6. Найти первообразную

6.1) $f(t) = \sin(at) + \cos(bt)$

$$d(f(t)) = a\cos(at) - b\sin(bt)$$

$\Rightarrow$ базис будет $\{\sin(at), \cos(bt), \sin(bt), \cos(at)\}$

• Матрица оператора:

- $d(\sin(at)) = a\cos(at)$
- $d(\cos(bt)) = -b\sin(bt)$
- $d(\sin(bt)) = b\cos bt$
- $d(\cos(at)) = -a\sin(at)$

$\rightarrow a\cos(at) = 0\sin(at) + 0\cos(bt) + 0\sin(bt) + a\cos(at)$

$\rightarrow b\sin(bt) = 0\sin(at) + 0\cos(bt) - b\sin(bt) + 0\cos(at)$

$\rightarrow b\cos(bt) = 0\sin(at) + b\cos(bt) + 0\sin(bt) + 0\cos(at)$

$\rightarrow -a\sin(at) = -a\sin at + 0\cos(bt) + 0\sin(bt) + 0\cos(at)$

$$\Rightarrow \quad D = \begin{bmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & +b & 0 \\ 0 & -b & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow D^{-1} = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -a & 1 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 & 1 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a \\ 0 & 1 & 0 & 0 & 0 & 0 & -1/b & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/b & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/a & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow D^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/a \\ 0 & 0 & -1/b & 0 \\ 0 & 1/b & 0 & 0 \\ -1/a & 0 & 0 & 0 \end{bmatrix}$$

$$f(t) = \sin(at) + \cos(bt) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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$$\Rightarrow D^{-} f(t) = \begin{bmatrix} 0 & 0 & 0 & 1/a \\ 0 & 0 & -1/b & 0 \\ 0 & 1/b & 0 & 0 \\ -1/a & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/b \\ -1/a \end{bmatrix}$$

$$\Rightarrow \int \left(\sin(at) + \cos(bt)\right) dt = \frac{1}{b} \sin(bt) - \frac{1}{a} \cos(bt) + C$$

6.2) $f(t) = t\sin(at)$

$d(t\sin at) = \sin at + t\cos(at)$

$\Rightarrow$ базис будет: $\{\sin at, \ t\sin(at), \ \cos(at), \ t\cos(at)\}$

· Матрица оператора:

$d(\sin(at)) = a\cos(at) \to [0 \ \ 0 \ \ a \ \ 0]$

$d(t\sin(at)) = \sin at + at\cos(at) \to [1 \ \ 0 \ \ 0 \ \ a]$

$d(\cos(at)) = -a\sin(at) \to [-a \ \ 0 \ \ 0 \ \ 0]$

$d(t\cos(at)) = \cos(at) - at\sin(at) \to [0 \ \ -a \ \ 1 \ \ 0]$

$$\Rightarrow D = \begin{bmatrix} 0 & 1 & -a & 0 \\ 0 & 0 & 0 & -a \\ a & 0 & 0 & 1 \\ 0 & a & 0 & 0 \end{bmatrix}$$

$$\Rightarrow D^{-1} = \begin{bmatrix} 0 & 1/a^2 & 1/a & 0 \\ 0 & 0 & 0 & 1/a \\ -1/a & 0 & 0 & 1/a^2 \\ 0 & -1/a & 0 & 0 \end{bmatrix}$$

$$f(t) = t\sin(at) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 0 & 1/a^2 & 1/a & 0 \\ 0 & 0 & 0 & 1/a \\ -1/a & 0 & 0 & 1/a^2 \\ 0 & -1/a & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/a^2 \\ 0 \\ 0 \\ -1/a \end{bmatrix}$$

$$\Rightarrow \int t\sin(at)\,dt = \frac{1}{a^2}\sin(at) - \frac{1}{a}t\cos(at) + C$$

6.3) $f(t) = e^{bt}\sin(at)$

$$d(f(t)) = be^{bt}\sin(at) + ae^{bt}\cos(at)$$

$\to$ базис будет: $\{ e^{bt}\sin(at),\ e^{bt}\cos(at) \}$

- Матрица оператора:

$$d(e^{bt}\sin(at)) = b e^{bt}\sin(at) + ae^{bt}\cos(at)$$

$\to [\,b \quad a\,]$

$$d(e^{bt}\cos(at)) = be^{bt}\cos(at) - a e^{bt}\sin(bt)$$

$\to [\,-a \quad b\,]$

$$\to D = \begin{bmatrix} b & -a \\ a & b \end{bmatrix}$$

$$\Rightarrow D^{-1} = \begin{bmatrix} \frac{b}{a^2-b^2} & \frac{a}{a^2-b^2} \\ -\frac{a}{a^2-b^2} & \frac{b}{a^2-b^2} \end{bmatrix}$$

$$f(t) = e^{bt}\sin(at) \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\to D^{-1}f(t) = \begin{bmatrix} \frac{b}{a^2-b^2} & \frac{a}{a^2-b^2} \\ -\frac{a}{a^2-b^2} & \frac{b}{a^2-b^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{b}{a^2-b^2} \\ \frac{-a}{a^2-b^2} \end{bmatrix}$$

$$\Rightarrow \int e^{bt}\sin(at)\,dt = \frac{b}{a^2-b^2}e^{bt}\sin(at) - \frac{a}{a^2-b^2}e^{bt}\cos(at) + C$$