AE460: Spacecraft Design

Attitude Dynamics & kinematics

T10a

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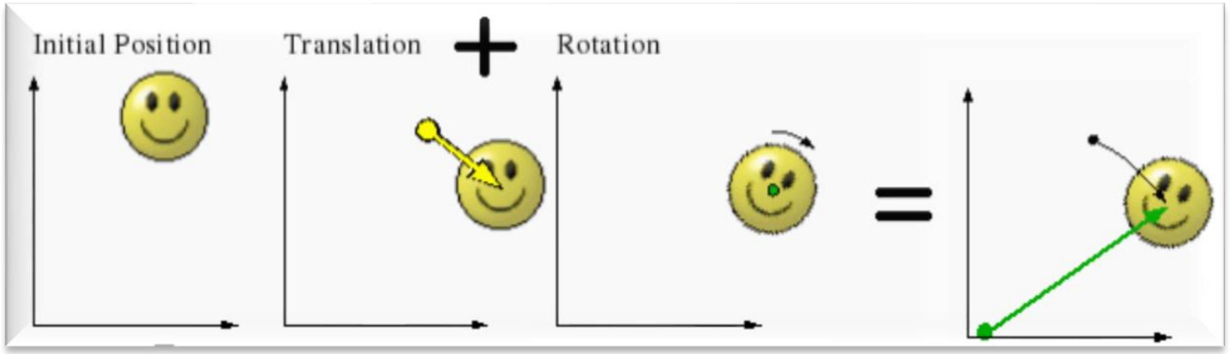
Rigid-body Dynamics



- We treat a rigid body as a system of particles, where the distance between any two particles is fixed
- We will assume that internal forces are generated to hold the relative positions fixed. These internal forces are all balanced out with Newton's third law, so that they all cancel out and have no effect on the total momentum or angular momentum
- The rigid body can actually have an infinite number of particles, spread out over a finite volume
- Instead of mass being concentrated at discrete points, we will consider the density as being variable over the volume
- Dynamics = Kinematics + Kinetics

Dynamics = Kinematics + Kinetics

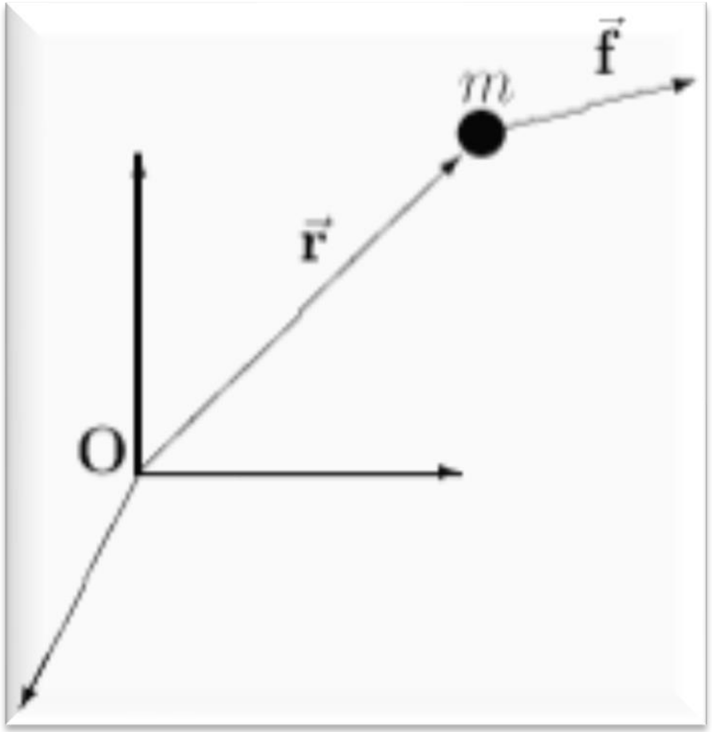
- **Translational** dynamics (<u>Newton's 2nd Law</u>)
  - 2nd order includes kinematics and kinetics

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}$$

  - 1st order separates the two

Kinematics $\dot{\vec{\mathbf{r}}} = \vec{\mathbf{p}} / m$

kinetics $\dot{\vec{\mathbf{p}}} = m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}$



Dynamics = Kinematics + Kinetics

- **Rotational** dynamics (Euler's Law)
  - Implies both kinematics and kinetics

$$\dot{\vec{\mathbf{h}}} = \vec{\mathbf{u}}$$

  - $\mathbf{h}$ is the angular momentum, $\mathbf{u}$ is the torque
  - 1st order separates the two

Kinematics **?**

kinetics $\dot{\boldsymbol{\omega}} = -I^{-1}\boldsymbol{\omega} \times I\boldsymbol{\omega} + I^{-1}\mathbf{u}$

$\boldsymbol{\omega} =$ Angular Velocity

$I =$ Moment of Inertia Tensor

Translational vs. Rotational

- Linear momentum
  =
  mass $\times$ velocity

- d/dt (linear momentum) =
  applied forces

- d/dt (position)
  =
  linear momentum/mass

- Angular momentum
  =
  inertia $\times$ angular velocity

- d/dt (angular momentum)
  =
  applied torques

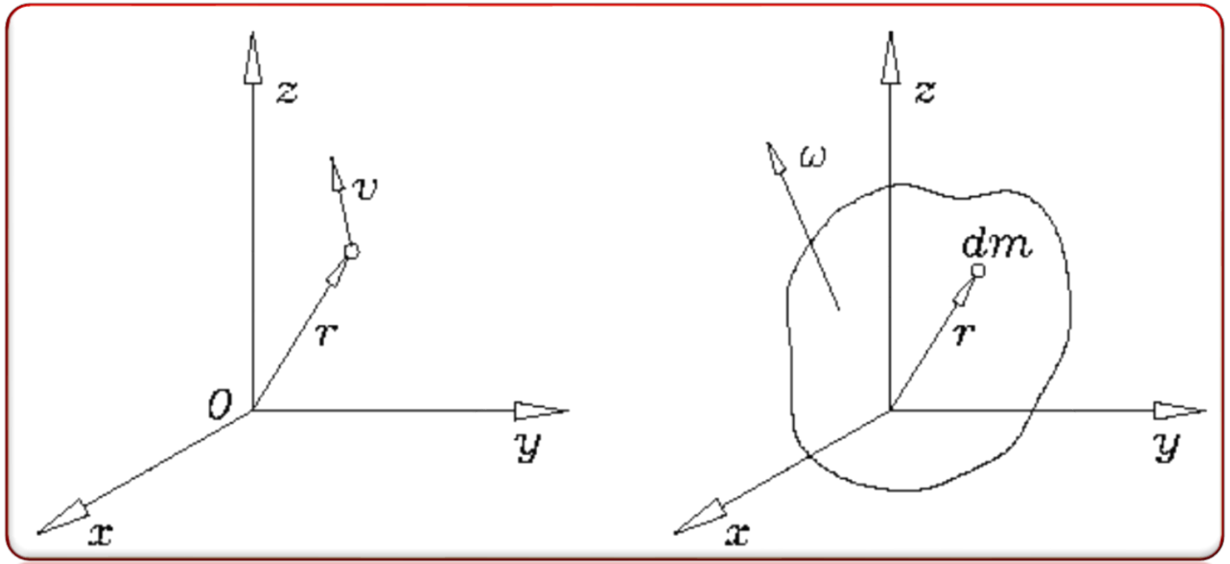
- d/dt (attitude)
  =
  "angular momentum/inertia"

Attitude Dynamics

Euler Equations, and Torque Free Motion of a Body of Revolution

Angular Momentum (1/3)

$$\vec{r} = r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3$$



$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad \text{where:} \begin{cases} I_{ii} = \int_B (r_j^2 + r_k^2)\, dm \quad (i, j, k = 1, 2, 3) \\ I_{ij} = -\int_B r_i r_j \, dm = I_{ji} \quad (i \neq j) \end{cases}$$

$\boxed{\textbf{\textit{I} is always symmetric}}$

Angular Momentum (2/3)

For a particle: $\vec{h} = \vec{r} \times (m\vec{v}) = (\vec{r} \times \vec{v})m$

For a Rigid Body: $\vec{h} = \int_M (\vec{r} \times \vec{v})dm = \int_M \vec{r} \times (\vec{\omega} \times \vec{r})dm$

$$\vec{r} \times (\omega \times \vec{r}) = (\vec{r} \cdot \vec{r})\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r} = r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}$$

Setting: $\vec{r} = \{x, y, z\}$ and $\vec{\omega} = \{\omega_1, \omega_2, \omega_3\}$, then we have

$$\vec{h} = \int_M r^2\vec{\omega}\, dm - \int_M (x\omega_1 + y\omega_2 + z\omega_3)\vec{r}\, dm$$

$$h = \int_M \begin{Bmatrix} (x^2 + y^2 + z^2)\omega_1 \\ (x^2 + y^2 + z^2)\omega_2 \\ (x^2 + y^2 + z^2)\omega_3 \end{Bmatrix} dm - \int_M \begin{Bmatrix} x^2\omega_1 + xy\omega_2 + xz\omega_3 \\ xy\omega_1 + y^2\omega_2 + yz\omega_3 \\ xz\omega_1 + yz\omega_2 + z^2\omega_3 \end{Bmatrix} dm$$

Angular Momentum (3/3)

$$h = \int_M \begin{Bmatrix} (y^2+z^2)\omega_1 \\ (x^2+z^2)\omega_2 \\ (x^2+y^2)\omega_3 \end{Bmatrix} dm - \int_M \begin{Bmatrix} xy\omega_2 + xz\omega_3 \\ xy\omega_1 + yz\omega_3 \\ xz\omega_1 + yz\omega_2 \end{Bmatrix} dm$$

Moment of inertia

$$\boxed{I_{11} = \int_M (y^2+z^2)\,dm \quad I_{22} = \int_M (x^2+z^2)\,dm \quad I_{33} = \int_M (y^2+x^2)\,dm}$$

Product of inertia

$$\boxed{I_{12} = I_{21} = -\int_M xy\,dm \quad I_{13} = I_{31} = -\int_M xz\,dm \quad I_{23} = I_{32} = -\int_M yz\,dm}$$

$$\boxed{h = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \begin{Bmatrix} \omega_1 I_{11} + \omega_2 I_{12} + \omega_3 I_{13} \\ \omega_1 I_{12} + \omega_2 I_{22} + \omega_3 I_{23} \\ \omega_1 I_{13} + \omega_2 I_{23} + \omega_3 I_{33} \end{Bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = I\omega}$$

I = Inertia tensor

Dynamics of a Rigid Body (1/3)

The angular momentum is

$$\vec{h}^{B/J} = I\vec{\omega}^{B/J} \quad \Rightarrow \begin{cases} h_1^{B/J} = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ h_2^{B/J} = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ h_3^{B/J} = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{cases}$$

The rotational kinetic energy is

$$E = \frac{1}{2}\vec{\omega}^{B/J} \cdot \vec{h}^{B/J} = \frac{1}{2}\omega^{\mathrm{T}} I \omega$$

The equations of motion are:

$$\frac{{}^{J}d\vec{h}^{B/J}}{dt} = \vec{T} = \text{sum of the torques on the body}$$

Dynamics of a Rigid Body (2/3)

$\vec{h}^{B/J}$ will always be the inertial angular momentum so we will drop the J.

If we want to evaluate the derivative in the BRF we will have:

$$\frac{{}^{J}d\vec{h}^{B}}{dt} = \frac{{}^{B}d\vec{h}^{B}}{dt} + \vec{\omega}^{B} \times \vec{h}^{B} = \vec{T} \quad \Rightarrow \quad \begin{cases} \dot{h}_1 + \omega_2 h_3 - \omega_3 h_2 = T_1 \\ \dot{h}_2 + \omega_3 h_1 - \omega_1 h_3 = T_2 \\ \dot{h}_3 + \omega_1 h_2 - \omega_2 h_1 = T_3 \end{cases}$$

These equations are derived for a rigid body but they apply to any system of particles and/or rigid bodies as long as the torques are computed about the reference point.
In this course our reference point for the angular momentum will always be the center of mass of the system.

Dynamics of a Rigid Body (3/3)

If the axes are principal axes then

$$h_1^B = I_{11}\omega_1, \quad h_2^B = I_{22}\omega_2, \quad \text{and} \quad h_3^B = I_{33}\omega_3$$

and the equations of motion become:
$$\begin{cases} I_{11}\dot{\omega}_1 + \omega_2\omega_3\left(I_{33} - I_{22}\right) = T_1 \\ I_{22}\dot{\omega}_2 + \omega_3\omega_1\left(I_{11} - I_{33}\right) = T_2 \\ I_{33}\dot{\omega}_3 + \omega_1\omega_2\left(I_{22} - I_{11}\right) = T_3 \end{cases}$$

These are called **Euler's equations of motion** for a rigid body.

In matrix form they can be written as: $I\dot{\omega} + \tilde{\omega}I\omega = T$

Where *I* is the symmetric tensor matrix (it becomes diagonal when BRF=Principal Inertia RF). If the matrix has to be evaluated in another RF, whose orientation is described by the matrix *A*, then ...

$$\boxed{I_1 A = A I_2 \quad \Rightarrow \quad I_1 = A\, I_2\, A^{\mathrm{T}} \quad \text{(tensor in the "2" RF)}}$$

T=0 for a Body of Revolution (1/3)

We want to consider spin about the axis of symmetry of an axial-symmetric body when there are no torques.

