On the existence of Auslander-Reiten (d+2)-angles in (d+2)-angulated categories $^{\bigstar}$

Panyue Zhou

Abstract

Let \mathcal{C} be a (d+2)-angulated category. In this note, we show that if \mathcal{C} is a locally finite, then \mathcal{C} has Auslander-Reiten (d+2)-angles. This extends a result of Xiao-Zhu for triangulated categories.

Key words: (d+2)-angulated categories; Auslander-Reiten (d+2)-angles; locally finite.

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1 Introduction

Auslander-Reiten theory was introduced by Auslander and Reiten in [AR1, AR2]. Since its introduction, Auslander-Reiten theory has become a fundamental tool for studying the representation theory of Artin algebras. Later it has been generalized to these situation of exact categories [Ji], triangulated categories [H, RV] and its subcategories [AS, J] and some certain additive categories [L, J, S] by many authors. Extriangulated categories were recently introduced by Nakaoka and Palu [NP] as a simultaneous generalization of exact categories and triangulated categories. Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Iyama, Nakaoka and Palu [INP] introduced the notion of almost split extensions and Auslander-Reiten-Serre duality for extriangulated categories, and gave explicit connections between these notions and also with the classical notion of dualizing k-varieties. Xiao and Zhu [XZ1, XZ2] showed that if a triangulated category $\mathcal C$ is locally finite, then $\mathcal C$ has Auslander-Reiten $\mathbb E$ -triangles.

In [GKO], Geiss, Keller and Oppermann introduced (d+2)-angulated categories. These are generalizations of triangulated categories, in the sense that triangles are replaced by (d+2)-angles, that is, morphism sequences of length d+2. Thus a 1-angulated category is precisely a triangulated category. Iyama and Yoshino [IY] defined Auslander-Reiten (d+2)-angle in special (d+2)-angulated categories. Later, Fedele [F] defined Auslander-Reiten (d+2)-angles in additive subcategories of (d+2)-angulated categories closed under d-extensions, an example

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of which are wide subcategories. He also proved that there are Auslander-Reiten (d+2)-angles in certain additive subcategories of (d+2)-angulated categories. Recently, the author [Z] showed that a (d+2)-angulated category $\mathcal C$ has Auslander-Reiten (d+2)-angles if and only if $\mathcal C$ has a Serre functor.

In this note, we continue to study Auslander-Reiten (d + 2)-angles in (d + 2)-angulated categories. We will generalise Xiao-Zhu's result into (d + 2)-angulated categories. Moreover, our proof is not far from the usual triangulated case.

Our main result is the following.

Theorem 1.1. (see Theorem 3.8 for details) Let C be a locally finite (d+2)-angulated category. If X is an indecomposable, then there exists an Auslander-Reiten (d+2)-angle ending at X, and if X is an indecomposable, then there exists an Auslander-Reiten (d+2)-angle starting at X. Thus C has Auslander-Reiten (d+2)-angles.

This article is organised as follows: In Section 2, we review some elementary definitions that we need to use, including (d+2)-angulated categories and Auslander-Reiten (d+2) angles. In Section 3, we show our main result.

2 Preliminaries

In this section, we first recall the definition and basic properties of (d+2)-angulated categories from [GKO]. Let \mathcal{C} be an additive category with an automorphism $\Sigma^d: \mathcal{C} \to \mathcal{C}$, and an integer d greater than or equal to one.

A (d+2)- Σ^d -sequence in \mathcal{C} is a sequence of objects and morphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_n \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

Its left rotation is the (d+2)- Σ^d -sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \sum^d A_0 \xrightarrow{(-1)^d \sum^d f_0} \sum^d A_1.$$

A morphism of (d+2)- Σ^d -sequences is a sequence of morphisms $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_{d+1})$ such that the following diagram commutes

$$A_{0} \xrightarrow{f_{0}} A_{1} \xrightarrow{f_{1}} A_{2} \xrightarrow{f_{2}} \cdots \xrightarrow{f_{d}} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^{d} A_{0}$$

$$\downarrow \varphi_{0} \qquad \downarrow \varphi_{1} \qquad \downarrow \varphi_{2} \qquad \qquad \downarrow \varphi_{d+1} \qquad \downarrow \Sigma^{d} \varphi_{0}$$

$$B_{0} \xrightarrow{g_{0}} B_{1} \xrightarrow{g_{1}} B_{2} \xrightarrow{g_{2}} \cdots \xrightarrow{g_{d}} B_{d+1} \xrightarrow{g_{d+1}} \Sigma^{d} B_{0}$$

where each row is a (d+2)- Σ^d -sequence. It is an *isomorphism* if $\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{d+1}$ are all isomorphisms in \mathcal{C} .

Definition 2.1. [GKO, Definition 2.1] A (d+2)-angulated category is a triple $(\mathcal{C}, \Sigma^d, \Theta)$, where \mathcal{C} is an additive category, Σ^d is an automorphism of \mathcal{C} (Σ^d is called the d-suspension functor),

and Θ is a class of (d+2)- Σ^d -sequences (whose elements are called (d+2)-angles), which satisfies the following axioms:

- (N1) (a) The class Θ is closed under isomorphisms, direct sums and direct summands.
 - (b) For each object $A \in \mathcal{C}$ the trivial sequence

$$A \xrightarrow{1_A} A \to 0 \to 0 \to \cdots \to 0 \to \Sigma^d A$$

belongs to Θ .

