University of Connecticut
School of Business

FNCE 5894 MSFRM — Capstone

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1 Default Probability and Credit Spreads

1.1 Default Probability and Recovery Rates

The cumulative default probability function $F(t)$ gives the cumulative probability of default occuring prior to time t, assuming, of course, that it has not already occured.[1]

The marginal default probability gives the probability of default between two dates in the future, and can be computed from the cumulative default probability function, for any $t_1 \leq t_2$:

$$q(t_1, t_2) = F(t_2) - F(t_1) \tag{1}$$

By definition, the survival probability is given as $S(t) = 1 - F(t)$.

1.2 Real World and Risk-Neutral Default Probabilities

Risk-neutral probabilities refer to probabilities that are implied by some arbitrage free pricing model. In the context of credit risk, typical sources of risk-neutral probabilities include the market for Credit Default Swaps (CDSs) and the market for corporate bonds, which pay an interest rate premium over the (approximately) risk-free bonds issued by the U.S. government.

Real world probabilities refer to the real probabilities that govern events like changes in asset prices or credit defaults that might happen with the passage of time. They are meant to represent actual probabilities of events happening in the future. Since real world probabilities "predict" the future, they are, by their nature, only forecasts or educated guesses. A typical way to compute real world probabilities is by looking at what happened in the past using knowledge of the past to predict what will happen next.

[1] These notes are based in part on Chapter 10 of the book "Counterparty Credit Risk and Credit Value Adjustment" by Jon Gregory.

When risk-neutral and real world probabilities of default are computed, it is typically observed that the risk-neutral default probabilities are higher than real world default probabilities. The basic idea established in the book is

$$\begin{aligned}
\text{Real world default probability} &= \text{default risk} \\
\text{Risk-neutral default probability} &= \text{default risk} \\
&\quad + \text{default risk premium} \\
&\quad + \text{liquidity premium}
\end{aligned}$$

2 Estimating Real Default Probabilities from Historical Data

2.1 Transition Probability Matrices

Moody's and Standard and Poor's (among others) publish ratings that are meant to represent default probabilities by letter grades. In addition to giving ratings which are meant to replicate historical default occurences, they publish matrices which show the probability of migrating from one rating to another over a set period of time. The time period is often selected to be 1 year. The essence of the methodology is to look at the various financial statements that a company publishes, compare those to financial statements from the past, and try to draw inferences about the likelihood of default.

Here is a simplified version of such a matrix, with only 3 letter grades (A, B, and C) as well as the grade of D for default. Notice that there are only 3 rows, which conforms to the idea that once a credit-rated entity goes into default, it stays there for the time period under consideration. We say that the default state is *absorbing*.

$$R_{1Y} = \begin{pmatrix} & A & B & C & D \\ A & .92 & .05 & .02 & .01 \\ B & .02 & .89 & .07 & .02 \\ C & .00 & .10 & .85 & .05 \end{pmatrix}$$

Each row of the matrix indicates the credit rating at the beginning of the period. The columns represent the rating at the end of the period. Notice that the sum of the elements in a row is 1, which means that every rating either stays the same or *migrates* to another rating. Also notice that the terms on the diagonal are the largest, which indicates that the credit ratings tend not to change from year to year. This is by design.

In some cases, we might choose to explicitly represent the fact that an entity which starts the period in default also ends the period in default by adding a row for D. This gives us

$$R_{1Y} = \begin{pmatrix} & A & B & C & D \\ A & .92 & .05 & .02 & .01 \\ B & .02 & .89 & .07 & .02 \\ C & .00 & .10 & .85 & .05 \\ D & .00 & .00 & .00 & 1.00 \end{pmatrix}$$

2.2 Multi-year Transition Probability Matrices

Suppose we are interested in the probability of default over the next 2 years. If we are willing to make the *big* assumption that the matrix R does not change from year to year, we can get the probabilities by squaring the matrix.

$$R_{2Y} = R * R = \begin{pmatrix} & A & B & C & D \\ A & \mathbf{.92} & \mathbf{.05} & .02 & .01 \\ B & .02 & .89 & .07 & .02 \\ C & .00 & .10 & .85 & .05 \\ D & .00 & .00 & .00 & 1.00 \end{pmatrix} * \begin{pmatrix} & A & B & C & D \\ A & \mathbf{.92} & .05 & .02 & .01 \\ B & \mathbf{.02} & .89 & .07 & .02 \\ C & .00 & .10 & .85 & .05 \\ D & .00 & .00 & .00 & 1.00 \end{pmatrix}$$

$$= \begin{pmatrix} & A & B & C & D \\ A & \mathbf{.8474} & .0925 & .0389 & .0212 \\ B & .0362 & .8001 & .1222 & .0415 \\ C & .0020 & .1740 & .7295 & .0945 \\ D & .0000 & .0000 & .0000 & 1.000 \end{pmatrix}$$

