University of Connecticut School of Business

FNCE 5894 MSFRM
Capstone Fall 2014

Ed Hayes

September 19, 2015

1 Basic Terms

1.1 Zero Coupon Bond Price

$P(t, T)$ is the amount to deposit at time t in order to receive \$1 at time T.

1.2 Forward Rate

$F(t, T)$ is the interest rate to be earned on a deposit made at time t that matures at time T. In what follows, $F(t, T)$ will be a simple interest rate, but it could also be a continuous rate or a rate with any other type of compounding.

2 Relationship between Zero Coupon Prices and Forward Rates

Since $P(0, t)$ is the amount to deposit at time 0 in order to receive \$1 at time t, \$1 deposited today grows to

$$\frac{1}{P(0,t)} \tag{1}$$

by time t.

Similarly, since $P(0, t + \Delta t)$ is the amount to deposit at time 0 in order to receive \$1 at time $t + \Delta t$, \$1 deposited today grows to

$$\frac{1}{P(0,t+\Delta t)} \tag{2}$$

by time $t + \Delta t$.

We define the forward rate for a deposit at time t that matures at time $t + \Delta t$ by

$$\frac{1}{P(0,t)}(1 + F(t, t+\Delta t)\Delta t) = \frac{1}{P(0,t+\Delta t)} \tag{3}$$

Solving for $F(t, T)$, we obtain a formula that will be useful below: We define the forward rate for a deposit at time t that matures at time $t + \Delta t$ by

$$F(t, t + \Delta t)\Delta t = \frac{P(0, t)}{P(0, t + \Delta t)} - 1 \tag{4}$$

3 The Price of an Interest Rate Swap

3.1 Price of the Fixed Leg

We will assume that the notional is \$1 and that we are pricing at the present time, so that $t = 0$. Let c be the fixed coupon. The present value of the fixed leg of the swap is

$$PV_{fixed} = c \sum_{i=1}^{n} P(0, t_i)\Delta t_i. \tag{5}$$

3.2 Price of the Floating Leg

We will assume that the notional is \$1. Let c be the fixed coupon. The present value of the float leg of the swap is

$$\begin{aligned} PV_{float} &= \sum_{i=1}^{n} P(0, t_i)F(t_{i-1}, t_i)\Delta t_i. \\ &= \sum_{i=1}^{n} P(0, t_i)(\frac{P(0, t_{i-1})}{P(0, t_i)} - 1) \\ &= \sum_{i=1}^{n} (P(0, t_{i-1}) - P(0, t_i)) \\ &= 1 - P(0, t_n). \end{aligned}$$

The last line follows because $P(0, 0) = 1$ and the sum is a telescoping sum.

4 Vasicek Model for the Short Rate

The Vasicek model is a mean reverting model for the short rate. The equation for the short rate is

$$dr_t = a(b - r_t)\,dt + \sigma\,dW_t \tag{6}$$

4.1 Zero Coupon Price under the Vasicek Model

$$Z_t^T = A(t, T)\exp(-B(t, T)r_t) \tag{7}$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \tag{8}$$

and

$$A(t,T) = \exp\left((B(t,T) - (T-t))(b - \frac{\sigma^2}{2a^2}) - \frac{\sigma^2 B(t,T)^2}{4a}\right) \tag{9}$$