SOME APPLICATIONS FOR FADELL-DOLD THEORM IN FIBRATION THEORY BY USING HOMOTOPY GROUPS

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Abstract

The purpose of this paper is to give some solutions for the classification problem in fibration theory by using the homotopy sequences of fibrations (sequences of n-th homotopy groups $\pi_n(S, s_o)$ of total spaces of fibrations). In particular, to show the role of homotopy sequence of n-th homotopy to get the required fiber map in Fadell-Dold theorem such that the restriction of this fiber map on some fiber spaces is a homotopy equivalence.

Keywords: Fibration; homotopy group; homotopy equivalence.

AMS classification: 55P10, 55R55, 55Q05

1 Introduction

The homotopy theory of topological spaces attempts to classify weak homotopy types of spaces and homotopy classes of maps. The classification of maps within a homotopy is a central problem in topology and several authors contributed in this area, see for example the related works in [19].

The concepts of Hurewicz fibrations have played very important roles for investigating the mutual relations of among the objects. For this purpose Coram and Duvall [5] introduced an approximate fibration as a map having the approximate homotopy lifting property for every space, which is a generalization of a Hurewicz fibration having valuable properties similar to the Hurewicz fibration and is widely applicable to the maps whose fibers are nontrivial shapes. Thus it is very essential to examine whether a given decomposition map is an approximate fibration, for exact homotopy sequence that will provide us structural informations about any one object by means of their interrelations with the others, Coram and Duvall [4] gave several characterizations for an approximate fibration.

In [7], Dwyer and Kan followed the simplicial model category and introduced weak equivalences between the objects. Further, Dwyer and Kan in [9] define a notion of equivalence of simplicial localizations by using simplicial sets for the diagrams, which provide an answer to the question that posed by Quillen on the equivalence of homotopy theories in [20]. In fact, the category of simplicial localizations together with this notion of equivalence gives rise to a "homotopy theory of homotopy theory", see [11].

There has also been some further developments leading to classification in the homotopy type of newer manifolds (Wall's manifolds, Milnor manifolds, etc.) which form generators of several different groups of manifolds, see more details [19], [24] and [25] by using the Π -algebras, see

[1].

Recall the problems of classifying Hurewicz fibrations whose fibres have just two non-zero homotopy groups which are very interesting study in homotopy theory. In what follows, S^I will denote the path apace (with the compact-open topology) of a space (Hausdorff space) S, $\Omega(S, s_o)$ will denote the loop space in S^I based a point s_o , \tilde{s} a constant path into $s \in S$, $\overline{\alpha}$ the inverse path of $\alpha \in S^I$, \star the usual path multiplication operation and \simeq the same homotopy type for spaces and homotopic for maps.

Let $f: S \longrightarrow O$ be a fibration with a base O, total space S and fiber space $F_{r_o} = f^{-1}(r_o)$, where $r_o \in O$. A map $L_f: \triangle f \longrightarrow S^I$ is called a *lifting function* for f if $L_f(s,\alpha)(0) = s$ and $f[L_f(s,\alpha)] = \alpha$ for all $(s,\alpha) \in \triangle f$, where $\triangle f = \{(s,\alpha) \in S \times O^I : f(s) = \alpha(0)\}$. If $L_f(s,f\circ \widetilde{s}) = \widetilde{s}$ for all $s \in S$, then the lifting function is called a regular lifting function. A fibration f is called regular fibration if it has regular lifting function.

Recall the Curtis-Hurewicz theorem, [15], which is one of the famous theorems in fibration theory which shows that any map is regular fibration if and only if it has regular lifting function. One of the main problems in fibration theory is a classification problem which is given by:

Under what conditions two fibrations, over a common base, will be fiber homotopy equivalent? Fadell-Dold theorem, [12], is one of the solutions of this problem which clarifies that if the common base O of two fibration $f_1: S_1 \longrightarrow O$ and $f_2: S_2 \longrightarrow O$ is a pathwise connected and an absolute neighborhood retract (ANR), then f_1 and f_2 are fiber homotopy equivalent if and only if there is a fiber map $h: S_1 \longrightarrow S_2$ such that the restriction map of h on $f_1^{-1}(r_o)$ is homotopy equivalence into $f_2^{-1}(r_o)$, for some $r_o \in O$.

In general it is difficult to find the required fiber map of Fadell-Dold theorem in Hurewicz fibration theory. Thus in this paper, we show the role of homotopy sequence of n-th homotopy groups $\pi_n(S, s_o)$ of two fibrations to get this required fiber map in Fadell-Dold theorem such that the restriction of this fiber map on some fiber spaces is a homotopy equivalence. That is, we give some solutions for the classification problem by using the n-th homotopy groups $\pi_n(S, s_o)$ of total spaces of fibrations.

2 Preliminaries

For the n-th homotopy groups $\pi_n(S, s_o)$ and n-th relative homotopy groups $\pi_n(S, A, s_o)$, recall [22] that:

- 1. If $h: S \longrightarrow O$ is a map then for a positive integer n > 0, there is a homomorphism $\widehat{h}: \pi_n(S, s_o) \longrightarrow \pi_n(O, h(s_o))$ defined by $\widehat{h}([\alpha]) = [h \circ \alpha]$. At n = 0, \widehat{h} sends the path-components of S into those of O. \widehat{h} is called a homomorphism induced by h.
- 2. For a positive integer n > 1, there is a homomorphism (called boundary operator) $\partial: \pi_n(S, A, s_o) \longrightarrow \pi_{n-1}(A, s_o)$ defined by $\partial([\alpha]) = [\alpha|_{I^{n-1} \times \{0\}}]$. At n = 1, $\alpha(I^n)$ is a point in A which determines a path-component $C \in \pi_0(S, s_o)$ and $\partial([\alpha]) = C$.

- 3. $\pi_n(S, s_o)$ is isomorphic to $\pi_{n-1}(\Omega(S, s_o), \widetilde{s_o})$, for a positive integer n > 0.
- 4. $\pi_n(S, s_o)$ is abelian group for a positive integer n > 1.

Theorem 2.1. [22] Let $h:(S,s_o) \longrightarrow (O,h(s_o))$ be a homotopy equivalence. Then the induced homomorphisms $\widehat{h}:\pi_n(S,s_o) \longrightarrow \pi_n(O,h(s_o))$ are isomorphisms for a positive integer n>0.

Recall [22] that if h is a fibration, then Theorem 2.1 remains valid. The following theorem is the consequences of Whitehead's [23] and Hurewicz's theorems, see [2].

Theorem 2.2. Let S and O be simply connected spaces which are dominated by ANR's. If there is a map $f: S \longrightarrow O$ induces isomorphism between the n-th homotopy groups of S and O, then f is homotopy equivalence.

Basically, Whitehead's Theorem says that for CW-complexes, if a map $f: X \to Y$ induces an isomorphism on all homotopy groups then it is a homotopy equivalence. But, as the example above shows, you need the map. Such a map is called a weak homotopy equivalence. We note that Whitehead's Theorem is not true for spaces wilder than CW-complexes for example, the Warsaw circle has all of its homotopy groups trivial but the unique map to a point is not a homotopy equivalence.

