CHAPTER 1

Set Theory

1.1 INTRODUCTION

The concept of a *set* appears in all mathematics. This chapter introduces the notation and terminology of set theory which is basic and used throughout the text. The chapter closes with the formal definition of mathematical induction, with examples.

1.2 SETS AND ELEMENTS, SUBSETS

A *set* may be viewed as any well-defined collection of objects, called the *elements* or *members* of the set. One usually uses capital letters, $A, B, X, Y, \dots$, to denote sets, and lowercase letters, $a, b, x, y, \dots$, to denote elements of sets. Synonyms for "set" are "class," "collection," and "family."

Membership in a set is denoted as follows:

$a \in S$ denotes that a belongs to a set S

$a, b \in S$ denotes that a and b belong to a set S

Here $\in$ is the symbol meaning "is an element of." We use $\notin$ to mean "is not an element of."

Specifying Sets

There are essentially two ways to specify a particular set. One way, if possible, is to list its members separated by commas and contained in braces { }. A second way is to state those properties which characterized the elements in the set. Examples illustrating these two ways are:

$$A = \{1, 3, 5, 7, 9\} \quad \text{and} \quad B = \{x \mid x \text{ is an even integer}, x > 0\}$$

That is, A consists of the numbers 1, 3, 5, 7, 9. The second set, which reads:

B is the set of x such that x is an even integer and x is greater than 0,

denotes the set B whose elements are the positive integers. Note that a letter, usually x, is used to denote a typical member of the set; and the vertical line | is read as "such that" and the comma as "and."

EXAMPLE 1.1

(a) The set A above can also be written as $A = \{x \mid x \text{ is an odd positive integer}, x < 10\}$.

(b) We cannot list all the elements of the above set B although frequently we specify the set by

$$B = \{2, 4, 6, \dots\}$$

where we assume that everyone knows what we mean. Observe that $8 \in B$, but $3 \notin B$.

(c) Let $E = \{x \mid x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$ and $G = \{1, 2, 2, 1\}$. Then $E = F = G$.

We emphasize that a set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged.

Even if we can list the elements of a set, it may not be practical to do so. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

Subsets

Suppose every element in a set A is also an element of a set B, that is, suppose $a \in A$ implies $a \in B$. Then A is called a *subset* of B. We also say that A is *contained* in B or that B *contains* A. This relationship is written

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is:

$$\boxed{A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A}$$

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A \not\subseteq B$.

EXAMPLE 1.2 Consider the sets:

$$A = \{1, 3, 4, 7, 8, 9\}, \quad B = \{1, 2, 3, 4, 5\}, \quad C = \{1, 3\}.$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 3, the elements of C, are also members of A and B. But $B \not\subseteq A$ since some of the elements of B, e.g., 2 and 5, do not belong to A. Similarly, $A \not\subseteq B$.

Property 1: It is common practice in mathematics to put a vertical line "|" or slanted line "/" through a symbol to indicate the opposite or negative meaning of a symbol.

Property 2: The statement $A \subseteq B$ does not exclude the possibility that $A = B$. In fact, for every set A we have $A \subseteq A$ since, trivially, every element in A belongs to A. However, if $A \subseteq B$ and $A \neq B$, then we say A is a proper subset of B (sometimes written $A \subset B$).

Property 3: Suppose every element of a set A belongs to a set B and every element of B belongs to a set C. Then clearly every element of A also belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The above remarks yield the following theorem.

Theorem 1.1: Let A, B, C be any sets. Then:

(i) $A \subseteq A$

(ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Special symbols

Some sets will occur very often in the text, and so we use special symbols for them. Some such symbols are:

$\mathbf{N}$ = the set of *natural numbers* or positive integers: $1, 2, 3, \ldots$
$\mathbf{Z}$ = the set of all integers: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbf{Q}$ = the set of rational numbers
$\mathbf{R}$ = the set of real numbers
$\mathbf{C}$ = the set of complex numbers

Observe that $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$.

Universal Set, Empty Set

All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the *universal set* which we denote by

$$\mathbf{U}$$

unless otherwise stated or implied.

Given a universal set **U** and a property P, there may not be any elements of **U** which have property P. For example, the following set has no elements:

$$S = \{x \mid x \text{ is a positive integer, } x^2 = 3\}$$

Such a set with no elements is called the *empty set* or *null set* and is denoted by

$$\emptyset$$

There is only one empty set. That is, if S and T are both empty, then $S = T$, since they have exactly the same elements, namely, none.

The empty set $\emptyset$ is also regarded as a subset of every other set. Thus we have the following simple result which we state formally.

Theorem 1.2: For any set A, we have $\emptyset \subseteq A \subseteq \mathbf{U}$.

Disjoint Sets

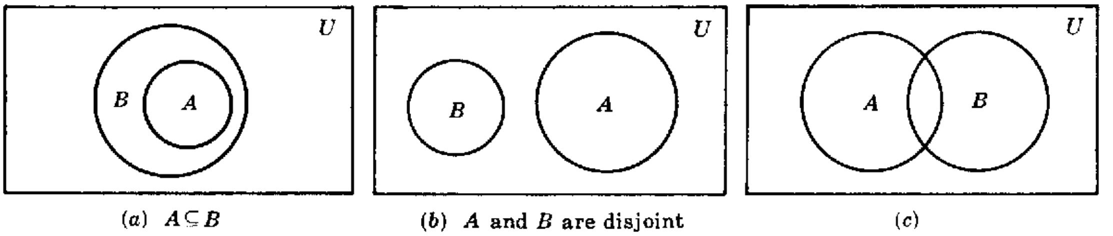
Two sets A and B are said to be *disjoint* if they have no elements in common. For example, suppose

$$A = \{1, 2\}, \quad B = \{4, 5, 6\}, \quad \text{and} \quad C = \{5, 6, 7, 8\}$$

Then A and B are disjoint, and A and C are disjoint. But B and C are not disjoint since B and C have elements in common, e.g., 5 and 6. We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).

1.3 VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane. The universal set **U** is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1-1(a). If A and B are disjoint, then the disk representing A will be separated from the disk representing B as in Fig. 1-1(b).



(a) $A \subsetneq B$ (b) A and B are disjoint (c)

Fig. 1-1

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. 1-1(c).

Arguments and Venn Diagrams

Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams. Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid.

EXAMPLE 1.3 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: All my tin objects are saucepans.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S : Your presents to me are not made of tin.

The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.

By S_1 the tin objects are contained in the set of saucepans, and by S_3 the set of saucepans and the set of useful things are disjoint. Furthermore, by S_2 the set of "your presents" is a subset of the set of useful things. Accordingly, we can draw the Venn diagram in Fig. 1-2.

The conclusion is clearly valid by the Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

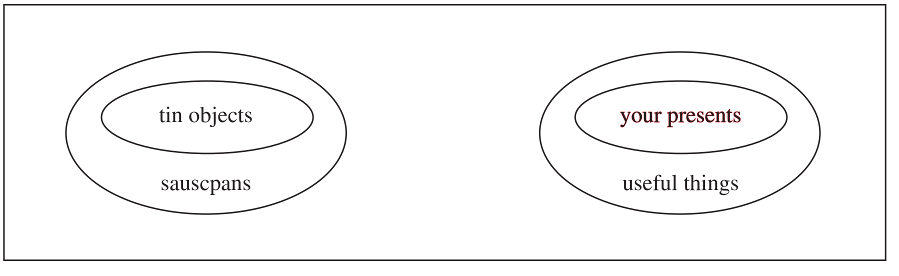


Fig. 1-2

1.4 SET OPERATIONS

This section introduces a number of set operations, including the basic operations of union, intersection, and complement.

