CHAPTER 5

Techniques of Counting

5.1 INTRODUCTION

This chapter develops some techniques for determining, without direct enumeration, the number of possible outcomes of a particular event or the number of elements in a set. Such sophisticated counting is sometimes called *combinatorial analysis*. It includes the study of permutations and combinations.

5.2 BASIC COUNTING PRINCIPLES

There are two basic counting principles used throughout this chapter. The first one involves addition and the second one multiplication.

Sum Rule Principle:

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m + n$ ways.

Product Rule Principle:

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in mn ways.

The above principles can be extended to three or more events. That is, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, and, following E_2; a third event E_3 can occur in n_3 ways, and so on. Then:

Sum Rule: If no two events can occur at the same time, then one of the events can occur in:

$$n_1 + n_2 + n_3 + \cdots \text{ ways.}$$

Product Rule: If the events occur one after the other, then all the events can occur in the order indicated in:

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \text{ ways.}$$

EXAMPLE 5.1 Suppose a college has 3 different history courses, 4 different literature courses, and 2 different sociology courses.

(a) The number m of ways a student can choose one of each kind of courses is:

$$m = 3(4)(2) = 24$$

(b) The number n of ways a student can choose just one of the courses is:

$$n = 3 + 4 + 2 = 9$$

There is a set theoretical interpretation of the above two principles. Specifically, suppose $n(A)$ denotes the number of elements in a set A. Then:

(1) ***Sum Rule Principle:*** Suppose A and B are disjoint sets. Then

$$n(A \cup B) = n(A) + n(B)$$

(2) ***Product Rule Principle:*** Let $A \times B$ be the Cartesian product of sets A and B. Then

$$n(A \times B) = n(A) \cdot n(B)$$

5.3 MATHEMATICAL FUNCTIONS

We discuss two important mathematical functions frequently used in combinatorics.

Factorial Function

The product of the positive integers from 1 to n inclusive is denoted by $n!$, read "n factorial." Namely:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)(n-1)n = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Accordingly, $1! = 1$ and $n! = n(n - l)!$. It is also convenient to define $0! = 1$.

EXAMPLE 5.2

(a) $3! = 3 \cdot 2 \cdot 1 = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24, \quad 5 = 5 \cdot 4! = 5(24) = 120.$

(b) $\dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{3 \cdot 2 \cdot 1 \cdot 9!} = \dfrac{12!}{3!\,9!}$ and, more generally,

$$\frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 3 \cdot 2 \cdot 1} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r(r-1)\cdots 3 \cdot 2 \cdot 1 \cdot (n-r)!} = \frac{n!}{r!(n-r)!}$$

(c) For large n, one uses Stirling's approximation (where $e = 2.7128...$):

$$n! = \sqrt{2\pi n}\, n^n e^{-n}$$

Binomial Coefficients

The symbol $\binom{n}{r}$, read "nCr" or "n Choose r," where r and n are positive integers with $r \leq n$, is defined as follows:

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\ldots 3\cdot 2\cdot 1} \qquad \text{or equivalently} \qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note that $n-(n-r)=r$. This yields the following important relation.

Lemma 5.1: $\binom{n}{n-r} = \binom{n}{r}$ or equivalently, $\binom{n}{a} = \binom{n}{b}$ where $a+b=n$.

Motivated by that fact that we defined $0! = 1$, we define:

$$\binom{n}{0} = \frac{n!}{0!n!} = 1 \quad \text{and} \quad \binom{0}{0} = \frac{0!}{0!\,0!} = 1$$

EXAMPLE 5.3

(a) $\binom{8}{2} = \frac{8\cdot 7}{2\cdot 1} = 28$; $\binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 126$; $\binom{12}{5} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 792$.

Note that $\binom{n}{r}$ has exactly r factors in both the numerator and the denominator.

(b) Suppose we want to compute $\binom{10}{7}$. There will be 7 factors in both the numerator and the denominator. However, $10-7=3$. Thus, we use Lemma 5.1 to compute:

$$\binom{10}{7} = \binom{10}{3} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$$

Binomial Coefficients and Pascal's Triangle

The numbers $\binom{n}{r}$ are called *binomial coefficients*, since they appear as the coefficients in the expansion of $(a+b)^n$. Specifically:

Theorem (Binomial Theorem) 5.2: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{r} a^{n-k} b^k$

The coefficients of the successive powers of $a+b$ can be arranged in a triangular array of numbers, called Pascal's triangle, as pictured in Fig. 5-1. The numbers in Pascal's triangle have the following interesting properties:

(i) The first and last number in each row is 1.

(ii) Every other number can be obtained by adding the two numbers appearing above it. For example:

$$10 = 4+6, \quad 15 = 5+10, \quad 20 = 10+10.$$

Since these numbers are binomial coefficients, we state the above property formally.

$$\begin{array}{rcl}
(a+b)^0 &=& 1 \\
(a+b)^1 &=& a + b \\
(a+b)^2 &=& a^2 + 2ab + b^2 \\
(a+b)^3 &=& a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &=& a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &=& a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &=& a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\
\cdots & & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccc}
&&&&&& 1 &&&&&& \\
&&&&& 1 && 1 &&&&& \\
&&&& 1 && 2 && 1 &&&& \\
&&& 1 && 3 && 3 && 1 &&& \\
&& 1 && 4 && 6 && 4 && 1 && \\
& 1 && 5 && 10 && 10 && 5 && 1 & \\
1 && 6 && 15 && 20 && 15 && 6 && 1 \\
\cdots &&&&&&&&&&&& \cdots
\end{array}$$

Fig. 5-1 Pascal's triangle

Theorem 5.3: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

5.4 PERMUTATIONS

Any arrangement of a set of n objects in a given order is called a *permutation* of the object (taken all at a time). Any arrangement of any $r \leq n$ of these objects in a given order is called an "r-permutation" or "a permutation of the n objects taken r at a time." Consider, for example, the set of letters A, B, C, D. Then:

(i) *BDCA*, *DCBA*, and *ACDB* are permutations of the four letters (taken all at a time).
(ii) *BAD*, *ACB*, *DBC* are permutations of the four letters taken three at a time.
(iii) *AD*, *BC*, *CA* are permutations of the four letters taken two at a time.

We usually are interested in the number of such permutations without listing them.
The number of permutations of n objects taken r at a time will be denoted by

$$P(n, r) \quad \text{(other texts may use } {}_nP_r,\ P_{n,r},\ \text{or } (n)_r).$$

The following theorem applies.

Theorem 5.4: $P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$

We emphasize that there are r factors in $n(n-1)(n-2)\cdots(n-r+1)$.

EXAMPLE 5.4 Find the number m of permutations of six objects, say, A, B, C, D, E, F, taken three at a time. In other words, find the number of "three-letter words" using only the given six letters without repetition.

