

# Faithful Teleportation with Partially Entangled States

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We write explicitly a general protocol for faithful teleportation of a  $d$ -state particle (qudit) via a partially entangled pair of (pure)  $n$ -state particles. The classical communication cost (CCC) of the protocol is  $\log_2(nd)$  bits, and it is implemented by a *projective* measurement performed by Alice, and a unitary operator performed by Bob (after receiving from Alice the measurement result). We prove the optimality of our protocol by a comparison with the concentrate and teleport strategy. We also show that if  $d > n/2$  or if there is no residual entanglement left after the faithful teleportation, the CCC of *any* protocol is at least  $\log_2(nd)$  bits. Furthermore, we find a lower bound on the CCC in the process transforming one bipartite state to another by means of local operation and classical communication (LOCC).

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In the process of quantum teleportation, one party, called Alice, transfers an unknown quantum state to a second party's system, operated by Bob. There are two distinctive resources for the process: (1) The classical information transmitted from one party to the other. (2) The two parties share an entangled state. In the original protocol [1], it has been shown that the resources of  $2\log_2 d$  bits of classical information and a pair of  $d$ -state particles in a maximally entangled state are sufficient for a faithful teleportation of a  $d$ -state object (or qudit). Since then there were several generalizations for the original protocol with more general channels [2, 3, 4, 5]. However, until now, faithful teleportation protocols (i.e. with unit fidelity and unit probability of success [6, 7]) of a  $d$ -state object have not been considered for the case when the entangled resource is a partially entangled pair of pure  $n$ -state particles (with  $n > d$ ). In this paper, we introduce a *protocol* for this scheme, and prove its optimality in a restricted sense by showing that the classical communication cost (CCC) in protocols that involves first concentration and then teleportation is at least  $\log_2(nd)$  bits (which is the CCC used in our protocol). Moreover, it is shown that if  $d > n/2$  or if there is no entanglement shared between Alice and Bob after Alice's measurement, the CCC of *any* strategy is at least  $\log_2(nd)$  bits [15]. We also find a lower bound on the classical information required in the process of deterministic entanglement concentration [8].

In the following, the qudit which is faithfully teleported from Alice to Bob is denoted by

$$|\psi_d\rangle_1 = \sum_{m=1}^d a_m |m\rangle_1, \quad (1)$$

and the entangled resource shared between Alice and Bob

is denoted by (not necessarily maximally entangled)

$$|\chi\rangle_{23} = \sum_{k=1}^n \sqrt{p_k} |k\rangle_2 |k\rangle_3, \quad (2)$$

where  $n = \text{Sch}(|\chi\rangle_{23})$  is the Schmidt number (we have included in the sum only the non-zero  $p_k$ 's). Thus, systems 1 and 2 belong to Alice's lab, and system 3 to Bob's lab.

The teleportation can be achieved by a protocol involving just the following steps [9]: Alice performs a single generalized measurement on her systems 1 and 2, and then sends the result to Bob, who performs a particular unitary operation on his system 3, according to Alice's message.

There are two interesting questions to ask. First, what are the conditions that the Schmidt numbers  $\{p_k\}$  must satisfy in order to achieve a faithful teleportation (i.e. with maximum fidelity,  $f = 1$ ). Second, what is the lower bound on the amount of classical bits that Alice must send to Bob.

The answer to the first question follows directly from Nielsen's theorem [10], and we have summarized it in the following theorem.

**Theorem 1:** Faithful teleportation is possible if, and only if,

$$E_t(|\chi\rangle_{23}) \equiv -\log_2 p_m \geq \log_2 d, \quad (3)$$

where  $p_m = \max\{p_k\}$ . That is, teleportation is possible if, and only if, none of the Schmidt coefficients are greater than  $1/d$ . This also implies that the Schmidt number  $n$  is greater or equal to  $d$ .

