On the existence of tight relative 2-designs on binary Hamming association schemes

Eiichi Bannai, Etsuko Bannai and Hideo Bannai

Abstract

It is known that there is a close analogy between "Euclidean t-designs vs. spherical t-designs" and "Relative t-designs in binary Hamming association schemes vs. combinatorial t-designs". In this paper, we want to prove how much we can develop a similar theory in the latter situation, imitating the theory in the former one. We first prove that the weight function is constant on each shell for tight relative t-designs on p shells on a wide class of Q-polynomial association schemes, including Hamming association schemes. In the theory of Euclidean t-designs on 2 concentric spheres (shells), it is known that the structure of coherent configurations is naturally attached. However, it seems difficult to prove this claim in a general context. In the case of tight 2-designs in combinatorial 2-designs, there are great many tight 2-designs, i.e., symmetric 2-designs, while there are very few tight $2e$-designs for $e \geq 2$. So, as a starting point, we concentrate our study to the existence problem of tight relative 2-designs, in particular on 2 shells, in binary Hamming association schemes $H(n,2)$. We prove that every tight relative 2-designs on 2 shells in $H(n,2)$ has the structure of coherent configuration. We determined all the possible parameters of coherent configurations attached to such tight relative 2-designs for $n \leq 30$. Moreover for each of them we determined whether there exists such a tight relative 2-design or not, either by constructing them from symmetric 2-designs or Hadamard matrices, or theoretically showing the non-existence. In particular, we show that for $n \equiv 6 \pmod 8$, there exist such tight relative 2-designs whose weight functions are not constant. These are the first examples of those with non-constant weight.

Keywords: tight design, relative t-design, Hamming scheme, regular t-wise balanced design, regular semi lattice.
2010 Mathematics Subject Classification: 05E30, 05B30

1 Introduction

As is well known, there is a close analogy between the theory of combinatorial t-designs (t-(v,k,λ) designs) and the theory of spherical t-designs. Furthermore, it is known that there is a close analogy between the theory of Euclidean t-designs and the theory of relative t-designs in binary Hamming association schemes $H(n,2)$. Although this last analogy is known, it is not very well known up to now. (See, Delsarte [14, 16], Delsarte-Seidel[17], Bannai-Bannai [6].) The purpose of the present paper is to dig into more on this analogy. The theory of spherical harmonics has been developed into a very elaborate stage, and the theory is extremely beautiful. On the other hand, the theory of spherical functions on (Q-polynomial) association schemes is also developed, but in a sense it is more sophisticated. So, we need more careful treatments in order to get similar results known in Euclidean t-designs, for relative t-designs on Q-polynomial, say, binary Hamming association schemes $H(n, 2)$.

For example, the tight spherical t-designs as well as the theory of tight combinatorial t-designs are both well studied, although complete classifications are not yet obtained at this stage. (Cf. [18], [8], [9], [11], [22], etc. for tight spherical t-designs, and Ray-Chaudhuri and Wilson [23], Enomoto-Ito-Noda [20], Bannai [1], Dukes-ShortGershman [19], etc. for tight combinatorial designs.) For tight Euclidean t-designs, the theory was developed in certain cases ([2, 3, 4, 5, 12, 13]). On the other hand, the theory of tight relative t-designs in $H(n, 2)$ is less developed, so far. So, we will try to see how much the methods in Euclidean t-designs can be applied in the study of tight relative t-designs in $H(n, 2)$. Here, we are mostly interested in the case where the number of spheres supporting the Euclidean designs or the number of shells supporting the combinatorial t-designs are relatively small, equal to 2 in most cases. Also, we must put some strong restrictions on t, in some cases. In this paper, we want to obtain the following explicit results.
(i) We prove that for tight relative $2e$-designs in a Q-polynomial association scheme, the weight function must be constant on each shell of the design, with a mild additional assumption. (See Theorem 2.1). In the latter part of this paper, we restrict our study to tight relative $2e$-designs in $H(n, 2)$, and also to $e = 1$. These are very strong restrictions, but still there are interesting examples and interesting theories. (ii) Using general theory of the study of tight relative $2e$-designs in $H(n, 2)$, we determine all the possible parameters of the tight relative 2-designs on 2-shells in $H(n, 2)$, for explicit small values of n, say $n \leq 30$. Then, (iii) we determine the existence and the non-existence with those parameters listed in (ii). Very interesting feature is that we did find some examples of tight relative 2-designs in $H(n, 2)$, where the weight functions are not constant. It seems such examples were not known explicitly before. Here, we use some results obtained in [21].

Our results obtained in this paper are only for special cases, but we expect that this approach will shed some light on the future studies of more general theory of (tight) relative $2e$-designs for bigger e in more general Q-polynomial association schemes. As for the information on association schemes, e.g., more general P-polynomial, Q-polynomial or P- and Q-polynomial schemes refer [10].

Now we introduce notation we use in this paper and some important definitions. Let $\mathfrak{X} = (X, \{R\}_{0\leq r\leq d})$ be a symmetric association scheme. Let $u_0 \in X$ fixed arbitrarily. Let $X_r = \{x \in X \mid (u_0, x) \in R_r\}$ for $r = 0, 1, \ldots, d$. $X_0, X_1, \ldots, X_d$ are called shells of $\mathfrak{X}$. $\mathcal{F}(X)$ be the vector space consists of all the real valued functions on X. In the following argument we often identify $\mathcal{F}(X)$ with the vector space $\mathbb{R}^{|X|}$ indexed by X. When we consider spherical designs or Euclidean designs, we use the properties of vector spaces of polynomials. For the usual polynomials in n variables defined on $\mathbb{R}^n$, it is convenient to consider the subspaces $Hom_j(\mathbb{R}^n)$ spanned by all the homogeneous polynomials of degree j. If $\mathfrak{X}$ is a P-polynomial scheme, it is natural to consider the following subspace of $\mathcal{F}(X)$.

For any $z \in X_j$, we define $f_z \in \mathcal{F}(X)$ by

$$f_z(x) = \begin{cases} 1 & \text{if } x \in X_i,\ i \geq j \text{ and } (x,z) \in R_{i-j}, \\ 0 & \text{other wise.} \end{cases} \tag{1.1}$$

Let $Hom_j(X) = \langle f_z \mid z \in X_j \rangle$. Then we have the following decomposition of $\mathcal{F}(X)$ into direct sum of subspaces.

$$\mathcal{F}(X) = \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_d(X). \tag{1.2}$$

Clearly we have

$$\dim(\mathrm{Hom}_j(X)) = |X_j| = k_j \ (0 \leq j \leq d).$$

When $\mathfrak{X}$ is a Q-polynomial scheme, it is natural to consider the following subspace of $\mathcal{F}(X)$. Let $E_0, E_1, \ldots, E_d$ be the primitive idempotents which give the Q-polynomial structure of $\mathfrak{X}$. For each E_j, let $L_j(X)$ be the subspace of $\mathcal{F}(X)$ spanned by all the column vectors of E_j. Then we have $\dim(L_j(X)) = rank(E_j) = m_j$ and we have the following decomposition of $\mathcal{F}(X)$ into orthogonal sum of subspaces.

$$\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X). \tag{1.3}$$

For each of the decomposition of $\mathcal{F}(X)$ given above we can develop theory of relative t-designs for weighted subset (Y, w) of X using the similar setting as for the Euclidean designs. We use the following notation. Let $\{r_1, r_2, \ldots, r_p\} = \{r \mid X_r \cap Y \neq \emptyset\}$. Let $S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$. We say (Y, w) is supported by p shells. Let $Y_{r_i} = X_{r_i} \cap Y$, $i = 1, 2, \ldots, p$. We also define $A(Y_{r_i}, Y_{r_j}) = \{\alpha \mid (x, y) \in R_\alpha, x \in Y_{r_i}, y \in Y_{r_j}, x \neq y\}$ for $1 \leq i, j \leq p$. We also use the notation $A(Y_{r_i}) = A(Y_{r_i}, Y_{r_i})$ for $1 \leq i \leq p$.

As for the decomposition given by (1.2) for $H(n, 2)$, Delsarte-Seidel [17] defined the design as regular t-wise balanced design.

In this paper we consider the decomposition (1.3) for Q-polynomial schemes. The concept of relative t-design with respect a fixed point $u_0 \in X$ is related to the decomposition given by (1.3) for Q-polynomial schemes. It was first defined by Delsarte in 1977 [16]. Without noticing his paper, we gave a definition of relative t-designs with respect to $u_0 \in X$ analyzing the concept of Euclidean t-designs [6]. Later H. Tanaka informed us the existence of the paper by Delsarte in 1977 [16]. In [6], we prove that our definition is equivalent to that of Delsarte. We found that the theory of relative t-designs with respect to a fixed point is very similar to the concept of Euclidean design, in which the origin $0 \in \mathbb{R}^n$ is a special point.

The following is the definition of relative t-design in the style of Euclidean t-design (see [6]).

Definition 1.1 *Let (Y, w) be a weighted subset of X with positive weight function w on Y. (Y, w) is called a relative t-design with respect to u_0 if the following condition holds.*

$$\sum_{i=1}^{p} \sum_{x \in X_{r_i}} \frac{W_{r_i}}{|X_{r_i}|} f(x) = \sum_{y \in Y} w(y) f(y) \tag{1.4}$$

for any function $f \in L_0(X) \perp L_1(X) \perp \cdots \perp L_t(X)$, where $W_{r_i} = \sum_{y \in Y_{r_i}} w(y)$, $i = 1, 2, \ldots, p$.

The following theorem is known [6].

Theorem 1.2 *Let (Y, w) be a relative $2e$-design of a Q-polynomial scheme. Then the following inequality holds.*

$$|Y| \geq \dim(L_0(S) + L_1(S) + \cdots + L_e(S)), \tag{1.5}$$

where $L_j(S) = \{f|_S \mid f \in L_j(X)\}$, $j = 0, 1, \ldots, e$.

Definition 1.3 (tight relative $2e$-design with respect to u_0) *If equality holds in (1.5) in Theorem 1.2, then (Y, w) is called a tight relative $2e$-design with respect to u_0.*

In the following we only consider the nontrivial tight $2e$-designs (Y, w). That is, Y does not contain X_r for any r, $0 \leq r \leq d$.

Remark 1.4 (1) *It is conjectured that*

$$\dim(L_0(S) + L_1(S) + \cdots + L_e(S)) = m_e + m_{e-1} + \cdots + m_{e-p+1}$$

holds for Q-polynomial schemes with some trivial exceptions. For binary hamming scheme it is proved that the conjecture is true [24].

(2) *In [17], it is proved that a regular $2e$-wise balanced design (Y, w) satisfies*

$$|Y| \geq \dim(Hom_{\,0}(S) + Hom_{\,1}(S) + \cdots + Hom_{\,e}(S)). \tag{1.6}$$

However Delsarte-Seidel [17] mentioned that the explicit computation of $\dim(Hom_{\,0}(S) + Hom_{\,1}(S) + \cdots + Hom_{\,e}(S))$ *will be difficult. Recently Xiang [24] proved that*

$$\dim(Hom_{\,0}(S) + Hom_{\,1}(S) + \cdots + Hom_{\,e}(S)) = k_e + k_{e-1} + \cdots + k_{e-p+1}$$

holds for $H(n, 2)$. It is also proved that

$$Hom_{\,0}(X) + Hom_{\,1}(X) + \cdots + Hom_{\,e}(X) = L_0(X) + L_1(X) + \cdots + L_e(X)$$

for some P-and Q-polynomial schemes including $H(n, 2)$ [7]. Hence conjecture is correct for $H(n, 2)$, if $S = X_{r_1} \cup X_{r_2} \cdots \cup X_{r_p}$ satisfies some suitable condition to avoid the cases which trivially do not satisfy the conjecture.

In §2, we give our main results. In §3 and §4, we give the proofs of the main results.

2 Main theorems

Theorem 2.1 *Let $\mathfrak{X} = (X, \{R_r\}_{0 \leq r \leq d})$ be a Q-polynomial scheme. Let G be the automorphism group of $\mathfrak{X}$. Let (Y, w) be a tight relative $2e$-design with respect to u_0 supported by p shells. Assume that the stabilizer G_{u_0} of u_0 acts transitively on every shell X_r, $1 \leq r \leq d$. Then the weight function w of any tight relative $2e$-design (Y, w) is constant on each Y_{r_i} $(1 \leq i \leq p)$.*

Theorem 2.2 *Let (Y, w) be a tight relative 2-design of the binary Hamming scheme $H(n, 2)$ supported by 2 shells, $S = X_{r_1} \cup X_{r_2}$. Let $N_{r_i} = |Y_{r_i}|$, $w(y) = w_{r_i}$ on $y \in Y_{r_i}$ for $i = 1, 2$.*

(1) *For any integers r_1, r_2 satisfying $1 \le r_1 < r_2 \le n-1$, the following holds*
$$|A(Y_{r_1})| = |A(Y_{r_2})| = |A(Y_{r_1}, Y_{r_2})| = 1.$$
This means that $Y = Y_{r_1} \cup Y_{r_2}$ has a structure of coherent configuration.

(2) *Assume $1 \le r_1 < r_2 \le n-1$ and $n \le 30$, then the set of parameters $\{n, r_1, r_2, N_{r_1}, N_{r_2}, \alpha_1, \alpha_2, \gamma, \frac{w_{r_2}}{w_{r_1}}\}$ is among those listed in §4.4, here $A(Y_{r_i}) = \{\alpha_i\}$ for $i = 1, 2$ and $A(Y_{r_1}, Y_{r_2}) = \{\gamma\}$.*

(3) *If $n \equiv 6 \pmod 8$, and there exists Hadamard matrix of size $\frac{1}{2}n + 1$, then there exists tight relative 2 design $Y \subset X_2 \cup X_{\frac{n}{2}}$ ($r_1 = 2$, $r_2 = \frac{n}{2}$) whose weights satisfy $\frac{w_{\frac{1}{2}n}}{w_2} = \frac{8}{n+2}$, that is, w is not constant on Y.*

(4) *If $n \le 30$, $w_{r_1} \ne w_{r_2}$ and $n \not\equiv 6 (mod\ 8)$ or $n \equiv 6 (mod\ 8)$ and Y is not related to the Hadamard matrices in (3), then there is no tight relative 2-designs with respect to u_0.*

Remark 2.3 *Since we consider only nontrivial tight designs and $|X_0| = |X_n| = 1$, we may assume $1 \le r_1 < r_2 \le n-1$.*

In §3, we give the proof of Theorem 2.1. In §4, we give the proof of Theorem 2.2. Proposition 4.3 in §4.1 gives the explicit formula for $\frac{w_{r_2}}{w_{r_2}}$, α_1, α_2, and γ in terms of n, r_1, r_2, N_{r_1}. This implies Theorem 2.2 (1). Proposition 4.1 and Proposition 4.2 in §4.1 give some useful formulas to prove Proposition 4.3. In §4.2 we give the proof of Propositions 4.1, 4.2 and 4.3. In §4.4 we give the table of possible sets of parameters $\{n, r_1, r_2, N_{r_1}, N_{r_2}, \alpha_1, \alpha_2, \gamma, \frac{w_{r_2}}{w_{r_1}}\}$ for tight relative 2-design (Y, w) with respect to u_0 for $n \le 30$. We give two kind of construction theorems. One is the construction from Hadamard matrices and the other is the construction from symmetric designs (Proposition 4.5 in §4.4).

