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We show that under suitable assumptions, Poincaré recurrences of a dynamical system determine its topology in phase space. Therefore, dynamical systems with the same recurrences are dynamically equivalent. This conclusion can be drawn from a theorem proved in this paper which states that the recurrence matrix determines the topology of closed sets. The theorem states that if a set of points M is mapped onto another set N, such that two points in N are closer than some prescribed fixed distance if and only if the corresponding points in M are closer than some, in general different, prescribed fixed distance, then both sets are homeomorphic, i.e., identical up to a continuous change in the coordinate system. The theorem justifies a range of methods in nonlinear dynamics which are based on recurrence properties. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3117151]

The theory of dynamical systems has revolutionized our understanding of natural systems. It has highlighted important concepts and quantities, which characterize the behavior of systems that change in time. The concept of *recurrence*, i.e., the idea that a system recurs to formerly visited states in its phase space, is known to be of fundamental importance for the theory of dynamical systems. It has found numerous applications, especially for the analysis of the experimental data. In this paper we prove a theorem that shows that recurrences are not only an important characteristic of a dynamical system but that—if properly used—they determine the dynamics of a system completely. Therefore, they yield an alternative description of a system's dynamics. Our theorem also provides a mathematical foundation for frequently used methods of data analysis, e.g., recurrence plots, surrogate data methods, and synchronization analysis. The theorem, however, is not limited to the realm of dynamical systems. It also is highly relevant for topology. We show that similar to distance-one-preserving maps, transformations, which preserve all distance inequalities, determine a set up to topological equivalence. This justifies methods that are used, e.g., for the reconstruction of the protein structure from NMR measurements.

I. INTRODUCTION

Recurrence is a fundamental characteristic of many dynamical systems. The concept of recurrence was introduced by Poincaré in a seminal work in 1890.[1] Therein, Poincaré not only described the "homoclinic tangle," which lies at the root of the chaotic behavior of orbits, but he also introduced the concept of recurrence. His well-known recurrence theorem states that in any dynamical system with a finite invariant measure almost every trajectory recurs infinitely many times, to any nonvanishing volume, which is visited at least once. Recurrence is a key concept in the theory of dynamical systems and one of the most pervasive characteristics of such systems. In this paper, we prove a theorem that shows that recurrences are not only pervasive but also contain all the information about the dynamics of a system.

In this paper we distinguish between geometrical, topological, and dynamical properties of a system. We first consider a geometrical aspect of a set of points M, e.g., an attractor, namely, the distances of all pairs of points on it. If these distances are smaller than a certain threshold, we will consider these points to be "neighbors." We next prove a theorem that shows that if we map the set M onto another set N by a function ϕ, such that two points are neighbors in N if and only if they are neighbors in M, then the transformation ϕ induces a homeomorphism, i.e., the sets M and N are the same up to a continuous change in the coordinate system. The two sets are then said to be topologically equivalent. Hence, the geometrical distance inequalities, i.e., whether points are further apart than a certain distance or not, determine the topology. Note that as a consequence this means that if the recurrence matrix, which holds all information on the distance inequalities, is identical for two sets, then the sets are topologically equivalent.

We next consider a dynamical system on M, i.e., we define a flow on M. We then map M onto N by ϕ and obtain a topologically equivalent flow on N. If we furthermore recognize that the distance inequalities determine the recurrences of the flows, we may conclude that identical recurrences determine a flow up to a continuous change in the coordinate system.

To state the Poincaré recurrence theorem mathematically, we first introduce the concept of a measure space. Let X be a set. A nonempty collection Σ of subsets of X is a σ-algebra if it (i) contains X, (ii) is closed under complements, and (iii) is closed under countable unions. A measure is a function $\mu:\Sigma\rightarrow\mathbb{R}_+\cup\{\infty\}$ that is σ-additive, i.e., $\mu(\cup_i A_i)=\Sigma_i\mu(A_i)$ if the A_i are pairwise disjoint. Then a *measure space* is a triple (X,Σ,μ) of a set, a σ-algebra, and a measure. Poincaré's recurrence theorem states that if T is a measure preserving transformation from a measure space

(X, Σ, μ) into itself and if $A \subset X$ is a measurable set, then for any $N \in \mathbb{N}$, we find that $\mu(\{x \in A | \{T^n(x)\}_{n \geq N} \subset \mathcal{X} \backslash A\}) = 0$.

