Nilpotency of self homotopy equivalences with coefficients

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Abstract

In this paper we study the nilpotency of certain groups of self homotopy equivalences. Our main goal is to extend, to localized homotopy groups and/or homotopy groups with coefficients, the general principle of Dror and Zabrodsky by which a group of self homotopy equivalences of a finite space which acts nilpotently on the homotopy groups is itself nilpotent.

1 Introduction

Given a pointed space X, denote by $\mathcal{E}(X)$ the group of (based) self homotopy equivalences, i.e., the group of automorphisms of X in the pointed homotopy category. From now on we shall consider connected complexes of finite type

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X which are either finite or with finitely many non trivial homotopy groups. We denote by $\dim X = N$ its topological or homotopical dimension. Unless explicitly stated otherwise, all spaces will be of this kind.

Although the computation of $\mathcal{E}(X)$ is known to be a hard task, there are two classical and key results that impose to this group important structural constraints:

On one hand, a theorem of Sullivan [18, Theorem 10.3] and Wilkerson [20, Theorem B] states that $\mathcal{E}(X)$ is finitely presented. This was originally proved for simply connected spaces and later on generalized to virtually nilpotent spaces by Dror, Dwyer and Kan [5, Theorem 1.1]. The main step in the proof is to show that $\mathcal{E}(X_{\mathbb{Q}})$ is an algebraic group and that $\mathcal{E}(X)$ is commensurable with an arithmetic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$. As a consequence, it can be shown that there exists a finite bound for the finite orders of elements of $\mathcal{E}(X)$.

On the other hand we have the following theorem due to Dror and Zabrodsky:

Theorem 1. [4, Theorem B] *Let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_{\leq N}(X)$. Then G is itself nilpotent. In particular, $\mathcal{E}_\sharp^m(X)$ is nilpotent.*

Recall that, for $0 \leq m \leq \infty$, $\mathcal{E}_\sharp^m(X)$ is the distinguished subgroup of $\mathcal{E}(X)$ formed by those classes inducing the identity on the homotopy groups up to m. In other words,

$$\mathcal{E}_\sharp^m(X) = \ker\big(\mathcal{E}(X) \longrightarrow \Pi_{i\leq m} \mathrm{aut}\, \pi_i X\big).$$

If $\dim X = N$ we shall denote $\mathcal{E}_\sharp^N(X)$ simply by $\mathcal{E}_\sharp(X)$.

Here, we present a slightly different proof for this well known result in which we use a broader study of self homotopy equivalences in the homotopy category $\mathcal{L}^*$ of (based) spaces with local coefficients. Recall (see [19, Chap.VI] for instance) that objects in this category are pairs $(X, \mathcal{M})$ in which X is a (based) topological space and $\mathcal{M} = \{M_x\}_{x\in X}$ is a local coefficient system in X. On the other hand, a morphism $(f, \Theta)\colon (X, \mathcal{M}) \to (Y, \mathcal{H})$ is a pair formed by a based map $f\colon X \to Y$ and a morphism $\Theta\colon f^*\mathcal{H} \to \mathcal{M}$ of local coefficient. By $f^*\mathcal{H}$ we denote, as usual, the local coefficient system on X induced by f, i.e., $(f^*\mathcal{H})_x = H_{f(x)}$. For each $x \in X$ we shall denote by $\Theta_x\colon H_{f(x)} \to M_x$ the corresponding group morphism at x. After considering the appropriate homotopy notion, one obtains the homotopy category $\mathcal{L}^*$. The group of self homotopy equivalences of an object $(X, \mathcal{M}) \in \mathcal{L}^*$ shall be denoted by $\mathcal{E}(X; \mathcal{M})$. Then, we prove:

Theorem 2. *Let X be a finite Postnikov piece and let $G \subset \mathcal{E}(X;\mathcal{M})$ be a subgroup which acts nilpotently on both $\pi_*(X)$ and $\mathcal{M}$. Then, G acts nilpotently on $H^*(X;\mathcal{M})$.*

At the sight of Theorem 1, and taking into account the bound of finite orders of elements of $\mathcal{E}(X)$ plus the existence of a "fracture lemma" for this group (see [15, Theorem 8.2]), it has been of interest to study whether $\mathcal{E}_\sharp(X)$ satisfies the same structural restrictions when taking p-localization, p-completion or considering $\mathcal{E}_{\sharp p}(X)$. This denotes the subgroup of $\mathcal{E}(X)$ formed by those classes which induce the identity on the homotopy groups of X with coefficients on $\mathbb{Z}/p$, up to the dimension of X, i.e.,

$$\mathcal{E}_{\sharp p}(X) = \ker\big(\mathcal{E}(X) \longrightarrow \Pi_{\leq N} \mathrm{aut}\, \pi_i(X;\mathbb{Z}/p)\big).$$

As examples of this, we mention two interesting results for a given nilpotent space X: Maruyama proved [11, Theorem 0.1] that $\mathcal{E}_\sharp(X)_{(p)} = \mathcal{E}_\sharp^N(X_{(p)})$ while, on the other hand, Møller showed [14, Theorem 4.3] that

$$\mathcal{E}_\sharp(X_{\mathbb{Z}_p}) = Ext(\mathbb{Z}/p^\infty, \mathcal{E}_\sharp(X)) = \mathcal{E}_\sharp(X_{\hat{p}}).$$

Here and henceforth, $(-)_{\mathbb{Z}_p}$ denotes $H_*(-;\mathbb{Z}/p)$-localization while $(-)_{(p)}$ and $(-)_{\hat{p}}$ are the classical localization and completion on the prime p.

