BOTT TOWERS, CROSSPOLYTOPES AND TORUS ACTIONS

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ABSTRACT. We study the geometry and topology of Bott towers in the context of toric geometry. We show that any kth stage of a Bott tower is a smooth projective toric variety associated to a fan arising from a crosspolytope; conversely, we prove that any toric variety associated to a fan obtained from a crosspolytope actually gives rise to a Bott tower. The former leads us to a description of the tangent bundle of the kth stage of the tower, considered as a complex manifold, which splits into a sum of complex line bundles. Applying Danilov-Jurkiewicz theorem, we compute the cohomology ring of any kth stage, and by way of construction, we provide all the monomial identities defining the related affine toric varieties.

1. Introduction

The theory of toric varieties offers a remarkable area for studying algebro-geometric and topological problems in the language of combinatorial objects, called fans, much like simplicial complexes. Many of the algebraic or topological properties of toric varieties are encoded in the associated fans. In this direction, our purpose here is to investigate the toric structure of some complex manifolds, known as Bott towers. These manifolds include such families of complex manifolds as the Bott-Samelson varieties, and as explained by Grossberg and Karshon [15], their study combines areas such as representation theory, combinatorics and algebraic geometry.

Since the theory of toric varieties has been widely studied in currently active areas of mathematics, it is impossible to find either a commonly accepted definition or a fixed notation. There are two common approaches for constructing a toric variety from given combinatorial data. The first is due to algebraic geometers in which it is formed by gluing affine algebraic varieties; it has the advantage of capturing the essence of turning a combinatorial object into an algebraic one, namely that its local structure resembles the whole. The second owes its existence to symplectic geometers, and it is more explicit in the sense that the variety can be constructed as the quotient of a complex space by the action of an algebraic torus. Several comprehensive textbooks are now available, by authors such as Ewald [13], Fulton [14], and Oda [17]. Throughout, however, we follow Batyrev and Ewald, so we refer readers to [2] and [13] for background notation.

By a definition, a Bott tower is defined inductively as an iterated projective bundle so that each stage of the tower is of the form $\mathbb{C}P(\mathbb{C} \oplus \xi)$ for an arbitrarily chosen line bundle ξ over the previous stage. It produces a sequence of fibered projective spaces with fibers

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isomorphic to $\mathbb{C}P^1$. A classic example is provided by taking ξ to be the trivial line bundle, so each stage of the tower, thus obtained, is the product of projective lines. In Section 2, by incorporating Grossberg and Karshon's construction, we present the kth stage N_k of any Bott tower of height n as a smooth projective toric variety for each $1 \leq k \leq n$, describing explicitly the associated smooth fan arising from a crossploytope. To achieve this, we introduce the notion of *Bott numbers*, which may be of particular interest to combinatorialists. Some properties of these numbers allow us to reveal the defining Laurent monomials of affine toric varieties associated to Bott towers. We also provide an example showing that Bott towers are not Fano varieties in general. We note that a Bott tower of height 2 is actually a Hirzebruch surface. Moreover, we show that bounded flag manifolds are also examples of Bott towers, and the results of [5] therefore suggest that they might have a role to play in complex bordism and cobordism theory that has yet to be revealed.

Toric geometry is something of a two-way study in the sense that we may start with a normal algebraic variety which contains the algebraic torus as a dense open subset, and then recover the associated combinatorial data. Conversely, beginning with a fan, we can construct such a variety. Following the latter pattern, we prove that any toric variety associated to a smooth fan arising from a crosspolytope is actually a Bott tower.

One of the advantage of having the toric structure of Bott towers is that we may easily describe their tangent bundles. According to Ehlers [12], the tangent bundle of a toric variety splits as a sum of line bundles (known as the *hyperplane bundles*), and each of these bundles is obtained as the pull-back of the normal bundle of a codimension-one subvariety corresponding to a one-dimensional cone of the associated fan. In a parallel work [8], we use such a splitting to assist our computation of real and complex K-groups of Bott towers and determine various structures (almost and stably complex structure, etc.) on them.

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2. Bott Towers

We begin with a precise description of Bott towers. The way we display these objects fits into the setting of [18], and we provide an alternative proof for Grossberg and Karshon's construction [15].

Let ξ_0 be a trivial line bundle over a single point $N_0 = *$, and let $N_1 := \mathbb{C}P(\mathbb{C} \oplus \xi_0) = \mathbb{C}P^1$. Similarly, we may choose any holomorphic line bundle ξ_1 over N_1, and take its direct sum with the trivial line bundle. By projectifying each fiber, we obtain a manifold $N_2 = \mathbb{C}P(\mathbb{C} \oplus \xi_1)$, which is a bundle over N_1 with fiber $\mathbb{C}P^1$. We may repeat this process n times, so that each N_k is a $\mathbb{C}P^1$-bundle over N_{k-1} (see Figure 1). We can consider each N_k as the space of lines in $\mathbb{C} \oplus \xi_{k-1}$. If we think of $\mathbb{C}P^1$ as a sphere with a south pole $[1, 0]$ and a north pole $[0, 1]$, then the zero section of ξ_{k-1} gives rise to a holomorphic section, the *south pole section*:

$$i_k^S \colon N_{k-1} \to N_k;$$

$$
\begin{array}{l}
\qquad\qquad\qquad \mathbb{C}P(\mathbb{C}\oplus\xi_{n-1}) = N_n \\
\qquad\qquad\quad \swarrow \pi_n \\
\qquad\qquad\ \ \vdots \\
\qquad\qquad \swarrow \pi_3 \\
\qquad \mathbb{C}P(\mathbb{C}\oplus\xi_1) = N_2 \\
\quad\ \ \swarrow \pi_2 \\
\mathbb{C}P(\mathbb{C}\oplus\xi_0) = N_1 \\
\ \swarrow \pi_1 \\
* = N_0
\end{array}
$$

FIGURE 1. A Bott tower of height n

similarly, we may obtain the *north pole section* $i_k^N\colon N_{k-1} \to N_k$ by allowing the first coordinate in $\mathbb{C}P(\mathbb{C}\oplus\xi_{k-1})$ to vanish.

Definition 2.1. A *Bott tower of height* n is defined to be a collection of complex manifolds $\{N_k\colon k \leq n\}$, constructed by the above process. The bundles $\xi_0, \ldots, \xi_{n-1}$ are called the *associated line bundles* of the tower.

Example 2.2. The collection $\{(\mathbb{C}P^1)^k := \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1(k \text{ times})\colon k \leq n\}$ is a Bott tower. In this case, each associated line bundle ξ_k is chosen to be trivial for every $0 \leq k \leq n-1$, and π_k is the obvious projection.

Our next example has a particular importance in complex cobordism theory and the resulting manifolds are known as bounded flag manifolds (see [5]). They were, in fact, introduced by Bott & Samelson [3].

We write $[n]$ for the set of natural numbers $\{1, 2, \ldots, n\}$ with the standard linear ordering and an interval in the poset $[n]$ has the form $[i, j]$ for some $1 \leq i \leq j \leq n$. Throughout, $\omega_1, \ldots, \omega_{n+1}$ will denote the standard basis vectors in $\mathbb{C}^{n+1}$, and we write $\mathbb{C}_I$ for the subspace spanned by the vectors $\{\omega_i\colon i \in I\}$, where $I \subset [n+1]$.

Definition 2.3. A flag $U\colon 0 < U_1 < \ldots < U_n < \mathbb{C}^{n+1}$ is called *bounded* if $\mathbb{C}_{[i-1]} < U_i$ for each $1 \leq i \leq n$. The space of all bounded flags in $\mathbb{C}^{n+1}$ is called *bounded flag manifold*, which is an n-dimensional smooth complex manifold and will be denoted by $B(\mathbb{C}^{n+1})$ (or simply by B_n when there is no confusion).