(c) Each morphism $f_0: A_0 \to A_1$ in \mathcal{C} can be extended to (d+2)- Σ^d -sequence:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

- (N2) A (d+2)- Σ^n -sequence belongs to Θ if and only if its left rotation belongs to Θ .
- (N3) Each solid commutative diagram

$$A_{0} \xrightarrow{f_{0}} A_{1} \xrightarrow{f_{1}} A_{2} \xrightarrow{f_{2}} \cdots \xrightarrow{f_{d}} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^{d} A_{0}$$

$$\downarrow^{\varphi_{0}} \qquad \downarrow^{\varphi_{1}} \qquad \downarrow^{\varphi_{2}} \qquad \qquad \downarrow^{\varphi_{d+1}} \qquad \downarrow^{\Sigma^{d} \varphi_{0}}$$

$$B_{0} \xrightarrow{g_{0}} B_{1} \xrightarrow{g_{1}} B_{2} \xrightarrow{g_{2}} \cdots \xrightarrow{g_{d}} B_{d+1} \xrightarrow{g_{d+1}} \Sigma^{d} B_{0}$$

with rows in Θ , the dotted morphisms exist and give a morphism of (d+2)- Σ^d -sequences.

(N4) In the situation of (N3), the morphisms $\varphi_2, \varphi_3, \dots, \varphi_{d+1}$ can be chosen such that the mapping cone

$$A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{d+1} & 0 \\ \varphi_{d+1} & g_d \end{pmatrix}} \Sigma^n A_0 \oplus B_{d+1} \xrightarrow{\begin{pmatrix} -\Sigma^d f_0 & 0 \\ \Sigma^d \varphi_1 & g_{d+1} \end{pmatrix}} \Sigma^d A_1 \oplus \Sigma^d B_0$$

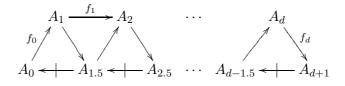
belongs to Θ .

Now we give an example of (d+2)-angulated categories.

Example 2.2. We recall the standard construction of (d+2)-angulated categories given by Geiss-Keller-Oppermann [GKO, Theorem 1]. Let \mathcal{C} be a triangulated category and \mathcal{T} a d-cluster tilting subcategory which is closed under Σ^d , where Σ is the shift functor of \mathcal{C} . Then $(\mathcal{T}, \Sigma^d, \Theta)$ is a (d+2)-angulated category, where Θ is the class of all sequences

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0$$

such that there exists a diagram



with $A_i \in \mathcal{T}$ for all $i \in \mathbb{Z}$, such that all oriented triangles are triangles in \mathcal{C} , all non-oriented triangles commute, and f_{d+1} is the composition along the lower edge of the diagram.

The following two lemmas are very useful which are needed later on.

Lemma 2.3. [F, Lemma 3.13] Let C be a (d+2)-angulated category, and

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$
 (2.1)

a (d+2)-angle in C. Then the following are equivalent:

- (1) α_0 is a section;
- (2) α_d is a retraction;
- (3) $\alpha_{d+1} = 0$.

If a (d+2)-angle (2.1) satisfies one of the above equivalent conditions, it is called split.

Lemma 2.4. [LZ, Corollary 3.4] Let C be a (d+2)-angulated category, and

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$

a (d+2)-angle in C. Then for any morphism $\varphi_0: A_0 \to B_0$, there exists the following commutative diagram of (d+2)-angles

$$A_{0} \xrightarrow{\alpha_{0}} A_{1} \xrightarrow{\alpha_{1}} A_{2} \xrightarrow{\alpha_{2}} \cdots \xrightarrow{\alpha_{d-1}} A_{d} \xrightarrow{\alpha_{d}} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^{d} A_{0}$$

$$\downarrow^{\varphi_{0}} \downarrow^{\varphi_{1}} \downarrow^{\varphi_{2}} \downarrow^{\varphi_{2}} \downarrow^{\varphi_{d}} \downarrow^{\varphi_{d}} \downarrow^{\varphi_{d}}$$

$$B_{0} \xrightarrow{\beta_{0}} B_{1} \xrightarrow{\beta_{1}} B_{2} \xrightarrow{\beta_{2}} \cdots \xrightarrow{\beta_{d-1}} B_{d} \xrightarrow{\beta_{d}} A_{d+1} \xrightarrow{\beta_{d+1}} \Sigma^{d} B_{0}$$

such that

$$A_0 \xrightarrow{\begin{pmatrix} -\alpha_0 \\ \varphi_0 \end{pmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -\alpha_1 & 0 \\ \varphi_1 & \beta_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -\alpha_{d-1} & 0 \\ \varphi_{d-1} & \beta_{d-2} \end{pmatrix}} A_d \oplus B_{d-1}$$

 $\xrightarrow{(\varphi_d, \beta_{d-1})} B_d \xrightarrow{(-1)^d \alpha_{d+1} \beta_d} \Sigma^d A_0 \text{ is a } (d+2)\text{-angle in } \mathcal{C}.$

Now we recall an Auslander-Reiten (d+2) theory in (d+2)-angulated categories.