In this paper we plan to continue this investigation extending Theorem 1 above, considering a subgroup of $\mathcal{E}(X)$ which acts nilpotently in the homotopy groups of X localized, completed or with coefficients in $\mathbb{Z}/p$. Concerning this purpose we prove:

Theorem 3. *Assume that $\pi_1(X)$ is a nilpotent group and let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_{\leq N}(X)_{(p)}$, for p any prime number and 0. If the nilpotency orders of all these actions are bounded by a fixed integer, then G is nilpotent.*

Remark 4. Observe that in the theorem above the condition of $\pi_1(X)$ being nilpotent is essential. Otherwise, choose any finite simple group G which is known to be generically trivial, i.e., $G_{(p)} = \{1\}$ for p any prime number or zero. On the other hand observe that the map $G \to \mathrm{aut}\, G$ given by inner automorphisms is a monomorphism. Indeed, its kernel is the center of G which is trivial since G is simple. This inclusion renders the non nilpotent group G as a subgroup of $\mathcal{E}\big(K(G,1)\big)$ which acts nilpotently in the localized homotopy group.

In a similar way, we may even produce an example of a solvable, non nilpotent group, of homotopy equivalences acting nilpotently in the localized homotopy groups of the space. Consider the symmetric group Σ_3 and observe that $\Sigma_{3(2)} = \mathbb{Z}/2$ while $\Sigma_{3(p)} = 1$ for $p \neq 2$. Again, $\Sigma_3 \subset \mathcal{E}\big(K(\Sigma_3, 1)\big)$ is a solvable non nilpotent group acting nilpotently on any $\Sigma_{3(p)}$.

A more subtle and slightly different situation is given when considering nilpotent actions of subgroups of self homotopy equivalences on the Frattini factor of the homotopy groups. Recall that given a group G, the Frattini subgroup $\Phi(G)$ is the intersection of all maximal proper subgroups of G. The quotient $G/\Phi(G)$ is called the Frattini factor.

Theorem 5. *Assume that $\pi_{\leq N}(X)$ is a finite nilpotent group and let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_{\leq N}(X)/\Phi\big(\pi_{\leq N}(X)\big)$. Then, G is nilpotent.*

In particular, taking into account that for an abelian p-group G its Frattini factor is precisely $G \otimes \mathbb{Z}/p$, we obtain the following:

Corollary 6. *Assume that $\pi_{\leq N}(X)$ is a finite abelian group and let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_{\leq N}(X) \otimes \mathbb{Z}/p$ for any prime p. Then, G is nilpotent.* $\square$

Notice that, by the Universal Coefficients Theorem for homotopy, $\pi_* X \otimes \mathbb{Z}/p = \operatorname{Ext}(\mathbb{Z}/p, \pi_* X)$ is a subgroup of $\pi_*(X; \mathbb{Z}/p)$. Hence, as an immediate consequence of Corollary 6 above we get:

Corollary 7. *Assume that $\pi_{\leq N}(X)$ is a finite abelian group and let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_{\leq N}(X; \mathbb{Z}/p)$ for any prime p. Then, G is nilpotent.* $\square$

Having studied the nilpotency of a general subgroup of $\mathcal{E}(X)$, we now focus on the group $\mathcal{E}_{\sharp p}(X)$ and give necessary conditions for it to be nilpotent.

Theorem 8. *Let X be a space for which $\pi_{\leq N}(X)$ is a finite abelian p-group. Then $\mathcal{E}_{\sharp p}(X)$ is nilpotent and $\mathcal{E}_{\sharp p}(X)/\mathcal{E}_{\sharp}(X)$ is a finite p-group.*

Theorem 9. *Let X be a space for which $\pi_{\leq N}(X)$ is a finitely generated abelian group. Then $\cap_{p\ prime} \mathcal{E}_{\sharp p}(X)$ is nilpotent.*