As a consequence of the definition, each factor U_i of any bounded flag $U \in B(\mathbb{C}^{n+1})$ is of the form $\mathbb{C}_{[i-1]} \oplus L_i$, where L_i is a line in $\mathbb{C}_i \oplus L_{i+1}$ for $1 \leq i \leq n$, and $L_{n+1} = \mathbb{C}_{n+1}$. Therefore, we may display U as

$$U\colon 0 < L_1 < \mathbb{C}_1 \oplus L_2 < \ldots < \mathbb{C}_{[n-1]} \oplus L_n < \mathbb{C}^{n+1}. \tag{2.4}$$

We define maps q_i and $r_i\colon B(\mathbb{C}^{n+1}) \to \mathbb{C}P_{[i,n+1]}$ by letting $q_i(U) = L_i$ and $r_i(U) = L_i^\perp$, where $L_i^\perp$ is the orthogonal complement of L_i in $\mathbb{C}_i \oplus L_{i+1}$ for each $U \in B(\mathbb{C}^{n+1})$, and

$1 \leq i \leq n$. We consider complex line bundles η_i and $\eta_i^{\perp}$ over B_n, classified respectively by the maps q_{n-i+1} and r_{n-i+1} for every $1 \leq i \leq n$, and we set η_0 to be the trivial line bundle with fiber $\mathbb{C}_{n+1}$. We sometimes refer them as the *canonical line bundles* on B_n.

Theorem 2.5. *The collection* $\{B(\mathbb{C}_{[n-k+1,n+1]})\colon k \leq n\}$ *of bounded flag manifolds is a Bott tower of height* n*, where* $B(\mathbb{C}_{[n-k+1,n+1]})$ *denotes the set of bounded flags in* $\mathbb{C}_{[n-k+1,n+1]}$.

Proof. As above, we may define maps

$$q_i \quad \text{and} \quad r_i\colon B(\mathbb{C}_{[n-k+1,n+1]}) \to \mathbb{C}P_{[i,n+1]} \quad \text{for } n-k+1 \leq i \leq n+1,$$

and the respective complex line bundles $\eta_0, \eta_1, \ldots, \eta_k$ over $B(\mathbb{C}_{[n-k+1,n+1]})$. We proved in [6] that the complex manifolds $\mathbb{C}P(\mathbb{C}_{n-k+1} \oplus \eta_{k-1})$ and $B(\mathbb{C}_{[n-k+1,n+1]})$ are diffeomorphic for each $1 \leq k \leq n$. Therefore, we can take $B(\mathbb{C}_{[n-k+1,n+1]})$ to be the kth stage of the tower; hence, we may abbreviate $B(\mathbb{C}_{[n-k+1,n+1]})$ to B_k. The projection $\pi_k\colon B_k \to B_{k-1}$ maps each flag

$$U_k\colon 0 < L_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}$$

in B_k to the flag

$$U_{k-1}\colon 0 < L_{n-k+2} < \mathbb{C}_{n-k+2} \oplus L_{n-k+3} < \ldots < \mathbb{C}_{[n-k+2,n-1]} \oplus L_n < \mathbb{C}_{[n-k+2,n+1]}$$

in B_{k-1}, whose fiber consists of the lines in $\mathbb{C}_{n-k+1} \oplus L_{n-k+2}$, and is therefore isomorphic to $\mathbb{C}P^1$. The south and north pole sections i_k^S and $i_k^N\colon B_{k-1} \to B_k$ are given respectively by

$$i_k^S(U_{k-1}) := 0 < \mathbb{C}_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]},$$

and

$$i_k^N(U_{k-1}) := 0 < L_{n-k+2} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}.$$

$\square$

3. Bott towers as toric varieties

As we explained earlier, one way of constructing toric varieties is to display them as the quotient of a complex space by the action of an algebraic torus (see [2]). To be more explicit, if Σ is a smooth and complete fan in $\mathbb{R}^n$ with the generating set $G(\Sigma) = \{x_1, \ldots, x_m\}$, then each primitive collection $\mathcal{P}$ in $G(\Sigma)$ defines an affine subspace in $\mathbb{C}^m$. Running over all primitive collections, we let $\mathcal{U}(\Sigma)$ be the complement of the union of corresponding affine subspaces. On the other hand, from the kernel $\mathcal{R}(\Sigma)$ of the map $\pi_\Sigma\colon \mathbb{Z}^m \to \mathbb{Z}^m$ defined by $\pi_\Sigma(e_i) := x_i$, where $e_1, \ldots, e_m$ are the standard basis vectors of $\mathbb{R}^m$, we obtain an algebraic subtorus $\mathcal{D}(\Sigma)$ that acts on $\mathcal{U}(\Sigma)$, so that the quotient $\mathcal{U}(\Sigma)/\mathcal{D}(\Sigma)$ is the associated toric variety X_Σ.

In the case of Bott towers, Grossberg and Karshon [15] have already been able to describe explicitly how to construct Bott towers as quotients. By incorporating their work, we may readily present any Bott tower as a sequence of smooth projective toric varieties. We first introduce some notation.

3.1. **Bott Numbers.** Let $\{c(i,j)\colon 1 \le i < j \le n\}$ be a collection of $n(n-1)/2$ arbitrary integers. Then, by setting $c(i,i) := 1$ for each $1 \le k \le n$, we denote the k-tuple $(c(1,k),\dots,c(k,k))$ in $\mathbb{Z}^k$ by c_k for each $1 \le k \le n$. Moreover, we let $\mathbf{c} := (c_1,\dots,c_n)$ be the resulting integral sequence.

We consider an upper triangular $n \times n$-matrix $\mathcal{C}(n)$, whose kth row is given by the vector $(0,\dots,0,c(k,k),c(k,k+1),\dots,c(k,n))$ for each $1 \le k \le n$. In otherwords, if $\mathcal{C}(n) = (c_{ij})$, then

$$c_{ij} := \begin{cases} c(i,j), & \text{if } i \le j \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.1. Let $\mathbf{c} = (c_1,\dots,c_n)$ be an integral sequence. We call the upper triangular matrix $\mathcal{B}(n)$ satisfying

$$\mathcal{C}(n)^{-1} = -\mathcal{B}(n) \tag{3.2}$$

the *Bott matrix* associated to $\mathbf{c}$. Inparticular, if we write $\mathcal{B}(n) = (b(i,j))$, we then have

$$\begin{aligned} b(i,i) &= -1 \quad \text{and} \quad b(i,j) = 0 \text{ if } i > j \\ b(i,j) &= -\sum_{i<k\le j} c(i,k)b(k,j) \quad \text{if } i < j. \end{aligned} \tag{3.3}$$

The numbers $b(i,j)$ for all $1 \le i < j \le n$ are said to be the *Bott numbers* associated to $\mathbf{c}$.

We may readily extend the definition of a Bott matrix to any subset of $[n]$ as follows. Let $I = \{i_1,\dots,i_m\} \subset [n]$ be given such that $1 \le i_1 < \dots < i_m \le n$. We then form an upper triangular $m \times m$-matrix $\mathcal{C}(I) := (c_{rs})$ by

$$c_{rs} := \begin{cases} c(i_r,i_s), & \text{if } r \le s \\ 0, & \text{otherwise,} \end{cases}$$

and obtain the associated *Bott matrix* $\mathcal{B}(I)$ of I satisfying $\mathcal{C}(I)^{-1} = -\mathcal{B}(I)$. In this case, if we denote the top-right-corner entry of $\mathcal{B}(I)$ by $b(I)$, we then call $b(I)$ the *Bott number* of I associated to $\mathbf{c}$.

There is also an alternative way to describe the Bott number $b(I)$ as follows. Let $J \subset [n]$ be given such that $\min(J) = i$ and $\max(J) = j$. Then, for any subset $L = \{l_1,\dots,l_k\} \subseteq J$ with $l_1 < \dots < l_k$, we associate an integer with it by

$$p(L,J) := c(i,l_1)c(l_1,l_2)\dots c(l_k,j), \tag{3.4}$$

and if $L = \emptyset$, we set $p(\emptyset,J) := c(i,j)$. Moreover, if we define $c(J) := \sum_{L\subseteq J}(-1)^{|L|}p(L,J)$, we may easily deduce that

$$b(I) = \sum_{J\subseteq I} c(J). \tag{3.5}$$

for any $I \subset [n]$ such that $|I| > 1$.