Union and Intersection

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

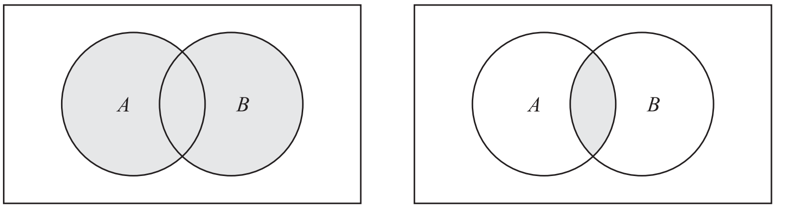
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Here "or" is used in the sense of and/or. Figure 1-3(a) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Figure 1-3(b) is a Venn diagram in which $A \cap B$ is shaded.



(a) $A \cup B$ is shaded (b) $A \cap B$ is shaded

Fig. 1-3

Recall that sets A and B are said to be *disjoint* or *nonintersecting* if they have no elements in common or, using the definition of intersection, if $A \cap B = \emptyset$, the empty set. Suppose

$$S = A \cup B \quad \text{and} \quad A \cap B = \emptyset$$

Then S is called the *disjoint union* of A and B.

EXAMPLE 1.4

(a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$. Then

$$\begin{array}{lll} A \cup B = \{1, 2, 3, 4, 5, 6, 7\}, & A \cup C = \{1, 2, 3, 4, 8, 9\}, & B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}, \\ A \cap B = \{3, 4\}, & A \cap C = \{2, 3\}, & B \cap C = \{3\}. \end{array}$$

(b) Let $\mathbf{U}$ be the set of students at a university, and let M denote the set of male students and let F denote the set of female students. The $\mathbf{U}$ is the disjoint union of M of F; that is,

$$\mathbf{U} = M \cup F \quad \text{and} \quad M \cap F = \emptyset$$

This comes from the fact that every student in $\mathbf{U}$ is either in M or in F, and clearly no student belongs to both M and F, that is, M and F are disjoint.

The following properties of union and intersection should be noted.

Property 1: Every element x in $A \cap B$ belongs to both A and B; hence x belongs to A and x belongs to B. Thus $A \cap B$ is a subset of A and of B; namely

$$A \cap B \subseteq A \quad \text{and} \quad A \cap B \subseteq B$$

Property 2: An element x belongs to the union $A \cup B$ if x belongs to A or x belongs to B; hence every element in A belongs to $A \cup B$, and every element in B belongs to $A \cup B$. That is,

$$A \subseteq A \cup B \quad \text{and} \quad B \subseteq A \cup B$$

We state the above results formally:

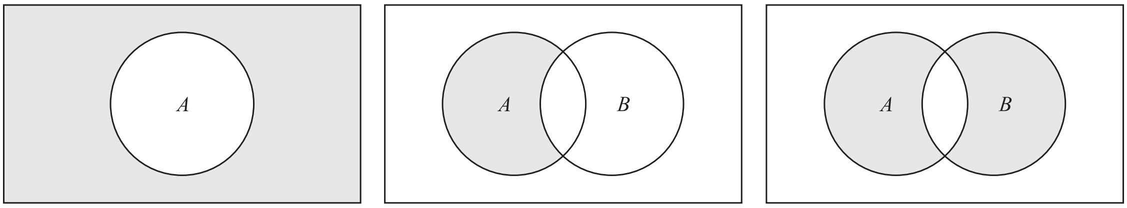
Theorem 1.3: For any sets A and B, we have:

(i) $A \cap B \subseteq A \subseteq A \cup B$ and (ii) $A \cap B \subseteq B \subseteq A \cup B$.

The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.

Theorem 1.4: The following are equivalent: $A \subseteq B$, $\quad A \cap B = A$, $\quad A \cup B = B$.

This theorem is proved in Problem 1.8. Other equivalent conditions to are given in Problem 1.31.



(a) A^C is shaded (b) $A \backslash B$ is shaded (c) $A \oplus B$ is shaded

Fig. 1-4

Complements, Differences, Symmetric Differences

Recall that all sets under consideration at a particular time are subsets of a fixed universal set **U**. The *absolute complement* or, simply, *complement* of a set A, denoted by A^C, is the set of elements which belong to **U** but which do not belong to A. That is,

$$A^C = \{x \mid x \in \mathbf{U}, x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$. Fig. 1-4(a) is a Venn diagram in which A^C is shaded.

The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A\backslash B$, is the set of elements which belong to A but which do not belong to B; that is

$$A\backslash B = \{x \mid x \in A, x \notin B\}$$

The set $A\backslash B$ is read "A minus B." Many texts denote $A\backslash B$ by $A - B$ or $A \sim B$. Fig. 1-4(b) is a Venn diagram in which $A\backslash B$ is shaded.

The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both. That is,

$$A \oplus B = (A \cup B)\backslash(A \cap B) \quad \text{or} \quad A \oplus B = (A\backslash B) \cup (B\backslash A)$$

Figure 1-4(c) is a Venn diagram in which $A \oplus B$ is shaded.

EXAMPLE 1.5 Suppose $\mathbf{U} = \mathbf{N} = \{1, 2, 3, \ldots\}$ is the universal set. Let

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{2, 3, 8, 9\}, \quad E = \{2, 4, 6, \ldots\}$$

(Here E is the set of even integers.) Then:

$$A^C = \{5, 6, 7, \ldots\}, \quad B^C = \{1, 2, 8, 9, 10, \ldots\}, \quad E^C = \{1, 3, 5, 7, \ldots\}$$

That is, E^C is the set of odd positive integers. Also:

$$\begin{array}{llll} A\backslash B = \{1, 2\}, & A\backslash C = \{1, 4\}, & B\backslash C = \{4, 5, 6, 7\}, & A\backslash E = \{1, 3\}, \\ B\backslash A = \{5, 6, 7\}, & C\backslash A = \{8, 9\}, & C\backslash B = \{2, 8, 9\}, & E\backslash A = \{6, 8, 10, 12, \ldots\}. \end{array}$$

Furthermore:

$$\begin{array}{ll} A \oplus B = (A\backslash B) \cup (B\backslash A) = \{1, 2, 5, 6, 7\}, & B \oplus C = \{2, 4, 5, 6, 7, 8, 9\}, \\ A \oplus C = (A\backslash C) \cup (B\backslash C) = \{1, 4, 8, 9\}, & A \oplus E = \{1, 3, 6, 8, 10, \ldots\}. \end{array}$$

Fundamental Products

Consider n distinct sets $A_1, A_2, \ldots, A_n$. A *fundamental product* of the sets is a set of the form

$$A_1^* \cap A_2^* \cap \ldots \cap A_n^* \quad \text{where} \quad A_i^* = A \quad \text{or} \quad A_i^* = A^C$$

We note that:

(i) There are $m = 2^n$ such fundamental products.
(ii) Any two such fundamental products are disjoint.
(iii) The universal set **U** is the union of all fundamental products.