To obtain the feasible parameters, $\{n, r_1, r_2, N_{r_1}, N_{r_2}, \alpha_1, \alpha_2, \gamma, \frac{w_{r_2}}{w_{r_1}}\}$ in the table given in §4, we mainly used the properties of Q-polynomial structure of $H(n, 2)$. In [16], Delsarte proved that if association scheme is attached to a regular semi lattice (then it is a P-polynomial scheme), and if it also has Q-polynomial structure with it's ordering, then (Y, w) is a relative t-design with respect to u_0 if and only if it is a geometric relative t-design with respect to u_0. For $H(n, 2)$, a geometric relative t-design with respect to u_0 is nothing but a regular t-wise balanced design. In §4.3 we briefly introduce regular semi-lattices and geometric relative t-designs. We also use the property of regular t-wise balanced design (Proposition 4.4 in §4) to show the non-existence of such a design for some feasible parameters in the table.

Remark 2.4 *We conclude this section by mentioning that Woodall [25], in particular Theorem 8 in [25] essentially discuss similar problem as ours under the additional assumption that the weight function is constant. It would be interesting to compare our approach with that of Woodall [25]*

3 Proof of Theorem 2.1

Let $L(S)$ be the vector space of real valued functions on S. We consider the inner product on $L(S)$ defined for $f, g \in L(S)$ by

$$\langle f, g\rangle = \sum_{i=1}^{p} \frac{W_{\nu_i}}{|X_{\nu_i}|} \sum_{x\in X_{\nu_i}} f(x)g(x). \tag{3.1}$$

Let $\{\varphi_1, \ldots, \varphi_N\} \subset L_0(X) \perp L_1(X) \perp \cdots \perp L_e(X)$. Assume that $\{\varphi_1|_S, \ldots, \varphi_N|_S\}$ is an orthonormal basis of $L_0(S) + L_1(S) + \cdots + L_e(S)$ with respect to this inner product. Let H be the matrix whose rows are indexed by Y with N columns whose (y, i)-entry is defined by $\sqrt{w(y)}\varphi_i(y)$. Since $fg \in L_0(X) \perp L_1(X) \perp \cdots \perp L_{2e}(X)$ holds for any $f, g \in L_0(X) \perp L_1(X) \perp \cdots \perp L_e(X)$, $\varphi_i\varphi_j \in L_0(X) \perp L_1(X) \perp \cdots \perp L_{2e}(X)$. Then we have the following

$$\begin{aligned}({}^tH\ H)(i,j) &= \sum_{y\in Y} w(y)\varphi_i(y)\varphi_j(y)\\ &= \sum_{i=1}^{p}\sum_{x\in X_{r_i}} \frac{W_{r_i}}{|X_{r_i}|}\varphi_i(x)\varphi_j(x) = \delta_{i,j}.\end{aligned} \tag{3.2}$$

This implies $\mathrm{rank}(H) = |Y| \geq N = \dim(L_0(S) + L_1(S) + \cdots + L_e(S)$. If $|Y| = N$, then H is an invertible matrix and $H\ {}^tH = I$ holds. Then we have

$$(H^tH)(y_1, y_2) = \sum_{i=1}^{N} \sqrt{w(y_1)w(y_2)}\varphi_i(y_1)\varphi_i(y_2) = \delta_{y_1,y_2}. \tag{3.3}$$

This implies

$$\sum_{i=1}^{N} \varphi_i(x)\varphi_i(y) = \delta(x, y)\frac{1}{w(y)}. \tag{3.4}$$

We introduce the following notation. Let $\phi_u^{(j)} \in L_j(X)$ whose x-entry is defined by $\phi_u^{(j)}(x) = \frac{1}{|X|}E_j(x, u)$ for $0 \leq j \leq d$. Let

$$\{\phi_{u_1}^{(j_1)}, \phi_{u_2}^{(j_2)}, \ldots, \phi_{u_N}^{(j_N)}\} \subset \bigcup_{j=0}^{e}\{\phi_u^{(j)} \mid u \in X\}$$

be a set of functions whose restrictions to S forms a basis of $L_0(S) + L_1(S) + \cdots + L_e(S)$. Let $u_s \in X_{l_s}$. For simplicity let us write $\phi_s = \phi_{u_s}^{(j_s)}$ for $s = 1, 2, \ldots, N$. From $\{\phi_1, \phi_2, \ldots, \phi_N\}$ we construct a set of $\{\varphi_1, \ldots, \varphi_N\}$ whose restrictions $\varphi_1|_S, \ldots, \varphi_N|_S$ to S forms an orthonormal system in $L(S)$. It is well known that Gram-Schmidt's method gives the following formula for $\varphi_1, \ldots, \varphi_N$.

$$\varphi_j = \frac{1}{\sqrt{D_{j-1}D_j}} \begin{vmatrix} \langle\phi_1, \phi_1\rangle & \langle\phi_2, \phi_1\rangle & \cdots & \langle\phi_j, \phi_1\rangle \\ \langle\phi_1, \phi_2\rangle & \langle\phi_2, \phi_2\rangle & \cdots & \langle\phi_j, \phi_2\rangle \\ \vdots & \cdots & \cdots & \vdots \\ \langle\phi_1, \phi_{j-1}\rangle & \langle\phi_2, \phi_{j-1}\rangle & \cdots & \langle\phi_j, \phi_{j-1}\rangle \\ \phi_1 & \phi_2 & \cdots & \phi_j \end{vmatrix}, \tag{3.5}$$

where D_j is the Gram determinant given by

$$D_j = \begin{vmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \cdots & \langle \phi_j, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \cdots & \langle \phi_j, \phi_2 \rangle \\ \vdots & \cdots & \cdots & \vdots \\ \langle \phi_1, \phi_{j-1} \rangle & \langle \phi_2, \phi_{j-1} \rangle & \cdots & \langle \phi_j, \phi_{j-1} \rangle \\ \langle \phi_1, \phi_j \rangle & \langle \phi_2, \phi_j \rangle & \cdots & \langle \phi_j, \phi_j \rangle \end{vmatrix}. \tag{3.6}$$

The formula (3.5) means φ_j is given by the linear sum of ϕ_l with coefficient given by the (j,l)-cofactor of the matrix given in (3.5). Let $y_1, y_2 \in Y \cap X_{r_i}$. By the assumption of this theorem G_{u_0} is transitive on X_{r_i}. Hence there exists $\sigma \in G_{u_0}$ satisfying $\sigma(y_1) = y_2$. Since $\sigma(u_0) = u_0$ and $u_s \in X_{l_s}$, we must have $\sigma(u_s) \in X_{l_s}$ for $s = 1, 2, \ldots, N$. Let $\phi_s^\sigma = \phi_{\sigma(u_s)}^{(j_s)}$. Since $\sigma(X_r) = X_r$ for $r = 0, 1, \ldots, d$ and

$$\begin{aligned} \sum_{s=1}^{N} c_s \phi_s^\sigma(x) &= \sum_{s=1}^{d} c_s \phi_{\sigma(u_s)}^{(j_s)}(x) = \sum_{s=1}^{N} c_s \frac{1}{|X|} E_{j_s}(x, \sigma(u_s)) \\ &= \sum_{s=1}^{N} c_s \frac{1}{|X|} E_{j_s}(\sigma^{-1}(x), u_s) = \sum_{s=1}^{N} c_s \phi_s(\sigma^{-1}(x)), \end{aligned} \tag{3.7}$$

$\{\phi_1^\sigma, \phi_2^\sigma, \ldots, \phi_N^\sigma\}$ is also a basis of $L_0(S) + L_1(S) + \cdots + L_e(S)$. Then we have

$$\begin{aligned} &\langle \phi_{l_1}^\sigma, \phi_{l_2}^\sigma \rangle = \langle \phi_{\sigma(u_{l_1})}^{(j_{l_1})}, \phi_{\sigma(u_{l_2})}^{(j_{l_2})} \rangle \\ &= \sum_{i=1}^{p} \frac{W_{r_i}}{|X_{r_i}|} \sum_{\nu_1=0}^{d} \sum_{\nu_2=0}^{d} \sum_{x \in X_{r_i} \cap \Gamma_{\nu_1}(\sigma(u_{l_1})) \cap \Gamma_{\nu_2}(\sigma(u_{l_2}))} \phi_{\sigma(u_{l_1})}^{(j_{l_1})}(x) \phi_{\sigma(u_{l_2})}^{(j_{l_2})}(x) \\ &= \sum_{i=1}^{p} \frac{W_{r_i}}{|X_{r_i}|} \sum_{\nu_1=0}^{d} \sum_{\nu_2=0}^{d} |\sigma(X_{r_i}) \cap \Gamma_{\nu_1}(\sigma(u_{l_1})) \cap \Gamma_{\nu_2}(\sigma(u_{l_2}))| Q_{j_{l_1}}(\nu_1) Q_{j_{l_2}}(\nu_2) \\ &= \sum_{i=1}^{p} \frac{W_{r_i}}{|X_{r_i}|} \sum_{\nu_1=0}^{d} \sum_{\nu_2=0}^{d} |X_{r_i} \cap \Gamma_{\nu_1}(u_{l_1}) \cap \Gamma_{\nu_2}(u_{l_2})| Q_{j_{l_1}}(\nu_1) Q_{j_{l_2}}(\nu_2) \\ &= \langle \phi_{u_{l_1}}^{(j_{l_1})}, \phi_{u_{l_2}}^{(j_{l_2})} \rangle = \langle \phi_{l_1}, \phi_{l_2} \rangle \end{aligned} \tag{3.8}$$

for any $l_1, l_2 \in \{1, \ldots, N\}$. Here $\Gamma_\nu(u) = \{x \in X \mid (u, x) \in R_\nu\}$. On the other hand if $y_1 \in \Gamma_\nu(u_{l_1})$, then we must have $y_2 = \sigma(y_1) \in \Gamma_\nu(\sigma(u_{l_1}))$. Hence we have

$$\phi_{l_1}^\sigma(y_2) = \phi_{\sigma(u_{l_1})}^{(j_{l_1})}(\sigma(y_1)) = \frac{1}{|X|} Q_{j_{l_1}}(\nu) = \phi_{u_{l_1}}^{(j_{l_1})}(y_1) = \phi_{l_1}(y_1). \tag{3.9}$$

Let $\{\varphi_1^\sigma, \ldots, \varphi_N^\sigma\}$ be the orthonormal system obtained from $\{\phi_1^\sigma, \ldots, \phi_N^\sigma\}$ by the formulas (3.5) and (3.6). Then we must have $\varphi_s(y_1) = \varphi_s^\sigma(y_2)$ for $s = 1, 2, \ldots, N$. Hence we have

$$\sum_{s=1}^{N} (\varphi_s(y_1))^2 = \sum_{s=1}^{N} (\varphi_s^\sigma(y_2))^2.$$

This implies $w(y_1) = w(y_2)$ and completes the proof of Theorem 1.1. ∎

4 Proof of Theorem 2.2

4.1 Important propositions

It is known that the binary Hamming scheme $H(n,2)$ satisfies the assumption of Theorem 2.1. First we introduce notation for $H(n,2)$. Let $F=\{0,1\}$ and $X=F^n$ and $H(n,2)=(X,\{R_i\}_{0\leq i\leq n})$. For $x=(x_1,x_2,\ldots,x_n)\in X$, we define $\overline{x}\subset\{1,2,\ldots,n\}$ by $\overline{x}=\{i\mid x_i=1,\ 1\leq i\leq n\}$.

Let (Y,w) be a relative tight $2e$-design of $H(n,2)$ supported by p shells, i.e., $S=X_{r_1}\cup\cdots\cup X_{r_p}$. Let $N=\dim(L_0(S)+L_1(S)+\cdots+L_e(S))$. Then by Theorem 2.1 we have $|Y|=N$ and $w(y)=w_{r_i}$ for any $y\in Y_{r_i}=Y\cap X_{r_i}$, $i=1,\ldots,p$, with positive real numbers $w_{r_1},\ldots,w_{r_p}$. In the proof of Theorem 2.1, we showed that for any orthonormal basis $\{\varphi_1,\ldots,\varphi_N\}$ of $L_0(S)+L_1(S)+\cdots+L_e(S)$ with respect to the inner product defined by (3.1) then

$$\sum_{i=1}^{N}\varphi_i(x)\varphi_i(y)=\delta_{x,y}\frac{1}{w(y)}$$

holds holds for any $x,y\in Y$. We use this property and investigate the relations between the constants $N(=|Y|)$, $r_1,\ldots,r_p$, $N_{r_i}=|Y\cap X_{r_i}|$ $(1\leq i\leq p)$) and $w_{r_1},\ldots,w_{r_p}$. It is known that the first and second eigen matrices P and Q of $H(n,2)$ coincide and given by

$$P_k(u)=Q_k(u)=\sum_{i=0}^{k}(-1)^i\binom{n-u}{k-i}\binom{u}{i}. \tag{4.1}$$

In particular $k_i=m_i=\binom{n}{i}$ holds for $i=0,1,\ldots,n$. We consider the relative 2-design (Y,w) with respect to u_0, on $S=X_{r_1}\cup X_{r_2}$. Without loss of generality we may assume $u_0=(0,0,\ldots,0)$. Then $x\in X_r$ if and only if $|\overline{x}|=r$. Let $X_1=\{u_1,\ldots,u_n\}$ (note that $k_1=m_1=n$ in this case). We use the following notation.

$$\phi_0(x)=\phi^{(0)}_{u_0}(x)=|X|E_0(x,u_0), \tag{4.2}$$
$$\phi_j(x)=\phi^{(1)}_{u_j}(x)=|X|E_1(x,u_j) \tag{4.3}$$

for any $x\in X$. By Proposition 2.2 (2) (b) in [21], $\{\phi_0|_S,\phi_1|_S,\ldots,\phi_n|_S\}$ is an basis of $L_0(S)+L_1(S)$, $S=X_{r_1}\cup X_{r_2}$, for any integers r_1,r_2 satisfying $1\leq r_1<r_2\leq n-1$. (The condition $(k,l)\neq(1,n-1)$ of Proposition 2.2 (2) (b) in [21] is not correct. It should be $(k,l)\neq(0,n),(n,0)$. So in our case we assume $1\leq r_1<r_2\leq n-1$, and then $\dim(L_0(S)+L_1(S))=n+1$ holds.) In this case the inner product $\langle f,g\rangle$, $f,g\in L_0(S)+L_1(S)$, $S=X_{r_1}\cup X_{r_2}$ is defined by

$$\langle f,g\rangle=\frac{W_{r_1}}{|X_{r_1}|}\sum_{x\in X_{r_1}}f(x)g(x)+\frac{W_{r_2}}{|X_{r_2}|}\sum_{x\in X_{r_2}}f(x)g(x).$$

By definition, $W_{r_i}=N_{r_i}w_{r_i}$ holds for $i=1,2$. The following propositions play the important role for the proof of Theorem 2.2.

Proposition 4.1

(1) $\langle \phi_i, \phi_0 \rangle = \frac{(n-2)}{n}\left((n-2r_1)W_{r_1} + (n-2r_2)W_{r_2}\right)$ *for* $1 \le i \le n$.

(2) $\langle \phi_i, \phi_i \rangle = \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{n}\Bigg(4(n-4)r_\nu^2 - 4n(n-4)r_\nu + n(n-2)^2\Bigg)$ *for* $1 \le i \le n$.

(3) $\langle \phi_i, \phi_j \rangle = \langle \phi_1, \phi_2 \rangle$
$= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{n(n-1)}\Bigg(4(n^2-5n+8)r_\nu^2 - 4n(n^2-5n+8)r_\nu + n(n-1)(n-2)^2\Bigg)$ *for any* $1 \le i \ne j \le n$.

We use the following notation: $d_0 = \langle \phi_1, \phi_0 \rangle$, $c_0 = \langle \phi_1, \phi_1 \rangle$ and $c_2 = \langle \phi_1, \phi_2 \rangle$.