The entanglement measure for faithful teleportation,  $E_t(|\chi\rangle_{23})$  (here called entanglement of teleportation), has been defined earlier [8] in the context of *deterministic* entanglement concentration. In [8], it has been shown (with different notations) that  $|\chi\rangle_{23}$  can be transformed (deterministically) by local operations and classical communications (LOCC) to a maximally entangled pair of qudits

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if, and only if, condition (3) is satisfied. This provides a proof for theorem 1, since a maximally entangled pair of qudits can be used for a teleportation of an unknown qudit [1].

In order to partially answer the second question, let us first consider protocols that involves two steps: Alice and Bob concentrate their entangled resource,  $|\chi\rangle_{23}$ , to a  $d \times d$  maximally entangled state and then teleport the state  $|\psi_d\rangle$  (using the Bennett *et al.* [1] protocol). It follows from theorem 2 (see below) that for these protocols the CCC is at least  $\log_2(nd)$  bits.

**Theorem 2:** Let  $n_1$  and  $n_2$  ( $n_1 \geq n_2$ ) be the Schmidt numbers of two bipartite states  $|\chi^{(1)}\rangle_{23}$  and  $|\chi^{(2)}\rangle_{23}$ , respectively. If  $|\chi^{(1)}\rangle_{23}$  can be transformed to  $|\chi^{(2)}\rangle$  by LOCC, then the CCC of the transformation is at least  $\log_2(n_1/n_2)$  bits.

**Proof:** Let us write the states  $|\chi^{(1)}\rangle_{23}$  and  $|\chi^{(2)}\rangle_{23}$  in their Schmidt decomposition

$$\begin{aligned} |\chi^{(1)}\rangle_{23} &= \sum_{k=1}^{n_1} \sqrt{p_k^{(1)}} |k\rangle_2 \otimes |k\rangle_3 \\ |\chi^{(2)}\rangle_{23} &= \sum_{m=1}^{n_2} \sqrt{p_m^{(2)}} |m\rangle_2 \otimes |m\rangle_3. \end{aligned} \quad (4)$$

As we have mentioned earlier, the transformation  $|\chi^{(1)}\rangle_{23} \rightarrow |\chi^{(2)}\rangle_{23}$  can be achieved by a single generalized measurement performed by Alice and a unitary operation performed by Bob (see [9]). Let us describe Alice's measurement by the measurement operators,  $\hat{M}^{(j)}$ , where  $j = 1, 2, \dots, s$ . That is,

$$\sum_{j=1}^s \hat{M}^{(j)\dagger} \hat{M}^{(j)} = I, \quad (5)$$

where  $I$  is the identity operator. Now, after Alice obtain the outcome  $j$ , the state of the system is proportional to

$$\hat{M}^{(j)} |\chi^{(1)}\rangle_{23} = \sum_{k=1}^{n_1} \sqrt{p_k^{(1)}} \left( \hat{M}^{(j)} |k\rangle_2 \right) \otimes |k\rangle_3. \quad (6)$$

Thus, after Bob perform a unitary operation,  $\hat{u}^{(j)}$ , the state of the system,  $|\chi^{(2)}\rangle_{23}$ , can be written as

$$|\chi^{(2)}\rangle_{23} = (N^j)^{-1/2} \sum_{k=1}^{n_1} \sqrt{p_k^{(1)}} \left( \hat{M}^{(j)} |k\rangle_2 \right) \otimes \left( \hat{u}^{(j)} |k\rangle_3 \right), \quad (7)$$

where  $N^j$  is the normalization coefficient. By a comparison of the above equation with the expression for  $|\chi^{(2)}\rangle_{23}$  in Eq. (4) we obtain

$$\hat{M}^{(j)} |k\rangle_2 = \sqrt{\frac{N^j}{p_k^{(1)}}} \sum_{m=1}^{n_2} {}_3\langle k | \hat{u}^{(j)\dagger} | m \rangle_3 \sqrt{p_m^{(2)}} |m\rangle_2. \quad (8)$$

That is, the operator  $\hat{M}^{(j)}$  (as well as  $\hat{M}^{(j)\dagger} \hat{M}^{(j)}$ ) projects the  $n_1$  states  $|k\rangle_2$ , into a  $n_2$  dimensional Hilbert

space. Thus, from the completeness equation (5) it follows that  $sn_2 \geq n_1$ , or equivalently,  $\log_2 s \geq \log_2(n_1/n_2)$   $\square$ .