Boltzmann gave rough estimates of the recurrence times for a gas which indicated how extremely long one would have to wait for a recurrence to occur in that system. The first mathematically rigorous results were reported by Birkhoff,[2] who proved that the mean recurrence time in an ergodic system is determined by the inverse of the volume of the set.

In the following years, research focused on return times and especially the *first return* of a *set*, which is defined as follows: if $A \subset \mathcal{X}$ is a measurable set of a measurable (probability) dynamical system $\{\mathcal{X}, \Sigma, \mu, T\}$, the first return of the set A is given by

$$\tau(A) = \min\{n > 0 : T^n A \cap A \neq \emptyset\}. \tag{1}$$

Generically, for hyperbolic systems the recurrence times appear to exhibit certain universal properties:[3]

(1) the recurrence time has an exponential limit distribution;
(2) successive recurrence times are independently distributed; and
(3) as a consequence of (1) and (2), the sequence of successive recurrence times has a limit law that is a Poisson distribution.

These properties, which are also well-known characteristics of certain stochastic systems, such as finite aperiodic Markov chains,[4–6] have been rigorously established for deterministic dynamical systems exhibiting sufficiently strong mixing.[7–9] They have also been shown to be valid for a wider class of systems which have to be hyperbolic.[10]

Recently, recurrences and return times have been studied with respect to their statistics[11,12] and have been linked to various other basic characteristics of dynamical systems, such as the Pesin's dimension,[13] the pointwise and local dimensions,[14–16] or the Hausdorff dimension.[17] Also the multifractal properties of return time statistics have been studied.[18,19] Furthermore, it has been shown that recurrences are related to Lyapunov exponents and to various entropies.[20,21] They have also been linked to rates of mixing,[22] and the relationship between the return time statistics of continuous and discrete systems has been investigated.[23]

These very fruitful studies have firmly established recurrences as a fundamental concept for the understanding of dynamical systems. However, recent numerical studies—see Sec. II—have indicated that not only the spectrum of return times but also the "recurrence matrix" might be an interesting object of further mathematical investigation. In the remaining sections of this paper, we discuss this matrix and prove a theorem, which states that this matrix contains all the dynamical information on a dynamical system. This central statement of the paper shows that the recurrence matrix is an alternative description of a dynamical system. This matrix contains information about fractal dimensions, entropies just like the spectrum of recurrence times, but it goes far beyond that. It is a "dynamically complete" description of the system.

II. RECURRENCE MATRIX

Instead of studying the first return time statistics, Eckmann *et al.*[24] introduced the recurrence matrix. For a dynamical system with flow $f: M \to M$ in an n-dimensional phase space $M \in \mathbb{R}^n$, the trajectory $f^i(x_0)$, which is initialized at the point $x_0 \in M$, yields the recurrence matrix

$$R_{i,j} = \Theta(\varepsilon - \|f^i(x_0) - f^j(x_0)\|), \tag{2}$$

where $f_i(x^0)$, $f^j(x_0)$ are the points on the continuous or discrete trajectory of a dynamical system at discrete times i and j. If the Euclidean distance between these two points is smaller than the arbitrary but small threshold distance ε, the Heaviside function $\Theta(\cdot)$ leads to $R_{i,j}=1$. If their distance is larger than ε, we have $R_{i,j}=0$. (This is why we use the term *distance inequalities*. We only know if the distance between two points is larger or smaller than the threshold ε.) Note that we adapt the definition of the Heaviside function for which $\Theta(0)=0$. In principle, the definition $\Theta(0)=1$ would also be possible. Based on these definitions, $R_{i,j}$ is a binary matrix, which contains information about the recurrences of the dynamical system. This means that the trajectory of the dynamical system is mapped to a binary matrix, which seems to indicate that, in a sense, much of the information contained in this trajectory is lost.

