CHAPTER 2

Relations

2.1 INTRODUCTION

The reader is familiar with many relations such as "less than," "is parallel to," "is a subset of," and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs."

An *ordered pair* of elements a and b, where a is designated as the first element and b as the second element, is denoted by (a, b). In particular,

$$(a, b) = (c, d)$$

if and only if $a = c$ and $b = d$. Thus $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets where the order of elements is irrelevant; for example, $\{3, 5\} = \{5, 3\}$.

2.2 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B." By definition,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

EXAMPLE 2.1 $\mathbf{R}$ denotes the set of real numbers and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 2-1. Here each point P represents an ordered pair (a, b) of real numbers and vice versa; the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

EXAMPLE 2.2 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a),\ (1, b),\ (1, c),\ (2, a),\ (2, b),\ (2, c)\}$$
$$B \times A = \{(a, 1),\ (b, 1),\ (c, 1),\ (a, 2),\ (b, 2),\ (c, 2)\}$$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

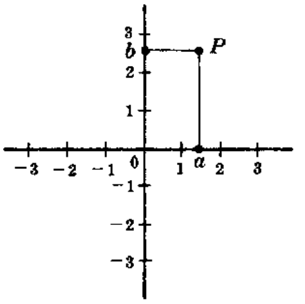


Fig. 2-1

There are two things worth noting in the above examples. First of all $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using $n(S)$ for the number of elements in a set S, we have:

$$n(A \times B) = 6 = 2(3) = n(A)n(B)$$

In fact, $n(A \times B) = n(A)n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1, A_2, \ldots, A_n$, the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$ is called the *product* of the sets $A_1, \ldots, A_n$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n \quad \text{or} \quad \prod_{i=1}^{n} A_1$$

Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

2.3 RELATIONS

We begin with a definition.

Definition 2.1: Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is R-related to b", written aRb.
(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a \not R b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation *on* A.

The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to R, and the *range* is the set of second elements.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 2.10, the term relation shall then mean binary relation unless otherwise stated or implied.

EXAMPLE 2.3

(a) $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, 1Rz, 3Ry, \quad \text{but} \quad 1\not{R}x, 2\not{R}x, 2\not{R}y, 2\not{R}z, 3\not{R}x, 3\not{R}z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of set A and B, either $A \subseteq B$ or $A \not\subseteq B$.

(c) A familiar relation on the set $\mathbf{Z}$ of integers is "m divides n." A common notation for this relation is to write $m \mid n$ when m divides n. Thus $6 \mid 30$ but $7 \nmid 25$.

(d) Consider the set L of lines in the plane. Perpendicularity, written "$\perp$," is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to," written "$\|$," is a relation on L since either $a \parallel b$ or $a \nparallel b$.

(e) Let A be any set. An important relation on A is that of *equality*,

$$\{(a, a) \mid a \in A\}$$

which is usually denoted by "=." This relation is also called the *identity* or *diagonal* relation on A and it will also be denoted by Δ_A or simply Δ.

(f) Let A be any set. Then $A \times A$ and $\emptyset$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then the inverse of

$$R = \{(1, y), (1, z), (3, y)\} \quad \text{is} \quad R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

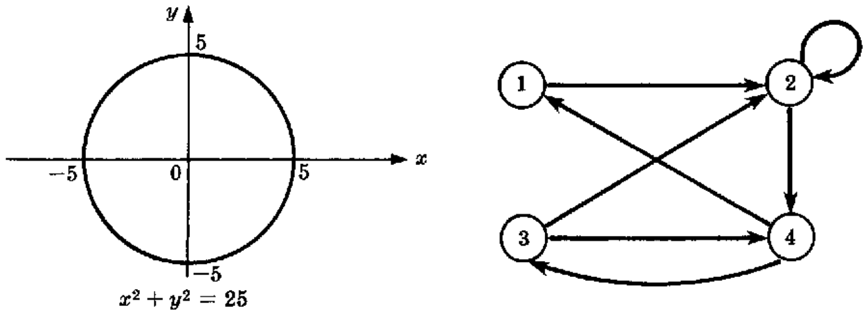
2.4 PICTORIAL REPRESENTATIVES OF RELATIONS

There are various ways of picturing relations.

Relations on R

Let S be a relation on the set $\mathbf{R}$ of real numbers; that is, S is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Frequently, S consists of all ordered pairs of real numbers which satisfy some given equation $E(x, y) = 0$ (such as $x^2 + y^2 = 25$).

Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is sometimes called the *graph* of the relation. For example, the graph of the relation $x^2 + y^2 = 25$ is a circle having its center at the origin and radius 5. See Fig. 2-2(a).



(a) (b)

Fig. 2-2

Directed Graphs of Relations on Sets

There is an important way of picturing a relation R on a finite set. First we write down the elements of the set, and then we draw an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation. Figure 2-2(b), for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under R.

These directed graphs will be studied in detail as a separate subject in Chapter 8. We mention it here mainly for completeness.

Pictures of Relations on Finite Sets

Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.