Thus **U** is the disjoint union of the fundamental products (Problem 1.60). There is a geometrical description of these sets which is illustrated below.

EXAMPLE 1.6 Figure 1-5(a) is the Venn diagram of three sets A, B, C. The following lists the $m = 2^3 = 8$ fundamental products of the sets A, B, C:

$$P_1 = A \cap B \cap C, \quad P_3 = A \cap B^C \cap C, \quad P_5 = A^C \cap B \cap C, \quad P_7 = A^C \cap B^C \cap C,$$
$$P_2 = A \cap B \cap C^C, \quad P_4 = A \cap B^C \cap C^C, \quad P_6 = A^C \cap B \cap C^C, \quad P_8 = A^C \cap B^C \cap C^C.$$

The eight products correspond precisely to the eight disjoint regions in the Venn diagram of sets A, B, C as indicated by the labeling of the regions in Fig. 1-5(b).

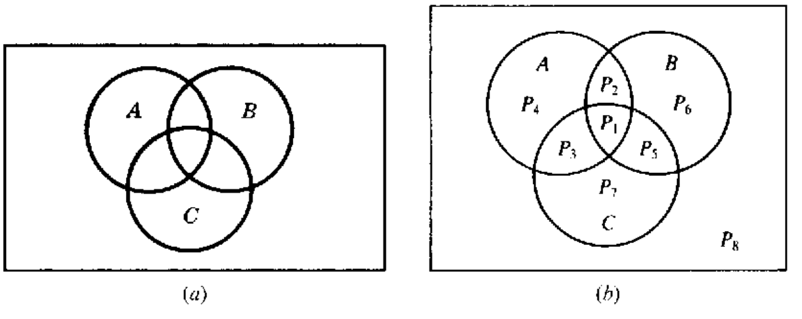


Fig. 1-5

1.5 ALGEBRA OF SETS, DUALITY

Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in Table 1-1. In fact, we formally state this as:

Theorem 1.5: Sets satisfy the laws in Table 1-1.

Table 1-1 Laws of the algebra of sets

Idempotent laws:	(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws:	(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	(5a) $A \cup \emptyset = A$	(5b) $A \cap \mathbf{U} = A$
	(6a) $A \cup \mathbf{U} = \mathbf{U}$	(6b) $A \cap \emptyset = \emptyset$
Involution laws:	(7) $(A^C)^C = A$	
Complement laws:	(8a) $A \cup A^C = \mathbf{U}$	(8b) $A \cap A^C = \emptyset$
	(9a) $\mathbf{U}^C = \emptyset$	(9b) $\emptyset^C = \mathbf{U}$
DeMorgan's laws:	(10a) $(A \cup B)^C = A^C \cap B^C$	(10b) $(A \cap B)^C = A^C \cup B^C$

Remark: Each law in Table 1-1 follows from an equivalent logical law. Consider, for example, the proof of DeMorgan's Law 10(a):

$$(A \cup B)^{\mathrm{C}} = \{x \mid x \notin (A \text{ or } B)\} = \{x \mid x \notin A \text{ and } x \notin B\} = A^{\mathrm{C}} \cap \mathrm{B}^{\mathrm{C}}$$

Here we use the equivalent (DeMorgan's) logical law:

$$\neg(p \vee q) = \neg p \wedge \neg q$$

where $\neg$ means "not," $\vee$ means "or," and $\wedge$ means "and." (Sometimes Venn diagrams are used to illustrate the laws in Table 1-1 as in Problem 1.17.)

Duality

The identities in Table 1-1 are arranged in pairs, as, for example, (2a) and (2b). We now consider the principle behind this arrangement. Suppose E is an equation of set algebra. The dual E^* of E is the equation obtained by replacing each occurrence of $\cup$, $\cap$, $\mathbf{U}$ and $\emptyset$ in E by $\cap$, $\cup$, $\emptyset$, and $\mathbf{U}$, respectively. For example, the dual of

$$(\mathbf{U} \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\emptyset \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that if any equation E is an identity then its dual E^* is also an identity.

1.6 FINITE SETS, COUNTING PRINCIPLE

Sets can be finite or infinite. A set S is said to be *finite* if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is *infinite*.

EXAMPLE 1.7

(a) The set A of the letters of the English alphabet and the set D of the days of the week are finite sets. Specifically, A has 26 elements and D has 7 elements.

(b) Let E be the set of even positive integers, and let $\mathbf{I}$ be the *unit interval*, that is,

$$E = \{2, 4, 6, \ldots\} \quad \text{and} \quad \mathbf{I} = [0, 1] = \{x \mid 0 \le x \le 1\}$$

Then both E and $\mathbf{I}$ are infinite.

A set $\boldsymbol{S}$ is *countable* if $\boldsymbol{S}$ is finite or if the elements of $\boldsymbol{S}$ can be arranged as a sequence, in which case $\boldsymbol{S}$ is said to be *countably infinite*; otherwise $\boldsymbol{S}$ is said to be *uncountable*. The above set E of even integers is countably infinite, whereas one can prove that the unit interval $\mathbf{I} = [0, 1]$ is uncountable.

Counting Elements in Finite Sets

The notation $n(S)$ or $|S|$ will denote the number of elements in a set S. (Some texts use #(S) or card(S) instead of $n(S)$.) Thus $n(A) = 26$, where A is the letters in the English alphabet, and $n(D) = 7$, where D is the days of the week. Also $n(\emptyset) = 0$ since the empty set has no elements.

The following lemma applies.

Lemma 1.6: Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

This lemma may be restated as follows:

Lemma 1.6: Suppose S is the disjoint union of finite sets A and B. Then S is finite and

$$n(S) = n(A) + n(B)$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

For any sets A and B, the set A is the disjoint union of $A \backslash B$ and $A \cap B$. Thus Lemma 1.6 gives us the following useful result.

Corollary 1.7: Let A and B be finite sets. Then

$$n(A \backslash B) = n(A) - n(A \cap B)$$

For example, suppose an art class A has 25 students and 10 of them are taking a biology class B. Then the number of students in class A which are not in class B is:

$$n(A \backslash B) = n(A) - n(A \cap B) = 25 - 10 = 15$$

Given any set A, recall that the universal set $\mathbf{U}$ is the disjoint union of A and A^{C}. Accordingly, Lemma 1.6 also gives the following result.

Corollary 1.8: Let A be a subset of a finite universal set $\mathbf{U}$. Then

$$n(A^{\mathrm{C}}) = n(\mathbf{U}) - n(A)$$

For example, suppose a class $\mathbf{U}$ with 30 students has 18 full-time students. Then there are $30 - 18 = 12$ part-time students in the class $\mathbf{U}$.

Inclusion–Exclusion Principle

There is a formula for $n(A \cup B)$ even when they are not disjoint, called the Inclusion–Exclusion Principle. Namely:

Theorem (Inclusion–Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.8) may be used to further generalize this result to any number of finite sets.

EXAMPLE 1.8 Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students: (a) only on list A, (b) only on list B, (c) on list A or B (or both), (d) on exactly one list.

(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

(d) By (a) and (b), $10 + 15 = 25$ names are only on one list; that is, $n(A \oplus B) = 25$.