We now discuss some properties of Bott numbers associated to some specific subsets of $[n]$, which we need in Section 3.3. Let $\mathcal{S}^k$ denote the set of binary codes of length k, where $\mathcal{S} = \{0, 1\}$. For a given $w = w_1 \dots w_k \in \mathcal{S}^k$, we define

$$\mathbb{I}_i^j := \{l \in (i, j) \colon w_l = 1\} \quad \text{and} \quad \mathbb{O}_i^j := \{l \in (i, j) \colon w_l = 0\} \tag{3.6}$$

for any $1 \leq i < j \leq k$. We note that since $\mathbb{O}_i^j \subset [n]$, the integer $b(\mathbb{O}_i^j)$ is defined as above.

Lemma 3.7. *Let $w \in \mathcal{S}^k$ be given. Then, for any $1 \leq i < j \leq k$, we have*

$$\textbf{(i)} \quad \sum_{l \in \mathbb{I}_i^j \cup \{i\}} b(\mathbb{O}_l^j) b(i, l) + b(i, j) = 0, \tag{3.8}$$

$$\textbf{(ii)} \quad \sum_{l \in \mathbb{I}_i^j \cup \{j\}} b(\mathbb{O}_i^l) b(l, j) + b(i, j) = 0. \tag{3.9}$$

Proof. It is obvious that proving (3.8) is equivalent to show that

$$b(i, j) = b(\mathbb{O}_i^j) - \sum_{l \in \mathbb{I}_i^j} b(\mathbb{O}_l^j) b(i, l) \tag{3.10}$$

which follows from (3.5). □

3.2. **Toric Structures.** Once we have described Bott numbers, we may easily provide the toric structure of Bott towers. We begin with recalling the definition of our central combinatorial objects, namely crosspolytopes (see [4]).

Definition 3.11. Let P^1 be a line segment in $\mathbb{R}^n$. We proceed by induction, and assume that P^k is defined for some $k > 1$. Let I_{k+1} be a line segment such that the intersection

$$(\text{relint } P^k) \cap (\text{relint } I_{k+1}) = \{q\},$$

is a single point. Then we define $P^{k+1} := \text{conv}(P^k \cup I_{k+1})$. The polytope thus obtained is called a $(k+1)$-*crosspolytope*. It easily follows that the vertex set of an n-crosspolytope P^n can be given as

$$V(P^n) = \{x_1, \dots, x_n, \widehat{x}_1, \dots, \widehat{x}_n\}$$

such that the line segment $[x_i, \widehat{x}_i]$ is not an edge of P^n for all $i = 1, \dots, n$.

For a given integral sequence $\mathbf{c} = (c_1, \dots, c_n)$, we consider a k-dimensional fan $\Sigma(k)$ in $\mathbb{R}^k$ for each $k \leq n$ with the generating set $G(k) = \{a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1}\}$ defined by

$$a_{i,\gamma} := \begin{cases} e_i, & \text{if } \gamma = 0, \\ (0, \dots, 0, b(i, i), b(i, i+1), \dots, b(i, k)), & \text{if } \gamma = 1 \end{cases} \tag{3.12}$$

for any $1 \leq i \leq k$. The k-dimensional cones of $\Sigma(k)$ are given by

$$\sigma(w) := \mathbb{R}_{\geq} a_{1,w_1} + \dots + \mathbb{R}_{\geq} a_{k,w_k} \tag{3.13}$$

for any $w := w_1 \dots w_k \in \mathcal{S}^k$.

Proposition 3.14. *The fan $\Sigma(k)$ is smooth and projective for the integral sequence $(c_1, \dots, c_k)$, which is the first k-tuple of $\mathbf{c}$.*

Proof. We consider the intersection points of a unit sphere centered at the origin with the one-dimensional cones of $\Sigma(k)$. The convex hull of those points is clearly a crosspolytope which spans $\Sigma(k)$. Smoothness follows from the construction. $\square$

We note that the smooth toric variety $X_{\Sigma(k)}$ associated to the fan $\Sigma(k)$ is not a Fano variety in general (see Figure 2). We may now apply Batyrev construction's (see [[2], p.4]

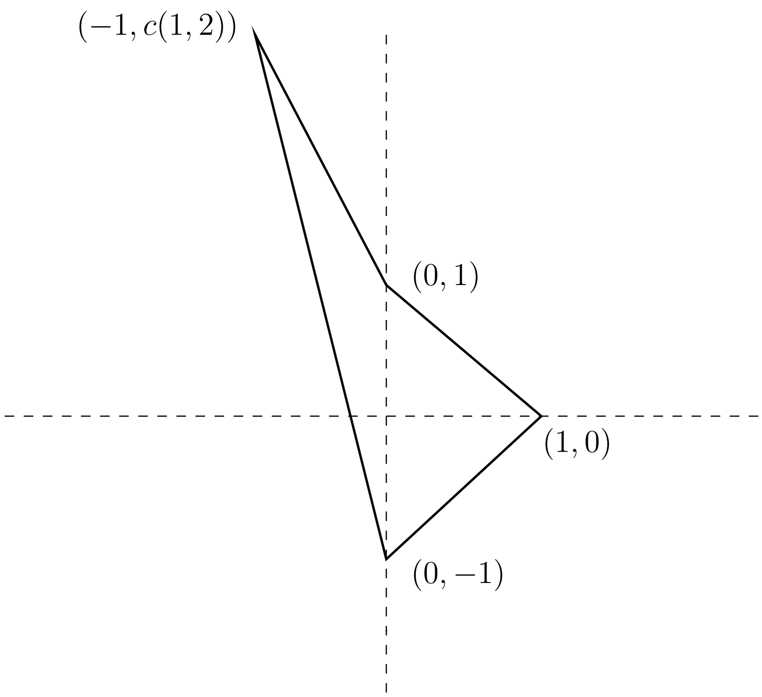

FIGURE 2. The fan of a Bott tower of height 2, which is not a Fano variety.

to the fan $\Sigma(k)$. Since $\Sigma(k)$ is spanned by a crosspolytope, the space $\mathcal{U}(\Sigma(k))$ is $(\mathbb{C}^2\backslash 0)^k$. Moreover, the group $\mathcal{R}(\Sigma(k))$ being the kernel of the map

$$\begin{aligned}\pi_k\colon \mathbb{Z}^{2k} &\to \mathbb{Z}^k,\\ e_{2r-1} &\mapsto e_r,\\ e_{2r} &\mapsto (0,\dots,0,b(r,r),b(r,r+1),\dots,b(r,k))\end{aligned}$$

for $1 \leq r \leq k$, is generated by the following k vectors

$$\begin{aligned}&(1,c(1,1),0,c(1,2),0,c(1,3),0,\dots,0,c(1,k)),\\ &(0,0,1,c(2,2),0,c(2,3),0,\dots,0,c(2,k)),\\ &\vdots\\ &(0,0,\dots,0,1,c(k-1,k-1),0,c(k-1,k)),\\ &(0,0,\dots,0,1,c(k,k)).\end{aligned}$$

Therefore, the corresponding k-dimensional algebraic torus $\mathcal{D}(\Sigma(k))$ is given by

$$\{(t_1, t_1; t_2, t_1^{c(1,2)} t_2; \ldots; t_k, t_1^{c(1,k)} t_2^{c(2,k)} \ldots t_{k-1}^{c(k-1,k)} t_k) \colon t_i \in \mathbb{C}_\times \text{ for } 1 \le i \le k\},$$

where $\mathbb{C}_\times$ is the algebraic torus. Furthermore the group $\mathcal{D}(\Sigma(k))$ acts on $(\mathbb{C}^2 \backslash 0)^k$ diagonally. We denote the resulting smooth and projective toric variety associated to $\Sigma(k)$ by $X_{\Sigma(k)}$.