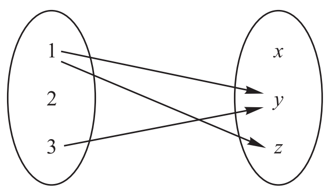
(i) Form a rectangular array (matrix) whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.

Figure 2-3 pictures the relation R in Example 2.3(a) by the above two ways.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Fig. 2-3

2.5 COMPOSITION OF RELATIONS

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by:

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc.$$

That is ,

$$R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always defined. Also, $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

Warning: Many texts denote the composition of relations R and S by $S \circ R$ rather than $R \circ S$. This is done in order to conform with the usual use of $g \circ f$ to denote the composition of f and g where f and g are functions. Thus the reader may have to adjust this notation when using this text as a supplement with another text. However, when a relation R is composed with itself, then the meaning of $R \circ R$ is unambiguous.

EXAMPLE 2.4 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} \quad \text{and} \quad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 2-4. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus:

$$2(R \circ S)z \quad \text{since } 2Rd \text{ and } dSz$$

Similarly there is a path from 3 to x and a path from 3 to z. Hence

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

Our first theorem tells us that composition of relations is associative.

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

We prove this theorem in Problem 2.8.

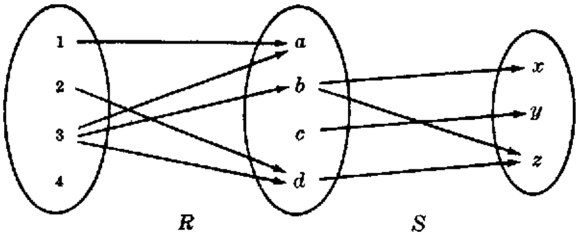


Fig. 2-4

Composition of Relations and Matrices

There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrix representations of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $M_{R \circ S}$ have the same nonzero entries.

2.6 TYPES OF RELATIONS

This section discusses a number of important types of relations defined on a set A.

Reflexive Relations

A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists $a \in A$ such that $(a, a) \notin R$.

EXAMPLE 2.5 Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$$\begin{aligned} R_1 &= \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\} \\ R_2 &= \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\} \\ R_3 &= \{(1, 3), (2, 1)\} \\ R_4 &= \emptyset, \text{ the empty relation} \\ R_5 &= A \times A, \text{the universal relation} \end{aligned}$$

Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, $(2, 2)$ does not belong to any of them.

EXAMPLE 2.6 Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

(5) Relation $\mid$ of divisibility on the set $\mathbf{N}$ of positive integers. (Recall $x \mid y$ if there exists z such that $xz = y$.)

Determine which of the relations are reflexive.

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every $x \in \mathbf{Z}$, $A \subseteq A$ for any set $A \in C$, and $n \mid n$ for every positive integer $n \in \mathbf{N}$.

Symmetric and Antisymmetric Relations

A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever $(a, b) \in R$ then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE 2.7

(a) Determine which of the relations in Example 2.5 are symmetric.

R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$. The other relations are symmetric.

(b) Determine which of the relations in Example 2.6 are symmetric.

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to line a. The other relations are not symmetric. For example:

$$3 \leq 4 \text{ but } 4 \not\leq 3; \quad \{1, 2\} \subseteq \{1, 2, 3\} \text{ but } \{1, 2, 3\} \not\subseteq \{1, 2\}; \quad \text{and} \quad 2 \mid 6 \text{ but } 6 \nmid 2.$$

A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$, that is, if $a \neq b$ and aRb then $b \not R a$. Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

EXAMPLE 2.8

(a) Determine which of the relations in Example 2.5 are antisymmetric.

R_2 is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_3 is not antisymmetric. All the other relations are antisymmetric.

(b) Determine which of the relations in Example 2.6 are antisymmetric.

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbf{N}$ is antisymmetric since whenever $m \mid n$ and $n \mid m$ then $m = n$. (Note that divisibility on $\mathbf{Z}$ is not antisymmetric since $3 \mid -3$ and $-3 \mid 3$ but $3 \neq -3$.) The relations $\perp$ and $\parallel$ are not antisymmetric.

Remark: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Transitive Relations

A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

EXAMPLE 2.9

(a) Determine which of the relations in Example 2.5 are transitive.

The relation R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

(b) Determine which of the relations in Example 2.6 are transitive.

The relations $\leq$, $\subseteq$, and $|$ are transitive, but certainly not $\perp$. Also, since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \parallel a$. Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

The property of transitivity can also be expressed in terms of the composition of relations. For a relation R on A we did define $R^2 = R \circ R$ and, more generally, $R^n = R^{n-1} \circ R$. Then we have the following result:

Theorem 2.2: A relation R is transitive if and only if, for every $n \geq 1$, we have $R^n \subseteq R$.

2.7 CLOSURE PROPERTIES

Consider a given set A and the collection of all relations on A. Let P be a property of such relations, such as being symmetric or being transitive. A relation with property P will be called a P-relation. The P-closure of an arbitrary relation R on A, written $P(R)$, is a P-relation such that

$$R \subseteq P(R) \subseteq S$$

for every P-relation S containing R. We will write

$$\text{reflexive}(R), \quad \text{symmetric}(R), \quad \text{and} \quad \text{transitive}(R)$$

for the reflexive, symmetric, and transitive closures of R.