Remark 10. Observe that in general $\mathcal{E}_{\sharp p}(X)$ is bigger than $\mathcal{E}_{\sharp}(X)$. For instance consider $X = K(\mathbb{Z}/p^r, n)$, $r, n \geq 2$. Obviously $\mathcal{E}_{\sharp}(X) = \{1\}$, while the automorphism ρ of $\mathbb{Z}/p^r$ given by $\rho(1) = p^{r-1} + 1$ induces a non trivial element of $\mathcal{E}_{\sharp p}(X)$. Indeed, by the Universal Coefficients Theorem for homotopy,

$$\pi_*(X, \mathbb{Z}/p) = \pi_n(X, \mathbb{Z}/p) \oplus \pi_{n-1}(X, \mathbb{Z}/p),$$

in which

$$\pi_n(X, \mathbb{Z}/p) = \hom(\mathbb{Z}/p, \mathbb{Z}/p^r) \text{ and } \pi_{n-1}(X, \mathbb{Z}/p) = \operatorname{Ext}(\mathbb{Z}/p, \mathbb{Z}/p^r) = \mathbb{Z}/p.$$

Trivially ρ induces the identity on both. Note that this example also shows that even $\cap_{p \text{ prime}} \mathcal{E}_{\sharp p}(X)$ can be bigger than $\mathcal{E}_{\sharp}(X)$.

The paper is organized as follows: in the next section we collect the results we shall need from group theory and from which Theorems 3 and 5 are immediately deduced. Theorem 2 and 1 are proved in section §3. Finally, in section §4 we establish Theorems 8 and 9.

2 From group theory

We begin by recalling some basic facts. If G is a group acting on another group A (i.e., A is a G-group), the n-th G-commutator subgroup $\Gamma_G^n(A) \subset A$ is the group generated by $\{(ga^{-1})a \mid g \in G,\, a \in \Gamma_G^{n-1}(A)\}$, being $\Gamma_G^0(A) = A$. The action is then nilpotent of nilpotency order r, $\operatorname{nil}_G A = r$, if this is the smallest integer for which $\Gamma_G^r(A) = \{1\}$. The group G also acts in each $\Gamma_G^n(A)$ and $\Gamma_G^m\big(\Gamma_G^n(A)\big) = \Gamma_G^{m+n}(A)$.

Statements of next sections shall heavily rely in the following results:

Lemma 11. *Let A be a G-group. Then:*

(i) $\Gamma_G^1(A)$ is a normal subgroup of A and the G-action induced on $A/\Gamma_G^1(A)$ is trivial.

(ii) The quotient morphism $A \xrightarrow{q} A/\Gamma_G^1(A)$ is equivariant and initial with respect to trivial actions, i.e., every equivariant morphism $A \xrightarrow{f} H$, in which the G-action on H is trivial, factors uniquely through q.

Proof. (i) is trivial. For (ii) observe that, for any f as in the lemma, $\Gamma_G^1(A) \subset \ker f$. $\square$

Lemma 12. *Let A be a G-group. If A is nilpotent then, for any m, $\Gamma^m_G(A)_{(p)} = \Gamma^m_G(A_{(p)})$.*

Proof. Since $\Gamma^m_G(A) = \Gamma^1_G\big(\Gamma^{m-1}_G(A)\big)$, once we show that $\Gamma^1_G(A)_{(p)} = \Gamma^1_G(A_{(p)})$ an easy induction proves the lemma. As localization is an exact functor in the category of nilpotent groups, the localization morphism $f : A \to A_{(p)}$ restricts to $f : \Gamma^1_G(A) \to \Gamma^1_G(A)_{(p)}$. Hence, we may consider $\Gamma^1_G(A)_{(p)}$, as well as $\Gamma^1_G(A_{(p)})$, as subgroups of $A_{(p)}$. Then, for any $g \in G$ and $a \in A$, the trivial identity $\big(gf(a)^{-1}\big)f(a) = f\big((ga^{-1})a\big)$ shows equality of both subgroups. □

Proposition 13. *The group G acts nilpotently on the nilpotent group A if and only if G acts nilpotently on $A_{(p)}$ for p any prime number or zero and all these nilpotency orders are bounded.*

Proof. Assume G acts nilpotently on A, i.e., $\Gamma^m_G(A) = \{1\}$ for some m. Hence, by Lemma 12 and for any p, $\Gamma^m_G(A_{(p)}) = \{1\}$.

Conversely, assume $\operatorname{nil}_G A_{(p)} \leq m$, for all p (p a prime number or 0), and let a be an element of $\Gamma^m_G(A)$. If a has finite order, say it is a q-element, then it obviously survives under the q-localization morphism $\Gamma^m_G\big(A\big) \to \Gamma^m_G(A)_{(q)}$. For a general group, elements of infinite order are not guaranteed to survive under rationalization (for instance, the rationalization of the free product of two finite groups is trivial while it contains elements of infinite order). However for a nilpotent group, which is our case, one can easily show by induction on the nilpotency order of the group, that any element of infinite order is not sent to zero under rationalization. Taking into account, again by Lemma 12, that $\Gamma^m_G(A)_{(p)} = \Gamma^m_G(A_{(p)}) = \{1\}$, it follows that $a = 1$ and the proof is complete. □

Proposition 14. *Let G be a group acting on a finite nilpotent group A in such a way that the induced action on the Frattini factor $A/\Phi(A)$ is nilpotent. Then, the G-action on A is also nilpotent.*