Let us represent the general three-letter word by the following three positions:

$$\_\_\_,\ \_\_\_,\ \_\_\_$$

The first letter can be chosen in 6 ways; following this the second letter can be chosen in 5 ways; and, finally, the third letter can be chosen in 4 ways. Write each number in its appropriate position as follows:

$$\underline{\ 6\ },\ \underline{\ 5\ },\ \underline{\ 4\ }$$

By the Product Rule there are $m = 6 \cdot 5 \cdot 4 = 120$ possible three-letter words without repetition from the six letters. Namely, there are 120 permutations of 6 objects taken 3 at a time. This agrees with the formula in Theorem 5.4:

$$P(6, 3) = 6 \cdot 5 \cdot 4 = 120$$

In fact, Theorem 5.4 is proven in the same way as we did for this particular case.

Consider now the special case of $P(n, r)$ when $r = n$. We get the following result.

Corollary 5.5: There are $n!$ permutations of n objects (taken all at a time).

For example, there are $3! = 6$ permutations of the three letters A, B, C. These are:

$$ABC, \quad ACB, \quad BAC, \quad BCA, \quad CAB, \quad CBA.$$

Permutations with Repetitions

Frequently we want to know the number of permutations of a multiset, that is, a set of objects some of which are alike. We will let

$$P(n;\ n_1,\ n_2,\ \ldots,\ n_r)$$

denote the number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike. The general formula follows:

Theorem 5.6: $P(n;\ n_1,\ n_2, \ldots,\ n_r) = \dfrac{n!}{n_1!\,n_2! \ldots n_r!}$

We indicate the proof of the above theorem by a particular example. Suppose we want to form all possible five-letter "words" using the letters from the word "*BABBY*." Now there are $5! = 120$ permutations of the objects B_1, A, B_2, B_3, Y, where the three B's are distinguished. Observe that the following six permutations

$$B_1B_2\,B_3AY, \quad B_2B_1\,B_3AY, \quad B_3B_1\,B_2AY, \quad B_1B_3\,B_2AY, \quad B_2B_3\,B_1AY, \quad B_3B_2\,B_1AY$$

produce the same word when the subscripts are removed. The 6 comes from the fact that there are $3! = 3{\cdot}2{\cdot}1 = 6$ different ways of placing the three B's in the first three positions in the permutation. This is true for each set of three positions in which the B's can appear. Accordingly, the number of different five-letter words that can be formed using the letters from the word "*BABBY*" is:

$$P(5;3) = \frac{5!}{3!} = 20$$

EXAMPLE 5.5 Find the number m of seven-letter words that can be formed using the letters of the word "*BENZENE*."

We seek the number of permutations of 7 objects of which 3 are alike (the three E's), and 2 are alike (the two N's). By Theorem 5.6,

$$m = P(7;\ 3,\ 2) = \frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 420$$

Ordered Samples

Many problems are concerned with choosing an element from a set S, say, with n elements. When we choose one element after another, say, r times, we call the choice an *ordered sample* of size r. We consider two cases.

(1) Sampling with replacement

Here the element is replaced in the set S before the next element is chosen. Thus, each time there are n ways to choose an element (repetitions are allowed). The Product rule tells us that the number of such samples is:

$$n \cdot n \cdot n \cdots n \cdot n (r \text{ factors}) = n^r$$

(2) Sampling without replacement

Here the element is not replaced in the set S before the next element is chosen. Thus, there is no repetition in the ordered sample. Such a sample is simply an r-permutation. Thus the number of such samples is:

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

EXAMPLE 5.6 Three cards are chosen one after the other from a 52-card deck. Find the number m of ways this can be done: (a) with replacement; (b) without replacement.

(a) Each card can be chosen in 52 ways. Thus $m = 52(52)(52) = 140\,608$.

(b) Here there is no replacement. Thus the first card can be chosen in 52 ways, the second in 51 ways, and the third in 50 ways. Therefore:

$$m = P(52, 3) = 52(51)(50) = 132\,600$$

5.5 COMBINATIONS

Let S be a set with n elements. A *combination* of these n elements taken r at a time is any selection of r of the elements where order does not count. Such a selection is called an *r-combination*; it is simply a subset of S with r elements. The number of such combinations will be denoted by

$$C(n, r) \qquad \text{(other texts may use } {}_nC_r,\ C_{n,r},\ \text{or } C_r^n\text{)}.$$

Before we give the general formula for $C(n, r)$, we consider a special case.

EXAMPLE 5.7 Find the number of combinations of 4 objects, A, B, C, D, taken 3 at a time.

Each combination of three objects determines $3! = 6$ permutations of the objects as follows:

$$\begin{array}{lllllll}
ABC: & ABC, & ACB, & BAC, & BCA, & CAB, & CBA \\
ABD: & ABD, & ADB, & BAD, & BDA, & DAB, & DBA \\
ACD: & ACD, & ADC, & CAD, & CDA, & DAC, & DCA \\
BCD: & BDC, & BDC, & CBD, & CDB, & DBC, & DCB
\end{array}$$

Thus the number of combinations multiplied by $3!$ gives us the number of permutations; that is,

$$C(4, 3)\cdot 3! = P(4, 3) \quad \text{or} \quad C(4, 3) = \frac{P(4, 3)}{3!}$$

But $P(4, 3) = 4\cdot 3\cdot 2 = 24$ and $3! = 6$; hence $C(4, 3) = 4$ as noted above.

As indicated above, any combination of n objects taken r at a time determines $r!$ permutations of the objects in the combination; that is,

$$P(n, r) = r!\, C(n, r)$$

Accordingly, we obtain the following formula for $C(n, r)$ which we formally state as a theorem.

Theorem 5.7: $C(n, r) = \dfrac{P(n,\ r)}{r!} = \dfrac{n!}{r!(n-r)!}$

Recall that the binomial coefficient $\dbinom{n}{r}$ was defined to be $\dfrac{n!}{r!(n-r)!}$; hence

$$\boxed{C(r, n) = \binom{n}{r}}$$

We shall use $C(n,\ r)$ and $\dbinom{n}{r}$ interchangeably.

EXAMPLE 5.8 A farmer buys 3 cows, 2 pigs, and 4 hens from a man who has 6 cows, 5 pigs, and 8 hens. Find the number m of choices that the farmer has.

The farmer can choose the cows in $C(6,\ 3)$ ways, the pigs in $C(5, 2)$ ways, and the hens in $C(8,\ 4)$ ways. Thus the number m of choices follows:

$$m = \binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 20 \cdot 10 \cdot 70 = 14\ 000$$

5.6 THE PIGEONHOLE PRINCIPLE

Many results in combinational theory come from the following almost obvious statement.

Pigeonhole Principle: If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

This principle can be applied to many problems where we want to show that a given situation can occur.

EXAMPLE 5.9

(a) Suppose a department contains 13 professors, then two of the professors (pigeons) were born in the same month (pigeonholes).

(b) Find the minimum number of elements that one needs to take from the set $S = \{1, 2, 3, \ldots, 9\}$ to be sure that two of the numbers add up to 10.
Here the pigeonholes are the five sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$. Thus any choice of six elements (pigeons) of S will guarantee that two of the numbers add up to ten.