Generally speaking, $P(R)$ need not exist. However, there is a general situation where $P(R)$ will always exist. Suppose P is a property such that there is at least one P-relation containing R and that the intersection of any P-relations is again a P-relation. Then one can prove (Problem 2.16) that

$$P(R) = \cap(S \mid S \text{ is a } P\text{-relation and } R \subseteq S)$$

Thus one can obtain $P(R)$ from the "top-down," that is, as the intersection of relations. However, one usually wants to find $P(R)$ from the "bottom-up," that is, by adjoining elements to R to obtain $P(R)$. This we do below.

Reflexive and Symmetric Closures

The next theorem tells us how to obtain easily the reflexive and symmetric closures of a relation. Here $\Delta_A = \{(a, a) \mid a \in A\}$ is the diagonal or equality relation on A.

Theorem 2.3: Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

In other words, reflexive(R) is obtained by simply adding to R those elements (a, a) in the diagonal which do not already belong to R, and symmetric(R) is obtained by adding to R all pairs (b, a) whenever (a, b) belongs to R.

EXAMPLE 2.10 Consider the relation $R = \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)\}$ on the set $A = \{1, 2, 3, 4\}$. Then

$$\text{reflexive}(R) = R \cup \{(2, 2), (4, 4)\} \quad \text{and} \quad \text{symmetric}(R) = R \cup \{(4, 2), (3, 4)\}$$

Transitive Closure

Let R be a relation on a set A. Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. We define

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

The following theorem applies:

Theorem 2.4: R^* is the transitive closure of R.

Suppose A is a finite set with n elements. We show in Chapter 8 on graphs that

$$R^* = R \cup R^2 \cup \ldots \cup R^n$$

This gives us the following theorem:

Theorem 2.5: Let R be a relation on a set A with n elements. Then

$$\text{transitive}\,(R) = R \cup R^2 \cup \ldots \cup R^n$$

EXAMPLE 2.11 Consider the relation $R = \{(1, 2), (2, 3), (3, 3)\}$ on $A = \{1, 2, 3\}$. Then:

$$R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\} \quad \text{and} \quad R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$$

Accordingly,

$$\text{transitive}\,(R) = \{(1, 2), (2, 3), (3, 3), (1, 3)\}$$

2.8 EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa. (2) If aRb, then bRa. (3) If aRb and bRc, then aRc.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike." In fact, the relation "=" of equality on any set S is an equivalence relation; that is:

(1) $a = a$ for every $a \in S$. (2) If $a = b$, then $b = a$. (3) If $a = b$, $b = c$, then $a = c$.

Other equivalence relations follow.

EXAMPLE 2.12

(a) Let L be the set of lines and let T be the set of triangles in the Euclidean plane.

  (i) The relation "is parallel to or identical to" is an equivalence relation on L.
  (ii) The relations of congruence and similarity are equivalence relations on T.

(b) The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

(c) Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod{m}$$

if m divides $a - b$. For example, for the modulus $m = 4$, we have

$$11 \equiv 3 \pmod{4} \quad \text{and} \quad 22 \equiv 6 \pmod{4}$$

since 4 divides $11 - 3 = 8$ and 4 divides $22 - 6 = 16$. This relation of congruence modulo m is an important equivalence relation.

Equivalence Relations and Partitions

This subsection explores the relationship between equivalence relations and partitions on a non-empty set S. Recall first that a partition P of S is a collection $\{A_i\}$ of nonempty subsets of S with the following two properties:

(1) Each $a \in S$ belongs to some A_i.
(2) If $A_i \neq A_j$ then $A_i \cap A_j = \emptyset$.

In other words, a partition P of S is a subdivision of S into disjoint nonempty sets. (See Section 1.7.)

Suppose R is an equivalence relation on a set S. For each $a \in S$, let [a] denote the set of elements of S to which a is related under R; that is:

$$[a] = \{x \mid (a, x) \in R\}$$

We call $[a]$ the *equivalence class* of a in S; any $b \in [a]$ is called a *representative* of the equivalence class.

The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R, that is,

$$S/R = \{[a] \mid a \in S\}$$

It is called the *quotient set* of S by R. The fundamental property of a quotient set is contained in the following theorem.

Theorem 2.6: Let R be an equivalence relation on a set S. Then S/R is a partition of S. Specifically:

(i) For each a in S, we have $a \in [a]$.
(ii) $[a] = [b]$ if and only if $(a, b) \in R$.
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

Conversely, given a partition $\{A_i\}$ of the set S, there is an equivalence relation R on S such that the sets A_i are the equivalence classes.

This important theorem will be proved in Problem 2.17.

EXAMPLE 2.13

(a) Consider the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ on $S = \{1, 2, 3\}$.

One can show that R is reflexive, symmetric, and transitive, that is, that R is an equivalence relation. Also:

$$[1] = \{1, 2\}, [2] = \{1, 2\}, [3] = \{3\}$$

Observe that $[1] = [2]$ and that $S/R = \{[1], [3]\}$ is a partition of S. One can choose either $\{1, 3\}$ or $\{2, 3\}$ as a set of representatives of the equivalence classes.