The numbers in $R * R$ are computing using the standard algorithm for matrix multiplication, and are a good example of why matrix multiplication is defined the way that it is. Consider the first element of the last matrix, 0.8474. This number says that if a company starts with rating A at time 0, the probability that it end up with rating A two years later is 0.8474.

Why is this so?

There are a couple of ways to arrive at the rating A after two years. One way is to start at A and end at A. Since the probability of staying in A for 1 year is 0.92, the probability of staying A for 2 years in a row is

$$\mathbf{.92} * \mathbf{.92} = \mathbf{.8464}$$

Alternately, you could start with rating A, migrate to rating B, and then migrate back to A. From A to B happens with probability 0.05. Once in B, the probability of migrating back to A is 0.02. Thus, the probability of going from A to B to A is given by

$$\mathbf{.05} * \mathbf{.02} = \mathbf{.0010}$$

Adding these two result together, we get the probablity of ending in A after having started in A is

$$\begin{aligned} \mathbf{.92} * \mathbf{.92} + \mathbf{.05} * \mathbf{.02} &= \mathbf{.8464} + \mathbf{.0010} \\ &= \mathbf{.8474} \end{aligned}$$

Notice that we did not consider starting in A and migrating to either C or D before migrating back to A, as the probability of getting from C to A is 0 in this case and the probability of getting from D to A is 0 by our assumption that default is an absorbing state.

We consider one more example. This time we will look at the probability of starting up in B and ending up in C. This time, there are three paths of interest:

$$B \to A \to C$$
$$B \to B \to C$$
$$B \to C \to C$$

$$R_{2Y} = R * R = \begin{pmatrix} & A & B & C & D \\ A & .92 & .05 & .02 & .01 \\ B & \mathbf{.02} & \mathbf{.89} & \mathbf{.07} & \mathbf{.02} \\ C & .00 & .10 & .85 & .05 \\ D & .00 & .00 & .00 & 1.00 \end{pmatrix} * \begin{pmatrix} & A & B & C & D \\ A & .92 & .05 & \mathbf{.02} & .01 \\ B & .02 & .89 & \mathbf{.07} & .02 \\ C & .00 & .10 & \mathbf{.85} & .05 \\ D & .00 & .00 & \mathbf{.00} & 1.00 \end{pmatrix}$$

$$= \begin{pmatrix} & A & B & C & D \\ A & .8474 & .0925 & .0389 & .0212 \\ B & .0362 & .8001 & \mathbf{.1222} & .0415 \\ C & .0020 & .1740 & .7295 & .0945 \\ D & .0000 & .0000 & .0000 & 1.000 \end{pmatrix}$$

As before, we multiply the two steps in each chain to get the two-year probabilities:

$$P(B \to A \to C) = \mathbf{.02} * \mathbf{.02} = 0.0004$$
$$P(B \to B \to C) = \mathbf{.89} * \mathbf{.07} = 0.0623$$
$$P(B \to C \to C) = \mathbf{.07} * \mathbf{.85} = 0.0595$$

Adding these three numbers up, we obtain the two-year probability of going from B to C:

$$\begin{aligned} P_{2Y}(B \to C) &= \mathbf{.02} * \mathbf{.02} \\ &+ \mathbf{.89} * \mathbf{.07} \\ &+ \mathbf{.07} * \mathbf{.85} \\ &= \mathbf{0.1222} \end{aligned}$$

As we noted, it is a big assumption that the transition matrix doesn't change from year-to-year. In reality, the transition probabilities are closely related to cycles of the economy. Sometimes the matrices are constructed to give ratings which hold, *on average* over a period of several years. In this case, the multiplication method works for computing multi-year probabilities. In other applications, like credit card rating models, we might be need to update the matrix every year in response to macroeconomic variables in order to get a more accurate forecast of potential losses in the upcoming year.

Spreadsheet 10.1 has a more detailed example of the above phenomenon.