rationally equivalent to $\mathcal{E}^U(X_\mathbb{Q})/\operatorname{aut}(X_\mathbb{Q})$, where $\mathcal{E}^G(X)$ acts by restriction of the left action of $\mathcal{E}^U(X_\mathbb{Q})$. We may identify $X_\mathbb{Q}$ with the nerve $\mathsf{MC}_\bullet(L)$. The combination of Lemma 3.16 and Lemma 3.22 shows that the $\operatorname{Aut} L$-space $\mathcal{E}^U(X_\mathbb{Q})/\operatorname{aut}(X_\mathbb{Q})$, where $\operatorname{Aut} L$ acts on the left via $\operatorname{Aut} L \to \mathcal{E}^U(X_\mathbb{Q})$, is weakly equivalent to the $\operatorname{Aut} L$-space $B\exp_\bullet(\operatorname{Der}^c_\mathfrak{u} L)$, where $\operatorname{Aut} L$ acts by conjugation. In particular, we get an $\operatorname{Aut} L$-equivariant isomorphism

$$H^*\big(\mathcal{E}^U(X_\mathbb{Q})/\operatorname{aut}(X_\mathbb{Q})\big) \cong H^*_{CE}(\operatorname{Der}^c_\mathfrak{u} L). \tag{16}$$

The right-hand side is manifestly an algebraic representation of the algebraic group $\mathcal{A}\mathrm{ut}\, L$. We may identify $\mathcal{E}^U(X_\mathbb{Q})$ with the $\mathbb{Q}$-points of the quotient of the affine algebraic group $\mathcal{A}\mathrm{ut}\, L$ by a normal unipotent algebraic subgroup $\mathcal{U}$. The action of $\operatorname{Aut} L$ factors through $\mathcal{E}^U(X_\mathbb{Q})$ in the left-hand side of (16). This means that the $\mathbb{Q}$-points $\mathcal{U}(\mathbb{Q})$ act trivially on this representation. Since $\mathcal{U}(\mathbb{Q})$ is Zariski dense in $\mathcal{U}$ (by *e.g.* [53, Theorem 17.93]), it follows that the representation is trivial as an algebraic representation of $\mathcal{U}$. Thus, it is an algebraic representation of $\mathcal{A}\mathrm{ut}\, L/\mathcal{U}$. This shows that the cohomology groups of $\mathcal{E}^U(X_\mathbb{Q})/\operatorname{aut}(X_\mathbb{Q})$ are algebraic representations of $\mathcal{E}^U(X_\mathbb{Q})$. □

Remark 3.35. Let us comment on the relation to [32]. If $G \leqslant \mathcal{E}(X)$ acts nilpotently on $H_*(X)$, then it acts unipotently on $H_*(X;\mathbb{Q})$ so by Lemma 3.33 we can find an algebraic unipotent subgroup $U \leqslant \mathcal{E}(X_\mathbb{Q})$ such that $q_*(G)$ includes into U by a $\mathbb{Q}$-isomorphism. One can check that $\operatorname{Der}^c_\mathfrak{u} L$ agrees with the model $\operatorname{Der}^\mathcal{G} L \widetilde{\rtimes} sL$ of [32, Theorem 0.1], where $\mathcal{G} \leqslant \operatorname{Aut}^h(L)$ corresponds to $U \leqslant \mathcal{E}(X_\mathbb{Q})$ under the isomorphism $\operatorname{Aut}^h(L) \cong \mathcal{E}(X_\mathbb{Q})$. The assumption that G and U are normal can be dropped in Theorem 3.34(iv) (see Remark 3.30). Thus, we recover the Lie model for $B\operatorname{aut}_G(X)$ of [32, Theorem 0.1]. Let us also remark that for $U = \mathcal{E}_u(X_\mathbb{Q})$, one can show that the dg Lie algebra $\operatorname{Der}^c_u L$ of Definition 3.15 agrees with the dg Lie algebra $\mathcal{D}er L$ considered in [32, Definition 7.4] if the filtration [32, (30)] is chosen to be a composition series as in (4).

A crucial advantage of our approach is that it lets us incorporate group actions. This aspect is not addressed in [32]. This is what allows us to prove the algebraicity statements in Theorem 3.34 and, in the next section, it will enable us to construct an algebraic model for the full space $B\operatorname{aut}(X)$, not only for the (virtually) nilpotent covers $B\operatorname{aut}_G(X)$.

Three interesting applications of Theorem 3.34 are obtained by choosing U to be the trivial subgroup, the kernel of the representation $H_*(X;\mathbb{Q})$, and the unipotent radical, respectively. Then, for suitable choices of G, the space $B\operatorname{aut}_G(X)$ may be identified with the universal cover of $B\operatorname{aut}(X)$, the homotopy Torelli space $B\operatorname{tor}(X)$, and the classifying space $B\operatorname{aut}_u(X)$ for normal unipotent fibrations, respectively.

Corollary 3.36. *Let X be a simply connected finite CW-complex. The representations of $\mathcal{E}(X)$ in the rational homotopy and (co)homology groups of the universal cover of $B\operatorname{aut}(X)$ are restrictions of finite-dimensional algebraic representations of the algebraic group $\mathcal{E}(X_\mathbb{Q})$.*

Proof. Apply Theorem 3.34 to the trivial subgroups $U = 1$ and $G = 1$.

Since the space $B\operatorname{aut}_1(X)$ is simply connected, there is also a natural holonomy action of $\mathcal{E}(X)$ on its rational homotopy groups. To show algebraicity of these representations, one argues as in the proof of Theorem 3.34 using the $\operatorname{Aut} L$-equivariant isomorphism

$$\pi_{k+1}(\mathcal{E}(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}}))\otimes\mathbb{Q}\cong H_k(\operatorname{Der}^c L)$$

for $k>0$. □

Remark 3.37. In the above case, the first two items of Theorem 3.34 are identical to the first two items of Theorem 2.23 and the fourth item recovers Tanré's model for $B\operatorname{aut}_1(X)$ [67, VII.4.(4)].

Kupers and Randal-Williams [49] recently established algebraicity and nilpotence results for Torelli groups of the manifolds W_g, *i.e.*, groups of diffeomorphisms that act trivially on integral homology. By analogy, we can define the *homotopy Torelli monoid* of a space X to be the monoid $\operatorname{tor}(X)$ of self-homotopy equivalences of X that act trivially on $H_*(X;\mathbb{Z})$. The following shows that a homotopical counterpart of the main result of [49] is valid for arbitrary simply connected finite complexes.

Note that the homotopy Torelli space sits in a homotopy fiber sequence

$$B\operatorname{tor}(X)\to B\operatorname{aut}(X)\to B\,\mathsf{GL}^{\mathcal{E}(X)}(H_*(X;\mathbb{Z})).$$

Corollary 3.38. *Let X be a simply connected finite CW-complex.*

(i) The group $\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ of isomorphisms of $H_*(X;\mathbb{Q})$ that are realizable by a self-homotopy equivalence of $X_{\mathbb{Q}}$ is a linear algebraic group,

(ii) the homomorphism $\mathsf{GL}^{\mathcal{E}(X)}(H_*(X;\mathbb{Z}))\to\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ has finite kernel and image an arithmetic subgroup,

(iii) the representations of $\mathsf{GL}^{\mathcal{E}(X)}(H_*(X;\mathbb{Z}))$ in the rational (co)homology groups of the homotopy Torelli space $B\operatorname{tor}(X)$ are restrictions of finite-dimensional algebraic representations of the linear algebraic group $\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$,

(iv) the homotopy Torelli space $B\operatorname{tor}(X)$ is nilpotent and has a Lie model described by the following: it agrees with $\operatorname{Der}^c L$ in positive degrees and in degree zero it consists of all derivations θ of L that commute with the differential and are decomposable in the sense that $\theta(L)\subseteq[L,L]$.

Proof. Clearly, $\operatorname{tor}(X)=\operatorname{aut}_G(X)$, where $G\leqslant\mathcal{E}(X)$ is the kernel of the action on $H_*(X;\mathbb{Z})$. Let $U\leqslant\mathcal{E}(X_{\mathbb{Q}})$ be the kernel of the algebraic representation $H_*(X;\mathbb{Q})$. Then U is unipotent by Theorem 2.23(iii). The cokernel of $G\to q_*^{-1}(U)$ may be identified with the kernel of

$$\mathsf{GL}^{\mathcal{E}(X)}(H_*(X;\mathbb{Z}))\to\mathsf{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q})),$$

which is clearly finite. Hence, the inclusion $q^*(G)\leqslant U$ is $\mathbb{Q}$-surjective by Lemma 3.31, so Theorem 3.34 applies.

The description of the Lie model follows from the fact that the action of a derivation $\theta\in\operatorname{Der} L$ on the reduced homology of X may be identified with the induced action on the indecomposables $L/[L,L]$. □

Corollary 3.39. *Let X be a simply connected finite CW-complex. Let $R(X)$ be the maximal reductive quotient of $\mathcal{E}(X_{\mathbb{Q}})$ and let $\Gamma(X)$ be the image of $\mathcal{E}(X)$ in $R(X)$.*

(i) $\Gamma(X)$ is an arithmetic subgroup of the linear algebraic group $R(X)$,

(ii) the classifying space $B\operatorname{aut}_u(X)$ for normal unipotent fibrations is virtually nilpotent with Lie model $\operatorname{nil}\operatorname{Der}^c L$,

(iii) the representations of $\Gamma(X)$ in the rational cohomology groups of $B\operatorname{aut}_u(X)$ are restrictions of finite-dimensional algebraic representations of $R(X)$.

Proof. Specialize Theorem 3.34 to the unipotent radical $U = \mathcal{E}_u(X_{\mathbb{Q}})$ and $G = \mathcal{E}_u(X) = q_*^{-1}(\mathcal{E}_u(X_{\mathbb{Q}}))$. The description of the Lie model follows from Lemma 3.20. □

3.5.2. *Algebraic Lie models.* Theorem 3.34 shows that the space $\mathcal{E}^G(X)/\operatorname{aut}(X)$ has Lie model $\operatorname{Der}^c_{\mathfrak{u}} L$, but this does not give an $\mathcal{E}^G(X)$-equivariant model, because there is no natural action of $\mathcal{E}^G(X)$ on $\operatorname{Der}^c_{\mathfrak{u}} L$ in general. However, as discussed before Theorem 3.23, when we pass to the unipotent radical, such an action exists and we can strengthen Theorem 3.34.

Theorem 3.40. *Let X be a simply connected finite CW-complex with minimal Quillen model L. Let $G \leqslant \mathcal{E}(X)$ be a normal subgroup such that $q_*(G) \leqslant \mathcal{E}_u(X_{\mathbb{Q}})$ and the inclusion is $\mathbb{Q}$-surjective. Then the $\mathcal{E}^G(X)$-space $\mathcal{E}^G(X)/\operatorname{aut}(X)$ is rationally equivalent to the nerve of the dg Lie algebra*

$$\mathfrak{g}(X) = \operatorname{nil}\operatorname{Der}^c L,$$

on which $\mathcal{E}^G(X)$ acts through the map $\mathcal{E}^G(X) \to R(X) = \mathcal{E}^U(X_{\mathbb{Q}})$ and through any choice of splitting of $\operatorname{Aut} L \to R(X)$.

As a consequence, the space $B\operatorname{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(X)\rangle_{h\,\mathcal{E}^G(X)}$.

Proof. Lemma 3.29 applied to the unipotent radical $U = \mathcal{E}_u(X_{\mathbb{Q}})$ and G shows that $\mathcal{E}^G(X)/\operatorname{aut}(X)$ is rationally equivalent to $R(X)/\operatorname{aut}(X_{\mathbb{Q}})$ as a $\mathcal{E}^G(X)$-space, where $\mathcal{E}^G(X)$ acts through the homomorphism $\mathcal{E}^G(X) \to \mathcal{E}^U(X_{\mathbb{Q}}) = R(X)$. By Theorem 3.23, the latter space is rationally equivalent to $B\exp_\bullet(\mathfrak{g}(X))$ as an $R(X)$-space. Hence,

$$B\operatorname{aut}(X) \sim \big(\mathcal{E}^G(X)/\operatorname{aut}(X)\big)_{h\,\mathcal{E}^G(X)} \sim_{\mathbb{Q}} \langle \mathfrak{g}(X)\rangle_{h\,\mathcal{E}^G(X)}.$$ □

Remark 3.41. Existence of an algebraic Lie model immediately implies algebraicity of (co)homology, because $H_n(\langle\mathfrak{g}\rangle)$ is naturally isomorphic $H_n^{CE}(\mathfrak{g})$, which is manifestly algebraic if $\mathfrak{g}$ is.

Remark 3.42. We expect that Theorem 3.34 can be strengthened to the existence of an $\mathcal{E}^U(X_{\mathbb{Q}})$-algebraic Lie model for $\mathcal{E}^U(X_{\mathbb{Q}})/\operatorname{aut}(X_{\mathbb{Q}})$ for arbitrary normal unipotent subgroups $U \leqslant \mathcal{E}(X_{\mathbb{Q}})$. The proof of Theorem 3.40 does not go through because one can not rectify the homotopy action of $\mathcal{E}^U(X_{\mathbb{Q}})$ on $\operatorname{Der}^c_{\mathfrak{u}} L$ to an algebraic action unless $\mathcal{E}^U(X_{\mathbb{Q}})$ is reductive (*cf.* §4.4 below). It is however conceivable that a suitable modification of $\operatorname{Der}^c_{\mathfrak{u}} L$ could accommodate such an action. Since the strongest applications of the existence of algebraic Lie models are obtained by passing to the reductive quotient and using the model in Theorem 1.2, we will not pursue this question further here. We leave it as a challenge to the interested reader.