We denote by $\operatorname{rad}_{\mathcal{C}}$ the Jacobson radical of \mathcal{C} . Namely, $\operatorname{rad}_{\mathcal{C}}$ is an ideal of \mathcal{C} such that $\operatorname{rad}_{\mathcal{C}}(A,A)$ coincides with the Jacobson radical of the endomorphism ring $\operatorname{End}(A)$ for any $A \in \mathcal{C}$.

Definition 2.5. [IY, Definition 3.8] and [F, Definition 5.1] Let C be a (d+2)-angulated category. A (d+2)-angle

$$A_{\bullet}: A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

in C is called an Auslander-Reiten (d+2)-angle if α_0 is left almost split, α_d is right almost split and when $d \geq 2$, also $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ are in rad_C.

Remark 2.6. [F, Remark 5.2] Assume A_{\bullet} as in the above definition is an Auslander-Reiten (d+2)-angle. Since α_0 is left almost split implies that $\operatorname{End}(A_0)$ is local and hence A_0 is indecomposable. Similarly, since α_d is right almost split, then $\operatorname{End}(A_{d+1})$ is local and hence A_{d+1} is indecomposable. Moreover, when d=1, we have α_0 and α_d in $\operatorname{rad}_{\mathcal{C}}$, so that α_d is right minimal and α_0 is left minimal. When $d \geq 2$, since $\alpha_{d-1} \in \operatorname{rad}_{\mathcal{C}}$, we have that α_d is right minimal and similarly α_0 is left minimal.

Remark 2.7. [F, Lemma 5.3] Let \mathcal{C} be a (d+2)-angulated category and

$$A_{\bullet}: A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

be a (d+2)-angle in C. Then the following statements are equivalent:

- (1) A_{\bullet} is an Auslander-Reiten (d+2)-angle;
- (2) $\alpha_0, \alpha_1, \dots, \alpha_{d-1}$ are in rad_C and α_d is right almost split;
- (3) $\alpha_1, \alpha_2, \dots, \alpha_d$ are in rad_C and α_0 is left almost split.

Lemma 2.8. [F, Lemma 5.4] Let C be a (d+2)-angulated category and

$$A_{\bullet}: A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

be a (d+2)-angle in C. Assume that α_d is right almost split and if $d \geq 2$, also that $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ are in rad_C. Then the following are equivalent:

- (1) A_{\bullet} is an Auslander-Reiten (d+2)-angle;
- (2) $\operatorname{End}(A_0)$ is local;
- (3) α_{d+1} is left minimal;
- (4) α_0 is in rad_C.

In the case d=1, so in the case of a triangulated category, a morphism can be extended to a triangle in a unique way up to isomorphism. On the other hand, for $d \geq 2$, a morphism can be extended to a (d+2)-angle in different non-isomorphic ways. However, we still have a unique "minimal" (d+2)-angle extending any given morphism.

Lemma 2.9. [OT, Lemma 5.18] and [F, Lemma 3.14] Let $d \ge 2$ and $h: A_{d+1} \to \Sigma^d A_0$ be any morphism in a (d+2)-angulated category C. Then, up to isomorphism, there exists a unique (d+2)-angle of the form

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{h} \Sigma^d A_0$$

with $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ in rad_C.

3 Proof of main result

In this section, let k be a field. We always assume that \mathcal{C} is a k-linear Hom-finite Krull-Schmidt (d+2)-angulated category. We denote by $\mathsf{ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in \mathcal{C} . For any $X \in \mathsf{ind}(\mathcal{C})$, we denote by $\mathsf{SuppHom}_{\mathcal{C}}(X,-)$ the subcategory of \mathcal{C} generated by objects Y in $\mathsf{ind}(\mathcal{C})$ with $\mathsf{Hom}_{\mathcal{C}}(X,Y) \neq 0$. Similarly, $\mathsf{SuppHom}_{\mathcal{C}}(-,X)$ denotes the subcategory generated by objects Y in $\mathsf{ind}(\mathcal{C})$ with $\mathsf{Hom}_{\mathcal{C}}(Y,X) \neq 0$. If $\mathsf{SuppHom}_{\mathcal{C}}(X,-)$ ($\mathsf{SuppHom}_{\mathcal{C}}(-,X)$, respectively) contains only finitely many indecomposables, we say that $|\mathsf{SuppHom}_{\mathcal{C}}(X,-)| < \infty$ ($|\mathsf{SuppHom}_{\mathcal{C}}(-,X)| < \infty$ respectively).

Based on the definition of locally finite triangulated categories [XZ1, XZ2], we define the notion of locally finite (d + 2)-angulated categories.

Definition 3.1. A (d+2)-angulated category \mathcal{C} is called *locally finite* if $|\mathsf{SuppHom}_{\mathcal{C}}(X,-)| < \infty$ and $|\mathsf{SuppHom}_{\mathcal{C}}(-,X)| < \infty$, for any object $X \in \mathsf{ind}(\mathcal{C})$.

