

Divide-and-Conquer: Polynomial Multiplication

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Data Structures and Algorithms
Algorithmic Toolbox

Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

Multiplying Polynomials

Example

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$

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$$B(x) = 5x^2 + x + 2$$

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$

$$B(x) = 5x^2 + x + 2$$

$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Multiplying polynomials

Input: Two $n - 1$ degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$$

Output:

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Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

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Multiplying Polynomials

Example

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$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Output: $C = (15, 13, 33, 9, 10)$

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MultPoly(A, B, n)

```
pair  $\leftarrow$  Array[ $n$ ][ $n$ ]  
for  $i$  from 0 to  $n - 1$ :  
  for  $j$  from 0 to  $n - 1$ :  
    pair[ $i$ ][ $j$ ]  $\leftarrow$   $A[i] * B[j]$ 
```

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product  $\leftarrow$  Array[ $2n - 1$ ]  
for  $i$  from 0 to  $2n - 1$ :  
    product[ $i$ ]  $\leftarrow$  0
```

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for  $i$  from 0 to  $2n - 1$ :  
    product[ $i$ ]  $\leftarrow$  0  
for  $i$  from 0 to  $n - 1$ :  
    for  $j$  from 0 to  $n - 1$ :  
        product[ $i + j$ ]  $\leftarrow$  product[ $i + j$ ] + pair[ $i$ ][ $j$ ]
```

MultPoly(A, B, n)

