A. Solve the following partial differential equations:

1) $u_x - u_y = 0.$ [Ans: $u(x,y) = c\, e^{k(x+y)}$]

2) $x\, u_x - 4y\, u_y = 0.$ [Ans: $u(x,y) = c\, x^k y^{\frac{k}{4}}$]

3) $u_x = 2u_t + u.$ Where, $u(x,0) = 6e^{-3x}$ [Ans: $u(x,y) = 6e^{-3x-2t}$]

4) $u_{xy} - u = 0.$ [Ans: $u(x,y) = c\, e^{(kx+\frac{y}{k})}$]

5) $u_{xy} + u_{yy} = 0.$ $\left[\text{Ans: } u(x,y) = \left(Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}\right)\left(C\cos\sqrt{k}y + D\sin\sqrt{k}y\right)\right]$

6) $x\, u_{xy} + 2y\, u = 0.$ [Ans: $u(x,y) = cx^k e^{-\frac{y^2}{k}}$]

7) $u_{xx} - 2u_x + u_y = 0.$ $\left[\text{Ans: } u(x,y) = \left(Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}\right)Ce^{-ky}\right]$

8) $y^2 u_x - x^2 u_y = 0.$

B. Problems related to one dimensional Wave equations:

1. A tight stretched string of length 20 cm is fastened at both ends is displaced from its position of equilibrium by imparting to each of its points and initial velocity, $g(x) = \begin{cases} x; & 0 \le x \le 10 \\ 20 - x; & 10 \le x \le 20 \end{cases}$. Find the deflection $u(x,t)$.

   $\left[\textbf{Ans}: u(x,t) = \frac{1600}{c\pi^3}\left[\sin\left(\frac{\pi x}{20}\right)\sin\left(\frac{\pi ct}{20}\right) - \frac{1}{3^3}\sin\left(\frac{3\pi x}{20}\right)\sin\left(\frac{3\pi ct}{20}\right) + \ldots\right]\right]$

2. Solve one dimensional wave equation with initial deflection is $0.01\ sin\ 3x$ and initial velocity is zero and $L = \pi$, $c^2 = 1$. [**Ans**: $u(x,t) = 0.01\cos 3t \sin 3x$]

3. Find the solution of one-dimensional wave equation corresponding to the triangular initial deflection

   $$u(x,0) = \begin{cases} \frac{2kx}{L}; & 0 \le x \le \frac{L}{2} \\ \frac{2k}{L}(L - x); & \frac{L}{2} \le x \le L \end{cases}$$ and initial its velocity zero and $c = 1$.

   $\left[\textbf{Ans}: u(x,t) = \frac{8k}{\pi^3}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi t}{L}\right) - \frac{1}{3^3}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi t}{L}\right) + \ldots\right]\right]$

4. A tight stretched string with two fixed ends points $x = 0$ and $x = l$, is initially the position given

   $$u(x,t) = \begin{cases} x; & 0 \le x \le \frac{L}{2} \\ L - x; & \frac{L}{2} \le x \le L \end{cases}$$. Also it is initially at rest and suddenly released. Find the displacement $u(x,t)$.

   $\left[\textbf{Ans}: u(x,t) = \frac{4L}{c\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$

C. Problems related to one dimensional heat equations:

5. A rod of length l has its ends A and B maintained at 0° C and 100° C respectively until steady state condition prevails. If B suddenly reduces to 0° C. Find the temperature $u(x,t)$ at a distance x from A at time t.

$\left[\textbf{Ans}: u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{c^2n^2\pi^2t}{l^2}}\sin\left(\frac{n\pi x}{l}\right)\right]$

6. A rod of length l with insulated sides is initially a uniform temperature u_0. Its ends are suddenly cooled at 0° C and are kept at that temperature. Find the temperature $u(x,t)$.

$\left[\textbf{Ans}: u(x,t) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^{n}}{n}e^{-\frac{c^2n^2\pi^2t}{l^2}}\sin\left(\frac{n\pi x}{l}\right)\right]$

7. Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is $u(x,t) = \begin{cases} x; & 0 \le x \le \frac{L}{2} \\ L-x; & \frac{L}{2} \le x \le L \end{cases}$.

$\left[\textbf{Ans}: u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}e^{-\frac{c^2n^2\pi^2t}{l^2}}\sin\left(\frac{n\pi x}{L}\right)\right]$

8. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the initial temperature is given $u(x,t) = \begin{cases} x; & 0 \le x \le 50 \\ 100-x; & 50 \le x \le 100 \end{cases}$ Find the temperature $u(x,t)$ at any time.

$\left[\textbf{Ans}: u(x,t) = -\sum_{n=1}^{\infty}\frac{200}{n\pi}\cos\left(\frac{n\pi}{2}\right)e^{-\frac{c^2n^2\pi^2t}{100^2}}\sin\left(\frac{n\pi x}{100}\right)\right]$