Torque Free Motion of a Body of Revolution

Torque free: $T_1 = T_2 = T_3 = 0$

Axial-symmetric body (B_3): $\begin{cases} I_{11} = I_{22} = I_T \\ I_{33} = I_S \end{cases}$ Define $\sigma = \dfrac{I_S}{I_T}$

Substituting in the Euler Eqns gives: $\begin{cases} \dot{\omega}_1 + \omega_2\omega_3(\sigma - 1) = 0 \\ \dot{\omega}_2 - \omega_1\omega_3(\sigma - 1) = 0 \\ \dot{\omega}_3 = 0 \quad \Rightarrow \quad \omega_3 = \omega_S \end{cases}$

Thus, the spin rate is a constant. Let $\lambda = (\sigma - 1)\omega_S$ (nutation frequency)

$$\begin{cases} \dot{\omega}_1 = -\lambda\omega_2 \\ \dot{\omega}_2 = \lambda\omega_1 \end{cases} \quad \Rightarrow \quad \ddot{\omega}_1 + \lambda^2\omega_1 = 0$$

T=0 for a Body of Revolution (2/3)

The solution is:

$$\begin{cases}\omega_1 = \omega_{10}\cos\lambda t - \omega_{20}\sin\lambda t \\ \omega_2 = \omega_{10}\sin\lambda t + \omega_{20}\cos\lambda t\end{cases} \Rightarrow \begin{cases}\omega_1 = \omega_T\cos(\lambda t + \alpha) \\ \omega_2 = \omega_T\sin(\lambda t + \alpha)\end{cases}$$

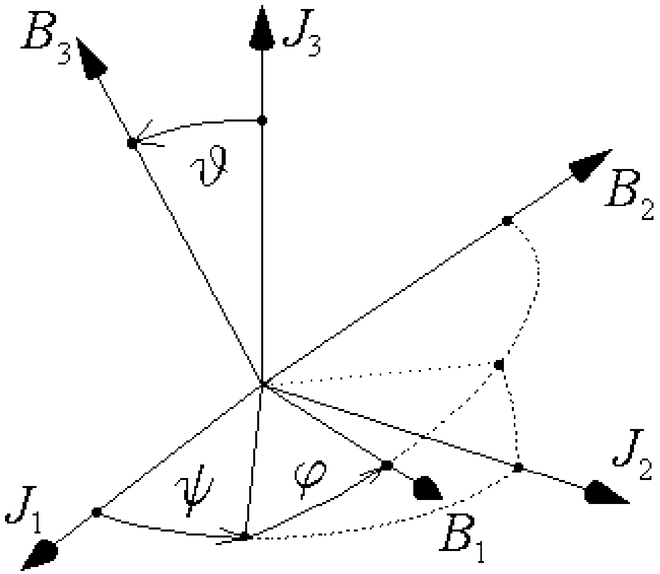
where $\omega_T = \sqrt{\omega_{10}^2 + \omega_{20}^2}$, and $\tan\alpha = \dfrac{\omega_{20}}{\omega_{10}}$

α is called *phase angle*; it depends on initial conditions only. ω_T is called the *transverse angular velocity.*

Now determine the angular velocity as a function of the "3-1-3" Euler sequence.

Since there are no external torques, the total angular momentum h is constant.

We will let the direction of the J_3 axis be along the angular momentum vector.



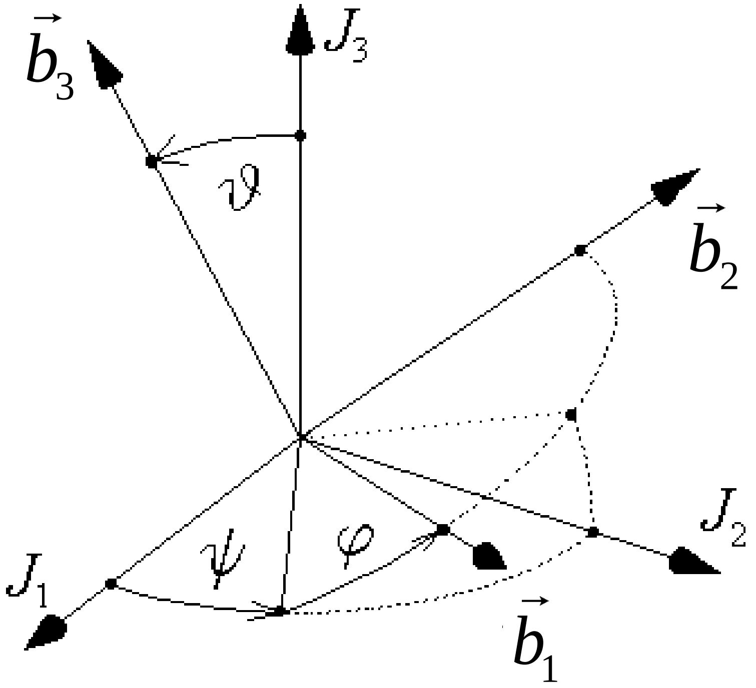
T=0 for a Body of Revolution (3/3)

$$\boxed{\vec{\omega}^B = (\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi)\vec{b}_1 + (-\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\phi)\vec{b}_2 + (\dot{\varphi} + \dot{\psi}\cos\vartheta)\vec{b}_3}$$

```
E313 =
E3rot(phi)*E1rot(th)*E3rot(psi);
E31  = E3rot(phi)*E1rot(th);
E3   = E3rot(phi);
omega =
E313*[0;0;psidot]+E31*[thdot;0;0]
+E3*[0;0;phidot]
```

$$\begin{cases} \omega_1 = \dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi \\ \omega_2 = -\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\varphi \\ \omega_3 = \dot{\varphi} + \dot{\psi}\cos\vartheta \end{cases}$$

$$\begin{cases} \dot{\psi} = (\omega_1\sin\varphi + \omega_2\cos\varphi)/\sin\vartheta \\ \dot{\vartheta} = \omega_1\cos\varphi - \omega_2\sin\varphi \\ \dot{\varphi} = \omega_3 - \dot{\psi}\cos\vartheta \end{cases}$$



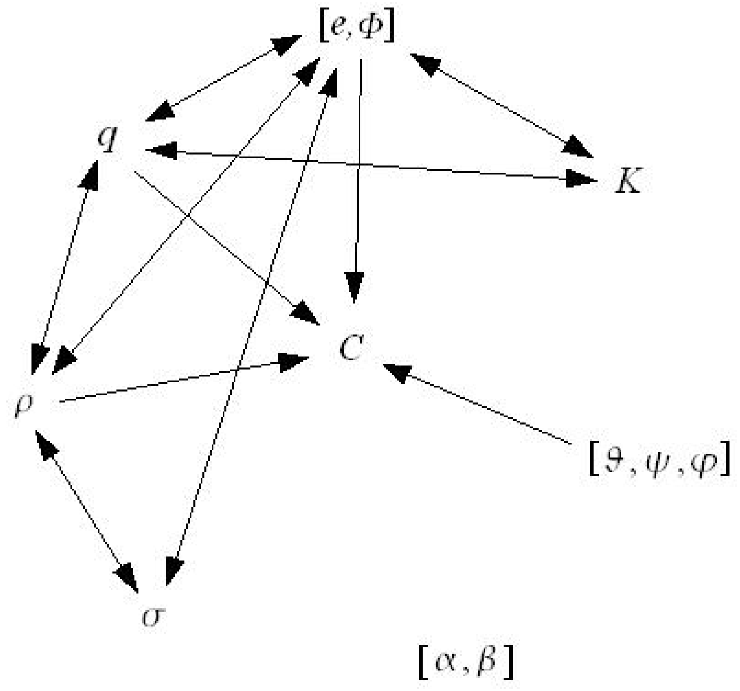
Also, since the spin rate, ω_S is a constant, then the component of h along the B_3 axis (h_3=$I_{33}\omega_S$), is constant.

Therefore $\cos\vartheta = h_3/h = I_S\omega_S/h$ is constant.

Thus, ϑ is constant and is called the *nutation angle*.

Attitude representations

- The rotation matrix represents the attitude
- A rotation matrix has 9 numbers, but they are not independent
- There are 6 constraints on the 9 elements of a rotation matrix (what are they?)
- Thus rotation has 3 degrees of freedom
- There are many different sets of parameters that can be used to represent or parameterize rotations
- Euler angles, Euler parameters (aka quaternions), Rodrigues parameters (aka Gibbs vectors), Modified Rodrigues parameters, …

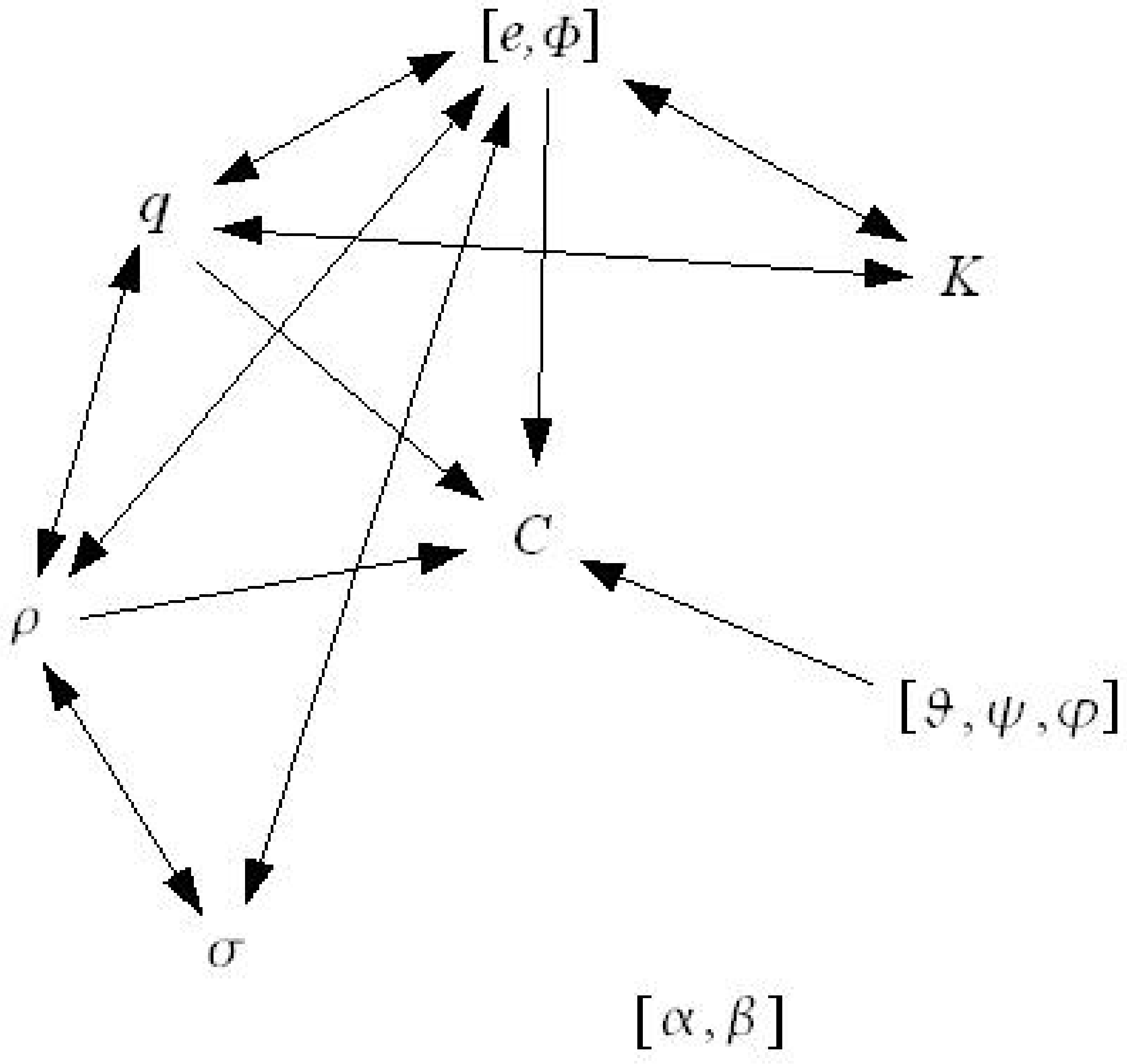


Attitude representations

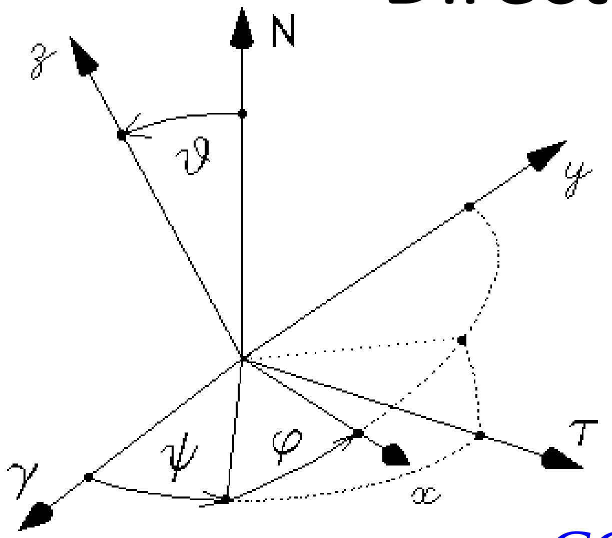
One angle
Two angles
Three angles

3-elem. vector
4-elem. vector

2×2 matrix
3×3 matrix



Direction cosine matrix (1/2)



$$C = [\gamma \quad \tau \quad N] \quad \leftrightarrow \quad C^{\mathrm{T}} = [x \quad y \quad z]$$

$$C = \begin{bmatrix} x^{\mathrm{T}} \\ y^{\mathrm{T}} \\ z^{\mathrm{T}} \end{bmatrix} \quad \leftarrow \quad \boxed{v_B = C_{BI} v_I} \quad \rightarrow \quad C^{\mathrm{T}} = \begin{bmatrix} \gamma^{\mathrm{T}} \\ \tau^{\mathrm{T}} \\ N^{\mathrm{T}} \end{bmatrix}$$

$$CC^{\mathrm{T}} = I \quad \leftrightarrow \quad C^{-1}CC^{\mathrm{T}} = C^{\mathrm{T}} = C^{-1} \quad \leftrightarrow \quad C^{\mathrm{T}}C = I$$

$$1 = \det(I) = \det(C^{\mathrm{T}}C) = \det(C^{\mathrm{T}})\det(C) = \det(C)\det(C) = [\det(C)]^2$$