We know that the derived categories of finite dimensional hereditary algebras of finite type and the stable module categories of finite dimensional self-injective algebras of finite type are examples of locally finite triangulated categories, see [XZ1, XZ2]. In those locally finite triangulated categories, we take a d-cluster titling subcategory which is closed under the d-th power of the shift functor. By Example 2.2, we can obtain locally finite (d + 2)-angulated categories.

Definition 3.2. Let \mathcal{C} be a (d+2)-angulated category and $X \in \operatorname{ind}(\mathcal{C})$. We define a set of (d+2)-angles as follows:

$$S(X) := \left\{ A_{\bullet} : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \, \middle| \, \begin{array}{l} A_{\bullet} \text{ is a non-split } (d+2)\text{-angle} \\ \text{with } A \in \mathsf{ind}(\mathcal{C}), \text{and when} \\ d \geq 2, \ \alpha_1, \alpha_2, \cdots, \alpha_{d-1} \text{ in } \mathrm{rad}_{\mathcal{C}}. \end{array} \right\}$$

Dually, we can define a set of (d+2)-angles as follows:

$$T(X) := \left\{ A_{\bullet} : X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \middle| \begin{array}{l} A_{\bullet} \text{ is a non-split } (d+2)\text{-angle} \\ \text{with } A \in \mathsf{ind}(\mathcal{C}), \text{ and when} \\ d \ge 2, \ \alpha_1, \alpha_2, \cdots, \alpha_{d-1} \text{ in rad}_{\mathcal{C}}. \end{array} \right\}$$

Lemma 3.3. Let C be a (d+2)-angulated category and $X \in \text{ind}(C)$. Then S(X) and T(X) are non-empty.

Proof. We only show that S(X) non-empty, dually one can show that T(X) is non-empty. Since $X \in \operatorname{ind}(\mathcal{C})$, there is an object $A \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(X, \Sigma^d A) \neq 0$. Thus there exists a non-split (d+2)-angle:

$$B_{\bullet}: A \xrightarrow{\alpha_0} B_1 \xrightarrow{\alpha_1} B_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} B_{d-1} \xrightarrow{\alpha_{d-1}} B_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d A$$
.

We decompose A into a direct sum of indecomposable objects $A = \bigoplus_{i=1}^{n} A_i$. Without loss of generality, we can assume that $A = U \oplus V$ where U and V are indecomposable. By Lemma

2.4, for the morphism (1,0): $U \oplus V \to U$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (1, 0) \qquad \downarrow \varphi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (1, 0)$$

$$U \xrightarrow{\beta_0} C_1 \xrightarrow{\beta_1} C_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} C_d \xrightarrow{\beta_d} X \xrightarrow{\beta_d} X$$

Similarly, for the morphism (0,1): $U \oplus V \to V$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (0, 1) \qquad \downarrow \psi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \downarrow (0, 1)$$

$$V \xrightarrow{\gamma_0} D_1 \xrightarrow{\gamma_1} D_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{d-1}} D_d \xrightarrow{\gamma_d} X \xrightarrow{\gamma_{d+1}} \Sigma^d V.$$

We claim that the at least one of the following two (d+2)-angles is non-split

$$U \xrightarrow{\beta_0} C_1 \xrightarrow{\beta_1} C_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} C_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d U,$$

$$V \xrightarrow{\gamma_0} D_1 \xrightarrow{\gamma_1} D_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{d-1}} D_d \xrightarrow{\gamma_d} X \xrightarrow{\gamma_{d+1}} \Sigma^d V$$

If not like this, by Lemma 2.3, we obtain that $\beta_{d+1} = 0 = \gamma_{d+1}$. By (N3), we have the following commutative diagram of (d+2)-angles:

where $\delta_i = \begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \end{pmatrix}$. It follows that h = 0. This is a contradiction since B_{\bullet} is non-split.

For the morphism $\beta_{d+1} \neq 0$ or $\gamma_{d+1} \neq 0$, by Lemma 2.9, we can find a (d+2)-angle as we want. This shows that S(X) is nonempty.

Definition 3.4. Let \mathcal{C} be a (d+2)-angulated category, and

$$A_{\bullet}: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

$$B_{\bullet}: B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

two (d+2)-angles in S(X). We say that $A_{\bullet} > B_{\bullet}$ if there are morphisms $\varphi_i \in \text{Hom}_{\mathcal{C}}(A_i, B_i)$, $(i=0,1,\cdots,d)$ such that the following diagram commutative:

We say that $A_{\bullet} \sim B_{\bullet}$ if φ_0 is an isomorphism.

Dually, let

$$A_{\bullet}: X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

$$B_{\bullet}: X \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

be two (d+2)-angles in T(X). We say that $A_{\bullet} > B_{\bullet}$ if there are morphisms $\varphi_i \in \text{Hom}_{\mathcal{C}}(A_i, B_i)$, $(i = 1, 2, \dots, d+1)$ such that the following diagram commutative:

We say that $A_{\bullet} \sim B_{\bullet}$ if φ_{d+1} is an isomorphism.

