University of Connecticut
School of Business

FNCE 5894 — MSFRM Capstone

Ed Hayes

1 Introduction to Default Probability Modeling

1.1 The Exponential Distribution

Probability Density Function

The exponential distribution has probability density function

$$f(t;\lambda) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

We are using t as the variable here because we will usually be modeling time in our applications. As required, we note that

$$\begin{aligned} \int_{-\infty}^{\infty} f(t;\lambda)dt &= \int_{0}^{\infty} \lambda e^{-\lambda t} dt \\ &= -e^{-\lambda t}\Big|_{0}^{\infty} \\ &= 1 \end{aligned}$$

Cumulative Distribution Function

The exponential distribution has cumulative distribution function

$$\begin{aligned} F(t;\lambda) &= \int_{-\infty}^{t} f(u;\lambda)du \\ &= \int_{0}^{t} \lambda e^{-\lambda u} du \\ &= -e^{-\lambda u}\Big|_{0}^{t} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Thus,

$$F(t;\lambda) = 1 - e^{-\lambda t} \tag{1}$$

is the distribution function for an exponentially distributed random variable. By the definition of the distribution function, we have

$$P(T < t) = 1 - e^{-\lambda t},$$

so that

$$\begin{aligned} P(T > t) &= 1 - P(T < t) \\ &= e^{-\lambda t}. \end{aligned}$$

Mean and variance of an exponentially distributed random variable

Let T be exponentially distributed. We compute its mean and variance by using integration by parts, with $u = t$ and $dv = \lambda e^{-\lambda t}$:

$$\begin{aligned} E(T) &= \int_0^\infty t\lambda e^{-\lambda t} dt \\ &= -te^{-\lambda u}\Big|_0^\infty + \int_0^\infty \lambda e^{-\lambda t} dt \\ &= 0 - \frac{1}{\lambda} e^{-\lambda u}\Big|_0^\infty \\ &= \frac{1}{\lambda} \end{aligned}$$

In the same manner, integrating by parts twice yields

$$E(T^2) = \frac{2}{\lambda^2}$$

so that

$$\begin{aligned} \mathrm{Var(T)} &= E(T^2) - (E(T))^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2}. \end{aligned}$$

The standard deviation is given by

$$\sigma(T) = \sqrt{\mathrm{Var(T)}} = \frac{1}{\lambda},$$

which means that the exponential distribution has the interesting property that its mean and its standard deviation are the same number:

$$\mu(T) = \sigma(T) = \frac{1}{\lambda}.$$

Memoryless property of the exponential distribution

Before we finish talking about the exponential distribution, we want to highlight one more important property, the "memoryless" property.

$$P\left(T > s + t \,|\, T > s\right) = P(T > t) \tag{2}$$

Equation (2) states that, if you have already waited s units of time, say minutes, for the event to occur, the probability that you will have to wait an additional t minutes for the event to occur is the same as if the process had just started. The time spent waiting does not mean that the event is any closer to occuring, in probability!

To see this, we recall the definition of conditional probability:

$$\begin{aligned} P(A \cap B) &= P(A) * P(B|A) \\ \Rightarrow P(B|A) &= \frac{P(A \cap B)}{P(A)} \end{aligned}$$

Applied here, we have

$$\begin{aligned} P\left(T > s + t \,|\, T > s)\right) &= \frac{P[(T > s + t) \cap (T > s)]}{P(T > s)} \\ &= \frac{P[(T > s + t)}{P(T > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(T > t). \end{aligned}$$

1.2 The Poisson Process

The Poisson process is a continuous time counting process $\{N(t), t \geq 0\}$. Since it is a *continuous* time stochastic process, it is a family of random variables indexed by the non-negative real number line $[0, \infty)$. Since it is a *counting* process, it takes values in the whole numbers, $W = \{0, 1, 2, 3, ...\}$.