(b) Let R_5 be the relation of congruence modulo 5 on the set $\mathbf{Z}$ of integers denoted by

$$x \equiv y \pmod 5$$

This means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. The quotient set $\mathbf{Z}/R_5$ contains the following five equivalence classes:

$$\begin{aligned} A_0 &= \{\ldots, -10, -5, 0, 5, 10, \ldots\} \\ A_1 &= \{\ldots, -9, -4, 1, 6, 11, \ldots\} \\ A_2 &= \{\ldots, -8, -3, 2, 7, 12, \ldots\} \\ A_3 &= \{\ldots, -7, -2, 3, 8, 13, \ldots\} \\ A_4 &= \{\ldots, -6, -1, 4, 9, 14, \ldots\} \end{aligned}$$

Any integer x, uniquely expressed in the form $x = 5q + r$ where $0 \le r < 5$, is a member of the equivalence class A_r, where r is the remainder. As expected, $\mathbf{Z}$ is the disjoint union of equivalence classes A_1, A_2, A_3, A_4. Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a set of representatives of the equivalence classes.

2.9 PARTIAL ORDERING RELATIONS

A relation R on a set S is called a *partial ordering* or a *partial order* of S if R is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set* or *poset*. Partially ordered sets will be studied in more detail in Chapter 14, so here we simply give some examples.

EXAMPLE 2.14

(a) The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A.
(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) The relation $\le$ on the set $\mathbf{R}$ of real numbers is reflexive, antisymmetric, and transitive. Thus $\le$ is a partial ordering on $\mathbf{R}$.

(c) The relation "a divides b," written $a \mid b$, is a partial ordering on the set $\mathbf{N}$ of positive integers. However, "a divides b" is not a partial ordering on the set $\mathbf{Z}$ of integers since $a \mid b$ and $b \mid a$ need not imply $a = b$. For example, $3 \mid -3$ and $-3 \mid 3$ but $3 \ne -3$.

2.10 *n*-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S. In particular, a subset of S^3 is called a *ternary relation* on S.

EXAMPLE 2.15

(a) Let L be a line in the plane. Then "betweenness" is a ternary relation R on the points of L; that is, $(a, b, c) \in R$ if b lies between a and c on L.

(b) The equation $x^2 + y^2 + z^2 = 1$ determines a ternary relation T on the set $\mathbf{R}$ of real numbers. That is, a triple (x, y, z) belongs to T if (x, y, z) satisfies the equation, which means (x, y, z) is the coordinates of a point in $\mathbf{R}^3$ on the sphere S with radius 1 and center at the origin $O = (0, 0, 0)$.

Solved Problems

PRODUCT SETS

2.1. Given: $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find: $A \times B \times C$.

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram (Fig. 2-5). The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

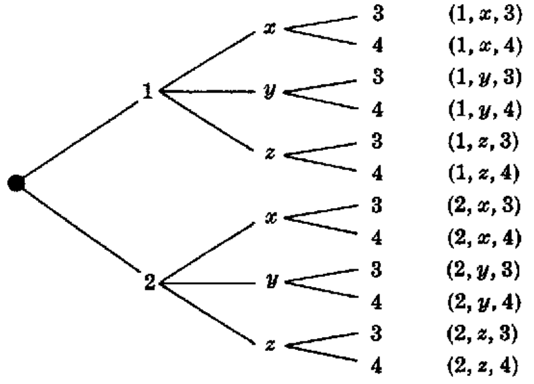


Fig. 2-5

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

2.2. Find x and y given $(2x, x + y) = (6, 2)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x = 6 \quad \text{and} \quad x + y = 2$$

from which we derive the answers $x = 3$ and $y = -1$.

RELATIONS AND THEIR GRAPHS

2.3. Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

There are $3(2) = 6$ elements in $A \times B$, and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.4. Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(*a*) Determine the matrix of the relation.

(*b*) Draw the arrow diagram of R.

(*c*) Find the inverse relation R^{-1} of R.

(*d*) Determine the domain and range of R.

(*a*) See Fig. 2-6(*a*) Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to $a \in A$ and $b \in B$ is 1 if a is related to b and 0 otherwise.

(*b*) See Fig. 2.6(*b*) Observe that there is an arrow from $a \in A$ to $b \in B$ iff a is related to b, i.e., iff $(a, b) \in R$.

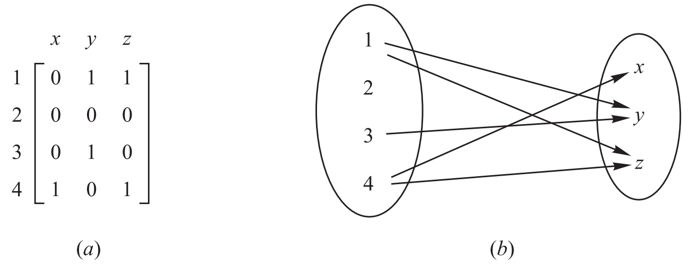


Fig. 2-6

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

Observe that by reversing the arrows in Fig. 2.6(b), we obtain the arrow diagram of R^{-1}.