Note that according to theorem 2, if Alice and Bob first transform the state  $|\chi\rangle_{23}$  to a  $d \times d$  maximally entangled state it will cost them at least  $\log_2(n/d)$  classical bits. Adding to it  $2 \log_2 d$  bits (see Bennett *et al.* [1]) will give a total of at least  $\log_2(nd)$  classical bits for the concentrate and teleport strategy. The CCC of our protocol (see the next section) is exactly  $\log_2(nd)$  bits. Therefore, in this sense our protocol is optimal. This, however, does not mean that there are no other strategies in which the CCC is less than  $\log_2(nd)$  bits.

For example, consider the case in which the entanglement resource shared between Alice and Bob is given by a product of two bell states, i.e.

$$\begin{aligned} |\chi\rangle_{23} &= |\text{Bell}\rangle_{23} |\text{Bell}\rangle_{23} \\ &\equiv \frac{1}{2} (|1, 1\rangle_{23} + |2, 2\rangle_{23} + |3, 3\rangle_{23} + |4, 4\rangle_{23}). \end{aligned} \quad (9)$$

If Alice wishes to teleport a qubit to Bob, she can do it with only two classical bits using one of the two Bell states. In this case, after the teleportation, there is a residual entanglement left. This simple example implies that the minimum amount of classical information that Alice must transmit to Bob, is depending on the residual entanglement left after the teleportation has been accomplished.

Let us denote by  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23})$  the *maximum* Schmidt entanglement (i.e. a logarithm of the Schmidt number) which can remain after a  $d$ -state has been faithfully teleported from Alice to Bob via  $|\chi\rangle_{23}$ . Note that if  $|\chi\rangle_{23}$  is a  $d$ -maximally entangled state, then  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = 0$ . However, there are many  $n$ -partially entangled states ( $n \geq d$ ) for which  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = 0$ . In particular, for  $d > n/2$  the residual entanglement,  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23})$ , must be zero.

The argument goes as follows: after the teleportation, the final state of Alice and Bob systems can be written in the form

$$|\text{final}\rangle_{123} = |\text{RE}\rangle_{12b_1} |\psi_d\rangle_{b_2}, \quad (10)$$

where  $b_1$  is the part of Bob's system 3 that is entangled with Alice systems 1 and 2. Therefore, the state  $|\text{RE}\rangle_{12b_1}$  represents the residual entanglement. The system  $b_2$  is the non-entangled part consisting of the teleported state in Bob's system 3. Let us now denote the Schmidt number of  $|\text{RE}\rangle_{12b_1}$  by  $n_s$ . Since the dimension of  $b_1$  is at least  $n_s$  and the dimension of  $b_2$  is at least  $d$ , the dimension of Bob's system  $n \geq n_s d$ . It is therefore clear that if  $d > n/2$  then  $n_s = 1$  (i.e. zero entanglement). Moreover,

$$\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = -\log_2 n_s \leq \log_2 n - \log_2 d. \quad (11)$$

Let us show now that if  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = 0$ , the lower bound on the amount of classical bits that Alice must send to Bob is given by  $\log_2(nd)$ .

Imagine teleporting a (full Schmidt number) entangled state corresponding to the system 0-1. Alice has the system 1, and 0 is the reference system. Alice and Bob share an entangled state,  $|\chi\rangle_{23}$ , corresponding to the system 2-3. Since Alice wants to teleport her state perfectly, she must completely destroy the entanglement with the reference system 0. Thus, if we assume  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = 0$ , she also needs to destroy all entanglement with Bob's system 3. The dimension of the system 1-2 is  $nd$ , so to disentangle it from 0-3 requires a measurement with at least  $nd$  linearly independent elements, i.e.  $\log_2(nd)$  classical bits.