Lemma 3.5. S(X) is a direct ordered set with the relation defined in Definition 3.4, and T(X) is a direct ordered set with the relation defined in Definition 3.4.

Proof. We just prove the first statement, the second statement proves similarly.

Assume that

$$A_{\bullet}: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

$$B_{\bullet}: B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

belong to S(X).

We first show that if $A_{\bullet} > B_{\bullet}$ and $B_{\bullet} > A_{\bullet}$, then $A_{\bullet} \sim B_{\bullet}$.

Since $A_{\bullet} > B_{\bullet}$ and $B_{\bullet} > A_{\bullet}$, we have the following two commutative diagrams

$$A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

$$\downarrow \varphi_0 \qquad \downarrow \varphi_1 \qquad \downarrow \varphi_2 \qquad \qquad \downarrow \varphi_d \qquad \qquad \downarrow \qquad \qquad \downarrow \Sigma^d \varphi_0$$

$$\downarrow \varphi_0 \qquad \downarrow \varphi_1 \qquad \downarrow \varphi_2 \qquad \qquad \downarrow \varphi_d \qquad \qquad \downarrow \Sigma^d \varphi_0$$

$$\downarrow \varphi_0 \qquad \downarrow \varphi_1 \qquad \downarrow \varphi_2 \qquad \qquad \downarrow \varphi_d \qquad$$

$$B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

$$\downarrow \psi_0 \qquad \downarrow \psi_1 \qquad \downarrow \psi_2 \qquad \qquad \downarrow \psi_d \qquad \qquad \downarrow \qquad \qquad \downarrow \Sigma^d \psi_0$$

$$\downarrow \psi \qquad \qquad \psi \qquad \qquad \psi \qquad \qquad \psi \qquad \qquad \downarrow \qquad \qquad \downarrow \Sigma^d \psi_0$$

$$A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$

Since A is an indecomposable, we have that $\operatorname{End}(A)$ is local implies that $\psi_0 \varphi_0$ is nilpotent or is an isomorphism. If $\psi_0 \varphi_0$ is nilpotent, there exists a positive integer m such that $(\psi_0 \varphi_0)^m = 0$. We write $\omega_i = \psi_i \varphi_i$. Thus we have the following commutative diagram

$$A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

$$\downarrow (\psi_0 \varphi_0)^m \downarrow (\omega_1)^m \qquad \downarrow (\omega_2)^m \qquad \downarrow (\omega_{d-1})^m \qquad \downarrow (\varphi_d)^m \qquad \downarrow \Sigma^d (\psi_0 \varphi_0)^m$$

$$A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$

Then $\alpha_{d+1} = \Sigma^d (\psi_0 \varphi_0)^m \alpha_{d+1} = 0$. This is a contradiction since A_{\bullet} is non-split. Hence $\psi_0 \varphi_0$ is

an isomorphism. By a similar argument we obtain that $\varphi_0\psi_0$ is an isomorphism. This shows that φ_0 is isomorphism. So $A_{\bullet} \sim B_{\bullet}$.

It is clear that if $A_{\bullet} > B_{\bullet}$ and $B_{\bullet} > C_{\bullet}$, then $A_{\bullet} \sim C_{\bullet}$.

Now we show that if $A_{\bullet}, B_{\bullet} \in S(X)$, then there exists $C_{\bullet} \in S(X)$ such that $A_{\bullet} > C_{\bullet}$ and $B_{\bullet} \sim C_{\bullet}$.

For the morphism $\beta_d \colon B_d \to X$, by the dual of Lemma 2.4, there exists the following commutative diagram of (d+2)-angles

such that

$$M_{\bullet}: D_1 \to M_1 \to M_2 \to \cdots \to M_{d-1} \to B_d \oplus A_d \xrightarrow{(\beta_d, \alpha_d)} X \xrightarrow{h} \Sigma^d D_1$$

is a (d+2)-angle in \mathcal{C} , where $M_i = D_{i+1} \oplus A_i$ $(i=1,2,\cdots,d-1)$. Since β_d and α_d are not retraction, we have that (β_d,α_d) is also not retraction. If not like this, there exists a morphism $\binom{s}{t}: X \to B_d \oplus A_d$ such that $(\beta_d,\alpha_d)\binom{s}{t} = 1_X$ and then $\beta_d s + \alpha_d t = 1_X$. Since X is an indecomposable, we have that $\operatorname{End}(X)$ is local implies that either $\beta_d s$ or $\alpha_d t$ is an isomorphism. Thus either β_d or α_d is an retraction, a contradiction. That is, M_{\bullet} is non-split.