The Pigeonhole Principle is generalized as follows.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

EXAMPLE 5.10 Find the minimum number of students in a class to be sure that three of them are born in the same month.

Here the $n = 12$ months are the pigeonholes, and $k + 1 = 3$ so $k = 2$. Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

5.7 THE INCLUSION–EXCLUSION PRINCIPLE

Let A and B be any finite sets. Recall Theorem 1.9 which tells us:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number $n(A \cup B)$ of elements in the union of A and B, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is, we "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $(A \cap B)$ twice.

The above principle holds for any number of sets. We first state it for three sets.

Theorem 5.8: For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

That is, we "include" $n(A)$, $n(B)$, $n(C)$, we "exclude" $n(A \cap B)$, $n(A \cap C)$, $n(B \cap C)$, and finally "include" $n(A \cap B \cap C)$.

EXAMPLE 5.11 Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian, given the following data:

65 study French,	20 study French and German,	
45 study German,	25 study French and Russian,	8 study all three languages.
42 study Russian,	15 study German and Russian,	

We want to find $n(F \cup G \cup R)$ where F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the Inclusion–Exclusion Principle,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Namely, 100 students study at least one of the three languages.

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_K})$$

of all possible k-tuple intersections of the given m sets. Then we have the following general Inclusion–Exclusion Principle.

Theorem 5.9: $n(A_1 \cup A_2 \cup \cdots \cup A_m) = s_1 - s_2 + s_3 - \cdots + (-1)^{m-1} s_m$.

5.8 TREE DIAGRAMS

A *tree diagram* is a device used to enumerate all the possible outcomes of a sequence of events where each event can occur in a finite number of ways. The construction of tree diagrams is illustrated in the following example.

EXAMPLE 5.12

(a) Find the product set $A \times B \times C$, where $A = \{1, 2\}$, $B = \{a, b, c\}$, $C = \{x, y\}$.

The tree diagram for $A \times B \times C$ appears in Fig. 5-2(a). Here the tree is constructed from left to right, and the number of branches at each point corresponds to the possible outcomes of the next event. Each endpoint (leaf) of the tree is labeled by the corresponding element of $A \times B \times C$. As noted previously, $A \times B \times C$ has $n = 2(3)(2) = 12$ elements.

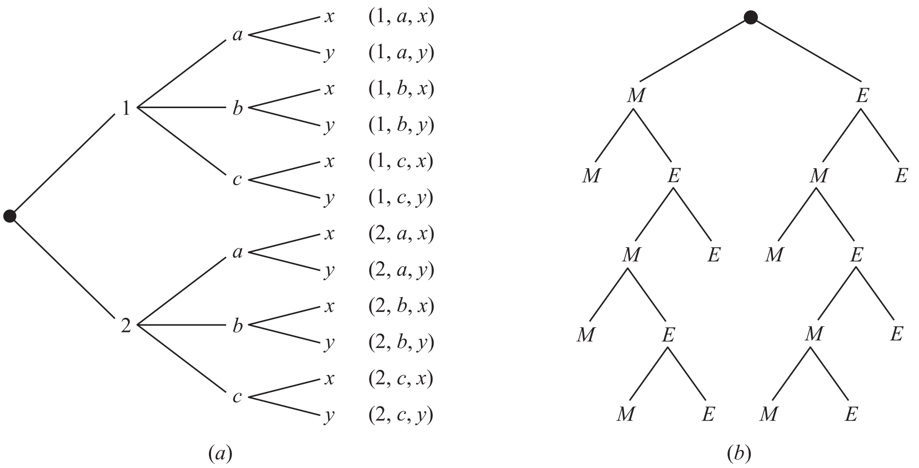


Fig. 5-2

(b) Mark and Erik are to play a tennis tournament. The first person to win two games in a row or who wins a total of three games wins the tournament. Find the number of ways the tournament can occur.

The tree diagram showing the possible outcomes of the tournament appears in Fig. 5-2(b). Here the tree is constructed from top-down rather than from left-right. (That is, the "root" is on the top of the tree.) Note that there are 10 endpoints, and the endpoints correspond to the following 10 ways the tournament can occur:

$$MM,\ MEMM,\ MEMEM,\ MEMEE,\ MEE,\ EMM,\ EMEMM,\ EMEME,\ EMEE,\ EE$$

The path from the beginning (top) of the tree to the endpoint describes who won which game in the tournament.

Solved Problems

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

5.1. Compute: (a) 4!, 5!; (b) 6!, 7!, 8!, 9!; (c) 50!.

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5(24) = 120$.
(b) Now use$(n+1)! = (n+1)n!$:

$$6! = 5(5!) = 6(120) = 720, \qquad 8! = 8(7!) = 8(5040) = 40\,320,$$
$$7! = 7(6!) = 7(720) = 5\,040, \qquad 9! = 9(8!) = 9(40\,320) = 362\,880.$$

(c) Since n is very large, we use Sterling's approximation: $n! = \sqrt{2\pi n}\, n^{\pi} e^{-n}$ (where $e \approx 2.718$). Thus:

$$50! \approx N = \sqrt{100\pi}\, 50^{50} e^{-50}$$

Evaluating N using a calculator, we get $N = 3.04 \times 10^{64}$ (which has 65 digits).

5.2. Compute: (a) $\dfrac{13!}{11!}$; (b) $\dfrac{7!}{10!}$.

(a) $\dfrac{13!}{11!} = \dfrac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 12 = 156.$

Alternatively, this could be solved as follows:

$$\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11!}{11!} = 13 \cdot 12 = 156.$$

(*b*) $\dfrac{7!}{10!} = \dfrac{7!}{10 \cdot 9 \cdot 8 \cdot 7!} = \dfrac{1}{10 \cdot 9 \cdot 8} = \dfrac{1}{720}$.

5.3. Simplify: (*a*) $\dfrac{n!}{(n-1)!}$; (*b*) $\dfrac{(n+2)!}{n!}$.

(*a*) $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} = n$; alternatively, $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$.

(*b*) $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2$.

5.4. Compute: (*a*) $\dbinom{16}{3}$; (*b*) $\dbinom{12}{4}$; (*c*) $\dbinom{8}{5}$.

Recall that there are as many factors in the numerator as in the denominator.

(*a*) $\dbinom{16}{3} = \dfrac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1} = 560$; (*b*) $\dbinom{12}{4} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$;

(*c*) Since $8 - 5 = 3$, we have $\dbinom{8}{5} = \dbinom{8}{3} = \dfrac{8 \cdot 7.6}{3 \cdot 2 \cdot 1} = 56$.