Theorem 2.3. [3] Let $f: S \longrightarrow O$ be a fibration and a fiber space F_{r_o} be a pathwise connected ANR for some $r_o \in O$. If $\Omega(O, r_o) \simeq ANR$, then $\Omega(S, s_o)$ is dominated by ANR for any $s_o \in F_{r_o}$. If S is a simply connected then $\Omega(S, s_o)$ is of the same homotopy type with ANR space.

The definition of homotopy sequence of a fibration $f: S \longrightarrow O$ is given as follow: Let $s_o \in F_{r_o}$. We can consider f as a map of a triple (S, F_{r_o}, s_o) into a pair (O, r_o) . Then the following sequence is called a *homotopy sequence* of a fibration f:

$$...\pi_{n}(S, s_{o}) \xrightarrow{\widehat{f}} \pi_{n}(O, r_{o}) \xrightarrow{\partial_{\bullet}} \pi_{n-1}(F_{r_{o}}, s_{o}) \xrightarrow{\widehat{i}} \pi_{n-1}(S, s_{o})...$$

$$...\pi_{2}(O, r_{o}) \xrightarrow{\partial_{\bullet}} \pi_{1}(F_{r_{o}}, s_{o}) \xrightarrow{\widehat{i}} \pi_{1}(S, s_{o}) \xrightarrow{\widehat{f}} \pi_{1}(O, r_{o})$$

$$\xrightarrow{\partial_{\bullet}} \pi_{0}(F_{r_{o}}, s_{o}) \xrightarrow{\widehat{i}} \pi_{0}(S, s_{o}),$$

where i, j are inclusion maps and $\partial_{\bullet} = \partial \circ (\widehat{f})^{-1}$. Recall [16] that this sequence is exact, that is, the kernel of each homomorphism is equal to the image of the previous one.

Theorem 2.4. [18] Let $f: S \longrightarrow O$ be a fibration with pathwise connected space O. Then $f^{-1}(r_1)$ and $f^{-1}(r_2)$ are of the same homotopy type for any $r_1, r_2 \in O$.

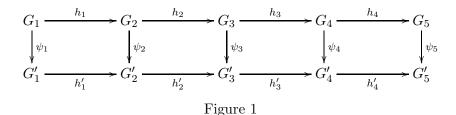
Lemma 2.5. [22] Let $R_i, M_j : S^I \longrightarrow S^I$ be maps, where i, j = 1, 2, 3, 4, which are defined by $R_1(\alpha) = \alpha(0), M_1(\alpha) = \alpha \star \overline{\alpha}, R_2(\alpha) = \alpha(1), M_2(\alpha) = \overline{\alpha} \star \alpha, M_3(\alpha) = \alpha, R_3(\alpha) = \alpha(0) \star \alpha, M_4(\alpha) = \alpha$ and $R_4(\alpha) = \alpha \star \alpha(1)$ for all $\alpha \in S^I$. Then R_1, M_1 are homotopic by homotopy H which has the following property:

$$[H(\alpha, r)](1) = \alpha(0) \quad \text{for } r \in I, \alpha \in S^I, \tag{1}$$

and R_i , M_i , (i = 2, 3, 4), are homotopic by homotopy G_i which has the following property:

$$[G_i(\alpha, r)](1) = \alpha(1) \quad \text{for } r \in I, \alpha \in S^I.$$
 (2)

Lemma 2.6. [14] Consider Figure 1 which involves abelian groups and homomorphisms



such that $\psi_{i+1} \circ h_i = h'_i \circ \psi_i$ for all i = 1, 2, 3, 4. If ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms, then ψ_3 is an isomorphism.

3 Fibrations $\Gamma(f, s_o)$ and $\Sigma(f)$

In this section, we shall introduce the notions of fibrations $\Sigma(f)$ and $\Gamma(f, s_o)$ which are induced by fibration $f: S \longrightarrow O$ with some results about their properties. The functors Γ and Σ are defined as follows:

$$\Gamma(S, F, s_o) = \{ \alpha \in S^I : \alpha(0) = s_o, \ \alpha(1) \in F \}$$

and

$$\Sigma(S, F) = \{ \alpha \in S^I : \alpha(0) \in F, \ \alpha(1) \in F \}$$

for any subspace F of any topological space S, where $s_o \in F$.

Let $f: S \longrightarrow O$ be a fibration with a fiber space F_{r_o} . We will define two fibrations $\Gamma(f, s_o)$ and $\Sigma(f)$ on the functors Γ and Σ , respectively, induced by f as follow: $\Gamma(f, s_o)$ will denote the fibration $\Psi_{s_o}: \Gamma(S, F_{r_o}, s_o) \longrightarrow F_{r_o}$ given by

$$\Psi_{s_o}(\alpha) = \alpha(1)$$
 for $\alpha \in \Gamma(S, F_{r_o}, s_o)$

and we say $\Gamma(f, s_o)$ is a fibration induced by f, which has fiber space $\Psi_{s_o}^{-1}(s_o) = \Omega(S, s_o)$ over a point $s_o \in F_{r_o}$.

 $\Sigma(f)$ will be denote the fibration $\Phi: \Sigma(S, F_{r_o}) \longrightarrow F_{r_o} \times F_{r_o}$ given by

$$\Phi(\alpha) = [\alpha(0), \alpha(1)] \quad \text{for } \alpha \in \Gamma(S, F_{r_o})$$

and we say $\Sigma(f)$ is a fibration induced by f, which has fiber space $\Phi^{-1}[(s_o, s_o)] = \Omega(S, s_o)$ over a point $(s_o, s_o) \in F_{r_o} \times F_{r_o}$.

Lemma 3.1. Let $f: S \longrightarrow O$ be a fibration. Then the maps $D, D_o: \triangle f \longrightarrow S$ defined by

$$D(s,\alpha) = L_f[L_f(s,\alpha)(1), \overline{\alpha}](1)$$
 and $D_o(s,\alpha) = s$,

for all $(s, \alpha) \in \triangle f$, are homotopic.

Proof. For $\alpha \in O^I$ and $r \in I$, define paths α_r, α'_r and α''_r in O by

$$\alpha_r(t) = \alpha(rt), \quad \alpha'_r(t) = \alpha[r + (1-r)t] \quad \text{and} \quad \alpha''_r(t) = \alpha[2r(1-t)],$$

for all $t \in I$. Define two homotopies $H : \triangle f \times I \longrightarrow S$ by

$$H[(s,\alpha),t] = L_f[L_f(s,\alpha_t)(1),\alpha_t'](1)$$
 for $t \in I, (s,\alpha) \in \triangle f$,

and a homotopy $G: O^I \times I \longrightarrow O^I$ by

$$[G(\alpha, r)](t) = \begin{cases} \alpha_r(t) & \text{for } 0 \le t \le 1/2, \\ \alpha_r''(t) & \text{for } 1/2 \le t \le 1, \end{cases}$$

for all $\alpha \in O^I, r \in I$. Hence define a homotopy $F: \triangle f \times I \longrightarrow S$ by

$$F[(s,\alpha),t] = H[(s,G(\alpha,t)),1/2]$$
 for $t \in I, (s,\alpha) \in \triangle f$.