Without loss of generality, we can assume that $D_1 = U \oplus V$ where U and V are indecomposable. By Lemma 2.4, for the morphism $(1,0): U \oplus V \to U$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow B_d \oplus A_d \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (1, 0) \qquad \downarrow \varphi_1 \qquad \downarrow \qquad \downarrow (1, 0)$$

$$\downarrow U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{h} \Sigma^d U.$$

Similarly, for the morphism (0,1): $U \oplus V \to V$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow B_d \oplus A_d \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (0, 1) \qquad \downarrow \psi_1 \qquad \downarrow \qquad \qquad \downarrow (0, 1)$$

$$V \xrightarrow{\eta_0} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow N_d \longrightarrow X \longrightarrow \Sigma^d V.$$

Using similar arguments as in the proof of Lemma 3.3, we conclude that the at least one of the following two (d + 2)-angles is non-split

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{h} \Sigma^d U,$$

$$V \xrightarrow{\eta_0} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow N_d \longrightarrow X \longrightarrow \Sigma^d V.$$

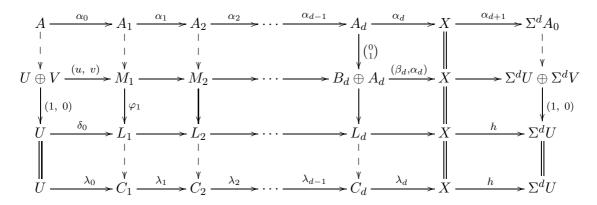
Without loss of generality, we assume that

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{h} \Sigma^d U$$

is non-split. By Lemma 2.9, we can find a non-split (d+2)-angle

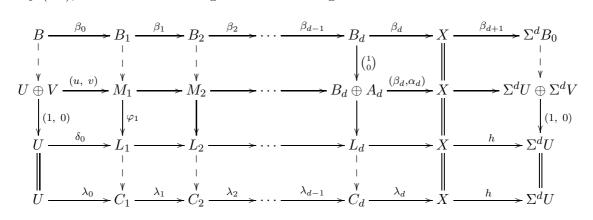
$$C_{\bullet}: U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{d-1}} C_d \xrightarrow{\lambda_d} X \xrightarrow{h} \Sigma^d U$$

with $\lambda_1, \lambda_2, \dots, \lambda_{d-1}$ in rad_C. By (N3), we have the following commutative diagram



of (d+2)-angles. This shows that $A_{\bullet} > C_{\bullet}$.

By (N3), we have the following commutative diagram



of (d+2)-angles. This shows that $B_{\bullet} > C_{\bullet}$.

Lemma 3.6. Let C be a locally finite (d+2)-angulated category and $X \in \text{ind}(C)$. Then S(X) has a minimal element, and T(X) has a minimal element.

Proof. We just prove the first statement, the second statement proves similarly.

Since $X \in \operatorname{ind}(\mathcal{C})$, there is an object $A \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(X, \Sigma^d A) \neq 0$. Then there exists a non-split (d+2)-angle:

$$A_{\bullet}: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d A.$$

We decompose B_d into a direct sum of indecomposable objects $A_d = \bigoplus_{k=1}^n B_k$. Thus A_{\bullet} can be

written as

$$A_{\bullet}: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} \bigoplus_{k=1}^n B_k \xrightarrow{(b_1, b_2, \cdots, b_n)} X \xrightarrow{h} \Sigma^d A$$

where $b_k \in \operatorname{rad}_{\mathcal{C}}(B_k, X), k = 1, 2, \dots, n$.

Since \mathcal{C} is locally finite, there are only finite many objects $X_i \in \operatorname{ind}(\mathcal{C}), i = 1, 2, \dots, m$ such that $\operatorname{Hom}_{\mathcal{C}}(X_i, X) \neq 0$. We assume that $\lambda_{ij}, 1 \leq j \leq q_i$ form a basis of the k-vector space $\operatorname{rad}_{\mathcal{C}}(B_k, X)$. Put $M := (\bigoplus_{k=1}^n B_k) \oplus (\bigoplus_{i=1}^n (X_i)^{\oplus q_i})$, we consider the morphism

$$\delta := (b_1, b_2, \cdots, b_n, \lambda_{11}, \cdots, \lambda_{ij}, \cdots, \lambda_{mq_m}) \in \operatorname{rad}_{\mathcal{C}}(M, X)$$

which is not retraction, it can be embedded in a (d+2)-angle:

$$M_{\bullet}: B \to M_1 \to M_2 \to \cdots \to M_{d-1} \to M \xrightarrow{\delta} X \to \Sigma^d B$$
.

Thus M_{\bullet} is non-split since δ is not retraction. Without loss of generality, we can assume that $B = U \oplus V$ where U and V are indecomposable. By Lemma 2.4, for the morphism $(1,0): U \oplus V \to U$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (1, 0) \qquad \downarrow \varphi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (1, 0)$$

$$U \xrightarrow{\theta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U.$$

Similarly, for the morphism (0,1): $U \oplus V \to V$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (0, 1) \qquad \downarrow \psi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (0, 1)$$

$$V \xrightarrow{\eta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \longrightarrow \Sigma^d V.$$

Using similar arguments as in the proof of Lemma 3.3, we conclude that the at least one of the following two (d + 2)-angles is non-split

$$U \xrightarrow{\theta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U,$$

$$V \xrightarrow{\eta_0} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow N_d \longrightarrow X \longrightarrow \Sigma^d V.$$

Without loss of generality, we assume that

$$U \xrightarrow{\theta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U,$$

is non-split. By Lemma 2.9, we can find a non-split (d+2)-angle

$$C_{\bullet}: U \xrightarrow{\omega_0} C_1 \xrightarrow{\omega_1} C_2 \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_{d-1}} C_d \xrightarrow{\omega_d} X \xrightarrow{f} \Sigma^d U$$

with $\omega_1, \omega_2, \dots, \omega_{d-1}$ in rad_C. Then $C_{\bullet} \in S(X)$. By (N3), we have the following commutative diagram

of (d+2)-angles.