Proof. Recall [8, 5.1] that the Frattini subgroup of a group A, $\Phi(A)$, is defined to be the intersection of all its maximal proper subgroups. The Frattini factor of A is $A/\Phi(A)$. Observe in the first place that, since $\Phi(A)$ is a characteristic subgroup of A, i.e., it is invariant under any automorphism of A, G in fact induces a natural action on the Frattini factor $A/\Phi(A)$ which, by hypothesis, is nilpotent. Hence, since $A/\Phi(A)$ is nilpotent, the induced action on $\big(A/\Phi(A)\big)_{(p)} = A_{(p)}/\Phi(A)_{(p)}$ is also nilpotent by Lemma 12. Next, observe that for any finite group A, $\Phi(A)_{(p)} = \Phi(A_{(p)})$. Indeed, this is immediate from the definition taking into account that localization commutes with limits, in particular, with intersections (see for instance [9]). Therefore, we conclude that G acts nilpotently on $A_{(p)}/\Phi(A_{(p)})$. Considering $\varphi\colon G \to \operatorname{aut}\big(A_{(p)}/\Phi(A_{(p)})\big)$ via this action, and taking into account that $A_{(p)}/\Phi(A_{(p)})$ is a finite p-group, we may apply [8, Corollary 5.3.3] to obtain that $\varphi(G)$ is also a p-group. But the action of a p-group on another p-group is always nilpotent, and therefore G acts nilpotently on $A_{(p)}$. Since this is the case for any p and A is finite we may apply Proposition 13 and the proposition follows. $\square$

From these results we immediately deduce:

Proof of Theorems 3 and 5. Apply directly Propositions 13 and 14 above to the subgroup G of $\mathcal{E}(X)$ to obtain that G acts nilpotently on $\pi_{\leq N}(X)$. Then, the result follows from Theorem 1. $\square$

Closely related to Proposition 14, we have the following:

Proposition 15. *Let G be a group acting on an abelian p-group A which has an exponent p^n. If G acts nilpotently on $A \otimes \mathbb{Z}/p$, then it does so on A and*

$$nil_G A \leq n \cdot nil_G A \otimes \mathbb{Z}/p.$$

Proof. Call $r = \operatorname{nil}_G A \otimes \mathbb{Z}/p$ and observe that $\Gamma_G^m(A \otimes \mathbb{Z}/p) = \Gamma_G^m(A) \otimes \mathbb{Z}/p$ for any m. Therefore, since $\Gamma_G^r(A \otimes \mathbb{Z}/p) = 0$, $\Gamma_G^r(A) \subset pA$. Assume, as induction hypothesis, that $\Gamma_G^{kr}(A) \subset p^k A$, for $k < n$. Hence,

$$\Gamma_G^{nr}(A) = \Gamma_G^{(n-1)r}\big(\Gamma_G^r(A)\big) \subset \Gamma_G^{(n-1)r}(pA) = p\Gamma_G^{(n-1)r}(A) \subset p^n A.$$

Since A has p^n as exponent, the proposition follows. $\square$

Proposition 16. *Let A be a finite abelian p-group and let $G \subset aut(A)$ be such that $\sigma \otimes \mathbb{Z}/p = 1_{A\otimes\mathbb{Z}/p}$ for each $\sigma \in G$. Then G is a p-group.*

Proof. As A is a finite abelian p-group, the Frattini factor $A/\Phi(A)$ (respec. the projection $A \to A/\Phi(A)$) is naturally identified with $A \otimes \mathbb{Z}/p$ (respec. the map $A \to A \otimes \mathbb{Z}/p$). Now, if G is not a p-group, there exists a non trivial p'-automorphism $\sigma \in G$ which, by hypothesis and using the identification above, induces the identity on the Frattini factor of A. But according to [8, Theorem 5.1.4], the only p'-automorphism that induces the identity on the Frattini factor of a p-group is the identity. Thus G must be a p-group. $\square$

As an immediate consequence we get:

Corollary 17. *In the conditions of the proposition above, the action of G on A is nilpotent.*

Proof. Indeed, recall that the action of a p-group H on another p-group is always nilpotent. $\square$

In which follows G is a group acting on another group A. Given $g, h \in G$ and $a \in A$, we use the following usual notation:

$$[a, g] = a^{-1}(ga), \quad [g, a] = (ga^{-1})a, \quad [g, h] = g^{-1}h^{-1}gh.$$

Hence, the following, which can be considered as a variation of the Witt-Hall identity [10, Theorem 5.1], is obtained by direct calculation.