By the regularity for L_f we observe that for $(s, \alpha) \in \triangle f$,

$$F[(s,\alpha),1] = H[(s,G(\alpha,1)),1/2] = H[(s,\alpha\star\overline{\alpha}),1/2]$$

$$= L_f\{K[s,(\alpha\star\overline{\alpha})_{1/2}],(\alpha\star\overline{\alpha})_{1/2}'\}(1)$$

$$= L_f[K(s,\alpha),\overline{\alpha}](1)$$

$$= L_f[L_f(s,\alpha)(1),\overline{\alpha}](1)$$

$$= D(s,\alpha)$$

for all $(s, \alpha) \in \triangle f$, and

$$\begin{split} F[(s,\alpha),0] &= H[(s,G(\alpha)),0](1/2) &= H[(s,\widetilde{\alpha(0)}),1/2] \\ &= L_f\{K[s,\widetilde{\alpha(0)}_{1/2}],\widetilde{\alpha(0)}_{1/2}'\}(1) \\ &= L_f\{K[s,\widetilde{\alpha(0)}],\widetilde{\alpha(0)}\}(1) \\ &= L_f[L_f(s,f\circ\widetilde{s})(1),f\circ\widetilde{s}](1) \\ &= L_f(s,f\circ\widetilde{s})(1) \\ &= s = D_o(s,\alpha) \end{split}$$

for all $(s, \alpha) \in \triangle f$. Hence D and D_o are homotopic.

In the proof of Lemma above we get that the homotopy F has the following property:

$$f\{F[(s,\alpha),t]\} = \alpha(0) \quad \text{for } (s,\alpha) \in \Delta f. \tag{3}$$

Proposition 3.2. For any fibration $f: S \longrightarrow O$ with fiber space F_{r_o} , the following statements are true:

1.
$$\Sigma(S, F_{r_o}) \simeq \Omega(O, r_o) \times F_{r_o};$$

2.
$$\Gamma(S, F_{r_o}, s_o) \simeq \Omega(O, r_o)$$
 for all $s_o \in F_{r_o}$.

Proof. 1. Define a map $N: \Sigma(S, F_{r_o}) \longrightarrow \Omega(O, r_o) \times F_{r_o}$ by

$$N(\alpha) = [f \circ \alpha, \alpha(0)] \text{ for } \alpha \in \Sigma(S, F_{r_0}),$$

and a map $M: \Omega(O, r_o) \times F_{r_o} \longrightarrow \Sigma(S, F_{r_o})$ by

$$M(\alpha, s) = L_f(s, \alpha)$$
 for $(\alpha, s) \in \Omega(O, r_o) \times F_{r_o}$.

Then we have that

$$(N \circ M)(\alpha, s) = N[L_f(s, \alpha)]$$

$$= \{f[L_f(s, \alpha)], L_f(s, \alpha)(0)\}$$

$$= (\alpha, s) = id_{\Omega(O, r_o) \times F_{r_o}}(\alpha, s)$$

for all $(\alpha, s) \in \Omega(O, r_o) \times F_{r_o}$. That is, $N \circ M = id_{\Omega(O, r_o) \times F_{r_o}}$. By Lemma 3.1, we have that the composition map $M \circ N : \Sigma(S, F_{r_o}) \longrightarrow \Sigma(S, F_{r_o})$ given by

$$(M \circ N)(\alpha) = L_f[\alpha(0), f \circ \alpha]$$
 for $\alpha \in \Sigma(S, F_{r_o})$

is homotopic to the identity map $id_{\Sigma(S,F_{r_o})}$. Therefore

$$\Sigma(S, F_{r_o}) \simeq \Omega(O, r_o) \times F_{r_o}$$

2. Let $s_o \in F_{r_o}$. Define a map $R: \Gamma(S, F_{r_o}, s_o) \longrightarrow \Omega(O, r_o)$ by

$$R(\alpha) = f \circ \alpha \quad \text{for } \alpha \in \Gamma(S, F_{r_0}, s_0),$$

and a map $D: \Omega(O, r_o) \longrightarrow \Gamma(S, F_{r_o}, s_o)$ by

$$D(\alpha) = L_f(s_o, \alpha)$$
 for $\alpha \in \Omega(O, r_o)$.

Then we have

$$(R \circ D)(\alpha) = R(L_f(s_o, \alpha))$$

= $f[L_f(s_o, \alpha)]$
= $\alpha = id_{\Omega(O, r_o)}(\alpha)$

for all $\alpha \in \Omega(O, r_o)$. That is, $R \circ D = id_{\Omega(O, r_o)}$. By Lemma 3.1, we get that the composition map $D \circ R : \Gamma(S, F_{r_o}, s_o) \longrightarrow \Gamma(S, F_{r_o}, s_o)$ given by

$$(D \circ R)(\alpha) = L_f(s_o, f \circ \alpha) = L_f(\alpha(0), f \circ \alpha)$$

for all $\alpha \in \Gamma(S, F_{r_o}, s_o)$ is homotopic to the identity map $id_{\Gamma(S, F_{r_o}, s_o)}$. Therefore

$$\Gamma(S, F_{r_o}, s_o) \simeq \Omega(O, r_o),$$

for all
$$s_o \in F_{r_o}$$
.

There are several fibrations $\Gamma(f, s_o)$ according to the number of points in F_{r_o} . But when we let F_{r_o} pathwise connected, then the set of fiber homotopy equivalence classes of the collection set of all these fibrations will be a single. As it is clear in the following theorem.

Theorem 3.3. Let $f: S \longrightarrow O$ be a fibration with a pathwise connected fiber space F_{r_o} . Then the fibration $\Gamma(f, s_o)$ is determined up to a fiber homotopy equivalence class. That is, $\Gamma(f, s_o)$ and $\Gamma(f, s'_o)$ are fiber homotopy equivalent for all $s_o, s'_o \in F_{r_o}$.

Proof. Let $s_o, s'_o \in F_{r_o}$. Since F_{r_o} is a pathwise connected then there is path $\beta: I \longrightarrow F_{r_o}$ between s_o and s'_o . Now let us to define two fiber maps, we can define the map $h: \Gamma(S, F_{r_o}, s_o) \longrightarrow \Gamma(S, F_{r_o}, s'_o)$ by

$$h(\alpha) = \overline{\beta} \star \alpha \quad \text{for } \alpha \in \Gamma(S, F_{r_0}, s_0),$$

and a map $g: \Gamma(S, F_{r_o}, s'_o) \longrightarrow \Gamma(S, F_{r_o}, s_o)$ by

$$g(\alpha) = \beta \star \alpha \quad \text{for } \alpha \in \Gamma(S, F_{r_o}, s'_o).$$

Then we have

$$\Psi_{s'_o}[h(\alpha)] = (\overline{\beta} \star \alpha)(1) = \alpha(1) = \Psi_{s_o}(\alpha)$$

and

$$\Psi_{s_o}[g(\alpha)] = (\beta \star \alpha)(1) = \alpha(1) = \Psi_{s_o'}(\alpha)$$

for all $\alpha \in \Gamma(S, F_{r_o}, s_o)$. That is, h and g are fiber maps.