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        product[ $i + j$ ]  $\leftarrow$  product[ $i + j$ ] + pair[ $i$ ][ $j$ ]  
return product
```


Multiplying Polynomials

Naïve Solution: $O(n^2)$

- Multiply all $a_i b_j$ pairs (n^2 multiplications)

Multiplying Polynomials

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- Multiply all $a_i b_j$ pairs (n^2 multiplications)
- Sum needed pairs (n^2 additions)

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- Let $A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$ where
$$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \dots + a_{\frac{n}{2}}$$
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- Let $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$ where
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- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$
$$= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$$

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- Calculate D_1E_1 , D_1E_0 , D_0E_1 , and D_0E_0

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- Calculate D_1E_1 , D_1E_0 , D_0E_1 , and D_0E_0

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

Polynomial Mult: Divide & Conquer

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

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$$D_1E_1 = 4x^2 + 11x + 6$$

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$$(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$$

$$6x^2 + 11x + 4$$

$$= 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4$$

Function Mult2(A, B, n, a_l, b_l)

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$R[0..n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l)$

Function Mult2(A, B, n, a_l, b_l)

$R = \text{array}[0..2n - 2]$

if $n = 1$:

$R[0] = A[a_l] * B[b_l]$; return R

$R[0..n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

Function Mult2(A, B, n, a_l, b_l)

$R = \text{array}[0..2n - 2]$

if $n = 1$:

$R[0] = A[a_l] * B[b_l]$; return R

$R[0..n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

$D_0E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