Remark 3.43. The results in this section have obvious analogs for simply connected finite Postnikov sections, where the minimal Quillen model L is replaced by the mimimal Sullivan model Λ.

By taking $G = \mathcal{E}_u(X)$, we get $\mathcal{E}^G(X) = \Gamma(X)$ so the combination of Theorem 3.34 and Theorem 3.40 specializes to the first part of Theorem 1.2. The last part of Theorem 1.2 is obtained by specializing the following result to the trivial $\mathbb{Q}\Gamma(X)$-module $M = \mathbb{Q}$.

Theorem 3.44. *For every cochain complex of $\mathbb{Q}\Gamma(X)$-modules M, there is a natural quasi-isomorphism*

$$\Omega^*(B\operatorname{aut}(X); M) \sim \Omega^*(\Gamma(X); C^*_{CE}(\mathfrak{g}(X)) \otimes M).$$

If M is a commutative cochain algebra, then this is a quasi-isomorphism of commutative cochain algebras.

Proof. It follows from Theorem 3.40 that $\Omega^*(B\operatorname{aut}(X);M)$ is quasi-isomorphic to $\Omega^*(\langle\mathfrak{g}(X)\rangle_{h\Gamma(X)};M)$. By Proposition 2.21, the latter is quasi-isomorphic to $\Omega^*(\Gamma(X),\Omega^*(\langle\mathfrak{g}(X)\rangle)\otimes M)$. The proof is completed by using the natural quasi-isomorphism $C^*_{CE}(\mathfrak{g}(X))\to\Omega^*(\langle\mathfrak{g}(X)\rangle)$. □

The most striking consequences of Theorem 3.40 come from combining it with semisimplicity of algebraic representations of reductive groups. The following result specialized to $M=\mathbb{Q}$ proves Corollary 1.3.

Corollary 3.45. *For any $\mathbb{Q}\Gamma(X)$-module M, there is an isomorphism*

$$H^*(B\operatorname{aut}(X);M)\cong H^*(\Gamma(X),H^*_{CE}(\mathfrak{g}(X))\otimes M).$$

When M is an algebra, this isomorphism is one of graded algebras.

Proof. The Chevalley–Eilenberg complex $C^*_{CE}(\mathfrak{g}(X))$ is a cochain complex of finite-dimensional algebraic representations of $R(X)$. Since $R(X)$ is reductive, every such representation is semisimple, and thus the cochain complex $C^*_{CE}(\mathfrak{g}(X))$ is split, which means that there is a contraction of cochain complexes in $\operatorname{Rep}_{\mathbb{Q}}(R(X))$,

$$\begin{tikzcd} C^*_{CE}(\mathfrak{g}(X)) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & H^*_{\mathrm{CE}}(\mathfrak{g}(X)) \arrow[l, shift left, "\nabla"] \end{tikzcd}$$

where p and ∇ are chain maps with $p(z)=[z]$ whenever $z\in C^*(\mathfrak{g}(X))$ is a cocycle, $p\nabla=\mathrm{id}$, and h is a chain homotopy: $\mathrm{id}-\nabla p=dh+hd$, see *e.g.* [15, Lemma B.1].

Applying the dg functor $\Omega^*(\Gamma(X);-\otimes M)$ yields a new contraction

$$\begin{tikzcd} \Omega^*(\Gamma(X),C^*_{CE}(\mathfrak{g})\otimes M) \arrow[loop left, "h_*"] \arrow[r, shift left, "p_*"] & \Omega^*(\Gamma(X),H^*_{\mathrm{CE}}(\mathfrak{g})\otimes M) \arrow[l, shift left, "\nabla_*"] \end{tikzcd}$$

Passing to cohomology proves the first claim.

When M is an algebra, we can use the homotopy transfer theorem (see *e.g.* [10]) to produce a C_∞-algebra structure $\{\mu_n\}_{n\geq 2}$ on $A=\Omega^*(\Gamma(X);H^*_{CE}(\mathfrak{g}(X))\otimes M)$ and an extension of ∇_* to a C_∞-quasi-isomorphism. The multiplication in this C_∞-algebra structure is given by $\mu_2=p_*m_2(\nabla_*\otimes\nabla_*)$, where m_2 is the multiplication on $\Omega^*(\Gamma(X);H^*_{CE}(\mathfrak{g}(X))\otimes M)$, and one checks that this agrees with the multiplication on $\Omega^*(\Gamma(X);H^*_{CE}(\mathfrak{g}(X))\otimes M)$, coming from viewing $H^*_{CE}(\mathfrak{g}(X))$ as a $\Gamma(X)$-cdga with trivial differential. The cdga (A,μ_2) is of course not equivalent to the C_∞-algebra $(A,\mu_2,\mu_3,\ldots)$ in general, but they obviously have isomorphic cohomology rings. □

The problem of determining an algebraic model for the space $B\operatorname{aut}(X)$ has been raised in [38, §7.3]. The approach taken in [38] is to study the fiberwise rationalization of $B\operatorname{aut}(X)\to B\,\mathcal{E}(X)$ by using local systems, in the sense of §2.4, of cdgas over $B\,\mathcal{E}(X)$. This notion of a local system is more general, and more involved, than cdgas with an action of $\mathcal{E}(X)$. Our approach to finding an algebraic model for $B\operatorname{aut}(X)$ is to instead consider the fiberwise rationalization of $B\operatorname{aut}(X)\to B\Gamma(X)$. This has two significant advantages: Firstly, we can work with cdgas of algebraic $\Gamma(X)$-representations instead of the more complicated local systems over $B\Gamma(X)$. Secondly, semisimplicity of these algebraic representations lets us construct a $\Gamma(X)$-equivariant minimal model explicitly using familiar methods, as we will now explain. This represents a possible solution to Problem 3 of [38, §7.3].

Corollary 3.46. *There exists a minimal Sullivan model $\mathfrak{M}(X)$ for $\Gamma(X)/\operatorname{aut}(X)$ whose graded components admit the structure of finite-dimensional algebraic $R(X)$-representations, such that the spatial realization $\langle \mathfrak{M}(X)\rangle$ is rationally equivalent to $\Gamma(X)/\operatorname{aut}(X)$ as a $\Gamma(X)$-space, whence $B\operatorname{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{M}(X)\rangle_{h\Gamma(X)}$.*

Proof. Let $\mathfrak{g}(X)$ be an algebraic Lie model for $\Gamma(X)/\operatorname{aut}(X)$ as in Theorem 1.2. By semisimplicity of $\operatorname{Rep}_{\mathbb{Q}}(R(X))$, we can find a contraction of chain complexes of algebraic $R(X)$-representations

$$h \circlearrowright \mathfrak{g}(X) \underset{g}{\overset{f}{\rightleftarrows}} H_*(\mathfrak{g}(X)).$$

Applying [10, Theorem 1.3], we get a contraction of cocommutative chain coalgebras in $\operatorname{Rep}_{\mathbb{Q}}(R(X))$,

$$H' \circlearrowright C_*^{CE}(\mathfrak{g}(X)) \underset{G'}{\overset{F'}{\rightleftarrows}} (\Lambda^c(V), d'),$$

where V is the suspension of $H_*(\mathfrak{g}(X))$, the differential d' is given by the explicit formula

$$d' = FtG + FtHtG + Ft(Ht)^2G + \cdots ,$$

and F', G', H' are given by similar formulas. Here, t is the quadratic part of the differential of $C_*^{CE}(\mathfrak{g}(X))$ that encodes the Lie bracket, F and G are the morphisms of dg coalgebras induced by f and g, and H is the 'symmetrized tensor trick homotopy' [10, §5]. The dual of $(\Lambda^c(V), d')$ will be the minimal model $\mathfrak{M}(X)$. We stress that this construction is completely explicit once the contraction between $\mathfrak{g}(X)$ and its homology has been fixed. □

4. Case studies

In this section we offer a few case studies. In addition to showcasing how the main results can be used in practice, they illustrate certain general points:

(i) Determining $\Gamma(X)$ typically entails some non-trivial integral homotopy theory, but is often easier than determining $\mathcal{E}(X)$. In fact, in many cases $\Gamma(X)$ is the group of automorphisms of $H_*(X;\mathbb{Z})$ that are induced by self-homotopy equivalences. In the literature, this group often appears as a stepping stone for computing $\mathcal{E}(X)$ or other groups of automorphisms of X.

(ii) Even in cases where $H^*_{CE}(\mathfrak{g}(X))$ is explicitly computable, a complete calculation of the cohomology $H^*(\Gamma(X), H^*_{CE}(\mathfrak{g}(X)))$ is in general out of reach, due to the difficulty of computing cohomology of arithmetic groups. However, in some cases the cohomology, or parts of it, can be understood via automorphic forms. A paradigmatic example is the Eichler–Shimura isomorphism.

(iii) By contrast, the calculation of the invariant ring

$$H^0(\Gamma(X), H^*_{CE}(\mathfrak{g}(X))) = H^*_{CE}(\mathfrak{g}(X))^{\Gamma(X)}$$

is more tractable. By employing structural results for affine algebraic groups over $\mathbb{Q}$ and density results [18], this can often be reduced to classical invariant theory for finite or reductive groups, which is well understood.

(iv) For a graded commutative algebra H in algebraic representations of a reductive group G and an arithmetic subgroup Γ of $G(\mathbb{Q})$, the split exact sequence

$$0 \to H^G \to H \to H/H^G \to 0$$

gives rise to a split exact sequence

$$0 \to H^*(\Gamma, \mathbb{Q}) \otimes H^G \cong H^*(\Gamma, H^G) \to H^*(\Gamma, H) \to H^*(\Gamma, H/H^G) \to 0.$$

In many cases of interest, stability and vanishing results for the cohomology of arithmetic groups with coefficients in algebraic representations, as in Borel's work [19, 20], show that the cokernel vanishes in a range of degrees. Thus, in some 'stable range', the cohomology of $B\operatorname{aut}(X)$ is isomorphic to

$$H^*(\Gamma(X), \mathbb{Q}) \otimes H^*_{CE}(\mathfrak{g}(X))^{R(X)}.$$

However, how non-trivial this 'stable range' is depends on the group $\Gamma(X)$ and the representations $H^*_{CE}(\mathfrak{g}(X))$.

4.1. **Products of spheres.** Consider the n-fold product of a d-dimensional sphere,

$$S^{d\times n} = S^d \times \ldots \times S^d.$$

We begin by describing the reductive group $R(S^{d\times n})$. The minimal Quillen model can be described explicitly, see [67, V.2.(3)], but the minimal Sullivan model is finitely generated and even easier to describe in this case, so we will work with the latter.

Proposition 4.1. *The automorphism group of the minimal Sullivan model Λ for $S^{d\times n}$ is given by*

$$\operatorname{Aut}\Lambda \cong \begin{cases} \mathsf{GL}_n(\mathbb{Q}), & d \text{ odd}, \\ \Sigma_n \ltimes (\mathbb{Q}^\times)^n, & d \text{ even}. \end{cases}$$

In particular, $\operatorname{Aut}\Lambda$ *is reductive, whence* $R(S^{d\times n}) \cong \operatorname{Aut}\Lambda$.

Proof. For d odd, the minimal model Λ is an exterior algebra $\Lambda(x_1, \ldots, x_n)$ on generators of degree d with zero differential. Clearly, $\operatorname{Aut}\Lambda \cong \mathsf{GL}_n(\mathbb{Q})$. For d even, the minimal model has the form

$$\Lambda = \big(\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n), d\big),$$

with $|x_i| = d$, $|y_i| = 2d-1$ and $dx_i = 0$, $dy_i = x_i^2$. Let φ be an automorphism of Λ. Then

$$\varphi(x_i) = \sum_j a_{ij} x_j$$

for some $A = (a_{ij}) \in \mathsf{GL}_n(\mathbb{Q})$. In cohomology, the equality

$$0 = \varphi(x_i^2) = \varphi(x_i)^2 = \sum_{j<k} 2a_{ij}a_{ik}x_jx_k,$$

implies $a_{ij}a_{ik} = 0$ for all $j \neq k$, whence exactly one entry in each row $(a_{i1}, \ldots, a_{in})$ must be non-zero. Therefore, $\varphi(x_i) = \lambda_i x_{\sigma(i)}$ for some permutation σ and some $\lambda_i \in \mathbb{Q}^\times$. Since $d\varphi(y_i) = \varphi(dy_i) = \lambda_i^2 x_{\sigma(i)}^2$, the only possibility is $\varphi(y_i) = \lambda_i^2 y_{\sigma(i)}$. This shows that every automorphism φ of Λ is of the form

$$\varphi(x_i) = \lambda_i x_{\sigma(i)}, \quad \varphi(y_i) = \lambda_i^2 y_{\sigma(i)},$$

for some $\sigma \in \Sigma_n$ and $\lambda_i \in \mathbb{Q}^\times$. One checks that this yields an isomorphism $\operatorname{Aut}\Lambda \cong \Sigma_n \ltimes (\mathbb{Q}^\times)^n$. □

Remark 4.2. The group $\Sigma_n \ltimes (\mathbb{Q}^\times)^n$ may be identified with the group of 'monomial matrices', i.e., invertible matrices with exactly one non-zero entry in each row. This is an example of a disconnected reductive group. The identity component is the torus $(\mathbb{Q}^\times)^n$ and the group of components is Σ_n.