Lemma 18. *For any $f, g \in G$ and $b \in A$, the following identity holds:*

$$\big[[f^{-1}, g^{-1}], gb\big]b^{-1}\big[[g, b^{-1}], f\big]b\big[[f, b], fgf^{-1}\big] = 1.$$

Lemma 19. *Let H be a subgroup of G and K a normal subgroup of H. Then,*

$$\big[[H, K], A\big] \subset \langle\big[K, [H, A]\big], \big[H, [K, A]\big]\rangle.$$

Proof. Making $f^{-1} = h$, $g^{-1} = k$ and $gb = a$ in Lemma 18, it follows that

$$\big[[h, k], a\big] = \big[h^{-1}k^{-1}h, [h^{-1}, g^{-1}a]\big]g^{-1}a^{-1}\big[h^{-1}, [k^{-1}, g^{-1}a]\big]g^{-1}a.$$

As K is normal in H, $\big[h^{-1}k^{-1}h, [h^{-1}, g^{-1}a]\big] \in \big[K, [H, A]\big]$. On the other hand, as commutators are normal subgroups,

$$g^{-1}a^{-1}\big[h^{-1}, [k^{-1}, g^{-1}a]\big]g^{-1}a \in g^{-1}a^{-1}\big[H, [K, A]\big]g^{-1}a = \big[H, [K, A]\big],$$

and the lemma follows. $\square$

Lemma 20. *If the action of G on A is nilpotent, then for each $n, m \geq 0$,*

$$[\Gamma^n(G), \Gamma^m_G(A)] \subset \Gamma^{n+m+1}_G(A).$$

In particular, $[\Gamma^n(G), A] \subset \Gamma^{n+1}_G(A)$.

Proof. Set $\mathrm{nil}_G A = r$. If $m \geq r$ the assertion is obvious. Assume the lemma holds for all n and $m \leq 1$ and let us prove it for $m = 0$ by induction on n:

Trivially, $[\Gamma^0(G), \Gamma^0_G(A)] = [G, A] = \Gamma^1_G(A)$. Finally,

$$
\begin{aligned}
[\Gamma^n(G), A] = \big[[G, \Gamma^{n-1}(G)], A\big] &\subset \text{(By Lemma 19)} \\
&\subset \langle \big[\Gamma^{n-1}(G), [G, A]\big], \big[G, [\Gamma^{n-1}(G), A]\big] \rangle \subset \text{(By induction)} \\
&\subset \langle [\Gamma^{n-1}(G), \Gamma^1_G(A)], [G, \Gamma^n_G(A)] \rangle \subset \text{(Again by induction)} \\
&\subset \Gamma^{n+1}_G(A).
\end{aligned}
$$

□

Proposition 21. *Let G be a subgroup of $aut(A)$. Then, $nil G \leq nil_G A - 1$.*

Proof. Assume $\mathrm{nil}_G A = r$. By Lemma 20, $[\Gamma^{r-1}(G), A] \subset \Gamma^r_G(A) = \{1\}$, and therefore $\Gamma^{r-1}(G) = \{1\}$. □

3 Self homotopy equivalence of spaces with local coefficients

As stated in the Introduction, and following the notation and approach of the standard reference [19, Chap. VI.2], in this section we consider self homotopy equivalences in the homotopy category $\mathcal{L}^*$ of based spaces with local coefficients. Observe that a self homotopy equivalence of an object $(X, \mathcal{M}) \in \mathcal{L}^*$ is given by $(f, \Theta)\colon (X, \mathcal{M}) \to (X, \mathcal{M})$ in which $f\colon X \to X$ is a based homotopy equivalence and $\Theta\colon \mathcal{M} \to \mathcal{M}$ is an isomorphism of the coefficient system $\mathcal{M}$. Note that such a self equivalence (f, Θ) acts in $\pi_*(X)$ by $\pi_* f$, in $\mathcal{M}$ by Θ, and in $H^*(X; \mathcal{M})$ by $H^*(f, \Theta)$.

It is also convenient to recall how cohomology classes with local coefficients are represented by maps into the "twisted Eilenberg-MacLane space" (see [3, Chapter 5.2], [7], [13] or [17] for precise details). Let $K(\mathcal{M}, n)$ be a fixed realization of the Eilenberg-MacLane space of type (M_{x_0}, n) being M_{x_0} the group of the system $\mathcal{M}$ at the base point. On the other hand, denote by $L(\mathcal{M}, n)$ the space obtained by applying the Borel construction to the universal fibration $\pi_1(X) \to \widetilde{K} \xrightarrow{q} K(\pi_1(X), 1)$ and the space $K(\mathcal{M}, n)$, i.e.,

$$L(\mathcal{M}, n) = \widetilde{K} \times_{\pi_1(X)} K(\mathcal{M}, n),$$

and it fits into the fibration

$$K(\mathcal{M},n) \longrightarrow L(\mathcal{M},n) \xrightarrow{p} K(\pi_1(X),1), \quad p(a,b)=q(a).$$

Then, for a given space X, $H^n(X;\mathcal{M})$ is in one to one correspondence with the set $[X, L(\mathcal{M},n)]_{K(\pi_1(X),1)}$ of homotopy classes of maps over $K(\pi_1(X),1)$ from X to $L(\mathcal{M},n)$.