$\det(C) = \pm 1$ $\quad C$ is a *proper* orthogonal matrix iff $\boxed{\det(C) = +1}$

orthogonal matrices have eigenvalues belonging to unit-radius circle

$$\rightarrow \quad \boxed{\lambda_k = \cos\phi_k \pm i \sin\phi_k} \quad \leftarrow$$

$\det(C)$ and $\mathrm{tr}(C)$ are invariants wrt similar trasformations

$$CW = W\Lambda \quad \rightarrow \quad \det(C) = \det(\Lambda), \text{ and } \mathrm{tr}(C) = \mathrm{tr}(\Lambda)$$

Direction cosine matrix (2/2)

$$\det(C)=\det(\Lambda)=\lambda_1\lambda_2\lambda_3=\lambda_1(\cos\phi+i\sin\phi)(\cos\phi-i\sin\phi)=\lambda_1(\cos^2\phi+\sin^2\phi)=1$$

$$\mathrm{tr}(C)=\mathrm{tr}(\Lambda)=\lambda_1+\lambda_2+\lambda_3=1+2\cos\phi$$

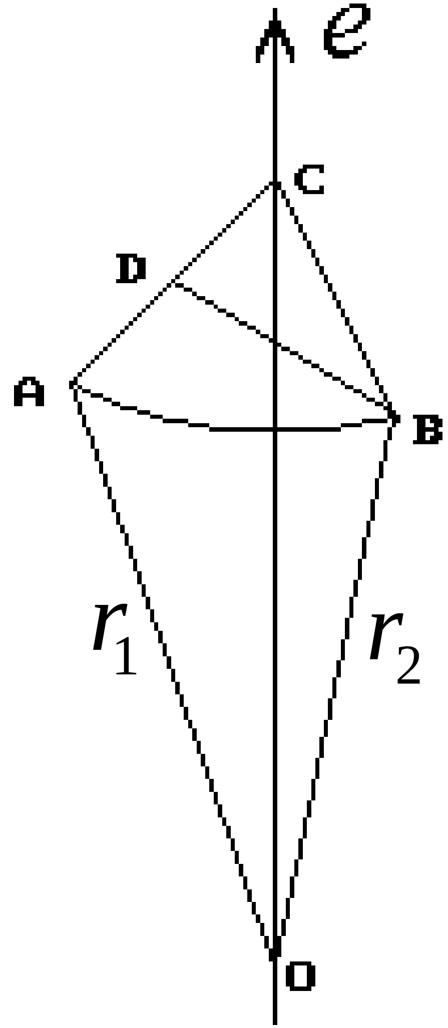
$$Ce=e \quad (\text{associated with } \lambda_1=1)$$

$$\overrightarrow{\mathrm{OB}}=\overrightarrow{\mathrm{OC}}+\overrightarrow{\mathrm{CD}}+\overrightarrow{\mathrm{DB}} \quad \text{where} \quad \begin{cases} \overrightarrow{\mathrm{OC}}=(e^{\mathrm{T}}r_1)e \\ \overrightarrow{\mathrm{CD}}=-e\times(e\times r_1)\cos\phi \\ \overrightarrow{\mathrm{DB}}=e\times r_1\sin\phi \end{cases}$$

$$r_2=(e^{\mathrm{T}}r_1)e-e\times(e\times r_1)\cos\phi+e\times r_1\sin\phi$$

$$r_2=\left[ee^{\mathrm{T}}+(I-ee^{\mathrm{T}})\cos\phi+\tilde{e}\sin\phi\right]r_1=C^{\mathrm{T}}r_1$$

$$\text{where } \tilde{e}\equiv[e\times]=\begin{bmatrix}0 & -e_3 & e_2\\ e_3 & 0 & -e_1\\ -e_2 & e_1 & 0\end{bmatrix}$$



Principal axis and angle (1/2)

$$C = ee^{\mathrm{T}} + (I - ee^{\mathrm{T}})\cos\phi - \tilde{e}\sin\phi$$

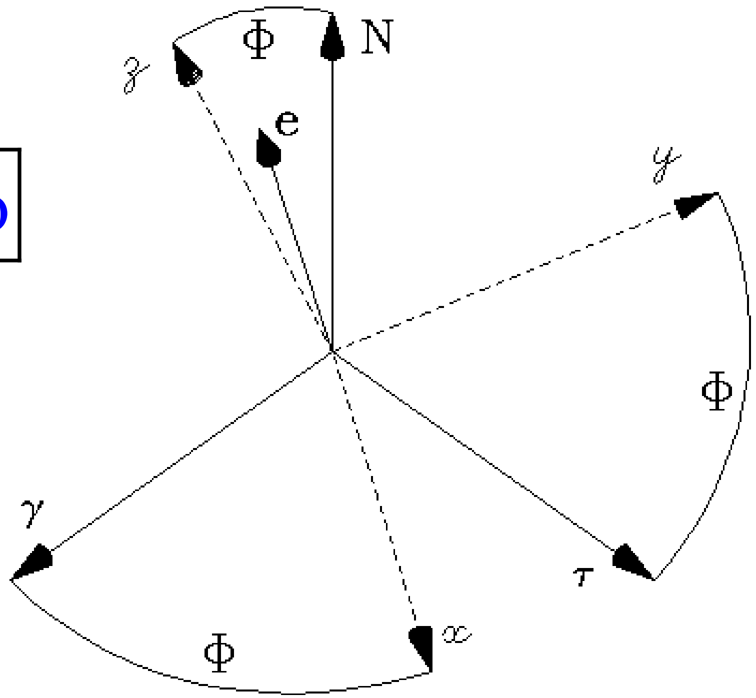
$$\boxed{C = I\cos\phi + (1-\cos\phi)ee^{\mathrm{T}} - \tilde{e}\sin\phi}$$

Orthonormality derived using:

$e^{\mathrm{T}}e = 1,\ \ \tilde{e}e = 0,$ and $\tilde{e}\tilde{e} = ee^{\mathrm{T}} - I$

e and ϕ are the

Euler's principal axis and angle



$$C = \begin{bmatrix} \cos\phi + (1-\cos\phi)e_1^2 & (1-\cos\phi)e_1e_2 + e_3\sin\phi & (1-\cos\phi)e_1e_3 - e_2\sin\phi \\ (1-\cos\phi)e_1e_2 - e_3\sin\phi & \cos\phi + (1-\cos\phi)e_2^2 & (1-\cos\phi)e_2e_3 + e_1\sin\phi \\ (1-\cos\phi)e_1e_3 + e_2\sin\phi & (1-\cos\phi)e_2e_3 - e_1\sin\phi & \cos\phi + (1-\cos\phi)e_3^2 \end{bmatrix}$$

Principal axis and angle (2/2)

$$\mathrm{tr}(C) = \mathrm{tr}(\Lambda) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\phi \quad \Rightarrow \quad \boxed{\cos\phi = \frac{1}{2}[\mathrm{trace}(C) - 1]}$$

$$\text{General property: } M = \frac{M + M^{\mathrm{T}}}{2} + \frac{M - M^{\mathrm{T}}}{2} = S + Q$$

$$C = I\cos\phi + (1-\cos\phi)ee^{\mathrm{T}} - \tilde{e}\sin\phi \quad \rightarrow \quad \boxed{\tilde{e} = \frac{C^{\mathrm{T}} - C}{2\sin\phi}}$$

$$\boxed{e_1 = \frac{C_{23} - C_{32}}{2\sin\phi}, \quad e_2 = \frac{C_{31} - C_{13}}{2\sin\phi}, \quad \text{and} \quad e_3 = \frac{C_{12} - C_{21}}{2\sin\phi},}$$

No ambiguity because (ϕ, e) and $(-\phi, -e)$ represent the same attitude!

However, the axis e has no meaning when $\phi \to 0$ (that is, when $C = I$)

The principal axis and angle (e, ϕ), constitutes a minimum set of parameters to describe the attitude

Any three-parameter attitude representation presents singularity.

Euler-Rodriguez symmetric parameters: the quaternion

In order to avoid singularity a four parameter attitude representation has been proposed.

$$q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} q_v \\ q_4 \end{Bmatrix} = \begin{Bmatrix} e \sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{Bmatrix} \quad \rightarrow \quad q^{\mathrm{T}} q = 1$$

From definition, the $(e, \phi) \leftrightarrow q$ conversion is easy

$$\begin{Bmatrix} q_v \\ q_4 \end{Bmatrix} = H(q^{(2)}) q^{(1)} = \begin{bmatrix} \tilde{q}_v^{(2)} + q_4^{(2)} I & q_v^{(2)} \\ -q_v^{(2)\mathrm{T}} & q_4^{(2)} \end{bmatrix} \begin{Bmatrix} q_v^{(1)} \\ q_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} q_4^{(1)} q_v^{(2)} + q_4^{(2)} q_v^{(1)} + \tilde{q}_v^{(2)} q_v^{(1)} \\ q_4^{(1)} q_4^{(2)} - q_v^{(2)\mathrm{T}} q_v^{(1)} \end{Bmatrix}$$

Subsequent rotations do not require computation of trigonometric functions

$$\boxed{C = [(q_4^2 - q_v^{\mathrm{T}} q_v) I + 2 q_v q_v^{\mathrm{T}} - 2 q_4 \tilde{q}_v]}$$

Quaternion: from C to q

$q \rightarrow C$ transformation is easy!

$$\boxed{C = [(q_4^2 - q_v^{\mathrm{T}} q_v) I + 2 q_v q_v^{\mathrm{T}} - 2 q_4 \tilde{q}_v]}$$

$C \rightarrow q$ transformation

requires some attention!

$$q_4 = \pm\frac{1}{2}\sqrt{1 + \mathrm{tr}[C]} \quad \text{and}$$

$$q_v = \frac{1}{4 q_4}\begin{Bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{Bmatrix}$$

Because of $q^{\mathrm{T}} q = 1$

$$\rightarrow \quad |q_k| \geq \frac{1}{2} \quad \leftarrow$$

```
function q=r_q(R)

%   q=r_q(R) translates the 3x3 direction cosine matrix R
%   into the corresponding quaternion q

tr=R(1,1)+R(2,2)+R(3,3);
b2(1)=(1+tr)/4;
b2(2)=(1+2*R(1,1)-tr)/4;
b2(3)=(1+2*R(2,2)-tr)/4;
b2(4)=(1+2*R(3,3)-tr)/4;

[v,i]=max(b2);
switch i,
case 1
   b(1)=sqrt(b2(1));
   b(2)=(R(2,3)-R(3,2))/4/b(1);
   b(3)=(R(3,1)-R(1,3))/4/b(1);
   b(4)=(R(1,2)-R(2,1))/4/b(1);
case 2
   b(2)=sqrt(b2(2));
   b(1)=(R(2,3)-R(3,2))/4/b(2);
   if b(1)<0, b(2)=-b(2); b(1)=-b(1); end
   b(3)=(R(1,2)+R(2,1))/4/b(2);
   b(4)=(R(3,1)+R(1,3))/4/b(2);
case 3
   b(3)=sqrt(b2(3));
   b(1)=(R(3,1)-R(1,3))/4/b(3);
   if b(1)<0, b(3)=-b(3); b(1)=-b(1); end
   b(2)=(R(1,2)+R(2,1))/4/b(3);
   b(4)=(R(2,3)+R(3,2))/4/b(3);
case 4
   b(4)=sqrt(b2(4));
   b(1)=(R(1,2)-R(2,1))/4/b(4);
   if b(1)<0, b(4)=-b(4); b(1)=-b(1); end
   b(2)=(R(3,1)+R(1,3))/4/b(4);
   b(3)=(R(2,3)+R(3,2))/4/b(4);
end

q=[b(2:4) b(1)]';
```