Proposition 4.2 *Let $h_1, h_2, \ldots, h_{n+1}$ be the orthogonal basis of $L_0(S) + L_1(S)$ obtained from the bases $\{\phi_1, \phi_2, \ldots, \phi_n, \phi_0\}$ by Gram-Schmidt's method with this ordering. Then we have the following formulas.*

$$h_1 = \phi_1, \tag{4.4}$$

$$h_i = \phi_i - \frac{c_2}{c_0 + (i-2)c_2} \sum_{j=1}^{i-1} \phi_j \qquad \text{for } i = 2, 3, \ldots, n, \tag{4.5}$$

$$h_{n+1} = \phi_0 - \frac{d_0}{c_0 + (n-1)c_2} \sum_{j=1}^{n} \phi_j, \tag{4.6}$$

$$\|h_1\|^2 = c_0, \tag{4.7}$$

$$\|h_i\|^2 = \frac{(c_0 - c_2)(c_0 + (i-1)c_2)}{(c_0 + (i-2)c_2)}, \quad \text{for } i = 2, \ldots, n, \tag{4.8}$$

$$\|h_{n+1}\|^2 = W_{r_1} + W_{r_2} - \frac{n d_0^2}{c_0 + (n-1)c_2}. \tag{4.9}$$

Proposition 4.3 (1) *$2 \le N_{r_1}, N_{r_2} \le n-1$ holds and*

$$\frac{w_{r_2}}{w_{r_1}} = \frac{N_{r_1} r_1 (n - N_{r_1})(n - r_1)}{r_2 (N_{r_1} - 1)(n + 1 - N_{r_1})(n - r_2)}. \tag{4.10}$$

(2) *If there exists an nonzero even integer α_ν satisfying $2 \le \alpha_\nu \le 2r_\nu$, and $x, y \in X_{r_\nu}$ $(\nu = 1, 2)$, with $(x, y) \in R_{\alpha_\nu}$, then the following holds.*

$$\alpha_1 = \frac{2(n - r_1) r_1 N_{r_1}}{n(N_{r_1} - 1)}, \tag{4.11}$$

$$\alpha_2 = \frac{2(n - r_2)(n + 1 - N_{r_1}) r_2}{n(n - N_{r_1})}. \tag{4.12}$$

(3) *If there exists an even integer γ satisfying $(x, y) \in R_\gamma$, for $x \in X_{r_1}$ and $y \in X_{r_2}$, then the following holds.*

$$\gamma = \frac{n(r_1 + r_2) - 2r_1 r_2}{n}. \tag{4.13}$$

4.2 Proof of the propositions

Proof of Proposition 4.1
(1):

$$
\begin{aligned}
\langle \phi_i, \phi_0 \rangle &= \frac{W_{r_1}}{|X_{r_1}|} \sum_{x \in X_{r_1}} \phi^{(1)}_{u_i}(x) \phi^{(0)}_{u_0}(x) + \frac{W_{r_2}}{|X_{r_2}|} \sum_{x \in X_{r_2}} \phi^{(1)}_{u_i}(x) \phi^{(0)}_{u_0}(x) \\
&= \frac{W_{r_1}}{|X_{r_1}|} \sum_{\nu=0}^{n} \sum_{x \in X_{r_1} \cap \Gamma_\nu(u_i)} Q_1(\nu) + \frac{W_{r_2}}{|X_{r_2}|} \sum_{x \in X_{r_2}} \phi_{u_i}(x) \\
&= \frac{W_{r_1}}{|X_{r_1}|} \Big(|X_{r_1} \cap \Gamma_{r_1-1}(u_i)| Q_1(r_1-1) + |X_{r_1} \cap \Gamma_{r_1+1}(u_i)| Q_1(r_1+1) \Big) \\
&+ \frac{W_{r_2}}{|X_{r_2}|} \Big(|X_{r_2} \cap \Gamma_{r_2-1}(u_i)| Q_1(r_2-1) + |X_{r_2} \cap \Gamma_{r_2+1}(u_i)| Q_1(r_2+1) \Big) \\
&= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{\binom{n}{r_\nu}} \left(\binom{n-1}{r_\nu - 1} Q_1(r_\nu - 1) + \binom{n-1}{r_\nu} Q_1(r_\nu + 1) \right) \\
&= \frac{(n-2)(n-2r_1)W_{r_1}}{n} + \frac{(n-2)(n-2r_2)W_{r_2}}{n}.
\end{aligned} \tag{4.14}
$$

This proves Proposition 4.1 (1).
(2) and (3):
Let $1 \leq i, j \leq n$. Then

$$
\begin{aligned}
\langle \phi_i, \phi_j \rangle &= \frac{W_{r_1}}{|X_{r_1}|} \sum_{x \in X_{r_1}} \phi^{(1)}_{u_i}(x) \phi^{(1)}_{u_j}(x) + \frac{W_{r_2}}{|X_{r_2}|} \sum_{x \in X_{r_2}} \phi^{(1)}_{u_i}(x) \phi^{(1)}_{u_j}(x) \\
&= \frac{W_{r_1}}{|X_{r_1}|} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} |X_{r_1} \cap \Gamma_{l_1}(u_i) \cap \Gamma_{l_2}(u_j)| Q_1(l_1) Q_1(l_2) \\
&+ \frac{W_{r_2}}{|X_{r_2}|} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} |X_{r_2} \cap \Gamma_{l_1}(u_i) \cap \Gamma_{l_2}(u_j)| Q_1(l_1) Q_1(l_2).
\end{aligned} \tag{4.15}
$$

If $u = u_i = u_j \in X_1$, then

$$
|X_{r_i} \cap \Gamma_\nu(u)| = p^1_{r_i,\nu} = \begin{cases} \binom{n-1}{r_i-1} & \text{if } \nu = r_i - 1, \\ \binom{n-1}{r_i+1} & \text{if } \nu = r_i + 1, \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore we have

$$\begin{aligned}
\langle \phi_u^{(1)}, \phi_u^{(1)} \rangle &= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{|X_{r_\nu}|} \left(\binom{n-1}{r_\nu - 1} Q_1(r_\nu - 1)^2 + \binom{n-1}{r_\nu} Q_1(r_\nu + 1)^2 \right) \\
&= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{\binom{n}{r_\nu}} \left(\binom{n-1}{r_\nu - 1} (n - 2r_\nu + 2)^2 + \binom{n-1}{r_\nu} (n - 2r_\nu - 2)^2 \right) \\
&= \sum_{\nu=1}^{2} \frac{r_\nu!(n-r_\nu)! W_{r_\nu}}{n!} \left(\frac{(n-1)!}{(r_\nu - 1)!(n-r_\nu)!} (n - 2r_\nu + 2)^2 \right. \\
&\left. + \frac{(n-1)!}{(r_\nu)!(n-r_\nu-1)!} (n - 2r_\nu - 2)^2 \right) \\
&= \sum_{\nu=1}^{2} W_{r_\nu} \left(\frac{r_\nu}{n} (n - 2r_\nu + 2)^2 + \frac{(n - r_\nu)}{n} (n - 2r_\nu - 2)^2 \right) \\
&= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{n} \left(4(n-4) r_\nu^2 - 4n(n-4) r_\nu + n(n-2)^2 \right). \qquad (4.16)
\end{aligned}$$

This implies (2).
If $u_i \neq u_j$, and $1 \leq r_1 < r_2 \leq n-1$, then

$$\begin{aligned}
\langle \phi_i, \phi_j \rangle &= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{|X_{r_\nu}|} \sum_{x \in X_{r_\nu}} \phi_{u_i}^{(1)}(x) \phi_{u_j}^{(1)}(x) \\
&= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{|X_{r_\nu}|} \sum_{l_1, l_2 = r_\nu - 1, r_\nu + 1} |X_{r_\nu} \cap \Gamma_{l_1}(u_i) \cap \Gamma_{l_2}(u_j)| Q_1(l_1) Q_1(l_2) \\
&= \sum_{\nu=1}^{2} \frac{W_{r_\nu}}{n(n-1)} \left(4(n^2 - 5n + 8) r_\nu^2 - 4n(n^2 - 5n + 8) r_\nu + n(n-1)(n-2)^2 \right). \qquad (4.17)
\end{aligned}$$

∎

Proof of Proposition 4.2
(4.4) and (4.7) are already shown. According to the Gram-Schmidt's method let $h_i = \phi_i + \sum_{j=1}^{i-1} a_{i,j} \phi_j$ for $i = 2, \ldots, n$. Then $\langle h_1, h_2 \rangle = 0$ implies

$$h_2 = \phi_2 - \frac{c_2}{c_0} \phi_1.$$

Then we must have

$$\begin{aligned}
\langle h_2, h_2 \rangle &= \langle \phi_2, \phi_2 \rangle - 2 \frac{c_2}{c_0} \langle \phi_1, \phi_2 \rangle + (\frac{c_2}{c_0})^2 \langle \phi_1, \phi_1 \rangle \\
&= c_0 - 2 \frac{c_2^2}{c_0} + (\frac{c_2}{c_0})^2 c_0 = \frac{(c_0 - c_2)(c_0 + c_2)}{c_0}.
\end{aligned}$$

Thus h_2 satisfies (4.5) and (4.8). We prove (4.5) and (4.8) by induction on i. Assume that (4.5) and (4.8) hold for any $i \leq s-1, s \leq n$ and we will show that they also hold for $i = s$.

$$\begin{aligned}
0 &= \langle h_s, h_{s-1} \rangle = \langle \phi_s + \sum_{j=1}^{s-1} a_{s,j}\phi_j, h_{s-1} \rangle = \langle \phi_s + a_{s,s-1}\phi_{s-1}, h_{s-1} \rangle \\
&= \langle \phi_s + a_{s,s-1}\phi_{s-1}, \phi_{s-1} \rangle - \frac{c_2}{c_0 + (s-3)c_2} \sum_{j=1}^{s-2} \langle \phi_s + a_{s,s-1}\phi_{s-1}, \phi_j \rangle \\
&= c_2 + a_{s,s-1}c_0 - \frac{c_2}{c_0 + (s-3)c_2}(s-2)(1 + a_{s,s-1})c_2. \qquad (4.18)
\end{aligned}$$

This implies $a_{s,s-1} = -\frac{c_2}{c_0+(s-2)c_2}$. By continuing such straight forwarded computation we obtain $a_{s,1} = a_{s,2} = \cdots = a_{s,s-1} = -\frac{c_2}{c_0+(s-2)c_2}$ and we can verify the formula (4.8) for $\|h_s\|^2$. This completes the proof for (4.5) and (4.8). Next let

$$h_{n+1} = \phi_0 + \sum_{j=1}^{n} a_j \phi_j.$$

Then we have the following.

$$\begin{aligned}
0 &= \langle h_{n+1}, h_n \rangle = \langle \phi_0 + \sum_{j=1}^{n} a_j\phi_j, h_n \rangle = \langle \phi_0 + a_n\phi_n, h_n \rangle \\
&= \langle \phi_0 + a_n\phi_n, \phi_n - \frac{c_2}{c_0 + (n-2)c_2} \sum_{j=1}^{n-1} \phi_j \rangle \\
&= \langle \phi_0, \phi_n \rangle + a_n \langle \phi_n, \phi_n \rangle - \frac{c_2}{c_0 + (n-2)c_2} \langle \phi_0, \sum_{j=1}^{n-1} \phi_j \rangle \\
&\quad - a_n \frac{c_2}{c_0 + (n-2)c_2} \langle \phi_n, \sum_{j=1}^{n-1} \phi_j \rangle \\
&= d_0\left(1 - \frac{(n-1)c_2}{c_0 + (n-2)c_2}\right) + \left(c_0 - \frac{(n-1)c_2^2}{c_0 + (n-2)c_2}\right)a_n. \qquad (4.19)
\end{aligned}$$

Hence

$$\begin{aligned}
a_n &= -\frac{d_0}{c_0 + (n-1)c_2} \\
&= -\frac{(n-2r_1)W_{r_1} + (n-2r_2)W_{r_2}}{(n-2)((n-2r_1)^2 W_{r_1} + (n-2r_2)^2 W_{r_2})}. \qquad (4.20)
\end{aligned}$$

By straight forwarded computation we obtain $a_1 = a_2 = \cdots = a_n$. This implies (4.6).

$$\begin{aligned}
&\|h_{n+1}\|^2 = \langle h_{n+1}, h_{n+1}\rangle \\
&= \langle \phi_0 - \frac{d_0}{c_0+(n-1)c_2}\sum_{j=1}^{n}\phi_j, \phi_0 - \frac{d_0}{c_0+(n-1)c_2}\sum_{l=1}^{n}\phi_l\rangle \\
&= \langle \phi_0, \phi_0\rangle - \frac{2d_0}{c_0+(n-1)c_2}\sum_{l=1}^{n}\langle \phi_0, \phi_l\rangle + \left(\frac{d_0}{c_0+(n-1)c_2}\right)^2 \sum_{j=1}^{n}\langle \phi_j, \sum_{l=1}^{n}\phi_l\rangle \\
&\quad = W_{r_1} + W_{r_2} - \frac{2nd_0^2}{c_0+(n-1)c_2} + \left(\frac{d_0}{c_0+(n-1)c_2}\right)^2 n(c_0+(n-1)c_2) \\
&\quad = W_{r_1} + W_{r_2} - \frac{nd_0^2}{c_0+(n-1)c_2}. \qquad (4.21)
\end{aligned}$$

This completes the proof of Proposition 4.2. ∎

Proof of Proposition 4.3

(1): Let $\nu = 1$ or 2. Let $x \in Y_{r_\nu} = X_{r_\nu} \cap Y$. Choose the ordering of the elements in X_1, we may assume $\overline{x} = \{1, 2, \ldots, r_\nu\}$. Then $\phi_i(x) = Q_1(r_\nu - 1)$ for $i = 1, 2, \ldots, r_\nu$ and $\phi_i(x) = Q_1(r_\nu + 1)$ for $i = r_\nu + 1, r_\nu + 2, \ldots, n$. Hence

$$h_i(x) = Q_1(r_\nu - 1) - \frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu - 1) \text{ for } 1 \le i \le r_\nu, \qquad (4.22)$$

$$h_i(x) = Q_1(r_\nu + 1) - \frac{r_\nu c_2}{c_0+(i-2)c_2}Q_1(r_\nu - 1) - \frac{(i-r_\nu-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)$$
$$\text{for } r_\nu + 1 \le i \le n, \qquad (4.23)$$

$$h_{n+1}(x) = 1 - \frac{d_0}{c_0+(n+1-2)c_2}(r_\nu Q_1(r_\nu - 1) + (n - r_\nu)Q_1(r_\nu + 1)). \qquad (4.24)$$