Now from Lemma 2.5 and Equations 1, 2, we observe that the composition map $g \circ h$: $\Gamma(S, F_{r_o}, s_o) \longrightarrow \Gamma(S, F_{r_o}, s_o)$ given by

$$(g \circ h)(\alpha) = (\beta \star \overline{\beta}) \star \alpha \text{ for } \alpha \in \Gamma(S, F_{r_o}, s_o)$$

is fiber homotopic to the identity map $id_{\Gamma(S,F_{r_o},s_o)}$ and the composition map $h \circ g: \Gamma(S,F_{r_o},s_o') \longrightarrow \Gamma(S,F_{r_o},s_o')$ given by

$$(h \circ g)(\alpha) = (\overline{\beta} \star \beta) \star \alpha \quad \text{for } \alpha \in \Gamma(S, F_{r_o}, s'_o)$$

is fiber homotopic to the identity map $id_{\Gamma(S,F_{r_o},s'_o)}$. Therefore $\Gamma(f,s_o)$ and $\Gamma(f,s'_o)$ are fiber homotopy equivalent.

Here we give some concepts which will be used in the next sections.

Definition 3.4. Let $f: S \longrightarrow O$ be a fibration with fiber space $F_{r_o} = f^{-1}(r_o)$, where $r_o \in O$. By the Lf-function for fibration f induced by a lifting function L_f we mean a map $\Theta_{L_f}: \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$ which is defined by

$$\Theta_{L_f}(\alpha, s) = L_f(s, \alpha)(1)$$
 for $s \in F_{r_o}, \alpha \in \Omega(O, r_o)$.

Henceforth, we will denote by $[S, f, O, F_{r_o}, \Theta_{L_f}]$ the regular fibration $f: S \longrightarrow O$ with an Lf-function $\Theta_{L_f}: \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$, induced by the lifting function L_f and with a fiber space $F_{r_o} = f^{-1}(r_o)$, where $r_o \in O$.

Definition 3.5. Let $[S, f, O, F_{r_o}, \Theta_{L_f}]$ be a fibration. For $s_o \in F_{r_o}$, the map $R : \Omega(O, r_o) \longrightarrow F_{r_o}$ defined by

$$R(\alpha) = \Theta_{L_f}(\alpha, s_o)$$
 for $\alpha \in \Omega(O, r_o)$

is called an Lf-restriction for the fibration f and we denote it by f^{s_o} .

Example 3.6. The first fibration $P_1: O \times S \longrightarrow O$ has a regular lifting function $L_{P_1}: \triangle P_1 \longrightarrow (O \times S)^I$ defined by

$$L_{P_1}[(b,s),\alpha](t) = (\alpha(t),s)$$
 for $t \in I, [(b,s),\alpha] \in \triangle L_{P_1}$.

Then the Lf-function $\Theta_{L_{P_1}}$ for fibration P_1 induced by L_{P_1} will be given by

$$\Theta_{L_{P_1}}(\alpha, x) = x \text{ for } x \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

The Lf-restriction $P_1^{s_o}$ for the fibration P_1 will be given by

$$P_1^{s_o}(\alpha) = s_o$$
 for all $\alpha \in \Omega(O, r_o)$.

Definition 3.7. Let $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ be two fibrations. The Lf-functions $\Theta_{L_{f_1}}$ and $\Theta_{L_{f_2}}$ are said to be *conjugate* if there is $g \in H(F_{r_o}^1, F_{r_o}^2)$ such that

$$\Theta_{L_{f_1}} \simeq_S \widetilde{\overline{g}} \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O,r_o)} \times g).$$

We say that f_1 and f_2 have conjugate Lf-restrictions if there is $g \in H(F_{r_0}^1, F_{r_0}^2)$ such that

$$f_1^{s_o} \simeq_S \widetilde{\overline{g}} \circ f_2^{g(s_o)}$$

where $s_o \in F_{r_o}$, $H(F_{r_o}^1, F_{r_o}^2)$ is the set of all homotopy equivalences from $F_{r_o}^1$ into $F_{r_o}^2$ and \tilde{g} is the inverse homotopy of g.

If two fibrations have conjugate Lf-functions, they also have conjugate Lf-restrictions.

4 Fibration $\Gamma(f, s_o)$ and Lf-restriction

In this section, we are going to introduce the role of homotopy sequences of fibrations (using Lf-restriction) in satisfying FHE between two fibrations $\Gamma(f_1, s_o)$ and $\Gamma(f_2, s'_o)$ which are induced by two fibrations $[S_1, f_1, O, F^1_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_o}, \Theta_{L_{f_2}}]$ over a common base O.

In the following theorem, we show that for two fibrations f_1 and f_2 with conjugate Lf-restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F_{r_o}^1, F_{r_o}^2)$, there are two fiber maps between two fibrations $\Gamma(f_1, s_o)$ and $\Gamma(f_2, g(s_o))$.

Theorem 4.1. Let $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ be two fibrations with conjugate Lf-restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F_{r_o}^1, F_{r_o}^2)$, where $s_o \in F_{r_o}^1$. Then there are two fiber maps

$$h: \Gamma(S_1, F_{r_o}^1, s_o) \longrightarrow \Gamma(S_2, F_{r_o}^2, g(s_o))$$

and

$$k: \Gamma(S_2, F_{r_o}^2, g(s_o)) \longrightarrow \Gamma(S_1, F_{r_o}^1, (\widetilde{\overline{g}} \circ g)(s_o))$$

over g and \widetilde{g} , respectively. That is, Figure 2 is commutative. Further the groups $\pi_n(S_1, s_o)$ and $\pi_n(S_2, g(s_o))$ are isomorphic for a positive integer n > 1.

$$\Gamma(S_{1}, F_{r_{o}}^{1}, s_{o}) \xrightarrow{h} \Gamma(S_{2}, F_{r_{o}}^{2}, g(s_{o})) \xrightarrow{k} \Gamma(S_{1}, F_{r_{o}}^{1}, (\widetilde{g} \circ g)(s_{o}))$$

$$\downarrow^{\Psi_{s_{o}}} \qquad \qquad \downarrow^{\Psi_{g(s_{o})}} \qquad \qquad \downarrow^{\Psi_{(\widetilde{g} \circ g)(s_{o})}}$$

$$F_{r_{o}}^{1} \xrightarrow{g} F_{r_{o}}^{1} \xrightarrow{\widetilde{g}} F_{r_{o}}^{1}$$