When  $n = d$ , our bound reduces to  $2\log_2 d$ , which has been proposed in [1] when the teleportation of a  $d$ -dimensional state is performed with a  $d$ -maximally entangled state (i.e. with a Schmidt number  $d$ ). For  $n > d$  the bound is stronger assuming there is no residual entanglement left. This means, that if Alice and Bob have to use *all* of their entanglement resource in order to teleport the qudit, Alice will need to send at least  $\log_2(nd)$  of classical bits. On the other hand, in the example above (see Eq. (9))  $n = 4$ , and therefore  $\log_2(nd) = 3$ . That is, after Alice transmitted the two classical bits to Bob, if she wishes also to destroy the residual entanglement she will need to perform one more measurement (that is equivalent to one more classical bit).

If  $d > n/2$ ,  $\mathcal{E}_r^{(d)}(|\chi\rangle_{23}) = 0$ , and therefore Alice will need to transmit Bob at least  $\log_2(nd)$  bits of classical information. This result is very interesting. It shows, for example, that if Alice and Bob share a  $n$ -maximally entangled state (with  $n < 2d$ ), Alice will have to send Bob *more* classical bits then she would have to if they shared a  $d$ -maximally entangled state. This simple example emphasizes that an increment in the entanglement of the resource will not necessarily reduce the amount of classical bits that are indispensable for a faithful teleportation of a qudit, but will more likely increase it.

Let us end this section, by showing how the lower bound of  $\log_2(nd)$  classical bits leads to another bound on the minimal amount of classical communication that is required for the process of *deterministic* entanglement concentration [8] (for the original asymptotic entanglement concentration see [11]). In this process, Alice and Bob share a  $d^n$ -dimensional state  $|\psi\rangle_{AB}^{\otimes n}$ , where  $|\psi\rangle_{AB}$  is a partially entangled state with a Schmidt number  $d \equiv \text{Sch}(|\psi\rangle_{AB})$ . Suppose that by LOCC Alice and Bob transform the state into  $m$ -copies of the Bell states. From [8], it follows that this transformation is possible if, and only if,

$$m \leq nE_t(|\psi\rangle_{AB}). \quad (12)$$

Therefore, if this condition is satisfied, after the transformation, the  $m$  copies of the Bell states could be used to teleport a  $2^m$ -dimensional state. Let us denote by  $C_1$  the minimum amount of classical bits that are required for the transformation  $|\psi\rangle_{AB}^{\otimes n} \rightarrow |\text{Bell}\rangle^{\otimes m}$ , and by  $C_2$  the amount that is required for the teleportation. Using the Bennett *et al.* protocol, we find that

$C_2 = 2\log_2 2^m = 2m$ . Now, since there is no residual entanglement left in this process, from our bound, it follows that  $C_1 + C_2 \geq \log_2(2^m d^n)$  and thus

$$C_1 \geq n\log_2 d - m \equiv nE_{Sch}(|\psi\rangle_{AB}) - m, \quad (13)$$

where  $E_{Sch}(|\psi\rangle_{AB}) = \log_2 \text{Sch}(|\psi\rangle_{AB})$  is the Schmidt entanglement. From Eq. (12) it follows that the minimum bound is

$$C_1 \geq n(E_{Sch}(|\psi\rangle_{AB}) - E_t(|\psi\rangle_{AB})). \quad (14)$$

Note that  $E_{Sch}(|\psi\rangle_{AB}) \geq E_t(|\psi\rangle_{AB})$  with equality if, and only if,  $|\psi\rangle_{AB}$  is a maximally entangled state.

### A general protocol for faithful teleportation

Let us now present a general protocol for teleportation of a qudit with maximum fidelity ( $f = 1$ ). The protocol consists of a projective local measurement performed by Alice and a subsequent unitary local operation performed by Bob. The protocol is a general one, in the sense that Alice teleports a qudit to Bob via  $\log_2(nd)$  classical bits and *any* partially entangled pair of pure  $n$ -state particles that satisfy Eq. (3) and Eq. (31) (see below).