Let $\varphi_i = \frac{1}{\|h_i\|}h_i$, $i = 1, 2, \ldots, n+1$. Then $\{\varphi_1, \ldots, \varphi_{n+1}\}$ is an orthnormal basis of $L_0(S) + L_1(S)$. Hence we have

$$\begin{aligned}
&\frac{1}{w_\nu} = \sum_{i=1}^{n+1}\varphi_i(x)^2 = \sum_{i=1}^{n+1}\frac{1}{\|h_i\|^2}h_i(x)^2 \\
&= \sum_{s=1}^{r_\nu}\frac{c_0+(s-2)c_2}{(c_0-c_2)(c_0+(s-1)c_2)}\left(1 - \frac{(s-1)c_2}{(c_0+(s-2)c_2)}\right)^2 Q_1(r_\nu-1)^2 \\
&+ \sum_{s=r_\nu+1}^{n}\frac{c_0+(s-2)c_2}{(c_0-c_2)(c_0+(s-1)c_2)}\Bigg(Q_1(r_\nu+1) - \frac{r_\nu c_2}{c_0+(s-2)c_2}Q_1(r_\nu-1) \\
&-\frac{(s-r_\nu-1)c_2}{c_0+(s-2)c_2}Q_1(r_\nu+1)\Bigg)^2 \\
&+\frac{c_0+(n-1)c_2}{(W_{r_1}+W_{r_2})(c_0+(n-1)c_2) - nd_0^2}\Bigg\{1 - \frac{d_0}{c_0+(n-1)c_2}\Bigg(r_\nu Q_1(r_\nu-1) \\
&+(n-r_\nu)Q_1(r_\nu+1)\Bigg)\Bigg\}^2. \qquad (4.25)
\end{aligned}$$

$$\sum_{s=1}^{r_\nu}\frac{c_0+(s-2)c_2}{(c_0-c_2)(c_0+(s-1)c_2)}\left(1-\frac{(s-1)c_2}{(c_0+(s-2)c_2)}\right)^2 Q_1(r_\nu-1)^2$$
$$=\frac{(c_0-c_2)(n-2r_\nu+2)^2}{c_2}\sum_{s=1}^{r_\nu}\left(\frac{1}{c_0+(s-2)c_2}-\frac{1}{c_0+(s-1)c_2}\right)$$
$$=\frac{(c_0-c_2)(n-2r_\nu+2)^2}{c_2}\left(\frac{1}{c_0-c_2}-\frac{1}{c_0+(r_\nu-1)c_2}\right). \tag{4.26}$$

$$\sum_{s=r_\nu+1}^{n}\frac{c_0+(s-2)c_2}{(c_0-c_2)(c_0+(s-1)c_2)}\Bigg(Q_1(r_\nu+1)-\frac{r_\nu c_2}{c_0+(s-2)c_2}Q_1(r_\nu-1)$$
$$-\frac{(s-r_\nu-1)c_2}{c_0+(s-2)c_2}Q_1(r_\nu+1)\Bigg)^2$$
$$=\frac{(nc_0-nc_2-2r_\nu c_0-2r_\nu c_2-2c_0+2c_2)^2}{(c_0-c_2)c_2}\sum_{s=r_\nu+1}^{n}\left(\frac{1}{c_0+(s-2)c_2}-\frac{1}{c_0+(s-1)c_2}\right)$$
$$=\frac{(nc_0-nc_2-2r_\nu c_0-2r_\nu c_2-2c_0+2c_2)^2}{(c_0-c_2)c_2}\left(\frac{1}{c_0+(r_\nu-1)c_2}-\frac{1}{c_0+(n-1)c_2}\right)$$
$$=\frac{(nc_0-nc_2-2r_\nu c_0-2r_\nu c_2-2c_0+2c_2)^2(n-r_\nu)}{(c_0-c_2)(c_0+r_\nu c_2-c_2)(c_0+nc_2-c_2))}. \tag{4.27}$$

$$\frac{c_0+(n-1)c_2}{(W_{r_1}+W_{r_2})(c_0+(n-1)c_2)-nd_0^2}\Bigg\{1-\frac{d_0}{c_0+(n-1)c_2}\Bigg(r_\nu Q_1(r_\nu-1)$$
$$+(n-r_\nu)Q_1(r_\nu+1)\Bigg)\Bigg\}^2=\frac{(c_0+nc_2-c_2+2d_0r_\nu n-4d_0r_\nu-d_0n^2+2d_0n)^2}{(c_0+(n-1)c_2)((W_{r_1}+W_{r_2})(c_0+(n-1)c_2)-nd_0^2)}. \tag{4.28}$$

Since $W_{r_1}=N_{r_1}w_{r_1}$ and $W_{r_2}=(n+1-N_{r_1})w_{r_2}$, (4.25), (4.26), (4.27) and (4.28) imply

$$\frac{1}{w_{r_2}}=\frac{nr_2(n+1-N_{r_1})(n-r_2)w_{r_2}+N_{r_1}r_1(n-r_1)w_{r_1}}{(n+1-N_{r_1})\Bigg(N_{r_1}w_{r_1}r_1(n-r_1)+r_2(n+1-N_{r_1})w_{r_2}(n-r_2)\Bigg)w_{r_2}}. \tag{4.29}$$

Therefore we have

$$w_{r_2}=\frac{N_{r_1}r_1(n-N_{r_1})(n-r_1)}{r_2(N_{r_1}-1)(n+1-N_{r_1})(n-r_2)}w_{r_1}.$$

This completes the proof for (1).
(2): Let $\nu=1$ or 2. Let $x,y\in Y_{r_\nu}$ and $x\neq y$. then $(x,y)\in R_{\alpha_\nu}$, $\alpha_\nu=2,\dots,2r_\nu$. Let $(x,y)\in R_{\alpha_\nu}$. Then take the ordering of $\{u_1,\dots,u_n\}$ so that $\overline{x}=\{1,2,\dots,r_\nu\}$ and $\overline{y}=\{\frac{1}{2}\alpha_\nu+1,\frac{1}{2}\alpha_\nu+2,\cdots,\frac{1}{2}\alpha_\nu+r_\nu\}$. Then for $1\le i\le\frac{1}{2}\alpha_\nu$ we have

$$h_i(x)=Q_1(r_\nu-1)-\frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1), \tag{4.30}$$
$$h_i(y)=Q_1(r_\nu+1)-\frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1), \tag{4.31}$$

If $\frac{1}{2}\alpha_\nu + 1 \le i \le r_\nu$, then

$$h_i(x) = Q_1(r_\nu - 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1), \tag{4.32}$$

$$h_i(y) = Q_1(r_\nu - 1) - \frac{\frac{1}{2}\alpha_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1) - \frac{(i - \frac{1}{2}\alpha_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1). \tag{4.33}$$

If $r_\nu + 1 \le i \le r_\nu + \frac{1}{2}\alpha_\nu$, then

$$\begin{aligned} h_i(x) &= Q_1(r_\nu + 1) - \frac{r_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) - \frac{(i - r_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1), \\ h_i(y) &= Q_1(r_\nu - 1) - \frac{\frac{1}{2}\alpha_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1) - \frac{(i - \frac{1}{2}\alpha_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1), \end{aligned} \tag{4.34}$$

If $r_\nu + \frac{1}{2}\alpha_\nu + 1 \le i \le n$, then

$$h_i(x) = Q_1(r_\nu + 1) - \frac{r_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) - \frac{(i - r_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1), \tag{4.35}$$

$$h_i(y) = Q_1(r_\nu + 1) - \frac{r_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) - \frac{(i - r_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1). \tag{4.36}$$

If $i = n + 1$, then

$$h_{n+1}(x) = h_{n+1}(y) = 1 - \frac{d_0}{c_0 + (n-1)c_2} \Big(r_\nu Q_1(r_\nu - 1) + (n - r_\nu) Q_1(r_\nu + 1) \Big). \tag{4.37}$$

Then

$$\begin{aligned} &\sum_{i=1}^{n+1} \varphi_i(x)\varphi_i(y) = \sum_{i=1}^{n+1} \frac{1}{\|h_i\|^2} h_i(x) h_i(y) \\ &= \sum_{i=1}^{\frac{1}{2}\alpha_\nu} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)} \left(Q_1(r_\nu - 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) \right) \times \\ &\left(Q_1(r_\nu + 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1) \right) \\ &+ \sum_{i=\frac{1}{2}\alpha_\nu + 1}^{r_\nu} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)} \left(Q_1(r_\nu - 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) \right) \times \\ &\left(Q_1(r_\nu - 1) - \frac{\frac{1}{2}\alpha_\nu c_2}{c_0 + (i-2)c_2} Q_1(r_\nu + 1) - \frac{(i - \frac{1}{2}\alpha_\nu - 1)c_2}{c_0 + (i-2)c_2} Q_1(r_\nu - 1) \right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=r_\nu+1}^{r_\nu+\frac{1}{2}\alpha_\nu}\frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)}\times\\
&\left(Q_1(r_\nu+1)-\frac{r_\nu c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)-\frac{(i-r_\nu-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)\right)\times\\
&\left(Q_1(r_\nu-1)-\frac{\frac{1}{2}\alpha_\nu c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)-\frac{(i-\frac{1}{2}\alpha_\nu-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)\right)\\
&+\sum_{i=r_\nu+\frac{1}{2}\alpha_\nu+1}^{n}\frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)}\times\\
&\left(Q_1(r_\nu+1)-\frac{r_\nu c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)-\frac{(i-r_\nu-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)\right)^2\\
&+\frac{c_0+(n-1)c_2}{(W_{r_1}+W_{r_2})(c_0+(n-1)c_2)-nd_0^2}\times\\
&\left\{1-\frac{d_0}{c_0+(n-1)c_2}\Big(r_\nu Q_1(r_\nu-1)+(n-r_\nu)Q_1(r_\nu+1)\Big)\right\}^2. \qquad (4.38)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{\frac{1}{2}\alpha_\nu}\frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)}\left(Q_1(r_\nu-1)-\frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)\right)\times\\
&\left(Q_1(r_\nu+1)-\frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)\right)\\
&=(c_0-c_2)Q_1(r_\nu-1)Q_1(r_\nu+1)\sum_{i=1}^{\frac{1}{2}\alpha_\nu}\frac{1}{(c_0+(i-2)c_2)(c_0+(i-1)c_2)}\\
&=\frac{(c_0-c_2)Q_1(r_\nu-1)Q_1(r_\nu+1)}{c_2}\sum_{i=1}^{\frac{1}{2}\alpha_\nu}\left(\frac{1}{c_0+(i-2)c_2}-\frac{1}{c_0+(i-1)c_2}\right)\\
&=\frac{(c_0-c_2)(n-2r_\nu+2)(n-2r_\nu-2)}{c_2}\left(\frac{1}{c_0-c_2}-\frac{1}{c_0+(\frac{\alpha_\nu}{2}-1)c_2}\right). \qquad (4.39)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=\frac{1}{2}\alpha_\nu+1}^{r_\nu}\frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)}\left(Q_1(r_\nu-1)-\frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)\right)\times\\
&\left(Q_1(r_\nu-1)-\frac{\frac{1}{2}\alpha_\nu c_2}{c_0+(i-2)c_2}Q_1(r_\nu+1)-\frac{(i-\frac{1}{2}\alpha_\nu-1)c_2}{c_0+(i-2)c_2}Q_1(r_\nu-1)\right)\\
&=(n-2(r_\nu-1))((c_0-c_2)(n-2(r_\nu-1))+2\alpha_\nu c_2)\times\\
&\sum_{i=\frac{1}{2}\alpha_\nu+1}^{r_\nu}\frac{1}{(c_0+(i-2)c_2)(c_0+(i-1)c_2)}\\
&=\frac{(n-2(r_\nu-1))((c_0-c_2)(n-2(r_\nu-1))+2\alpha_\nu c_2)}{c_2}\left(\frac{1}{c_0+(\frac{\alpha_\nu}{2}-1)c_2}-\frac{1}{c_0+(r_\nu-1)c_2}\right)
\end{aligned}
$$

$$= \frac{(n-2r_\nu+2)(c_0(n-2r_\nu+2)-c_2(n-2r_\nu+2)+2\alpha_\nu c_2)(2r_\nu-\alpha_\nu)}{(2c_0+(\alpha_\nu-2)c_2)(c_0+(r_\nu-1)c_2)}. \quad (4.40)$$

$$\begin{aligned}
&\sum_{i=r_\nu+1}^{r_\nu+\frac{1}{2}\alpha_\nu} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \times \\
&\left(Q_1(r_\nu+1) - \frac{r_\nu c_2}{c_0+(i-2)c_2} Q_1(r_\nu-1) - \frac{(i-r_\nu-1)c_2}{c_0+(i-2)c_2} Q_1(r_\nu+1)\right) \times \\
&\left(Q_1(r_\nu-1) - \frac{\frac{1}{2}\alpha_\nu c_2}{c_0+(i-2)c_2} Q_1(r_\nu+1) - \frac{(i-\frac{1}{2}\alpha_\nu-1)c_2}{c_0+(i-2)c_2} Q_1(r_\nu-1)\right) \\
&= \frac{(c_0(n-2r_\nu+2)-c_2(n-2r_\nu+2)+2\alpha_\nu c_2)(c_0(n-2r_\nu-2)-c_2(n+2r_\nu-2))}{c_2(c_0-c_2)} \times \\
&\sum_{i=r_\nu+1}^{r_\nu+\frac{1}{2}\alpha_\nu} \left(\frac{1}{c_0+(i-2)c_2} - \frac{1}{c_0+(i-1)c_2}\right) \\
&= \frac{(c_0(n-2r_\nu+2)-c_2(n-2r_\nu+2)+2\alpha_\nu c_2)(c_0(n-2r_\nu-2)-c_2(n+2r_\nu-2))}{c_2(c_0-c_2)} \times \\
&\left(\frac{1}{c_0+(r_\nu-1)c_2} - \frac{1}{c_0+(r_\nu+\frac{\alpha_\nu}{2}-1)c_2}\right). \qquad (4.41)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=r_\nu+\frac{1}{2}\alpha_\nu+1}^{n} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \times \\
&\left(Q_1(r_\nu+1) - \frac{r_\nu c_2}{c_0+(i-2)c_2} Q_1(r_\nu-1) - \frac{(i-r_\nu-1)c_2}{c_0+(i-2)c_2} Q_1(r_\nu+1)\right)^2 \\
&= \frac{(c_0(n-2r_\nu-2)-c_2(n+2r_\nu-2))^2}{(c_0-c_2)c_2} \left(\frac{1}{c_0+(i-2)c_2} - \frac{1}{c_0+(i-1)c_2}\right) \\
&= \frac{(c_0(n-2r_\nu-2)-c_2(n+2r_\nu-2))^2}{(c_0-c_2)c_2} \left(\frac{1}{c_0+(r_\nu+\frac{1}{2}\alpha_\nu-1)c_2} - \frac{1}{c_0+(n-1)c_2}\right). \\
&\qquad (4.42)
\end{aligned}$$

(4.38), (4.39), (4.40), (4.41), and (4.42) imply

$$\begin{aligned}
&-\frac{8\alpha_\nu}{c_0-c_2} + \frac{c_0(4(n-4)r_\nu^2-4n(n-4)r_\nu+n(n-2)^2)-nc_2(4r_\nu^2-4nr_\nu+(n-2)^2)}{(c_0+(n-1)c_2)(c_0-c_2)} \\
&+\frac{c_0+(n-1)c_2}{(W_{r_1}+W_{r_2})(c_0+(n-1)c_2)-nd_0^2}\left\{1-\frac{(n-2)(n-2r_\nu)d_0}{c_0+(n-1)c_2}\right\}^2 = 0. \qquad (4.43)
\end{aligned}$$

Then using the formula in Proposition 4.1, we have

$$\frac{2nW_{r_1}r_1(n-r_1)+2W_{r_2}r_2(n-r_2)-n\alpha_\nu W_{r_\nu}(n-1)}{2W_{r_\nu}(W_{r_2}r_2(n-r_2)+W_{r_1}r_1(n-r_1))} = 0.$$

This implies

$$\alpha_1 = \frac{2(n(n-r_1)r_1W_{r_1} + (n-r_2)r_2W_{r_2})}{n(n-1)W_{r_1}}$$

and

$$\alpha_2 = \frac{2(n(n-r_2)r_2W_{r_2} + (n-r_1)r_1W_{r_1})}{n(n-1)W_{r_2}}.$$

Then substitute $W_{r_1} = N_{r_1}w_{r_1}$, $W_{r_2} = (n+1-N_{r_1})w_{r_2}$, where w_{r_2} is given in (1), we obtain (2).