(d) The domain of R, Dom(R), consists of the first elements of the ordered pairs of R, and the range of R, Ran(R), consists of the second elements. Thus,

$$\text{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \text{Ran}(R) = \{x, y, z\}$$

2.5. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S, and $M_{R \circ S}$ of the respective relations R, S, and $R \circ S$, and compare $M_{R \circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2-7(a). Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$.
We have

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$

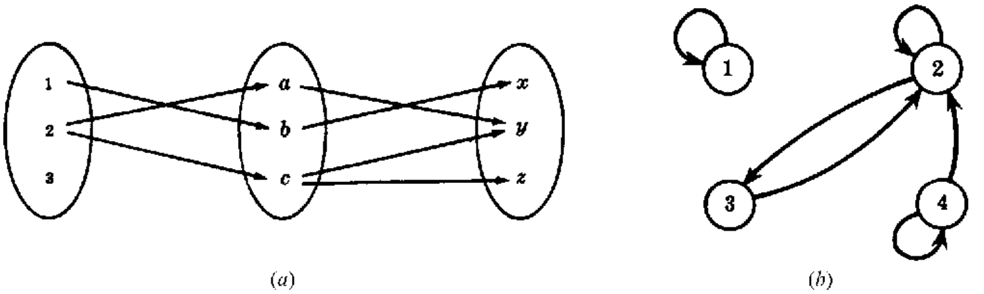


Fig. 2-7

(*b*) The matrices of M_R, M_S, and $M_{R\circ S}$ follow:

$$M_R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad M_S = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \quad M_{R\circ S} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that $M_{R\circ S}$ and $M_R M_S$ have the same zero entries.

2.6. Consider the relation $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$ on $A = \{1, 2, 3, 4\}$.
(*a*) Draw its directed graph. (*b*) Find $R^2 = R\circ R$.

(*a*) For each $(a, b) \in R$, draw an arrow from a to b as in Fig. 2-7(*b*).

(*b*) For each pair $(a, b) \in R$, find all $(b, c) \in R$. Then $(a, c) \in R^2$. Thus

$$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$

2.7. Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \quad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find (*a*) $R \cup S$, $R \cap S$, R^{C}; (*b*) $R\circ S$; (*c*) $S^2 = S\circ S$.

(*a*) Treat R and S simply as sets, and take the usual intersection and union. For R^{C}, use the fact that $A \times A$ is the universal relation on A.

$$\begin{aligned} R \cap S &= \{(1, 2), (3, 3)\} \\ R \cup S &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\} \\ R^{\mathrm{C}} &= \{(1, 3), (2, 1), (2, 2), (3, 2)\} \end{aligned}$$

(*b*) For each pair $(a, b) \in R$, find all pairs $(b, c) \in S$. Then $(a, c) \in R\circ S$. For example, $(1, 1) \in R$ and $(1, 2)$, $(1, 3) \in S$; hence $(1, 2)$ and $(1, 3)$ belong to $R\circ S$. Thus,

$$R\circ S = \{(1, 2), (1, 3), (1, 1), (2, 3), (3, 2), (3, 3)\}$$

(*c*) Following the algorithm in (*b*), we get

$$S^2 = S\circ S = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3)\}$$

2.8. Prove Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C and T is a relation from C to D. Then $(R\circ S)\circ T = R\circ(S\circ T)$.

We need to show that each ordered pair in $(R\circ S)\circ T$ belongs to $R\circ(S\circ T)$, and vice versa.

Suppose (a, d) belongs to $(R\circ S)\circ T$. Then there exists $c \in C$ such that $(a, c) \in R\circ S$ and $(c, d) \in T$. Since $(a, c) \in R\circ S$, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$ and $(c, d) \in T$, we have $(b, d) \in S\circ T$; and since $(a, b) \in R$ and $(b, d) \in S\circ T$, we have $(a, d) \in R \circ (S\circ T)$. Therefore, $(R\circ S)\circ T \subseteq R\circ(S\circ T)$. Similarly $R\circ(S\circ T) \subseteq (R\circ S)\circ T$. Both inclusion relations prove $(R\circ S)\circ T = R\circ(S\circ T)$.

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.9. Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$\begin{array}{ll} R = \{(1,1), (1,2), (1,3), (3,3)\}, & \emptyset = \text{empty relation} \\ S = \{(1,1)(1,2), (2,1)(2,2), (3,3)\}, & A \times A = \text{universal relation} \\ T = \{(1,1), (1,2), (2,2), (2,3)\} & \end{array}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

(*a*) R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A \times A$ are reflexive.

(*b*) R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A \times A$ are symmetric.

(*c*) T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

(*d*) S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.10. Give an example of a relation R on $A = \{1, 2, 3\}$ such that:

(*a*) R is both symmetric and antisymmetric.

(*b*) R is neither symmetric nor antisymmetric.