Remark 4.3. The preceding result, as well as the remainder of this section, goes through even for $d = 1$. The cautious reader will object on the grounds that the space $S^{1\times n} = \big(S^1\big)^n$ is not simply connected. It is, however, nilpotent (indeed a topological group), and therefore, by the remark made in §3.4.1, amenable to analysis by our methods.

We now turn to the determination of the group $\Gamma(S^{d\times n})$. For this, we first need to work out some elementary homotopy theory of maps between products of spheres.

Definition 4.4. Let us call an integer vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ *realizable* if there is a map

$$S^d \times \ldots \times S^d \to S^d$$

such that the restriction to the ith factor is a degree a_i self-map of S^d. This is equivalent to asking the n-fold higher order Whitehead product

$$\big[a_1\iota_d, \dots, a_n\iota_d\big] \subseteq \pi_{nd-1}(S^d)$$

to be defined and contain 0, where $\iota_d \in \pi_d(S^d)$ is the class of the identity map.

We would be surprised if the following has not been observed before, but we have not found a reference (except for the simplest case $n = 2$, which is discussed in *e.g.* [3, Example 5.1]), so we supply a proof.

Proposition 4.5.

(i) For $d = 1, 3, 7$, every integer vector $(a_1, \dots, a_n)$ is realizable.
(ii) For d odd $\neq 1, 3, 7$, $(a_1, \dots, a_n)$ is realizable if and only if at most one a_i is odd.
(iii) For d even, $(a_1, \dots, a_n)$ is realizable if and only if at most one a_i is non-zero.

Lemma 4.6. *If $(a_1, \dots, a_n)$ is realizable, then so are the vectors*

$$(a_{\sigma_1}, \dots, a_{\sigma_k}), \quad (\lambda_1 a_1, \dots, \lambda_n a_n), \quad (a_1, \dots, a_n, 0),$$

for all injective maps $\sigma\colon \{1, \dots, k\} \to \{1, \dots, n\}$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$.

Proof. Precompose the given realizable map $S^d \times \ldots \times S^d \to S^d$ with the map that includes the k-fold product of S^d according to σ and inserts the basepoint in the other factors, or with the map $\lambda_1 \times \ldots \times \lambda_n$, or with the projection onto the first n factors, respectively. □

Lemma 4.7. *If $(1, a_2, \dots, a_n)$ and $(b_1, \dots, b_n)$ are realizable, then so is*

$$(b_1, b_2 + a_2, \dots, b_n + a_n).$$

Proof. By hypothesis, the matrices on the left-hand side of the equation

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 + a_2 & \cdots & b_n + a_n \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

can be realized as self-maps of $S^{d\times n}$. It follows that the same is true of the matrix in the right-hand side. In particular its first row is realizable. □

Lemma 4.8. *If $(1, a)$ is realizable, then so is $(1, a, \dots, a) \in \mathbb{Z}^n$ for every $n \geq 2$.*

Proof. Assume by induction that $(1, a, \dots, a) \in \mathbb{Z}^{n-1}$ is realizable. Then both $(1, a \dots, a, 0) \in \mathbb{Z}^n$ and $(1, 0, \dots, 0, a) \in \mathbb{Z}^n$ are realizable by Lemma 4.6, and hence $(1, a, \dots, a) \in \mathbb{Z}^n$ is realizable by Lemma 4.7. □

Proof of Proposition 4.5. For $d = 1, 3, 7$, the fact that S^d is an H-space means precisely that $(1, 1)$ is realizable. It follows from Lemma 4.8 that $(1, \ldots, 1) \in \mathbb{Z}^n$ is realizable and then from Lemma 4.6 that $(a_1, \ldots, a_n)$ is realizable for all $a_1, \ldots, a_n \in \mathbb{Z}$.

For d odd $\neq 1, 3, 7$, it is well known that the Whitehead product

$$[\iota_d, \iota_d] \in \pi_{2d-1}(S^d) \tag{17}$$

is a non-zero class of order 2 (this can be seen, *e.g.*, by inspecting the EHP sequence). As noted above, (a_1, a_2) is realizable precisely when the Whitehead product $[a_1\iota_d, a_2\iota_d]$ is trivial. Since the binary Whitehead product is bilinear, this happens if and only if a_1a_2 is even. By the first part of Lemma 4.6, this implies that $(a_1, \ldots, a_n)$ is realizable only if a_ia_j is even for all $i \neq j$, which implies that at most one a_i is odd. Conversely, we have that $(1, 2)$ is realizable since $[\iota_d, 2\iota_d] = 0$. Hence so is $(1, 2, \ldots, 2)$ by Lemma 4.8. Now one can use Lemma 4.6 to deduce that $(a_1, \ldots, a_n)$ is realizable if at most one of the entries is odd.

For d even, the Whitehead product (17) is a non-zero element of infinite order. As above, this implies that $(a_1, \ldots, a_n)$ is realizable only if at most one a_i is non-zero. Conversely, Lemma 4.6 shows that $(a_1, \ldots, a_n)$ is realizable if at most one entry is non-zero. □

Now we are ready to compute $\Gamma(S^{d\times n})$.

Proposition 4.9. *We have*

$$\Gamma(S^{d\times n}) \cong \begin{cases} \mathsf{GL}_n(\mathbb{Z}), & d = 1,3,7, \\ \mathsf{GL}_n^{\Sigma}(\mathbb{Z}), & d \text{ odd} \neq 1,3,7, \\ \Sigma_n^{\pm}, & d \text{ even}, \end{cases}$$

where $\mathsf{GL}_n^{\Sigma}(\mathbb{Z})$ denotes the group of invertible $n \times n$ integer matrices with exactly one odd entry in each row, and $\Sigma_n^{\pm}$ denotes the group of $n \times n$ signed permutation matrices.

Proof. By Corollary 1.7, we may identify $\Gamma(S^{d\times n})$ with the group of automorphisms of $H_*(S^{d\times n}; \mathbb{Z})$ that are realizable by a self-homotopy equivalence.

Given $A \in \mathsf{GL}_n(\mathbb{Z})$, it is clear how to write down a map

$$S^d \vee \cdots \vee S^d \to S^d \times \cdots \times S^d$$

that realizes A on $H_d(-; \mathbb{Z})$. This extends to the product if and only if the projection to each factor S^d does, which is precisely the condition that each row in A is realizable. When an extension exists, it follows from the Whitehead theorem that it is a homotopy equivalence. Thus, $\Gamma(S^{d\times n})$ may be identified with the group of invertible $n\times n$ integer matrices in which each row is realizable. To finish the proof, invoke Proposition 4.5. (In the case d odd $\neq 1, 3, 7$, note that invertibility of the matrix implies that at least one entry in each row of must be odd. Similarly, in the case d even, invertibility of the matrix implies that there is a unique non-zero entry in each row and that this must be a unit.) □

Remark 4.10. The group $\mathsf{GL}_n^{\Sigma}(\mathbb{Z})$ may be identified with the semidirect product,

$$\mathsf{GL}_n^{\Sigma}(\mathbb{Z}) \cong \Sigma_n \ltimes \mathsf{GL}_n(\mathbb{Z}, 2),$$

where $\mathsf{GL}_n(\mathbb{Z}, 2) \leqslant \mathsf{GL}_n(\mathbb{Z})$ denotes the principal level 2 congruence subgroup, i.e., the kernel of the homomorphism $\mathsf{GL}_n(\mathbb{Z}) \to \mathsf{GL}_n(\mathbb{Z}/2\mathbb{Z})$ that reduces the entries mod 2, and where the symmetric group Σ_n acts by simultaneous permutation of the rows and columns.

The signed permutation group $\Sigma_n^{\pm}$, also known as the hyperoctahedral group, admits a similar decomposition,

$$\Sigma_n^{\pm} \cong \Sigma_n \ltimes D_n(\mathbb{Z}),$$

where $D_n(\mathbb{Z}) \cong (\mathbb{Z}^\times)^n$ is the group of diagonal matrices in $\mathsf{GL}_n(\mathbb{Z})$.

Remark 4.11. The group of homology isomorphisms that are realizable by a self-homotopy equivalence of $S^{d\times n}$ has also been determined by Basu–Farrell [6, §2]. The proof given here is simpler because we do not need to argue using generators for the groups involved.

Work of Lucas–Saeki [50] shows that $\Gamma(S^{d\times n})$ also agrees with the group of homology isomorphisms of $S^{d\times n}$ that are realizable by a diffeomorphism.

The group $\Gamma(S^{d\times 2})$ agrees with the group G_d that is used as a stepping stone in Baues' computation of the group of self-homotopy equivalences of $S^d \times S^d$ [7, §6].

Remark 4.12. If we let $\Gamma^+(S^{d\times n})$ denote the image in $R(S^{d\times n})$ of the orientation preserving self-homotopy equivalences, then it is easily seen that $\Gamma^+(S^{d\times n}) = \Gamma(S^{d\times n}) \cap \mathsf{SL}_n(\mathbb{Z})$ for d odd, and that $\Gamma^+(S^{d\times n}) \leqslant \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ is the subgroup of all (σ, λ) such that $\lambda_1 \dots \lambda_n = 1$ for d even.

Next, we determine an algebraic Lie model for $\Gamma(S^{d\times n})/\operatorname{aut}(S^{d\times n})$.

Proposition 4.13. *Let d be odd. The space $\Gamma(S^{d\times n})/\operatorname{aut}(S^{d\times n})$ admits an algebraic Lie model of the form*

$$\mathfrak{g}(S^{d\times n}) = V_n^*[-d],$$

where the right-hand side is the dual of the standard representation $V_n = \mathbb{Q}^n$ concentrated in homological degree d. The differential and the Lie bracket are trivial.

Proof. When d is odd, the Sullivan minimal model of $S^{d\times n}$ is the exterior algebra $\Lambda = \Lambda(x_1, \dots, x_n)$ with zero differential and with x_i in cohomological degree d. By Theorem 1.10, the space $\Gamma(S^{d\times n})/\operatorname{aut}(S^{d\times n})$ has algebraic Lie model

$$\mathfrak{g}(S^{d\times n}) = \operatorname{nil}\operatorname{Der}\Lambda.$$

The differential is trivial. The degree 0 component is zero, because by Lemma 3.25 it is isomorphic to the Lie algebra of the unipotent radical of the algebraic group $\operatorname{Aut}\Lambda$, which is isomorphic to $\mathsf{GL}_n(\mathbb{Q})$ and hence reductive. Recall that we are using the convention that cohomological degrees are regarded as negative homological degrees. The only derivations of Λ of positive homological degree are therefore

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}. \tag{18}$$

Thus, $\mathfrak{g}(S^{d\times n})$ is an abelian dg Lie algebra with zero differential with basis (18) in degree d. As a representation of GL_n, it is dual to the standard representation. □

Remark 4.14. The Chevalley–Eilenberg cochain algebra of the abelian Lie algebra $V_n^*[-d]$ may be identified with $\operatorname{Sym}(V_n[d+1])$, the polynomial algebra on the standard $\mathsf{GL}_n(\mathbb{Q})$-representation $V_n = \mathbb{Q}^n$ concentrated in cohomological degree $d+1$ with trivial differential. Hence, there are isomorphisms of graded algebras of $\Gamma(S^{d\times n})$-modules,

$$H^*(B\operatorname{aut}_u(S^{d\times n}); \mathbb{Q}) \cong H^*_{CE}(\mathfrak{g}(S^{d\times n})) \cong \operatorname{Sym}(V_n[d+1]).$$

This can be given a more geometric interpretation. The above implies that the evident map

$$B\operatorname{aut}_u(S^d) \times \dots \times B\operatorname{aut}_u(S^d) \to B\operatorname{aut}_u(S^d \times \dots \times S^d)$$

is a rational equivalence. This may be interpreted as a splitting principle of sorts: every normal unipotent $S^{d\times n}$-fibration is rationally equivalent to the 'Whitney sum' of n normal unipotent S^d-fibrations. Since $H^*(B\operatorname{aut}_u(S^d);\mathbb{Q})$ is a polynomial ring in the Euler class, we can say that the ring of rational characteristic classes of normal unipotent $S^{d\times n}$-fibration is a polynomial ring in the Euler classes $e_1,\dots,e_n$ of the associated S^d-fibrations.

By combining Proposition 4.13, Proposition 4.9, and Corollary 1.3, we obtain

Theorem 4.15. *For d odd, there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}^+(S^{d\times n});\mathbb{Q}) \cong \begin{cases} H^*\big(\mathsf{SL}_n(\mathbb{Z}), \operatorname{Sym}^\bullet(V_n[d+1])\big), & d=1,3,7,\\ H^*\big(\mathsf{SL}_n^\Sigma(\mathbb{Z}), \operatorname{Sym}^\bullet(V_n[d+1])\big), & d\neq 1,3,7,\end{cases} \tag{19}$$

where $V_n=\mathbb{Q}^n$ is the standard representation of $\mathsf{GL}_n(\mathbb{Q})$ and $\mathsf{SL}_n^\Sigma(\mathbb{Z}) = \mathsf{GL}_n^\Sigma(\mathbb{Z})\cap \mathsf{SL}_n(\mathbb{Z})$. □

For $n=2$ the right-hand side of (19) can be computed in terms of modular forms via the Eichler–Shimura isomorphism, as we now will discuss.