Euler's angles

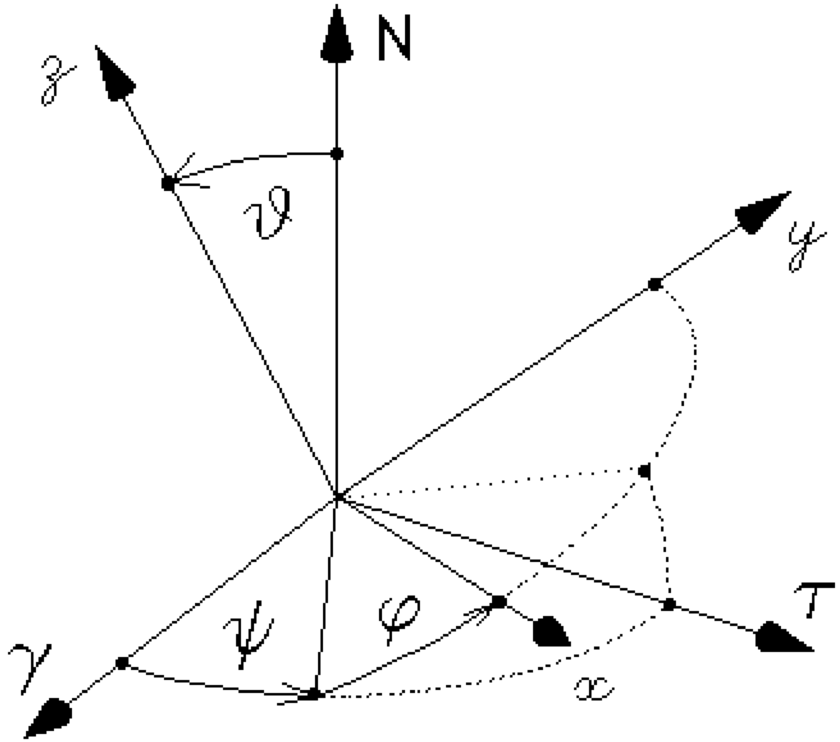
"3-1-3" Orbital Mechanics

$$\begin{cases} \theta_1 = \Omega & \text{(RAAN angle)} \\ \theta_2 = i & \text{(inclination)} \\ \theta_3 = \omega & \text{(Pericenter argument)} \end{cases}$$

$$\boxed{C = R_3(\theta_3)\, R_2(\theta_2)\, R_1(\theta_1)}$$

12 possible sequences: 6 classic + 6 modified

$$\text{Euler angle sequence: } \begin{cases} \text{classic (two rotations on the same axis)} \\ \text{modified (otherwise, as the "3-1-2" sequence)} \end{cases}$$



$$\boxed{C = \begin{bmatrix} +C_\varphi C_\psi - S_\varphi S_\psi C_\vartheta & +C_\varphi S_\psi + S_\varphi C_\psi C_\vartheta & +S_\varphi S_\vartheta \\ -S_\varphi C_\psi - C_\varphi S_\psi C_\vartheta & -S_\varphi S_\psi + C_\varphi C_\psi C_\vartheta & +C_\varphi S_\vartheta \\ +S_\psi S_\vartheta & -C_\psi S_\vartheta & +C_\vartheta \end{bmatrix}}$$

"3-1-3" Attitude dynamics

$$\begin{cases} \theta_1 = \psi & \text{(precession angle)} \\ \theta_2 = \vartheta & \text{(nutation angle)} \\ \theta_3 = \varphi & \text{(rotation, or spin angle)} \end{cases}$$

Gibbs vector

Gibbs vector is derived from quaternion

$$\left\{ e_1 \tan\frac{\phi}{2} \quad e_2 \tan\frac{\phi}{2} \quad e_3 \tan\frac{\phi}{2} \quad 1 \right\}^{\mathrm{T}} = \left\{ \rho^{\mathrm{T}} \quad 1 \right\}^{\mathrm{T}} = \frac{1}{q_4} q$$

Called also Rodrigues vector or Euler-Gibbs vector

$$\rho = e \tan\frac{\phi}{2}$$

$$\cos\frac{\phi}{2} \neq 0 \qquad \Rightarrow \qquad \phi \neq (2k-1)\pi$$

A three-parameter attitude representation has always a singularity!

Quaternion is easily derived from the Gibbs vector

$$q = \frac{1}{\sqrt{1+\rho^{\mathrm{T}}\rho}} \begin{Bmatrix} \rho \\ 1 \end{Bmatrix}$$

Cayley-Klein parameters

Cayley-Klein parameters are derived from quaternion

$$K = \begin{bmatrix} q_4 + iq_3 & q_2 + iq_1 \\ -q_2 + iq_1 & q_4 - iq_3 \end{bmatrix}$$

that can be written as

$$K = q_4 I + i(q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3) = \begin{bmatrix} q_4 + iq_3 & q_2 + iq_1 \\ -q_2 + iq_1 & q_4 - iq_3 \end{bmatrix}$$

where $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, and $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

are the Pauli' spin matrices

K is orthonormal: $K^{\mathrm{T}}K = KK^{\mathrm{T}} = I$

Small rotation representation (2/2)

When using infinitesimal rotation, the formal differences between different parameterizations also disappear!

(Euler axis and angle) $C = I - \tilde{\Phi}$ where $\Phi = \phi e$

$$\boxed{\text{(quaternion)} \quad C = I - 2\tilde{q}_v = \begin{bmatrix} 1 & 2q_3 & -2q_2 \\ -2q_3 & 1 & 2q_1 \\ 2q_2 & -2q_1 & 1 \end{bmatrix}}$$

$$\boxed{\text{(Euler's angle "1-2-3")} \quad C = I - \tilde{\theta} = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}}$$

Summary: Attitude Representations

FROM	TO	RELATIONSHIP
Principal angle/axis	DCM	$C = I\cos\phi + (1-\cos\phi)ee^T - \tilde{e}\sin\phi$
DCM	Principal angle/axis	$\cos\phi = \frac{1}{2}[\mathrm{trace}(C) - 1], \quad \tilde{e} = \frac{C^T - C}{2\sin\phi}$
Quaternions	DCM	$C = [(q_4^2 - q_v^T q_v)I + 2q_v q_v^T - 2q_4\tilde{q}_v]$
DCM	Quaternions	$q_4 = \pm\frac{1}{2}\sqrt{1+\mathrm{tr}[C]}$ and $q_v = \frac{1}{4q_4}\begin{Bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{Bmatrix}$ $\lvert q_k \rvert \geq \frac{1}{2}$
Euler Angles	DCM	$C_{313} = \begin{bmatrix} +C_\varphi C_\psi - S_\varphi S_\psi C_\vartheta & +C_\varphi S_\psi + S_\varphi C_\psi C_\vartheta & +S_\varphi S_\vartheta \\ -S_\varphi C_\psi - C_\varphi S_\psi C_\vartheta & -S_\varphi S_\psi + C_\varphi C_\psi C_\vartheta & +C_\varphi S_\vartheta \\ +S_\psi S_\vartheta & -C_\psi S_\vartheta & +C_\vartheta \end{bmatrix}$ *Different sequences have different C matrices…*
DCM	Euler Angles	$\vartheta = \cos^{-1}(C(3,3)), \quad \psi = \tan^{-1}(C(3,1), -C(3,2)),$ $\varphi = \tan^{-1}(C(1,3), C(2,3))$ *Different sequences have different C matrices…*
Principal angle/axis	Gibbs vector	$\rho = e\tan\frac{\phi}{2}$
Gibbs vector	Quaternions	$q = \frac{1}{\sqrt{1+\rho^T\rho}}\begin{Bmatrix} \rho \\ 1 \end{Bmatrix}$

⋮

And more…

Example:

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

a. Is C really an orientation matrix? Why?
b. Determine the principal angle and principal line.
c. Determine the Euler Parameters/Quaternion.
d. Determine the Euler angles corresponding to the sequence 2-3-1.
e. Determine the Gibbs vector.
f. Determine the Cayley-Klein parameters.
g. Determine the skew-symmetric matrix Q that results in Cayley transformation.

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

a. Is C really an orientation matrix? Why?

YES!

$$\boxed{\begin{aligned} &\det(C) = +1 \\ &CC^T = C^T C = I \end{aligned}}$$

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

b. Determine the principal angle and principal line.

$$\cos\phi = \frac{1}{2}[\mathrm{trace}(C) - 1] \quad \Rightarrow \quad \phi = 77.3409^\circ$$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2\sin\phi}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} = \begin{bmatrix} 0.9291 \\ -0.1879 \\ -0.3184 \end{bmatrix}$$

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

c. Determine the Euler Parameters/Quaternion.

$$\boxed{q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} q_v \\ q_4 \end{Bmatrix} = \begin{Bmatrix} e\sin\dfrac{\phi}{2} \\ \cos\dfrac{\phi}{2} \end{Bmatrix} = \begin{Bmatrix} 0.5806 \\ -0.1174 \\ -0.1990 \\ 0.7808 \end{Bmatrix}}$$

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

d. Determine the Euler angles corresponding to the sequence 2-3-1.

$$\boxed{\begin{bmatrix} \psi \\ \vartheta \\ \varphi \end{bmatrix}_{231} = \begin{bmatrix} 3.0565 \\ -26.5514 \\ 73.9904 \end{bmatrix} \text{deg.}}$$

You know how to do it ☺

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

e. Determine the Gibbs vector.

$$\boxed{\rho = e\tan\frac{\phi}{2} = \begin{bmatrix} 0.7436 \\ -0.1504 \\ -0.2548 \end{bmatrix}}$$

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

f. Determine the Cayley-Klein parameters.

$$\boxed{K = \begin{bmatrix} q_4 + iq_3 & q_2 + iq_1 \\ -q_2 + iq_1 & q_4 - iq_3 \end{bmatrix} = \begin{bmatrix} 0.7808 - 0.1990i & -0.1174 + 0.5806i \\ 0.1174 + 0.5806i & 0.7808 + 0.1990i \end{bmatrix}}$$

Example (1): SOLUTION

Suppose a rigid body orientation is described by the orientation matrix given below

$$C = \begin{bmatrix} 0.8933 & -0.4470 & -0.0477 \\ 0.1744 & 0.2467 & 0.9533 \\ -0.4144 & -0.8598 & 0.2983 \end{bmatrix}$$

g. Determine the skew-symmetric matrix Q that results in Cayley transformation.

$$\begin{cases} C = (I-Q)(I+Q)^{-1} = (I+Q)^{-1}(I-Q) & \text{(Forward)} \\ Q = (I-C)(I+C)^{-1} = (I+C)^{-1}(I-C) & \text{(Inverse)} \end{cases}$$

$$Q = \begin{bmatrix} 0 & 0.2548 & -0.1504 \\ -0.2548 & 0 & -0.7436 \\ 0.1504 & 0.7436 & 0 \end{bmatrix} \equiv [\tilde{\rho}] \quad \text{"Gibbs Vector"}$$

How does attitude vary with time?

Differential Equations of Kinematics

- Given the velocity of a point and initial conditions for its position, we can compute its position as a function of time by integrating the differential equation

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}}$$

- We now need to develop the equivalent differential equations for the attitude when the angular velocity is known

$$\dot{\mathbf{u}} = [A(\mathbf{u})]\boldsymbol{\omega}$$

$$\mathbf{u} := \text{attitude parameter}$$

$$\dot{\mathbf{u}} = [A(\mathbf{u})]\boldsymbol{\omega}$$

$$\mathbf{u} := \text{attitude parameter}$$

Attitude Parameter	Attitude Kinematics
DCM	$\dot{C} = -[\boldsymbol{\omega}\times]C$
Quaternion	$\dot{q} = \frac{1}{2}\begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} q_v \\ q_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$
CRPs	$\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[[I_{3\times 3}] + [\boldsymbol{\rho}\times] + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]\boldsymbol{\omega}$
MRPs	$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1-\boldsymbol{\sigma}^T\boldsymbol{\sigma})[I_{3\times 3}] + 2[\boldsymbol{\sigma}\times] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega}$
Euler Angles*	$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}_{ijk} = \mathbf{S}^{-1}(\theta_1, \theta_2, \theta_3)\boldsymbol{\omega}$
Principal angle/axis	$\dot{\phi} = \boldsymbol{\omega}^T\hat{\boldsymbol{e}}, \quad \dot{\boldsymbol{e}} = \frac{1}{2}\left[[\hat{\boldsymbol{e}}\times] - \cot(\phi/2)[\hat{\boldsymbol{e}}\times][\hat{\boldsymbol{e}}\times]\right]\boldsymbol{\omega}$
Cayley-Klein	$\mathrm{col}(\dot{K}) = \frac{1}{2}\Psi_0\begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}\Psi_0^{-1}\mathrm{col}(K)$, where $\Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$

*See Ref.[2] for $\mathbf{S}^{-1}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ definition.

[2] P. Hughes, Spacecraft Attitude Dynamics, J. Wiley, 1986.

$$\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\boldsymbol{\omega}$$

The short hand notation $c\theta_i = \cos\theta_i$ and $s\theta_i = \sin\theta_i$ is used again here.