(3): Let $x \in Y_{r_1}$ and $y \in Y_{r_2}$. Then $(x,y) \in R_{r_2-r_1+2a}$ with an integer a satisfying $0 \leq a \leq r_1$. Choose the ordering of the elements in X_1 so that $\overline{x} = \{1,2,\ldots,r_1\}$ and $\overline{y} = \{a+1, a+2, \ldots, a+r_2\}$. hold.

Then for $1 \leq i \leq a$ we have

$$h_i(x) = Q_1(r_1-1) - \frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_1-1), \tag{4.44}$$

$$h_i(y) = Q_1(r_2+1) - \frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_2+1). \tag{4.45}$$

If $a+1 \leq i \leq r_1$, then

$$h_i(x) = Q_1(r_1-1) - \frac{(i-1)c_2}{c_0+(i-2)c_2}Q_1(r_1-1), \tag{4.46}$$

$$h_i(y) = Q_1(r_2-1) - \frac{ac_2}{c_0+(i-2)c_2}Q_1(r_2+1) - \frac{(i-a-1)c_2}{c_0+(i-2)c_2}Q_1(r_2-1). \tag{4.47}$$

If $r_1+1 \leq i \leq r_2+a$, then

$$h_i(x) = Q_1(r_1+1) - \frac{r_1c_2}{c_0+(i-2)c_2}Q_1(r_1-1) - \frac{(i-r_1-1)c_2}{c_0+(i-2)c_2}Q_1(r_1+1), \tag{4.48}$$

$$h_i(y) = Q_1(r_2-1) - \frac{ac_2}{c_0+(i-2)c_2}Q_1(r_2+1) - \frac{(i-a-1)c_2}{c_0+(i-2)c_2}Q_1(r_2-1). \tag{4.49}$$

If $r_2+a \leq i \leq n$, then

$$h_i(x) = Q_1(r_1+1) - \frac{r_1c_2}{c_0+(i-2)c_2}Q_1(r_1-1) - \frac{(i-r_1-1)c_2}{c_0+(i-2)c_2}Q_1(r_1+1), \tag{4.50}$$

$$h_i(y) = Q_1(r_2+1) - \frac{r_2c_2}{c_0+(i-2)c_2}Q_1(r_2-1) - \frac{(i-r_2-1)c_2}{c_0+(i-2)c_2}Q_1(r_2+1). \tag{4.51}$$

If $i = n+1$, then

$$h_{n+1}(x) = 1 - \frac{d_0}{c_0 + (n-1)c_2}\Big(r_1 Q_1(r_1 - 1) + (n - r_1)Q_1(r_1 + 1)\Big), \tag{4.52}$$

$$h_{n+1}(y) = 1 - \frac{d_0}{c_0 + (n-1)c_2}\Big(r_2 Q_1(r_2 - 1) + (n - r_2)Q_1(r_2 + 1)\Big). \tag{4.53}$$

Hence we have

$$\begin{aligned}
&\sum_{i=1}^{n+1} \varphi_i(x)\varphi_i(y) \\
&= \sum_{i=1}^{a} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)}\Big(Q_1(r_1 - 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2}Q_1(r_1 - 1)\Big) \times \\
&\Big(Q_1(r_2 + 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2}Q_1(r_2 + 1)\Big) \\
&+ \sum_{i=a+1}^{r_1} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)}\Big(Q_1(r_1 - 1) - \frac{(i-1)c_2}{c_0 + (i-2)c_2}Q_1(r_1 - 1)\Big) \times \\
&\Big(Q_1(r_2 - 1) - \frac{ac_2}{c_0 + (i-2)c_2}Q_1(r_2 + 1) - \frac{(i-a-1)c_2}{c_0 + (i-2)c_2}Q_1(r_2 - 1)\Big) \\
&+ \sum_{i=r_1+1}^{r_2+a} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)} \times \\
&\Big(Q_1(r_1 + 1) - \frac{r_1 c_2}{c_0 + (i-2)c_2}Q_1(r_1 - 1) - \frac{(i-r_1-1)c_2}{c_0 + (i-2)c_2}Q_1(r_1 + 1)\Big) \times \\
&\Big(Q_1(r_2 - 1) - \frac{ac_2}{c_0 + (i-2)c_2}Q_1(r_2 + 1) - \frac{(i-a-1)c_2}{c_0 + (i-2)c_2}Q_1(r_2 - 1)\Big) \\
&+ \sum_{i=r_2+a+1}^{n} \frac{c_0 + (i-2)c_2}{(c_0 - c_2)(c_0 + (i-1)c_2)} \times \\
&\Big(Q_1(r_1 + 1) - \frac{r_1 c_2}{c_0 + (i-2)c_2}Q_1(r_1 - 1) - \frac{(i-r_1-1)c_2}{c_0 + (i-2)c_2}Q_1(r_1 + 1)\Big) \times \\
&\Big(Q_1(r_2 + 1) - \frac{r_2 c_2}{c_0 + (i-2)c_2}Q_1(r_2 - 1) - \frac{(i-r_2-1)c_2}{c_0 + (i-2)c_2}Q_1(r_2 + 1)\Big) \\
&+ \frac{c_0 + (n-1)c_2}{(W_{r_1} + W_{r_2})(c_0 + (n-1)c_2) - nd_0^2} \times \\
&\Big\{1 - \frac{d_0}{c_0 + (n-1)c_2}\Big(r_1 Q_1(r_1 - 1) + (n - r_1)Q_1(r_1 + 1)\Big)\Big\} \times \\
&\Big\{1 - \frac{d_0}{c_0 + (n-1)c_2}\Big(r_2 Q_1(r_2 - 1) + (n - r_2)Q_1(r_2 + 1)\Big)\Big\}.
\end{aligned} \tag{4.54}$$

$$
\begin{aligned}
&\sum_{i=1}^{a} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \left(Q_1(r_1-1) - \frac{(i-1)c_2}{c_0+(i-2)c_2} Q_1(r_1-1) \right) \times \\
&\left(Q_1(r_2+1) - \frac{(i-1)c_2}{c_0+(i-2)c_2} Q_1(r_2+1) \right) \\
&= (c_0-c_2)(n-2(r_1-1))(n-2(r_2+1)) \sum_{i=1}^{a} \frac{1}{(c_0+(i-2)c_2)(c_0+(i-1)c_2)} \\
&= \frac{1}{c_2}(c_0-c_2)(n-2(r_1-1))(n-2(r_2+1)) \left(\frac{1}{c_0-c_2} - \frac{1}{c_0+(a-1)c_2} \right) \\
&= \frac{(n-2(r_1-1))(n-2(r_2+1))a}{(c_0-c_2)+ac_2}. \qquad (4.55)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=a+1}^{r_1} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \left(Q_1(r_1-1) - \frac{(i-1)c_2}{c_0+(i-2)c_2} Q_1(r_1-1) \right) \times \\
&\left(Q_1(r_2-1) - \frac{ac_2}{c_0+(i-2)c_2} Q_1(r_2+1) - \frac{(i-a-1)c_2}{c_0+(i-2)c_2} Q_1(r_2-1) \right) \\
&= \frac{1}{c_2}(n-2r_1+2)((n-2r_2+2)(c_0-c_2)+4c_2a) \sum_{i=a+1}^{r_1} \left(\frac{1}{c_0+(i-2)c_2} - \frac{1}{c_0+(i-1)c_2)} \right) \\
&= \frac{1}{c_2}(n-2r_1+2)((n-2r_2+2)(c_0-c_2)+4c_2a) \left(\frac{1}{c_0+(a-1)c_2} - \frac{1}{c_0+(r_1-1)c_2} \right) \\
&= \frac{(n-2r_1+2)((c_0-c_2)(n-2r_2+2)+4c_2a)(r_1-a)}{(c_0+(a-1)c_2)(c_0+(r_1-1)c_2)}. \qquad (4.56)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=r_1+1}^{r_2+a} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \times \\
&\left(Q_1(r_1+1) - \frac{r_1c_2}{c_0+(i-2)c_2} Q_1(r_1-1) - \frac{(i-r_1-1)c_2}{c_0+(i-2)c_2} Q_1(r_1+1) \right) \times \\
&\left(Q_1(r_2-1) - \frac{ac_2}{c_0+(i-2)c_2} Q_1(r_2+1) - \frac{(i-a-1)c_2}{c_0+(i-2)c_2} Q_1(r_2-1) \right) \\
&= \frac{((c_0-c_2)(n-2r_2+2)+4c_2a)((c_0-c_2)(n-2)-2r_1(c_0+c_2))}{c_2(c_0-c_2)} \times \\
&\sum_{i=r_1+1}^{r_2+a} \left(\frac{1}{c_0+(i-2)c_2} - \frac{1}{c_0+(i-1)c_2} \right) \\
&= \frac{(r_2-r_1+a)\big((c_0-c_2)(n-2r_2+2)+4c_2a\big)\big((n-2)(c_0-c_2)-2r_1(c_0+c_2)\big)}{(c_0-c_2)(c_0+(r_1-1)c_2)(c_0+(r_2-1)c_2+c_2a)}. \qquad (4.57)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=r_2+a+1}^{n} \frac{c_0+(i-2)c_2}{(c_0-c_2)(c_0+(i-1)c_2)} \times \\
&\left(Q_1(r_1+1)-\frac{r_1c_2}{c_0+(i-2)c_2}Q_1(r_1-1)-\frac{(i-r_1-1)c_2}{c_0+(i-2)c_2}Q_1(r_1+1)\right) \times \\
&\left(Q_1(r_2+1)-\frac{r_2c_2}{c_0+(i-2)c_2}Q_1(r_2-1)-\frac{(i-r_2-1)c_2}{c_0+(i-2)c_2}Q_1(r_2+1)\right) \\
&=\frac{((n-2)(c_0-c_2)-2r_2(c_0+c_2))((n-2)(c_0-c_2)-2r_1(c_0+c_2))}{(c_0-c_2)c_2} \times \\
&\sum_{i=r_2+a+1}^{n}\left(\frac{1}{c_0+(i-2)c_2}-\frac{1}{c_0+(i-1)c_2}\right) \\
&=\frac{\big((n-r_2)-a\big)\big((n-2)(c_0-c_2)-2r_2(c_0+c_2)\big)\big((n-2)(c_0-c_2)-2r_1(c_0+c_2)\big)}{(c_0-c_2)\big(c_0+c_2(r_2-1)+c_2a\big)\big(c_0+(n-1)c_2\big)}.
\end{aligned} \tag{4.58}
$$

Then (4.54), (4.55), (4.56), (4.57) and (4.58) implies

$$
\begin{aligned}
&\frac{1}{(c_2-c_0)((n-1)c_2+c_0)}\Big\{16((n-1)c_2+c_0)a-2r_1\Big((n^2+4n-4)c_2 \\
&-(n^2-4n-4)c_0\Big)-4r_1r_2\Big((n-4)c_0-nc_2\Big)+(n-2)^2(n-2r_2)(c_2-c_0)\Big\} \\
&+\frac{\Big((n-2)(n-2r_1)d_0-c_0-(n-1)c_2\Big)\Big((n-2)(n-2r_2)d_0-c_0-(n-1)c_2\Big)}{\Big((n-1)c_2+c_0\Big)\Big((W_{r_1}+W_{r_2})(c_0+(n-1)c_2)-nd_0^2\Big)} \\
&=0.
\end{aligned} \tag{4.59}
$$

Then we have

$$
\frac{(n-1)((n-r_2)r_1-na)}{r_1(n-r_1)W_{r_1}+r_2(n-r_2)W_{r_2}}=0,
$$

and $a=\frac{(n-r_2)r_1}{n}$. This implies (3). ∎

4.3 Regular semi-lattices and Geometric relative t-designs

There is one more important property satisfied by geometric relative t-designs of association schemes attached to regular semi-lattices. In [16], Delsarte proved that if P-polynomial association scheme has the property of regular semi-lattice and also satisfies the Q-polynomial property, then (Y, w) is a relative t-design with respect to a point u_0 if and only if (Y, w) is a geometric relative t-design with respect to the regular semi-lattice. Let $\mathfrak{X}=(X,\{R_i\}_{0\le i\le n})$ be the P-polynomial scheme associated with a regular semi-lattice Λ. Let h be the hight function of Λ with $0\le h(x)\le n$. Let $\Lambda_j=\{x\in\Lambda \mid h(x)=j\}$ for $j=0,1,\ldots,n$. Then X is the top fiber $\Lambda_n=\{x\in\Lambda\mid h(x)=n\}$ of Λ. Let $\chi\in\mathcal{F}(X)$.

Assume $\chi(x) \geq 0$ for any $x \in X$. Let $j \in \{1, 2, \ldots, n\}$ and we define the following function $\lambda_{j,\chi}$ on Λ_j by

$$\lambda_{j,\chi}(z) = \sum_{\substack{x \in \Lambda_n \\ x \geq z}} \chi(x), \ \ z \in \Lambda_j. \tag{4.60}$$

If the following condition satisfied, then χ is called geometric relative t-design with respect to a point $u_0 \in X$. For any integer satisfying $0 \leq j \leq t$, there exists a constant $\lambda_{u_0,j}$ and

$$\lambda_{t,\chi}(z) = \lambda_{u_0,j} \tag{4.61}$$

holds for any $z \in \Lambda_t$ satisfying $h(z \wedge u_0) = j$. Now we consider the semi-lattice structure which gives $H(n, 2)$. Let $\Lambda = \{(x_1, \ldots, x_n) \mid x_i \in \{0, 1\} \text{ or } x_i = \cdot\}$. For $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \Lambda$, we deine $x \leq y$ if $x_i = y_i$ or $x_i = \cdot$ for $1 \leq i \leq n$. We defined $h(x) = |\{i \mid x_i \in \{0, 1\}\}|$. Then Λ is a regular semi-lattice with the hight function h. Clearly the top fiber is $\Lambda_n = F^n$ and Λ_n gives association scheme $H(n, 2)$. Now we consider the geometric relative t-design with respect to $u_0 = (0, 0, \ldots, 0) \in X$. Let $z = (z_1, \ldots, z_n) \in \Lambda_t,\ h(z \wedge u_0) = t - j$. Then $|\{i \mid z_i \in \{0, 1\}\}| = t$, $|\{i \mid z_i = 0\}| = t - j$, $|\{i \mid z_i = 1\}| = j$. Then for $z \in \Lambda_t,\ h(z \wedge u_0) = t - j$ we have

$$\lambda_{u_0,t-j} = \lambda_{t,\chi}(z) = \sum_{\substack{x \in \Lambda_n \\ x \geq z}} \chi(x). \tag{4.62}$$

For any $u \in X_j$, we define $\overline{u} = \{i \mid u_i = 1\} = \{i_1, \ldots, i_j\}$. Then there exists $z = (z_1, \ldots, z_n) \in \Lambda_t$ satisfying $z_{i_1} = z_{i_2} = \cdots = z_{i_j} = 1$ and $|\{i \mid z_i = 0\}| = t - j$. Then $x \in \Lambda_n$ satisfies $x \geq z$ if and only if $\overline{u} \subset \overline{x}$. Let $Y = \{y \in X \mid \chi(y) > 0\}$ and $w(y) = \chi(y)$ for $y \in Y$, then (Y, w) is a relative t-design in the style of Definition 1.1 (see more information in [15, 16, 6]). The argument given above implies the following proposition.