For any $D_{\bullet} \in S(X)$, it can be written as

$$D_{\bullet}: D \to D_1 \to D_2 \to \cdots \to D_{d-1} \to \bigoplus_{i=1}^p L_i \xrightarrow{d=(d_1,d_2,\cdots,d_p)} X \to \Sigma^d D$$

with $d_i \in \operatorname{rad}_{\mathcal{C}}(L_i, X)$, $i = 1, 2, \dots, p$. Since $D_{\bullet} \in S(X)$ is non-split, d is not retraction implies that $d \in \operatorname{rad}_{\mathcal{C}}(\bigoplus_{i=1}^p L_i, X)$. By the definitions of λ_{ij} and δ , there exists a morphism $\rho \colon \bigoplus_{i=1}^p L_i \to M$ such that $d = \delta \rho$. By (N3), we have the following commutative diagram

$$D \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \cdots \longrightarrow D_{d-1} \longrightarrow \bigoplus_{i=1}^p L_i \xrightarrow{d} X \longrightarrow \Sigma^d D$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

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of (d+2)-angles, where $B=U\oplus V$. Thus we get the following commutative diagram

$$D \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \cdots \longrightarrow D_{d-1} \longrightarrow \bigoplus_{i=1}^p L_i \xrightarrow{d} X \longrightarrow \Sigma^d D$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$U \xrightarrow{\omega_0} C_1 \xrightarrow{\omega_1} C_2 \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_{d-2}} C_{d-1} \xrightarrow{\omega_{d-1}} C_d \xrightarrow{\omega_d} X \xrightarrow{f} \Sigma^d U$$

of (d+2)-angles. This shows that C_{\bullet} is a minimal element in S(X).

Remark 3.7. If there exists a minimal element S(X) or T(X), then it is unique up to isomorphism by Lemma 2.9.

We are now ready to state and prove our main result.

Theorem 3.8. Let C be a locally finite (d+2)-angulated category. If $X \in \operatorname{ind}(C)$, then there exists an Auslander-Reiten (d+2)-angle ending at X, and if $X \in \operatorname{ind}(C)$, then there exists an Auslander-Reiten (d+2)-angle starting at X. Thus C has Auslander-Reiten (d+2)-angles.

Proof. Since $X \in \operatorname{ind}(\mathcal{C})$, we know that S(X) is non-empty by Lemma 3.3. By Lemma 3.6,

there exists a (d+2)-angle

$$A_{\bullet}: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

which is a minimal element in S(X). Since $A_{\bullet} \in S(X)$, we have that $\alpha_1, \alpha_2, \dots, \alpha_{d_{d-1}} \in \operatorname{rad}_{\mathcal{C}}$ and A is an indecomposable. Then $\operatorname{End}(A)$ is local.

We want to prove that A_{\bullet} is an Auslander-Reiten (d+2)-angle, by Lemma 2.8, it suffices to show that α_d is right almost split.

Assume that $g: M \to X$ is not retraction. By the dual of Lemma 2.4, there exists the following commutative diagram of (d+2)-angles

such that

$$N_{\bullet}: B_1 \to N_1 \to N_2 \to \cdots \to N_{d-1} \to M \oplus A_d \xrightarrow{(g, \alpha_d)} X \xrightarrow{h} \Sigma^d B_1$$

is a (d+2)-angle in \mathcal{C} , where $N_i = B_{i+1} \oplus A_i$, $i = 1, 2, \dots, d-1$. Since g and α_d are not retraction, we have that (g, α_d) is also not retraction by using similar arguments as in the proof of Lemma 3.5. That is, N_{\bullet} is non-split.

Without loss of generality, we can assume that $B_1 = U \oplus V$ where U and V are indecomposable. By Lemma 2.4, for the morphism (1,0): $U \oplus V \to U$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow M \oplus A_d \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (1, 0) \qquad \downarrow \varphi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (1, 0)$$

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \longrightarrow X \longrightarrow \Sigma^d U.$$

Similarly, for the morphism (0,1): $U \oplus V \to V$, there exists the following commutative diagram of (d+2)-angles

$$U \oplus V \xrightarrow{(u, v)} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow M \oplus A_d \longrightarrow X \longrightarrow \Sigma^d U \oplus \Sigma^d V$$

$$\downarrow (0, 1) \qquad \downarrow \psi_1 \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (0, 1)$$

$$V \xrightarrow{\eta_0} Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_d \longrightarrow X \longrightarrow \Sigma^d V.$$

Using similar arguments as in the proof of Lemma 3.3, we conclude that the at least one of the following two (d + 2)-angles is non-split