The protocol presented below involves  $(nd)^2$  coefficients,  $V_{mk}^{(j)}$  (where  $j = 1, 2, \dots, nd$ ,  $m = 1, 2, \dots, d$  and  $k = 1, 2, \dots, n$ ), that satisfy the following two conditions:

$$\delta_{j'j} = \sum_{m=1}^d \sum_{k=1}^n V_{mk}^{(j)*} V_{mk}^{(j')} \quad (15)$$

$$\delta_{m,m'} = nd \sum_{k=1}^n p_k V_{m'k}^{(j)*} V_{mk}^{(j)}. \quad (16)$$

As we will see later, such coefficients can be found in many cases. We write now the steps of the protocol in terms of these coefficients:

(1) The initial state is:

$$|I\rangle_{123} \equiv |\psi\rangle_1 |\chi\rangle_{23}. \quad (17)$$

(2) Alice performs a joint *projective* measurement on systems 1 and 2; the corresponding projectors  $P^{(j)} \equiv |M^{(j)}\rangle_{12} \langle M^{(j)}|$  ( $j = 1, 2, \dots, nd$ ) are given in terms of the coefficients  $V_{mk}^{(j)}$ :

$$|M^{(j)}\rangle_{12} = \sum_{m=1}^d \sum_{k=1}^n V_{mk}^{(j)} |m\rangle_1 |k\rangle_2. \quad (18)$$

Note that Eq. (15) guaranties that the  $nd$  states  $|M^{(j)}\rangle_{12}$  are orthonormal.

(3) The state of the system after Alice obtained the measurement  $j$  (up to normalization):

$$\begin{aligned} P^{(j)} |I\rangle_{123} &= \sum_{m=1}^d \sum_{k=1}^n a_m \sqrt{p_k} V_{mk}^{(j)*} |M^{(j)}\rangle_{12} |k\rangle_3 \\ &\equiv \frac{1}{\sqrt{s}} \sum_{m=1}^d a_m |M^{(j)}\rangle_{12} \otimes \hat{u}^{(j)} |m\rangle_3, \end{aligned} \quad (19)$$

where

$$\hat{u}^{(j)}|m\rangle_3 \equiv \sqrt{s} \sum_{k=1}^n V_{mk}^{(j)*} \sqrt{p_k} |k\rangle_3. \quad (20)$$

Eq. (16) guaranties that  $\hat{u}^{(j)}$  (as defined in the above equation) is a unitary operator; its domain of definition can be extended to *all* the  $n$ -dimensional Hilbert space of Bob ( $\mathcal{H}_3^{(n)}$ ).

(4) After Bob performs on his system 3, the unitary operation,  $\hat{u}^{(j)\dagger}$ , the final (normalized) state is:

$$\begin{aligned} |F\rangle_{123} &= \sum_{m=1}^d a_m |M^{(j)}\rangle_{12} \otimes |m\rangle_3 \\ &= |M^{(j)}\rangle_{12} \otimes |\psi_d\rangle_3, \end{aligned} \quad (21)$$

where the teleported qudit,  $|\psi_d\rangle_3$ , is given by (cf Eq. (1))

$$|\psi_d\rangle_3 = \sum_{k=1}^d a_k |k\rangle_3. \quad (22)$$

(Note that although in the above sum  $k$  runs from 1 to  $d$ ,  $\mathcal{H}_3^{(n)}$  is an  $n$ -dimensional Hilbert space ( $n \geq d$ )). Thus, our protocol works if there are  $(nd)^2$  parameters that satisfy both Eq. (15) and Eq. (16).

Let us first define the  $s^2$  parameters for the case  $d = 2$  and  $n \geq 2$ . This case represents a general faithful teleportation of a qubit. It implies that teleportation of a qubit, if possible, can always be implemented by a projective measurement performed by Alice and a unitary operation performed by Bob. Furthermore, for the case  $n = 2$  we will see below that our protocol reduces to the original one given in [1].