(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several such examples. One possible set of examples follows:

(*a*) $R = \{(1, 1), (2, 2)\}$; (*b*) $R = \{(1, 2), (2, 3)\}$; (*c*) $R = \{(1, 2)\}$.

2.11. Suppose C is a collection of relations S on a set A, and let T be the intersection of the relations S in C, that is, $T = \cap(S \mid S \in C)$ Prove:

(*a*) If every S is symmetric, then T is symmetric.

(*b*) If every S is transitive, then T is transitive.

(*a*) Suppose $(a, b) \in T$. Then $(a, b) \in S$ for every S. Since each S is symmetric, $(b, a) \in S$ for every S. Hence $(b, a) \in T$ and T is symmetric.

(*b*) Suppose (a, b) and (b, c) belong to T. Then (a, b) and (b, c) belong to S for every S. Since each S is transitive, (a, c) belongs to S for every S. Hence, $(a, c) \in T$ and T is transitive.

2.12. Let R be a relation on a set A, and let P be a property of relations, such as symmetry and transitivity. Then P will be called *R-closable* if P satisfies the following two conditions:

(1) There is a P-relation S containing R.

(2) The intersection of P-relations is a P-relation.

(*a*) Show that symmetry and transitivity are R-closable for any relation R.

(*b*) Suppose P is R-closable. Then $P(R)$, the P-closure of R, is the intersection of all P-relations S containing R, that is,

$$P(R) = \cap(S \mid S \text{ is a } P\text{-relation and } R \subseteq S)$$

(*a*) The universal relation $A \times A$ is symmetric and transitive and $A \times A$ contains any relation R on A. Thus (1) is satisfied. By Problem 2.11, symmetry and transitivity satisfy (2). Thus symmetry and transitivity are R-closable for any relation R.

(*b*) Let $T = \cap(S \mid S$ is a P-relation and $R \subseteq S)$. Since P is R-closable, T is nonempty by (1) and T is a P-relation by (2). Since each relation S contains R, the intersection T contains R. Thus, T is a P-relation containing R. By definition, $P(R)$ is the smallest P-relation containing R; hence $P(R) \subseteq T$. On the other hand, $P(R)$ is one of the sets S defining T, that is, $P(R)$ is a P-relation and if $R \subseteq P(R)$. Therefore, $T \subseteq P(R)$. Accordingly, $P(R) = T$.

2.13. Consider the relation $R = \{(a,a), (a,b), (b,c), (c,c)\}$ on the set $A = \{a,b,c\}$. Find: (*a*) reflexive(R); (*b*) symmetric(R); (*c*) transitive(R).

(*a*) The reflexive closure on R is obtained by adding all diagonal pairs of $A \times A$ to R which are not currently in R. Hence,

$$\text{reflexive}(R) = R \cup \{(b,b)\} = \{(a,a), (a,b), (b,b), (b,c), (c,c)\}$$

(*b*) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$$\text{symmetric}(R) = R \cup \{(b,a), (c,b)\} = \{(a,a), (a,b), (b,a), (b,c), (c,b), (c,c)\}$$

(*c*) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$. Note that

$$R^2 = R \circ R = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$$
$$R^3 = R \circ R \circ R = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$$

Hence

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$$

EQUIVALENCE RELATIONS AND PARTITIONS

2.14. Consider the **Z** of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod{m}$$

if $x - y$ is divisible by m. Show that this defines an equivalence relation on **Z**.

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in Z we have $x \equiv x \pmod{m}$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod{m}$, so $x - y$ is divisible by m. Then $-(x-y) = y - x$ is also divisible by m, so $y \equiv x \pmod{m}$. Thus the relation is symmetric.

(iii) Now suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod{m}$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on **Z** is an equivalence relation.

2.15. Let A be a set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined by

$$(a,b) \approx (c,d) \quad \text{whenever} \quad ad = bc$$

Prove that $\approx$ is an equivalence relation.

We must show that $\approx$ is reflexive, symmetric, and transitive.

(i) *Reflexivity*: We have $(a,b) \approx (a,b)$ since $ab = ba$. Hence $\approx$ is reflexive.

(ii) *Symmetry*: Suppose $(a,b) \approx (c,d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c,d) = (a,b)$. Thus, $\approx$ is symmetric.

(iii) *Transitivity*: Suppose $(a,b) \approx (c,d)$ and $(c,d) \approx (e,f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a,b) \approx (e,f)$. Thus $\approx$ is transitive. Accordingly, $\approx$ is an equivalence relation.

2.16. Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Find the partition of A induced by R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3, and 6, hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly, the following is the partition of A induced by R:

$$[\{1, 5\}, \{2, 3, 6\}, \{4\}]$$

2.17. Prove Theorem 2.6: Let R be an equivalence relation in a set A. Then the quotient set A/R is a partition of A. Specifically,

(i) $a \in [a]$, for every $a \in A$.
(ii) $[a] = [b]$ if and only if $(a, b) \in R$.
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

(*a*) *Proof of (i)*: Since R is reflexive, $(a, a) \in R$ for every $a \in A$ and therefore $a \in [a]$.