Proposition 4.4 *Let (Y, w) be a relative t-design in $H(n, 2)$ with respect to $u_0 = (0, 0, \ldots, 0)$. Then for any $u \in X_j$*

$$\sum_{y \in Y, \overline{u} \subset \overline{y}} w(y) = \lambda_j$$

is a constant depends only on j $(0 \leq j \leq t)$. Here $\overline{x} = \{i \mid x_i = 1, 1 \leq i \leq n\}$ for $x = (x_1, \ldots, x_n) \in X$.

Please refer [15] for more information on regular semi-lattices.

4.4 Proof of Theorem 2.2

Proposition 4.3 (2) and (3) imply Theorem 2.2 (1).

List of possible parameters for $n \leq 30$

We first determined the parameter set $\{n, r_1, r_2, N_{r_1}, N_{r_2}, \alpha_1, \alpha_2, \gamma, \frac{w_{r_2}}{w_{r_1}}\}$ according to the formula given in Proposition 4.3. If (Y, w) is a relative t-design with respect to u_0, then $(Y, \mu w)$, $(\mu w)(y) = \mu w(y)$ for $y \in Y$, is also a relative t-design with respect to u_0 for any positive real number μ. Therefore in the following argument we assume $w_{r_1} = 1$. We apply Proposition 4.4 and determine λ_1 and λ_2. For this purpose we count the elements in the set $\{(x, y) \mid x \in X_i, y \in Y\}$ for $i = 1, 2$. Then we have

$$w_{r_1}\binom{r_1}{i}N_{r_1} + w_{r_2}\binom{r_2}{i}N_{r_2} = \binom{n}{i}\lambda_i, \text{ for } i = 1, 2. \tag{4.63}$$

We note that if $w_{r_1} = w_{r_2} = 1$, then $\lambda_i = |\{y \in Y \mid \overline{u} \subset \overline{y}\}|$ for any $u \in X_i$ for $i = 1, 2$ and λ_1, λ_2 must be integers. We list the feasible parameters $n, r_1, r_2, N_{r_1}, N_{r_2}, \alpha_1, \alpha_2, \gamma, w = w_{r_2}$ $(w_{r_1} = 1), \lambda_1, \lambda_2$ for a tight relative 2-designs with respect to a point u_0 for $6 \le n \le 30$ below.

n	r_1	r_2	N_{r_1}	N_{r_2}	α_1	α_2	γ	w	λ_1	λ_2	
6(1)	2	3	3	4	4	4	3	1	3	1	○
6(2)	3	4	4	3	4	4	3	1	4	2	○
10(1)	2	5	5	6	4	6	5	$\frac{2}{3}$	3	1	×
10(2)	4	5	5	6	6	6	5	1	5	2	○
10(3)	5	6	6	5	6	6	5	1	6	3	○
10(4)	5	8	6	5	6	4	5	$\frac{3}{2}$	9	6	×
12(1)	3	4	4	9	6	6	5	1	4	1	○
12(2)	3	8	4	9	6	6	7	1	7	4	○
12(3)	4	6	9	4	6	8	6	$\frac{3}{4}$	$\frac{9}{2}$	$\frac{3}{2}$	×
12(4)	4	9	9	4	6	6	7	1	6	3	○
12(5)	6	8	4	9	8	6	6	$\frac{4}{3}$	10	6	×
12(6)	8	9	9	4	6	6	5	1	9	6	○
14(1)	2	7	7	8	4	8	7	$\frac{1}{2}$	3	1	○
14(2)	6	7	7	8	8	8	7	1	7	3	○
14(3)	7	8	8	7	8	8	7	1	8	4	○
14(4)	7	12	8	7	8	4	7	2	16	12	○
15(1)	5	6	6	10	8	8	7	1	6	2	○
15(2)	5	9	6	10	8	8	8	1	8	4	○
15(3)	6	10	10	6	8	8	8	1	8	4	○
15(4)	9	10	10	6	8	8	7	1	10	6	○
18(1)	2	9	9	10	4	10	9	$\frac{2}{5}$	3	1	×
18(2)	8	9	9	10	10	10	9	1	9	4	○
18(3)	9	10	10	9	10	10	9	1	10	5	○
18(4)	9	16	10	9	10	4	9	$\frac{5}{2}$	25	20	×
20(1)	4	5	5	16	8	8	7	1	5	1	○
20(2)	4	15	5	16	8	8	13	1	13	9	○
20(3)	5	8	16	5	8	12	9	$\frac{2}{3}$	$\frac{16}{3}$	$\frac{4}{3}$	×
20(4)	5	12	16	5	8	12	11	$\frac{2}{3}$	6	2	×
20(5)	5	16	16	5	8	8	13	1	8	4	○
20(6)	8	15	5	16	12	8	11	$\frac{3}{2}$	20	14	×
20(7)	12	15	5	16	12	8	9	$\frac{3}{2}$	21	15	×
20(8)	15	16	16	5	8	8	7	1	16	12	○

n	r_1	r_2	N_{r_1}	N_{r_2}	α_1	α_2	γ	w	λ_1	λ_2	
21(1)	3	7	7	15	6	10	8	$\frac{3}{5}$	4	1	×
21(2)	3	14	7	15	6	10	13	$\frac{3}{5}$	7	4	×
21(3)	6	7	7	15	10	10	9	1	7	2	○
21(4)	6	14	7	15	10	10	12	1	12	7	○
21(5)	7	9	15	7	10	12	10	$\frac{5}{6}$	$\frac{15}{2}$	$\frac{5}{2}$	×
21(6)	7	12	15	7	10	12	11	$\frac{5}{6}$	$\frac{25}{3}$	$\frac{10}{3}$	×
21(7)	7	15	15	7	10	10	12	1	10	5	○
21(8)	7	18	15	7	10	6	13	$\frac{5}{3}$	15	10	×
21(9)	9	14	7	15	12	10	11	$\frac{6}{5}$	15	9	×
21(10)	12	14	7	15	12	10	10	$\frac{6}{5}$	16	10	×
21(11)	14	15	15	7	10	10	9	1	15	10	○
21(12)	14	18	15	7	10	6	8	$\frac{5}{3}$	20	15	×
22(1)	2	11	11	12	4	12	11	$\frac{1}{3}$	3	1	○
22(2)	10	11	11	12	12	12	11	1	11	5	○
22(3)	11	12	12	11	12	12	11	1	12	6	○
22(4)	11	20	12	11	12	4	11	3	36	30	○
24(1)	8	9	9	16	12	12	11	1	9	3	○
24(2)	8	15	9	16	12	12	13	1	13	7	○
24(3)	9	16	16	9	12	12	13	1	12	6	○
24(4)	15	16	16	9	12	12	11	1	16	10	○
26(1)	2	13	13	14	4	14	13	$\frac{2}{7}$	3	1	×
26(2)	6	13	13	14	10	14	13	$\frac{5}{7}$	8	3	×
26(3)	8	13	13	14	12	14	13	$\frac{6}{7}$	10	4	×
26(4)	12	13	13	14	14	14	13	1	13	6	○
26(5)	13	14	14	13	14	14	13	1	14	7	○
26(6)	13	18	14	13	14	12	13	$\frac{7}{6}$	$\frac{35}{2}$	$\frac{21}{2}$	×
26(7)	13	20	14	13	14	10	13	$\frac{7}{5}$	21	14	×
26(8)	13	24	14	13	14	4	13	$\frac{7}{2}$	49	42	×
27(1)	9	15	7	21	14	14	14	1	14	7	×
27(2)	12	18	21	7	14	14	14	1	14	7	×
28(1)	7	8	8	21	12	12	11	1	8	2	○
28(2)	7	20	8	21	12	12	17	1	17	11	○
28(3)	8	14	21	8	12	16	14	$\frac{3}{4}$	9	3	×
28(4)	8	21	21	8	12	12	17	1	12	6	○
28(5)	14	20	8	21	16	12	14	$\frac{4}{3}$	24	16	×
28(6)	20	21	21	8	12	12	11	1	21	15	○
30(1)	2	15	15	16	4	16	15	$\frac{1}{4}$	3	1	○
30(2)	3	10	10	21	6	14	11	$\frac{3}{7}$	4	1	×
30(3)	3	20	10	21	6	14	19	$\frac{3}{7}$	7	4	×
30(4)	5	6	6	25	10	10	9	1	6	1	○
30(5)	5	24	6	25	10	10	21	1	21	16	○
30(6)	6	10	25	6	10	16	12	$\frac{5}{8}$	$\frac{25}{4}$	$\frac{5}{4}$	×
30(7)	6	15	25	6	10	18	15	$\frac{5}{9}$	$\frac{20}{3}$	$\frac{5}{3}$	×
30(8)	6	20	25	6	10	16	18	$\frac{5}{8}$	$\frac{15}{2}$	$\frac{5}{2}$	×

n	r_1	r_2	N_{r_1}	N_{r_2}	α_1	α_2	γ	w	λ_1	λ_2	
30(9)	6	25	25	6	10	10	21	1	10	5	○
30(10)	9	10	10	21	14	14	13	1	10	3	○
30(11)	9	20	10	21	14	14	17	1	17	10	○
30(12)	10	12	21	10	14	16	14	$\frac{7}{8}$	$\frac{21}{2}$	$\frac{7}{2}$	×
30(13)	10	18	21	10	14	16	16	$\frac{7}{8}$	$\frac{49}{4}$	$\frac{21}{4}$	×
30(14)	10	21	21	10	14	14	17	1	14	7	○
30(15)	10	24	6	25	16	10	18	$\frac{8}{5}$	34	26	×
30(16)	10	27	21	10	14	6	19	$\frac{7}{3}$	28	21	×
30(17)	12	20	10	21	16	14	16	$\frac{8}{7}$	20	12	×
30(18)	14	15	15	16	16	16	15	1	15	7	○
30(19)	15	16	16	15	16	16	15	1	16	8	○
30(20)	15	24	6	25	18	10	15	$\frac{9}{5}$	39	30	×
30(21)	15	28	16	15	16	4	15	4	64	56	○
30(22)	18	20	10	21	16	14	14	$\frac{8}{7}$	22	14	×
30(23)	20	21	21	10	14	14	13	1	21	14	○
30(24)	20	24	6	25	16	10	12	$\frac{8}{5}$	36	28	×
30(25)	20	27	21	10	14	6	11	$\frac{7}{3}$	35	28	×
30(26)	24	25	25	6	10	10	9	1	25	20	○

The last column in the table given above, " ○ " indicates existence, " × " indicates non-existence of the tight relative 2-design with the corresponding parameters. For the cases with " ○ ", complete classification problem is still open.

Constructions

First we give two kind of construction theorem. First one is the construction by Hadamard matrices.

Let $m \equiv -1 \pmod 4$, and $n = 2m$. Suppose there is an Hadamard matrix H_{m+1} of size $(m+1)\times(m+1)$. Let $h_1, h_2, \ldots, h_{m+1}$ be the row vectors of H_{m+1}. We may assume that each vector h_j is of the following form by normalization, i.e., $h_j = (+, a_{j,1}, a_{j,2}, \ldots, a_{j,m})$ with $a_{j,\nu} \in \{+,-\},\ 1 \le \nu \le m$. First we define $Y_2 \subset X_2$ in the following way.

$$\begin{aligned}Y_2 = \{&(y_{i,1}, y_{i,2}, \ldots, y_{i,2\nu-1}, y_{i,2\nu}, \ldots, y_{i,2m-1}, y_{i,2m}) \in X_2 \mid \\ &1 \le i \le m,\ y_{i,2i-1} = y_{i,2i} = 1, y_{i,\nu} = 0, \nu \neq 2i-1,\ 2i\}.\end{aligned}$$

Then $|Y_2| = m = \frac{n}{2}$. Next we define Y_m in X_m using $m+1$ row vectors $h_1, \ldots, h_{m+1}$ of H_{m+1}. For each h_j $(1 \le j \le m+1)$, we define

$$y(h_j) = (y_{j,1}, y_{j,2}, \ldots, y_{j,2\nu+1}, y_{j,2\nu+2}, \ldots, y_{j,2m-1}, y_{j,2m}) \in X_m$$

as follws: the $(2\nu-1)$-th and 2ν-th entries $y_{j,2\nu-1}, y_{j,2\nu}$ of $y(h_j)$ are given by

$$(y_{j,2\nu-1}, y_{j,2\nu}) = \begin{cases} (1,0) & \text{if } a_{j,\nu} = + \\ (0,1) & \text{if } a_{j,\nu} = - \end{cases}$$

for $\nu = 1, \ldots, m$. Let $Y_m = \{y(h_1), \ldots, y(h_{m+1})\}$. Thens we have $|Y_m| = m+1$ and $Y = Y_2 \cup Y_m$ satisfies the conditions of relative 2-design with respect to $u_0 = (0,0,\ldots,0)$ in

$H(n,2)$. We can also easily check that the set $Y' = \{(1,1,\dots,1) - y \mid y \in Y\}$ which is the complement of Y also is a relative 2-design with respect to u_0 with $w'(y') = \frac{1}{w((1,1,\dots,1)-y')}$ for $y' \in Y'$. This completes the proof of Theorem 2.2 (3).

Next one is the constructions from symmetric designs. The following proposition is known.

Proposition 4.5 (Woodall [25]) *Let $(V,\mathcal{B})$ be a symmetric design 2-$(n+1,k,\lambda)$ design. Let the point set $V = \{0,1,2,\dots,n\}$.*

(1) *Let $Y_{k-1} = \{y \in F^n \mid \overline{y} = B\backslash\{0\}, B \in \mathcal{B}, 0 \in B\}$ and $Y_k = \{y \in F^n \mid \overline{y} = B, B \in \mathcal{B}, 0 \notin B\}$. Then $Y = Y_{k-1} \cup Y_k$ is a tight relative 2-design of $H(n,2)$ with respect to $u_0 = (0,0,\dots,0)$.*

(2) *Let $2k \neq n+1$. Let $Y_{n-k+1} = \{y \in F^n \mid \overline{y} = V\backslash B, B \in \mathcal{B}, 0 \in B\}$ and $Y_k = \{y \in F^n \mid \overline{y} = B, B \in \mathcal{B}, 0 \notin B\}$. Then $Y = Y_{n-k+1} \cup Y_k$ is a tight relative 2-design withrespect to u_0.*

It is known that the complement of a symmetric design is also a symmetric design. Therefore using Proposition 4.5, we can construct tight relative 2-design of $H(n,2)$ with respect to $u_0 = (0,0,\dots,0)$ for each set of parameters in the table satisfying $w = 1$ except for $n = 27$.

Remark 4.6 (1) *The tight relative 2-designs with the parameters 14(1), 14(4), 22(1), 22(4), 30(1) and 30(21) are constructed by using Hadamard matrices according to the method given above (Theorem 2.2 (3)).*

(2) *For $n = 27$ the parameters do not correspond to symmetric 2-$(n+1,k,\lambda)$ designs.*

Non-existence

In the following we prove that for each set of parameters with " $\times$ " in the last column, tight relative 2-design does not exist.