Figure 2

Proof. By hypothesis, $f_1^{s_o} \simeq \frac{\widetilde{g}}{\widetilde{g}} \circ f_2^{g(s_o)}$. This implies $g \circ f_1^{s_o} \simeq f_2^{g(s_o)}$. Then there is a homotopy $G: \Omega(O, r_o) \times I \longrightarrow F_{r_o}^2$ such that

$$G(\alpha, 0) = f_2^{g(s_o)}(\alpha) = \Theta_{L_{f_2}}(\alpha, g(s_o))$$

and

$$G(\alpha, 1) = (g \circ f_1^{s_o})(\alpha) = g[\Theta_{L_{f_1}}(\alpha, s_o)]$$

for all $\alpha \in \Omega(O, r_o)$. For $\alpha \in S_1^I$ and $r \in I$, we can define a path $\alpha_r \in S_1^I$ by

$$\alpha_r(t) = \begin{cases} \alpha(t) & \text{for } 0 \le t \le r, \\ \alpha(r) & \text{for } r \le t \le 1. \end{cases}$$

For $\beta = f_1 \circ \alpha$ and $r \in I$, we can define the path $\beta^{1-r} \in O^I$ by

$$\beta^{1-r}(t) = \begin{cases} \beta(r+t) & \text{for } 0 \le t \le 1-r, \\ \beta(1) & \text{for } 1-r \le t \le 1. \end{cases}$$

Define a homotopy $H: S_1^I \times I \longrightarrow S_1^I$ by

$$[H(\alpha,r)](t) = \left\{ \begin{array}{cc} \alpha_r(t), & \text{for} \quad 0 \leq t \leq r, \\ L_{f_1}(\alpha(r),\beta^{1-r})(t-r), & \text{for} \quad r \leq t \leq 1, \end{array} \right.$$

for all $r \in I, \alpha \in S_1^I$. We get that

$$H(\alpha,0) = L_{f_1}(\alpha(0), f_1 \circ \alpha)$$
 and $H(\alpha,1) = \alpha$ for $\alpha \in S_1^I$.

For $\alpha \in \Gamma(S_1, F_{r_o}^1, s_o)$, let $M(\alpha)$ be a path in $\Gamma(S_1, F_{r_o}^1, s_o)$ defined by

$$M(\alpha)(r) = [H(\alpha,r)](1) \quad \text{for } r \in I.$$

Hence we can define a map $M': \Gamma(S_1, F_{r_o}^1, s_o) \longrightarrow (F_{r_o}^2)^I$ by

$$M'(\alpha) = g[M(\alpha)]$$
 for $\alpha \in \Gamma(S_1, F_{r_o}^1, s_o)$,

and a map $L'_{f_2}: \Gamma(S_1, F^1_{r_o}, s_o) \longrightarrow \Gamma(S_2, F^2_{r_o}, g(s_o))$ by

$$L'_{f_2}(\alpha) = L_{f_2}(g(s_o), f_1 \circ \alpha)$$
 for $\alpha \in \Gamma(S_1, F_{r_o}^1, s_o)$.

Now define the map $h: \Gamma(S_1, F_{r_o}^1, s_o) \longrightarrow \Gamma(S_2, F_{r_o}^2, g(s_o))$ by

$$h(\alpha) = [L'_{f_2}(\alpha) \star G(f_1 \circ \alpha)] \star M'(\alpha) \text{ for } \alpha \in \Gamma(S_1, F^1_{r_o}, s_o).$$

Then h is a well-defined as a continuous map. Since

$$(\Psi_{g(s_o)} \circ h)(\alpha) = \{ [L'_{f_2}(\alpha) \star G(f_1 \circ \alpha)] \star M'(\alpha) \}(1)$$

$$= M'(\alpha)(1) = g[M(\alpha)(1)] = g[\alpha(1)]$$

$$= (g \circ \Psi_{s_o})(\alpha)$$

for all $\alpha \in \Gamma(S_1, F_{r_o}^1, s_o)$. That is, $\Psi_{g(s_o)} \circ h = g \circ \Psi_{s_o}$. Hence h is fiber map over g.

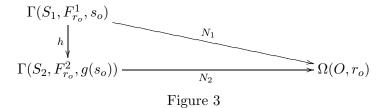
Now we prove that $\pi_n(S_1, s_o)$ and $\pi_n(S_2, g(s_o))$ are isomorphic for a positive integer n > 1. In Figure 3, define the map $N_1 : \Gamma(S_1, F_{r_o}^1, s_o) \longrightarrow \Omega(O, r_o)$ by

$$N_1(\alpha) = f_1 \circ \alpha \quad \text{for } \alpha \in \Gamma(S_1, F_{r_o}^1, s_o),$$

and a map $N_2: \Gamma(S_2, F_{r_o}^2, g(s_o)) \longrightarrow \Omega(O, r_o)$ by

$$N_2(\alpha) = f_2 \circ \alpha$$
 for $\alpha \in \Gamma(S_2, F_{r_o}^2, g(s_o))$.

From the definition of fiber map h, we observe that $[(N_2 \circ h)(\alpha)](t) = r_o$ at t = 1/4 and for



 $t \in [0, 1/4],$

$$[(N_2 \circ h)(\alpha)](t) = [N_2(h(\alpha))](t) = f_2[L'_{f_2}(\alpha)](4t)$$

$$= f_2[L_{f_2}(g(s_o), f_1 \circ \alpha)](4t)$$

$$= (f_1 \circ \alpha)(4t) = [N_1(\alpha)](4t).$$

That is, $(N_2 \circ h)(\alpha) \neq N_1(\alpha)$ concentrated on the interval [0,1] but $(N_2 \circ h)(\alpha) = N_1(\alpha)$ concentrated on the interval [0,1/4]. This implies that $N_2 \circ h \simeq N_1$. Hence Figure 3 is not commutative in the usual sense but it is a homotopy commutative. That is, Figure 4 is a commutative i.e.,

$$\pi_{n}(\Gamma(S_{1}, F_{r_{o}}^{1}, s_{o}), \widetilde{s_{o}}) \xrightarrow{\widehat{N}_{1}} \pi_{n}(\Gamma(S_{2}, F_{r_{o}}^{2}, g(s_{o})), \widetilde{g(s_{o})}) \xrightarrow{\widehat{N}_{2}} \pi_{n}(\Omega(O, r_{o}), \widetilde{r_{o}})$$
Figure 4

$$\widehat{N}_2 \circ \widehat{h} = \widehat{N}_1. \tag{4}$$

By the part 2 in Proposition 3.2, the maps N_1 and N_2 are homotopy equivalences of $\Gamma(S_1, F_{r_o}^1, s_o)$ into $\Omega(O, r_o)$ and of $\Gamma(S_2, F_{r_o}^2, g(s_o))$ into $\Omega(O, r_o)$, respectively. Hence