Mapping Between Body Angular Velocity Vector and the Euler Angle Rates

Euler angle rates $\dot{\boldsymbol{\theta}} = (\dot\theta_1, \dot\theta_2, \dot\theta_3)^T$

	$[B(\boldsymbol{\theta})]$	$[B(\boldsymbol{\theta})]^{-1}$
1-2-1	$\frac{1}{s\theta_2}\begin{bmatrix} 0 & s\theta_3 & c\theta_3 \\ 0 & s\theta_2 c\theta_3 & -s\theta_2 s\theta_3 \\ s\theta_2 & -c\theta_2 s\theta_3 & -c\theta_2 c\theta_3 \end{bmatrix}$	$\begin{bmatrix} c\theta_2 & 0 & 1 \\ s\theta_2 s\theta_3 & c\theta_3 & 0 \\ s\theta_2 c\theta_3 & -s\theta_3 & 0 \end{bmatrix}$
1-2-3	$\frac{1}{c\theta_2}\begin{bmatrix} c\theta_3 & -s\theta_3 & 0 \\ c\theta_2 s\theta_3 & c\theta_2 c\theta_3 & 0 \\ -s\theta_2 c\theta_3 & s\theta_2 s\theta_3 & c\theta_2 \end{bmatrix}$	$\begin{bmatrix} c\theta_2 c\theta_3 & s\theta_3 & 0 \\ -c\theta_2 s\theta_3 & c\theta_3 & 0 \\ s\theta_2 & 0 & 1 \end{bmatrix}$
1-3-1	$\frac{1}{s\theta_2}\begin{bmatrix} 0 & -c\theta_3 & s\theta_3 \\ 0 & s\theta_2 s\theta_3 & s\theta_2 c\theta_3 \\ s\theta_2 & c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 \end{bmatrix}$	$\begin{bmatrix} c\theta_2 & 0 & 1 \\ -s\theta_2 c\theta_3 & s\theta_3 & 0 \\ s\theta_2 s\theta_3 & c\theta_3 & 0 \end{bmatrix}$
1-3-2	$\frac{1}{c\theta_2}\begin{bmatrix} c\theta_3 & 0 & s\theta_3 \\ -c\theta_2 s\theta_3 & 0 & c\theta_2 c\theta_3 \\ s\theta_2 c\theta_3 & c\theta_2 & s\theta_2 s\theta_3 \end{bmatrix}$	$\begin{bmatrix} c\theta_2 c\theta_3 & -s\theta_3 & 0 \\ -s\theta_2 & 0 & 1 \\ c\theta_2 s\theta_3 & c\theta_3 & 0 \end{bmatrix}$
2-1-2	$\frac{1}{s\theta_2}\begin{bmatrix} s\theta_3 & 0 & -c\theta_3 \\ s\theta_2 c\theta_3 & 0 & s\theta_2 s\theta_3 \\ -c\theta_2 s\theta_3 & s\theta_2 & c\theta_2 c\theta_3 \end{bmatrix}$	$\begin{bmatrix} s\theta_2 s\theta_3 & c\theta_3 & 0 \\ c\theta_2 & 0 & 1 \\ -s\theta_2 c\theta_3 & s\theta_3 & 0 \end{bmatrix}$
2-1-3	$\frac{1}{c\theta_2}\begin{bmatrix} s\theta_3 & c\theta_3 & 0 \\ c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 & 0 \\ s\theta_2 s\theta_3 & s\theta_2 c\theta_3 & c\theta_2 \end{bmatrix}$	$\begin{bmatrix} c\theta_2 s\theta_3 & c\theta_3 & 0 \\ c\theta_2 c\theta_3 & -s\theta_3 & 0 \\ -s\theta_2 & 0 & 1 \end{bmatrix}$
2-3-1	$\frac{1}{c\theta_2}\begin{bmatrix} 0 & c\theta_3 & -s\theta_3 \\ 0 & c\theta_2 s\theta_3 & c\theta_2 c\theta_3 \\ c\theta_2 & -s\theta_2 c\theta_3 & s\theta_2 s\theta_3 \end{bmatrix}$	$\begin{bmatrix} s\theta_2 & 0 & 1 \\ c\theta_2 c\theta_3 & s\theta_3 & 0 \\ -c\theta_2 s\theta_3 & c\theta_3 & 0 \end{bmatrix}$

	$[B(\boldsymbol{\theta})]$	$[B(\boldsymbol{\theta})]^{-1}$
2-3-2	$\frac{1}{s\theta_2}\begin{bmatrix} c\theta_3 & 0 & s\theta_3 \\ -s\theta_2 s\theta_3 & 0 & s\theta_2 c\theta_3 \\ -c\theta_2 c\theta_3 & s\theta_2 & -c\theta_2 s\theta_3 \end{bmatrix}$	$\begin{bmatrix} s\theta_2 c\theta_3 & -s\theta_3 & 0 \\ c\theta_2 & 0 & 1 \\ s\theta_2 s\theta_3 & c\theta_3 & 0 \end{bmatrix}$
3-1-2	$\frac{1}{c\theta_2}\begin{bmatrix} -s\theta_3 & 0 & c\theta_3 \\ c\theta_2 c\theta_3 & 0 & c\theta_2 s\theta_3 \\ s\theta_2 s\theta_3 & c\theta_2 & -s\theta_2 c\theta_3 \end{bmatrix}$	$\begin{bmatrix} -c\theta_2 s\theta_3 & c\theta_3 & 0 \\ s\theta_2 & 0 & 1 \\ c\theta_2 c\theta_3 & s\theta_3 & 0 \end{bmatrix}$
3-1-3	$\frac{1}{s\theta_2}\begin{bmatrix} s\theta_3 & c\theta_3 & 0 \\ s\theta_2 c\theta_3 & -s\theta_2 s\theta_3 & 0 \\ -c\theta_2 s\theta_3 & -c\theta_2 c\theta_3 & s\theta_2 \end{bmatrix}$	$\begin{bmatrix} s\theta_3 s\theta_2 & c\theta_3 & 0 \\ s\theta_2 c\theta_3 & -s\theta_3 & 0 \\ c\theta_2 & 0 & 1 \end{bmatrix}$
3-2-1	$\frac{1}{c\theta_2}\begin{bmatrix} 0 & s\theta_3 & c\theta_3 \\ 0 & c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 \\ c\theta_2 & s\theta_2 s\theta_3 & s\theta_2 c\theta_3 \end{bmatrix}$	$\begin{bmatrix} -s\theta_2 & 0 & 1 \\ c\theta_2 s\theta_3 & c\theta_3 & 0 \\ c\theta_2 c\theta_3 & -s\theta_3 & 0 \end{bmatrix}$
3-2-3	$\frac{1}{s\theta_2}\begin{bmatrix} -c\theta_3 & s\theta_3 & 0 \\ s\theta_2 s\theta_3 & s\theta_2 c\theta_3 & 0 \\ c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 & s\theta_2 \end{bmatrix}$	$\begin{bmatrix} -s\theta_2 c\theta_3 & s\theta_3 & 0 \\ s\theta_2 s\theta_3 & c\theta_3 & 0 \\ c\theta_2 & 0 & 1 \end{bmatrix}$

Kinematic Singularity in the Differential Equation for Euler Angles

- For the "symmetric" Euler angle sequences (3-1-3, 2-1-2, 1-3-1, etc) the singularity occurs when $\theta_2 = 0$ or π
- For the "asymmetric" Euler angle sequences (3-2-1, 2-3-1, 1-3-2, etc) the singularity occurs when $\theta_2 = \pi/2$ or $3\pi/2$
- This kinematic singularity is a major disadvantage of using Euler angles for large-angle motion

Typical Problem Involving Angular Velocity and Attitude

- Given initial conditions for the attitude (in any form), and a time history of angular velocity, compute $\mathbf{R}$ or any other attitude representation as a function of time
  - Requires integration of one of the sets of differential equations involving angular velocity

Stability of Spin (1/4)

We want to investigate the stability of the spin of a rigid body about a principal axis. The body is no longer axial-symmetric. Let the rigid body be rotating about the B_3 axis, and let the three axes be the principal axes. We want to determine the stability of this spin.

The Eqs. of motion are:
$$\begin{cases} \dot{\omega}_1 = \omega_2 \omega_3 \left(I_{22} - I_{33} \right) / I_{11} \\ \dot{\omega}_2 = \omega_1 \omega_3 \left(I_{33} - I_{11} \right) / I_{22} \\ \dot{\omega}_3 = \omega_1 \omega_2 \left(I_{11} - I_{22} \right) / I_{33} \end{cases}$$

An equilibrium point of this system is any one of the ω_i unequal to zero, and the other two ω_i equal to zero, i.e., spin about anyone of the coordinate axes. Let us consider a spin about the B_3 axis. The equilibrium point is (ω_1=0, ω_2=0, ω_3=ω_s).
Consider now a small perturbation on each component.

Stability of Spin (2/4)

Linearizing about the equilibrium point: $$\begin{cases} \omega_1 \cong \delta\omega_1 \ll 1 \\ \omega_2 \cong \delta\omega_2 \ll 1 \\ \omega_3 \cong \omega_S + \delta\omega_3 \cong \omega_S \end{cases}$$

The equations of motion become: $$\begin{cases} \delta\dot{\omega}_1 = \left[\omega_S \left(I_{22} - I_{33} \right) / I_{11} \right] \delta\omega_2 \\ \delta\dot{\omega}_2 = \left[\omega_S \left(I_{33} - I_{11} \right) / I_{22} \right] \delta\omega_1 \\ \delta\dot{\omega}_3 = 0 \end{cases}$$

Differentiating the first equation and substituting for $\delta\dot{\omega}_2$

$$\delta\ddot{\omega}_1 + \left[\omega_S^2 \left(I_{33} - I_{11} \right)\left(I_{33} - I_{22} \right) / \left(I_{11} I_{22} \right) \right] \delta\omega_1 = 0$$

Depending on the sign of the coefficient multiplying $\delta\omega_1$,
the solution $\delta\omega_1(t)$ oscillates or diverges.

$$\delta\ddot{\omega}_1 + a\delta\omega_1 = 0$$

2^{nd} ord. eq. with const. coeff.

$$\ddot{y} + ay = 0 \quad \text{the solution is: } y = c_1 e^{s_1 x} + c_2 e^{s_2 x}$$

where s_1 and s_2 are the evaluated from the

$$\text{characteristic equation:} \quad s^2 + a = 0 \quad \Rightarrow \quad s_{1,2} = \pm\sqrt{-a}$$

$$\text{if } \; a < 0, \text{ then } s_{1,2} = \pm c, \; \Rightarrow (+c) \text{ positive exponential!}$$

$$\text{if } \; a > 0, \text{ then } s_{1,2} = \pm ic, \; \Rightarrow \text{ armonic oscillator}$$

$$\delta\ddot{\omega}_1 + \left[\omega_S^2 \left(I_{33} - I_{11} \right)\left(I_{33} - I_{22} \right) / \left(I_{11} I_{22} \right) \right] \delta\omega_1 = 0$$

If the coefficient multiplying $\delta\omega_1$ is positive, then the motion is described by a harmonic oscillator. Therefore, it is stable if

$$(I_{33} > I_{11} \; \text{ and } \; I_{33} > I_{22}) \quad \text{or} \quad (I_{33} < I_{11} \; \text{ and } \; I_{33} < I_{22})$$

Stability of Spin (3/4)

In the other cases the spin is unstable. Therefore, we get the result that **spin about the axis of minimum or maximum moment of inertia is stable**, and **<u>spin about the axis of middle moment of inertia is unstable</u>**.

However, if there is any energy dissipation in the body, which is always the case, then spin about the axis of minimum moment of inertia is also unstable.

Heuristic demonstration:

For spin about a principal axis the kinetic energy is:

$$E = \frac{1}{2}\omega^{\mathrm{T}} I \omega = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \frac{1}{2} I_3 \omega_S^2$$

The overall angular momentum is:

$$\vec{h} = I\vec{\omega} = I_1\omega_1\vec{b}_1 + I_2\omega_2\vec{b}_2 + I_3\omega_3\vec{b}_3 = I_3\omega_S\vec{b}_3 \quad \Rightarrow \quad h = I_3\omega_S$$

From the above we derive: $E = \dfrac{h^2}{2I_3}$

Stability of Spin (4/4)

$$\text{if} \quad I_3 > I_2 > I_1 \quad \text{then} \quad E_1 = \frac{h^2}{2I_1} > E_2 = \frac{h^2}{2I_2} > E_3 = \frac{h^2}{2I_3}$$

Since the energy dissipation is internal (no external forces or torques) the angular momentum is constant. Spin about a principal axis is an equilibrium configuration and energy cannot dissipate. However, let there be a small perturbation from this configuration. Then energy will dissipate and the system will seek out the equilibrium configuration with the minimum kinetic energy. This is **spin about the axis of maximum moment of inertia**. This fact, although well known to astronomers, was not known by some of the early spacecraft designers and became known by a satellite (Explorer 1 – 1958) becoming unstable. The time constant for the unstable motion may be minutes or hours depending upon the source of dissipation.

Reaction or Momentum Wheels (1/5)

Consider a satellite with a set of N reaction or momentum wheels with spin axes defined in the B-frame by the unit vectors $\vec{n}_i$

The angular momentum of the system is: $\vec{h} = \vec{h}_b + \Sigma_i \vec{h}_{wi}$

where the $\vec{h}_b$ *is* the angular momentum of the rigid body and $\vec{h}_{wi}$ are the angular momenta of the wheels relative to the body. They are given by

$$\vec{h}_{wi} = h_{wi}\vec{n}_i = I_{wi}\omega_{wi}\vec{n}_i$$

Euler's equations are applicable since they are for a system.

$$\dot{\vec{h}} + \vec{\omega}\times\vec{h} = \vec{T} \quad \rightarrow \quad \begin{cases} \dot{h}_1 + \omega_2 h_3 - \omega_3 h_2 = T_1 \\ \dot{h}_2 + \omega_3 h_1 - \omega_1 h_3 = T_2 \\ \dot{h}_3 + \omega_1 h_2 - \omega_2 h_1 = T_3 \end{cases}$$

Reaction or Momentum Wheels (2/5)

For axis-symmetric wheels we can write $\dot{\vec{h}}_{wi} = \vec{T}_{mi}$

where $\vec{T}_{mi}$ are the wheel motor torques applied to the wheels.

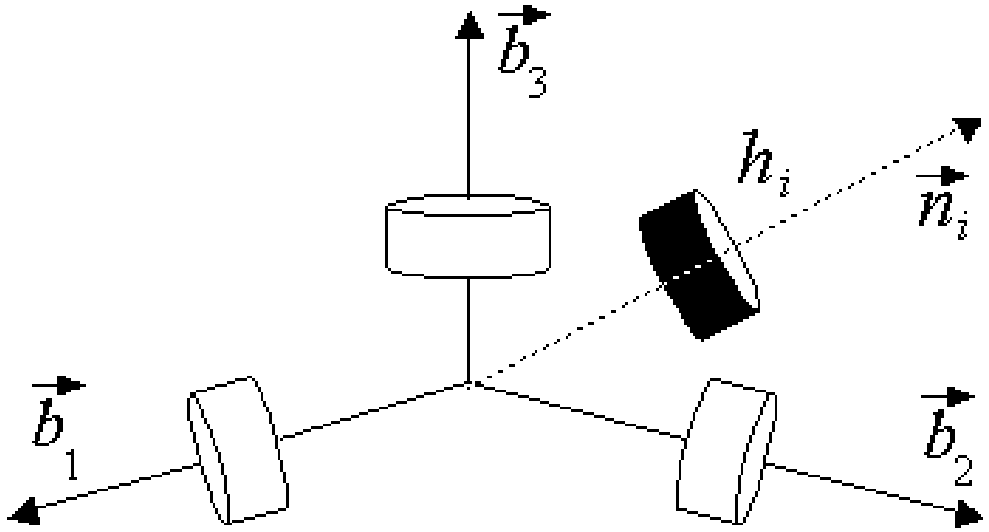
An equal and opposite torque is applied to the satellite.