The recurrence matrix and the derived recurrence plots, which are a graphical representation of the recurrence matrix, have been applied very successfully to analyze time series.[25] In these cases usually only one observable is known, instead of the exact state of the system. One then reconstructs the phase space, e.g., by Takens' theorem.[26] Many dynamical invariants, such as generalized Renyi entropies and dimensions, and also the mutual information can be estimated from the recurrence matrix.[27]

The recurrence matrix has also been used to detect generalized synchronization, where two systems are identical up to a homeomorphic transformation. To detect generalized synchronization, one basically considers simultaneous recurrences in the two systems,[28,29] and thereby effectively compares the recurrence matrices of both systems. It has been conjectured in the past that a high similarity of the matrices indicates that the systems are homeomorphic. The main theorem of this paper proves that, in fact, two systems are homeomorphic if their recurrence matrices are identical.

Recently, it has been shown that for one-dimensional systems the time series can be reconstructed from the recurrence matrix up to a smooth observation function.[30] This gave a first indication that the recurrence matrix contains more complete information about a dynamical system than the spectrum of return times. Later a numerical algorithm was used to reconstruct n-dimensional attractors from the recurrence matrices of various systems, e.g., the Lorenz system.[31] Note that such algorithms for the reconstruction of n-dimensional structures from distance inequalities have been used before and are highly relevant, e.g., for the reconstruction of the structure of proteins from NMR measurements.[32] The efficiency of such reconstruction algorithms is discussed in Refs. 30–32. A further indication of the information content of the recurrence matrix is given by the method of twin surrogates,[33] where the recurrence matrix is used to generate dynamically equivalent "realizations" of trajectories of a dynamical system, i.e., trajectories that exhibit the same dynamics but are initialized at different points. This algorithm is further evidence that the recurrence matrix contains all dynamically relevant information.

All these numerical and analytical results suggest that once the recurrences of a system are known in the form of the recurrence matrix in the limit of long time series, its dynamics is determined. This conjecture lies at the heart of all the work outlined in this section.[25,27–33] The theorem proved in this paper validates this conjecture showing that under rather weak assumptions it is, in fact, possible to "reconstruct the geometry" of an attractor from the recurrence matrix, i.e., when the distance inequalities are known for each pair of points. This theorem links geometrical to topological properties of the attractor and allows us to conclude that, if the recurrences are known in the form of the recurrence matrix, this determines the dynamics of a given dynamical system. We therefore prove the above mentioned crucial conjecture; we mathematically justify the techniques given in Refs. 25 and 27–33, and we establish the recurrence matrix as an important mathematical object for the study of dynamical systems.

III. THE RECONSTRUCTION THEOREM

To investigate how much the recurrences tell us about the dynamics of a system, we first consider two compact sets of points which can be mapped onto each other and which have the same distance inequalities, i.e., the same recurrence matrix. Hence, two points are close in one set if and only if the corresponding points are close in the other set. We then ask, how similar are these two sets in such a case. The answer to this question is that they are topologically identical.

Note that we do *not* consider dynamical systems $\{\mathcal{X},\Sigma,\mu,T\}$ at this point, but only subsets of a metric space, i.e., we only consider geometrical/topological structures rather than dynamics. In this sense our theorem is rather general and not confined to dynamical systems. We will see later in this paper that the topological equivalence of the two sets then allows us to conclude that two dynamical systems with the same recurrence matrix are, in fact, dynamically equivalent.

We first assume that we have a subset M of $\mathbb{R}^n$ for some n, and that for each pair of points, their distance inequality is known, i.e., we know whether $d(x,y)<\varepsilon$ or $d(x,y)\geq\varepsilon$. These might be the points on the attractor of a dynamical system. Suppose that we have another set N, onto which the points of M are mapped by a function ϕ, such that the resulting distance inequalities are the same, i.e., we have $d(\phi(x),\phi(y))<\varepsilon$ if and only if $d(x,y)<\varepsilon$. Note that in general we could have different but fixed thresholds ε and ε' before and after the transformation. However, this would only change the size of the reconstructed attractor. Hence, we can without loss of generality assume that the thresholds are the same, i.e., $\varepsilon=\varepsilon'$. Our theorem shows that under weak assumptions, the function ϕ, which maps M onto N, induces a homeomorphism, i.e., that the two systems are identical up to a continuous change in the coordinate system.

Now we have two sets with an induced topology inherited from $\mathbb{R}^n$, which have the same distance inequalities, or from the point of view of dynamical systems theory, the same recurrences. Furthermore, we impose a fairly natural additional condition that any two elements of M can be distinguished by comparing their distances relative to ε from enough elements of M. More precisely, if the "separation" condition of the theorem and its corollary below fails for elements of M, then there are distinct points x, $y\in M$ such that the closed ball in M of radius ε with center x is contained in the corresponding closed ball in M with center y. In the following proofs, we need to distinguish carefully between strict inequalities and cases where equality might be allowed. In the practical applications, this distinction seems unlikely to matter.