Function Mult2(A, B, n, a_l, b_l)

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if $n = 1$:

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$D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

$D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2} \dots n + \frac{n}{2} - 2] += D_1 E_0 + D_0 E_1$

Function Mult2(A, B, n, a_l, b_l)

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$D_0E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

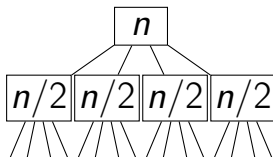
$D_1E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2} \dots n + \frac{n}{2} - 2] += D_1E_0 + D_0E_1$

return R

$$n$$

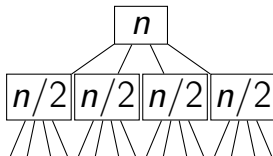
level



level

0

1



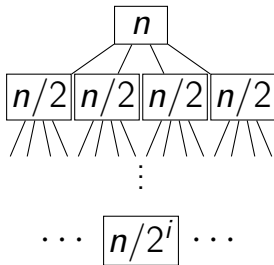
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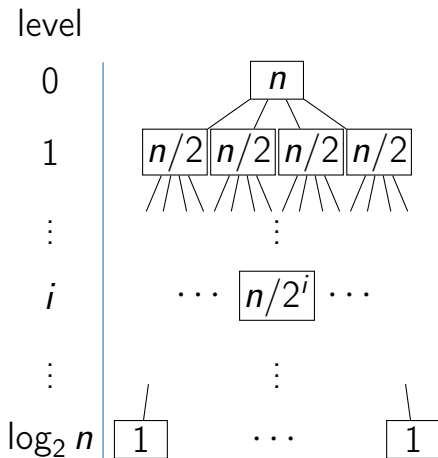
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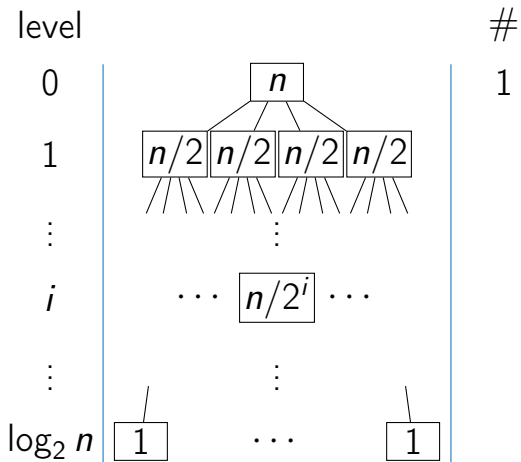
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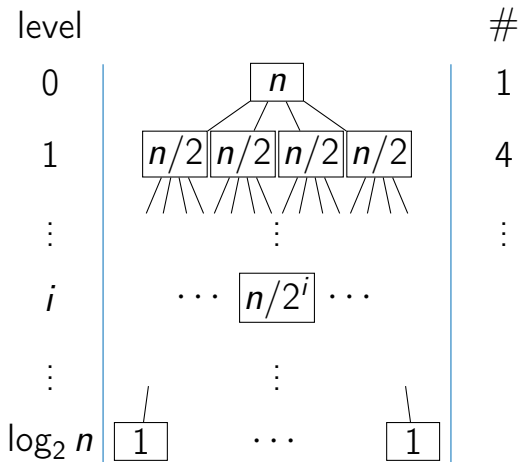
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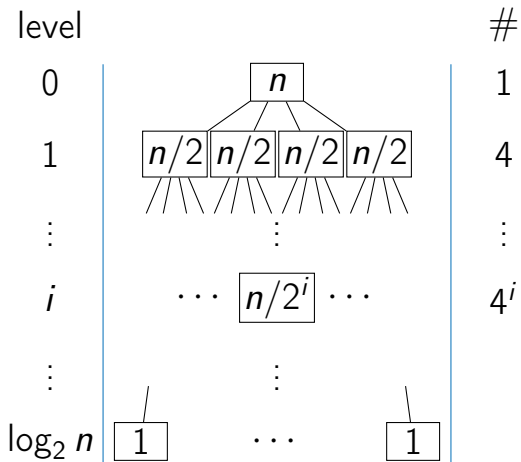
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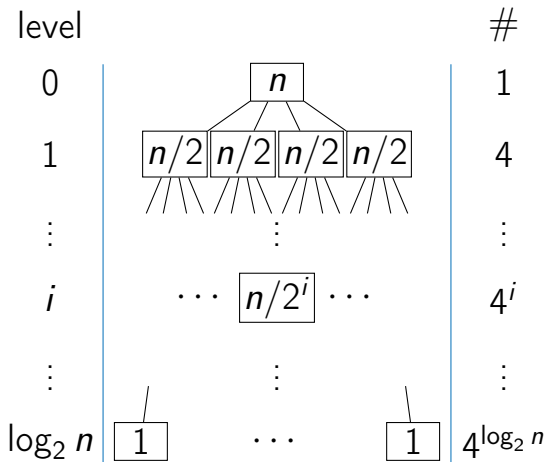


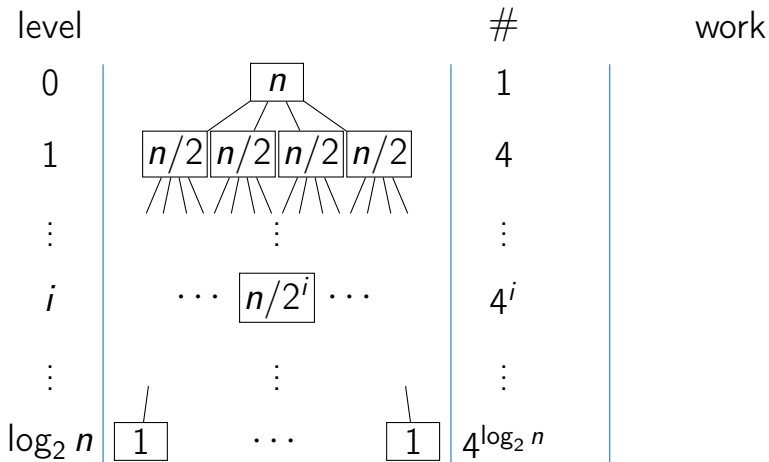


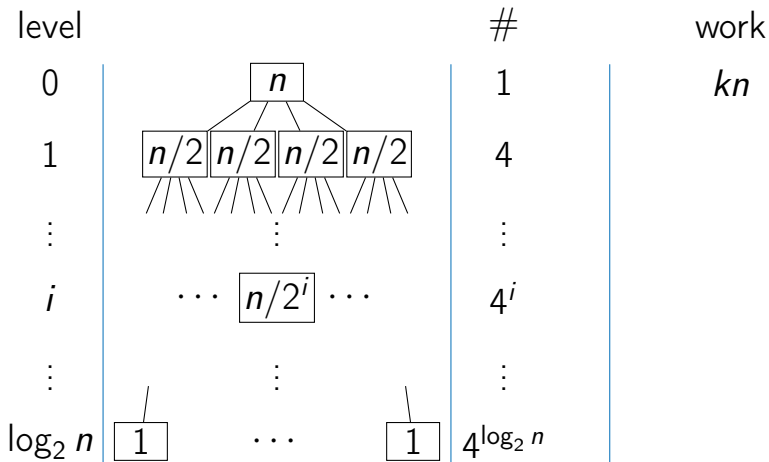


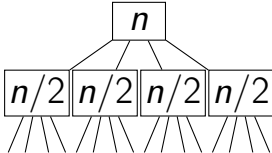



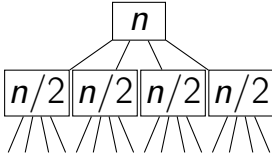



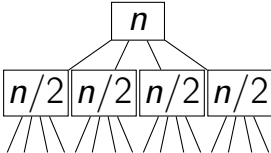



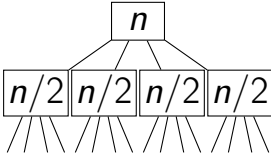




level		#	work
0		1	kn
1		4	$4k\frac{n}{2} = k2n$
\vdots		\vdots	
i	$\dots \boxed{n/2^i} \dots$	4^i	
\vdots		\vdots	
$\log_2 n$		$4^{\log_2 n}$	

level		#	work
0		1	kn
1		4	$4k\frac{n}{2} = k2n$
\vdots		\vdots	\vdots
i	$\dots \boxed{n/2^i} \dots$	4^i	$4^i k\frac{n}{2^i} = k2^i n$
\vdots		\vdots	
$\log_2 n$		$4^{\log_2 n}$	

level		#	work
0		1	kn
1		4	$4k\frac{n}{2} = k2n$
\vdots		\vdots	\vdots
i	$\dots \boxed{n/2^i} \dots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
\vdots		\vdots	\vdots
$\log_2 n$		$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

level		#	work
0		1	kn
1		4	$4k\frac{n}{2} = k2n$
\vdots		\vdots	\vdots