Therefore: $\vec{T} = \sum_i \vec{T}_{mi}$ and $T_k = \sum_{i=1}^{N} (\vec{b}_k \cdot \vec{n}_i) T_{mi}$

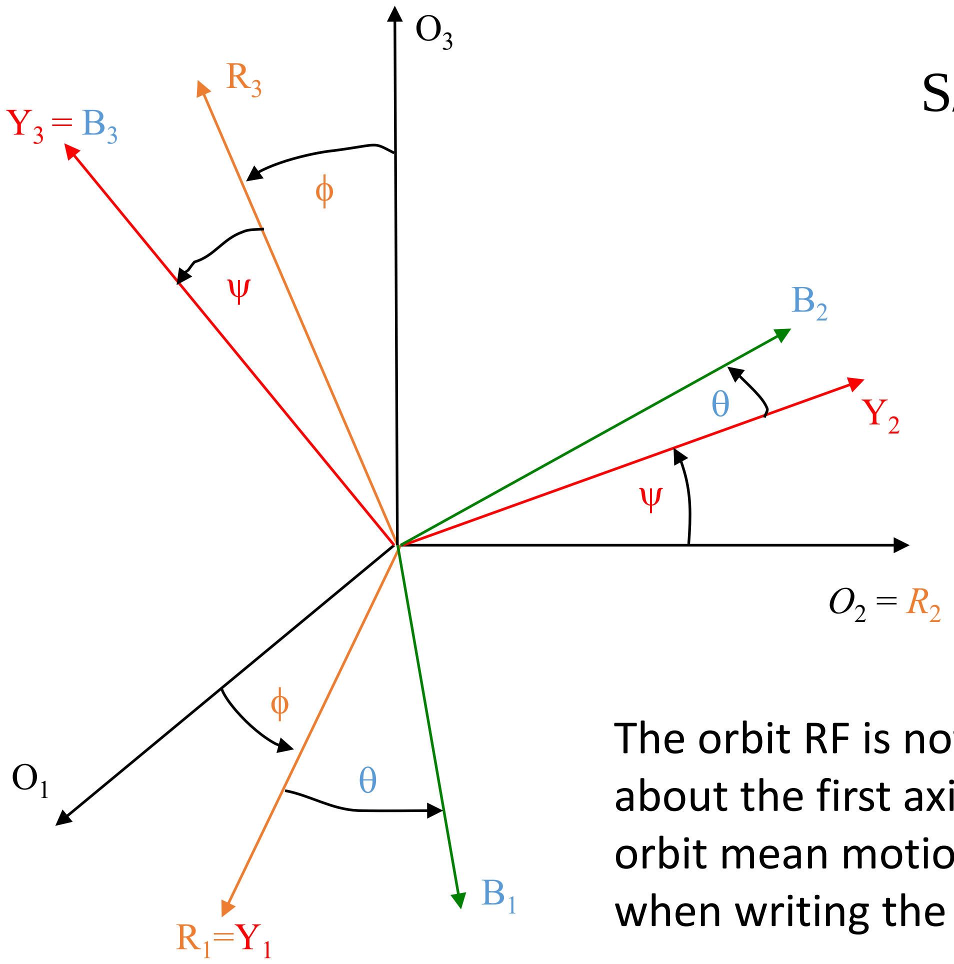
$\vec{h}_{wi} = I_{wi} \vec{\omega}_{wi}$ and $\dot{\vec{h}}_{wi} = \vec{T}_{mi}$

The angular momentum:

$$\begin{cases} h_1 = h_{b1} + \Sigma_i (\vec{n}_i \cdot \vec{b}_1) h_{wi} \\ h_2 = h_{b2} + \Sigma_i (\vec{n}_i \cdot \vec{b}_2) h_{wi} \\ h_3 = h_{b3} + \Sigma_i (\vec{n}_i \cdot \vec{b}_3) h_{wi} \end{cases}$$



"1-2-3" Euler angles sequence



S/C attitude with respect to the ORF

$$R = R_3(\theta) R_2(\phi) R_1(\psi)$$

$$b = Ro$$

$\vec{o}_3 \equiv$ pointing to Earth

$\vec{o}_1 \equiv$ pointing $\perp$ to orbit

The orbit RF is not inertial. For a circular orbit it rotates about the first axis with an angular velocity equal to the orbit mean motion *n*. This must be taken into account when writing the angular velocity.

Reaction or Momentum Wheels (3/5)

The satellite angular momentum can be expressed as

$$\vec{h}_b = I\vec{\omega}$$

With a "1-2-3" Euler angles (yaw, roll, pitch) sequence of rotations the angular velocity is:

$$\vec{\omega} = R_3(\theta)R_2(\phi)R_1(\psi)\begin{Bmatrix}\dot{\psi}\\ 0\\ n\end{Bmatrix} + R_3(\theta)R_2(\phi)\begin{Bmatrix}0\\ \dot{\phi}\\ 0\end{Bmatrix} + R_3(\theta)\begin{Bmatrix}0\\ 0\\ \dot{\theta}\end{Bmatrix}$$

$$\vec{\omega} = \begin{Bmatrix}\dot{\psi}\cos\theta\cos\phi - \dot{\phi}\sin\theta\\ \dot{\psi}\sin\theta\cos\phi + \dot{\phi}\cos\theta\\ \dot{\theta} + \dot{\psi}\sin\phi\end{Bmatrix} + n\begin{Bmatrix}-\sin\theta\sin\psi - \cos\theta\sin\phi\cos\psi\\ \cos\theta\sin\psi - \sin\theta\sin\phi\cos\psi\\ \cos\phi\cos\psi\end{Bmatrix}$$

where n is the orbital angular velocity or mean motion. This term is needed since we are referencing the attitude with respect to a rotating (Earth pointing) reference frame.

Reaction or Momentum Wheels (4/5)

The angular rates are

$$\begin{cases} \dot{\phi} = \left(-\omega_1 \sin\theta + \omega_2 \cos\theta - n\sin\psi\right) \\ \dot{\psi} = \left(\omega_1 \cos\theta + \omega_2 \sin\theta + n\sin\phi\cos\psi\right) / \cos\phi \\ \dot{\theta} = \omega_3 - n\cos\phi\cos\psi - \dot{\psi}\sin\phi = \omega_3 - \left(\omega_1 \cos\theta + \omega_2 \sin\theta\right)\tan\phi - n\cos\psi / \cos\phi \end{cases}$$

Note the singularity when $\phi = \pm\dfrac{\pi}{2}$

There is always a singularity with any set of Euler angles.

Evaluate the S/C attitude with respect to the ORF

$$R = R_3(\theta) R_2(\phi) R_1(\psi)$$

For GEO S/C sometimes the mean motion *n* is neglected (small)

Summary of equations (5/5)

Dynamics of rigid body: $\dot{\vec{h}} + \vec{\omega} \times \vec{h} = \vec{T}$

Equations of the wheels: $\vec{h}_{wi} = I_{wi} \vec{\omega}_{wi}$ and $\dot{\vec{h}}_{wi} = \vec{T}_{mi}$

Overall angular momentum: $\vec{h} = \vec{h}_b + \Sigma_i \vec{h}_{wi}$

Body angular momentum: $\vec{h}_b = I \vec{\omega}$

Torque exchange: $\vec{T} = \Sigma_i \vec{T}_{mi}$

From the above we can derive:

$$\dot{\vec{h}}_b + \vec{\omega} \times \vec{h}_b = -\vec{\omega} \times \Sigma_i \vec{h}_{wi}$$

Kynematics eqs.: $\vec{\omega} = f(\dot{X}, X)$ or $\dot{X} = g(\vec{\omega}, X)$

Attitude Error (definition)

True attitude: *T*

Estimated attitude: *C*

Corrective matrix

$$\Delta = TC^{\mathrm{T}}$$

$$\Delta C = TC^{\mathrm{T}}C = T$$

$$\begin{cases} b = Cr \\ \hat{b} = Tr \end{cases}$$

$$\hat{b} = TC^{\mathrm{T}}b = \Delta b$$

Body direction *b* is affected by the error ε

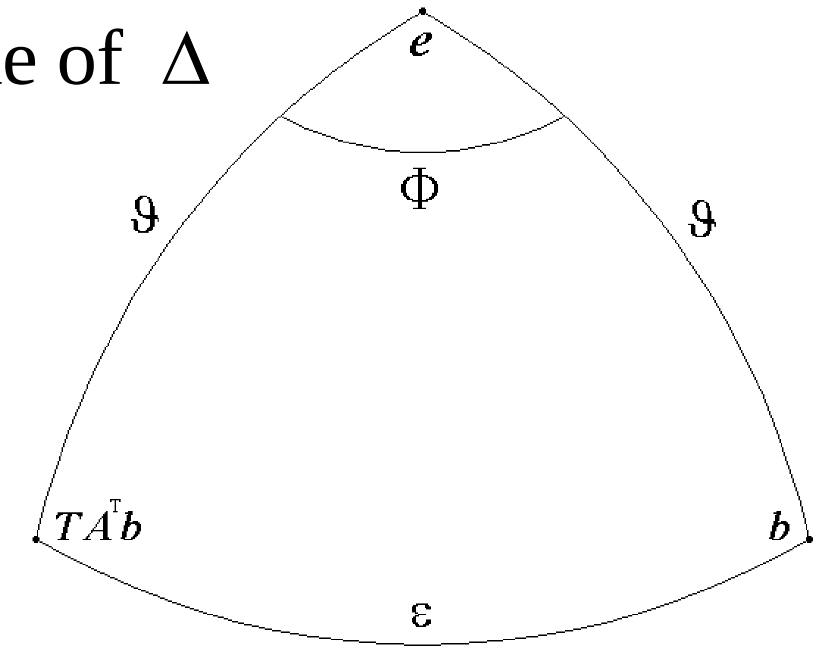
$$\cos\varepsilon = b^{\mathrm{T}}\hat{b} = b^{\mathrm{T}}\Delta b$$

Attitude Error (geometry)

$[e, \phi]$ principal axis and angle of Δ

$$\cos\varepsilon = b^{\mathrm{T}}\Delta b = b^{\mathrm{T}} T C^{\mathrm{T}} b$$

$$\cos\vartheta = e^{\mathrm{T}} b = e^{\mathrm{T}}\Delta b = e^{\mathrm{T}} T C^{\mathrm{T}} b$$



sine theorem $\rightarrow$ $\sin\vartheta = \dfrac{\sin(\varepsilon/2)}{\sin(\phi/2)}$

$$\cos\varepsilon = (1-\cos\phi)\cos^2\vartheta + \cos\phi$$

Maximum error: $(\varepsilon = \phi)$ for $\vartheta = \pi/2$

Errorless directions: $(\varepsilon = 0)$ for $\vartheta = 0, \pi$

Attitude Error (statistics)

Usually ϕ is small $\rightarrow$ $\varepsilon_{\max} = \phi \cong \sqrt{3 - \mathrm{tr}[TA^{\mathrm{T}}]}$

$$\varepsilon \cong \phi \sin\vartheta = \varepsilon_{\max} \sin\vartheta$$

Probability density function $\Rightarrow$ $P(\vartheta) = \frac{1}{2}\sin\vartheta$

$$E\{\varepsilon\} = \overline{\varepsilon} = \int_0^{\pi} \varepsilon(\vartheta) P(\vartheta) d\vartheta \cong \frac{\pi}{4}\varepsilon_{\max}$$

$$E\{\varepsilon^2\} = \overline{\varepsilon^2} \cong \int_0^{\pi} \varepsilon_{\max}^2 \sin^2\vartheta \frac{\sin\vartheta}{2} d\vartheta = \frac{2}{3}\varepsilon_{\max}^2$$

$$V\{\varepsilon\} = E\{(\varepsilon - \overline{\varepsilon})^2\} = \overline{\varepsilon^2} - (\overline{\varepsilon})^2 \cong [2/3 - (\pi/4)^2]\varepsilon_{\max}^2$$

$$D\{\varepsilon\} = \sqrt{V\{\varepsilon\}} \cong \varepsilon_{\max}\sqrt{2/3 - (\pi/4)^2}$$

Example: attitude error between quaternion, principal axis and angle, and Gibbs vector.

$$\text{True attitude described by the quaternion } \; q_{True} = \begin{Bmatrix} 0.58564493386788 \\ 0.20407106724448 \\ 0.02456847506594 \\ 0.78407359411060 \end{Bmatrix}$$

First method estimate principal axis and angle:

$$e_1 = \begin{Bmatrix} 0.94361701274567 \\ 0.32866855166594 \\ 0.03954638292888 \end{Bmatrix} \quad \text{and} \quad \phi_1 = 1.33922635256429 \text{ rad}$$

$$\text{Second method estimates the Gibbs vector: } \; \rho_2 = \begin{Bmatrix} 0.74692597514675 \\ 0.26027029704522 \\ 0.03133439928404 \end{Bmatrix}$$

Question: which method provides the best result?