Example 3.15. We consider an integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$ given for each $1 \le k \le n$ by $c_k = (0, 0, \ldots, 0, 1) \in \mathbb{Z}^k$, and let $\{M_k \colon k \le n\}$ be the collection of associated complex manifolds. The fan $\Sigma(k)$ constructed from $(c_1, \ldots, c_k)$ has the generating set $G(\Sigma(k)) = \{e_1, \ldots, e_k, -e_1, \ldots, -e_k\}$; hence, the associated toric variety is $(\mathbb{C}P^1)^k$. Therefore, we have $X_{\Sigma(k)} = (\mathbb{C}P^1)^k$ for any $1 \le k \le n$.

We next provide an alternative proof of Grossberg and Karshon's construction in a toric setting. Our method basically applies to the relation between linear support functions of fans and holomorphic line bundles over toric varieties (see [17]).

Theorem 3.16. *For a given integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$, the family of manifolds $\{X_{\Sigma(k)} \colon k \le n\}$ is a Bott tower, and any Bott tower arises in this way.*

Proof. Let $\{X_{\Sigma(k)} \colon k \le n\}$ be given. We proceed by induction on k. When $k = 1$, there is nothing to prove, since $X_{\Sigma(1)} = \mathbb{C}P^1$. Therefore, we may assume that there exist line bundles $\xi_0, \ldots, \xi_{k-1}$ for some $k < n$ such that $X_{\Sigma(l)} = \mathbb{C}P(\mathbb{C} \oplus \xi_{l-1})$ for any $1 \le l \le k$. Then, it can be easily verified that the function $h_k \colon |\Sigma(k)| \to \mathbb{R}$ given by

$$h_k(a_{i,\gamma}) := \begin{cases} 0, & \text{if } \gamma = 0, \\ b(i, k+1), & \text{if } \gamma = 1 \end{cases} \tag{3.17}$$

for each $1 \le i \le k$ is a $\Sigma(k)$-linear support function. However, each such function h_k defines an equivariant line bundle $L(h_k)$ over $X_{\Sigma(k)}$. On the other hand, the space $\mathbb{C}P(\mathbb{C} \oplus L(h_k))$ is a smooth projective toric variety whose associated fan may be described as the join $\Sigma \cdot \widetilde{\Sigma}$, where $\Sigma := \{\mathbb{R}_{\ge} e_{k+1}, \mathbb{R}_{\ge}(-e_{k+1}), \{0\}\}$ and $\widetilde{\Sigma} := \Psi(\Sigma)$, and the linear map $\Psi \colon \mathbb{R}^k \to \mathbb{R}^{k+1}$ is given by $y \mapsto (y, h_k(y))$. Since we have $\Sigma(k+1) = \Sigma \cdot \widetilde{\Sigma}$, we deduce that $X_{\Sigma(k+1)} \cong \mathbb{C}P(\mathbb{C} \oplus L(h_k))$, which completes the first part of the assertion.

Let a Bott tower $\{N_k \colon k \le n\}$ of height n be given with the associated line bundles $\xi_0, \ldots, \xi_{n-1}$. Once again, we proceed by induction on k. Since the base case is obvious, we may assume that there exists an integral sequence $(c_1, \ldots, c_k)$ for which $N_k = \mathbb{C}P(\mathbb{C} \oplus \xi_{k-1})$ is of the form $X_{\Sigma(k)}$ for some $1 < k < n$. Let ξ_k be any holomorphic line bundle over $N_k = X_{\Sigma(k)}$, and without loss of generality, we assume that it is nontrivial. Then, there exists a $\Sigma(k)$-linear support function $h \colon |\Sigma(k)| \to \mathbb{R}$ such that the bundles $L(h)$ and ξ_k are isomorphic. Since h is $\Sigma(k)$-linear, there exists $r(w) = (r_1, \ldots, r_k) \in \mathbb{Z}^k$ for any $w = w_1 \ldots w_k \in \mathcal{S}^k$ such that

$$h(x) = \langle r(w), x \rangle \tag{3.18}$$

for all $x \in \sigma(w)$. Specifically, we consider the binary code $w(1) := 11 \ldots 1$ and its associated integral vector $r(1) := (r_1^1, \ldots, r_k^1) \in \mathbb{Z}^k$ satisfying (3.18). On the other hand, if $w_i = 0$ for any given $w \in \mathcal{S}^k$ and for some $1 \le i \le k$, then $h(a_{i,0}) = \langle r(w), e_i \rangle = r_i =: X_i \in \mathbb{Z}$, where $r(w) = (r_1, \ldots, r_k)$ is the associated integral vector of w. If we define

$$c(1, k+1) := X_1 - r_1^1, \ldots, c(k, k+1) := X_k - r_k^1,$$

then we necessarily have

$$h(a_{i,\gamma}) = \begin{cases} X_i, & \text{if } \gamma = 0, \\ b(i, k+1) - X_i + X_{i+1} b(i, i+1) + \ldots + X_k b(i, k), & \text{if } \gamma = 1 \end{cases}$$

for each $1 \le i \le k$ by (3.3). Let $\Psi_k \colon \mathbb{R}^k \to \mathbb{R}^{k+1}$ be a linear map given by $y \mapsto (y, h(y))$. We define

$$\widetilde{\Sigma} := \{\Psi_k(\sigma) \colon \sigma \in \Sigma(k)\} \quad \text{and} \quad \Sigma' := \widetilde{\Sigma} \cdot \Sigma_1,$$

where $\Sigma_1 := \{\mathbb{R}_{\ge} e_{k+1}, \mathbb{R}_{\ge}(-e_{k+1}), \{0\}\}$. Then, any $(k+1)$-dimensional cone of Σ' is of the form

$$\sigma(w) = \mathbb{R}_{\ge} A_{1,w_1} + \ldots + \mathbb{R}_{\ge} A_{k+1,w_{k+1}},$$

where

$$A_{i,\gamma} := \begin{cases} (0, \ldots, 0, 1, 0, \ldots, X_i), & \text{if } \gamma = 0 \\ (0, \ldots, 0, -1, f(i, i+1), \ldots, f(i, k), h(a_{i,1})), & \text{if } \gamma = 1. \end{cases}$$

Let $\Sigma(k+1)$ be the fan associated to the integral sequence $(c_1, \ldots, c_{k+1})$ constructed as in (3.12). We consider the following map

$$\begin{aligned} L \colon \mathbb{R}^{k+1} &\to \mathbb{R}^{k+1}, \\ e_i &\mapsto (0, \ldots, 0, 1, 0, \ldots, 0, X_i) \quad \text{for } 1 \le i \le k, \\ e_{k+1} &\mapsto e_{k+1}, \end{aligned}$$

which is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ X_1 & X_2 & \ldots & X_k & 1 \end{pmatrix}.$$

It easily follows that L is unimodular, and it maps the cones of $\Sigma(k+1)$ bijectively onto the cones of Σ'; hence, $\Sigma(k+1) \cong \Sigma'$. However, this guarantees that the space $\mathbb{C}P(\mathbb{C} \oplus L(h)) \cong \mathbb{C}P(\mathbb{C} \oplus \xi_k)$ is a toric variety associated to the fan $\Sigma(k+1)$ with a generating set $\{a_{1,0}, a_{1,1}, \ldots, a_{k+1,0}, a_{k+1,1}\}$ defined by (3.12) for the integral sequence $(c_1, \ldots, c_{k+1})$. $\square$

We note that every finite poset defines a Bott tower in the following way. Let P be a finite poset with n elements and let $Z(P)$ be its Zeta matrix (see [19]). By taking $\mathcal{C}(n) := Z(P)$, then the corresponding Bott matrix is given by $\mathcal{B}(n) = -M(P)$, where $M(P)$ is the Moebius matrix of P, from which we can construct a Bott tower of height n.