5.5. Prove: $\dbinom{17}{6} = \dbinom{16}{5} + \dbinom{16}{6}$.

Now $\dbinom{16}{5} + \dbinom{16}{6} = \dfrac{16!}{5!11!} + \dfrac{16!}{6!10!}$. Multiply the first fraction by $\frac{6}{6}$ and the second by $\frac{11}{11}$ to obtain the same denominator in both fractions; and then add:

$$\begin{aligned}\binom{16}{5} + \binom{16}{6} &= \frac{6 \cdot 16!}{6 \cdot 5! \cdot 11!} + \frac{11 \cdot 16!}{6! \cdot 11 \cdot 10!} = \frac{6 \cdot 16!}{6! \cdot 11!} + \frac{11 \cdot 16!}{6! \cdot 11!}\\ &= \frac{6 \cdot 16! + 11 \cdot 16!}{6! \cdot 11!} = \frac{(6+11) \cdot 16!}{6! \cdot 11!} = \frac{17 \cdot 16!}{6! \cdot 11!} = \frac{17!}{6! \cdot 11!} = \binom{17}{6}\end{aligned}$$

5.6. Prove Theorem 5.3: $\dbinom{n+1}{r} = \dbinom{n}{r-1} + \dbinom{n}{r}$.

(The technique in this proof is similar to that of the preceding problem.)

Now $\dbinom{n}{r-1} + \dbinom{n}{r} = \dfrac{n!}{(r-1)! \cdot (n-r+1)!} + \dfrac{n!}{r! \cdot (n-r)!}$.

To obtain the same denominator in both fractions, multiply the first fraction by $\frac{r}{r}$ and the second fraction by $\dfrac{n-r+1}{n-r+1}$. Hence

$$\begin{aligned}\binom{n}{r-1} + \binom{n}{r} &= \frac{r \cdot n!}{r \cdot (r-1)! \cdot (n-r+1)!} + \frac{(n-r+1) \cdot n!}{r! \cdot (n-r+1) \cdot (n-r)!}\\ &= \frac{r \cdot n!}{r!(n-r+1)!} + \frac{(n-r+1) \cdot n!}{r!(n-r+1)!}\\ &= \frac{r \cdot n! + (n-r+1) \cdot n!}{r!(n-r+1)!} = \frac{[r + (n-r+1)] \cdot n!}{r!(n-r+1)!}\\ &= \frac{(n+1)n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}\end{aligned}$$

COUNTING PRINCIPLES

5.7. Suppose a bookcase shelf has 5 History texts, 3 Sociology texts, 6 Anthropology texts, and 4 Psychology texts. Find the number n of ways a student can choose:

(*a*) one of the texts; (*b*) one of each type of text.

(*a*) Here the Sum Rule applies; hence, $n = 5 + 3 + 6 + 4 = 18$.

(*b*) Here the Product Rule applies; hence, $n = 5 \cdot 3 \cdot 6 \cdot 4 = 360$.

5.8. A history class contains 8 male students and 6 female students. Find the number n of ways that the class can elect: (a) 1 class representative; (b) 2 class representatives, 1 male and 1 female; (c) 1 president and 1 vice president.

(a) Here the Sum Rule is used; hence, $n = 8 + 6 = 14$.

(b) Here the Product Rule is used; hence, $n = 8 \cdot 6 = 48$.

(c) There are 14 ways to elect the president, and then 13 ways to elect the vice president. Thus $n = 14 \cdot 13 = 182$.

5.9. There are four bus lines between A and B, and three bus lines between B and C. Find the number m of ways that a man can travel by bus: (a) from A to C by way of B; (b) roundtrip from A to C by way of B; (c) roundtrip from A to C by way of B but without using a bus line more than once.

(a) There are 4 ways to go from A to B and 3 ways from B to C; hence $n = 4 \cdot 3 = 12$.

(b) There are 12 ways to go from A to C by way of B, and 12 ways to return. Thus $n = 12 \cdot 12 = 144$.

(c) The man will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A \to B \to C \to B \to A$$

The man can travel four ways from A to B and three ways from B to C, but he can only travel two ways from C to B and three ways from B to A since he does not want to use a bus line more than once. Enter these numbers above the corresponding arrows as follows:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Thus, by the Product Rule, $n = 4 \cdot 3 \cdot 2 \cdot 3 = 72$.

PERMUTATIONS

5.10. State the essential difference between permutations and combinations, with examples.

Order counts with permutations, such as words, sitting in a row, and electing a president, vice president, and treasurer. Order does not count with combinations, such as committees and teams (without counting positions). The product rule is usually used with permutations, since the choice for each of the ordered positions may be viewed as a sequence of events.

5.11. Find: (a) $P(7, 3)$; (b) $P(14, 2)$.

Recall $P(n, r)$ has r factors beginning with n.

(a) $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$; ($b$) $P(14, 2) = 14 \cdot 13 = 182$.

5.12. Find the number m of ways that 7 people can arrange themselves:

(a) In a row of chairs; (b) Around a circular table.

(a) Here $m = P(7, 7) = 7!$ ways.

(b) One person can sit at any place at the table. The other 6 people can arrange themselves in $6!$ ways around the table; that is $m = 6!$.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n - 1)!$ ways.

5.13. Find the number n of distinct permutations that can be formed from all the letters of each word:

(a) *THOSE*; (b) *UNUSUAL*; (c) *SOCIOLOGICAL.*

This problem concerns permutations with repetitions.

(a) $n = 5! = 120$, since there are 5 letters and no repetitions.

(b) $n = \dfrac{7!}{3!} = 840$, since there are 7 letters of which 3 are U and no other letter is repeated.

(c) $n = \dfrac{12!}{3!2!2!2!}$, since there are 12 letters of which 3 are O, 2 are C, 2 are I, and 2 are L. (We leave the answer using factorials, since the number is very large.)

5.14. A class contains 8 students. Find the number n of samples of size 3:
(a) With replacement; (b) Without replacement.

(a) Each student in the ordered sample can be chosen in 8 ways; hence, there are

$$n = 8 \cdot 8 \cdot 8 = 8^3 = 512 \text{ samples of size 3 with replacement.}$$

(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n = 8 \cdot 7 \cdot 6 = 336$ samples of size 3 without replacement.

5.15. Find n if $P(n, 2) = 72$.

$$P(n,\ 2) = n(n-1) = n^2 - n. \quad \text{Thus, we get}$$

$$n^2 - n = 72 \qquad \text{or} \qquad n^2 - n - 72 = 0 \quad \text{or} \quad (n-9)(n+8) = 0$$

Since n must be positive, the only answer is $n = 9$.

COMBINATIONS

5.16. A class contains 10 students with 6 men and 4 women. Find the number n of ways to:

(a) Select a 4-member committee from the students.
(b) Select a 4-member committee with 2 men and 2 women.
(c) Elect a president, vice president, and treasurer.

(a) This concerns combinations, not permutations, since order does not count in a committee. There are "10 choose 4" such committees. That is:

$$n = C(10,\ 4) = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

(b) The 2 men can be chosen from the 6 men in $C(6, 2)$ ways, and the 2 women can be chosen from the 4 women in $C(4, 2)$ ways. Thus, by the Product Rule:

$$n = \binom{6}{2}\binom{4}{2} = \frac{6 \cdot 5}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} = 15(6) = 90$$

(c) This concerns permutations, not combinations, since order does count. Thus,

$$n = P(6, 3) = 6 \cdot 5 \cdot 4 = 120$$

5.17. A box contains 8 blue socks and 6 red socks. Find the number of ways two socks can be drawn from the box if:
(a) They can be any color. (b) They must be the same color.