1.7 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a *set of sets*. Whenever such a situation occurs, to avoid confusion, we will speak of a *class* of sets or *collection* of sets rather than a *set* of sets. If we wish to consider some of the sets in a given class of sets, then we speak of *subclass* or *subcollection*.

EXAMPLE 1.9 Suppose $S = \{1, 2, 3, 4\}$.

(a) Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

That is, the elements of A are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.

(b) Let B be the class of subsets of S, each which contains 2 and two other elements of S. Then

$$B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of B are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S , we may speak of the class of all subsets of S. This class is called the *power set* of S , and will be denoted by $P(S)$. If S is finite, then so is $P(S)$. In fact, the number of elements in $P(S)$ is 2 raised to the power $n(S)$. That is,

$$n(P(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

EXAMPLE 1.10 Suppose $S = \{1, 2, 3\}$. Then

$$P(S) = [\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$$

Note that the empty set $\emptyset$ belongs to $P(S)$ since $\emptyset$ is a subset of S. Similarly, S belongs to $P(S)$. As expected from the above remark, $P(S)$ has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A *partition* of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.
(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_j \neq A_k \quad \text{then} \quad A_j \cap A_k = \emptyset$$

The subsets in a partition are called *cells*. Figure 1-6 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, A_5.

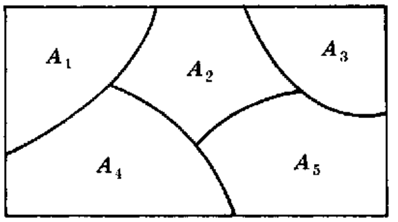


Fig. 1-6

EXAMPLE 1.11 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

Generalized Set Operations

The set operations of union and intersection were defined above for two sets. These operations can be extended to any number of sets, finite or infinite, as follows.

Consider first a finite number of sets, say, $A_1, A_2, \ldots, A_m$. The union and intersection of these sets are denoted and defined, respectively, by

$$A_1 \cup A_2 \cup \ldots \cup A_m = \bigcup_{i=1}^{m} A_i = \{x \mid x \in A_i \text{ for some } A_i\}$$
$$A_1 \cap A_2 \cap \ldots \cap A_m = \bigcap_{i=1}^{m} A_i = \{x \mid x \in A_i \text{ for every } A_i\}$$

That is, the union consists of those elements which belong to at least one of the sets, and the intersection consists of those elements which belong to all the sets.

Now let $\mathscr{A}$ be any collection of sets. The union and the intersection of the sets in the collection A is denoted and defined, respectively, by

$$\bigcup (A|A \in \mathscr{A}) = \{x \mid x \in A_i \text{ for some } A_i \in \mathscr{A}\}$$
$$\bigcap (A|A \in \mathscr{A}) = \{x \mid x \in A_i \text{ for every } A_i \in \mathscr{A}\}$$

That is, the union consists of those elements which belong to at least one of the sets in the collection $\mathscr{A}$ and the intersection consists of those elements which belong to every set in the collection A.

EXAMPLE 1.12 Consider the sets

$$A_1 = \{1, 2, 3, \ldots\} = \mathbf{N}, \quad A_2 = \{2, 3, 4, \ldots\}, \quad A_3 = \{3, 4, 5, \ldots\}, \quad A_n = \{n, n+1, n+2, \ldots\}.$$

Then the union and intersection of the sets are as follows:

$$\bigcup (A_k \mid k \in \mathbf{N}) = \mathbf{N} \quad \text{and} \quad \bigcap (A_k \mid k \in \mathbf{N}) = \emptyset$$

DeMorgan's laws also hold for the above generalized operations. That is:

Theorem 1.11: Let $\mathscr{A}$ be a collection of sets. Then:

(i) $\left[\bigcup (A \mid A \in \mathscr{A})\right]^{C} = \bigcap (A^{C} \mid A \in \mathscr{A})$

(ii) $\left[\bigcap (A \mid A \in \mathscr{A})\right]^{C} = \bigcup (A^{C} \mid A \in \mathscr{A})$

1.8 MATHEMATICAL INDUCTION

An essential property of the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers follows:

Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers $\mathbf{N}$; that is, $P(n)$ is either true or false for each $n \in \mathbf{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(k + 1)$ is true whenever $P(k)$ is true.

Then P is true for every positive integer $n \in \mathbf{N}$.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when $\mathbf{N}$ is developed axiomatically.

EXAMPLE 1.13 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The kth odd number is $2k - 1$, and the next odd number is $2k + 1$.) Observe that $P(n)$ is true for $n = 1$; namely,

$$P(1) = 1^2$$

Assuming $P(k)$ is true, we add $2k + 1$ to both sides of $P(k)$, obtaining

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) - k^2 + (2k + 1) = (k + 1)^2$$

which is $P(k + 1)$. In other words, $P(k + 1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

There is a form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the above principle of induction.

Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers $\mathbf{N}$ such that:

(i) $P(1)$ is true.

(ii) $P(k)$ is true whenever $P(j)$ is true for all $1 \leq j < k$.

Then P is true for every positive integer $n \in \mathbf{N}$.

Remark: Sometimes one wants to prove that a proposition P is true for the set of integers

$$\{a, a + 1, a + 2, a + 3, \ldots\}$$

where a is any integer, possibly zero. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.

Solved Problems

SETS AND SUBSETS

1.1 Which of these sets are equal: $\{x, y, z\}$, $\{z, y, z, x\}$, $\{y, x, y, z\}$, $\{y, z, x, y\}$?

They are all equal. Order and repetition do not change a set.

1.2 List the elements of each set where $\mathbf{N} = \{1, 2, 3, \ldots\}$.

(*a*) $A = \{x \in \mathbf{N} \mid 3 < x < 9\}$

(*b*) $B = \{x \in \mathbf{N} \mid x \text{ is even}, x < 11\}$

(*c*) $C = \{x \in \mathbf{N} \mid 4 + x = 3\}$

(*a*) A consists of the positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.

(*b*) B consists of the even positive integers less than 11; hence $B = \{2, 4, 6, 8, 10\}$.

(*c*) No positive integer satisfies $4 + x = 3$; hence $C = \emptyset$, the empty set.

1.3 Let $A = \{2, 3, 4, 5\}$.

(*a*) Show that A is not a subset of $B = \{x \in \mathbf{N} \mid x \text{ is even}\}$.

(*b*) Show that A is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

(*a*) It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

(*b*) Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

SET OPERATIONS

1.4 Let $\mathbf{U} = \{1, 2, \ldots, 9\}$ be the universal set, and let

$$A = \{1, 2, 3, 4, 5\}, \quad C = \{5, 6, 7, 8, 9\}, \quad E = \{2, 4, 6, 8\},$$
$$B = \{4, 5, 6, 7\}, \quad D = \{1, 3, 5, 7, 9\}, \quad F = \{1, 5, 9\}.$$

Find: (*a*) $A \cup B$ and $A \cap B$; (*b*) $A \cup C$ and $A \cap C$; (*c*) $D \cup F$ and $D \cap F$.

Recall that the union $X \cup Y$ consists of those elements in either X or Y (or both), and that the intersection $X \cap Y$ consists of those elements in both X and Y.