We will make the assumption that the sets N and M are closed, i.e., that they contain their boundaries. This assumption is crucial for the proof, as it will allow us to prove the compactness of certain sets $K=\overline{B'(\phi(x),\varepsilon)}$. The compactness will make it possible to find a *finite* subcover of K and finally to show that the transformation ϕ must be continuous.

Reconstruction theorem: *Let M, N be closed subsets of $\mathbb{R}^n$ for some $n<\infty$. Let $\phi: M\to N$ be a surjection with the property that for some fixed $\varepsilon>0$, we have $d(\phi(x),\phi(y))<\varepsilon$ if and only if $d(x,y)<\varepsilon$, where d is the usual metric. Suppose also that, for each $x\neq y\in M$, there is an element $z\in M$ with $d(x,z)>\varepsilon$ and $d(y,z)<\varepsilon$, and similarly, whenever $u\neq v\in N$, there is an element $w\in N$ with $d(u,w)>\varepsilon$ and $d(v,w)<\varepsilon$. Then ϕ is a homeomorphism.*

Proof: Notice that ϕ is injective since for x, y, z above, $d(\phi(x),\phi(z))\geq\varepsilon$ and $d(\phi(y),\phi(z))<\varepsilon$, so $\phi(x)$ and $\phi(y)$ are distinct. Hence, by symmetry, it suffices to prove that ϕ is continuous.

Notice, then, that for each $x\in M$,

$$M\setminus\{x\}=\bigcup_{\{z\in M:d(x,z)>\varepsilon\}} B(z,\varepsilon),$$

where $B(z,\varepsilon)$ denotes the open ball in M with center z and radius ε since x is outside this union, but, by hypothesis, any y different from x is in the union [it is at distance less than ε from some z such that $d(x,z)>\varepsilon$].

Also,

$$N\setminus\{\phi(x)\}=\bigcup_{\{z\in M:d(x,z)>\varepsilon\}} B'(\phi(z),\varepsilon),$$

where $B'(v,\varepsilon)$ denotes the open ball in N with center v and radius ε. To see this, we apply ϕ to each side of the equality

$$M\setminus\{x\}=\bigcup_{\{z\in M:d(x,z)>\varepsilon\}} B(z,\varepsilon)$$

and note that

$$\phi(B(z,\varepsilon))=B'(\phi(z),\varepsilon)$$

by the hypotheses.

Take a sequence (x_k) of elements of M having limit $x\in M$. We will prove that $(\phi(x_k))$ has limit $\phi(x)$. Let $K=\overline{B'(\phi(x),\varepsilon)}$ (since N is closed in $\mathbb{R}^n$, the closures in N and $\mathbb{R}^n$ coincide). Notice that K is closed and bounded, so is compact. For large enough k, we have $x_k \in B(x,\varepsilon)$ and $\phi(x_k) \in B'(\phi(x),\varepsilon)$ using the hypotheses for the latter inclusion.

Choose $\delta > 0$. Now

$$N = B'(\phi(x),\delta) \cup \bigcup_{\{z\in M: d(x,z)>\varepsilon\}} B'(\phi(z),\varepsilon)$$

and

$$K = (K \cap B'(\phi(x),\delta)) \cup \bigcup_{\{z\in M: d(x,z)>\varepsilon\}} (K \cap B'(\phi(z),\varepsilon)),$$

a cover of K by open sets (in K). Since K is compact, there is a finite subcover, which must include $(K \cap B'(\phi(x),\delta))$, as none of the open sets $B'(\phi(z),\varepsilon)$, where $d(x,z) > \varepsilon$, contains $\phi(x)$, for $d(x,z) > \varepsilon$ yields $d(\phi(x),\phi(z)) \geq \varepsilon$ by hypothesis.