(a) There are "14 choose 2" ways to select 2 of the 14 socks. Thus:

$$n = C(14,\ 2) = \binom{14}{2} = \frac{14 \cdot 13}{2 \cdot 1} = 91$$

(b) There are $C(8, 2) = 28$ ways to choose 2 of the 8 blue socks, and $C(6, 2) = 15$ ways to choose 2 of the 4 red socks. By the Sum Rule, $n = 28 + 15 = 43$.

5.18. Find the number m of committees of 5 with a given chairperson that can be selected from 12 people.

The chairperson can be chosen in 12 ways and, following this, the other 4 on the committee can be chosen from the 11 remaining in $C(11, 4)$ ways. Thus $m = 12 \cdot C(11, 4) = 12 \cdot 330 = 3960$.

PIGEONHOLE PRINCIPLE

5.19. Find the minimum number n of integers to be selected from $S = \{1, 2, \ldots, 9\}$ so that: (a) The sum of two of the n integers is even. (b) The difference of two of the n integers is 5.

(a) The sum of two even integers or of two odd integers is even. Consider the subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$ of S as pigeonholes. Hence $n = 3$.

(b) Consider the five subsets $\{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5\}$ of S as pigeonholes. Then $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

5.20. Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

Here the $n = 4$ classes are the pigeonholes and $k + 1 = 5$ so $k = 4$. Thus among any $kn + 1 = 17$ students (pigeons), five of them belong to the same class.

5.21. Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants).

(a) Show that L has a sublist consisting of four or more consecutive consonants.

(b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

(a) The five letters partition L into $n = 6$ sublists (pigeonholes) of consecutive consonants. Here $k + 1 = 4$ and so $k = 3$. Hence $nk + 1 = 6(3) + 1 = 19 < 21$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into $n = 5$ sublists. Here $k + 1 = 5$ and so $k = 4$. Hence $kn + 1 = 21$. Thus some sublist has at least five consecutive consonants.

INCLUSION–EXCLUSION PRINCIPLE

5.22. There are 22 female students and 18 male students in a classroom. Find the total number t of students.

The sets of male and female students are disjoint; hence $t = 22 + 18 = 40$.

5.23. Suppose among 32 people who save paper or bottles (or both) for recycling, there are 30 who save paper and 14 who save bottles. Find the number m of people who:

(a) save both; (b) save only paper; (c) save only bottles.

Let P and B denote the sets of people saving paper and bottles, respectively. Then:

$$\begin{array}{ll} (a) & m = n(P \cap B) = n(P) + n(B) - n(P \cup B) = 30 + 14 - 32 = 12 \\ (b) & m = n(P \backslash B) = n(P) - n(P \cap B) = 30 - 12 = 18 \\ (c) & m = n(B \backslash P) = n(B) - n(P \cap B) = 14 - 12 = 2 \end{array}$$

5.24. Let A, B, C, D denote, respectively, art, biology, chemistry, and drama courses. Find the number N of students in a dormitory given the data:

$$\begin{array}{llll} 12 \text{ take } A, & 5 \text{ take } A \text{ and } B, & 4 \text{ take } B \text{ and } D, & 2 \text{ take } B, \mathrm{C}, D, \\ 20 \text{ take } B, & 7 \text{ take } A \text{ and } C, & 3 \text{ take } C \text{ and } D, & 3 \text{ take } A,\ C,\ D, \\ 20 \text{ take } C, & 4 \text{ take } A \text{ and } D, & 3 \text{ take } A,\ B, C, & 2 \text{ take all four}, \\ 8 \text{ take } D, & 16 \text{ take } B \text{ and } C, & 2 \text{ take } A,\ B,\ D, & 71 \text{ take none}. \end{array}$$

Let T be the number of students who take at least one course. By the Inclusion–Exclusion Principle Theorem 5.9, $T = s_1 - s_2 + s_3 - s_4$ where:

$$s_1 = 12 + 20 + 20 + 8 = 60, \quad s_2 = 5 + 7 + 4 + 16 + 4 + 3 = 39,$$
$$s_3 = 3 + 2 + 2 + 3 = 10, \quad s_4 = 2.$$

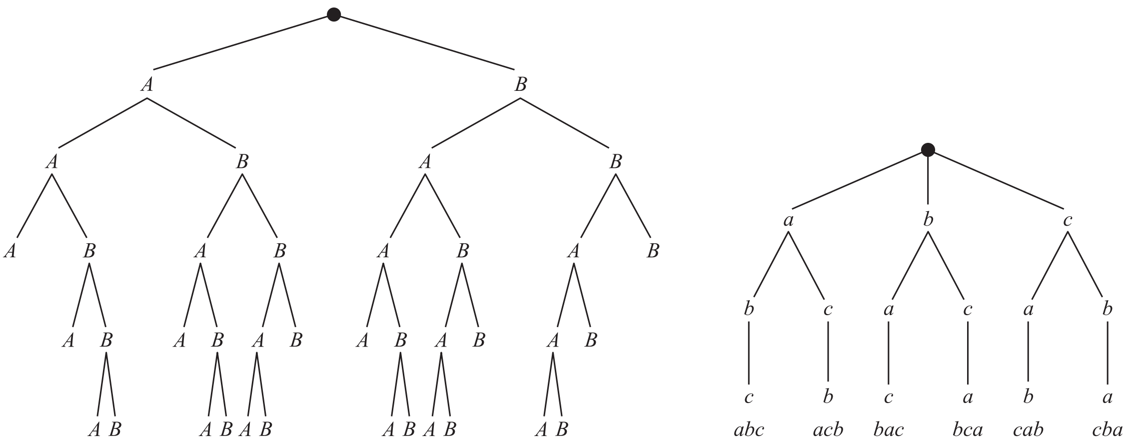
Thus $T = 29$, and $N = 71 + T = 100$.

TREE DIAGRAMS

5.25. Teams A and B play in a tournament. The first team to win three games wins the tournament. Find the number n of possible ways the tournament can occur.

Construct the appropriate tree diagram in Fig. 5-3(a). The tournament can occur in 20 ways:

AAA, $AABA$, $AABBA$, $AABBB$, $ABAA$, $ABABA$, $ABABB$, $ABBAA$, $ABBAB$, $ABBB$,
BBB, $BBAB$, $BBAAB$, $BBAAA$, $BABB$, $BABAB$, $BABAA$, $BAABB$, $BAABA$, $BAAA$



(a) (b)

Fig. 5-3

5.26. Construct the tree diagram that gives the permutations of $\{a, b, c\}$.

The tree diagram appears in Fig. 5-3(b). There are six permutations, and they are listed on the bottom of the diagram.