(*b*) *Proof of (ii)*: Suppose $(a, b) \in R$. We want to show that $[a] = [b]$. Let $x \in [b]$; then $(b, x) \in R$. But by hypothesis $(a, a) \in R$ and so, by transitivity, $(a, x) \in R$. Accordingly $x \in [a]$. Thus $[b] \subseteq [a]$. To prove that $[a] \subseteq [b]$ we observe that $(a, b) \in R$ implies, by symmetry, that $(b, a) \in R$. Then, by a similar argument, we obtain $[a] \subseteq [b]$. Consequently, $[a] = [b]$.
On the other hand, if $[a] = [b]$, then, by (i), $b \in [b] = [a]$; hence $(a, b) \in R$.

(*c*) *Proof of (iii)*: We prove the equivalent contrapositive statement:

$$\text{If } [a] \cap [b] \neq \emptyset \quad \text{then} \quad [a] = [b]$$

If $[a] \cap [b] \neq \emptyset$, then there exists an element $x \in A$ with $x \in [a] \cap [b]$. Hence $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$ and by transitivity, $(a, b) \in R$. Consequently by (ii), $[a] = [b]$.

PARTIAL ORDERINGS

2.18. Let ℓ be any collection of sets. Is the relation of set inclusion $\subseteq$ a partial order on ℓ?

Yes, since set inclusion is reflexive, antisymmetric, and transitive. That is, for any sets A, B, C in ℓ we have: (i) $A \subseteq A$; (ii) if $A \subseteq B$ and $B \subseteq A$, then $A = B$; (iii) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2.19. Consider the set $\mathbf{Z}$ of integers. Define aRb by $b = a^r$ for some positive integer r. Show that R is a partial order on $\mathbf{Z}$, that is, show that R is: (*a*) reflexive; (*b*) antisymmetric; (*c*) transitive.

(*a*) R is reflexive since $a = a^1$.

(*b*) Suppose aRb and bRa, say $b = a^r$ and $a = b^s$. Then $a = (a^r)^s = a^{rs}$. There are three possibilities: (i) $rs = 1$, (ii) $a = 1$, and (iii) $a = -1$. If $rs = 1$ then $r = 1$ and $s = 1$ and so $a = b$. If $a = 1$ then $b = 1^r = 1 = a$, and, similarly, if $b = 1$ then $a = 1$. Lastly, if $a = -1$ then $b = -1$ (since $b \neq 1$) and $a = b$. In all three cases, $a = b$. Thus R is antisymmetric.

(*c*) Suppose aRb and bRc say $b = a^r$ and $c = b^s$. Then $c = (a^r)^s = a^{rs}$ and, therefore, aRc. Hence R is transitive.

Accordingly, R is a partial order on $\mathbf{Z}$.

Supplementary Problems

RELATIONS

2.20. Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Find $S \times T \times W$.

2.21. Find x and y where: (a) $(x + 2, 4) = (5, 2x + y)$; (b) $(y - 2, 2x + 1) = (x - 1, y + 2)$.

2.22. Prove: (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$; (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

2.23. Consider the relation $R = \{(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$ on $A = \{1, 2, 3, 4\}$.

(a) Find the matrix M_R of R. (d) Draw the directed graph of R.

(b) Find the domain and range of R. (e) Find the composition relation $R \circ R$.

(c) Find R^{-1}. (f) Find $R \circ R^{-1}$ and $R^{-1} \circ R$.

2.24. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{x, y, z\}$. Consider the relations R from A to B and S from B to C as follows:

$$R = \{(1, b), (3, a), (3, b), (4, c)\} \quad \text{and} \quad S = \{(a, y), (c, x), (a, z)\}$$

(a) Draw the diagrams of R and S.

(b) Find the matrix of each relation R, S (composition) $R \circ S$.

(c) Write R^{-1} and the composition $R \circ S$ as sets of ordered pairs.

2.25. Let R and S be the following relations on $B = \{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (c, b), (c, d), (d, b)\} \quad \text{and} \quad S = \{(b, a), (c, c), (c, d), (d, a)\}$$

Find the following composition relations: (a) $R \circ S$; (b) $S \circ R$; (c) $R \circ R$; (d) $S \circ S$.

2.26. Let R be the relation on $\mathbf{N}$ defined by $x + 3y = 12$, i.e. $R = \{(x, y) \mid x + 3y = 12\}$.

(a) Write R as a set of ordered pairs. (c) Find R^{-1}.

(b) Find the domain and range of R. (d) Find the composition relation $R \circ R$.

PROPERTIES OF RELATIONS

2.27. Each of the following defines a relation on the positive integers $\mathbf{N}$:

(1) "x is greater than y." (3) $x + y = 10$

(2) "xy is the square of an integer." (4) $x + 4y = 10$.

Determine which of the relations are: (a) reflexive; (b) symmetric; (c) antisymmetric; (d) transitive.

2.28. Let R and S be relations on a set A. Assuming A has at least three elements, state whether each of the following statements is true or false. If it is false, give a counterexample on the set $A = \{1, 2, 3\}$:

(a) If R and S are symmetric then $R \cap S$ is symmetric.