$$\widehat{N}_1: \pi_n(\Gamma(S_1, F_{r_o}^1, s_o), \widetilde{s_o}) \longrightarrow \pi_n(\Omega(O, r_o), \widetilde{r_o})$$

and

$$\widehat{N}_2: \pi_n(\Gamma(S_2, F_{r_o}^2, g(s_o)), \widetilde{g(s_o)}) \longrightarrow \pi_n(\Omega(O, r_o), \widetilde{r_o})$$

are isomorphisms for a positive integer n > 0. By Equation 4, we get that

$$\widehat{h}: \pi_n(\Gamma(S_1, F_{r_o}^1, s_o)) \longrightarrow \pi_n(\Gamma(S_2, F_{r_o}^2, g(s_o)))$$

is an isomorphism for a positive integer n > 0. Consider

$$h_o: \Omega(S_1, s_o) \longrightarrow \Omega(S_2, g(s_o))$$

is a restriction of h on $\Psi_{s_o}^{-1}(s_o) = \Omega(S_1, s_o)$. Now we can integrate the homotopy sequences of fibrations Ψ_{s_o} and $\Psi_{g(s_o)}$ in Figure 5,

where j_1, j_2 are inclusion maps, ∂_1 and ∂_2 are boundary operators and

$$(\partial_{1})_{\bullet} = \partial_{1} \circ (\widehat{\Psi_{so}})^{-1},$$

$$(\partial_{2})_{\bullet} = \partial_{2} \circ (\widehat{\Psi_{g(so)}})^{-1},$$

$$\pi_{n}(\Gamma(S_{1})) = \pi_{n}(\Gamma(S_{1}, F_{r_{o}}^{1}, s_{o}), \widetilde{s_{o}}),$$

$$\pi_{n}(\Gamma(S_{2})) = \pi_{n}(\Gamma(S_{2}, F_{r_{c}}^{2}, g(s_{o})), \widetilde{g(s_{o})}),$$

for a positive integer n > 0.

In Figure 5, we observe that $\widehat{j_1}$, $\widehat{j_2}$, $(\partial_1)_{\bullet}$, $(\partial_2)_{\bullet}$, $\widehat{\Psi_{s_o}}$ and $\widehat{\Psi_{g(s_o)}}$ are homomorphisms. Since \widehat{h} and \widehat{g} are isomorphisms, then Lemma 2.6 shows that for a positive integer n > 0,

$$\widehat{h_o}: \pi_n(\Omega(S_1, s_o), \widetilde{s_o}) \longrightarrow \pi_n(\Omega(S_2, g(s_o)), \widetilde{g(s_o)})$$

is an isomorphism. Since $\pi_{n+1}(S_1, s_o)$ is isomorphic to $\pi_n(\Omega(S_1, s_o), \widetilde{s_o})$ and $\pi_{n+1}(S_2, g(s_o))$ is isomorphic to $\pi_n(\Omega(S_2, g(s_o)), g(s_o))$ for a positive integer n > 0, then $\pi_n(S_1, s_o)$ is isomorphic to $\pi_n(S_2, g(s_o))$ for a positive integer n > 1.

Finally, since $g \circ f_1^{s_o} \simeq f_2^{g(s_o)} \Longrightarrow g \circ f_1^{(\widetilde{g} \circ g)(s_o)} \simeq f_2^{g(s_o)}$, then similarly, there is a fiber map k satisfying above requirement properties for h.

Corollary 4.2. If two fibrations $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ have conjugate Lf-functions by $g \in H(F_{r_o}^1, F_{r_o}^2)$, then Theorem 4.1 holds for any $s_o \in F_{r_o}^1$.

Proof. It is clear that if two fibrations f_1 and f_2 have conjugate Lf-functions by $g \in H(F_{r_o}^1, F_{r_o}^2)$, then they have conjugate Lf-restrictions f^{s_o} and $f^{g(s_o)}$ by $g \in H(F_{r_o}^1, F_{r_o}^2)$, for any $s_o \in F_{r_o}^1$. Hence Theorem 4.1 holds for any $s_o \in F_{r_o}^1$.

We explain in the following corollary that if S_1 , S_2 are simply connected in Theorem 4.1, then two loop spaces $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are of the same homotopy type.

Corollary 4.3. Let $[S_1, f_1, O, F^1_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F^2_{r_o}, \Theta_{L_{f_2}}]$ be fibrations with conjugate Lf-restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by $g \in H(F^1_{r_o}, F^2_{r_o})$ and S_1, S_2 be simply connected spaces. Let $\Omega(O, r_o) \simeq ANR$. If $F^1_{r_o}$ and $F^2_{r_o}$ are pathwise connected and ANR's, then

$$\Omega(S_1, s_o) \simeq \Omega(S_2, g(s_o)).$$

Proof. Since S_1 and S_2 are simply connected, then it is clear that $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are pathwise connected. Since $\Omega(O, r_o) \simeq ANR$ then Theorem 2.3 shows that $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are dominated by ANR's. By Theorem 4.1, there is a map

$$h_o: \Omega(S_1, s_o) \longrightarrow \Omega(S_2, g(s_o))$$

induces isomorphisms between the homotopy groups. Hence by Theorem 2.2, h_o is homotopy equivalence.

In the next step, we employ Theorem 4.1 to satisfy FHE relation for fibrations $\Gamma(f,s_o)$. Figure 2 in Theorem 4.1 suggests that perhaps in some sense there is FHE relation between $\Gamma(f_1,s_o)$ and $\Gamma(f_2,g(s_o))$. But the notion of the FHE relation applied to fibrations having a common base. One might try fibering $\Gamma(S_1,F^1_{r_o},s_o)$ over $F^1_{r_o}$ using the map $g\circ\Psi_{s_o}$ but in general this will not give rise to fibering since it might happen that $g(F^1_{r_o})=g(s_o)$ and in this case $g\circ\Psi_{s_o}$ would not be onto if $F^2_{r_o}$ consisted of more than one point. Hence we give the restrictions such as $F^1_{r_o}=F^2_{r_o}=F_{r_o}$ and $g:F_{r_o}\longrightarrow F_{r_o}$ is a homeomorphism map.

Let $[S, f, O, F_{r_o}, \Theta_{L_f}]$ be a fibration and g be a homeomorphism map of O onto a topological semigroup O'. Then the composition $g \circ f : S \longrightarrow O'$ is also a fibration denoted by $[f]_g$.

Theorem 4.4. Let $[S_1, f_1, O, F_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}, \Theta_{L_{f_2}}]$ be fibrations with conjugate Lf-restrictions $f_1^{s_o}$ and $f_2^{g(s_o)}$ by a homeomorphism $g \in H(F_{r_o}, F_{r_o})$, where $s_o \in F_{r_o}$, and F_{r_o} be a pathwise connected ANR. If S_1 and S_2 are simply connected and $\Omega(O, r_o) \simeq ANR$, then $[\Gamma(f_1, s_o)]_g$ and $\Gamma(f_2, g(s_o))$ are fiber homotopy equivalent.