i	$\dots \boxed{n/2^i} \dots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
\vdots		\vdots	\vdots
$\log_2 n$	$\boxed{1} \dots \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

Total: $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

Outline

- ① Problem Overview
- ② Naïve Algorithm
- ③ Naïve Divide and Conquer Algorithm
- ④ Faster Divide and Conquer

Karatsuba approach

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$$A(x) = a_1x + a_0$$

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$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

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Needs 4 multiplications

Karatsuba approach

$$A(x) = a_1x + a_0$$

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Needs 4 multiplications

Rewrite as:

Karatsuba approach

$$A(x) = a_1x + a_0$$

$$B(x) = b_1x + b_0$$

$$C(x) = a_1b_1x^2 + (\textcolor{red}{a_1b_0} + \textcolor{red}{a_0b_1})x + a_0b_0$$

Needs 4 multiplications

Rewrite as:

$$\begin{aligned} C(x) = & a_1b_1x^2 + \\ & ((\textcolor{red}{a_1} + \textcolor{red}{a_0})(\textcolor{red}{b_1} + \textcolor{red}{b_0}) - \textcolor{red}{a_1b_1} - \textcolor{red}{a_0b_0})x + \\ & a_0b_0 \end{aligned}$$

Karatsuba approach

$$A(x) = a_1x + a_0$$

$$B(x) = b_1x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1b_1x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x + a_0b_0$$

Needs 3 multiplications

Karatsuba approach

$$A(x) = a_1x + a_0$$

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Rewrite as:

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Needs 3 multiplications

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$D_1E_1 = 4x^2 + 11x + 6$$

Karatsuba Example

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$$(D_1 + D_0)(E_1 + E_0) =$$

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$$D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)$$

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$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= 24x^2 + 52x + 24$$

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$$E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB &= (4x^2 + 11x + 6)x^4 + \\ &\quad (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ &\quad - (6x^2 + 11x + 4))x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$$