Step1: transforms all representations in DCMs

$$\begin{cases} T = [(q_4^2 - q_v^{\mathrm{T}} q_v) I + 2 q_v q_v^{\mathrm{T}} - 2 q_4 \tilde{q}_v] \\ T = \begin{bmatrix} 0.91550277909327 & 0.27755335845501 & -0.29123666438628 \\ 0.20049938826795 & 0.31283280293564 & 0.92840488619546 \\ 0.34879027620718 & -0.90835002648636 & 0.23075002189717 \end{bmatrix} \end{cases}$$

$$\begin{cases} C_1 = I \cos\phi_1 + (1 - \cos\phi_1) e_1 e_1^{\mathrm{T}} - \tilde{e}_1 \sin_1 \phi \\ C_1 = \begin{bmatrix} 0.91556391153999 & 0.27744970182937 & -0.29114324110373 \\ 0.20046813731259 & 0.31273697315168 & 0.92844391944067 \\ 0.34864774459045 & -0.90841468976383 & 0.23071086149827 \end{bmatrix} \end{cases}$$

$$\begin{cases} q(\rho_2) = \dfrac{1}{\sqrt{1 + \rho_2^{\mathrm{T}} \rho_2}} \begin{Bmatrix} \rho_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} q_v \\ q_4 \end{Bmatrix} \quad \text{and} \quad C_2 = [(q_4^2 - q_v^{\mathrm{T}} q_v) I + 2 q_v q_v^{\mathrm{T}} - 2 q_4 \tilde{q}_v] \\ C_2 = \begin{bmatrix} -0.88781557416783 & 0.38252415395964 & -0.25584913113496 \\ 0.45195298210484 & 0.82951610817929 & -0.32808768376400 \\ 0.08672951188373 & -0.40691313312370 & -0.90934025197385 \end{bmatrix} \end{cases}$$

Step2: evaluate maximum and mean error

$$\begin{cases} \cos\varepsilon_{1\max} = \dfrac{1}{2}\left(\mathrm{tr}[TC_1^{\mathrm{T}}]-1\right) = 0.99999998476913 \\ \cos\varepsilon_{2\max} = \dfrac{1}{2}\left(\mathrm{tr}[TC_2^{\mathrm{T}}]-1\right) = -0.69827824738985 \end{cases}$$

$$\begin{cases} \varepsilon_{1\max} = 0.01\,\mathrm{deg} \\ \varepsilon_{2\max} = 134.2890302532824\ \mathrm{deg} \end{cases}$$

$$\begin{cases} \overline{\varepsilon}_{1\max} = \dfrac{\pi}{4}\varepsilon_{1\max} = 0.00785398176348\,\mathrm{deg} \\ \overline{\varepsilon}_{2\max} = \dfrac{\pi}{4}\varepsilon_{2\max} = 105.4703577253524\ \mathrm{deg} \end{cases}$$

First method is accurate, the second is completely wrong!

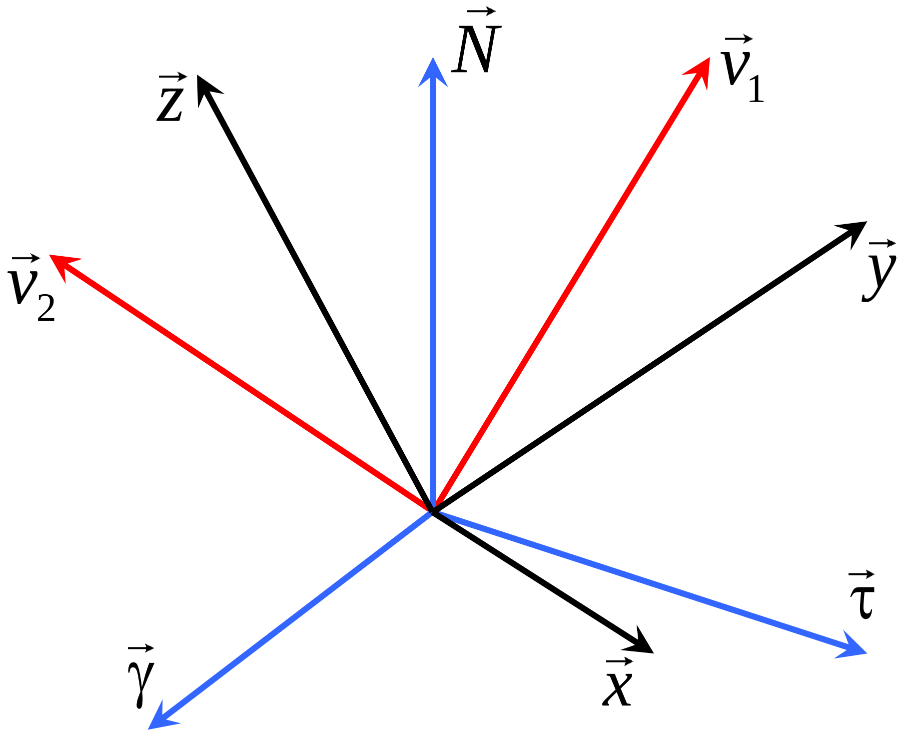
Attitude determination

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = r \quad \leftarrow \quad \vec{v} \quad \rightarrow \quad b = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Two measurements are needed!

$$\rightarrow \quad Cr_k = b_k \quad \leftarrow$$

$$\begin{cases} \text{Single-point attitude estimation} \\ \text{Multiple-point attitude estimation} \end{cases}$$



$$\begin{cases} r_k \text{ are in IRF (evaluated by codes)} \\ b_k \text{ are in BRF (measured by sensors)} \end{cases}$$

Ideal sensors

$$\begin{cases} Cr_i = b_i \\ Cr_j = b_j \end{cases} \quad \rightarrow \quad r_i^{\mathrm{T}} r_j = b_i^{\mathrm{T}} b_j$$

Harold D. Black, 1964

TRIAD

$$\begin{cases} Cr_1 = b_1 \\ Cr_2 = b_2 \end{cases} \Rightarrow \quad C[r_1 \quad r_2] = [b_1 \quad b_2]$$

Let f be any vectorial function, and a set of n identities $Cr_k = b_k$

$$\rightarrow \qquad \boxed{Cr = Cf(r_1, r_2, \cdots, r_n) = b = f(b_1, b_2, \cdots, b_n)} \qquad \leftarrow$$

In particular consider f be vector cross-product function

$$\rightarrow \qquad \boxed{Cr_3 = C(r_1 \times r_2) = b_3 = b_1 \times b_2} \qquad \leftarrow$$

$$C\left[r_1 \;\; r_2 \;\; (r_1 \times r_2)\right] = \left[b_1 \;\; b_2 \;\; (b_1 \times b_2)\right]$$

$$C = \left[b_1 \;\; b_2 \;\; (b_1 \times b_2)\right]\left[r_1 \;\; r_2 \;\; (r_1 \times r_2)\right]^{-1}$$

Example:

Consider a spacecraft inertially fixed with attitude assigned by quaternion
q = {0.2 -0.5 1 0.5}.
Write a code that simulates input data for TRIAD single-point attitude determination algorithms.

```
close all; clear all;clc; format long
% Parameters Definitions
q        = [0.2 -0.5 1 0.5]';  % Attitude - Quaternion
q        = q/sqrt(q'*q);       % Normalized Quaternion
T        = EP2C(q) ;           % DCM -- inertially fixed "True Attitude"
r1       = rand(3,1) ; r1 = r1/sqrt(r1'*r1);  % Reference Direction
r2       = rand(3,1) ; r2 = r2/sqrt(r2'*r2);  % Reference Direction
sqm1     = 0.00001616 ;        % Gaussian noise - sensor 1 in rad
sqm2     = deg2rad(0.5/3) ;    % Gaussian noise - sensor 2 in rad
alpha1   = sqm2/(sqm1+sqm2);   % weights
alpha2   = sqm1/(sqm1+sqm2);   % weights
    phi1     = sqm1*randn ;    % Random principal angle
    phi2     = sqm2*randn ;  % Random principal angle
    m1       = cross(T*r1,rand(3,1)) ; m1 = m1/sqrt(m1'*m1);
    m2       = cross(T*r2,rand(3,1)) ; m2 = m2/sqrt(m2'*m2);
    R1       = eye(3)*cos(phi1)+(1-cos(phi1))*m1*m1'-Tilde(m1)*sin(phi1);
    R2       = eye(3)*cos(phi2)+(1-cos(phi2))*m2*m2'-Tilde(m2)*sin(phi2);
    b1       = R1*T*r1;      % Observation Direction "multiplicative model"
    b2       = R2*T*r2;      % Observation Direction "multiplicative model"
%-------------------------- TRIAD -----------------------------------
C     = [b1 b2 Tilde(b1)*b2]*inv([r1 r2 Tilde(r1)*r2]);
C     = (C*C')^-0.5 * C;              % Orthogonal Procrustes Problem
th_c = acos(0.5*(trace(T*C')-1))   % Principle angle
```

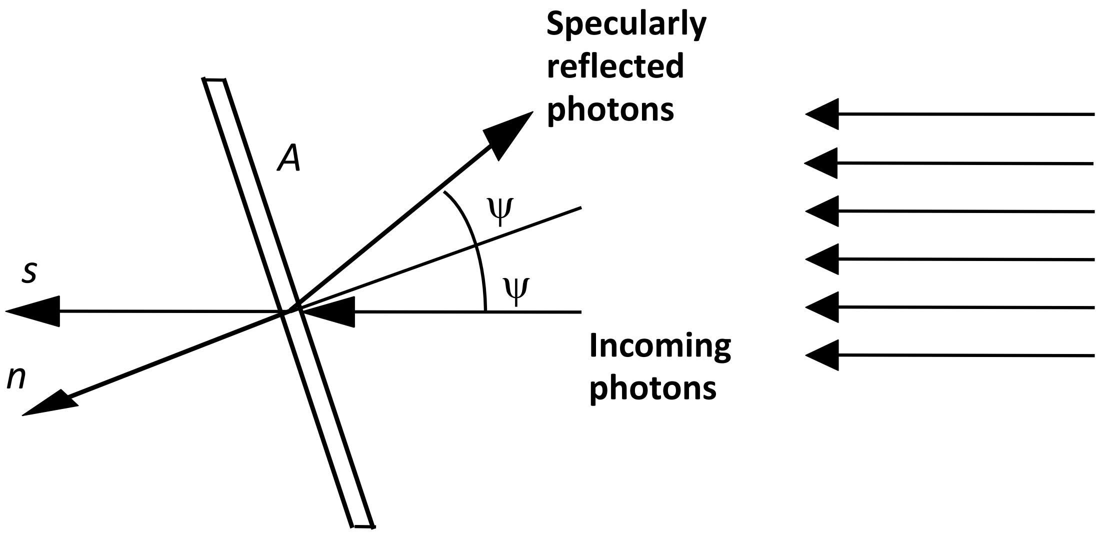
Types of Disturbance Torques

There are a number of external torques acting on a spacecraft which disturb the attitude motion:

- Solar Radiation Pressure
  - ✓ Dominant torque for geosynchronous satellites
- Gravity Gradient
  - ✓ Can be disturbance or control torque
- Atmospheric drag
  - ✓ drag coefficient C_D=2.2 for a spherical shape satellite; C_D=3 for a cylinder. Only a factor for LEO.
- Magnetic Field
  - ✓ Used for control

Solar Pressure Torque (1/3)

The torque resulting from solar pressure is the major long-term disturbance torque for geosynchronous spacecraft. The dominant torque on LEO spacecraft is the gravity gradient or aerodynamic. The solar radiation force results from the impingement of photons on the spacecraft. A fraction, ρ_s, are specularly reflected, a fraction, ρ_d, are diffusely reflected and a fraction, ρ_a, are absorbed by the surface



The force created by the absorbed photons results from the transfer of momentum of the photons to the spacecraft and is given by

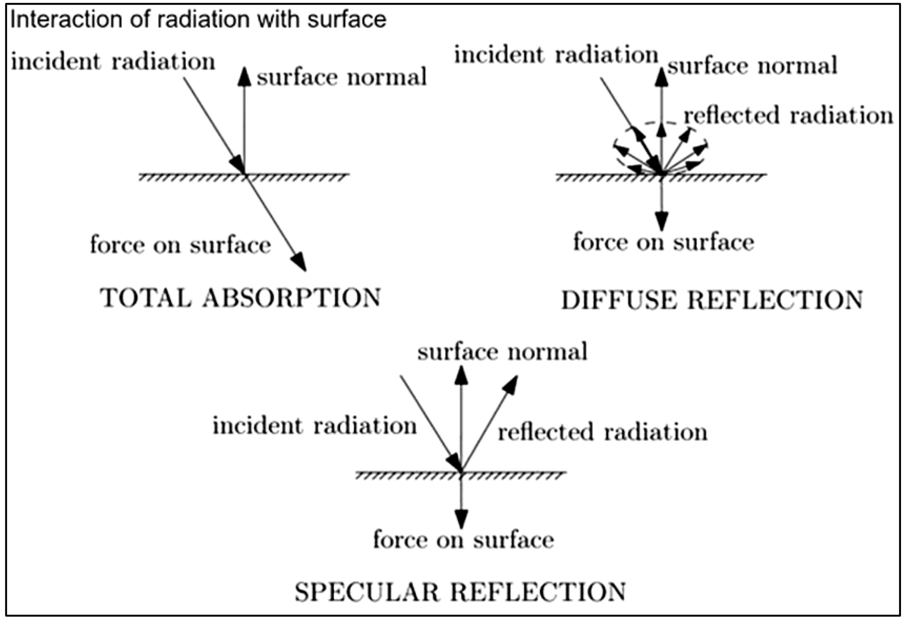
$$\vec{F}_a = \rho_a PA(\vec{n} \cdot \vec{s})\vec{s} = \rho_a P(A\cos\psi)\vec{s} \quad \text{(absorbed); } P \text{ is the solar flux}$$

Note that the force is in the direction along the sun line.