Hence there is an integer s and elements $v_1,\ldots,v_s \in M$ such that $d(x,v_j) > \varepsilon$ and $d(\phi(x),\phi(v_j)) \geq \varepsilon$ for each j and

$$K = (K \cap B'(\phi(x),\delta)) \cup \bigcup_{j=1}^{s} (K \cap B'(\phi(v_j),\varepsilon)).$$

Let

$$\alpha = \min\{d(x,v_j) - \varepsilon : 1 \leq j \leq s\} > 0.$$

For all large enough m, we have $x_m \in B(x,\varepsilon)$, $\phi(x_m) \in B'(\phi(x),\varepsilon)$, and $d(x,x_m) < \alpha$. Then for each j

$$d(x,v_j) \leq d(x,x_m) + d(x_m,v_j) < \alpha + d(x_m,v_j),$$

so that

$$d(x_m,v_j) > d(x,v_j) - \alpha \geq d(x,v_j) - (d(x,v_j) - \varepsilon) = \varepsilon,$$

as $\alpha \leq d(x,v_j) - \varepsilon$. Hence $d(\phi(x_m),\phi(v_j)) \geq \varepsilon$ for each j.

Since

$$K = (K \cap B'(\phi(x),\delta)) \cup \bigcup_{j=1}^{s} (K \cap B'(\phi(v_j),\varepsilon))$$

and $d(\phi(x_m),\phi(v_j)) \geq \varepsilon$ for each j, it must be the case that $d(\phi(x),\phi(x_m)) < \delta$. Since δ is arbitrarily chosen, $(\phi(x_k))$ has limit $\phi(x)$. Thus ϕ is continuous, as required. □

The following useful corollary is immediate since compact subsets of $\mathbb{R}^n$ are closed, and because the separation property required in N follows automatically from the properties of ϕ and the fact that the corresponding property holds in M.

Corollary: *Let M, N be compact subsets of $\mathbb{R}^n$ for some $n < \infty$. Let $\phi: M \to N$ be a surjection with the property that for some fixed $\varepsilon > 0$, we have $d(\phi(x),\phi(y)) < \varepsilon$ if and only if $d(x,y) < \varepsilon$ and $d(\phi(x),\phi(y)) > \varepsilon$ if and only if $d(x,y) > \varepsilon$, where d is the usual metric. Suppose also that, for each $x \neq y \in M$, there is an element $z \in M$ with $d(x,z) > \varepsilon$ and $d(y,z) < \varepsilon$. Then ϕ is a homeomorphism.*

Note that the circle S^1, the sphere S^2, and the torus T^2 are all examples of sets for which the conditions of the theorem hold if ε is suitably chosen. If for a compact set, the separation condition does not hold, considering the ε-interior often makes the application of the theorem possible.

IV. INTERPRETATION

The theorem and the corollary given in Sec. III show that the distance inequalities restrict the possible structure of a set up to homeomorphic transformations. To understand the relevance to the theory of dynamical systems, suppose that we have the recurrence matrix of a dynamical system. We then have to consider two problems: (i) is it possible to place all the points, described by the matrix, in space such that all distance inequalities are respected (existence of a solution); (ii) if two or more structures have the same distance inequalities, are they equivalent up to a homeomorphism ("uniqueness")? The first problem (i) is trivial as the recurrence matrix was computed from a trajectory, i.e., a sequence of points, in space. This set of points is obviously a solution. Furthermore, our theorem addresses problem (ii) since it guarantees that if we had two solutions, i.e., two distinct ways of placing points in space that both respect the distance inequalities, these two solutions are homeomorphic. Note that this statement is a purely geometrical one.

However, the recurrence matrix also contains information on the time evolution of the system, i.e., the sequence in which the points are visited. This fact allows us to follow the trajectory once the topology of the set is reconstructed. The indices i and j in Eq. (2) correspond to the times at which the points $f^i(x_0)$ and $f^j(x_0)$ are close to each other. The recurrence matrix therefore defines the sequence in which the points are visited and hence, allows to determine the time evolution of the system. We therefore can reconstruct a topologically conjugate dynamical system from the knowledge on the recurrences. This means that the reconstructed system is dynamically conjugate to the original one, in the sense of Ref. 34. Both systems can then be considered to be identical up to a continuous change in the coordinate system.

As a direct consequence of our remark that the reconstruction theorem can be applied to S^1, we note that the recurrences of, e.g., the chaotic Bernoulli shift map, which can be interpreted as shift on a circle, suffice to reconstruct the dynamics. As the conditions for the theorem also hold for T^2, the recurrences determine the system's dynamics of, for example, Anosov diffeomorphisms, if the norm is appropriately chosen.