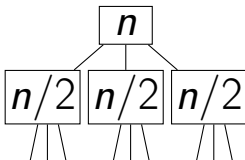
$$- (6x^2 + 11x + 4))x^2 +$$

$$6x^2 + 11x + 4$$

$$= 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4$$

n

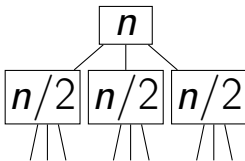
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level

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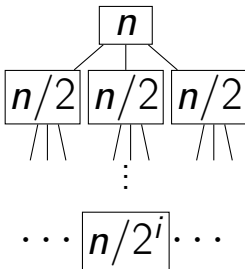
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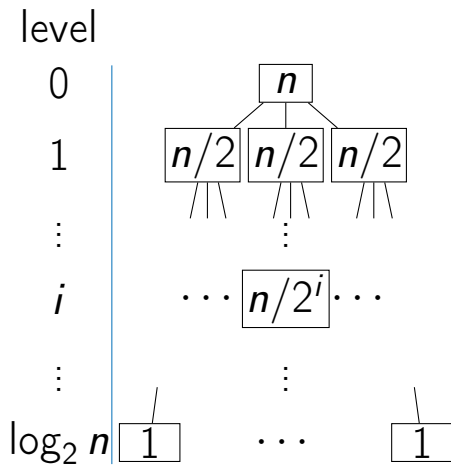
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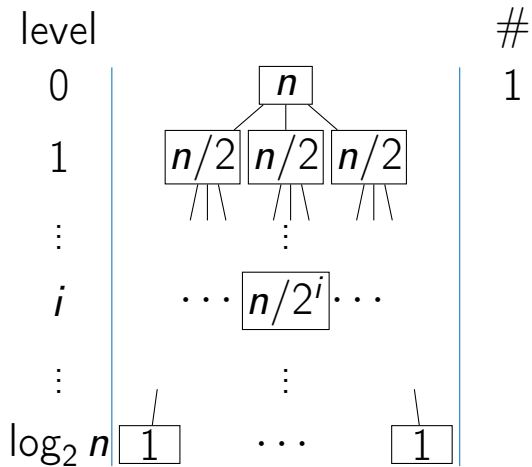
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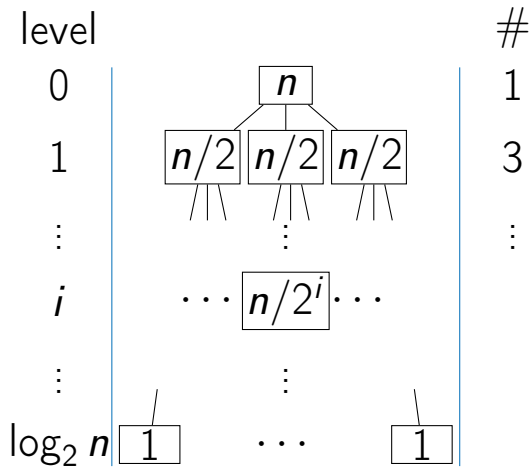
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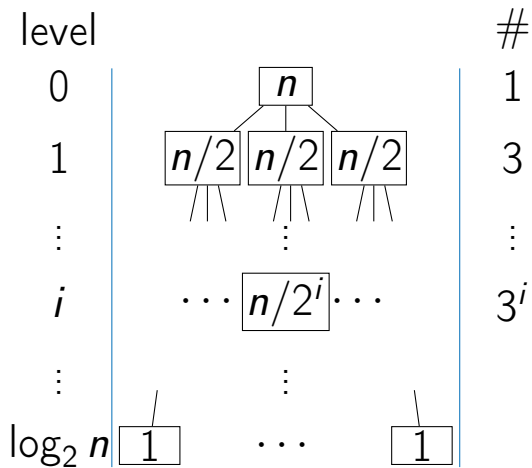
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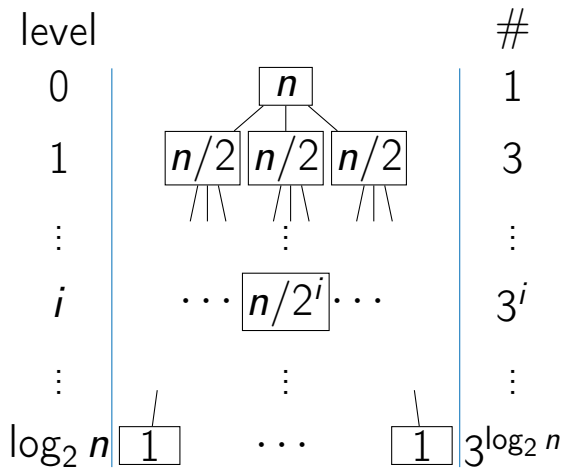


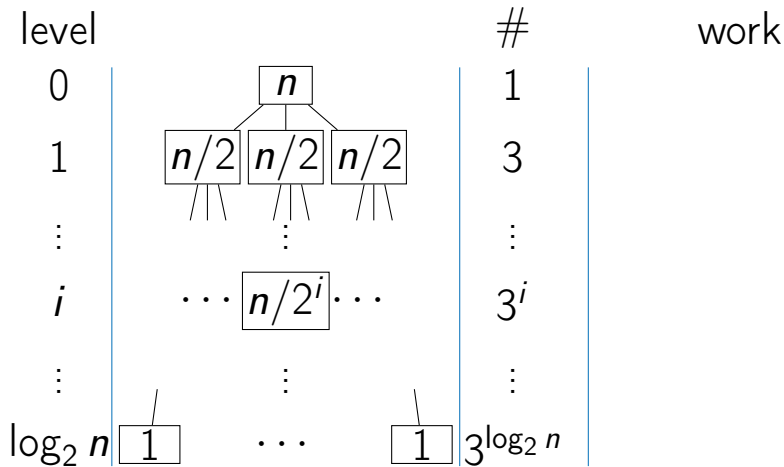


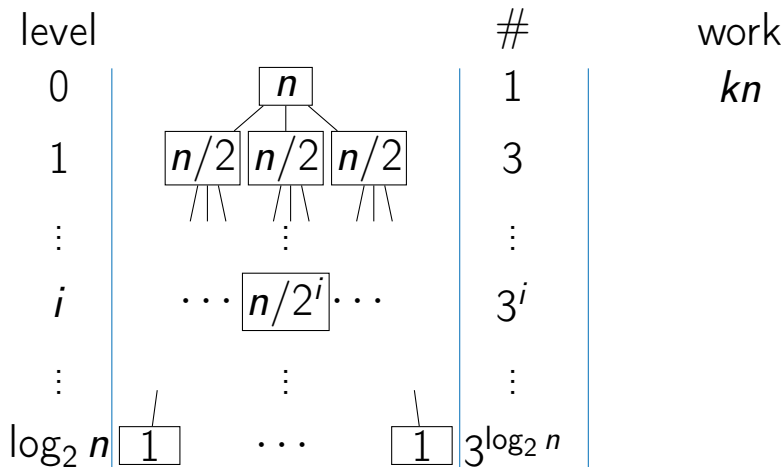


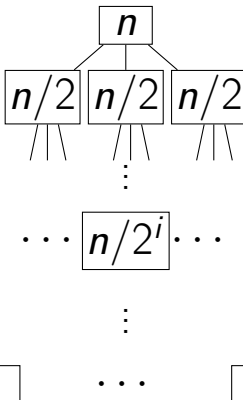



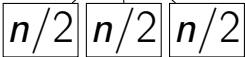





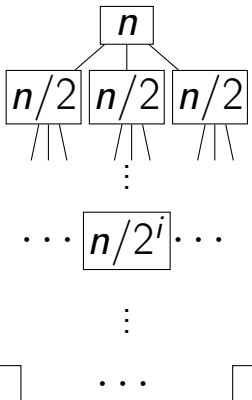



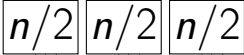







level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
\vdots		\vdots	
i		3^i	
\vdots		\vdots	
$\log_2 n$		$3^{\log_2 n}$	

level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
\vdots	\vdots	\vdots	\vdots
i	\dots  \dots	3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
\vdots	\vdots	\vdots	
$\log_2 n$	 \dots 	$3^{\log_2 n}$	

level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
\vdots		\vdots	\vdots
i		3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
\vdots		\vdots	\vdots
$\log_2 n$		$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
\vdots	\vdots	\vdots	\vdots
i	\dots  \dots	3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
\vdots	\vdots	\vdots	\vdots
$\log_2 n$	 \dots 	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

$$\begin{aligned}
 \text{Total: } \sum_{i=0}^{\log_2 n} 3^i k \frac{n}{2^i} &= \Theta(n^{\log_2 3}) \\
 &= \Theta(n^{1.58})
 \end{aligned}$$