Solar Pressure Torque (2/3)

The fraction of the photons who are **specularly reflected** transfer twice the momentum and the direction is normal to the surface.
There is no momentum transfer tangent to the surface

$$\vec{F}_s = \rho_s PA(\vec{n}\cdot\vec{s}_{in})\vec{s}_{in} + \rho_s PA(\vec{n}\cdot\vec{s}_{out})\vec{s}_{out}$$
$$\vec{n}\cdot\vec{s}_{in} = \vec{n}\cdot\vec{s}_{out} = \cos\psi$$
$$\vec{F}_s = 2\rho_s PA\cos^2\psi\,\vec{n} \quad \text{(Specularly reflected)}$$



For that portion that is **diffusely reflected** the photon's momentum may be considered stopped at the surface, and re-radiated uniformly into the hemisphere. Thus the force has a component due to the transfer of momentum plus a component due to the re-radiation. Since it is re-radiated uniformly, the re-radiation component will be normal to the surface.

$$\text{The force is } \vec{F}_d = \rho_d PA(\vec{n}\cdot\vec{s})\left(\vec{s} + \frac{2}{3}\vec{n}\right) \quad \text{(Diffusely reflected)}$$

Solar Pressure Torque (3/3)

The total solar radiation force is

$$\vec{F} = \vec{F}_a + \vec{F}_s + \vec{F}_d = PA(\vec{n}\cdot\vec{s})\left\{(1-\rho_s)\vec{s} + \left[2\rho_s(\vec{n}\cdot\vec{s}) + \frac{2}{3}\rho_d\right]\vec{n}\right\}$$

where $\rho_a + \rho_s + \rho_d = 1$ has been used.

The solar pressure is usually assumed to be constant with value

$$\boxed{P = 4.644\times 10^{-6}\ \text{N/m}^2}$$

The total solar radiation force may be written as

$$\vec{F} = PA(\vec{n}\cdot\vec{s})(F_{S1}\vec{s} + F_{S2}\vec{n})$$

where: $F_{S1} = 1-\rho_s$ and $F_{S2} = 2\rho_s(\vec{n}\cdot\vec{s}) + \frac{2}{3}\rho_d$

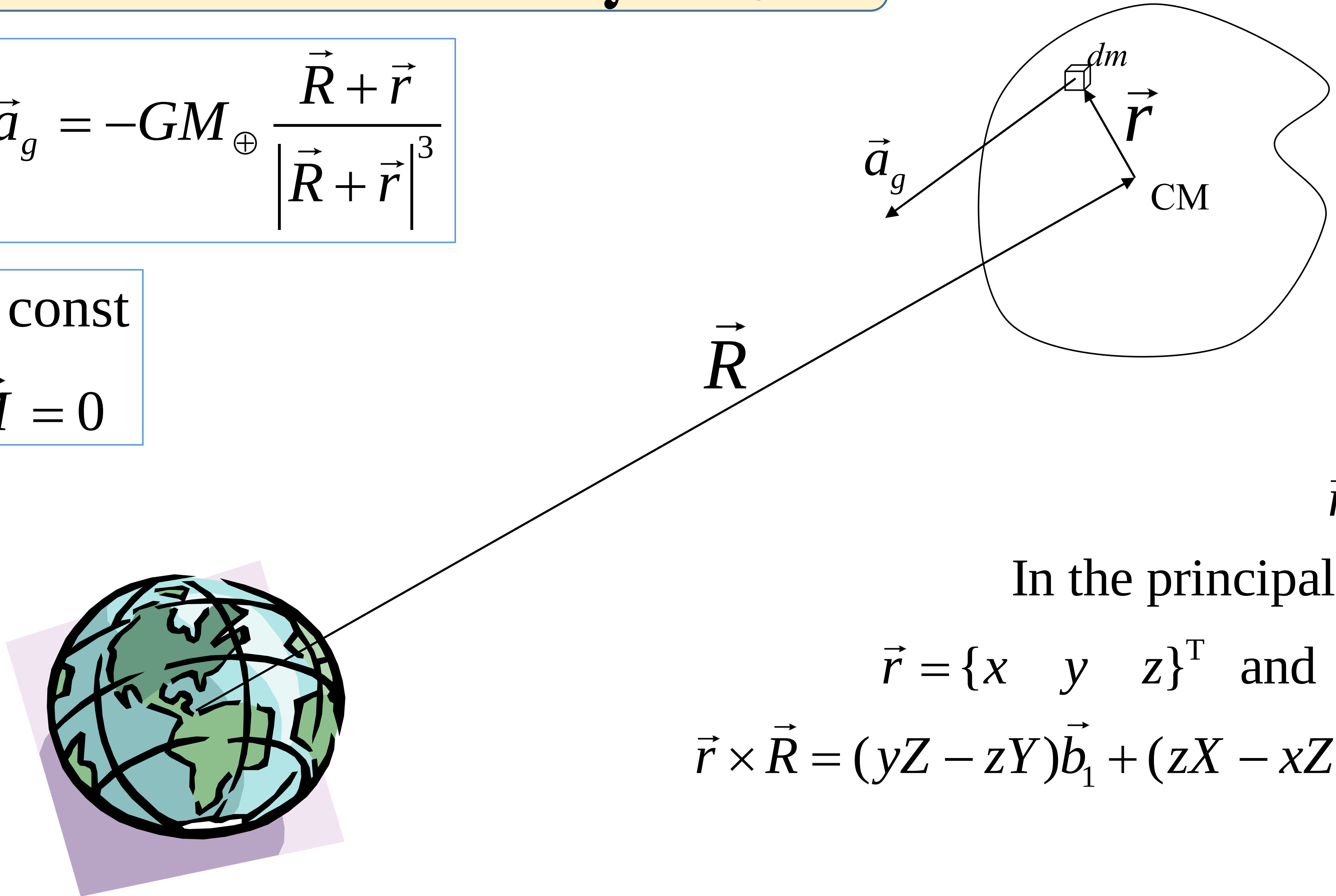
Gravity Gradient Torque (1/4)

The gravitational torque is: $\vec{M} = \int \vec{r} \times \vec{a}_g \, dm$

where: $\vec{a}_g = -GM_{\oplus} \dfrac{\vec{R}+\vec{r}}{\left|\vec{R}+\vec{r}\right|^3}$

if $\vec{a}_g = \text{const}$

then $\vec{M} = 0$



$$\vec{r} \times (\vec{R}+\vec{r}) = \vec{r} \times \vec{R}$$

In the principal-axis body frame:

$$\vec{r} = \{x \quad y \quad z\}^{\mathrm{T}} \text{ and } \vec{R} = \{X \quad Y \quad Y\}^{\mathrm{T}}$$

$$\vec{r} \times \vec{R} = (yZ - zY)\vec{b}_1 + (zX - xZ)\vec{b}_2 + (xY - yX)\vec{b}_3$$

Gravity Gradient Torque (2/4)

Since $r << R$

$$\left|\vec{R}+\vec{r}\right|^{-3} = (R^2 + 2\vec{R}\cdot\vec{r} + r^2)^{-3/2} = R^{-3}\left(1+\frac{2\vec{R}\cdot\vec{r}}{R^2}+\cdots\right)^{-3/2}$$

using the binomial theorem

$$\left|\vec{R}+\vec{r}\right|^{-3} \cong R^{-3}\left[1-\frac{3(xX+yY+zZ)}{R^2}+\cdots\right]$$

non zero

$$\begin{aligned} T_1 \quad &= -\frac{GM_\oplus}{R^3}\left\{Z\int ydm - Y\int zdm\right\} + \\ &-\frac{3GM_\oplus}{R^5}\left\{-XZ\int xydm - YZ\int (y^2-z^2)dm\right\} + \\ &-\frac{3GM_\oplus}{R^5}\left\{-Z^2\int yzdm + XY\int xzdm + Y^2\int zydm\right\} \end{aligned}$$

Gravity Gradient Torque (3/4)

The products of inertia integrals are also zero because we have chosen the principal axis body frame. Therefore, it remains ...

$$T_1 = \frac{3GM_{\oplus}YZ}{R^5}\int (y^2 - z^2)dm = \frac{3GM_{\oplus}YZ}{R^5}\int \left[(x^2+y^2)-(x^2+z^2)\right]dm =$$

$$= \frac{3GM_{\oplus}}{R^5}YZ(I_3 - I_2)$$ and the other two torque components are:

$$T_2 = \frac{3GM_{\oplus}}{R^5}XZ(I_1 - I_3) \quad \text{and} \quad T_3 = \frac{3GM_{\oplus}}{R^5}XY(I_2 - I_1)$$

Note that a fully symmetric body will NOT experience any gravitational torque.

Gravity Gradient Torque (4/4)

The Euler's equations for a rigid body subject to gravity gradient Torque are

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \dfrac{3GM_\oplus}{R^5} YZ(I_3 - I_2) \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = \dfrac{3GM_\oplus}{R^5} XZ(I_1 - I_3) \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = \dfrac{3GM_\oplus}{R^5} XY(I_2 - I_1) \end{cases}$$

Note that *X*, *Y* and *Z* are expressed in the BRF, while they are more conveniently evaluated in the IRF. These two different representations of the same vector are related through the attitude matrix

$$R_B = \{X \quad Y \quad Z\}^T = C\{X_I \quad Y_I \quad Z_I\}^T = CR_I$$

Aerodynamic drag (1/3)

The torque is originated by the interaction between the satellite's surfaces and the upper atmosphere.
Model: elastic impact without reflection (energy completely absorbed).
Mechanism: the particle arrived with v_0, reaches thermal equilibrium (velocity v_1) with molecular surface and escapes with $v_1 << v_0$.
The torque is about the center of mass.
Aerodynamic drag is the dominant perturbation for h<400 Km «LEO».

$$\boxed{d\vec{f}_{\text{drag}} = -\frac{1}{2} C_D \rho V^2 (\hat{n} \cdot \hat{v}) \hat{v}\, dA = -\frac{1}{2} C_D \rho V^2 \cos\alpha\, \hat{v}\, dA}$$

$$\boxed{\text{where: } C_D = f(\text{surface structure, local angle of attack } \alpha)}$$

$$\boxed{\vec{T}_{\text{drag}} = \int \vec{r} \times d\vec{f}_{\text{drag}} \quad (\text{only where } \hat{n} \cdot \hat{v} = \cos\alpha > 0 \text{ is satisfied})}$$

Aerodynamic drag (2/3)

The wind velocity, $\vec{V} = V\hat{v},$ is wrt the atmosphere, therefore:

$$\vec{V} = \vec{V}_0 + \vec{\omega}\times\vec{r} \quad \text{where} \begin{cases} \vec{V}_0 = V_0\hat{v}_0 = \text{CM velocity} & \hat{v} = \hat{v}_0 + \dfrac{\vec{\omega}\times\vec{r}}{V_0} \\ \vec{\omega} = \text{relative to the atmosphere} \end{cases}$$

$$\begin{aligned} \vec{T}_{\text{drag}} = \quad & \frac{1}{2}C_D\rho V_0^2\int(\hat{n}\cdot\hat{v}_0)(\hat{v}_0\times\vec{r})\,dA + \\ & +\frac{1}{2}C_D\rho V_0\int\{[\hat{n}\cdot(\vec{\omega}\times\vec{r})](\hat{v}_0\times\vec{r}) + (\hat{n}\cdot\hat{v}_0)[(\vec{\omega}\times\vec{r})\times\vec{r}]\}dA \end{aligned}$$

usually: $\left|\vec{V}_0\right| \gg \left|\vec{\omega}\times\vec{r}\right|$

First term: displacement between CM and center of pressure.
Second term: torque due to the S/C spin (wrt the atmosphere).

Aerodynamic drag (3/3)

For an Earth orbiting satellite, since $\left|\vec{V}_0\right| \gg \left|\vec{\omega} \times \vec{r}\right|$,

the 2nd term is four order of magnitude less than the first one.

$$\vec{T}_{\text{drag}} \simeq \frac{1}{2} C_D \rho V_0^2 \int (\hat{n} \cdot \hat{v}_0)(\hat{v}_0 \times \vec{r})\, dA$$

Aerodynamic drag is evaluated similarly to the solar pressure. Satellite surface is split in smaller geometrical shapes. Evaluate the torque for each shape. The overall torque is then the vectorial sum. Shadowing is very important, especially, at lower altitudes (because ρ increases).

FIGURE	FORCE
Sphere (radius *r*)	$F_{drag} = -0.5\pi C_D \rho (VR)^2$
Plane (area *A* and normal *n*)	$F_{drag} = -0.5 C_D \rho A V^2 (\hat{n} \cdot \hat{v})$
Cylinder (length *l*, diameter *d*, axis *a*)	$F_{drag} = -0.5 C_D \rho V^2 DL \sqrt{1 - (\vec{a} \cdot \hat{v})^2}$

Magnetic Torque

Torque caused by the interaction between the geomagnetic field *B* and the satellite's residual magnetic fields *m*.

$$\boxed{\vec{T}_{\text{mag}} = \vec{m} \times \vec{B}}$$

B is the geocentric magnetic flux density (Wb/m^2)
m is the sum of permanent and induced magnetism, and current loops.