Note that our theorem shows that if two systems have the same recurrence matrix, they are topologically equivalent. This is a sufficient condition and not a necessary one. Two systems can be topologically equivalent but have different recurrence matrices. To our knowledge, there is no algorithm to decide whether two different recurrence matrices correspond to two different but homeomorphic sets.

V. CONCLUSIONS

Several conclusions can be drawn from these considerations. First of all, the theorem shows that under some rather weak assumptions, recurrences determine a system up to topological conjugacy. Two systems with the same recurrence matrix are identical up to a continuous change in the coordinate system. One can now conclude from this work that recurrences provide an alternative description of a dynamical system. In fact, the reconstruction theorem is more general, as it only assumes a metric space and not a dynamical system; it is topological as opposed to dynamical. The topology

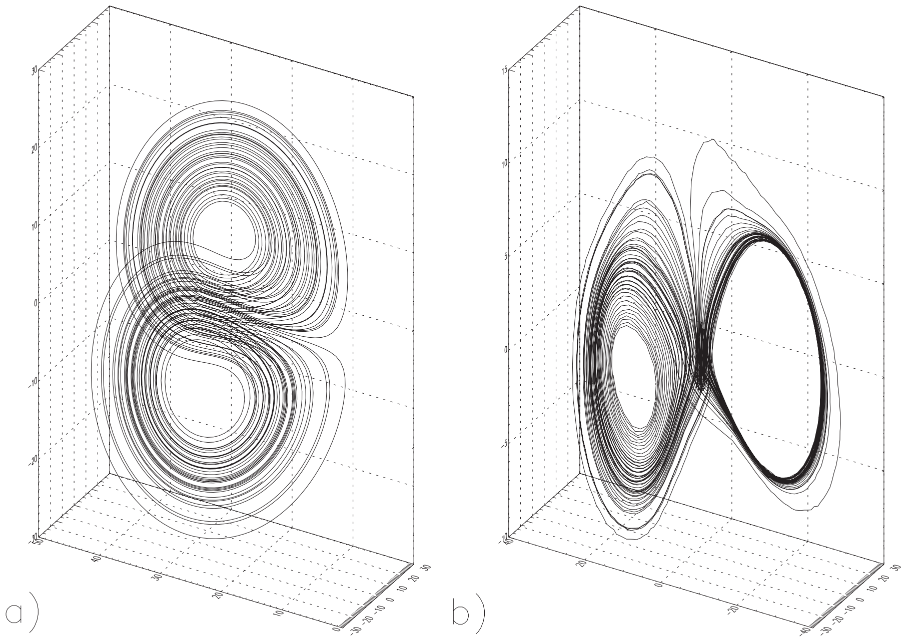

FIG. 1. (a) Trajectory of a Lorenz attractor in phase space: parameters $\sigma=10$, $r=28$, and $b=8/3$, for equations see Ref. 34. (b) Trajectory reconstructed from the recurrence matrix of the trajectory in (a).

of a compact set is determined once the distance inequalities for all its pairs of points are known.

Our theorem provides the mathematical foundation for several standard techniques, e.g., those which are used to detect generalized synchronization and for the numerical algorithm used in Ref. 31 to reconstruct the trajectory of the underlying system from its recurrence matrix. It was observed that such a reconstruction is possible for finite sets of points and a rather large interval of values of the threshold ε. In most applications, the finite sets used are dense approximations to an infinite idealized configuration, and the theorem applies best to the idealized situation. However, the theorem justifies the belief that as more and more points are used in finite reconstructions, the true topology of the attractor is revealed. The quality of the reconstruction is then practically independent of ε, and even spatial structures much smaller than ε could be reconstructed for long time series. A typical reconstruction of a Lorenz attractor based on the algorithm described in Refs. 32 and 31 is shown in Fig. 1.

These numerical findings are confirmed by the *theorem*. Furthermore, our theorem also supports the method proposed in Ref. 33 to generate twin surrogate data from the recurrence matrix.

Our considerations also suggest that it be might possible to show that ϕ together with its inverse ϕ^{-1} has to be differentiable, i.e., at least $\mathcal{C}^1$, and therefore induces a diffeomorphism between M and N.

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