MISCELLANEOUS PROBLEMS

5.27. There are 12 students in a class. Find the number n of ways that the 12 students can take 3 tests if 4 students are to take each test.

There are $C(12, 4) = 495$ ways to choose 4 of the 12 students to take the first test. Following this, there are $C(8, 4) = 70$ ways to choose 4 of the remaining 8 students to take the second test. The remaining students take the third test. Thus:

$$n = 70(495) = 34\,650$$

5.28. Prove Theorem (Binomial Theorem) 5.2: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$.

The theorem is true for $n = 1$, since

$$\sum_{r=0}^{1} \binom{1}{r} a^{1-r} b^r = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b = (a+b)^1$$

We assume the theorem is true for $(a+b)^n$ and prove it is true for $(a+b)^n + 1$.

$$(a+b)^{n+1} = (a+b)\,(a+b)^n$$

$$= (a+b)[a^n + \tbinom{n}{1} a^{n-1} b + \cdots + \tbinom{n}{r-1} a^{n-r+1} b^{r-1} + \tbinom{n}{r} a^{n-r} b^r + \cdots \tbinom{n}{1} a\, b^{n-1} + b^n]$$

Now the term in the product which contains b^r is obtained from

$$\begin{aligned} b[\tbinom{n}{r-1}\, a^{n-r+1}\, b^{r-1}] + a[\tbinom{n}{r}\, a^{n-r}\, b^r] &= \tbinom{n}{r-1}\, a^{n-r+1}\, b^r + \tbinom{n}{r} a^{n-r+1}\, b^r \\ &= [\tbinom{n}{r-1} + \tbinom{n}{r}] a^{n-r+1}\, b^r \end{aligned}$$

But, by Theorem 5.3, $\binom{n}{r-1} = \binom{n}{r} = \binom{n+1}{r}$. Thus, the term containing b^r is:

$$\binom{n+1}{r}\, a^{n-r+1}\, b^r$$

Note that $(a+b)(a+b)^n$ is a polynomial of degree $n+1$ in b. Consequently:

$$(a+b)^{n+1} = (a+b)(a+b)^n = \sum_{r=0}^{n+1} \binom{n+1}{r}\, a^{n-r+1}\, b^r$$

which was to be proved.

5.29. Let n and $n_1, n_2, \ldots, n_r$ be nonnegative integers such that $n_1 + n_2 + \cdots + n_r = n$. The *multinomial coefficients* are denoted and defined by:

$$\binom{n}{n_1,\ n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Compute the following multinomial coefficients:

(*a*) $\binom{6}{3,\ 2,\ 1}$; (*b*) $\binom{8}{4,\ 2,\ 2,\ 0}$; (*c*) $\binom{10}{5,\ 3,\ 2,\ 2}$.

(*a*) $\binom{6}{3,\ 2,\ 1} = \dfrac{6!}{3!\,2!\,1!} = \dfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1\cdot1} = 60$

(*b*) $\binom{8}{4,\ 2,\ 2,\ 0} = \dfrac{8\,!}{4!\,2!\,2!\,0!} = \dfrac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot2\cdot1\cdot2\cdot1\cdot1} = 420$

(*c*) $\binom{10}{5,\ 3,\ 2,\ 2}$ has no meaning, since $5+3+2+2 \neq 10$.

5.30. A student must take five classes from three areas of study. Numerous classes are offered in each discipline, but the student cannot take more than two classes in any given area.

(*a*) Using the pigeonhole principle, show that the student will take at least two classes in one area.

(*b*) Using the Inclusion–Exclusion Principle, show that the student will have to take at least one class in each area.

(*a*) The three areas are the pigeonholes and the student must take five classes (pigeons). Hence, the student must take at least two classes in one area.

(*b*) Let each of the three areas of study represent three disjoint sets, A, B, and C. Since the sets are disjoint, $m(A \cup B \cup C) = 5 = n(A) + n(B) + n(C)$. Since the student can take at most two classes in any area of study, the sum of classes in any two sets, say A and B, must be less than or equal to four. Hence, $5 - [n(A) + n(B)] = n(C) \geq 1$. Thus, the student must take at least one class in any area.

Supplementary Problems

FACTORIAL NOTATION, BINOMIAL COEFFICIENTS

5.31. Find: (a) 10!, 11!, 12!; (b) 60!. (Hint: Use Sterling's approximation to $n!$.)

5.32. Evaluate: (a) 16!/14!; (b) 14!/11!; (c) 8!/10!; (d) 10!/13!.

5.33. Simplify: (a) $\dfrac{(n+1)!}{n!}$; (b) $\dfrac{n!}{(n-2)!}$; (c) $\dfrac{(n-1)!}{(n+2)!}$; (d) $\dfrac{(n-r+1)!}{(n-r-1)!}$.

5.34. Find: (a) $\binom{5}{2}$; (b) $\binom{7}{3}$; (c) $\binom{14}{2}$; (d) $\binom{6}{4}$; (e) $\binom{20}{17}$; (f) $\binom{18}{15}$.

5.35. Show that: (a) $\binom{n}{0}+\binom{n}{n}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n$

(b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+\binom{n}{n}=0$

5.36. Given the following eighth row of Pascal's triangle, find: (a) the ninth row; (b) the tenth row.

1 8 28 56 70 56 28 8 1

5.37. Evaluate the following multinomial coefficients (defined in Problem 5.29):

(a) $\binom{6}{2,\ 3,\ 1}$; (b) $\binom{7}{3,\ 2,\ 2,\ 0}$; (c) $\binom{9}{3,\ 5,\ 1}$; (d) $\binom{8}{4,\ 3,\ 2}$.

COUNTING PRINCIPLES

5.38. A store sells clothes for men. It has 3 kinds of jackets, 7 kinds of shirts, and 5 kinds of pants. Find the number of ways a person can buy: (a) one of the items; (b) one of each of the three kinds of clothes.

5.39. A class has 10 male students and 8 female students. Find the number of ways the class can elect: (a) a class representative; (b) 2 class representatives, one male and one female; (c) a class president and vicepresident.

5.40. Suppose a code consists of five characters, two letters followed by three digits. Find the number of: (a) codes; (b) codes with distinct letter; (c) codes with the same letters.

PERMUTATIONS

5.41. Find the number of automobile license plates where: (a) Each plate contains 2 different letters followed by 3 different digits. (b) The first digit cannot be 0.

5.42. Find the number m of ways a judge can award first, second, and third places in a contest with 18 contestants.

5.43. Find the number of ways 5 large books, 4 medium-size books, and 3 small books can be placed on a shelf where: (a) there are no restrictions; (b) all books of the same size are together.

5.44. A debating team consists of 3 boys and 3 girls. Find the number of ways they can sit in a row where:
(a) there are no restrictions; (b) the boys and girls are each to sit together; (c) just the girls are to sit together.

5.45. Find the number of ways 5 people can sit in a row where: (a) there are no restrictions; (b) two of the people insist on sitting next to each other.