Let $V=\mathbb{C}^2$ denote the standard representation of $\mathsf{GL}_2(\mathbb{C})$ and let $\widetilde{\Gamma}\leqslant \mathsf{GL}_2(\mathbb{Z})$ be a congruence subgroup containing $-I$ and strictly containing $\Gamma=\widetilde{\Gamma}\cap\mathsf{SL}_2(\mathbb{Z})$, so that we have an exact sequence

$$1\to\Gamma\to\widetilde{\Gamma}\xrightarrow{\det}\mathbb{Z}^\times\to 1.$$

This gives rise to an action of $\mathbb{Z}^\times$ on $H^*(\Gamma,\operatorname{Sym}^\bullet(V))$, i.e. an involution, such that

$$H^*(\widetilde{\Gamma},\operatorname{Sym}^\bullet(V))\cong H^*(\Gamma,\operatorname{Sym}^\bullet(V))_+,$$

where $+$ indicates the $+1$ eigenspace of the involution.

Lemma 4.16. *There is an isomorphism of graded vector spaces with involution,*

$$H^*(\Gamma,\operatorname{Sym}^\bullet(V))\cong \mathbb{C}[0]^+\oplus M_{\bullet+2}(\Gamma)[1]^-\oplus S_{\bullet+2}(\Gamma)[1]^+, \tag{20}$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ, and where $\mathbb{C}[0]$ is $\mathbb{C}$ concentrated in $\bullet=0$ and $*=0$. A superscript $\pm$ indicates how the involution acts. In particular, extraction of the $+1$ eigenspace yields

$$H^*(\widetilde{\Gamma},\operatorname{Sym}^\bullet(V))\cong\mathbb{C}[0]\oplus S_{\bullet+2}(\Gamma)[1]. \tag{21}$$

Proof. The Eichler–Shimura isomorphism gives an isomorphism

$$ES\colon M_k(\Gamma)\oplus\overline{S_k(\Gamma)}\xrightarrow{\cong}H^1(\Gamma,\operatorname{Sym}^{k-2}(V)), \tag{22}$$

where $M_k(\Gamma)$ is the space of modular forms of weight k for Γ and $\overline{S_k(\Gamma)}$ is the space of antiholomorphic cusp forms of weight k, see *e.g.* [70, §6]. Identifying the involution on the left-hand side of (22) requires a little care.

The space of modular forms decomposes as $M_k(\Gamma)=E_k(\Gamma)\oplus S_k(\Gamma)$, where $E_k(\Gamma)$ denotes the Eisenstein space. Letting r denote the automorphism of the upper half plane given by $r(z)=-\overline{z}$, one can check that the composite

$$E_k(\Gamma)\oplus S_k(\Gamma)\oplus S_k(\Gamma)\xrightarrow{\varphi}M_k(\Gamma)\oplus\overline{S_k(\Gamma)}\xrightarrow{ES}H^1(\Gamma,\operatorname{Sym}^{k-2}(V)),$$

where

$$\varphi(e,f,g)=(e+f+g,fr-gr)$$

is an isomorphism of vector spaces with involution, where the involution on the left-hand side is given by $(e,f,g)\mapsto(-e,-f,g)$. This explains the isomorphism (20) for $*=1$. Restriction to the $+1$ eigenspace yields an isomorphism

$$S_k(\Gamma)\to H^1(\Gamma,\operatorname{Sym}^{k-2}(V))_+$$

given by $g \mapsto ES(g, -gr)$.

For the other cohomological degrees, one checks that

$$H^0(\Gamma, \mathrm{Sym}^{k-2}(V)) = \mathrm{Sym}^{k-2}(V)^\Gamma \cong \mathbb{C},$$

and $H^i(\Gamma, \mathrm{Sym}^{k-2}(V)) = 0$ for $i > 1$, because the virtual cohomological dimension of any finite index subgroup of $\mathsf{SL}_2(\mathbb{Z})$ is 1. □

Assembly of the above considerations yields

Theorem 4.17. *For d odd, there is an isomorphism of graded vector spaces with involution*

$$\widetilde{H}^*(B\,\mathrm{aut}^+(S^d \times S^d); \mathbb{Q}) \cong \bigoplus_k \big(M_k(\Gamma)^- \oplus S_k(\Gamma)^+\big)[(k-2)(d+1)+1],$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ, and where $\Gamma = \mathsf{SL}_2(\mathbb{Z})$ for $d = 1,3,7$ and $\Gamma = \mathsf{SL}_2^\Sigma(\mathbb{Z})$ for $d \neq 1,3,7$.

In particular, since the reduced cohomology is concentrated in odd degrees, it follows that the space $B\,\mathrm{aut}^+(S^d \times S^d)$ is formal with trivial cohomology ring. □

Define the Poincaré series of a graded vector space H^* with involution to be the formal power series in z with coefficients in $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$ given by

$$\sum_{k \geq 0} \Big(\dim(H_+^k) + \dim(H_-^k)\epsilon\Big) z^k.$$

There are well-known dimension formulas for spaces of modular forms, so as a corollary we get a formula for the Poincaré series of the cohomology of $B\,\mathrm{aut}^+(S^d \times S^d)$.

Corollary 4.18. *The Poincaré series of the graded vector space with involution*

$$H^*(B\,\mathrm{aut}^+(S^d \times S^d); \mathbb{Q})$$

is given by the following formulas, where we set $\ell = d+1$. For $d = 1,3,7$:

$$1 + z^{2\ell+1}\frac{\epsilon\big(1 + z^{4\ell} - z^{6\ell}\big) + z^{8\ell}}{(1 - z^{4\ell})(1 - z^{6\ell}).}$$

For d odd $\neq 1,3,7$:

$$1 + z\frac{\epsilon\big(1 + z^{2\ell} - z^{4\ell}\big) + z^{6\ell}}{(1 - z^{2\ell})(1 - z^{4\ell}).}$$

Proof. As is well known, the ring of modular forms $M(\mathsf{SL}_2(\mathbb{Z}))$ is freely generated by the Eisenstein series E_4 and E_6, and the space of cusp forms $S(\mathsf{SL}_2(\mathbb{Z}))$ is the principal ideal generated by the discriminant Δ, which is of weight 12. In particular, the Poincaré series are given by

$$\sum_{k \geq 0} \dim M_k(\mathsf{SL}_2(\mathbb{Z}))t^k = \frac{1}{(1-t^4)(1-t^6)},$$

$$\sum_{k \geq 0} \dim S_k(\mathsf{SL}_2(\mathbb{Z}))t^k = \frac{t^{12}}{(1-t^4)(1-t^6)}.$$

The group $\mathsf{SL}_2^\Sigma(\mathbb{Z})$ is an index 3 subgroup of $\mathsf{SL}_2(\mathbb{Z})$, sometimes referred to as the 'theta group'. It is generated by the two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The Poincaré series of the modular forms and the cusp forms for the theta group are

$$\sum_{k\geq 0} \dim M_k(\mathsf{SL}_2^\Sigma(\mathbb{Z}))t^k = \frac{1}{(1-t^2)(1-t^4)},$$
$$\sum_{k\geq 0} \dim S_k(\mathsf{SL}_2^\Sigma(\mathbb{Z}))t^k = \frac{t^8}{(1-t^2)(1-t^4)}.$$

This can be seen from the dimension formulas in [47, Proposition 1], or alternatively by observing that the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ conjugates $\mathsf{SL}_2^\Sigma(\mathbb{Z})$ to the congruence subgroup $\Gamma_0(2) = \Gamma_1(2)$; dimension formulas for the latter can be found in [28, p.108]. Theorem 4.17 together with the above formulas yield the desired result after some manipulations with generating functions. $\square$

Remark 4.19. Computations of $H^*(B\operatorname{Diff}^+(T^2);\mathbb{Q})$ have been carried out in [54] and [35] (the latter using the Eichler–Shimura isomorphism). The inclusion of $\operatorname{Diff}^+(T^2)$ into $\operatorname{aut}^+(T^2)$ is a weak homotopy equivalence, so we recover this computation by setting $d = 1$ and $\epsilon = 1$ in the above.

The cohomology of $\mathsf{GL}_3(\mathbb{Z})$ and $\mathsf{SL}_3(\mathbb{Z})$ with coefficients in irreducible algebraic representations has recently been computed in many cases [2]. Let $V = \mathbb{C}^3$ be the standard representation of $\mathsf{GL}_3(\mathbb{C})$. The combination of [2, Corollary 18] and [2, Theorem 16] specialized to $\mathcal{M}_{k,0} = \operatorname{Sym}^k(V)$ (which is not self-dual for $k > 0$) shows

$$H^q(\mathsf{SL}_3(\mathbb{Z}), \operatorname{Sym}^k(V)) \cong \begin{cases} S_{k+2}, & \text{for } q = 3 \text{ and } k > 0 \text{ even,} \\ S_{k+3} \oplus \mathbb{C}, & \text{for } q = 2 \text{ and } k \text{ odd,} \\ \mathbb{C}, & \text{for } q = 0 \text{ and } k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where S_k denotes the space of cusp forms of weight k for $\mathsf{SL}_2(\mathbb{Z})$. If we take the liberty of writing M_{k+3} for the isomorphic vector space $S_{k+3} \oplus \mathbb{C}$ in the second case above, we can summarize the calculation as an isomorphism of bigraded vector spaces with involution:

$$H^*(\mathsf{SL}_3(\mathbb{Z}), \operatorname{Sym}^\bullet(V)) \cong \mathbb{C}[0]^+ \oplus M_{\bullet+3}[2]^- \oplus S_{\bullet+2}[3]^+. \tag{23}$$

The action of the involution on the right-hand side is implicit in [2, Lemma 17]; this lemma asserts that

$$H^*(\mathsf{GL}_3(\mathbb{Z}), \operatorname{Sym}^\bullet(V)) \cong \mathbb{C}[0] \oplus S_{\bullet+2}[3]. \tag{24}$$

As before, this leads to

Theorem 4.20. *For $d = 1, 3, 7$, there is an isomorphism of graded vector spaces with involution*

$$\widetilde{H}^*(B\operatorname{aut}^+(S^d \times S^d \times S^d);\mathbb{Q}) \cong \bigoplus_k M_k^-[(k-3)(d+1)+2] \oplus S_k^+[(k-2)(d+1)+3)],$$

and the Poincaré series is given by

$$z^{\ell+2}\frac{\epsilon\big(1 + z^{2\ell} - z^{6\ell}\big) + z^{9\ell+1}}{(1-z^{4\ell})(1-z^{6\ell})}. \qquad \square$$

Remark 4.21. It seems plausible that computations similar to those of [2] can be carried out for the group $\mathsf{SL}_3^\Sigma(\mathbb{Z})$, but this is beyond the scope of this paper. However, even if we currently lack complete calculations, certain qualitative conclusions can be drawn. By Theorem 3.44, the cdga $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to

$$\Omega^*(\Gamma(X), C^*_{CE}(\mathfrak{g}(X))).$$

If $C^*_{CE}(\mathfrak{g}(X))$ is formal as a cdga in $\mathrm{Rep}_{\mathbb{Q}}(R(X))$, then this is quasi-isomorphic to

$$\Omega^*(\Gamma(X); H^*_{CE}(\mathfrak{g}(X)))$$

as a cdga. An application of the homotopy transfer theorem for C_∞-algebras yields a C_∞-algebra structure $\{m_n\}$ on $H^*(\Gamma(X); H^*_{CE}(\mathfrak{g}(X)))$ that is bigraded in the sense that m_n has bidegree $(2-n, 0)$, and that is C_∞-quasi-isomorphic to $\Omega^*(B\operatorname{aut}(X))$.

If the cohomology of $\Gamma(X)$ with coefficients in algebraic representations is concentrated in a small range of degrees, this can force these C_∞-operations to be trivial, implying that the cdga $\Omega^*(B\operatorname{aut}(X))$ is formal. For example, this happens if there is an r such that $\widetilde{H}^i(\Gamma(X), V) = 0$ unless $r \leq i \leq 3r-2$, for all $V \in \mathrm{Rep}_{\mathbb{Q}}(R(X))$.

For example, if $\Gamma \leqslant \mathsf{SL}_3(\mathbb{Z})$ is a finite index subgroup, then $H^1(\Gamma, V) = 0$ for any finite-dimensional $\mathbb{Q}$-vector space V with an action of Γ by [5], and $H^i(\Gamma, V) = 0$ for $i > 3$, because Γ has virtual cohomological dimension 3.

These considerations lead to

Theorem 4.22. *For d odd, the space $B\operatorname{aut}^+(S^d \times S^d \times S^d)$ is formal and the rational cohomology has trivial ring structure.* □

Another remark is that the invariant ring $H^*(\Gamma(S^{d\times n})/\operatorname{aut}(S^{d\times n}); \mathbb{Q})^{\Gamma(S^{d\times n})}$ is the trivial ring $\mathbb{Q}$ in this case, because $\mathrm{Sym}^k(V_n)^{\Gamma(S^{d\times n})}$ is easily seen to be trivial. In particular, $H^*(B\operatorname{aut}(S^{d\times n}); \mathbb{Q})$ consists entirely of nilpotent elements.

For $n > 3$, calculations become increasingly difficult, but one can at least say something about the limit as $n \to \infty$.