Corollary 3.19. *Let β be a holomorphic line bundle over the kth stage of a Bott tower $\{N_k\colon k \leq n\}$. Then there exist integers $b_1, \ldots, b_k$ such that*

$$\beta \cong (\mathbb{C}^2\backslash 0)^k \times_{\mathbb{C}^k_\times} \mathbb{C}, \tag{3.20}$$

where the action of $\mathbb{C}^k_\times$ is given by

$$((x,y),v)\cdot t := ((x,y)\cdot t, t_1^{b_1}\ldots t_k^{b_k}v) \tag{3.21}$$

for any $t = (t_1, \ldots, t_k) \in \mathbb{C}^k_\times$.

One of the important consequence of Corollary 3.19 is that each bundle ξ_k associated to the $(k+1)$th stage N_{k+1} of the tower for $1 \leq k \leq n-1$ is given over N_k by the action

$$((x,y),v)\cdot t := ((x,y)\cdot t, t_1^{c(1,k+1)}\ldots t_k^{c(k,k+1)}v) \tag{3.22}$$

for all $t = (t_1, \ldots, t_k) \in \mathbb{C}^k_\times$.

Corollary 3.23. *The integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$ associated to Bott tower of bounded flag manifolds $\{B_k\colon k \leq n\}$ is given by $c_k := (0, \ldots, 0, -1, 1) \in \mathbb{Z}^k$ for each $1 \leq k \leq n$.*

Proof. In order to retain the notation that was already introduced, a slight modification of the quotient description of the varieties $X_{\Sigma(k)}$ is required to construct bounded flag manifolds as toric varieties. The obvious reason is that the first stage of the tower $\{B(\mathbb{C}_{[n-k+1,n+1]})\colon k \leq n\}$ is the projectivization of $\mathbb{C}_n \oplus \mathbb{C}_{n+1}$. We therefore define $X_{\Sigma(k)}$ for each $k \leq n$ to be the quotient of $(\mathbb{C}^2\backslash 0)^k$ by the k-fold algebraic torus $\mathbb{C}^k_\times$, under the action

$$\begin{aligned}(x_1, y_1; \ldots; x_k, y_k)\cdot(t_1, \ldots, t_k) :=&\\ &(x_1t_1, y_1t_1t_2^{-1}; \ldots; x_{k-1}t_{k-1}, y_{k-1}t_{k-1}t_k^{-1}; x_kt_k, y_kt_k).\end{aligned} \tag{3.24}$$

Therefore, the integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$ associated to $\{X_{\Sigma(k)}\colon k \leq n\}$ is given by $c_k := (0, \ldots, 0, -1, 1) \in \mathbb{Z}^k$ for each $1 \leq k \leq n$.

For a given vector $(x,y) \in (\mathbb{C}^2\backslash 0)^k$, we define

$$\begin{aligned} l_{n+1} &:= \omega_{n+1},\\ l_n &:= x_k\omega_n + y_kl_{n+1},\\ &\vdots\\ l_{n-k+1} &:= x_1\omega_{n-k+1} + y_1l_{n-k+2}.\end{aligned} \tag{3.25}$$

If L_j denotes the line in $\mathbb{C}_{[n-k+1,n+1]}$ spanned by the vector l_j for $n-k+1 \leq j \leq n$, then

$$U(x,y)\colon 0 < L_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}$$

is a bounded flag in $B(\mathbb{C}[n-k+1, n+1])$ determined by the vector (x,y). Conversely, for any bounded flag $U \in B(\mathbb{C}[n-k+1, n+1])$, we can find $(x,y) \in (\mathbb{C}^2\backslash 0)^k$ such that $U = U(x,y)$; however, such a vector is not always unique. The claim follows from the fact that if we denote the orbit through $(x,y) \in (\mathbb{C}^2\backslash 0)^k$ by $[x,y]$, then the map $\Gamma_k\colon X_{\Sigma(k)} \to B(\mathbb{C}[n-k+1, n+1])$ defined by $\Gamma_k([x,y]) := U(x,y)$ is an equivariant diffeomorphism of complex manifolds for $1 \leq k \leq n$ (see [6]). $\square$

Corollary 3.26. *The associated smooth fan $\Sigma(n)$ of the bounded flag manifold B_n has the generating set $G(\Sigma(n)) = \{a_{1,0}, a_{1,1}, \ldots, a_{n,0}, a_{n,1}\}$, where*

$$a_{j,\gamma} := \begin{cases} e_j & \text{if } \gamma = 0, \\ -e_1 - \ldots - e_j & \text{if } \gamma = 1 \end{cases} \tag{3.27}$$

for $1 \le j \le n$.

3.3. **Local Structures.** Even though we have displayed Bott towers as quotients, since we know their associated smooth fans, we may also describe their local structures in a toric setting, namely affine toric varieties forming each $X_{\Sigma(k)}$. In particular, we note that the collection of k-dimensional affine toric varieties provides an atlas for the manifold $X_{\Sigma(k)}$.

Definition 3.28. For a given $w = w_1 \ldots w_k \in \mathcal{S}^k$, we define

$$\mathbb{I}(j) := \{l \in [1, j) \colon w_l = 1\} \tag{3.29}$$

for any $1 \le j \le k$.

Definition 3.30. Let a binary code $w = w_1 \ldots w_k \in \mathcal{S}^k$ be given. We define vectors in $\mathbb{R}^k$ for each $1 \le j \le k$ by

$$\upsilon_{j,w_j} := \begin{cases} \sum_{i\in\mathbb{I}(j)} (-1)^{w_j}\, b(\mathbb{O}_i^j)\, e_i + (-1)^{w_j}\, e_j, & \text{if } \mathbb{I}(j) \neq \emptyset, \\ (-1)^{w_j} e_j, & \text{otherwise.} \end{cases} \tag{3.31}$$

Proposition 3.32. *For any $w = w_1 \ldots w_k \in \mathcal{S}^k$, the dual cone $\check{\sigma}_w$ of σ_w in $\Sigma(k)$ is given by*

$$\check{\sigma}_w = \mathbb{R}_{\geq}\upsilon_{1,w_1} + \ldots + \mathbb{R}_{\geq}\upsilon_{k,w_k}. \tag{3.33}$$

Proof. We recall that $\check{\sigma}_w = \{x \in \mathbb{R}^k \colon \langle x, y\rangle \ge 0, \forall y \in \sigma_w\}$. If we set

$$d\sigma_w := \mathbb{R}_{\geq}\upsilon_{1,w_1} + \ldots + \mathbb{R}_{\geq}\upsilon_{k,w_k},$$

then we need to show that $\check{\sigma}_w = d\sigma_w$.

So, let $x = x_1\upsilon_{1,w_1} + \ldots + x_k\upsilon_{k,w_k}$ be any vector in $d\sigma_w$. It follows that

$$\langle x, y\rangle = \sum_{i=1}^{k}\sum_{j=1}^{k} x_i y_j \langle \upsilon_{i,w_i}, a_{j,w_j}\rangle$$

for all $y = y_1 a_{1,i_1} + \ldots + y_n a_{n,i_n} \in \sigma_w$. By using (3.8), it can be verified that

$$\langle \upsilon_{i,w_i}, a_{j,w_j}\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$$

for any $1 \le i, j \le k$. Therefore $\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i \ge 0$, providing $x \in \check{\sigma}_w$.