5.46. Repeat Problem 5.45 if they sit around a circular table.

5.47. Consider all positive integers with three different digits. (Note that zero cannot be the first digit.) Find the number of them which are: (a) greater than 700; (b) odd; (c) divisible by 5.

5.48. Suppose repetitions are not permitted. (a) Find the number of three-digit numbers that can be formed from the six digits 2, 3, 5, 6, 7, and 9. (b) How many of them are less than 400? (c) How many of them are even?

5.49. Find the number m of ways in which 6 people can ride a toboggan if one of 3 of them must drive.

5.50. Find n if: (a) $P(n,4)=42P(n,2)$; (b) $2P(n,2)+50=P(2n,2)$.

PERMUTATIONS WITH REPETITIONS, ORDERED SAMPLES

5.51. Find the number of permutations that can be formed from all the letters of each word: (a) QUEUE; (b) COMMITTEE; (c) PROPOSITION; (d) BASEBALL.

5.52. Suppose we are given 4 identical red flags, 2 identical blue flags, and 3 identical green flags. Find the number m of different signals that can be formed by hanging the 9 flags in a vertical line.

5.53. A box contains 12 lightbulbs. Find the number n of ordered samples of size 3:
(a) with replacement; (b) without replacement.

5.54. A class contains l0 students. Find the number n of ordered samples of size 4:
(a) with replacement; (b) without replacement.

COMBINATIONS

5.55. A restaurant has 6 different desserts. Find the number of ways a customer can choose:
(a) 1 dessert; (b) 2 of the desserts; (c) 3 of the desserts.

5.56. A class contains 9 men and 3 women. Find the number of ways a teacher can select a committee of 4 from the class where there is:
(a) no restrictions; (b) 2 men and 2 women; (c) exactly one woman; (d) at least one woman.

5.57. A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:
(a) There are no restrictions.
(b) Two of the friends are married to each other and will not attend separately.
(c) Two of the friends are not speaking with each other and will not attend together.

5.58. A class contains 8 men and 6 women and there is one married couple in the class. Find the number m of ways a teacher can select a committee of 4 from the class where the husband or wife but not both can be on the committee.

5.59. A box has 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where:
(a) There are no restrictions. (b) They are different colors. (c) They are the same color.

5.60. A women student is to answer 10 out of 13 questions. Find the number of her choices where she must answer:
(a) the first two questions; (c) exactly 3 out of the first 5 questions;
(b) the first or second question but not both; (d) at least 3 of the first 5 questions.

INCLUSION–EXCLUSION PRINCIPLE

5.61. Suppose 32 students are in an art class A and 24 students are in a biology class B, and suppose 10 students are in both classes. Find the number of students who are:
(a) in class A or in class B; (b) only in class A; (c) only in class B.

5.62. A survey of 80 car owners shows that 24 own a foreign-made car and 60 own a domestic-made car. Find the number of them who own:
(a) both a foreign made car and a domestic made car;
(b) only a foreign made car;
(c) only a domestic made car.

5.63. Consider all integers from 1 up to and including 100. Find the number of them that are:
(a) odd or the square of an integer; (b) even or the cube of an integer.

5.64. In a class of 30 students, 10 got A on the first test, 9 got A on a second test, and 15 did not get an A on either test. Find: the number of students who got:

(a) an A on both tests;
(b) an A on the first test but not the second;
(c) an A on the second test but not the first.

5.65. Consider all integers from 1 up to and including 300. Find the number of them that are divisible by:
(a) at least one of 3, 5, 7; (c) by 5, but by neither 3 nor 7;
(b) 3 and 5 but not by 7; (d) by none of the numbers 3, 5, 7.

5.66. In a certain school, French (F), Spanish (S), and German (G) are the only foreign languages taught. Among 80 students:

(i) 20 study F, 25 study S, 15 study G.
(ii) 8 study F and S, 6 study S and G, 5 study F and G.
(iii) 2 study all three languages.

Find the number of the 80 students who are studying:
(a) none of the languages; (b) only French; (c) only one language; (d) only Spanish and German; (e) exactly two of the languages.

5.67. Find the number m of elements in the union of sets A, B, C, D where:

(i) A, B, C, D have 50, 60, 70, 80 elements, respectively.
(ii) Each pair of sets has 20 elements in common.
(iii) Each three of the sets has 10 elements in common.
(iv) All four of the sets have 5 elements in common.

PIGEONHOLE PRINCIPLE

5.68. Find the minimum number of students needed to guarantee that 4 of them were born: (a) on the same day of the week; (b) in the same month.

5.69. Find the minimum number of students needed to guarantee that 3 of them:
(a) have last names which begin with the same first letter;
(b) were born on the same day of a month (with 31 days).

5.70. Consider a tournament with n players where each player plays against every other player. Suppose each player wins at least once. Show that at least 2 of the players have the same number of wins.

5.71. Suppose 5 points are chosen at random in the interior of an equilateral triangle T where each side has length two inches. Show that the distance between two of the points must be less than one inch.

5.72. Consider any set $X = \{x_1, x_2, \ldots, x_7\}$ of seven distinct integers. Show that there exist $x, y \in X$ such that $x + y$ or $x - y$ is divisible by 10.

MISCELLANEOUS PROBLEMS

5.73. Find the number m of ways 10 students can be divided into three teams where one team has 4 students and the other teams have 3 students.

5.74. Assuming a cell can be empty, find the number n of ways that a set with 3 elements can be partitioned into:
(a) 3 ordered cells; (b) 3 unordered cells.

5.75. Assuming a cell can be empty, find the number n of ways that a set with 4 elements can be partitioned into:
(a) 3 ordered cells; (b) 3 unordered cells.

5.76. The English alphabet has 26 letters of which 5 are vowels. Consider only 5-letter "words" consisting of 3 different consonants and 2 different vowels. Find the number of such words which:

(a) have no restrictions; (b) contain the letter B; (c) contain the letters B and C; (d) begin with B and contain the letter C.

5.77. Teams A and B play in the World Series of baseball, where the team that first wins four games wins the series. Suppose A wins the first game, and that the team that wins the second game also wins the fourth game.

(a) Find and list the number n of ways the series can occur.
(b) Find the number of ways that B wins the series.
(c) Find the number of ways the series lasts seven games.

5.78. Find the number of ways a coin can be tossed:

(a) 6 times so that there is exactly 3 heads and no two heads occur in a row.
(b) $2n$ times so that there is exactly n heads and no two heads occur in a row.

5.79. Find the number of ways 3 elements a, b, c, can be assigned to 3 cells, so exactly 1 cell is empty.

5.80. Find the number of ways n distinct elements can be assigned to n cells so exactly 1 cell is empty.

Answers to Supplementary Problems

5.31. (a) 3 628 800; 39 916 800; 479 001 600;
(b) $\log(60!) = 81.92$, so $60! = 6.59 \times 10^{81}$.

5.32. (a) 240; (b) 2 184; (c) 1/90; (d) 1/1716.