Consider the maps

$$\sigma\colon B\operatorname{aut}(S^{d\times n}) \to B\operatorname{aut}(S^{d\times(n+1)}),$$
$$\pi\colon B\operatorname{aut}(S^{d\times n}) \to B\Gamma(S^{d\times n}),$$

where σ is induced by extending a self-homotopy equivalence of $S^{d\times n}$ by the identity on the second factor of $S^{d\times(n+1)} = S^{d\times n} \times S^d$ and π is the evident map. It is a consequence of Borel's results on the stable cohomology of arithmetic groups [19, 20] that these maps induce an isomorphism in $H^i(-;\mathbb{Q})$ for $n \gg i$. The explicit ranges stated below rely on more recent results due to Kupers–Miller–Patzt [48] and Putman [56].

Theorem 4.23. *Let d be odd. The map*

$$\sigma^*\colon H^i(B\operatorname{aut}(S^{d\times(n+1)}); \mathbb{Q}) \to H^i(B\operatorname{aut}(S^{d\times n}); \mathbb{Q})$$

is an isomorphism for $n \geq i+1$ if $d = 1, 3, 7$ and for $n \geq 2i+6$ for all odd d and the map

$$\pi^*\colon H^i(\Gamma(S^{d\times n}), \mathbb{Q}) \to H^i(B\operatorname{aut}(S^{d\times n}); \mathbb{Q}).$$

is an isomorphism for $n \geq i-d+1$ if $d = 1, 3, 7$ and for $n \geq 2i-2d+6$ for all odd d.

Proof. By Theorem 4.15, the map π^* is injective and its cokernel is isomorphic to

$$\bigoplus_{\substack{j+k(d+1)=i\\ k>0}} H^j(\Gamma(S^{d\times n}), \mathrm{Sym}^k(V_n)). \tag{25}$$

By Borel's vanishing theorem [20, Theorem 4.4], the summands vanish for n large. Explicitly, [48, Theorem 7.6] implies vanishing for $n \geq j+k+1$ for $d = 1, 3, 7$, and [56, Theorem C] implies vanishing for $n \geq 2j+2k+6$ for $d \neq 1, 3, 7$. It follows that (25) vanishes for $n \geq i-d+1$ for $d = 1, 3, 7$ and for $n \geq 2i-2d+6$ for $d \neq 1, 3, 7$.

The statement about σ^* now follows from the corresponding statement for

$$H^i(\Gamma(S^{d\times(n+1)}),\mathbb{Q}) \to H^i(\Gamma(S^{d\times n}),\mathbb{Q}).$$

By Borel's work [19] this is an isomorphism for n large. According to [48, Theorem A] it is an isomorphism for $n \geq i+1$ for $d = 1,3,7$, and [56, Theorem C] implies it is an isomorphism for $n \geq 2i+6$ for $d \neq 1,3,7$. $\square$

By Borel's calculation [19], the stable cohomology may be identified with an exterior algebra

$$\varprojlim_n H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q}) \cong \varprojlim_n H^*(\Gamma(S^{d\times n}),\mathbb{Q}) \cong \Lambda[x_5, x_9, x_{13}, \ldots].$$

We now turn to the case d even, which behaves very differently.

Proposition 4.24. *For d even, the dg Lie algebra $\mathfrak{g}(S^{d\times n})$ is formal with homology the abelian Lie algebra with basis*

$$\alpha_1,\ldots,\alpha_n, \quad \beta_{ij}, \quad 1 \leq i \neq j \leq n,$$

in degrees $|\alpha_i| = 2d-1$ and $|\beta_{ij}| = d-1$. The action of $(\sigma,\lambda) \in R(S^{d\times n}) = \Sigma_n \ltimes (\mathbb{Q}^\times)^n$ is given by

$$(\sigma,\lambda)\cdot\alpha_i = \lambda_i^{-2}\alpha_{\sigma(i)}, \quad (\sigma,\lambda)\cdot\beta_{ij} = \lambda_i\lambda_j^{-2}\beta_{\sigma(i)\sigma(j)}.$$

Proof. Let Λ denote the minimal model. Since $\operatorname{Aut}\Lambda$ is reductive, the positive truncation $\operatorname{Der}\Lambda\langle 1\rangle$ is a Lie model for $B\operatorname{aut}_u(X)$. This is spanned by derivations of the form

$$\frac{\partial}{\partial x_i}, \quad x_i\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial y_i},$$

and the only non-trivial differential is given by

$$\left[d, \frac{\partial}{\partial x_i}\right] = -2x_i\frac{\partial}{\partial y_i}.$$

The action of the group $\operatorname{Aut}\Lambda$ is easily computed, *e.g.*, if φ corresponds to $(\sigma,\lambda) \in \Sigma_n \ltimes (\mathbb{Q}^\times)^n$, then $\varphi\big(x_i\frac{\partial}{\partial y_j}\big) = \lambda_i\lambda_j^{-2}x_{\sigma(i)}\frac{\partial}{\partial y_{\sigma(j)}}$. A basis for the homology is represented by the cycles $\alpha_i = \frac{\partial}{\partial y_i}$ and $\beta_{ij} = x_i\frac{\partial}{\partial y_i}$ for $i \neq j$. These span an abelian dg Lie subalgebra stable under the action of $\operatorname{Aut}\Lambda$, and the inclusion into $\operatorname{Der}\Lambda\langle 1\rangle$ is a quasi-isomorphism. $\square$

Corollary 4.25. *For d even, the space $B\operatorname{aut}(S^{d\times n})$ is formal and there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma(S^{d\times n})},$$

where $|a_i| = 2d$, $|b_{ij}| = d$ and $(\sigma,\lambda) \in \Gamma(S^{d\times n}) = \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ acts by

$$(\sigma,\lambda)\cdot a_i = a_{\sigma(i)}, \quad (\sigma,\lambda)\cdot b_{ij} = \lambda_i b_{\sigma(i)\sigma(j)}.$$

Similarly, the space $B\operatorname{aut}^+(S^{d\times n})$ is formal and there is an isomorphism of algebras

$$H^*(B\operatorname{aut}^+(S^{d\times n});\mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma^+(S^{d\times n})},$$

where $\Gamma^+(S^{d\times n}) \leqslant \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ is the subgroup of all (σ,λ) such that $\lambda_1\ldots\lambda_n = 1$.

Proof. Clearly, $C^*_{CE}(\mathfrak{g}(S^{d\times n})) = \mathbb{Q}[a_i, b_{ij}]$, where a_i and b_{ij} are the dual 1-cochains of α_i and β_{ij}, and the differential is trivial since $\mathfrak{g}(S^{d\times n})$ is abelian with trivial differential. In particular, $C^*_{CE}(\mathfrak{g}(S^{d\times n}))$ is formal, so Remark 4.21 shows that the C_∞-algebra structure on $H^*(\Gamma(S^{d\times n}); H^*_{CE}(\mathfrak{g}(S^{d\times n})))$ respects the bigrading. But since the group $\Gamma(S^{d\times n})$ is finite, the cohomology reduces to the invariants concentrated in bidegree $(0,*)$, whence the C_∞-operations m_n vanish for $n > 2$. $\square$

The computation of invariant subrings $\mathbb{Q}[V]^G$ for finite-dimensional representations V of finite or reductive groups G is classical and well-understood in principle (see *e.g.* [27]). The invariant subring is a Cohen–Macaulay ring, which means that one can find a regular sequence $\theta_1, \ldots, \theta_m$ of invariant polynomials (the 'primary invariants') such that $\mathbb{Q}[V]^G$ is a finitely generated free module over $\mathbb{Q}[\theta_1, \ldots, \theta_m]$ on certain invariant polynomials $\eta_1, \ldots, \eta_t$ (the 'secondary invariants').

Corollary 4.26. *For d even, the algebra $H^*(B\operatorname{aut}^+(S^{d\times n}); \mathbb{Q})$ is a Cohen–Macaulay ring of Krull dimension n^2, concentrated in degrees that are multiples of d.* □

Let us examine the case $n = 2$ closer to illustrate. The invariant theory calculation can be carried out in two steps using

$$\mathbb{Q}[a_1, a_2, b_{12}, b_{21}]^{\Gamma(S^{d\times 2})} = \Big(\mathbb{Q}[a_1, a_2, b_{12}, b_{21}]^{(\mathbb{Z}^\times)^2}\Big)^{\Sigma_2}.$$

The invariant ring with respect to the action of $(\mathbb{Z}^\times)^2$ is easily seen to be a polynomial ring in $a = a_1$, $b = b_{12}^2$, $c = b_{21}^2$, $d = a_2$, so we are left to identify the invariant ring

$$\mathbb{Q}[a, b, c, d]^{\Sigma_2},$$

where the non-trivial element of Σ_2 acts as the permutation $(ad)(bc)$. By using Molien's theorem (specialized to permutation representations as in [64, Proposition 4.3.4]), the Poincaré series with respect to word-length in the generators can be computed to be

$$\frac{1 + t^2}{(1-t)^2(1-t^2)^2}.$$

The invariant polynomials

$$a + d, \quad b + c, \quad ad, \quad bc,$$

form a regular sequence of length equal to the Krull dimension, so these form a set of primary invariants. A choice of secondary invariants is given by the two polynomials

$$1, \quad ab + cd.$$

The ring structure is determined by writing $(ab+cd)^2$ in the basis; one checks that

$$(ab + cd)^2 = \Big((a + d)(b + c)\Big) \cdot (ab + cd) - \Big((a + d)^2 bc + ad(b + c)^2 - 4(ad)(bc)\Big) \cdot 1.$$

The calculation for $\Gamma^+(X_2)$ is similar. We omit the details; the only difference is that there is one additional invariant $\alpha_0 = b_{12}b_{21}$, which squares to $bc = b_{12}^2 b_{21}^2$. Writing α_1, α_2, β_1, β_2, η for $a + d$, $b + c$, ad, bc, $ab + cd$, respectively, these considerations lead to

Theorem 4.27. *For d even, there is an isomorphism of graded algebras with involution*

$$H^*(B\operatorname{aut}^+(S^d \times S^d); \mathbb{Q}) \cong \mathbb{Q}[\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \eta]/I,$$

where the generators $\alpha_0, \alpha_1, \alpha_2$ are of degree $2d$ and β_1, β_2, η are of degree $4d$, where I is the ideal generated by the two elements

$$\eta^2 - \alpha_1\alpha_2\eta + \big(\alpha_1^2\beta_2 + \beta_1\alpha_2^2 - 4\beta_1\beta_2\big), \quad \alpha_0^2 - \beta_2,$$

and where the involution acts by $\alpha_0 \mapsto -\alpha_0$ and trivially on the other generators. The Poincaré series of this graded vector space with involution is given by

$$(1 + \epsilon z^{2d})\frac{1 + z^{4d}}{(1 - z^{2d})^2(1 - z^{4d})^2}.$$

A presentation for the cohomology ring of $B\operatorname{aut}(S^d \times S^d)$ is obtained by removing the generator α_0 and the relation $\alpha_0^2 - \beta_2$ from the above presentation, and the Poincaré series is obtained by removing the term involving ϵ. □

The case $n > 2$ can in principle be treated similarly but we stop here.

4.2. **Highly connected even-dimensional manifolds.** The rational homotopy theory of self-homotopy equivalences of highly connected even-dimensional manifolds has been thoroughly studied in [15, §5]. We will now discuss how the methods of the present paper lead to simplifications of certain arguments in [15], as well as to some new results.

Let M be an $(n-1)$-connected $2n$-dimensional manifold ($n > 1$). We will consider the space $\operatorname{aut}^+(M)$ of orientation preserving self-homotopy equivalences. Let $\Gamma^+(M)$ denote the image of $\mathcal{E}^+(M) = \pi_0 \operatorname{aut}^+(M)$ in $R(M)$, and let $\operatorname{Aut}^+ \mathbb{L}$ denote the group of automorphisms of the minimal Quillen model $\mathbb{L}$ of M that represent orientation preserving self-homotopy equivalences.

As discussed in [15, §3.5], the minimal Quillen model of M can be presented as

$$\mathbb{L} = \big(\mathbb{L}(\alpha_1, \ldots, \alpha_r, \gamma), \delta\gamma = \omega\big),$$

where $\alpha_1, \ldots, \alpha_r$ is a basis for $H_n(M;\mathbb{Q})[1-n]$, where γ corresponds to the fundamental class of M, and where ω is dual to the cup product pairing

$$\langle\, , \rangle \colon H^n(M;\mathbb{Q}) \otimes H^n(M;\mathbb{Q}) \to \mathbb{Q}, \quad \langle x, y\rangle = \langle x \smile y, [M]\rangle.$$

Explicitly,

$$\omega = \frac{1}{2}\sum_i \,[\alpha_i^{\#}, \alpha_i],$$

where $\alpha_i^{\#}$ is the dual basis with respect to the intersection form.