Proof of Theorem 2. To avoid excessive notation we shall not distinguish between a homotopy class and a map which represents it. First, observe that, if (f,Θ) is a self homotopy equivalence of $(X,\mathcal{M})$ and $\alpha\colon X \to L(\mathcal{M},n)$ represents a class of $H^*(X;\mathcal{M})$, $H^n(f,\Theta)(\alpha)$ is represented by the map

$$X \xrightarrow{f} X \xrightarrow{\alpha} L(\mathcal{M},n) \xrightarrow{\xi} L(\mathcal{M},n)$$

in which ξ is defined by the action of Θ_{x_0} on M_{x_0}. Explicitly, for $(a,b) \in L(\mathcal{M},n)$, $\xi(a,b) = (a, \overline{\Theta}_{x_0} b)$ where and $\overline{\Theta}_{x_0}$ is the realization of Θ_{x_0}. Observe that ξ is well defined as $\Theta\colon \mathcal{M} \to \mathcal{M}$ is a morphism of local coefficient systems.

Moreover, if $\alpha, \beta\colon X \to L(\mathcal{M},n)$ are in $H^n(X;\mathcal{M})$, they coincides after composing with the fibration $p\colon L(\mathcal{M},n) \to K(\pi(X),1)$, and therefore, for each $x \in X$, $\alpha(x), \beta(x)$ live in the same fiber $K(\mathcal{M},n)$ of p. Hence, α and β can be added up on $K(\mathcal{M},n)$ and the resulting map $\alpha+\beta$ represents precisely their sum as cohomology classes with twisted coefficients.

We shall prove the theorem by induction on the length of the Postnikov decomposition of X. Assume $X = K(\pi,m)$ and let $(f,\Theta)\colon (X,\mathcal{M}) \to (X,\mathcal{M})$ be a self equivalence. Then, in view of the above, for any n-cohomology class $\alpha\colon K(\pi,m) \to L(\mathcal{M},n)$, $H^n(f,\Theta)(\alpha) - \alpha$ is represented by the map $K(\pi,m) \to L(\mathcal{M},n)$ which, fiberwise, is $\overline{\Theta}_{x_0}\alpha f - \alpha$. Writing $\overline{\Theta}_{x_0}\alpha f - \alpha = \overline{\Theta}_{x_0}\alpha f - \overline{\Theta}_{x_0}\alpha + \overline{\Theta}_{x_0}\alpha - \alpha$ it is straightforward, using the nilpotency hypothesis, to show that the s-th commutator of the action of G on $H^*(X;\mathcal{M})$ vanishes as long as $s \le \max\{\operatorname{nil}_G \pi_*(X), \operatorname{nil}_G \mathcal{M}\}$.

Assume the theorem holds for $X = X^{r-1}$ and let $X = X^r$ be a r-dimensional Postnikov piece. Consider the Serre spectral sequence with local coefficients on $\mathcal{M}$ associated to the fibration

$$K(\pi_r(X),r) \to X \to X^{r-1}.$$

whose E_2-term is

$$E_2^{*,*} = H^*\big(X^{r-1}; \mathcal{H}^*(K(\pi_r(X),r);\mathcal{M})\big).$$

Note that G acts naturally in the base, total space and fiber of this fibration, and hence, it does so in all the terms of the spectral sequence. The same argument used for $r = 1$ shows that G acts nilpotently on the local coefficient system $\mathcal{H}^*(K(\pi_r(X), r); \mathcal{M})$ and therefore, by induction hypothesis, G acts nilpotently on $H^*\big(X^{r-1}; \mathcal{H}^*(K(\pi_r(X), r); \mathcal{M})\big)$.

As the spectral sequence converges, the action of G on the associated graded module of $H^*(X; \mathcal{M})$ is nilpotent. Finally, reasoning by induction on the filtration degree we deduce that the G-action on $H^*(X; \mathcal{M})$ is also nilpotent. □

In particular, for any space X and any j we may consider the local coefficient system given by $\pi_j X$. In this case, any self homotopy equivalence $f \in \mathcal{E}(X)$ can be seen as a self homotopy equivalence $(f, \pi_j f^{-1}) \in \mathcal{E}(X; \pi_j X)$. Hence, any subgroup of $\mathcal{E}(X)$ may be considered as a subgroup of $\mathcal{E}(X, \pi_j X)$ which then acts naturally on $H^*(X; \pi_j X)$ when considering local coefficients. In this context, the theorem above reads:

Corollary 22. *Let X be a finite Postnikov piece and let G be a subgroup of $\mathcal{E}(X)$ which acts nilpotently on $\pi_*(X)$. Then, for any j, G acts nilpotently on $H^*(X; \pi_j)$.* □

This result is used in the proof of Theorem 1 that we now present:

Proof of Theorem 1. Consider the restriction to G of the exact sequence $1 \to \mathcal{E}_\sharp(X) \to \mathcal{E}(X) \to \Pi_{i \leq N} \mathrm{aut}\, \pi_i(X)$:

$$1 \to \mathcal{E}_\sharp(X) \cap G \to G \to \Pi_{i \leq N} \mathrm{aut}\, \pi_i(X).$$