Proposition 4.7 (1) *Let $u \in X_1$ and $\lambda_1^{(i)}(u) = |\{y \in Y_{r_i} \mid \overline{u} \subset \overline{y}\}|$ for $i = 1,2$. Then $\lambda_1^{(i)}(u)$ does not depend on the choice of $u \in X_1$ and given by the following formulas.*

(a)

$$\lambda_1^{(1)} = \lambda_1^{(1)}(u) = \frac{(r_2-1)\lambda_1 - (n-1)\lambda_2}{(r_2-r_1)w_1}, \tag{4.64}$$

$$\lambda_1^{(2)} = \lambda_1^{(2)}(u) = \frac{(n-1)\lambda_2 - (r_1-1)\lambda_1}{(r_2-r_1)w_2}. \tag{4.65}$$

(b) The following holds.

$$\sum_{i=1}^{2} \frac{N_{r_i} w_{r_i}}{|X_{r_i}|}\left(\binom{n-1}{r_i-1} Q_1(r_i-1) + \binom{n-1}{r_i} Q_1(r_i+1)\right)$$
$$= \sum_{i=1}^{2} w_{r_i}\left(\lambda_1^{(i)} Q_1(r_i-1) + (N_{r_i} - \lambda_1^{(i)}) Q_1(r_i+1)\right). \tag{4.66}$$

(2) *Let* $u \in X_2$*, and let* $\lambda_2^{(i)}(u) = |\{y \in Y_{r_i} \mid \overline{u} \subset \overline{y}\}$ *and* $\lambda_{2,C}^{(i)}(u) = |\{y \in Y_{r_i} \mid \overline{u} \cap \overline{y} = \emptyset\}$ *for* $i = 1, 2$*. Then the following holds.*

$$\sum_{i=1}^{2} \frac{N_{r_i} w_{r_i}}{|X_{r_i}|} \left(\binom{n-2}{r_i-2} Q_2(r_i-2) + \binom{n-2}{r_i} Q_2(r_i+2) + 2\binom{n-2,}{r_i-1} Q_2(r_i) \right)$$
$$= \sum_{i=1}^{2} w_{r_i} \Big(\lambda_2^{(i)}(u) Q_2(r_i-2) + \lambda_{2,C}^{(i)}(u) Q_2(r_i+2)$$
$$+ \big(N_{r_i} - \lambda_2^{(i)}(u) - \lambda_{2,C}^{(i)}(u)\big) Q_2(r_i) \Big). \tag{4.67}$$

Proof (1) (a) Let $u \in X_1$ be fixed arbitrarily and consider the following sum.

$$\sum_{\substack{y \in Y,\ \overline{u} \subset \overline{y} \\ \{\overline{x}, \overline{u}\} \subset \overline{y},\ x \in X_1,\ x \neq u}} w(y) = \sum_{\substack{x \in X_1, \\ x \neq u}} \sum_{\substack{y \in Y, \\ \{\overline{x}, \overline{u}\} \subset \overline{y}}} w(y) = \sum_{\substack{x \in X_1, \\ x \neq u}} \lambda_2 = (n-1)\lambda_2. \tag{4.68}$$

The left side of (4.68) has the following reformation.

$$\sum_{\substack{y \in Y,\ \overline{u} \subset \overline{y} \\ \{\overline{x}, \overline{u}\} \subset \overline{y},\ x \in X_1,\ x \neq u}} w(y) = \sum_{i=1}^{2} \sum_{\substack{y \in Y_{r_i}, \\ \overline{u} \subset \overline{y}}} \sum_{\substack{x \in X_1, x \neq u, \\ \overline{x} \subset \overline{y}}} w(y) = \sum_{i=1}^{2} (r_i - 1) w_{r_i} |\{y \in Y_{r_i} \mid \overline{u} \subset \overline{y}\}|$$
$$= \sum_{i=1}^{2} (r_i - 1) w_{r_i} \lambda_1^{(i)}(u). \tag{4.69}$$

Therefore we must have

$$\sum_{i=1}^{2} (r_i - 1) w_{r_i} \lambda_1^{(i)}(u) = (n-1)\lambda_2. \tag{4.70}$$

On the other hand Proposition 4.4 implies

$$\lambda_1^{(1)}(u) w_{r_1} + \lambda_1^{(2)}(u) w_{r_2} = \lambda_1. \tag{4.71}$$

Since the coefficient matrix of equations (4.70) and (4.71) with variable $\lambda_1^{(1)}(u), \lambda_1^{(2)}(u)$ is nonsingular $\lambda_1^{(1)}(u), \lambda_1^{(2)}(u)$ are determined by the formulas in (1) (a).
(1) (b) and (2): The equation (1.4) for $\phi_u^{(1)}$, $u \in X_1$ and $\phi_u^{(2)}$, $u \in X_2$ imply equation (4.66) and (4.67) respectively. ∎

Proposition 2.2 (2) in [21] implies that if (4.66) and (4.67) are satisfied for each $u \in X_1$ and $u \in X_2$ respectively, then $(Y_{r_1} \cup Y_{r_2}, w)$ is a relative 2-design. On the other hand Proposition 4.4 implies

$$\lambda_2^{(1)}(u) w_{r_1} + \lambda_2^{(2)}(u) w_{r_2} = \lambda_2 \tag{4.72}$$

for any $u \in X_2$. In the following we will show that for each case marked " $\times$ ", there is no set of integers $\{\lambda_1^{(1)}, \lambda_1^{(2)}\} \cup \{\lambda_2^{(1)}(u), \lambda_{2,C}^{(1)}(u), \lambda_2^{(2)}(u), \lambda_{2,C}^{(2)}(u) \mid u \in X_2\}$ satisfying (4.66), (4.67) and (4.72).

$n = 10$: non-existence for 10(1) and 10(4)

• 10(1): Equation (4.67) implies

$$2\lambda_2^{(1)}(u) + 2 - \lambda_{2,C}^{(1)}(u) + \frac{1}{3}\lambda_2^{(2)}(u) + \frac{1}{3}\lambda_{2,C}^{(2)}(u) = 0.$$

On the other hand (4.72) implies $\lambda_2^{(1)}(u) + \frac{2}{3}\lambda_2^{(2)}(u) = \lambda_2 = 1$. Since $0 \leq \lambda_2^{(1)}, \lambda_{2,C}^{(1)} \leq 5$, $\lambda_2^{(1)} + \lambda_{2,C}^{(1)} \leq 5$, $0 \leq \lambda_2^{(2)}, \lambda_{2,C}^{(2)} \leq 6$, and $\lambda_2^{(2)} + \lambda_{2,C}^{(2)} \leq 6$, only solution for these equations is $\lambda_2^{(1)}(u) = 1, \lambda_{2,C}^{(1)}(u) = 4, \lambda_2^{(2)}(u) = \lambda_{2,C}^{(2)}(u) = 0$. This contradict $r_2 = 5$.

• 10(4): We have 2 solutions $\lambda_2^{(1)}(u) = \lambda_{2,C}^{(1)}(u) = 0, \lambda_2^{(2)}(u) = 4, \lambda_{2,C}^{(2)}(u) = 1$ and $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 6, \lambda_2^{(2)}(u) = 4, \lambda_{2,C}^{(2)}(u) = 0$. This contradict $r_1 = 2$.

$n = 12$: non existence for 12(3) and 12(5)

• 12(3): We have $24\lambda_2^{(1)}(u) - \frac{40}{11} - 8\lambda_{2,C}^{(1)}(u) + 8\lambda_2^{(2)}(u) + 8\lambda_{2,C}^{(1)}(u) = 0$ and $25\lambda_2^{(1)}(u) + \frac{3}{4}\lambda_2^{(2)}(u) = \lambda_2 = \frac{3}{2}$. Then $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 3, \lambda_2^{(2)}(u) = 2, \lambda_{2,C}^{(1)}(u) = 2$ is the unique solution of these equations. This contradicts $r_1 = 4$.

•12(5): $r_1 = 6$, $r_2 = 8$, $N_{r_1} = 4$, $N_{r_2} = 9$. We obtain $\lambda_1^{(1)} = 2, \lambda_1^{(2)} = 6$ and $\lambda_2^{(1)}(u) = 2, \lambda_2^{(2)}(u) = 3$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_4\}$. Since $r_1 = 6$ and $\alpha_1 = 8$, we have $|\overline{y}_i \cap \overline{y}_j| = 2$ for $i \neq j$. Hence we may assume $\overline{y}_1 = \{1, 2, 3, 4, 5, 6\}$ and $\overline{y}_2 = \{1, 2, 7, 8, 9, 10\}$. Since $\lambda_1^{(1)} = 2$, we must have $1, 2 \notin \overline{y}_3, \overline{y}_4$. Then $\{1, 3\} \not\subset \overline{y}_2$, $\overline{y}_3$, $\overline{y}_4$. This contradicts $\lambda_2^{(1)}(\{1, 3\}) = 2$. ∎

$n = 18$: non existence for 18(1) and 18(4).

• 18(1): Similar computation shows that there is unique solution $\lambda_2^{(1)}(u) = 1, \lambda_{2,C}^{(1)}(u) = 8, \lambda_2^{(2)}(u) = \lambda_{2,C}^{(2)}(u) = 0$. This contradicts $r_2 = 9$.

• 18(4): Similar computation shows that there is unique solution $\lambda_2^{(1)}(u) = \lambda_{2,C}^{(1)}(u) = 0, \lambda_2^{(2)}(u) = 8, \lambda_{2,C}^{(2)}(u) = 1$. This contradicts $r_1 = 9$.

$n = 20$: non existence for 20(3), 20(4), 20(6), 20(7).

• 20(3): Similar computation shows that there is unique solution $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 8, \lambda_2^{(2)}(u) = 2, \lambda_{2,C}^{(2)}(u) = 3$. This contradicts $r_1 = 5$.

• 20(4): Similar computation shows that there $\lambda_2^{(1)}(u) = 0$ or $\lambda_2^{(1)}(u) = 2$. On the other hand $\alpha_1 = 8$ implies $|y_1 \cap y_2| = 1$ for any $y_1, y_2 \in Y_{r_1}$. Then we must have $\lambda_2^{(1)}(u) = 0$ and contradicts $r_1 = 5$.

• 20(6): $r_1 = 8$, $r_2 = 15$, $N_1 = 5$, $N_2 = 16$. We have $\lambda_1^{(1)} = 2$ and $\lambda_1^{(2)} = 12$, and $\lambda_2^{(1)}(u) = 2$, $\lambda_{2,C}^{(1)}(u) = 3$, $\lambda_2^{(2)}(u) = 8$, $\lambda_{2,C}^{(2)}(u) = 0$. Let $Y_8 = \{y_1, \ldots, y_5\}$. Since $\alpha_1 = 12$, we must have $|\overline{y}_i \cap \overline{y}_j| = 2$ for any distinct $y_i, y_j \in Y_{r_1}$. Then we may assume $\overline{y}_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\overline{y}_2 = \{1, 2, 9, 10, 11, 12, 13, 14\}$.
Since $\lambda_1^{(1)} = 2$, we must have $1 \notin \overline{y}_j$ for $j = 3, 4, 5$. Hence $\lambda_2^{(1)}(\{1, 3\}) = 2$ implies $\{1, 3\} \subset \overline{y}_1, \overline{y}_2$. But this is impossible.

• 20(7): $r_1 = 12$, $N_{r_1} = 5$. In this case we have the following solutions. $\lambda_1^{(1)} = 3$, $\lambda_1^{(2)} = 12$, and $\lambda_2^{(1)}(u) = 0$, or 3 for any $u \in X_2$. Let $Y_{r_1} = \{y_1, y_2, \ldots, y_5\}$. Since $\alpha_1 = 12$, we must have $|\overline{y}_i \cap \overline{y}_j| = 6$ for any distinct y_i, $y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $\overline{y}_2 = \{1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18\}$.
Since $\lambda_1^{(1)} = 3$, we may assume $1 \in \overline{y}_3$, then we must have $1 \notin \overline{y}_4, \overline{y}_5$. On the other hand, $\lambda_2^{(1)}(u) = 0$, or 3 and $\{1, 7\} \subset \overline{y}_1$ implies $\lambda_2^{(1)}(\{1, 7\}) = 3$. Since $1 \notin \overline{y}_4, \overline{y}_5$, we must have $\{1, 7\} \subset \overline{y}_1, \overline{y}_2, \overline{y}_3$. This is impossible.

$n = 21$: non existence for 21(1), 21(2), 21(5), 21(6), 21(8), 21(9),21(10),21(12).
• 21(1): We have $\lambda_2^{(2)}(u) = 0$ for any $u \in X_2$. This contradicts $r_2 = 7$.
• 21(2): $r_1 = 3$ and $N_{r_1} = 7$. Similar computation implies there is unique solution $\lambda_2^{(1)}(u) = 1, \lambda_{2,C}^{(1)}(u) = 6, \lambda_2^{(2)}(u) = 5, \lambda_{2,C}^{(2)}(u) = 0$. However in this case we have $\alpha_1 = 6$ implies $|\overline{y_1} \cap \overline{y_2}| = 0$ for any distinct $y_1, y_2 \in Y_{r_1}$. This shows that there exists $u \in X_2$ satisfying $\lambda_2^{(1)}(u) = 0$. This is a contradiction.
• 21(5): There is a unique solution $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 5, \lambda_2^{(2)}(u) = 3, \lambda_{2,C}^{(2)}(u) = 4$. This contradicts $r_1 = 7$.
• 21(6). There is a unique solution $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 5, \lambda_2^{(2)}(u) = 4, \lambda_{2,C}^{(2)}(u) = 3$. This contradicts $r_1 = 7$.
• 21(8): There is a unique solution $\lambda_2^{(1)}(u) = 0, \lambda_{2,C}^{(1)}(u) = 5, \lambda_2^{(2)}(u) = 6, \lambda_{2,C}^{(2)}(u) = 1$. This contradicts $r_1 = 7$.
• 21(9): $r_1 = 9$ and $N_{r_1} = 7$. We have $\lambda_1^{(1)} = 3, \lambda_1^{(2)} = 10$ and $\lambda_2^{(1)}(u) = 3, \lambda_2^{(2)}(u) = 5$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, y_2, \ldots, y_7\}$. Since $\alpha_1 = 12$, $|\overline{y}_i \cap \overline{y}_j| = 3$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $\overline{y}_2 = \{1, 2, 3, 10, 11, 12, 13, 14, 15\}$. Since $\lambda_1^{(1)} = 3$, we may assume $1 \in \overline{y}_3$ and $1 \notin \overline{y}_j$, $j = 4, 5, 6, 7$. Hence $\lambda_2^{(1)}(\{1, 4\}) = 3$ implies $\{1, 4\} \subset \overline{y}_j$, for $j = 1, 2, 3$. This is impossible.
• 21(10): $r_1 = 12$, $N_{r_1} = 7$. We have $\lambda_1^{(1)} = 4, \lambda_1^{(2)} = 10$ and $\lambda_2^{(1)}(u) = 4, \lambda_2^{(2)}(u) = 5$ for any $u \in X_2$. Let $Y_{r_1} = \{y_i \mid 1 \le i \le 7\}$. Since $r_1 = 12$ and $\alpha_1 = 12$, $|\overline{y}_i \cap \overline{y}_j| = 6$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{i \mid 1 \le i \le 12\}$ and $\overline{y}_2 = \{1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18\}$. Since $\lambda_1^{(1)} = 4$ we may assume $1 \in \overline{y}_3, \overline{y}_4$ and $1 \notin \overline{y}_j$, $j = 5, 6, 7$. Then $\lambda_2^{(1)}(\{1, 7\}) = 4$ and $1 \notin \overline{y}_j$, $j = 5, 6, 7$ imply $\{1, 7\} \subset \overline{y}_j$, $j = 1, 2, 3, 4$. This is a contradiction.
• 21(12): $r_2 = 18$, $N_{r_2} = 7$. We have $\lambda_1^{(1)} = 10, \lambda_1^{(2)} = 6$ and $\lambda_2^{(1)}(u) = 5, \lambda_2^{(2)}(u) = 6$ for any $u \in X_2$. Let $Y_{r_2} = \{y_i \mid 1 \le i \le 7\}$. Since $r_2 = 18$ and $\alpha_2 = 6$, $|\overline{y}_i \cap \overline{y}_j| = 15$ for any distinct $y_i, y_j \in Y_{r_2}$. We may assume $\overline{y}_1 = \{i \mid 1 \le i \le 18\}$ and $\overline{y}_2 = \{1, 2, \ldots, 15, 19, 20, 21\}$. Since $\lambda_1^{(2)} = 6$, we may assume $1 \in \overline{y}_j$, for $j = 1, 2, 3, 4, 5, 6$ and $1 \notin \overline{y}_7$. Then it is imposible to have $\lambda_2^{(2)}(\{1, 16\}) = 6$.