Proposition 4.28. *Let M be an $(n-1)$-connected $2n$-dimensional manifold ($n > 1$). The group $\operatorname{Aut}^+ \mathbb{L}$ is isomorphic to the group of automorphisms of $H^n(M;\mathbb{Q})$ that preserve the cup product pairing. In particular, $\operatorname{Aut}^+ \mathbb{L}$ is reductive, whence $\operatorname{Aut}^+ \mathbb{L} \cong \mathcal{E}^+(M_{\mathbb{Q}}) \cong R^+(M)$.*

Proof. An automorphism of $\mathbb{L}$ represents an orientation preserving self-homotopy equivalence if and only if it fixes γ. Using this, one sees that $\operatorname{Aut}^+ \mathbb{L}$ is isomorphic to the group of automorphisms of the space spanned by $\alpha_1, \ldots, \alpha_r$ that fix ω. Since ω is dual to the cup product pairing, this is in turn isomorphic to $\operatorname{Aut}\big(H^n(M;\mathbb{Q}), \langle\, , \rangle\big)$. The group of automorphisms of a non-degenerate symmetric or skew-symmetric bilinear form is well-known to be reductive. □

Corollary 1.7 implies that the group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H_*(M;\mathbb{Z})$ that are induced by an orientation preserving homotopy equivalence of M. This group is known, *cf.* [7, Theorem 8.14], [14, Theorem 2.12] or [15, §5.1]. To describe it, recall the cohomology operation $\psi \colon H^n(M) \to \pi_{2n-1}(S^n)$ defined by Kervaire–Milnor [46, §8]; the class $\psi(x)$ is the obstruction for the existence of a map $f \colon M \to S^n$ such that $f^*(s) = x$, where s is a generator for $H^n(S^n)$.

Proposition 4.29. *Let M be an $(n-1)$-connected $2n$-dimensional manifold ($n > 1$). The group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H^n(M)$ that preserve the cup product pairing*

$$\langle -, - \rangle \colon H^n(M) \otimes H^n(M) \to \mathbb{Z}$$

and the Kervaire–Milnor cohomology operation

$$\psi \colon H^n(M) \to \pi_{2n-1}(S^n).$$ □

For example, for the manifold $W_g = \#^g S^n \times S^n$, the group $\Gamma^+(W_g)$ coincides with the group denoted Γ_g in [15]. As explained in [15, Example 5.5] it may be described by

$$\Gamma_g \cong \begin{cases} \mathsf{O}_{g,g}(\mathbb{Z}), & n \text{ even}, \\ \mathsf{Sp}_{2g}(\mathbb{Z}), & n = 1,3,7, \\ \mathsf{Sp}^q_{2g}(\mathbb{Z}), & n \text{ odd} \neq 1,3,7, \end{cases}$$

where $\mathsf{Sp}^q_{2g}(\mathbb{Z})$ denotes the group of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}_{2g}(\mathbb{Z}),$$

such that the diagonal entries of the $g \times g$-matrices $C^t A$ and $D^t B$ are even.

The next result should be viewed as an upgrade of [15, Theorem 5.8], which asserts that the Lie model for the space $B\operatorname{aut}_\circ(M)$ is formal. The novelty is that the formality can be made equivariant.

Proposition 4.30. *Let M be an $(n-1)$-connected $2n$-dimensional manifold ($n > 1$) with $\dim H^n(M;\mathbb{Q}) > 2$. The algebraic Lie model $\mathfrak{g}(M)$ for $\Gamma^+(M)/\operatorname{aut}^+(M)$ is formal as a dg Lie algebra of algebraic representations of $R(M)$ and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \operatorname{Der} L/\operatorname{ad} L\langle 1\rangle,$$

where $L = \pi_*(\Omega M)\otimes\mathbb{Q}$ is the rational homotopy Lie algebra of M. This Lie algebra admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \ldots, \alpha_r)/(\omega),$$

with $\alpha_1, \ldots, \alpha_r$ and ω as above. The action of $R^+(M)$ is induced by the action on $H_*(M;\mathbb{Q})$.

Proof. Since $\operatorname{Aut}^+\mathbb{L}$ is reductive, the algebraic Lie model $\operatorname{Der}^c_u\mathbb{L}$ for the space $\Gamma^+(M)/\operatorname{aut}^+(M)$ of Theorem 1.10 is zero in degree 0 by Lemma 3.19, whence $\operatorname{Der}^c\mathbb{L}\langle 1\rangle \to \operatorname{Der}^c_u\mathbb{L}$ is a quasi-isomorphism. Now that we know that $\operatorname{Der}^c\mathbb{L}\langle 1\rangle$ is a $\Gamma^+(M)$-equivariant Lie model for $\Gamma^+(M)/\operatorname{aut}^+(M)$, it is straightforward to inspect that the zig-zag of quasi-isomorphisms connecting this dg Lie algebra with its homology given in the proof of Theorem 5.9 of [15] can be made into a zig-zag of algebraic $R(M)$-representations. □

The action of $\Gamma^+(M)$ on the rational homotopy groups was also identified in [15], but the methods of [15] were insufficient for proving the following result, which is a direct consequence of Corollary 1.3 and Proposition 4.30.

Theorem 4.31. *Let M be an $(n-1)$-connected $2n$-dimensional manifold ($n > 1$) with $\dim H^n(M;\mathbb{Q}) > 2$. There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}^+(M);\mathbb{Q}) \cong H^*(\Gamma^+(M), H^*_{CE}(\mathfrak{g})),$$

where $\Gamma^+(M) = \operatorname{Aut}(H^n(M;\mathbb{Z}), \langle\,,\,\rangle, \psi)$, $\mathfrak{g} = \operatorname{Der} L/\operatorname{ad} L\langle 1\rangle$ and $L = \pi_*(\Omega M)\otimes\mathbb{Q}$. □

When adapted to the space $\operatorname{aut}_\partial(W_{g,1})$ of self-homotopy equivalences of

$$W_{g,1} = W_g \setminus \operatorname{int} D^{2n}$$

relative to the boundary, our methods yield a significant simplification of the computation of the stable cohomology of $B\operatorname{aut}_\partial(W_{g,1})$ of [15]. By applying our methods to the homotopy fiber sequence

$$\Gamma_g/\operatorname{aut}_\partial(W_{g,1}) \to B\operatorname{aut}_\partial(W_{g,1}) \to \Gamma_g, \tag{26}$$

one can upgrade [15, Theorem 3.12] to show that the Γ_g-space $\Gamma_g/\operatorname{aut}_\partial(W_{g,1})$ has R_g-algebraic Lie model

$$\mathfrak{g}_g = \operatorname{Der}_\omega \mathbb{L}\langle 1\rangle,$$

where $\operatorname{Der}_\omega \mathbb{L}$ denotes the graded Lie algebra of derivations of of the free graded Lie algebra $\mathbb{L} = \mathbb{L}(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ that annihilate $\omega = [\alpha_1, \beta_1] + \dots + [\alpha_g, \beta_g]$. As above, this implies

Theorem 4.32. *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, H^*_{CE}(\mathfrak{g}_g)). \tag{27}$$ □

Remark 4.33. In the stable range, i.e., for g large compared to the cohomological degree, the right-hand side can be simplified. Let $H = H^*_{CE}(\mathfrak{g}_g)$. By semisimplicity of $\operatorname{Rep}_{\mathbb{Q}}(R_g)$ there is a split exact sequence

$$0 \to H^{R_g} \to H \to H/H^{R_g} \to 0.$$

This gives rise to a ring homomorphism

$$H^*(\Gamma_g, \mathbb{Q}) \otimes H^{R_g} \cong H^*(\Gamma_g, H^{R_g}) \to H^*(\Gamma_g, H),$$

whose cokernel may be identified with

$$H^*(\Gamma_g, H/H^{R_g}).$$

Since H/H^{R_g} is a direct sum of non-trivial irreducible algebraic representations of R_g, the cokernel vanishes in degrees that are small compared to g by Borel's vanishing theorem [20, Theorem 4.4]. Note that this entails the statement that $H^{R_g} = H^{\Gamma_g}$ for g large (set $* = 0$). Thus, in the stable range, we obtain an isomorphism

$$H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, \mathbb{Q}) \otimes H^{\Gamma_g}.$$

This essentially recovers Theorem [15, Theorem 1.3].

Remark 4.34. The isomorphism (27), and in particular the collapse of the spectral sequence of (26), is only obtained in the stable range (cohomological degrees below $g/2-2$) in [15]. Moreover, the argument in [15] depends on several extraneous ingredients. One step in the argument is to show that the map

$$B\operatorname{aut}_\partial(W_{g,1}) \to B\Gamma_g$$

is injective on indecomposables in rational cohomology in the stable range [15, Theorem 8.6]. This is proved by establishing the stronger statement [15, Theorem 8.2] that

$$B\operatorname{Diff}_\partial(W_{g,1}) \to B\Gamma_g$$

is injective on indecomposables in stable rational cohomology, which in turn relies on deep results on the stable cohomology of $B\operatorname{Diff}_\partial(W_{g,1})$ due to Galatius–Randal-Williams [36], as well as on non-vanishing results for the coefficients of the Hirzebruch L-polynomials due to Berglund–Bergström [13]. Our proof of Theorem 4.32 does not depend on these extraneous ingredients. In fact, Theorem 4.32 implies a strengthening of [15, Theorem 8.6], see Corollary 4.35 below.

Furthermore, many arguments of [15] rely on deep results on almost algebraicity of finite-dimensional representations of Γ_g due to Bass–Milnor–Serre [5, §16] (see also [62, §1.3(9)] and [15, Appendix A]). The existence of algebraic Lie models shows that the representations in question are in fact algebraic, which in particular removes the necessity to invoke such results.

Corollary 4.35. *The ring homomorphism*

$$H^*(\Gamma_g, \mathbb{Q}) \to H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q})$$

is split injective for all g. In particular, the induced map on indecomposables

$$QH^*(\Gamma_g, \mathbb{Q}) \to QH^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q})$$

is injective for all g. □

4.3. **Highly connected odd-dimensional manifolds.** The geometry and rational homotopy theory of $(n-1)$-connected $(2n+1)$-dimensional manifolds has also been thoroughly studied (*e.g.* in Wall's classification [69]), and we collect some relevant results in this section.

Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold ($n > 8$). Using the techniques of [15, §3.5], one can show that the minimal Quillen model of M is given by

$$\mathbb{L} = (\mathbb{L}(\alpha_1, \ldots, \alpha_r, \alpha_1^\#, \ldots, \alpha_r^\#, \gamma), \delta\gamma = \omega)$$

where the α_i are a basis for $\mathrm{s}^{-1}\widetilde{H}_n(M;\mathbb{Q})$, the $\alpha_i^\#$ are the dual basis of the space $\mathrm{s}^{-1}\widetilde{H}_{n+1}(M;\mathbb{Q})$ with respect to the intersection form, and

$$\omega = \sum_i [\alpha_i^\#, \alpha_i]$$

is dual to the intersection form. (So up to rational equivalence, such manifolds are completely classified by the torsion-free rank of $H_n(M)$.) We use the notation of the preceding section freely.

Lemma 4.36. *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold with $n > 1$. The group $\operatorname{Aut}^+\mathbb{L}$ is isomorphic to the group $\mathsf{GL}(H^n(M;\mathbb{Q}))$ of linear automorphisms of $H^n(M;\mathbb{Q})$. In particular, it is reductive, wherefore $\operatorname{Aut}^+\mathbb{L} \cong \mathcal{E}^+(M_\mathbb{Q}) \cong R^+(M)$.*

Proof. An automorphism of $\mathbb{L}$ is orientation-preserving precisely when it fixes γ. Thus an automorphism $\phi \in \operatorname{Aut}^+\mathbb{L}$ is uniquely determined by its action on $\mathbb{L}_{n-1} = \langle \alpha_1, \ldots, \alpha_r\rangle$: it preserves the element $\omega = \delta\gamma$, so its action on the $\alpha_i^\#$ is ω-dual to the action on the α_i. □

The identification of the group $\Gamma^+(M)$ is somewhat more involved than in the preceding case. We will instead determine the closely related group

$$\Gamma^+_\mathbb{Z}(M) = \operatorname{Aut}^{\mathcal{E}^+(M)}(H_*(M;\mathbb{Z}))$$

of orientation-preserving automorphisms of *integral* homology that are realisable by homotopy automorphisms of M, rather than the group

$$\Gamma^+(M) = \mathsf{GL}^{\mathcal{E}^+(M)}(H_*(M;\mathbb{Q})).$$

This makes no matter in light of Theorem 3.40, whose notation we use freely: as shown in the preceding lemma, the group $\mathcal{E}^+(M_\mathbb{Q})$ is reductive, so $q_*^{-1}(U) \leqslant \mathcal{E}^+(M)$ consists precisely of the homotopy automorphisms of M that act trivially on $H_*(M;\mathbb{Q})$. Let G be the kernel of the action of $\mathcal{E}^+(M)$ on $H_*(M;\mathbb{Z})$, so G has finite index in $q_*^{-1}(U)$ and the group $\mathcal{E}^G(M)$ agrees with $\Gamma^+_\mathbb{Z}(M)$. Thus by Theorem 3.40, the space $B\operatorname{aut}^+(M)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(M)\rangle_{\mathrm{h}\Gamma^+_\mathbb{Z}(M)}$.

The identification of $\Gamma^+_\mathbb{Z}(M)$ was achieved by Floer [34] in the course of his classification of the homotopy types of $(n-1)$-connected $(2n+1)$-dimensional Poincaré complexes. We briefly recall the necessary ingredients, *cf.* also Wall's classification [69] of diffeomorphism types of $(n-1)$-connected $(2n+1)$-dimensional manifolds. We assume $n > 8$ throughout (although calculations are also possible for smaller values of n; see *e.g.* Barden [4] and Baues–Buth [8] for the case $n = 2$, and Crowley–Nordström [26] for the case $n = 3$).