Conversely, let $x = (x_1, \ldots, x_k)$ be any vector in $\check{\sigma}_w$ so that $\langle x, y\rangle \ge 0$ for all $y \in \sigma_w$. If we define

$$u_j := \begin{cases} x_j & \text{if } w_j = 0, \\ b(j,j)x_j + \ldots + b(j,k)x_k & \text{if } w_j = 1, \end{cases}$$

then since $a_{j,w_j} \in \sigma_w$, it follows that

$$\langle x, a_{j,w_j} \rangle = u_j \geq 0 \tag{3.34}$$

for any $1 \leq j \leq k$. We claim that $x = u_1 v_{1,w_1} + \ldots + u_k v_{k,w_k}$, providing $x \in d\sigma_w$. To see that we let

$$v = (v_1, \ldots, v_k) := u_1 v_{1,w_1} + \ldots + u_k v_{k,w_k}.$$

If $w_j = 0$ for some $1 \leq j \leq k$, then $v_j = u_j = x_j$. On the other hand, if $w_j = 1$, then

$$\begin{aligned} v_j &= -\, b(j,j)\; u_j + (-1)^{w_{j+1}} b(\mathbb{O}_j^{j+1})\; u_{j+1} + (-1)^{w_k} b(\mathbb{O}_j^k)\; u_k, \\ &= x_j - \sum_{i=j+1}^{k} \left(\sum_{l \in \mathbb{I}_j^i} b(\mathbb{O}_j^l)\; b(l,i) + b(j,i) \right) x_i. \end{aligned} \tag{3.35}$$

However, by (3.9), the coefficient of x_i in (3.35) for any $j+1 \leq i \leq k$ equals to zero; hence, $v_j = x_j$. □

Corollary 3.36. *For any $w \in \mathcal{S}^k$, the coordinate ring of the affine toric variety $X_{\check{\sigma}_w}$ is given by $R_{\check{\sigma}_w} = \mathbb{C}[\phi_1, \ldots, \phi_k] \subset \mathbb{C}[z, z^{-1}]$, where $\phi_j := z^{v_{j,w_j}}$ for each $1 \leq j \leq k$.*

4. Classification Problem

We devote this section to smooth toric varieties arising from crosspolytopes and show that any such variety actually gives rise to a Bott tower. This may be thought of as the generalization of Hirzebruch surfaces [17].

Throughout we will assume that P^n is an n-crosspolytope in $\mathbb{R}^n$ with a set of vertices $V = \{x_1, \ldots, x_n, \widehat{x}_1, \ldots, \widehat{x}_n\}$, which spans a smooth projective fan $\Sigma^n = \Sigma(P^n)$. We therefore identify the generating set of Σ^n with the vertex set of P^n, i.e., $G(\Sigma^n) = V$.

We recall that any primitive collection of Σ^n is of the form $\mathcal{P} = \{x_i, \widehat{x}_i\}$ for $i = 1, \ldots, n$. We then combine this fact with Proposition 3.2 of [1] to obtain the following result.

Corollary 4.1. *Let $\Sigma^n = \Sigma(P^n)$ be given as above. Then there exists a primitive collection $\mathcal{P} = \{x_i, \widehat{x}_i\}$ for some $1 \leq i \leq n$ such that $x_i + \widehat{x}_i = 0$.*

Let us assume that $\mathcal{P}^n = \{x_n, \widehat{x}_n\}$ is the primitive collection of Σ^n satisfying $x_n + \widehat{x}_n = 0$, which exists by Corollary 4.1, and let $\sigma_n := \mathbb{R}_{\geq} x_n$ be the one-dimensional smooth cone of Σ^n generated by x_n. We then consider the orthogonal projection $p_n \colon \mathbb{R}^n \to \sigma_n^{\perp} = (\mathbb{R}x_n)^{\perp}$, and define

$$\Sigma^{n-1} := \{p_n(\sigma) \colon \sigma \in \Sigma^n \text{ and } \sigma_n \text{ is a face of } \sigma\}. \tag{4.2}$$

In fact, Σ^{n-1} is the quotient fan Σ^n/σ_n (see [13]). Since Σ^n is smooth, so is Σ^{n-1}. We may proceed to define Σ^k and p_k for any $1 \leq k < n-1$ in a similar way in order to obtain a sequence of smooth fans:

$$\Sigma^n \xrightarrow{p_n} \Sigma^{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_3} \Sigma^2 \xrightarrow{p_2} \Sigma^1 \xrightarrow{p_1} \{0\}. \tag{4.3}$$

Proposition 4.4. *For each $1 \leq k \leq n-1$, the fan Σ^k is a projective fan arising from a crosspolytope.*

Proof. We prove the claim when $k = n-1$ and a similar argument applies to the other cases. Assume that $\mathcal{P}^n = \{x_n, \widehat{x}_n\}$ is the primitive collection for Σ^n satisfying $x_n + \widehat{x}_n = 0$. We observe that Σ^{n-1} is a fan in $\sigma_n^\perp$ with the generating set $G(\Sigma^{n-1}) = \{y_1, \dots, y_{n-1}, \widehat{y}_1, \dots, \widehat{y}_{n-1}\}$, where y_i (respectively $\widehat{y}_i$) is the image of x_i (resp. $\widehat{x}_i$) under p_n for each $1 \le i \le n{-}1$. By definition of a crosspolytope, each $P^i := \operatorname{conv}\{x_1, \dots, x_i, \widehat{x}_1, \dots, \widehat{x}_i\}$ is an i-crosspolytope and satisfies

$$(\operatorname{relint} P^i) \cap (\operatorname{relint}[x_{i+1}, \widehat{x}_{i+1}]) = \{q_i\}, \tag{4.5}$$

a single point, for $1 \le i \le n-1$. By induction, we can obtain that the set $R^i := \operatorname{conv}\{y_1, \dots, y_i, \widehat{y}_1, \dots, \widehat{y}_i\}$ is an i-crosspolytope in $\sigma_n^\perp$ and satisfies

$$(\operatorname{relint} R^i) \cap (\operatorname{relint}[y_{i+1}, \widehat{y}_{i+1}]) = \{p_n(q_i)\}, \tag{4.6}$$

for each $1 \le i \le n-2$; therefore, $\Sigma^{n-1} = \Sigma(R^{n-1})$. □

Without any confusion we will assume that each fan Σ^k is given by $\Sigma^k = \Sigma(P^k)$, where P^k is a k-crosspolytope with $V(P^k) = \{x_1, \dots, x_k, \widehat{x}_1, \dots, \widehat{x}_k\}$, and equivalently, $\mathcal{P}^k = \{x_k, \widehat{x}_k\}$ is the primitive collection of Σ^k satisfying $x_k + \widehat{x}_k = 0$ for each $1 \le k \le n$.

We define Σ_1^k and Σ_2^k to be fans in $|\Sigma^k| = \bigcup_{\sigma \in \Sigma^k} \sigma \cong \mathbb{R}^k$ generated by the set of vectors $\{x_k, \widehat{x}_k\}$ and $\{x_1, \dots, x_{k-1}, \widehat{x}_1, \dots, \widehat{x}_{k-1}\}$ respectively. It can be easily seen that Σ^k is the join of Σ_1^k and Σ_2^k; that is,

$$\Sigma^k = \Sigma_1^k \cdot \Sigma_2^k. \tag{4.7}$$

Moreover, the map $p_k \colon \Sigma^k \to \Sigma^{k-1}$ induces a bijection $p_k|_{\Sigma_2^k} \colon \Sigma_2^k \to \Sigma^{k-1}$; that is, Σ^{k-1} is the projective fan of Σ^k. Since the fan Σ_1^k is unimodular equivalent to the fan of $\mathbb{C}P^1$, we obtain the following.

Theorem 4.8. *Let X^k denote the toric variety associated to the fan Σ^k for each $1 \le k \le n$. Then, the induced toric morphism $\bar{p}_k \colon X^k \to X^{k-1}$ is a fiber bundle with fibers isomorphic to $\mathbb{C}P^1$.*

Therefore, we can exhibit the associated smooth projective toric variety X^n as a sequence of smooth toric varieties of lower dimensions as a counterpart of (4.3):

$$X^n \xrightarrow{\bar{p}_n} X^{n-1} \xrightarrow{\bar{p}_{n-1}} \dots \xrightarrow{\bar{p}_3} X^2 \xrightarrow{\bar{p}_2} X^1 \xrightarrow{\bar{p}_1} *, \tag{4.9}$$

where each map $\bar{p}_k$ is a fiber bundle with fibers isomorphic to $\mathbb{C}P^1$. However, this is very close to $\{X^k \colon k \le n\}$ being a Bott tower; in fact, we can say the following.