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U,$$

$$V \xrightarrow{\eta_0} Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_d \longrightarrow X \longrightarrow \Sigma^d V.$$

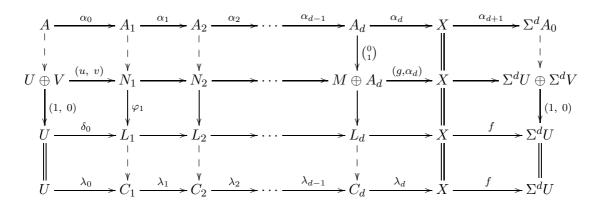
Without loss of generality, we assume that

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U,$$

is non-split. By Lemma 2.9, we can find a non-split (d+2)-angle

$$C_{\bullet}: U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{d-1}} C_d \xrightarrow{\lambda_d} X \xrightarrow{f} \Sigma^d U$$

with $\lambda_1, \lambda_2, \dots, \lambda_{d-1}$ in rad_C. By (N3), we have the following commutative diagram



of (d+2)-angles. We obtain that $A_{\bullet} > C_{\bullet}$ implies that $A_{\bullet} \sim C_{\bullet}$ since A_{\bullet} is the minimal element in S(X). Thus there exists the following commutative diagram

$$U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{d-1}} C_d \xrightarrow{\lambda_d} X \xrightarrow{f} \Sigma^d U$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

of (d+2)-angles. Hence we get the following commutative diagram

of (d+2)-angles. It follows that $g = a\alpha_d$. This shows that α_d is right almost split.

Similarly, we can show that if $X \in \operatorname{ind}(\mathcal{C})$, then there exists an Auslander-Reiten (d+2)-angle starting at X. Thus \mathcal{C} has Auslander-Reiten (d+2)-angles.

Remark 3.9. As a special case of Theorem 3.8 when d = 1, that is, if C is a locally finite triangulated category, then C has Auslander-Reiten triangles, see [XZ1, XZ2].

References

[AR1] M. Auslander, I. Reiten. Representation theory of Artin algebras. III. Almost split sequences. Comm. Algebra 3: 239-294, 1975.

- [AR2] M. Auslander, I. Reiten. Representation theory of Artin algebras. IV. Invariants given by almost split sequences. Comm. Algebra 5(5): 443-518, 1977.
- [AS] M. Auslander, S. Smalø. Almost split sequences in subcategories. J.Algebra 69(2): 426-454, 1981.
- [F] F. Fedele. Auslander-Reiten (d+2)-angles in subcategories and a (d+2)-angulated generalisation of a theorem by Brüning. J. Pure Appl. Algebra, 223(8): 3554-3580, 2019.
- [GKO] C. Geiss, B. Keller and S. Oppermann. *n*-angulated categories. J. Reine Angew. Math. 675: 101-120, 2013.
- [H] D. Happel. Triangulated categories in the representation theory of finite-dimensional algebras. London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.
- [INP] O. Iyama, H. Nakaoka, Y. Palu. Auslander-Reiten theory in extriangulated categories. arXiv: 1805.03776, 2018.
- [IY] O. Iyama, Y. Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. 172(1): 117-168, 2008.
- [Ji] P. Jiao. The generalized Auslander-Reiten duality on an exact category. J. Algebra Appl. 17(12): 1850227, 14 pp. 2018.
- [J] P. Jørgensen. Auslander-Reiten triangles in subcategories. J. K-Theory 3(3): 583-601, 2009.
- [L] S. Liu. Auslander-Reiten theory in a Krull-Schmidt category. São Paulo J. Math. Sci. 4(3): 425-472, 2010.
- [LZ] Z. Lin. Y. Zheng. Homotopy Cartesian diagrams in *n*-angulated categories. Homology Homotopy Appl. 21(2): 377-394, 2019.
- [NP] H. Nakaoka, Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Différ. Catég. 60(2): 117-193, 2019.
- [RV] I. Reiten, M. Van den Bergh. Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. 15(2): 295-366, 2012.
- [S] A. Shah. Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories. J. Pure Appl. Algebra 224(1): 98-124, 2020.
- [OT] S. Oppermann, H. Thomas. Higher-dimensional cluster combinatorics and representation theory. J. Eur. Math. Soc. 14(6): 1679-1737, 2012.
- [XZ1] J. Xiao, B. Zhu. Relations for the Grothendieck groups of triangulated categories. J. Algebra 257(1): 37-50, 2002.

[XZ2] J. Xiao, B. Zhu. Locally finite triangulated categories. J. Algebra 290(2): 473C490, 2005.

- [Z] P. Zhou. On the relation between Auslander-Reiten (d+2)-angles and Serre duality. arXiv: 1910.01454, 2019.
- [ZZ] B. Zhu, X. Zhuang. Grothendieck groups in extriangualted categories. arXiv: 1912.00621, 2019.

Panyue Zhou

College of Mathematics, Hunan Institute of Science and Technology, 414006, Yueyang, Hunan, P. R. China.

and

Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Québec J1K 2R1, Canada.

E-mail: panyuezhou@163.com