(*a*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ and $A \cap B = \{4, 5\}$

(*b*) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathbf{U}$ and $A \cap C = \{5\}$

(*c*) $D \cup F = \{1, 3, 5, 7, 9\} = D$ and $D \cap F = (1, 5, 9) = F$

Observe that $F \subseteq D$, so by Theorem 1.4 we must have $D \cup F = D$ and $D \cap F = F$.

1.5 Consider the sets in the preceding Problem 1.4. Find:

(*a*) A^C, B^C, D^C, E^C; (*b*) $A \backslash B, B \backslash A, D \backslash E$; (*c*) $A \oplus B,\ C \oplus D,\ E \oplus F$.

Recall that:

(1) The complements X^C consists of those elements in $\mathbf{U}$ which do not belong to X.

(2) The difference $X \backslash Y$ consists of the elements in X which do not belong to Y.

(3) The symmetric difference $X \oplus Y$ consists of the elements in X or in Y but not in both.

Therefore:

(*a*) $A^C = \{6, 7, 8, 9\}$; $B^C = \{1, 2, 3, 8, 9\}$; $D^C = \{2, 4, 6, 8\} = E$; $E^C = \{1, 3, 5, 7, 9\} = D$.

(*b*) $A \backslash B = \{1, 2, 3\}$; $B \backslash A = \{6, 7\}$; $D \backslash E = \{1, 3, 5, 7, 9\} = D$; $F \backslash D = \emptyset$.

(*c*) $A \oplus B = \{1, 2, 3, 6, 7\}$; $C \oplus D = \{1, 3, 6, 8\}$; $E \oplus F = \{2, 4, 6, 8, 1, 5, 9\} = E \cup F$.

1.6 Show that we can have: (*a*) $A \cap B = A \cap C$ without $B = C$; (*b*) $A \cup B = A \cup C$ without $B = C$.

(*a*) Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$; but $B \neq C$.

(*b*) Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and $A \cup C = \{1, 2, 3\}$ but $B \neq C$.

1.7 Prove: $B \backslash A = B \cap A^C$. Thus, the set operation of difference can be written in terms of the operations of intersection and complement.

$$B \backslash A = \{x \mid x \in B,\ x \notin A\} = \{x \mid x \in B,\ x \in A^C\} = B \cap A^C.$$

1.8 Prove Theorem 1.4. The following are equivalent: $A \subseteq B$, $A \cap B = A$, $A \cup B = B$.

Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Theorem 1.3, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$; hence $x \in A$ and $x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

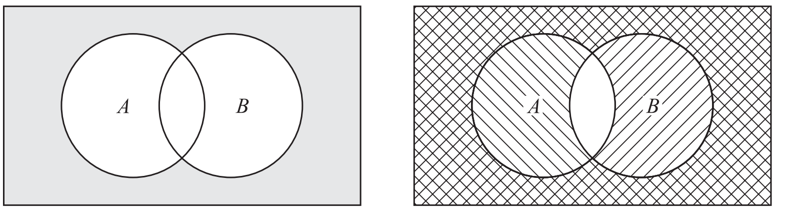
Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In either case, $x \in B$. Therefore $A \cup B \subseteq B$. By Theorem 1.3, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B = B$ and let $x \in A$. Then $x \in A \cup B$ by definition of the union of sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cup B = A$ and $A \cup B = B$ are equivalent.

VENN DIAGRAMS, ALGEBRA OF SETS, DUALITY

1.9 Illustrate DeMorgan's Law $(A \cup B)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}}$ using Venn diagrams.

Shade the area outside $A \cup B$ in a Venn diagram of sets A and B. This is shown in Fig. 1-7(*a*); hence the shaded area represents $(A \cup B)^{\mathrm{C}}$. Now shade the area outside A in a Venn diagram of A and B with strokes in one direction (////), and then shade the area outside B with strokes in another direction (\\\\\\\\). This is shown in Fig. 1-7(*b*); hence the cross-hatched area (area where both lines are present) represents $A^{\mathrm{C}} \cap B^{\mathrm{C}}$. Both $(A \cup B)^{\mathrm{C}}$ and $A^{\mathrm{C}} \cap B^{\mathrm{C}}$ are represented by the same area; thus the Venn diagram indicates $(A \cup B)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}}$. (We emphasize that a Venn diagram is not a formal proof, but it can indicate relationships between sets.)



(*a*) (*b*)

Fig. 1-7

1.10 Prove the Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A, x \in (B \cup C)\} \\ &= \{x \mid x \in A, x \in B \text{ or } x \in A, x \in C\} = (A \cap B) \cup (A \cap C) \end{aligned}$$

Here we use the analogous logical law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ where $\wedge$ denotes "and" and $\vee$ denotes "or."

1.11 Write the dual of: (*a*) $(\mathbf{U} \cap A) \cup (B \cap A) = A$; (*b*) $(A \cap \mathbf{U}) \cap (\emptyset \cup A^{\mathrm{C}}) = \emptyset$.

Interchange $\cup$ and $\cap$ and also $\mathbf{U}$ and $\emptyset$ in each set equation:

(*a*) $(\emptyset \cup A) \cap (B \cup A) = A$; (*b*) $(A \cup \emptyset) \cup (\mathbf{U} \cap A^{\mathrm{C}}) = \mathbf{U}$.

1.12 Prove: $(A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$. (Thus either one may be used to define $A \oplus B$.)

Using $X \backslash Y = X \cap Y^{\mathrm{C}}$ and the laws in Table 1.1, including DeMorgan's Law, we obtain:

$$\begin{aligned} (A \cup B) \backslash (A \cap B) &= (A \cup B) \cap (A \cap B)^{\mathrm{C}} = (A \cup B) \cap (A^{\mathrm{C}} \cup B^{\mathrm{C}}) \\ &= (A \cup A^{\mathrm{C}}) \cup (A \cap B^{\mathrm{C}}) \cup (B \cap A^{\mathrm{C}}) \cup (B \cap B^{\mathrm{C}}) \\ &= \emptyset \cup (A \cap B^{\mathrm{C}}) \cup (B \cap A^{\mathrm{C}}) \cup \emptyset \\ &= (A \cap B^{\mathrm{C}}) \cup (B \cap A^{\mathrm{C}}) = (A \backslash B) \cup (B \backslash A) \end{aligned}$$

1.13 Determine the validity of the following argument:

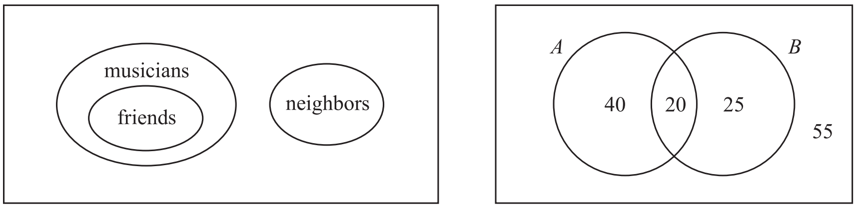
S_1: All my friends are musicians.

S_2: John is my friend.

S_3: None of my neighbors are musicians.

S : John is not my neighbor.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-8(a). By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.



(a) (b)

Fig. 1-8

FINITE SETS AND THE COUNTING PRINCIPLE

1.14 Each student in Liberal Arts at some college has a mathematics requirement A and a science requirement B. A poll of 140 sophomore students shows that:

60 completed A, 45 completed B, 20 completed both A and B.