In the determination of the parameters  $V_{mk}^{(j)}$  we will make use of the following notations. First,

$$e_{k,k'} \equiv \exp\left(i \frac{2\pi}{n} k k'\right), \quad (23)$$

where  $k, k' = 1, 2, \dots, n$  (note that  $e_{k,k'}$  is a unitary matrix). Second, we define  $n$  angles  $\theta_1, \theta_2, \dots, \theta_n$  such that

$$\sum_{k=1}^n p_k \exp(i\theta_k) = 0. \quad (24)$$

Such phase factors can always be found when all the  $n$  Schmidt probabilities  $p_k \leq 1/2$  (compare with Eq. (24) in [12]). According to Theorem 1, for  $d = 2$  we have  $E_t(|\chi\rangle_{23}) \geq 1$  and therefore  $p_k \leq 1/2$  for all  $k = 1, 2, \dots, n$ .

With these definitions, the protocol for  $d = 2$  is given by

$$V_{1k}^{(j)} = \frac{1}{\sqrt{s}} e_{j,k} \quad \text{and} \quad V_{2k}^{(j)} = \frac{1}{\sqrt{s}} e_{j,k} \exp(i\theta_k), \quad (25)$$

for  $1 \leq j \leq n$ , and

$$V_{1k}^{(j)} = -\frac{1}{\sqrt{s}} e_{j,k} \exp(-i\theta_k) \quad \text{and} \quad V_{2k}^{(j)} = \frac{1}{\sqrt{s}} e_{j,k}, \quad (26)$$

for  $n < j \leq 2n$ . It can be shown that these  $s^2 = 4n^2$  parameters satisfy both Eqs. (15,16) and thus define a general protocol for faithful teleportation of a qubit.

Consider the case in which  $n = 2$ , and thus  $p_1 = p_2 = 1/2$ . Two angles that satisfy Eq. (24) are  $\theta_1 = 0$  and  $\theta_2 = \pi$ . With this choice, Eqs. (25,26) yields  $V_{11}^{(2)} = V_{11}^{(3)} = V_{12}^{(2)} = V_{21}^{(2)} = V_{21}^{(4)} = V_{22}^{(3)} = V_{22}^{(4)} = 1/2$ , where all the other  $V_{mk}^{(j)} = -1/2$ . The 4 orthonormal measurement states are given by (see Eq. (18))

$$\begin{aligned} |M^{(j)}\rangle_{12} &= V_{11}^{(j)} |1\rangle_1 |1\rangle_2 + V_{12}^{(j)} |1\rangle_1 |2\rangle_2 \\ &\quad + V_{21}^{(j)} |2\rangle_1 |1\rangle_2 + V_{22}^{(j)} |2\rangle_1 |2\rangle_2. \end{aligned} \quad (27)$$

After Bob receives the message  $j$  from Alice's measurement, he performs a unitary operation with matrix elements  $(\hat{u}^{(j)\dagger})_{mk} = \sqrt{2} V_{mk}^{(j)}$ . This protocol is identical to the Bennett *et al.* one [1], if  $|\downarrow_1\rangle, |\uparrow_1\rangle$  in [1] are identified with  $(|1\rangle_1 \pm |2\rangle_1)/\sqrt{2}$  and  $|\downarrow_2\rangle, |\uparrow_2\rangle$  are identified with  $|1\rangle_2, |2\rangle_2$ .

Let us now consider another example, in which the state shared between Alice and Bob is given by

$$|\chi\rangle_{23} = \sqrt{\frac{1}{2}} |1\rangle_1 |1\rangle_2 + \sqrt{\frac{1}{3}} |2\rangle_1 |2\rangle_2 + \sqrt{\frac{1}{6}} |3\rangle_1 |3\rangle_2 \quad (28)$$