(b) If R and S are symmetric then $R \cup S$ is symmetric.

(c) If R and S are reflexive then $R \cap S$ is reflexive.

(d) If R and S are reflexive then $R \cup S$ is reflexive.

(e) If R and S are transitive then $R \cup S$ is transitive.

(f) If R and S are antisymmetric then $R \cup S$ is antisymmetric.

(g) If R is antisymmetric, then R^{-1} is antisymmetric.

(h) If R is reflexive then $R \cap R^{-1}$ is not empty.

(i) If R is symmetric then $R \cap R^{-1}$ is not empty.

2.29. Suppose R and S are relations on a set A, and R is antisymmetric. Prove that $R \cap S$ is antisymmetric.

EQUIVALENCE RELATIONS

2.30. Prove that if R is an equivalence relation on a set A, than R^{-1} is also an equivalence relation on A.

2.31. Let $S = \{1, 2, 3, \ldots, 18, 19\}$. Let R be the relation on S defined by "xy is a square," (a) Prove R is an equivalence relation. (b) Find the equivalence class $[1]$. (c) List all equivalence classes with more than one element.

2.32. Let $S = \{1, 2, 3, \ldots, 14, 15\}$. Let R be the equivalence relation on S defined by $x \equiv y \pmod 5$, that is, $x - y$ is divisible by 5. Find the partition of S induced by R, i.e. the quotient set S/R.

2.33. Let $S = \{1, 2, 3, \ldots, 9\}$, and let $\sim$ be the relation on $A \times A$ defined by

$$(a, b) \sim (c, d) \quad \text{whenever} \quad a + d = b + c.$$

(a) Prove that $\sim$ is an equivalence relation.

(b) Find $[(2, 5)]$, that is, the equivalence class of $(2, 5)$.

Answers to Supplementary Problems

2.20. $\{(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d), (b, c, a), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c, d, d)\}$

2.21. (a) $x = 3$, $y = -2$; (b) $x = 2$, $y = 3$.

2.23. (a) $M_R = [0,\ 0,\ 1,\ 1;\ 0,\ 0,\ 0,\ 0;\ 0,\ 1,\ 1,\ 1; 0,\ 0,\ 0,\ 0]$;

(b) Domain $= \{1, 3\}$, range $= \{2, 3, 4\}$;

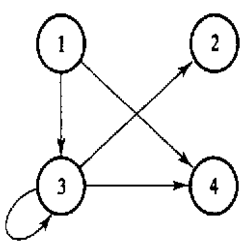
(c) $R^{-1} = \{(3,\ 1),\ (4,\ 1),\ (2,\ 3),\ (3,\ 3),\ (4,\ 3)\}$;

(d) See Fig. 2-8(a);

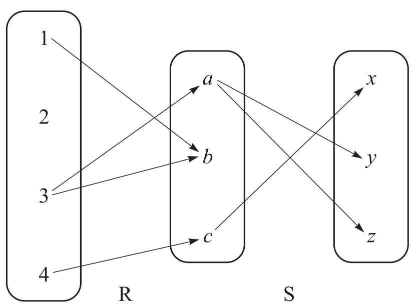
(e) $R\ o\ R = \{(1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\}$.

2.24. (a) See Fig. 2-8(b);
(b) $R = [0, 1, 0;\ 0, 0, 0;\ 1, 1, 0;\ 0, 0, 1]$,
$S = [0, 1, 1;\ 0, 0, 0;\ 1, 0, 0]$,
$R \circ S = [0, 0, 0;\ 0, 0, 0;\ 0, 1, 1;\ 1, 0, 0]$;
(c) $\{(b, 1),\ (a, 3),\ (b, 3),\ (c, 4)\}$, $\{(3, y),\ (3, z),\ (4, x)\}$.

2.25. (a) $R \circ S = \{(a, c),\ (a, d), (c, a), (d, a)\}$
(b) $S \circ R = \{(b, a), (b, c), (c, b), (c, d), (d, a), (d, c)\}$
(c) $R \circ R = \{(a, a), (a, b), (a, c), (a, d), (c, b)\}$
(d) $S \circ S = \{(c, c), (c, a), (c, d)\}$



(a)



(b)

Fig. 2-8

2.26. (a) $\{(9, 1), (6, 2), (3, 3)\}$; (b) (i) $\{9, 6, 3\}$, (ii) $\{1, 2, 3\}$, (iii) $\{(1, 9), (2, 6), (3, 3)\}$; (c) $\{(3, 3)\}$.

2.27. (a) None; (b) (2) and (3); (c) (1) and (4); (d) all except (3).

2.28. All are true except: (e) $R = \{(1, 2)\}$, $S = \{(2, 3)\}$; (f) $R = \{(1, 2)\}$, $S = \{(2, 1)\}$.

2.31. (b) $\{1, 4, 9, 16\}$; (c) $\{1, 4, 9, 16\}$, $\{2, 8, 18\}$, $\{3, 12\}$.

2.32. $[\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 133\}, \{4, 9, 14\}, \{5, 10, 15\}]$

2.33. (b) $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.