5.33. (a) $n+1$; (b) $n(n-1)$; (c) $1/[n(n+1)(n+2)$; (d) $(n-r)(n-r+1)$.

5.34. (a) 10; (b) 35; (c) 91; (d) 15; (e) 1140; (f) 816.

5.35. Hints: (a) Expand $(1+1)^n$; (b) Expand $(1-1)^n$.

5.36. (a) 1, 9, 36, 84, 126, 126, 84, 36, 9, 1;
(b) 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.

5.37. (a) 60; (b) 210; (c) 504; (d) not defined.

5.38. (a) 15; (b) 105.

5.39. (a)18; (b) 80; (c) 306.

5.40. (a) $26^2 \cdot 10^3$; (b) $26 \cdot 25 \cdot 10^3$; (b); (c) $26 \cdot 10^3$.

5.41. (a) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468\,000$; (b) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421\,200$.

5.42. $m = 18 \cdot 17 \cdot 16 = 4896$.

5.43. (a) 12!; (b) $3!5!4!3! = 103\,680$.

5.44. (a) $6! = 720$; (b) $2 \cdot 3! \cdot 3! \cdot = 72$; (c) $4 \cdot 3! \cdot 3! = 144$.

5.45. (a) 120; (b) 48.

5.46. (a) 24; (b) 12.

5.47. (a) $3 \cdot 9 \cdot 8$; (b) $9 \cdot 8 \cdot 5$; (c) $9 \cdot 8 \cdot 7/2$; (d) $9 \cdot 8 \cdot 7/5$.

5.48. (a) $P(6, 3) = 120$; (b) $2 \cdot 5 \cdot 4 = 40$; (c) $2 \cdot 5 \cdot 4 = 40$.

5.49. $m = 360$.

5.50. (a) 9; (b) 5.

5.51. (a) 30; (b) $9!/[2!2!2!] = 45\,360$; (c) $11!/[2!3!2!] = 1\,663\,200$; (d) $8!/[2!2!2!] = 5040$.

5.52. $m = 9!/[4!2!3!] = 1260$.

5.53. (a) $12^3 = 1\,728$; (b) $P(12, 3) = 1320$.

5.54. (a) $10^4 = 10\,000$; (b) $P(10, 4) = 5040$.

5.55. (a) 6; (b) 15; (c) 20.

5.56. (a) $C(12, 4)$; (b) $C(9, 2) \cdot C(3, 2) = 108$;
(c) $C(9, 3) \cdot 3 = 252$; (d) $9 + 108 + 252 = 369$ or $C(12, 4) - C(9, 4) = 369$.

5.57. (a) $C(11, 5) = 462$; (b) $126 + 84 = 210$;
(c) $C(9, 5) + 2C(9, 4) = 378$.

5.58. $m = C(12, 4) + 2C(12, 3) = 935$.

5.59. (a) $C(10, 2) = 45$; (b) $6 \cdot 4 = 24$; (c) $C(6, 2) + C(4, 2) = 21$ or $45 - 24 = 21$.

5.60. (a) 165; (b) 110; (c) 80; (d) 276.

5.61. (a) 46; (b) 22; (c) 14.

5.62. (a) 4; (b) 20; (c) 56.

5.63. (a) 55; (b) 52.

5.64. (a) 4; (b) 6; (c) 5.

5.65. (a) $100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$;
(b) $20 - 2 = 18$; (c) $60 - 20 - 8 + 2 = 34$;
(d) $300 - 162 = 138$.

5.66. (a) 37; (b) 9; (c) 28; (d) 4; (e) 13.

5.67. $m = 175$

5.68. (a) 22; (b) 37.

5.69. (a) 53; (b) 63.

5.70. Each player will win anywhere from 1 up to $n - 1$ games (pigeonholes). There are n players (pigeons).

5.71. Draw three lines between the midpoints of the sides of T. This partitions T into 4 equilateral triangles (pigeonholes) where each side has length 1. Two of the 5 points (pigeons) must lie in one of the triangles.

5.72. Let r_i be the remainder when x_i is divisible by 10. Consider the six pigeonholes: $H_1 = \{x_i \mid r_i = 0\}$, $H_2 = \{x_i \mid r_i = 5\}$, $H_3 = \{x_i \mid r_i = 1 \text{ or } 9\}$, $H_4 = \{x_i \mid r_i = 2 \text{ or } 8\}$, $H_5 = \{x_i \mid r_i = 3 \text{ or } 7\}$, $H_6 = \{x_i \mid r_i = 4 \text{ or } 6\}$. Then some x and y belong to some H_k.

5.73. $m = C(10, 4) \cdot C(6, 3) = 420$

5.74. (a) $n = 3^3 = 27$ (Each element can be placed in any of the three cells.) (b) The number of elements in three cells can be distributed as follows: [3, 0, 0], [2,1,0], or [1,1,1]. Thus $n = 1 + 3 + 1 = 5$.

5.75. (a) $n = 3^4 = 81$ (Each element can be placed in any of the three cells.) (b) The number of elements in three cells can be distributed as follows: [4, 0, 0], [3, 1, 0], [2, 2, 0], or [2, 1, 1]. Thus $n = 1 + 4 + 3 + 6 = 14$.

5.76. (a) $C(21, 3) \cdot C(5, 2) \cdot 5!$; (b) $C(20, 2) \cdot C(5, 2) \cdot 5!$; (c) $19 \cdot C(5, 2) \cdot 5!$; (d) $19 \cdot C(5, 2) \cdot 4!$.

5.77. Draw tree diagram T as in Fig. 5-4. Note T begins at A, the winner of the first game, and there is only one choice in the fourth game, the winner of the second game.
(a) $n = 15$ as listed below; (b) 6; (c) 8:
AAAA, *AABAA*, *AABABA*, *AABABBA*, *AABABBB*, *ABABAA*, *ABABABA*, *ABABABB*, *ABABBAA*, *ABABBAB*, *ABABBB*, *ABBBAAA*, *ABBBAAB*, *ABBBAB*, *ABBBB*.

5.78. (a) 4, *HTHTHT*, *HTTHTH*, *HTHTTH*, *THTHTH*; (b) $n + 1$.

5.79. 18.

5.80. $n!C(n, 2)$.

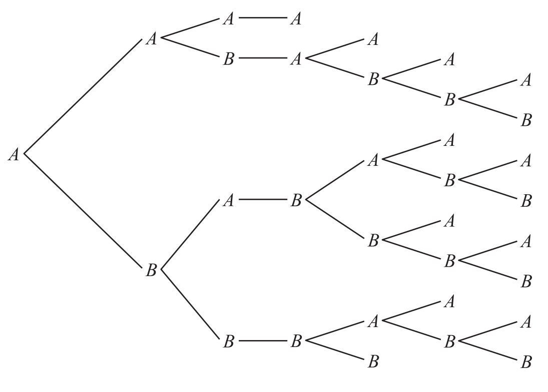


Fig. 5-4