Use a Venn diagram to find the number of students who have completed:

(a) At least one of A and B; (b) exactly one of A or B; (c) neither A nor B.

Translating the above data into set notation yields:

$n(A) = 60,\ n(B) = 45,\ n(A \cap B) = 20,\ n(\mathbf{U}) = 140$

Draw a Venn diagram of sets A and B as in Fig. 1-1(c). Then, as in Fig. 1-8(b), assign numbers to the four regions as follows:

20 completed both A and B, so $n(A \cap B) = 20$.

$60 - 20 = 40$ completed A but not B, so $n(A \backslash B) = 40$.

$45 - 20 = 25$ completed B but not A, so $n(B \backslash A) = 25$.

$140 - 20 - 40 - 25 = 55$ completed neither A nor B.

By the Venn diagram:

(a) $20 + 40 + 25 = 85$ completed A or B. Alternately, by the Inclusion–Exclusion Principle:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 60 + 45 - 20 = 85$$

(b) $40 + 25 = 65$ completed exactly one requirement. That is, $n(A \oplus B) = 65$.

(c) 55 completed neither requirement, i.e. $n(A^{\mathrm{C}} \cap B^{\mathrm{C}}) = n[(A \cup B)^{\mathrm{C}}] = 140 - 85 = 55$.

1.15 In a survey of 120 people, it was found that:

65 read *Newsweek* magazine, 20 read both *Newsweek* and *Time*,
45 read *Time*, 25 read both *Newsweek* and *Fortune*,
42 read *Fortune*, 15 read both *Time* and *Fortune*,
8 read all three magazines.

(a) Find the number of people who read at least one of the three magazines.

(b) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig. 1-9(a) where N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune*, respectively.

(c) Find the number of people who read exactly one magazine.

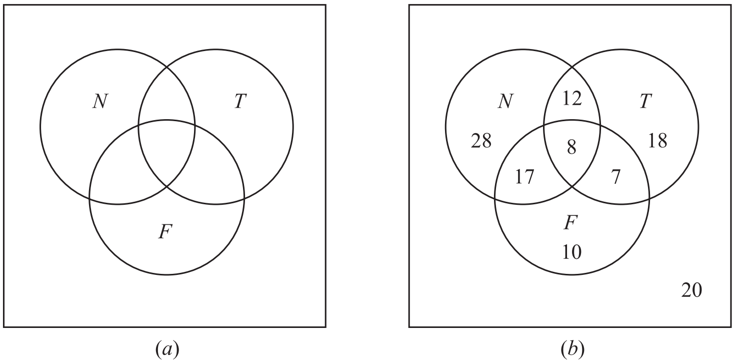


Fig. 1-9

(a) We want to find $n(N \cup T \cup F)$. By Corollary 1.10 (Inclusion–Exclusion Principle),

$$\begin{aligned} n(N \cup T \cup F) &= n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

(b) The required Venn diagram in Fig. 1-9(b) is obtained as follows:

8 read all three magazines,

$20 - 8 = 12$ read *Newsweek* and *Time* but not all three magazines,

$25 - 8 = 17$ read *Newsweek* and *Fortune* but not all three magazines,

$15 - 8 = 7$ read *Time* and *Fortune* but not all three magazines,

$65 - 12 - 8 - 17 = 28$ read only *Newsweek*,

$45 - 12 - 8 - 7 = 18$ read only *Time*,

$42 - 17 - 8 - 7 = 10$ read only *Fortune*,

$120 - 100 = 20$ read no magazine at all.

(c) $28 + 18 + 10 = 56$ read exactly one of the magazines.

1.16 Prove Theorem 1.9. Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are finite then, clearly, $A \cup B$ and $A \cap B$ are finite.

Suppose we count the elements in A and then count the elements in B.

Then every element in $A \cap B$ would be counted twice, once in A and once in B. Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

CLASSES OF SETS

1.17 Let $A = [\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}]$. (a) List the elements of A; (b) Find $n(A)$.

(a) A has three elements, the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.

(b) $n(A) = 3$.

1.18 Determine the power set $P(A)$ of $A = \{a, b, c, d\}$.

The elements of $P(A)$ are the subsets of A. Hence

$$P(A) = [A, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset]$$

As expected, $P(A)$ has $2^4 = 16$ elements.

1.19 Let $S = \{a, b, c, d, e, f, g\}$. Determine which of the following are partitions of S:

(a) $P_1 = [\{a, c, e\}, \{b\}, \{d, g\}]$, (c) $P_3 = [\{a, b, e, g\}, \{c\}, \{d, f\}]$,
(b) $P_2 = [\{a, e, g\}, \{c, d\}, \{b, e, f\}]$, (d) $P_4 = [\{a, b, c, d, e, f, g\}]$.

(a) P_1 is not a partition of S since $f \in S$ does not belong to any of the cells.

(b) P_2 is not a partition of S since $e \in S$ belongs to two of the cells.

(c) P_3 is a partition of S since each element in S belongs to exactly one cell.

(d) P_4 is a partition of S into one cell, S itself.

1.20 Find all partitions of $S = \{a, b, c, d\}$.

Note first that each partition of S contains either 1, 2, 3, or 4 distinct cells. The partitions are as follows:

(1) $[\{a, b, c, d\}]$

(2) $[\{a\}, \{b, c, d\}], [\{b\}, \{a, c, d\}], [\{c\}, \{a, b, d\}], [\{d\}, \{a, b, c\}]$,
$[\{a, b\}, \{c, d\}], [\{a, c\}, \{b, d\}], [\{a, d\}, \{b, c\}]$

(3) $[\{a\}, \{b\}, \{c, d\}], [\{a\}, \{c\}, \{b,d\}], [\{a\}, \{d\}, \{b, c\}]$,
$[\{b\}, \{c\}, \{a, d\}], [\{b\}, \{d\}, \{a, c\}], [\{c\}, \{d\}, \{a, b\}]$

(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$

There are 15 different partitions of S.

1.21 Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and, for each $n \in \mathbf{N}$, Let $A_n = \{n, 2n, 3n, \ldots\}$. Find:

(a) $A_3 \cap A_5$; (b) $A_4 \cap A_5$; (c) $\bigcup_{i \in Q} A_i$ where $Q = \{2, 3, 5, 7, 11, \ldots\}$ is the set of prime numbers.

(a) Those numbers which are multiples of both 3 and 5 are the multiples of 15; hence $A_3 \cap A_5 = A_{15}$.

(b) The multiples of 12 and no other numbers belong to both A_4 and A_6, hence $A_4 \cap A_6 = A_{12}$.

(c) Every positive integer except 1 is a multiple of at least one prime number; hence

$$\bigcup_{i \in Q} A_i = \{2, 3, 4, \ldots\} = \mathbf{N} \backslash \{1\}$$

1.22 Let $\{A_i \mid i \in I\}$ be an indexed class of sets and let $i_0 \in I$. Prove

$$\bigcap_{i \in I} A_i \subseteq A_{i_0} \subseteq \bigcup_{i \in I} A_i.$$

Let $x \in \bigcap_{i \in I} A_i$ then $x \in A_i$ for every $i \in I$. In particular, $x \in A_{i_0}$. Hence $\bigcap_{i \in I} A_i \subseteq A_{i_0}$. Now let $y \in A_{i_0}$. Since $i_0 \in I$, $y \in \bigcap_{i \in I} A_i$. Hence $A_{i_0} \subseteq \bigcup_{i \in I} A_i$.