$n = 26$: nonexistence for 26(i), $i \neq 4, 5$
• 26(1), 26(2) and 26(3): We have $\lambda_2^{(2)}(u) = 0$. This contradicts $r_2 = 13$.
• 26(6), 26(7) and 26(8): We have $\lambda_2^{(1)}(u) = 0$. This contradicts $r_1 = 13$.

$n = 27$: nonexistence for 27(1), 27(2)
• 27(1): $\lambda_1^{(1)} = \frac{7}{3}$ and $\lambda_1^{(2)} = \frac{35}{3}$. This is a contradiction.

• 27(2): $\lambda_1^{(1)} = \frac{28}{3}$ and $\lambda_1^{(2)} = \frac{14}{3}$. This is a contradiction. ∎

$n = 28$: nonexistence for 28(3) and 28(5)

• 28(3): $r_2 = 14$, $N_{r_2} = 8$. We have $\lambda_1^{(1)} = 6, \lambda_1^{(2)} = 4$ and $\lambda_2^{(1)}(u) = 0, \lambda_2^{(2)}(u) = 4$; or $\lambda_2^{(1)}(u) = 3, \lambda_2^{(2)}(u) = 0$ for any $u \in X_2$. Let $Y_{r_2} = \{y_i \mid 1 \le i \le 8\}$. Then $r_2 = 14$ and $\alpha_2 = 16$ implies $|\overline{y}_i \cap \overline{y}_j| = 6$ for any distinct $y_i, y_j \in Y_{r_2}$. We may assume $\overline{y}_1 = \{i \mid 1 \le i \le 14\}$ and $\overline{y}_2 = \{1,2,3,4,5,6,15,16,17,18,19,20,21,22\}$. Since $\lambda_2^{(2)}(\{1,2\}) \ge 1$ we must have $\lambda_2^{(2)}(\{1,2\}) = 4$. Therefore we may assume $\{1,2\} \subset \overline{y}_3, \overline{y}_4$. Then $\lambda_1^{(2)} = 4$ implies $1 \notin \overline{y}_j$ for $j = 5,6,7,8$. $\{1,7\} \subset \overline{y}_7$ implies $\lambda_2^{(2)}(\{1,7\}) = 4$. Therefore we must have $\{1,7\} \subset \overline{y}_j$ for $j = 1,2,3,4$. But this is impossible.

• 28(5): $r_1 = 14$ and $N_{r_1} = 8$. We have $\lambda_1^{(1)} = 4, \lambda_1^{(2)} = 15$ and $\lambda_2^{(1)}(u) = 0, \lambda_2^{(2)}(u) = 12$; or $\lambda_2^{(1)}(u) = 4, \lambda_2^{(2)}(u) = 9$ for any $u \in X_2$. Let $Y_{r_1} = \{y_i \mid 1 \le i \le 8\}$. Then $r_1 = 14$ and $\alpha_1 = 16$ implies $|\overline{y}_i \cap \overline{y}_j| = 6$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{1,2,\ldots,14\}$ and $\overline{y}_2 = \{1,2,\ldots,6,15,16,17,18,19,20,21,22\}$. Since $\lambda_1^{(1)} = 4$, we may assume $1 \in \overline{y}_3, \overline{y}_4$ and $1 \notin \overline{y}_j$, $j = 5,4,3,8$. Since $\{1,14\} \subset \overline{y}_1$, we must have $\lambda_2^{(1)}(\{1,14\}) = 4$. However since $14 \notin \overline{y}_2$ this is impossible.

$n = 30$: non existences for " $\times$ "

• 30(2): We have $\lambda_2^{(2)}(u) = 0$ for any $u \in X_2$. This contradicts $r_2 = 10$.

• 30(3): We have $\lambda_2^{(1)}(u) = 1$. Since $r_1 = 3$ and $\alpha_1 = 6$, there exists $u \in X_2$ with $\lambda_2^{(1)}(u) = 0$. This is a contradiction.

• 30(6), 30(7) and 30(8): We have $\lambda_2^{(1)}(u) = 0$ for any $u \in X_2$. This contradicts $r_1 = 6$.

• 30(12), 30(13) and 30(16): We have $\lambda_2^{(1)}(u) = 0$ for any $u \in X_2$. This contradicts $r_1 = 10$.

• 30(15): $r_1 = 10$ and $N_{r_1} = 6$. We have $\lambda_1^{(1)} = 2, \lambda_1^{(2)} = 20$ and $\lambda_2^{(1)}(u) = 2, \lambda_2^{(2)}(u) = 15$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_6\}$. Then $r_1 = 6$ and $\alpha_1 = 16$, implies $|\overline{y}_i \cap \overline{y}_j| = 8$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{1,2,3,4,5,6,7,8,9,10\}$, $\overline{y}_2 = \{1,2,3,4,5,6,7,8,11,12\}$. Since $\lambda_1^{(1)} = 2$, we must have $1, 2 \notin \overline{y}_i$, for $i = 3,4,5,6$. Then it is impossible to have $\lambda_2^{(1)}(\{1,9\}) = 2$.

• 30(17): $r_1 = 12$ and $N_{r_1} = 10$. We have $\lambda_1^{(1)} = 4, \lambda_1^{(2)} = 14$ and $\lambda_2^{(1)}(u) = 4, \lambda_2^{(2)}(u) = 7$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_{10}\}$. Then $r_1 = 12$ and $\alpha_1 = 16$, implies $|\overline{y}_i \cap \overline{y}_j| = 4$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{1,2,3,4,5,6,7,8,9,10,11,12\}$, $\overline{y}_2 = \{1,2,3,4,13,14,15,16,17,18,19,20\}$. Since $\lambda_2^{(1)} = 4$ and $\lambda_1^{(1)} = 4$, we may assume $\{1,2\} \subset \overline{y}_j$ for $j = 3,4$ and $1,2 \notin \overline{y}_j$ for $j = 5,6,7,8,9,10$. On the other hand $\lambda_2^{(1)}(\{1,5\}) = 4$. Hence we must have $\{1,5\} \subset \overline{y}_j$ for $j = 1,2,3,4$. But this is impossible.

• 30(20): $r_1 = 15$ and $N_{r_1} = 6$. We have $\lambda_1^{(1)} = 3, \lambda_1^{(2)} = 20$ and $\lambda_2^{(1)}(u) = 3, \lambda_2^{(2)}(u) = 15$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_6\}$. Then $r_1 = 15$ and $\alpha_1 = 18$, implies $|\overline{y}_i \cap \overline{y}_j| = 4$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{i \mid 1 \le i \le 15\}$, $\overline{y}_2 = \{1,2,3,4,16,17,18,19,20,21,22,23,24,$
$25,26\}$. Since $\lambda_2^{(1)} = 3$ and $\lambda_1^{(1)} = 3$, we may assume $\{1,2\} \subset \overline{y}_3$ and $1,2 \notin \overline{y}_j$ for $j = 4,5,6$. Then it is impossible to have $\lambda_2^{(1)}(\{1,5\}) = 3$.

• 30(22): $r_1 = 18$ and $N_{r_1} = 10$. We have $\lambda_1^{(1)} = 6, \lambda_1^{(2)} = 14$ and $\lambda_2^{(1)}(u) = 6, \lambda_2^{(2)}(u) =$

7 for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_{10}\}$. Then $r_1 = 18$ and $\alpha_1 = 16$, implies $|\overline{y}_i \cap \overline{y}_j| = 10$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{i \mid 1 \leq i \leq 18\}$, $\overline{y}_2 = \{1, 2, \ldots, 10, 19, \ldots, 26\}$. Since $\lambda_2^{(1)} = \lambda_1^{(1)} = 6$, we may assume $\{1, 2\} \subset \overline{y}_i$ for $i = 3, 4, 5, 6$ and $1, 2 \notin \overline{y}_i$ for $i = 7, 8, 9, 10$. Then we must have $j \in \overline{y}_i$ for $j = 1, 2, \ldots, 10$ and $i = 1, 2, \ldots, 6$. Then it is impossible to have $\lambda_2^{(1)}(\{1, 11\}) = 6$.

• 30(24): $r_1 = 20$ and $N_{r_1} = 6$. We have $\lambda_1^{(1)} = 4, \lambda_1^{(2)} = 20$ and $\lambda_2^{(1)}(u) = 4, \lambda_2^{(2)}(u) = 15$ for any $u \in X_2$. Let $Y_{r_1} = \{y_1, \ldots, y_6\}$. Then $r_1 = 20$ and $\alpha_1 = 16$, implies $|\overline{y}_i \cap \overline{y}_j| = 12$ for any distinct $y_i, y_j \in Y_{r_1}$. We may assume $\overline{y}_1 = \{i \mid 1 \leq i \leq 20\}$, $\overline{y}_2 = \{1, 2, \ldots, 12, 21, \ldots, 28\}$. Since $\lambda_1^{(1)} = 4$, we may assume $1 \in \overline{y}_j$ for $j = 3, 4$ and $1 \notin \overline{y}_j$ for $j = 5, 6$. Then we must have $\{1, 13\} \subset \overline{y}_j$ for $j = 1, 2, 3, 4$. This impossible.

• 30(25): $r_2 = 27$ and $N_{r_2} = 10$. We have $\lambda_1^{(1)} = 14,\ \lambda_1^{(2)} = 9$ and $\lambda_2^{(1)}(u) = 7, \lambda_2^{(2)}(u) = 9$ for any $u \in X_2$. Let $Y_{r_2} = \{y_1, \ldots, y_{10}\}$. Then $r_2 = 27$ and $\alpha_2 = 6$, implies $|\overline{y}_i \cap \overline{y}_j| = 24$ for any distinct $y_i, y_j \in Y_{r_2}$. We may assume $\overline{y}_1 = \{i \mid 1 \leq i \leq 27\}$, $\overline{y}_2 = \{1, 2, \ldots, 24, 28, 29, 30\}$. Since $\lambda_1^{(2)} = 9$, we may assume $1 \in \overline{y}_j$ for $3 \leq j \leq 9$ and $1 \notin \overline{y}_{10}$. Then $\lambda_2^{(2)}(\{1, 30\}) = 9$ implies $\{1, 30\} \subset \overline{y}_j$ for $j = 1, \ldots, 9$. But this is impossible since $30 \notin \overline{y}_1$. ∎

Acknowledgment: Eiichi Bannai was supported in part by NSFC grant No. 11271257. Hideo Bannai was supported in part by Kakenhi No. 22680013 and No. 25280086.
The authors thank Professor Woodall for kindly answering some questions by the authors on his paper in 1970.

References

[1] E. Bannai, *On tight designs*, Quart. J. Math. (Oxford), 28 (1977), 433-448.

[2] Ei. Bannai and Et. Bannai, *On Euclidean tight 4-designs*, J. Math. Soc. Japan, **58** (2006), 775–804.

[3] Ei. Bannai and Et. Bannai, *Spherical designs and Euclidean designs*, in: *Recent Developments in Algebra and Related Areas* (Beijing, 2007), 1–37, Adv. Lect. Math. 8, Higher Education Press, Beijing; International Press, Boston, 2009.

[4] Ei. Bannai and Et. Bannai, *A survey on spherical designs and algebraic combinatorics on spheres*, European J. Combin., **30** (2009), 1392–1425.

[5] Ei. Bannai and Et. Bannai, *Euclidean designs and coherent configurations*, Contemporary Mathematics Volume 531, (2010), 59–93.

[6] Ei. Bannai and Et. Bannai, *Remarks on the concepts of t-designs*, J. Appl Math Comput. 40 no. 1–2,(2012), 195–207. (Proceedings of AGC2010 (13 pages).)

[7] Ei. Bannai, Et. Bannai, S. Suda and H. Tanaka, *On relative t-designs in polynomial association schemes*, arXiv:1303.7163

[8] Ei. Bannai and R. M. Damerell, *Tight spherical designs* I, J. Math. Soc. Japan, **31** (1979), 199–207.

[9] Ei. Bannai and R. M. Damerell, *Tight spherical designs* II, J. London Math. Soc. (2), **21** (1980), 13–30.

[10] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, California, (1984).

[11] Ei. Bannai, A. Munemasa and B. Venkov, *The nonexistence of certain tight spherical designs*, Algebra i Analiz 16 (2004), 1-23. MR 2090848 (2005e:05022) (translation in St. Petersburg Math. J., 16 (2005), 609–625.)

[12] Et. Bannai, *On antipodal Euclidean tight* $(2e+1)$*-designs*, J. Algebraic Combin., **24** (2006), 391–414.

[13] Et Bannai, *New examples of Euclidean tight* 4*-designs*, European J. Combin., **30** (2009), 655–667.

[14] P. Delsarte, *An algebraic approach to the association schemes of the coding theory*, Thesis, Universite Catholique de Louvain (1973) Philips Res. Repts Suppl. 10 (1973).

[15] P. Delsarte, *Association Schemes and t-Designs in Regular Semilattices*, Journal of Combinatorial Theory (A) 20 (1976), 230–243.

[16] P. Delsarte, Pairs of vectors in the space of an association scheme, Philips Res. Repts 32 (1977), 373–411.

[17] P. Delsarte and J. J. Seidel, *Fisher type inequalities for Euclidean t-designs*, Linear Algebra Appl. 114-115 (1989), 213–230.

[18] P. Delsarte, J. M. Goethals and J. J. Seidel, *Bounds for systems of lines and Jacobi polynomials*, Philips Research Reports 30 (1975).

[19] P. Dukes and J. Short-Gershman, *Nonexistence results for tight block designs*, J. Algebraic Comb., (on line) DOI 10.1007/s10801-012-0395-8, arXiv: 1110.3463.

[20] H. Enomoto, N. Ito and R. Noda, *Tight 4-designs*, Osaka J. Math. 16 no. 1(1979), 39–43.

[21] Z. Li, Ei. Bannai and Et Bannai, *Tight Relative 2- and 4-Designs on Binary Hamming Association Schemes*, Graphs and Combinatorics, (on line) DOI 10.1007/s00373-012-1252-1

[22] G. Nebe and B. Venkov, *On tight spherical designs*, to appear in St. Petersburg Mathematical Journal.

[23] D. K. RayChaudhuri and R. M. Wilson, *On t-Designs*, Osaka J. Math. 12 (1975), 737–744.

[24] Z. Xiang, *A Fisher type inequality for weighted regular t-wise balanced designs*, Journal of Combinatorial Theory A, 119 (2012) 1523–1527.

[25] D. R. Woodall, *Square λ-linked designs*, Proc. London Math. Soc. (3) 20 (1970), 669–687.

Eiichi Bannai: Department of Mathematics, Shanghai Jiao Tong University,
800 Dongchuan Road, Shanghai, 200240, China
e-mail: bannai@sjtu.edu.cn

Etsuko Bannai: Misakigaoka 2-8-21, Itoshima-shi, Fukuoka, 819-1136, Japan
e-mail: et-ban@rc4.so-net.ne.jp

Hideo Bannai:
Department of Informatics, Kyushu University
744 Moto-oka, Nishi-ku, Fukuoka 819-0395, Japan.
e-mail: bannai@inf.kyushu-u.ac.jp