The image of G under this morphism, call it $\widetilde{G}$, is a subgroup of automorphism of the group $\pi_{\leq N}(X)$ in which G acts nilpotently by hypothesis. Then, by Proposition 21, $\widetilde{G}$ is itself nilpotent and $\mathrm{nil}\, \widetilde{G} < \mathrm{nil}_G \pi_{\leq N}(X)$. Therefore, if we prove that G acts nilpotently on $\mathcal{E}_\sharp(X)$, then (see for instance [9, Proposition 4.1]) G would be nilpotent and

$$\mathrm{nil}\, G < \mathrm{nil}_G \mathcal{E}_\sharp(X) + \mathrm{nil}_G \pi_{\leq N}(X). \tag{1}$$

For that, observe in the first place that $[X, X] \cong [X^N, X^N]$, where X^N denotes the N-th Postnikov stage of X, and this bijection restricts to an isomorphism $\mathcal{E}_\sharp(X) \cong \mathcal{E}_\sharp(X^N)$. On the other hand, consider the exact sequence

$$1 \to A_j \to \mathcal{E}_\sharp(X^j) \to \mathcal{E}_\sharp(X^{j-1}) \tag{2}$$

where $\mathcal{E}_\sharp(X^j) \to \mathcal{E}_\sharp(X^{j-1})$ is just the obvious restriction and A_j its kernel. Since G acts on any $\mathcal{E}_\sharp(X^j)$ and $\mathcal{E}_\sharp(X^1) = 1$, it will be enough to show that G acts nilpotently on every A_j to conclude, by an easy induction, that it does so on $\mathcal{E}_\sharp(X^N) = \mathcal{E}_\sharp(X)$.

By classical obstruction theory of liftings (see [19, Chapter 6.6]) recall that, for $j \geq 2$, there is a bijection $\varphi\colon B_j \to H^j(X^j;\pi_j)$ where

- The cohomology is taken with local coefficients.
- B_j is the set of homotopy classes of $[X^j, X^j]$ which restrict to the identity on X^{j-1}, i.e., homotopy classes of liftings of $X^j \to X^{j-1}$ to X^j.
- $\varphi(g) = \delta(g,1)$ is the difference cochain of degree j between g and the identity on X^j.

Recall also that, in general, $\delta(g,f) = \delta(g,1) + \delta(1,f)$ and that $\delta(gh,fh) = H^j(h)\big(\delta(g,f)\big)$. Moreover, if $h \in \mathcal{E}_\sharp(X^j)$, $\delta(hg,hf)$ is the image of $\delta(g,f)$ under the map $H^j(X^j;\pi_j) \to H^j(X^j;\pi_j)$ induced by h on π_j.

From now on, as in Corollary 22, any $f \in \mathcal{E}(X)$, and thus in G, shall be considered as a self homotopy class $(f, \pi_j f^{-1}) \in \mathcal{E}(X,\pi_j)$. Hence, restricting φ to A_j we obtain a map $\varphi\colon A_j \hookrightarrow H^j(X^j;\pi_j)$ which is a G-map with respect to the action $g{\cdot}f = g^{-1}fg$, $g \in G$, $f \in A_j$, and the usual action on $H^j(X^j;\pi_j)$: if $\alpha \in H^j(X_j,\pi_j)$, and $g \in G$, $g\cdot\alpha$ is the cohomology class represented by the map

$$X^j \xrightarrow{g} X^j \xrightarrow{\alpha} L(\pi_j,j) \xrightarrow{\xi} L(\pi_j,j), \qquad \alpha \in H^j(X^j;\pi_j),$$

with $\xi\colon L(\pi_j,j) \to L(\pi_j,j)$ induced by $\pi_*(g^{-1})$.

Moreover, this restriction is a group morphism. Indeed, given $f,h \in A_j$, $\varphi(fh) = \delta(fh,1) = \delta(f^{-1}fh, f^{-1}) = \delta(h,f^{-1}) = \delta(h,1) + \delta(1,f^{-1}) = \delta(h,1) + \delta(f,1) = \varphi(f) + \varphi(h)$. As an immediate consequence we then obtain that A_j is an (abelian!) subgroup of $H^j(X^j;\pi_j)$.

Finally, by Corollary 22, G acts nilpotently on $H^j(X^j;\pi_j)$ for any j. Hence, it does so on A_j and

$$\operatorname{nil}_G A_j \leq \operatorname{nil}_G H^j(X^j;\pi_j).$$

Thus, by induction on j using repeatedly [9, Proposition 4.1], one easily sees in view of (2) that $\operatorname{nil}_G \mathcal{E}_\sharp(X) = \operatorname{nil}_G \mathcal{E}_\sharp(X^N) \leq \sum_{j=2}^N \operatorname{nil}_G A_j \leq \sum_{j=2}^N \operatorname{nil}_G H^j(X^j;\pi_j)$ and the theorem follows. $\square$

4 Groups which fix the homotopy groups

In this section we establish Theorems 8 and 9.