1.23 Prove (De Morgan's law): For any indexed class $\{A_i \mid i \in I\}$, we have $\left(\bigcup_i A_i\right)^C = \bigcap_i A_i^C$.

Using the definitions of union and intersection of indexed classes of sets:

$$\begin{aligned}\left(\bigcup_i A_i\right)^C &= \{x \mid x \notin \bigcup_i A_i\} = \{x \mid x \notin A_i \text{ for every } i\} \\ &= \{x \mid x \in A_i^C \text{ for every } i\} = \bigcap_i A_i^C\end{aligned}$$

Supplementary Problems

SETS AND SUBSETS

1.26 Which of the following sets are equal?

$$\begin{array}{llll} A = \{x \mid x^2 - 4x + 3 = 0\}, & C = \{x \mid x \in \mathbf{N}, x < 3\}, & E = \{1, 2\}, & G = \{3, 1\}, \\ B = \{x \mid x^2 - 3x + 2 = 0\}, & D = \{x \mid x \in \mathbf{N}, x \text{ is odd}, x < 5\}, & F = \{1, 2, 1\}, & H = \{1, 1, 3\}. \end{array}$$

1.27 List the elements of the following sets if the universal set is $\mathbf{U} = \{a, b, c, \ldots, y, z\}$.

Furthermore, identify which of the sets, if any, are equal.

$$\begin{array}{ll} A = \{x \mid x \text{ is a vowel}\}, & C = \{x \mid x \text{ precedes } f \text{ in the alphabet}\}, \\ B = \{x \mid x \text{ is a letter in the word "little"}\}, & D = \{x \mid x \text{ is a letter in the word "title"}\}. \end{array}$$

1.28 Let $A = \{1, 2, \ldots, 8, 9\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 3, 5, 7, 9\}$, $D = \{3, 4, 5\}$, $E = \{3, 5\}$.
Which of the these sets can equal a set X under each of the following conditions?

(a) X and B are disjoint.

(b) $X \subseteq D$ but $X \not\subseteq B$.

(c) $X \subseteq A$ but $X \not\subset C$.

(d) $X \subseteq C$ but $X \not\subset A$.

SET OPERATIONS

1.29 Consider the universal set $\mathbf{U} = \{1, 2, 3, \dots, 8, 9\}$ and sets $A = \{1, 2, 5, 6\}$, $B = \{2, 5, 7\}$, $C = \{1, 3, 5, 7, 9\}$. Find:

(a) $A \cap B$ and $A \cap C$ (c) A^C and C^C (e) $A \oplus B$ and $A \oplus C$

(b) $A \cup B$ and $B \cup C$ (d) $A \backslash B$ and $A \backslash C$ (f) $(A \cup C) \backslash B$ and $(B \oplus C) \backslash A$

1.30 Let A and B be any sets. Prove:

(a) A is the disjoint union of $A \backslash B$ and $A \cap B$.

(b) $A \cup B$ is the disjoint union of $A \backslash B$, $A \cap B$, and $B \backslash A$.

1.31 Prove the following:

(a) $A \subseteq B$ if and only if $A \cap B^C = \emptyset$ (c) $A \subseteq B$ if and only if $B^C \subseteq A^C$

(b) $A \subseteq B$ if and only if $A^C \cup B = \mathbf{U}$ (d) $A \subseteq B$ if and only if $A \backslash B = \emptyset$

(Compare the results with Theorem 1.4.)

1.32 Prove the Absorption Laws: (a) $A \cup (A \cap B) = A$; (b) $A \cap (A \cup B) = A$.

1.33 The formula $A \backslash B = A \cap B^C$ defines the difference operation in terms of the operations of intersection and complement. Find a formula that defines the union $A \cup B$ in terms of the operations of intersection and complement.

VENN DIAGRAMS

1.34 The Venn diagram in Fig. 1-5(a) shows sets A, B, C. Shade the following sets:

(a) $A \backslash (B \cup C)$; (b) $A^C \cap (B \cup C)$; (c) $A^C \cap (C \backslash B)$.

1.35 Use the Venn diagram in Fig. 1-5(b) to write each set as the (disjoint) union of fundamental products:

(a) $A \cap (B \cup C)$; (b) $A^C \cap (B \cup C)$; (c) $A \cup (B \backslash C)$.

1.36 Consider the following assumptions:

S_1: All dictionaries are useful.

S_2: Mary owns only romance novels.

S_3: No romance novel is useful.

Use a Venn diagram to determine the validity of each of the following conclusions:

(a) Romance novels are not dictionaries.

(b) Mary does not own a dictionary.

(c) All useful books are dictionaries.

ALGEBRA OF SETS AND DUALITY

1.37 Write the dual of each equation:

(a) $A = (B^C \cap A) \cup (A \cap B)$

(b) $(A \cap B) \cup (A^C \cap B) \cup (A \cap B^C) \cup (A^C \cap B^C) = \mathbf{U}$

1.38 Use the laws in Table 1-1 to prove each set identity:

(a) $(A \cap B) \cup (A \cap B^C) = A$

(b) $A \cup B = (A \cap B^C) \cup (A^C \cap B) \cup (A \cap B)$

FINITE SETS AND THE COUNTING PRINCIPLE

1.39 Determine which of the following sets are finite:

(a) Lines parallel to the x axis. (c) Integers which are multiples of 5.

(b) Letters in the English alphabet. (d) Animals living on the earth.

1.40 Use Theorem 1.9 to prove Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

1.41 A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning (A),	5 had A and P,	
12 had radio (R),	9 had A and R,	3 had all three options.
11 had power windows (W),	4 had R and W,	

Find the number of cars that had: (a) only W; (b) only A; (c) only R; (d) R and W but not A; (e) A and R but not W; (f) only one of the options; (g) at least one option; (h) none of the options.

CLASSES OF SETS

1.42 Find the power set $P(A)$ of $A = \{1, 2, 3, 4, 5\}$.

1.43 Given $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

(a) List the elements of A. (b) Find $n(A)$. (c) Find the power set of A.

1.44 Suppose A is finite and $n(A) = m$. Prove the power set $P(A)$ has 2^m elements.

PARTITIONS

1.45 Let $S = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of S:

(a) $[\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}]$ (c) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$

(b) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$ (d) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

1.46 Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of S:

(a) $P_1 = [\{1, 2, 3\}, \{1, 4, 5, 6\}]$ (c) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$

(b) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$ (d) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

1.47 Determine whether or not each of the following is a partition of the set $\mathbf{N}$ of positive integers:

(a) $[\{n \mid n > 5\}, \{n \mid n < 5\}]$; (b) $[\{n \mid n > 6\}, \{1, 3, 5\}, \{2, 4\}]$;

(c) $[\{n \mid n^2 > 11\}, \{n \mid n^2 < 11\}]$.

1.48 Let $[A_1, A_2, \ldots, A_m]$ and $[B_1, B_2, \ldots, B_n]$ be partitions of a set S.

Show that the following collection of sets is also a partition (called the *cross partition*) of S:

$$P = [A_i \cap B_j | i = 1, \ldots, m,\ j = 1, \ldots, n] \backslash \emptyset$$

Observe that we deleted the empty set $\emptyset$.