Definition A Poisson process is a continuous time counting process denoted by $\{N(t), t \geq 0\}$. It has the following properties:

1. N(0) = 0
2. The times T between arrivals of the phenomena being counted are independent.
3. The interarrival times are exponentially distributed:

$$T(N + 1) - T(N) \sim \lambda e^{-\lambda t}.$$

We have included the exponential distribution between waiting times as part of the definition here, but it is also possible to derive it if we instead assume that the number of arrivals that occur in a time interval Δt follows a Poisson distribution:

$$P(N(t+\Delta t) - N(t) = k) = e^{-\lambda \Delta t}\frac{(\lambda \Delta t)^k}{k!},\ k = 0, 1, 2, ... \tag{3}$$

Note that the Poisson distribution has total probability of 1, as required:

$$\begin{aligned}
&\sum_{k=0}^{\infty} e^{-\lambda \Delta t}\frac{(\lambda \Delta t)^k}{k!} \\
&= e^{-\lambda \Delta t}\sum_{k=0}^{\infty}\frac{(\lambda \Delta t)^k}{k!} \\
&= e^{-\lambda \Delta t}e^{\lambda \Delta t} \\
&= 1,
\end{aligned}$$

where we have used the Taylor series for the exponential function to simplify the second line:

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

With the assumption (3), we can derive the exponential distribution of interarrival times. Let T_k be the time of the k^{th} arrival. The above definitions imply

$$P(T_k > t) = P(N(t) < k).$$

This says that if the k^{th} arrival happens after time t, then the counting process $N(t)$ has counted less than k arrivals. In particular,

$$\begin{aligned}
P(T_1 > t) &= P(N(t) < 1) \\
&= P(N(t) = 0) \\
&= P(N(t) - N(0) = 0) \\
&= e^{-\lambda t}\frac{(\lambda t)^0}{0!} \\
&= e^{-\lambda t}.
\end{aligned}$$

Thus,

$$P(T_1 > t) = e^{-\lambda t}.$$

and

$$P(T_1 \leq t) = 1 - e^{-\lambda t}. \tag{4}$$

Since (4) is the cumulative distribution function of the exponential distribution, we have shown that the if the number of arrivals in an interval has a Poisson distribution, then the interarrival times have an exponential distribution with the same parameter λ.

1.3 Application of the Poisson process to credit default monitoring

We model credit default as the first arrival of a Poisson process[1], which means that the cumulative default probability is given by the cumulative distribution function of the exponential function

$$F(t) = 1 - e^{-ht},$$

where we have switched to using the letter h for the intensity parameter as suggestive of the *hazard* rate. The cumulative survival probability is the probability of no default by time t and is given by

$$\begin{aligned} S(t) &= 1 - F(t) \\ &= e^{-ht} \end{aligned} \tag{5}$$

The instantaneous default probability is given by the probability density function of the exponential

$$\frac{dF}{dt} = he^{-ht} \tag{6}$$

or

$$\frac{dF}{dt} = hS(t) \tag{7}$$

Writing (7) in differential form, we have

$$\begin{aligned} dF &= he^{-ht}dt \\ F(t+dt) - F(t) &= he^{-ht}dt \\ F(t+dt) - F(t) &= S(t) * h\,dt \end{aligned} \tag{8}$$

We interpret (8) in the following manner. The term $F(t+dt) - F(t)$ represents the probability of default occuring in the interval $(t, t+dt)$. The term on the right hand side says that this probability is the product of $S(t)$, which is the probability of default not occuring by time t, and $h\,dt$. For this reason, h is called the *hazard* rate, or the rate per unit time at which default occurs. As Gregory puts it, "the probability of default in an infinitely small period dt, conditional on no prior default (S(t)), is $h\,dt$.

Valuation of a risky cash flow

The provider of protection in a CDS receives payments up until the time that a credit default happens. Although the payments are made periodically, they are earned continuously, conditional on survival of the credit. The value of such a risky annuity can be

[1] These notes are based in part on Chapter 10 of the book "Counterparty Credit Risk and Credit Value Adjustment" by Jon Gregory. See especially Appendix 10B for this section.

written as

$$\int_0^T B(t)S(t)dt$$

This is the value of the payment stream paid by the protection buyer, or the ***unit cost of protection***. In the same manner, we can value the protection provided to the buyer of the CDS. We start by writing it as a Riemann sum:

$$\begin{aligned}\text{unit value of protection} &\approx (1-R)\sum_{j=1}^{n} B(t_j)(F(t_j)-F(t_{j-1})) \\ &= \text{LGD}\sum_{j=1}^{n} B(t_j)(F(t_j)-F(t_{j-1}))\end{aligned}$$