Proof. By Theorem 4.1, in Figure 6, there is a fiber map

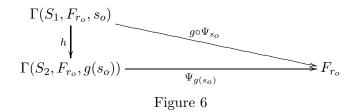
$$h: \Gamma(S_1, F_{r_o}, s_o) \longrightarrow \Gamma(S_2, F_{r_o}, g(s_o)).$$

Let

$$A_1 =: (g \circ \Psi_{s_o})^{-1}(s_o) = \{\alpha \in S_1^I : \alpha(0) = s_o, \ \alpha(1) = g^{-1}(s_o)\},$$

and

$$A_2 =: \Psi_{g(s_o)}^{-1}(s_o) = \{ \alpha \in S_2^I : \alpha(0) = g(s_o), \ \alpha(1) = s_o \}.$$



We observe that A_1 and A_2 are fiber spaces for fibrations $[\Gamma(f_1, s_o)]_g$ and $\Gamma(f_2, g(s_o))$ over s_o , respectively. Hence homotopy sequences for the fibrations $[\Gamma(f_1, s_o)]_g$ and $\Gamma(f_2, g(s_o))$ in Theorem 4.1 show that $h|_{A_1}: A_1 \longrightarrow A_2$ induces isomorphisms between $\pi_n(A_1)$ and $\pi_n(A_2)$ for a positive integer n > 0.

Now since S_1 and S_2 are simply connected spaces, then $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are pathwise connected. Since $\Omega(O, r_o) \simeq ANR$, then by Theorem 2.3, $\Omega(S_1, s_o)$ and $\Omega(S_2, g(s_o))$ are dominated by ANR's. Also these loop spaces are fiber spaces for the fibrations $\Gamma(f_1, s_o)$ and $\Gamma(f_2, g(s_o))$ over s_o , respectively. Since F_{r_o} is a pathwise connected and by Theorem 2.4, all fiber spaces are of the same homotopy type, then A_1 and A_2 are pathwise connected and dominated by ANR's. Since $h|_{A_1}: A_1 \longrightarrow A_2$ induces isomorphisms between $\pi_n(A_1)$ and $\pi_n(A_2)$ for a positive integer n > 0, then by Theorem 2.2, $h|_{A_1}: A_1 \longrightarrow A_2$ is a homotopy equivalence. Therefore since F_{r_o} is pathwise connected ANR, then by Fadell-Dold theorem, we get that $[\Gamma(f_1, s_o)]_g$ and $\Gamma(f_2, g(s_o))$ are fiber homotopy equivalent.

Corollary 4.5. Let $[S, f, O, F_{r_o}, \Theta_{L_f}]$ be a fibration with simply connected ANR fiber space F_{r_o} and with simply connected base O such that $\Omega(O, r_o) \simeq ANR$. If there is a map $k : O \longrightarrow S$ such that $f \circ k = id_O$, then $\Gamma(f, k(r_o))$ and $\Gamma(P_1, (r_o, k(r_o)))$ are fiber homotopy equivalent, where $P_1 : O \times F_{r_o} \longrightarrow O$ is the first fibration.

Proof. Since $k(r_o) \in S$ and $f \circ k = id_O$, then

$$f^{k(r_o)}(\alpha) = \Theta_{L_f}(\alpha, k(r_o)) = L_f(k(r_o), \alpha)(1)$$
$$= L_f[k(r_o), f \circ (k \circ \alpha)](1)$$

for all $\alpha \in \Omega(O, r_o)$. We observe easily that the Lf-restriction $f^{k(r_o)}$ is homotopic to the map $L: \Omega(O, r_o) \longrightarrow F_{r_o}$ which is defined by

$$\Omega(\alpha) = (k \circ \alpha)(1) = k(r_o)$$
 for $\alpha \in \Omega(O, r_o)$

by using the form of homotopy H in the proof of Theorem 4.1. Consider F_{r_o} as fiber space for P_1 because there is a homeomorphism map between it and $P_1^{-1}(r_o) = \{r_o\} \times F_{r_o}$. Now from Example 3.6, we get that fibration P_1 has Lf-restriction $P_1^{(r_o,k(r_o))}: \Omega(O,r_o) \longrightarrow F_{r_o}$ given by

$$P_1^{(r_o,k(r_o))}(\alpha) = k(r_o)$$
 for $\alpha \in \Omega(O,r_o)$.

Hence $f^{k(r_o)} \simeq L = P_1^{k(r_o)}$, that is, fibrations f and P_1 have conjugate Lf-restrictions $f^{k(r_o)}$ and $P_1^{(r_o,k(r_o))}$ by a homeomorphism $g = id_{F_{r_o}} \in H(F_{r_o},F_{r_o})$. Hence by theorem above, $\Gamma(f,k(r_o))$ and $\Gamma(P_1,(r_o,k(r_o)))$ are fiber homotopy equivalent.

5 Fibration $\Sigma(f)$ and Lf-function

Here, we will introduce the role of homotopy sequences of fibrations (using Lf-function) in satisfying FHE between two fibrations $\Sigma(f_1)$ and $\Sigma(f_2)$ which are induced by two fibrations $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ over a common base O.

In the following theorem, we show that for two fibrations f_1 and f_2 with conjugate Lf-functions, there are two fiber maps between two fibrations $\Sigma(f_1)$ and $\Gamma(f_2)$.

Theorem 5.1. Let $[S_1, f_1, O, F_{r_o}^1, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}^2, \Theta_{L_{f_2}}]$ be fibrations with conjugate Lf-functions by $g \in H(F_{r_o}^1, F_{r_o}^2)$. Then there are two fiber maps

$$D: \Sigma(S_1, F_{r_o}^1) \longrightarrow \Sigma(S_2, F_{r_o}^2)$$
 and $R: \Sigma(S_2, F_{r_o}^2) \longrightarrow \Sigma(S_1, F_{r_o}^1)$

over $g \times g$ and $\widetilde{\overline{g}} \times \widetilde{\overline{g}}$, respectively. That is, Figure 7 is a commutative

$$\Sigma(S_{1}, F_{r_{o}}^{1}) \xrightarrow{D} \Sigma(S_{2}, F_{r_{o}}^{2}) \xrightarrow{R} \Sigma(S_{1}, F_{r_{o}}^{1})$$

$$\downarrow^{\Phi_{1}} \qquad \downarrow^{\Phi_{2}} \qquad \downarrow^{\Phi_{1}}$$

$$F_{r_{o}}^{1} \times F_{r_{o}}^{1} \xrightarrow{g \times g} F_{r_{o}}^{2} \times F_{r_{o}}^{2} \xrightarrow{\tilde{g} \times \tilde{g}} F_{r_{o}}^{1} \times F_{r_{o}}^{1}$$

Proof. Firstly, we will define fiber map D. By the hypothesis we get that

$$\Theta_{L_{f_1}} \simeq \widetilde{\overline{g}} \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O,r_o)} \times g).$$