Theorem 4.10. *If $\Sigma^n = \Sigma(P^n)$ is a smooth projective fan with P^n being an n-crosspolytope, then $\{X^k \colon k \le n\}$ is a Bott tower.*

Proof. We proceed by induction. If $k = 1$, then P^1 is just a line segment so that X^1 is isomorphic to $\mathbb{C}P^1$. So, assume for some $1 < k < n$, there exist line bundles $\xi_0, \dots, \xi_{k-1}$ such that $X^j = \mathbb{C}P(\mathbb{C} \oplus \xi_{j-1})$ for each $1 \le j \le k$. Suppose Σ^{k+1} is the associated fan to X^{k+1} with the generating set $G(\Sigma^{k+1}) = \{x_1, \dots, x_{k+1}, \widehat{x}_1, \dots, \widehat{x}_{k+1}\}$ and the orthogonal projection $p_{k+1} \colon |\Sigma^{k+1}| \to (\mathbb{R}x_{k+1})^\perp$ such that $\Sigma^k = \{p_{k+1}(\sigma) \colon \sigma \in \Sigma^{k+1}$ and σ_{k+1} is a face of $\sigma\}$, where $\sigma_{k+1} = \mathbb{R}_{\ge} x_{k+1}$. Let $\{u_1, \dots, u_k\}$ be an orthonormal basis of $(\mathbb{R}x_{k+1})^\perp$. Then the projection p_{k+1} is given by $p_{k+1}(v) = \sum_{i=1}^{k}\langle v, u_i\rangle u_i$, and satisfies $v = p_{k+1}(v)+(v-p_{k+1}(v))$ for each $v \in |\Sigma^{k+1}|$. We let $\{y_1, \ldots, y_k, \widehat{y}_1, \ldots, \widehat{y}_k\}$ be the generating set of Σ^k so that $y_i = p_{k+1}(x_i)$ and $\widehat{y}_i = p_{k+1}(\widehat{x}_i)$ for $1 \leq i \leq k$. If we define a map $h\colon |\Sigma^k| \to \mathbb{R}x_{k+1} \cong \mathbb{R}$ by

$$h(y_i) := x_i - p_{k+1}(x_i) \text{ and } h(\widehat{y}_i) := \widehat{x}_i - p_{k+1}(\widehat{x}_i), \tag{4.11}$$

the image of Σ^k under the map $\Psi\colon |\Sigma^k| \to |\Sigma^{k+1}|$ given by $y \mapsto (y, h(y))$ is exactly the fan obtained from Σ^{k+1} by removing its two one-dimensional cones $\mathbb{R}_{\geq}x_{k+1}$ and $\mathbb{R}_{\geq}\widehat{x}_{k+1}$. On the other hand, the map h gives rise to an equivariant line bundle $L(h)$ over $X^k = \mathbb{C}P(\mathbb{C}\oplus\xi_{k-1})$ such that the space $\mathbb{C}P(\mathbb{C}\oplus L(h))$ is a toric variety with a fan being the join of $\Sigma_1 := \{\mathbb{R}_{\geq}x_{k+1}, \mathbb{R}_{\geq}\widehat{x}_{k+1}, \{0\}\}$ and $\widetilde{\Sigma} := \Psi(\Sigma^k)$. However, the fan $\Sigma_1 \cdot \widetilde{\Sigma}$ is in fact the fan Σ^{k+1} associated to the toric variety X^{k+1}. Therefore, we have $X^{k+1} = \mathbb{C}P(\mathbb{C} \oplus L(h))$, which completes the proof. □

5. Tangent Bundle of Bott Towers

In this section, we investigate the tangent bundle of each kth stage N_k of a Bott tower. Following [12], the tangent bundle of any smooth toric variety has a natural decomposition into sum of line bundles, each of which arises from a one-dimensional cone of the associated fan. By Theorem 3.16, we will not distinguish a Bott tower from its quotient description. One of the advantage of having such a description is that we may consider the action as an equivalence relation, that is, any two vectors in $(\mathbb{C}^2\backslash 0)^k$ are equivalent if they lie in the same orbit. Therefore, if $[x, y]$ denotes the equivalence class of the vector $(x, y) \in (\mathbb{C}^2\backslash 0)^k$, then we may write

$$N_k = X_{\Sigma(k)} = \{[x, y]\colon (x, y) \in (\mathbb{C}^2\backslash 0)^k\} \tag{5.1}$$

for each $k \leq n$.

Definition 5.2. Let a Bott tower $\{N_k\colon k \leq n\}$ of height n be given and let $\xi_0, \ldots, \xi_{n-1}$ be the associated line bundles. We then define $\lambda(k)$ to be the canonical line bundle over the projective space $N_k = \mathbb{C}P(\mathbb{C} \oplus \xi_{k-1})$ for each $1 \leq k \leq n$.

Corollary 5.3. *For any $1 \leq k \leq n$, there exists a line bundle $\lambda(k)^\perp$ over N_k satisfying*

$$\lambda(k) \oplus \lambda(k)^\perp \cong \mathbb{C} \oplus \xi_{k-1}. \tag{5.4}$$

Proof. Using the standard inner product in $\mathbb{C} \oplus \xi_{k-1}$, choose as a fiber for $\lambda(k)^\perp$ over $L \in \mathbb{C}P(\mathbb{C} \oplus \xi_{k-1})$ the orthogonal complement $L^\perp$ in $\mathbb{C} \oplus \xi_{k-1}$. □

Without any confusion we will continue to denote the line bundles over N_k obtained by pulling back $\lambda_1, \ldots, \lambda_{k-1}$ along the projections $p_j\colon N_j \to N_{j-1}$ for all $1 \leq j \leq k$ and $1 \leq k \leq n$.

Example 5.5. Consider the Bott tower $\{B_k\colon k \leq n\}$ arising from bounded flag manifolds and recall that $\eta_1, \ldots, \eta_{n-1}$ are the associated line bundles. By the definition of the bundles η_k, we have $\lambda(k) = \eta_k$ and $\lambda(k)^\perp = \eta_k^\perp$ for each $1 \leq k \leq n$.

Before proceeding further, we would like to give a detailed description of the bundles $\lambda(k)$ and $\lambda(k)^\perp$ over N_k.

Proposition 5.6. *For each $1 \leq k \leq n$, the canonical bundle $\lambda(k)$ over N_k is given by*

$$\lambda(k) \cong (\mathbb{C}^2 \backslash 0)^k \times_{\mathbb{C}_\times^k} \mathbb{C}, \tag{5.7}$$

where the action of $\mathbb{C}_\times^k$ is defined by

$$((x,y),v)) \cdot t := ((x,y) \cdot t, t_k^{-1} v)$$

for all $t = (t_1, \ldots, t_k) \in \mathbb{C}_\times^k$.

Proof. Since $N_k = \mathbb{C}P(\mathbb{C} \oplus \xi_{k-1})$, we may interpret an equivalence class $[x,y] \in N_k$ as a line in $\mathbb{C} \oplus \xi_{k-1}$; hence, any vector (x', y') in $[x,y]$ represents a point on this line. Therefore, the total space of $\lambda(k)$ can be given as

$$E(\lambda(k)) = \{([x,y],(x',y'))\colon [x,y] \in N_k \ \text{ and } \ (x',y') \in [x,y]\}.$$

Now the map $g_k \colon (\mathbb{C}^2 \backslash 0)^k \times_{\mathbb{C}_\times^k} \mathbb{C} \to E(\lambda(k))$ defined by

$$g_k([x,y;v]) := \begin{cases} ([x,y],(x_1,y_1;\ldots;x_{k-1},y_{k-1};x_k v^{-1}, y_k v^{-1})), & \text{if } v \neq 0, \\ ([x,y],(x_1,y_1;\ldots;x_k,y_k)), & \text{if } v = 0 \end{cases}$$

provides the desired isomorphism. □

Corollary 5.8. *Let a Bott tower $\{N_k \colon k \leq n\}$ be given with the integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$, and let $\xi_0, \ldots, \xi_{n-1}$ be the associated line bundles. Then,*

$$\xi_k \cong \lambda(1)^{-c(1,k+1)} \otimes \ldots \otimes \lambda(k)^{-c(k,k+1)} \tag{5.9}$$

for all $1 \leq k \leq n-1$.