Proof of Theorem 8. Let $\alpha \in \mathcal{E}_{\sharp p}(X)$. Then, for each $i \leq \dim X$, the morphism $\pi_i(\alpha; \mathbb{Z}/p) \colon \pi_i(X, \mathbb{Z}/p) \to \pi_i(X, \mathbb{Z}/p)$ is just the identity. On the other hand, the Universal Coefficients Theorem for homotopy yields the following split short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/p, \pi_{i+1}X) \longrightarrow \pi_i(X; \mathbb{Z}/p) \longrightarrow \hom(\mathbb{Z}/p; \pi_i X) \longrightarrow 0.$$

Thus, both $\mathrm{Ext}(\mathbb{Z}/p, \pi_{i+1}\alpha)$ and $\hom(\mathbb{Z}/p; \pi_i\alpha)$ are the identity. But observe that $\mathrm{Ext}(\mathbb{Z}/p, \pi_{i+1}X) = \pi_{i+1}X \otimes \mathbb{Z}/p$ so that $\pi_{i+1}\alpha \otimes \mathbb{Z}/p = 1_{\pi_{i+1}X}$.

Hence, by Proposition 16, for each $i \leq \dim X$, the image of $\mathcal{E}_{\sharp p}(X)$ in $\mathrm{aut}\,(\pi_i X)$ is a p-group and then, by Corollary 17, the action of $\mathcal{E}_{\sharp p}(X)$ on $\pi_i X$ is nilpotent. Thus, by Theorem 1, $\mathcal{E}_{\sharp p}(X)$ is nilpotent. On the other hand, notice that $\mathcal{E}_{\sharp}(X)$ is precisely the kernel of the obvious map $\mathcal{E}_{\sharp p}(X) \to \Pi_{i \leq \dim X} \mathrm{aut} \pi_i(X)$. Hence, as we just proved that the image of this map is a p-group, $\mathcal{E}_{\sharp p}(X)/\mathcal{E}_{\sharp}(X)$ is a finite p-group, and the proof is complete. $\square$

Proof of Theorem 9. Write $\pi_i X = \mathbb{Z}^{n_i} \oplus \big(\oplus_{p \text{ prime}} T_p(\pi_i X)\big)$ in which $T_p(\pi_i X)$ is the group of p-torsion elements in $\pi_i X$. Now, if $\alpha \in \cap_{p \text{ prime}} \mathcal{E}_{\sharp p}(X)$, then for each $i \leq \dim X$, $\pi_i(\alpha)|_{T_p(\pi_i X)} \in \mathrm{aut}\,\big(T_p(\pi_i X)\big)$. Let $z_1, \ldots, z_{n_i}$ be generators of $\mathbb{Z}^{n_i} \subset \pi_i X$. Then $\pi_i(\alpha)(z_k) = \sum_{j=1}^{1} n_i a_{k,j} z_j + \omega$, where ω is the torsion part. But this element has to coincide with z_k mod p, for all prime p. Therefore, also for any p, $a_{k,j} = 0 (\mathrm{mod}\, p)$ for $k \neq j$ and $a_{k,k} = 1 (\mathrm{mod}\, p)$, for $1 \leq k \leq n_i$. The only possible solution is $a_{k,j} = 0$, $k \neq j$, and $a_{k,k} = 1$. In other words, $\pi_i(\alpha)(z_k) = z_k + \omega$, in which ω is a torsion element.

Adding up, for any element $\gamma \in \pi_i X$, $\pi_i(\alpha)\gamma - \gamma$ is a torsion element in $\pi_i X$. This is equivalent to say that the 1-commutators of the action of $\cap_{p \text{ prime}} \mathcal{E}_{\sharp p}(X)$ on $\pi_i X$ live in the torsion part of $\pi_i X$. However, by Corollary 17, the action of $\cap_{p \text{ prime}} \mathcal{E}_{\sharp p}(X)$ on the torsion part is nilpotent and therefore, the action on $\pi_i X$ is also nilpotent. Apply Theorem 1 and the proof is complete. $\square$

Remark 23. We end up by noting that the hypothesis of Theorem 8 are necessary. Indeed, consider $X = K\big((\mathbb{Z}/2)^2, n\big)$ and observe that, for a prime p different from 2, $\mathcal{E}_{\sharp p}(X) = \mathcal{E}(X) = GL_2(\mathbb{Z}/2) \cong \Sigma_3$ which is not nilpotent.

On the other hand, take $X = K(\mathbb{Z}^2, n)$ for which $\mathcal{E}(X) = GL_2(\mathbb{Z})$. In this case $\mathcal{E}_\sharp(X) = \{1\}$ and, for any prime p, $\mathcal{E}_{\sharp p}(X)$ fits in the following short exact sequence

$$\{1\} \to \mathcal{E}_{\sharp p}(X) \to GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/p) \to \{1\}$$

where the surjection $GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/p)$ is just the mod-p reduction. Hence $\mathcal{E}_{\sharp p}(X) = \mathcal{E}_{\sharp p}(X)/\mathcal{E}_\sharp(X)$ is an infinite, non nilpotent group.

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