Writing $H_n(M;\mathbb{Z})_{\mathrm{tor}}$ for the torsion subgroup of $H_n(M;\mathbb{Z})$, the linking form is a non-singular bilinear form

$$b\colon H_n(M;\mathbb{Z})_{\mathrm{tor}} \otimes H_n(M;\mathbb{Z})_{\mathrm{tor}} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

When n is even, $b(x,x) \equiv 0$, whereas for n odd, $2b$ admits a quadratic refinement

$$q\colon H_n(M;\mathbb{Z})_{\mathrm{tor}} \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

Every element of $\pi_n(M) \cong H_n(M;\mathbb{Z})$ may be represented by an embedded sphere. The stable normal bundle associated to such an embedding yields a homomorphism

$$\alpha\colon H_n(M;\mathbb{Z}) \longrightarrow \pi_{n-1}(\mathsf{SO}).$$

Following Floer, we will denote the composite $J\alpha\colon H_n(M;\mathbb{Z}) \to \pi^s_{n-1}$ of α with the stable J-homomorphism by ψ. Note that whereas α sees the smooth structure of M, ψ is only a homotopy invariant, and may be defined in terms of a cell decomposition of M, *cf.* [34, p. IV.iii].

To make the discussion simpler, we now restrict to the "non-exceptional" case of Wall, which amounts to requiring that the homomorphism $\eta_*\colon \pi^s_{n-1} \to \pi^s_n$ determined by the generator $\eta \in \pi^s_1$ annihilates the image of ψ, *cf.* [34, p. 60]. (We remark that all manifolds are non-exceptional if $n \not\equiv 0, 1 \mod 8$.) In this case, the group $KG(M)$ of stable spherical fibrations over M sits in a split short exact sequence

$$0 \longrightarrow H^{n+1}(M;\pi^s_n) \longrightarrow KG(M) \longrightarrow H^n(M;\pi^s_{n-1}) \longrightarrow 0,$$

and thus the class of the stable spherical fibration determined by the tangent bundle of M is determined by its image in $H^n(M;\pi^s_{n-1}) \cong \mathrm{Hom}(H_n(M;\mathbb{Z}),\pi^s_{n-1})$, which can be shown to be precisely ψ, and a cohomology class $\beta \in H^{n+1}(M;\pi^s_n) \cong H_n(M;\mathbb{Z})\otimes\pi^s_n$. The class β is the image of Wall's class $\widehat{\beta} \in H^{n+1}(M;\pi_n\mathsf{SO})$ under the change of coefficients effected by the stable J-homomorphism.

Finally, for n even, Floer constructs a homology class $\Delta \in H_n(M;\mathbb{Z}/2) \cong H_n(M;\mathbb{Z})\otimes\mathbb{Z}/2$. The construction proceeds via a rather delicate analysis of the homotopy groups of Moore spaces, and we refer the reader to Floer's text for details. We remark that if $H_n(M;\mathbb{Z})$ contains no 2-torsion, then $\Delta = 0$.

Proposition 4.37 ([34, p. 57, Satz])**.** *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold with $n > 8$ which is non-exceptional in the sense of Wall. The group $\Gamma^+_{\mathbb{Z}}(M)$ of orientation-preserving realisable integral homology automorphisms of M may be identified with the group of automorphisms of $H_n(M;\mathbb{Z})$ that preserve the linking form*

$$b\colon H_n(M;\mathbb{Z})_{\mathrm{tor}} \otimes H_n(M;\mathbb{Z})_{\mathrm{tor}} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

the homomorphism

$$\psi\colon H_n(M;\mathbb{Z}) \longrightarrow \pi^s_{n-1},$$

the class

$$\beta \in H_n(M;\mathbb{Z}) \otimes \pi^s_n,$$

the quadratic form

$$q\colon H_n(M;\mathbb{Z})_{\mathrm{tor}} \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

if n is odd, and the class

$$\Delta \in H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/2$$

if n is even. ∎

Remark 4.38. Floer's list of invariants is slightly different: he includes the class $\nu^\circ(M) \in KG(M)$ of the stable spherical fibration, but in the non-exceptional case, this is equivalently captured by the data of ψ and β by the discussion preceding the statement of our proposition. Likewise, Floer's invariant $v\colon H^{n+1}(M;\mathbb{Z}) \to \mathbb{Q}/2\mathbb{Z}$ is Poincaré dual to q, see [34, p. 65, Satz]. Finally, the obstruction ζ mentioned in Floer's classification theorem (and defined at the end of its proof) trivially vanishes because in the non-exceptional case, it lies in the zero $\mathbb{Z}/2$-vector space.

Just like in the case of even-dimensional manifolds, $\mathfrak{g}(M)$ enjoys a close connection to the derivations of the rational homotopy Lie algebra $\pi_*(\Omega M) \otimes \mathbb{Q}$ of M.

Proposition 4.39. *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold ($n > 1$) with $\dim H^n(M;\mathbb{Q}) > 1$. The algebraic Lie model $\mathfrak{g}(M)$ for the space $\Gamma^+(M)/\operatorname{aut}^+(M)$ is formal as a dg Lie algebra of algebraic representations of $R(M)$ and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \operatorname{Der} L/\operatorname{ad} L\langle 1\rangle$$

where $L = \pi_*(\Omega M) \otimes \mathbb{Q}$ is the rational homotopy Lie algebra of M, which admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^\#, \dots, \alpha_r^\#)/(\omega)$$

with α_i, $\alpha_i^\#$, and ω as above. The action of $R^+(M)$ is induced by the action on $H_*(M;\mathbb{Q})$.

Proof. We need to verify that the minimal Quillen model $f\colon \mathbb{L} \to L$, with

$$\mathbb{L} = \Big(\mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^\#, \dots, \alpha_r^\#, \gamma), \delta\gamma = \omega\Big),$$

satisfies the hypotheses of [15, Theorem 5.9], namely that L is centre-free and that the map $f^*\colon \operatorname{Der} L \to \operatorname{Der}_f(\mathbb{L}, L)$ induces isomorphisms on homology in non-negative degrees.

To show that L has trivial centre, we proceed like in the proof of [15, Proposition 5.10]. Namely, we need to prove that the graded Lie algebra L has finite global dimension and that its Euler characteristic

$$\chi(L) = \sum_i (-1)^i \dim \operatorname{Ext}^i_{UL}(\mathbb{Q}, \mathbb{Q})$$

is non-zero, where UL is the universal enveloping algebra of L. By [11, Theorem 3], L is a Koszul Lie algebra and its Koszul dual graded commutative algebra is $H^*(M;\mathbb{Q})$. Thus $\operatorname{Ext}^i_{UL}(\mathbb{Q}, \mathbb{Q})$ is the weight i part of $H^*(M;\mathbb{Q})$, where the generators α_i and $\alpha_i^\#$ are given weight 1. Explicitly, $\operatorname{Ext}^1_{UL}(\mathbb{Q}, \mathbb{Q}) = H^n(M;\mathbb{Q}) \oplus H^{n+1}(M;\mathbb{Q})$ and $\operatorname{Ext}^2_{UL}(\mathbb{Q}, \mathbb{Q}) = H^{2n+1}(M;\mathbb{Q})$; all higher cohomology groups vanish. Thus L has global dimension 2, and $\chi(L) = 2 - 2r$ is non-zero as soon as $r > 1$.

To verify the second condition, we mimic the proof of [15, Lemma 5.11]. Note that the differential D on $\operatorname{Der}_f(\mathbb{L}, L)$ is given by

$$D(\theta) = \pm\theta(\omega)\frac{\partial}{\partial\gamma} \tag{28}$$

since the differential δ on $\mathbb{L}$ is just $\omega\frac{\partial}{\partial\gamma}$. The chain complex $\operatorname{Der}_f(\mathbb{L}, L)$ is spanned by elements of the form $x\frac{\partial}{\partial\alpha_i}$, $x\frac{\partial}{\partial\alpha_i^\#}$, and $x\frac{\partial}{\partial\gamma}$, where $x \in L$. The image of f^* is evidently contained in the span of the first two classes of generators since $f(\gamma) = 0$, and in view of (28), it consists precisely of the cycles in this span. Every element of the form $x\frac{\partial}{\partial\gamma}$ is also a cycle, but if $|x| > n$, then $x \in L$ is decomposable, and hence $x\frac{\partial}{\partial\gamma}$ is a boundary, since

$$D\left(y\frac{\partial}{\partial\alpha_i}\right) = \pm\left[y, f(\alpha_i^{\#})\right]\frac{\partial}{\partial\gamma}$$
$$D\left(y\frac{\partial}{\partial\alpha_i^{\#}}\right) = \pm\left[y, f(\alpha_i)\right]\frac{\partial}{\partial\gamma}$$

If $x \in L_{\leqslant n}$, then the degree of $x\frac{\partial}{\partial\gamma}$ is negative, and thus the (non-trivial) cycle does not prevent f^* from inducing an isomorphism in non-negative degrees. □

Thus we obtain the following consequence of Theorem 3.40.

Theorem 4.40. *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold with $\dim H^n(M;\mathbb{Q}) > 1$. There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}^+(M);\mathbb{Q}) \cong H^*(\Gamma^+_{\mathbb{Z}}(M), H^*_{\mathrm{CE}}(\mathfrak{g}))$$

where $\mathfrak{g} = \operatorname{Der} L/\operatorname{ad} L\langle 1\rangle$ is the truncated dg Lie algebra of outer derivations of $L = \pi_*(\Omega M)\otimes\mathbb{Q}$, and $\Gamma^+_{\mathbb{Z}}(M) = \operatorname{Aut}^{\mathcal{E}^+(M)}(H^*(M;\mathbb{Z}))$. □

We conclude this section by making a couple of remarks about the manifolds

$$Z_g = \#^g S^n \times S^{n+1}$$
$$Z_{g,1} = Z_g \setminus \operatorname{int} D^{2n+1}$$

for $n > 1$. By Lemma 4.36, we have that $R^+(Z_g) \cong \mathsf{GL}(H_n(Z_g;\mathbb{Q})) \cong \mathsf{GL}_g(\mathbb{Q})$. Let

$$\Gamma_\partial(Z_{g,1}) \cong \mathsf{GL}^{\operatorname{aut}_\partial(Z_{g,1})}(H_*(Z_{g,1};\mathbb{Q}))$$

be the group of automorphisms of the rational homology of $Z_{g,1}$ which can be realized by a boundary-preserving homotopy automorphism of $Z_{g,1}$. Each such homotopy automorphism can be extended by the identity on the interior of the removed disc to an orientation-preserving homotopy automorphism of Z_g, yielding an injection

$$\Gamma_\partial(Z_{g,1}) \hookrightarrow \Gamma^+(Z_g). \tag{29}$$

Proposition 4.41. *The homomorphism* (29) *is an isomorphism, and $\Gamma^+(Z_g)$ corresponds to the subgroup $\operatorname{Aut}(H_n(Z_g;\mathbb{Z}))$ of $\mathsf{GL}(H_n(Z_g;\mathbb{Q})) \cong R^+(Z_g)$.*

Proof. The second claim is an easy consequence of Floer's result, at least when $n > 8$, since the invariants ψ and β vanish as Z_g is stably parallelisable. However, in the case of Z_g, a more hands-on argument that works for all $n > 1$ is possible:

It is evident that $\Gamma^+(Z_g)$ injects into $\operatorname{Aut}(H_n(Z_g;\mathbb{Z}))$. Both claims can be established simultaneously by showing that every automorphism of $H_n(Z_g;\mathbb{Z}) \cong \mathbb{Z}^g$ may be realized by a homotopy automorphism of Z_g that fixes the given embedded disc $D^{2n+1} \subset Z_g$ pointwise. This is an easy consequence of the Hilton–Milnor theorem once one notes that $Z_{g,1} \simeq \bigvee^g S^n \vee \bigvee^g S^{n+1}$ and that Z_g is obtained from $Z_{g,1}$ by attaching a $(2n+1)$-cell along

$$\omega = \sum_{i=1}^{g}[\alpha_i, \alpha_i^{\#}] \in \pi_{2n}(Z_{g,1}),$$

where $\alpha_i \in \pi_n(Z_{g,1})$ are the classes of the S^n-factors of $Z_{g,1}$ and $\alpha_i^{\#} \in \pi_{n+1}(Z_{g,1})$ are the classes of the S^{n+1}-factors, suitably indexed. See [39, §5] where the more general case of connected sums of products of spheres of varying dimensions is investigated. □

Our methods also apply to the homotopy fiber sequence

$$\Gamma_\partial(Z_{g,1})/\operatorname{aut}_\partial(Z_{g,1}) \longrightarrow B\operatorname{aut}_\partial(Z_{g,1}) \longrightarrow B\Gamma_\partial(Z_{g,1})$$

to show that the $\Gamma_\partial(Z_{g,1})$-space $\Gamma_\partial(Z_{g,1})/\operatorname{aut}_\partial(Z_{g,1})$ has an $R^+(Z_g)$-algebraic Lie model

$$\mathfrak{g}_\partial(Z_{g,1}) = \Big(\operatorname{Der}_\omega \mathbb{L}(\alpha_1,\ldots,\alpha_g,\alpha_1^\#,\ldots,\alpha_g^\#)\Big)\langle 1\rangle$$

where $\operatorname{Der}_\omega$ indicates derivations that annihilate the element $\omega = \sum_i [\alpha_i,\alpha_i^\#]$. The action of $R^+(Z_g)$ (and thus by restriction also the action of $\Gamma_\partial(Z_{g,1})$) comes from the action on $H_*(Z_g;\mathbb{Q})$. Thus we obtain

Theorem 4.42. *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}_\partial(Z_{g,1});\mathbb{Q}) \cong H^*(\Gamma_\partial(Z_{g,1}); H^*_{\mathrm{CE}}(\mathfrak{g}_\partial(Z_{g,1}))). \qquad \square$$

Using this result, the stable cohomology of $B\operatorname{aut}_\partial(Z_{g,1})$ is computed by Stoll [65].

