Intro: Fibonacci Numbers

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Data Structures and Algorithms
Algorithmic Toolbox

Learning Objectives

- Understand the definition of the Fibonacci numbers.
- Show that the naive algorithm for computing them is slow.
- Efficiently compute large Fibonacci numbers.

Outline

1. Problem Overview
2. Naive Algorithm
3. Efficient Algorithm

Definition

$$F_n = \begin{cases} 0, & n = 0\,, \\ 1, & n = 1\,, \\ F_{n-1} + F_{n-2}, & n > 1\,. \end{cases}$$

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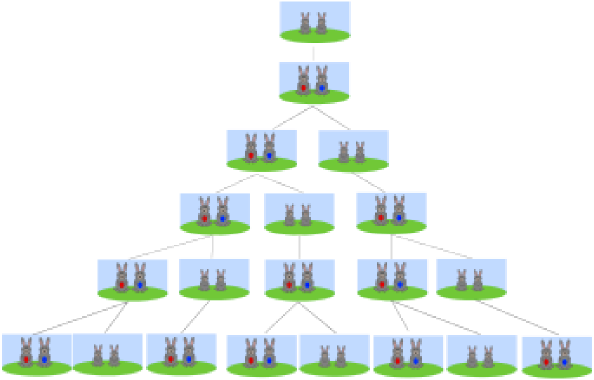
$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Developed to Study Rabbit Populations



Rapid Growth

Lemma

$F_n \geq 2^{n/2}$ for $n \geq 6$.

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By induction

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By induction

Base case: $n = 6, 7$ (by direct computation).

Inductive step:

$$F_n = F_{n-1} + F_{n-2} \geq 2^{(n-1)/2} + 2^{(n-2)/2} \geq 2 \cdot 2^{(n-2)/2} = 2^{n/2}. \qquad \square$$

Formula

Theorem

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Example

$$F_{20} \; = \; 6765$$

Example

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\begin{aligned}
F_{20} &= 6765 \\
F_{50} &= 12586269025
\end{aligned}
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F_{100} &= 354224848179261915075
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F_{100} &= 354224848179261915075 \\
F_{500} &= 139423224561697880139724382 8\\
&\ 7040728395007025658769730726 \\
&\ 4108962948325571622863290691 \\
&\ 557658876222521294125
\end{aligned}
$$

Computing Fibonacci numbers

Compute F_n

Input: An integer $n \geq 0$.

Output: F_n.

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Algorithm

FibRecurs(n)

```
if n ≤ 1:
  return n
```

Algorithm

FibRecurs(n)

```
if n ≤ 1:
    return n
else:
    return FibRecurs(n − 1) + FibRecurs(n − 2)
```

Running time

Let $T(n)$ denote the number of lines of code executed by `FibRecurs`(n).

If $n \leq 1$

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```

$T(n) = 2$.

If $n \geq 2$

FibRecurs(n)

```
if n ≤ 1:
  return n
else:
  return FibRecurs(n − 1) + FibRecurs(n − 2)
```

If $n \geq 2$

FibRecurs(n)

```
if n ≤ 1:
  return n
else:
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```

$T(n) = 3$

If $n \geq 2$

```
FibRecurs(n)