According to Theorem 1, this state can be used for a teleportation of a qubit. According to our protocol, there are 6 possible outcomes in the projective measurement performed by Alice. Three angles that satisfy Eq. (24) are  $\theta_1 = 0$  and  $\theta_2 = \theta_3 = \pi$ . Substituting these values for  $\theta_k$  in Eqs. (25,26) gives

$$\begin{aligned} V_{11}^{(j)} &= V_{21}^{(j)} = \frac{1}{\sqrt{6}} \exp\left(\frac{2\pi j}{3}\right) \\ V_{12}^{(j)} &= -V_{22}^{(j)} = \frac{1}{\sqrt{6}} \exp\left(\frac{4\pi j}{3}\right) \\ V_{13}^{(j)} &= -V_{23}^{(j)} = \frac{1}{\sqrt{6}}, \end{aligned} \quad (29)$$

for  $j = 1, 2, 3$ , and for  $j = 4, 5, 6$ ,

$$\begin{aligned} V_{11}^{(j)} &= -V_{21}^{(j)} = -\frac{1}{\sqrt{6}} \exp\left(\frac{2\pi j}{3}\right) \\ V_{12}^{(j)} &= V_{22}^{(j)} = \frac{1}{\sqrt{6}} \exp\left(\frac{4\pi j}{3}\right) \\ V_{13}^{(j)} &= V_{23}^{(j)} = \frac{1}{\sqrt{6}}. \end{aligned} \quad (30)$$

Thus, substitution of the above values in Eq. (18) and Eq. (20) yields the 6 orthonormal states,  $|M^{(j)}\rangle_{12}$ , and the 6 unitary operators,  $\hat{u}^{(j)}$ . This determines the protocol explicitly. We now present the more general scheme with general  $d \geq 2$  and  $n \geq d$ .

We first define  $s = nd$  angles,  $\theta_{mk}$ , such that

$$\sum_{k=1}^n p_k \exp[i(\theta_{mk} - \theta_{m'k})] = \delta_{mm'} \quad (31)$$

(recently, these factors have been used in the construction of general deterministic protocols for dense coding [13]). It can be shown that if such phase factors can be found, then  $p_k \leq 1/d$  for all  $k = 1, 2, \dots, n$ . For  $d = 2$  and  $d = n$  such phase factors can always be found as long as  $p_k \leq 1/d$ . For  $2 < d < n$ , in general, it is not *always* possible to find such phase factors [14], but there are several cases in which one can calculate them explicitly [16]. Now, according to Theorem 1,  $E_t(|\chi\rangle_{23}) \geq \log_2 d$ , and therefore,  $p_k \leq 1/d$  for all  $k = 1, 2, \dots, n$ .

With these notations (and with the assumption that the phase factors in Eq. (31) can be found) the protocol is given by

$$V_{mk}^{(j)} = \frac{1}{\sqrt{s}} \exp(i\theta_{mk}) \exp\left[ij\left(\frac{2\pi}{s}m + \frac{2\pi}{n}k\right)\right]. \quad (32)$$

It can be shown that these  $s^2 = (nd)^2$  parameters satisfy

both Eqs. (15,16) and thus define a protocol for faithful teleportation of a qudit.

In conclusion, we have found lower bounds on the amount of classical information that are required for general faithful teleportation schemes and a deterministic entanglement concentration. We have also found a specific protocol for faithful teleportation of a qudit, which generalizes the protocol given in [1] for the case in which Alice and Bob share a partially entangled resource. The protocol requires no more classical communication than is conceivable with a 'concentrate and teleport' strategy. The next step in this direction would be to find a protocol for teleportation using a *mixed* state entangled resource.

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  - [14] S. Mozes, private communication
  - [15] Here we assume that Alice transmits her entire measurement outcome, or equivalently that Alice and Bob do not discard any part of their state at any time.
  - [16] For example, consider the case in which the set of  $n$  probabilities  $\{p_k\}$  can be divided into  $d$  subgroups such that the sum of the probabilities in each subgroup is  $1/d$ . Then,  $\theta_{mk} = \frac{2\pi}{d}ml$  if  $k$  belong to the subgroup  $l$  ( $l = 1, 2, \dots, d$ ).