4.4. **A counterexample.** In all the examples discussed so far, the space X was always formal, the group $\mathcal{E}(X_\mathbb{Q})$ was reductive (hence equal to $R(X)$) and isomorphic to the group of algebra automorphisms of $H^*(X;\mathbb{Q})$, the group $\Gamma(X)$ was the group of automorphisms of $H_*(X;\mathbb{Q})$ that are induced by a self-homotopy equivalence of X, and the dg Lie algebra $\mathfrak{g}(X)$ was formal. In this section, we will discuss an example of a space X that has none of these properties. This example is taken from [38, Example 6.7].

Let X be the space $F\times K(\mathbb{Z},3)\times K(\mathbb{Z},6)$, where F is the homotopy fiber of the cup product

$$K(\mathbb{Z},3)\times K(\mathbb{Z},3) \xrightarrow{\smile} K(\mathbb{Z},6).$$

This is evidently a finite Postnikov section with finitely generated homotopy groups, and its Sullivan model is the cdga

$$\Lambda = \big(\Lambda(x,y,z,u,w),d\big),$$

whose underlying graded commutative algebra is freely generated by cocycles x,y,z in degree 3, a cocycle w in degree 6, and an element u in degree 5 which satisfies $du = yz$.

Remark 4.43. Note that X is indeed not rationally formal: the homomorphism

$$\operatorname{Aut}\Lambda \longrightarrow \operatorname{Aut}_{alg} H^*(\Lambda)$$

is not surjective. For any $\lambda\in\mathbb{Q}^\times$, there is a graded algebra automorphism ϕ of $H^*(\Lambda)$ given by $\phi(a) = \lambda^{|a|}a$. But ϕ affords no lift to $\operatorname{Aut}\Lambda$ unless $\lambda = 1$: any such lift $\widetilde{\phi}$ would have to have $\widetilde{\phi}(u) = \lambda^6 u$ so as to commute with d. But then $\widetilde{\phi}(uy) = \lambda^9 uy$ whereas $\phi(uy) = \lambda^8 uy$.

Proposition 4.44. *The group $R(X)$ is isomorphic to*

$$\mathsf{GL}_1^2 \times \mathsf{GL}_2\,.$$

A section of the surjection $\operatorname{Aut}\Lambda \to R(X)$ is given by letting the two GL_1 factors act by scalar multiplication on x and on w, respectively, and by letting $A\in\mathsf{GL}_2$ act via the standard representation on the linear span of y and z, and by scalar multiplication by $\det A$ on u. The subgroup $\Gamma(X)$ of $R(X)$ corresponds to

$$\big(\mathbb{Z}^\times\big)^2 \times \mathsf{GL}_2^\Sigma(\mathbb{Z})$$

under this isomorphism.

Remark 4.45. In [38, Example 6.7] the authors construct two automorphisms $\phi, \psi \in \operatorname{Aut}\Lambda$. ϕ is the identity on all generators but w, where $\phi(w) = w + xy$, and ψ is the identity on all generators but x, where $\psi(x) = x + z$. As observed in *loc. cit.*, these two automorphisms span a copy of $\mathbb{Z}^2$ in $\operatorname{Aut}^h \Lambda \cong \mathcal{E}(X_{\mathbb{Q}})$, but there exists no pair (ϕ', ψ') of homotopic automorphisms (or even automorphisms with the same effect on $H^*(\Lambda)$ as ϕ and ψ, respectively) which would already commute in $\operatorname{Aut}\Lambda$. This shows that we cannot in general expect to be able to lift the action of $\mathcal{E}(X_{\mathbb{Q}})$ on $\pi_* X \otimes \mathbb{Q}$ or on $H^*(X;\mathbb{Q})$ to an action on the minimal Sullivan model. It is clear from the proof of Proposition 4.44 that both ϕ and ψ lie in the kernel of $\operatorname{Aut}\Lambda V \to R(X)$, so they do not obstruct the existence of an action of $R(X)$ on Λ.

Proposition 4.46. *The dg Lie algebra* $\mathfrak{g}(X)$ *is not formal: the Massey product*

$$\left\langle y\frac{\partial}{\partial x}, x\frac{\partial}{\partial u}, z\frac{\partial}{\partial w} \right\rangle$$

is non-trivial.

The remainder of the section is devoted to proving Propositions 4.44 and 4.46.

We write V for the graded vector subspace of Λ which is spanned by the generators x, y, z, u and w, and we write ΛV for the underlying graded commutative algebra of Λ. The group $\mathsf{GL}(V)$ of graded linear automorphisms of V evidently injects into the group $\operatorname{Aut}\Lambda V$ of graded commutative automorphisms of ΛV, and we will abuse notation by referring to its image by $\mathsf{GL}(V)$ as well.

Lemma 4.47. *The automorphism group* $\operatorname{Aut}\Lambda V$ *of the underlying graded commutative algebra is a semidirect product*

$$\mathsf{GL}(V) \ltimes \operatorname{Hom}(V^6, \Lambda^2 V^3)$$

where the abelian group $\operatorname{Hom}(V^6, \Lambda^2 V^3)$ *injects into* $\operatorname{Aut}\Lambda V$ *by sending* $f\colon V^6 \to \Lambda^2 V^3$ *to the automorphism that is the identity on all generators except* w *and that takes* w *to* $w + f(w)$.

Proof. There is an obvious identification of V with the indecomposables $Q\Lambda V = I/I^2$, where $I = (\Lambda V)^{>0}$ is the augmentation ideal. This yields a retraction $\operatorname{Aut}\Lambda V \twoheadrightarrow \mathsf{GL}(Q\Lambda V) \cong \mathsf{GL}(V)$. Let $\operatorname{Aut}_1 \Lambda V$ be the kernel of this retraction. Every $\phi \in \operatorname{Aut}_1 \Lambda V$ fixes x, y, z and u, and is determined by the value $\phi(w) - w \in \Lambda^2 V^3 \subset (\Lambda V)^6$. It can be seen that there is an isomorphism

$$\operatorname{Aut}_1 \Lambda V \cong \operatorname{Hom}(V^6, \Lambda^2 V^3)$$

of groups with a $\mathsf{GL}(V)$-action which sends $\phi \in \operatorname{Aut}_1 \Lambda V$ to the homomorphism $w \mapsto \phi(w) - w$. □

The group $\operatorname{Aut}\Lambda$ of dgla automorphisms of Λ is the isotropy subgroup of the differential d inside $\operatorname{Aut}\Lambda V$. The subgroup $\operatorname{Aut}_1 \Lambda V$ of the preceding lemma clearly commutes with d, so it suffices to determine $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$.

Lemma 4.48. *Let* $W = \langle y, z\rangle \subset V^3$ *be the* $\mathbb{Q}$*-linear span of* y *and* z*. An automorphism* $\phi \in \mathsf{GL}(V)$ *commutes with* d *if and only if* W *is* ϕ*-stable, and* $\phi(u) = (\det \phi|_W) \cdot u$.

Proof. Note that the line $\Lambda^2 W$ is precisely the span of yz, and also the image of the differential $d\colon (\Lambda V)^5 \hookrightarrow (\Lambda V)^6$. Hence ϕ commutes with d if and only if it carries $\Lambda^2 W$ to itself and it acts on $\Lambda^2 W$ and $(\Lambda V)^5$ by the same scalar. But $\phi(\Lambda^2 W) = \Lambda^2 \phi(W)$ is equal to $\Lambda^2 W$ if and only if $\phi(W) = W$. If this is the case, then ϕ scales $\Lambda^2 W$ by $\det \phi|_W$. □

Proof of Proposition 4.44. The unipotent radical of $\operatorname{Aut}\Lambda V$, which we identified in Lemma 4.47, is contained in $\operatorname{Aut}\Lambda$, and hence in its unipotent radical. Thus the maximal reductive quotient of $\operatorname{Aut}\Lambda$ agrees with that of $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$. From Lemma 4.48 it is easily seen that the unipotent radical of $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$ consists of the automorphisms that fix all generators except x and that send x to x plus an element of W, and the maximal reductive quotient is as described in the statement of the proposition.

To determine $\Gamma(X)$, recall that X is the homotopy fiber of the 6th integral cohomology class

$$K(\mathbb{Z},3)^3 \times K(\mathbb{Z},6) \longrightarrow K(\mathbb{Z},6)$$

given by the cup product of the fundamental classes of two of the $K(\mathbb{Z},3)$ factors.

The integral cohomology ring of X agrees up to dimension 6 with

$$\Lambda_{\mathbb{Z}}(\bar{x},\bar{y},\bar{z},\bar{w})/(\bar{y}\bar{z}, 2\bar{x}^2, 2\bar{y}^2, 2\bar{z}^2)$$

where the generators $\bar{x},\bar{y}$ and $\bar{z}$ live in degree 3, $\bar{w}$ lives in degree 6, and each generator maps to the corresponding element of $H^*(\Lambda) \cong H^*(X;\mathbb{Q})$ under the change-of-coefficient homomorphism. Explicitly, $H^3(X) \cong \mathbb{Z}^3$ is spanned by $\bar{x},\bar{y},\bar{z}$, and $H^6(X) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3$ is spanned by $\bar{x}\bar{y}, \bar{x}\bar{z}, \bar{w}$ and $\bar{x}^2, \bar{y}^2, \bar{z}^2$.

Thus the homotopy class of an endomorphism $f\colon X \to X$ is uniquely determined by the quadruple $(f^*\bar{x}, f^*\bar{y}, f^*\bar{z}, f^*\bar{w}) \in H^3(X)^3 \times H^6(X)$, and a quadruple $(\bar{x}', \bar{y}', \bar{z}', \bar{w}')$ is induced by an endomorphism as soon as $\bar{y}' \smile \bar{z}' = 0$.

Thus an element $(\lambda, A, \mu) \in \mathsf{GL}_1 \times \mathsf{GL}_2 \times \mathsf{GL}_1 \cong R(X)$ can be realised by a homotopy automorphism of X if and only if $\lambda, \mu \in \mathbb{Z}$, the matrix A has integer entries, and each row of A contains at least one even number (since $(a\bar{y} + b\bar{z})(c\bar{y} + d\bar{z}) = ac\bar{y}^2 + bd\bar{z}^2 \in H^6(X)$ is non-zero if ac or bd are odd). □

Proof of Proposition 4.46. We begin by describing in some detail the structure of the dg Lie algebra $\mathfrak{g}(X)$. In positive degrees, $\mathfrak{g}(X)$ is just the full dg Lie algebra of derivations of the free cdga ΛV, and it is easy to list the basis in each degree.

In degree 0, $\mathfrak{g}(X)$ is the Lie algebra of the unipotent radical of $\operatorname{Aut}\Lambda$. This radical sits in a (split) short exact sequence

$$1 \longrightarrow \operatorname{Hom}(V^6, \Lambda^2 V^3) \longrightarrow \operatorname{Aut}_u \Lambda \longrightarrow \operatorname{Hom}(\langle x\rangle, W) \longrightarrow 1$$

where the left-hand term is the unipotent radical of $\operatorname{Aut}\Lambda V$ and the right-hand term is the unipotent radical of $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$. From this, a basis for $\mathfrak{g}_0(X)$ can be read off. We list bases in all degrees in the table below.

degree	basis
0	$y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ xy\frac{\partial}{\partial w},\ xz\frac{\partial}{\partial w},\ yz\frac{\partial}{\partial w}$
1	$u\frac{\partial}{\partial w}$
2	$x\frac{\partial}{\partial u},\ y\frac{\partial}{\partial u},\ z\frac{\partial}{\partial u}$
3	$x\frac{\partial}{\partial w},\ y\frac{\partial}{\partial w},\ z\frac{\partial}{\partial w},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$
5	$\frac{\partial}{\partial u}$
6	$\frac{\partial}{\partial w}$

The differential δ is given by

$$u\frac{\partial}{\partial w} \longmapsto yz\frac{\partial}{\partial w}, \qquad \frac{\partial}{\partial y} \longmapsto -z\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial z} \longmapsto y\frac{\partial}{\partial u};$$

it vanishes on all the other generators.

Note that

$$\left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial u}\right] = y\frac{\partial}{\partial u} = \delta\left(\frac{\partial}{\partial z}\right)$$

while the other two pairs bracket to zero. Thus the Massey product contains

$$\left[\frac{\partial}{\partial z}, z\frac{\partial}{\partial w}\right] = \frac{\partial}{\partial w}.$$

The indeterminacy is equal to the subset

$$\left[y\frac{\partial}{\partial x}, H_6(\mathfrak{g}(X))\right] + \left[x\frac{\partial}{\partial u}, H_4(\mathfrak{g}(X))\right] + \left[z\frac{\partial}{\partial w}, H_3(\mathfrak{g}(X))\right]$$

of $H_6(\mathfrak{g}(X))$. This is easily seen to only contain 0. Thus the Massey product is indeed non-trivial as $\frac{\partial}{\partial w}$ is a non-trivial cycle, presenting an obstruction to formality of $\mathfrak{g}(X)$. $\square$

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