if n ≤ 1:
  return n
else:
  return FibRecurs(n − 1) + FibRecurs(n − 2)
```

$$T(n) = 3 + T(n-1) + T(n-2).$$

Running Time

$$T(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + 3 & \text{else.} \end{cases}$$

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$T(100) \approx 1.77 \cdot 10^{21}$ (1.77 sextillion)

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Therefore $T(n) \geq F_n$

$T(100) \approx 1.77 \cdot 10^{21}$ (1.77 sextillion)

Takes 56,000 years at 1GHz.

Why so slow?

$$F_n$$

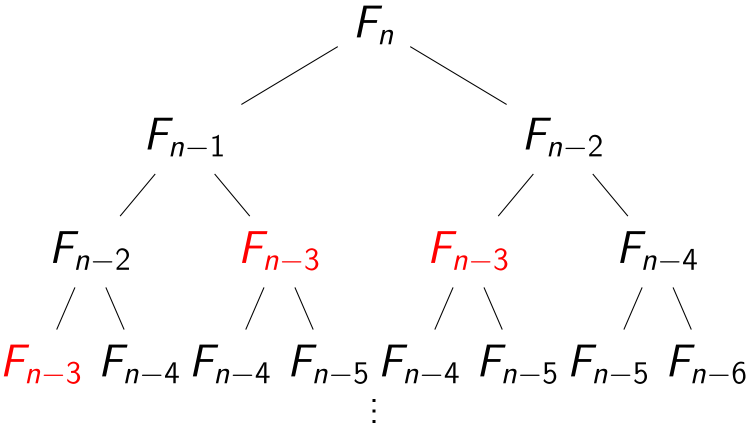
$$F_{n-1} \qquad\qquad F_{n-2}$$

$$F_{n-2} \qquad F_{n-3} \qquad F_{n-3} \qquad F_{n-4}$$

$$F_{n-3} \; F_{n-4} \; F_{n-4} \; F_{n-5} \; F_{n-4} \; F_{n-5} \; F_{n-5} \; F_{n-6}$$

$$\vdots$$

Why so slow?



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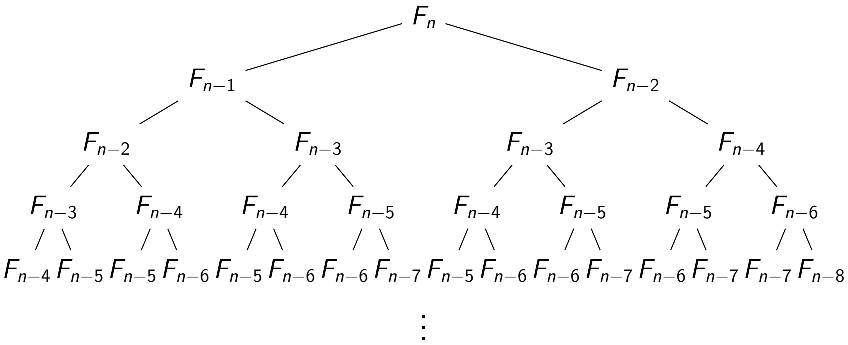
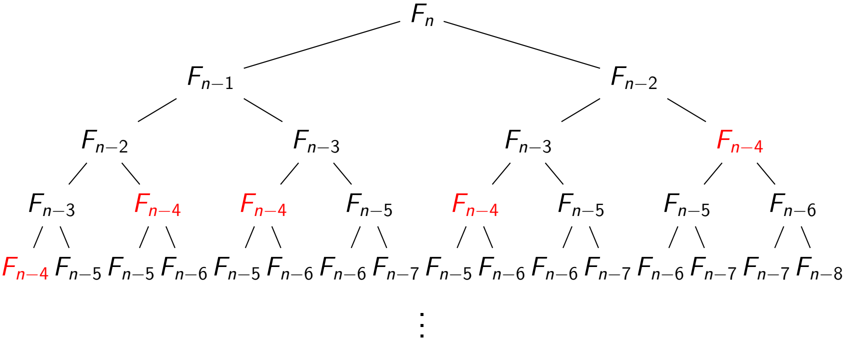


F_n
F_{n-1} F_{n-2}
F_{n-2} F_{n-3} F_{n-3} F_{n-4}
F_{n-3} F_{n-4} F_{n-4} F_{n-5} F_{n-4} F_{n-5} F_{n-5} F_{n-6}
F_{n-4} F_{n-5} F_{n-5} F_{n-6} F_{n-5} F_{n-6} F_{n-6} F_{n-7} F_{n-5} F_{n-6} F_{n-6} F_{n-7} F_{n-6} F_{n-7} F_{n-7} F_{n-8}
$\vdots$

Why so slow?


F_n
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$\vdots$

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Another Algorithm

Imitate hand computation:

0, 1

Another Algorithm

Imitate hand computation:

0, 1, 1

$0 + 1 = 1$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2

$$0 + 1 = 1$$
$$1 + 1 = 2$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3

$0 + 1 = 1$

$1 + 1 = 2$

$1 + 2 = 3$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3, 5

$$0 + 1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 = 3$$
$$2 + 3 = 5$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3, 5, 8

$0 + 1 = 1$

$1 + 1 = 2$

$1 + 2 = 3$

$2 + 3 = 5$

$3 + 5 = 8$

New Algorithm

FibList(n)

create an array $F[0 \ldots n]$
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for i from 2 to n:
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

New Algorithm

FibList(n)

```
create an array F[0...n]
F[0] ← 0
F[1] ← 1
for i from 2 to n:
    F[i] ← F[i − 1] + F[i − 2]
return F[n]
```

- $T(n) = 2n + 2$. So $T(100) = 202$.
- Easy to compute.

Summary

- Introduced Fibonacci numbers.
- Naive algorithm takes ridiculously long time on small examples.
- Improved algorithm incredibly fast.

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Moral: The right algorithm makes all the difference.