Figure 7

This implies

$$g \circ \Theta_{L_{f_1}} \simeq \Theta_{L_{f_2}} \circ (id_{\Omega(O,r_o)} \times g).$$

Hence there is a homotopy $T: \Omega(O, r_o) \times F_{r_o}^1 \longrightarrow (F_{r_o}^2)^I$ such that

$$\begin{array}{lcl} T(\alpha,s)(0) & = & [\Theta_{L_{f_2}} \circ (id_{\Omega(O,r_o)} \times g)](\alpha,s) \\ & = & \Theta_{L_{f_2}}(\alpha,g(s)) \end{array}$$

and

$$T(\alpha, s)(1) = [g \circ \Theta_{L_{f_1}}](\alpha, s)$$
$$= g[\Theta_{L_{f_1}}(\alpha, s)]$$

for all $\alpha \in \Omega(O, r_o), s \in F^1_{r_o}$. Define a map $L''_{f_2} : \Sigma(S_1, F^1_{r_o}) \longrightarrow \Sigma(S_2, F^2_{r_o})$ by

$$L_{f_2}''(\alpha) = L_{f_2}(g(\alpha(0)), f_1 \circ \alpha)$$
 for $\alpha \in \Sigma(S_1, F_{r_o}^1),$

and for $\alpha \in \Sigma(S_1, F_{r_o}^1)$, we can use the homotopy H (which is defined in the proof of Theorem 4.1) to define the path $W(\alpha) \in (F_{r_o}^2)^I$ by

$$W(\alpha)(t) = g\{[H(\alpha, t)](1)\}$$
 for $t \in I$.

Now we can define a map $D: \Sigma(S_1, F_{r_o}^1) \longrightarrow \Sigma(S_2, F_{r_o}^2)$ by

$$D(\alpha) = [L_{f_2}''(\alpha) \star T(f_1 \circ \alpha)] \star W(\alpha) \quad \text{for } \alpha \in \Sigma(S_1, F_{r_2}^1). \tag{5}$$

Hence it is clear that D is well defined as a continuous map. We get that

$$\begin{split} [\Phi_2 \circ D](\alpha) &= [D(\alpha)(0), D(\alpha)(1)] &= [L''_{f_2}(\alpha)(0), W(\alpha)(1)] \\ &= [g(\alpha(0)), g(\alpha(1))] \\ &= (g \times g)(\alpha(0), \alpha(1)) \\ &= [(g \times g) \times \Phi_1](\alpha) \end{split}$$

for all $\alpha \in \Sigma(S_1, F_{r_o}^1)$. That is, D is a fiber map over $g \times g$. Secondly, we can find a fiber map R by above similar manner.

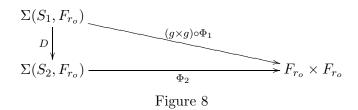
In the proof of Theorem 5.1, the two fiber maps have properties:

$$D|_{\Omega(S_1,s_o)} = h_o$$
 and $R|_{\Omega(S_2,g(s_o))} = k_o$,

where h_o and k_o are defined in Theorem 4.1 and $s_o \in F_{r_o}^1$. In proof of Theorem 4.1, it is clear that the map G is a restriction of a map T on $\Omega(O, r_o)$, the map L'_{f_2} is a restriction of a map L'_{f_2} on $\Gamma(S_1, F_{r_o}^1, s_o)$, and the map M' is a restriction of a map M on $\Gamma(S_1, F_{r_o}^1, s_o)$. Hence from Equations 4 and 5, we get that the map M is a restriction of a map M on M on

Also we introduce theorem about the functor Σ which is similar of Theorem 4.4.

Theorem 5.2. Let $[S_1, f_1, O, F_{r_o}, \Theta_{L_{f_1}}]$ and $[S_2, f_2, O, F_{r_o}, \Theta_{L_{f_2}}]$ be fibrations with conjugate Lf-functions by a homeomorphism $g \in H(F_{r_o}, F_{r_o})$, where $s_o \in F_{r_o}$, and F_{r_o} be a common pathwise connected ANR. If S_1 , S_2 are simply connected and $\Omega(O, r_o) \simeq ANR$, then $[\Sigma(f_1)]_{g \times g}$ and $\Sigma(f_2)$ are fiber homotopy equivalent.



Proof. By Theorem above, there is a fiber map $D: \Sigma(S_1, F_{r_o}) \longrightarrow \Sigma(S_2, F_{r_o})$ in Figure 8. Let $B_1 = [(g \times g) \circ \Phi_1]^{-1}(s_o, s_o)$ and $B_2 = \Phi_2^{-1}(s_o, s_o))$, then

$$B_1 = [(g \times g) \circ \Phi_1]^{-1}(s_o, s_o)$$

$$= \Phi_1^{-1}[(g^{-1} \times g^{-1})(s_o, s_o)]$$

$$= \{\alpha \in S_1^I : \alpha(0) = g^{-1}(s_o), \ \alpha(1) = g^{-1}(s_o)\}$$

$$= \Omega(S_1, g^{-1}(s_o)),$$

and

$$B_2 = \Phi_2^{-1}(s_o, s_o) = \{ \alpha \in S_2^I : \alpha(0) = s_o, \ \alpha(1) = s_o \} = \Omega(S_2, s_o).$$

We observe that B_1 and B_2 are fiber spaces for two fibrations $[\Sigma(f_1)]_{g\times g}$ and $\Sigma(f_2)$ over (s_o, s_o) , respectively. Hence homotopy sequences for two fibrations $[\Sigma(f_1)]_{g\times g}$ and $\Sigma(f_2)$ in Theorem 4.1 show that $D|_{B_1}: B_1 \longrightarrow B_2$ induces isomorphisms between $\pi_n(B_1)$ and $\pi_n(B_2)$ for a positive integer n>0. Since S_1 and S_2 are simply connected spaces, then B_1 and B_2 are pathwise $S_{\mathcal{N}_i}$ -connected. Since $\Omega(O, r_o) \simeq ANR$, then by Theorem 2.3, we get that B_1 and B_2 are dominated by ANR's. And since $D|_{B_1}: B_1 \longrightarrow B_2$ induces isomorphisms between $\pi_n(B_1)$ and $\pi_n(B_2)$ for a positive integer n>0, then by Theorem 2.2, $D|_{B_1}: B_1 \longrightarrow B_2$ is a homotopy equivalence. Hence since $F_{r_o} \times F_{r_o}$ is pathwise connected ANR, then by Fadell-Dold theorem, we get that $[\Sigma(f_1)]_{g\times g}$ and $\Sigma(f_2)$ are fiber homotopy equivalent.

Conclusion: Further we also prove some theorems related to fiber homotopy equivalent classes by using the fiber homotopy sequences of homotopy groups. Thus we show the role of these fiber homotopy sequences in order to get the required fiber map in Fadell-Dold theorem. Further, the possible practical use of our theorems as applications will provide some solutions for the classification problem in Hurewicz fibration theory by using Fadell-Dold theorem.

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