This says that, at time $F(t_j) = F(j * dt)$, we will be paid the present value of the notional, adjusted by the recovery value, if default occurs in the time window between $F(t_j)$ and $F(t_{j-1})$. Here, $B(t_j)$ is the discount factor to date t_j and R is the recovery rate, so that $1-R$ is the **Loss Given Default** or **LGD**. The total CDS value is the sum of the value of all the possible payments that might occur. Letting $\Delta t = t_j - t_{j-1}$ Passing to the limit of the Riemann sum (actually a Riemann-Stieltjes sum), we obtain

$$\begin{aligned}\text{unit value of protection} &= (1-R)\int\limits_0^T B(t)dF(t) \\ &= (1-R)\,h\int\limits_0^T B(t)S(t)dt.\end{aligned}$$

where we have used (7) in the second line.

Relationship between CDS spread, hazard rate, and recovery

Paraphrasing Gregory, we note that the fair CDS spread is the ratio of the value of default protection divided by the risky annuity (the unit cost of paying for protection). Thus,

$$\text{CDS Spread} = \frac{\text{unit value of protection}}{\text{unit cost of protection}}$$

$$= \frac{(1-R)\,h \int\limits_0^T B(t)S(t)dt}{\int\limits_0^T B(t)S(t)dt}$$
$$= (1-R)\,h,$$

so that

$$h = \frac{\text{CDS Spread}}{1-R}.$$

Term structure of hazard rates

Here, we follow Gregory's Appendix 10B (iii), which references work by Li (1998). If the hazard rate is not constant, we can express the survival probability as

$$S(t) = \exp(-\int\limits_0^t h(s)ds). \tag{9}$$

Notice that if $h(s) \equiv h$ is a constant, this reduces to

$$S(t) = \exp(-ht),$$

as in (5).

For the intution behind this formula, start by considering time periods t_{01}, t_{02} and t_{12} and hazard rates h_{01}, h_{02}, and h_{12}. Here, t_{12}, for example, is the time period between t_1 and t_2 and h_{12} is the hazard rate over that interval. We will usually write t_1 and h_1 when time is measured as starting from today, $t = 0$. Survival until time t_1 is given by

$$S(t_1) = \exp(-h_1 t_1),$$

Survival from time t_1 to t_2, conditional on having survived until time t_1, is then given by

$$S(t_{12}) = \exp(-h_{12}(t_2 - t_1)).$$

Putting these together, we have

$$\begin{aligned} S(t_2) &= \exp(-h_2 t_2) \\ &= \exp(-h_1 t_1) * \exp(-h_{12}(t_2 - t_1)) \\ &= \exp(-(h_1 t_1 + h_{12} t_{12})). \end{aligned}$$

The middle equation here states that there is a probability of surviving from the current time t_0 until time t_1, and that there is then a probability of continuing to survive until time t_2. We could also write this as

$$\begin{aligned} S(t_{02}) &= \exp(-h_1 t_1) * \exp(-h_{12}(t_2 - t_1)) \\ &= S(t_{01}) * S(t_{12}). \end{aligned}$$

If we continue in this fashion for additional subdivisions of the time interval $[0, t]$, we obtain

$$S(t) = \exp(-\sum_{j=1}^{n} h(t_{j-1}, t_j)\Delta t_j).$$

Passing to the limit, we obtain (9).

1.4 Application: Credit Valuation Adjustment

The key formula that we will implement in problem 3 of homework assignment 2 is given at the bottom of page 3 in Appendix 12B. It is modified here to make the starting time $t = 0$ and to make the discounting explicit with the $df()$ instead of implied by the subscript in EE_d:

$$\text{CVA}(0, T) \approx (1 - \bar{R}) \sum_{j=1}^{n} EE(t_j)(F(t_j) - F(t_{j-1})) * df(j) \tag{10}$$

This formula can also be written in using the common terms "loss given default" (LGD), "exposure at default" (EAD), and "probability of default" (PD). The result is

$$\text{CVA}(0, T) \approx \sum_{j=1}^{n} \left(\text{EAD}(t_j) * \text{LGD}(t_j) * \text{PD}(t_j) * \text{df}(j)\right),$$

where we have replaced $1 - \bar{R}$ term by LGD and moved the LGD term inside the summation to more accurately reflect the term structure of LGD. The $1 - \bar{R}$ approximation given in (10) is used in the spreadsheet, but it is a rough approximation that would not be used in practice.