Corollary 5.10. *For each $1 \leq k \leq n$, the line bundle $\lambda(k)^\perp$ over M_k can be constructed as a quotient $(\mathbb{C}^2 \backslash 0)^k \times_{\mathbb{C}_\times^k} \mathbb{C}$, where the action of $\mathbb{C}_\times^k$ is given by*

$$((x,y),v)) \cdot t := ((x,y) \cdot t, t_1^{c(1,k)} \ldots t_{k-1}^{c(k-1,k)} t_k v)$$

Proof. When the isomorphism $\lambda(k) \oplus \lambda(k)^\perp \cong \mathbb{C} \oplus \xi_{k-1}$ is combined with (5.9), we deduce that

$$\lambda(k)^\perp \cong \lambda(1)^{-c(1,k)} \otimes \ldots \otimes \lambda(k-1)^{-c(k-1,k)} \otimes \lambda(k)^{-1},$$

from which the result follows. □

We are now ready to describe the tangent bundle of each kth stage N_k of a Bott tower. To do this, we recall the description of N_k as a toric variety $N_k \cong X_{\Sigma(k)}$, and the fact that the tangent bundle of a toric variety has a decomposition into a sum of line bundles, each of which is constructed from a one-dimensional cone of the associated fan (see [12]). Therefore, our next task is to identify these bundles for $X_{\Sigma(k)}$.

Let us denote the one-dimensional cones of $\Sigma(k)$ by $\sigma_j^\gamma := \mathbb{R}_{\geq} a_{j,\gamma}$ for each $1 \leq j \leq k$ and $\gamma = 0, 1$, where

$$a_{j,\gamma} := \begin{cases} e_j & \text{if } \gamma = 0, \\ (0, 0, \ldots, 0, b(j,j), \ldots, b(j,k)) & \text{if } \gamma = 1. \end{cases}$$

To simplify the notation, we abbreviate $\Sigma_{\sigma_j^\gamma}$ and $X_{\Sigma_{\sigma_j^\gamma}}$ to Σ_j^γ and X_j^γ respectively, where $\Sigma_{\sigma_j^\gamma}$ is the quotient fan of $\Sigma(k)$ by σ_j^γ. Then, it follows that the orthogonal complement of σ_j^γ in $\mathbb{R}^k$ is $\mathbb{R}_{[k]\setminus\{j\}}$ if $\gamma = 0$ and $V_j^k \times \mathbb{R}_{[j+1,k]}$ if $\gamma = 1$, where V_j^k is a $(j-1)$-dimensional subspace of $\mathbb{R}^k$.

Proposition 5.11. *Under the identification* (5.1)*, the toric subvarieties X_j^γ correspond to the codimension-one submanifolds N_j^γ of N_j defined by*

(i) $N_j^0 := \{[x,y] \in N_j \mid x_j = 0\}$ *if* $\gamma = 0$,
(ii) $N_j^1 := \{[x,y] \in N_j \mid y_j = 0\}$ *if* $\gamma = 1$

for each $1 \leq j \leq k$.

Proof. Let $\gamma = 0$ and let $\sigma_w = \mathbb{R}_{\geq} a_{1,w_1} + \ldots + \mathbb{R}_{\geq} a_{k,w_k}$ be any k-dimensional cone of $\Sigma(k)$ for some $w = w_1 \ldots w_k \in \mathcal{S}^k$ such that σ_j^0 is a face of it. Note that we then necessarily have $w_j = 0$ in w. We see that any cone in $\check{\Sigma}_j^0$ is of the form

$$\beta_w := \check{\sigma}_w \cap \mathbb{R}_{[k]\setminus\{j\}} = \mathbb{R}_{\geq} \upsilon_{1,w_1} + \ldots + \mathbb{R}_{\geq} \upsilon_{j-1,w_{j-1}} + \mathbb{R}_{\geq} \upsilon_{j+1,w_{j+1}} + \ldots + \mathbb{R}_{\geq} \upsilon_{k,w_k}.$$

Therefore, the associated affine toric subvariety X_{β_w} of X_j^0 is embedded in $X_{\check{\sigma}_w} = \{(\phi_1, \ldots, \phi_k)\}$ by assigning $\phi_j = 0$.

On the other hand, N_j^0 being a codimension-one submanifold of N_k has an open covering by the sets $X_{\check{\sigma}_w} \cap N_j^0$ for any $w \in \mathcal{S}^k$ for which $w_j = 0$. Moreover, the intersection $X_{\check{\sigma}_w} \cap N_j^0$ may be thought of the geometric realization of the affine variety X_{β_w} providing the desired result. A similar argument applies to the case $\gamma = 1$. □

Example 5.12. When we consider the Bott tower $\{B_k \colon k \leq n\}$ arising from bounded flag manifolds, the submanifold N_j^0 of B_n is a copy of B_{n-1} whose flags lie in $\mathbb{C}_{[n+1]\setminus\{j\}}$, and similarly, N_j^1 is a copy of $B_{j-1} \times B_{n-j}$ whose flags lie in $\mathbb{C}_{[j]} \times \mathbb{C}_{[j+1,n+1]}$.

Proposition 5.13. *If we denote the bundle over $X_{\Sigma(k)}$ determined by the codimension-one subvarieties X_j^γ by $\nu(\sigma_j^\gamma)$, there exist isomorphisms of complex line bundles:*

$$\nu(\sigma_j^0) \cong \bar{\lambda}(j) \quad \textit{and} \quad \nu(\sigma_j^1) \cong \lambda(j)^\perp, \tag{5.14}$$

for each $1 \leq j \leq k$, where $\bar{\lambda}(j)$ denotes the conjugate of $\lambda(j)$.

Proof. This may be achieved directly by examining the corresponding transition functions of these bundles (see [16]). □

Theorem 5.15. *Let $\{N_k\colon k \le n\}$ be a Bott tower of height n. Then the tangent bundle $\tau(N_k)$ of each kth stage N_k of the tower satisfies*

$$\tau(N_k) \oplus \mathbb{C}^k \cong \bigoplus_{j=1}^{k} \bar{\lambda}(j) \oplus \lambda(j)^{\perp}. \tag{5.16}$$

Corollary 5.17. *When the bounded flag manifold B_n is considered as a complex manifold, there exists an isomorphism:*

$$\tau(B_n) \oplus \mathbb{C}^n \cong \bigoplus_{i=1}^{n} \bar{\eta}_j \oplus \eta_j^{\perp}. \tag{5.18}$$

Corollary 5.19. *If we denote the first Chern classes of line bundles $\lambda(j)$ and $\lambda(j)^{\perp}$ by t_j and z_j respectively, then the top Chern class of N_k, the kth stage of the tower $\{N_k\colon k \le n\}$ is given by*

$$c(N_k) = \prod_{j=1}^{k} (1 - t_j + z_j).$$

We conclude our discussion with the cohomology ring of an arbitrary Bott tower by combining Theorem 3.16 and Danilov-Jurkiewicz theorem (see for example [13]).

Theorem 5.20. *Let $\{N_k\colon k \le n\}$ be a Bott tower associated with the integral sequence $\mathbf{c} = (c_1, \ldots, c_n)$. Then for each $k \le n$, the cohomology ring $H^*(N_k;\mathbb{Z})$ is isomorphic to the quotient $\mathbb{Z}[x_1, \ldots, x_k]/I_k$, where I_k denotes the ideal*

$$\Big(x_j(x_j + c(1,j)x_1 + \ldots + c(j-1,j)x_{j-1})\colon 1 \le j \le k\Big).$$

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