1.49 Let $S = \{1, 2, 3, \ldots, 8, 9\}$. Find the cross partition P of the following partitions of S:

$$P_1 = [\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}] \quad \text{and} \quad P_2 = [\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}]$$

Miscellaneous Problems

1.58 Suppose $\mathbf{N} = \{1, 2, 3, \ldots\}$ is the universal set, and

$$A = \{n \mid n \leq 6\}, \quad B = \{n \mid 4 \leq n \leq 9\}, \quad C = \{1, 3, 5, 7, 9\}, \quad D = \{2, 3, 5, 7, 8\}.$$

Find: (a) $A \oplus B$; (b) $B \oplus C$; (c) $A \cap (B \oplus D)$; (d) $(A \cap B) \oplus (A \cap D)$.

1.59 Prove the following properties of the symmetric difference:

(a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (Associative Law).

(b) $A \oplus B = B \oplus A$ (Commutative Law).

(c) If $A \oplus B = A \oplus C$, then $B = C$ (Cancellation Law).

(d) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ (Distributive Law).

1.60 Consider m nonempty distinct sets $A_1, A_2, \ldots, A_m$ in a universal set $\mathbf{U}$. Prove:

(a) There are 2^m fundamental products of the m sets.

(b) Any two fundamental products are disjoint.

(c) $\mathbf{U}$ is the union of all the fundamental products.

Answers to Supplementary Problems

1.26 $B = C = E = F$, $A = D = G = H$.

1.27 $A = \{a, e, i, o, u\}$, $B = D = \{l, i, t, e\}$, $C = \{a, b, c, d, e\}$.

1.28 (a) C and E; (b) D and E; (c) A, B, and D; (d) None.

1.29 (a) $A \cap B = \{2, 5\}$, $A \cap C = \{1, 5\}$;
(b) $A \cup B = \{1, 2, 5, 6, 7\}$, $B \cup C = \{1, 2, 3, 5, 7, 9\}$;
(c) $A^C = \{3, 4, 7, 8, 9\}$, $C^C = \{2, 4, 6, 8\}$;
(d) $A \backslash B = \{1, 6\}$, $A \backslash C = \{2, 6\}$;
(e) $A \oplus B = \{1, 6, 7\}$, $A \oplus C = \{2, 3, 6, 7, 9\}$;
(f) $(A \cup C) \backslash B = \{1, 3, 6, 9\}$, $(B \oplus C) \backslash A = \{3, 9\}$.

1.33 $A \cup B = (A^C \cap B^C)^C$.

1.34 See Fig. 1-10.

1.35 (a) $(A \cap B \cap C) \cup (A \cap B \cap C^C) \cup (A \cap B^C \cap C)$

(b) $(A^C \cap B \cap C^C) \cup (A^C \cap B \cap C) \cup (A^C \cap B^C \cap C)$

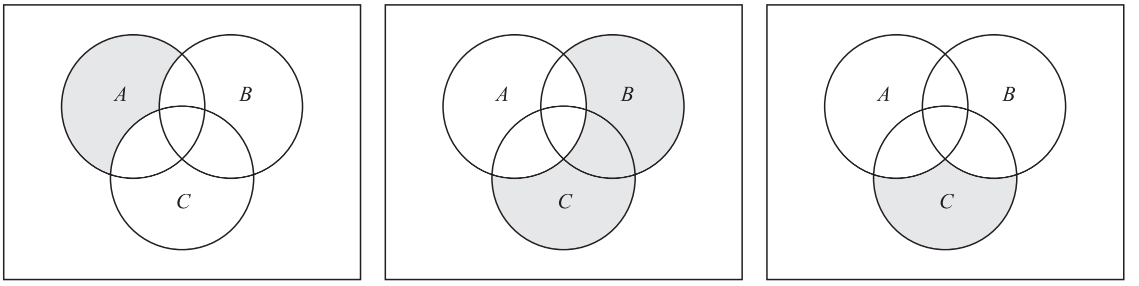
(c) $(A \cap B \cap C) \cup (A \cap B \cap C^C) \cup (A \cap B^C \cap C)$
$\cup (A^C \cap B \cap C^C) \cup (A \cap B^C \cap C^C)$

1.36 The three premises yield the Venn diagram in Fig. 1-11(*a*). (a) and (b) are valid, but (c) is not valid.

1.37 (a) $A = (B^C \cup A) \cap (A \cup B)$

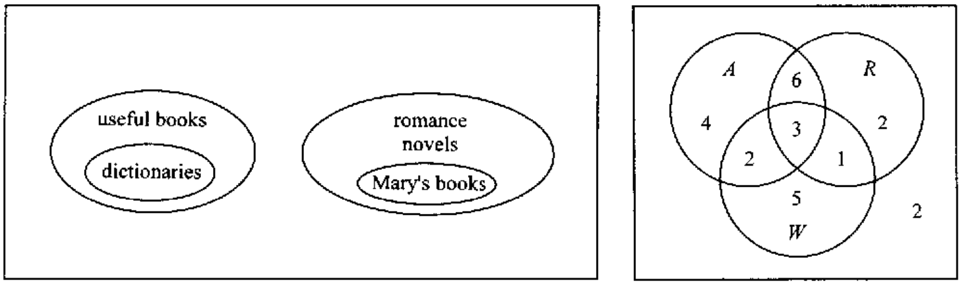
(b) $(A \cup B) \cap (A^C \cup B) \cap (A \cup B^C) \cap (A^C \cup B^C) = \emptyset$

1.39 (a) Infinite; (b) finite; (c) infinite; (d) finite.



(a) (b) (c)

Fig. 1-10



(a) (b)

Fig. 1-11

1.41 Use the data to fill in the Venn diagram in Fig. 1-11(b). Then:

(a) 5; (b) 4; (c) 2; (d) 1; (e) 6; (f) 11; (g) 23; (h) 2.

1.42 $P(A)$ has $2^5 = 32$ elements as follows:

[∅, {1}, {2}, {3}, {4}, {5}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}, {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {2, 3, 4}, {2, 3, 5}, {3, 4, 5}, {1, 3, 4}, {1, 3, 5}, {1, 4, 5}, {2, 4, 5}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5}, A]

1.43 (a) Three elements: $[a, b]$, (c), and $\{d, e, f\}$. (b) 3. (c) $P(A)$ has $2^3 = 8$ elements as follows:

$$P(A) = \{A, [\{a, b\}, \{c\}], [\{a, b\}, \{d, e, f\}],$$
$$[\{c\}, \{d, e, f\}], [\{a, b\}], [\{c\}], [\{d, e, f\}], \emptyset\}$$

1.44 Let X be an element in $P(A)$. For each $a \in A$, either $a \in X$ or $a \notin X$. Since $n(A) = m$, there are 2^m different sets X. That is $|P(A)| = 2^m$.

1.45 (a) No, (b) no, (c) yes, (d) yes.

1.46 (a) No, (b) no, (c) yes, (d) no.

1.47 (a) No, (b) no, (c) yes.

1.49 [{1,3}, {2,4}, {5,7}, {9}, {6,8}]