University of Connecticut
School of Business

FNCE 5894 MSFRM
Capstone Lecture 1

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1 The PFE, EE, and EPE of a normally-distributed value

1.1 Potential Future Exposure (PFE)

This formula is given as Appendix 8A(i) in the book. We prove the formula here. Denote the value of the exposure to a given counterparty by V and assume that

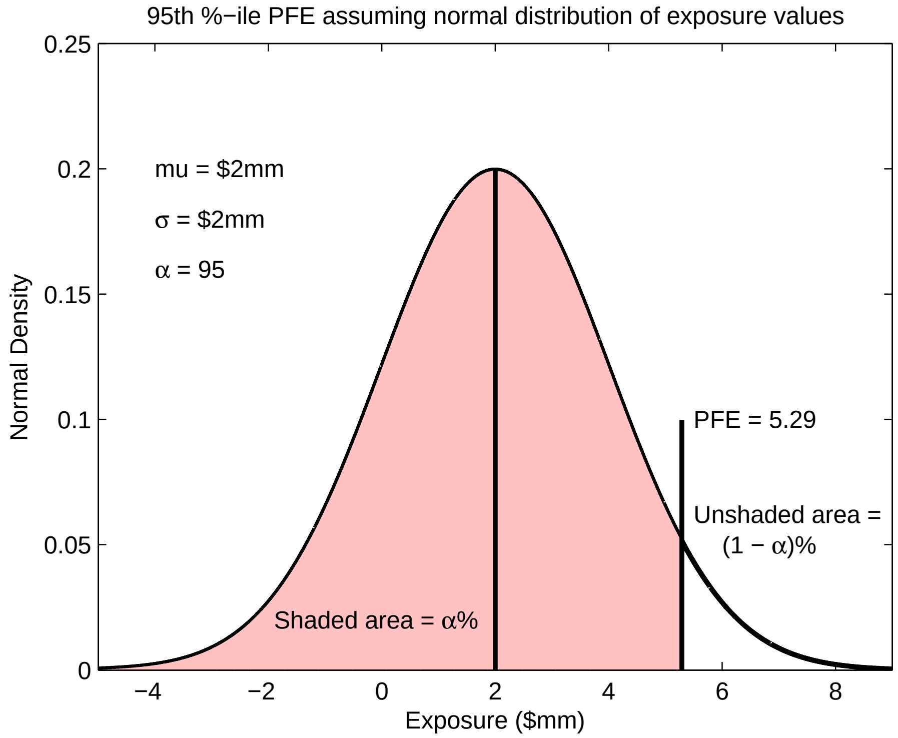
$$V = \mu + \sigma Z. \tag{1}$$

By the definition of PFE, we are interested in the level of exposure L associated with a given confidence interval α such that L satisfies

$$P(\text{exposure} > L) = 1 - \alpha. \tag{2}$$

We can express the level L in terms of the cumulative distribution function $\Phi(z)$ of the standard normal distribution, as follows:

$$\begin{aligned}
P(\text{exposure} > L) &= 1 - \alpha \\
P(V > L) &= 1 - \alpha \\
P(V < L) &= \alpha \\
P(\mu + \sigma Z < L) &= \alpha \\
P(Z < \frac{L - \mu}{\sigma}) &= \alpha \\
\Phi(\frac{L - \mu}{\sigma}) &= \alpha \\
\frac{L - \mu}{\sigma} &= \Phi^{-1}(\alpha) \\
L &= \mu + \sigma \Phi^{-1}(\alpha) \qquad (3)
\end{aligned}$$



1.2 Expected Exposure (EE)

This formula is given as Appendix 8A(ii) in the book. We prove the formula here. By definition, the exposure is always non-negative:

$$\text{Exposure} = \max(V, 0),$$

where V is the future value of a position.

Expected Exposure is defined to be the average value over the distribution of exposures:

$$\text{Expected Exposure} = \int_{-\infty}^{\infty} \max(v, 0)\, f(v)\, dv$$

where $f(v)$ is the distribution of exposures.

As in the previous section, we assume the value of the exposure to a given counterparty to be

$$V = \mu + \sigma Z.$$

where $Z = N(0, 1)$ is a standard normal variable.

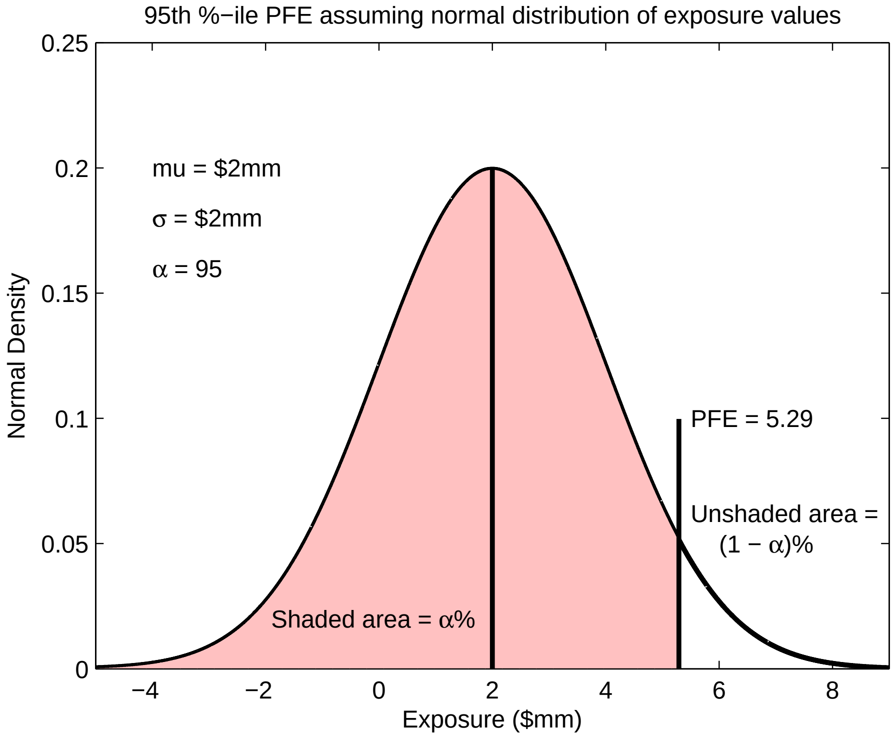
By the definition of EE, we need the average level of exposure, which is given by an expectation against the normal distribution:

$$
\begin{aligned}
\mathrm{EE} &= \int_{-\infty}^{\infty} \max(v,0)\, f(v)\, dv \\
&= \int_{-\infty}^{\infty} \max(\mu+\sigma z)\, \phi(z) dz
\end{aligned}
$$

where

$$
\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) \tag{4}
$$

is the density function for the standard normal distribution



We can evaluate this expression in terms of $\Phi(z)$ and $\phi(z)$. We begin by removing the *max* function and modifying the limits of integration:

$$
\begin{aligned}
\mu + \sigma z > 0 &\Rightarrow \\
z &> \frac{-\mu}{\sigma}
\end{aligned}
$$

Thus

$$\begin{aligned}\mathrm{EE} &= \int_{\frac{-\mu}{\sigma}}^{\infty} (\mu + \sigma z)\, \phi(z) dz \\ &= \mu \int_{\frac{-\mu}{\sigma}}^{\infty} \phi(z) dz + \sigma \int_{\frac{-\mu}{\sigma}}^{\infty} z\, \phi(z) dz \\ &= \mu \Phi \left(\frac{\mu}{\sigma}\right) + \sigma \phi \left(\frac{\mu}{\sigma}\right) \end{aligned} \tag{5}$$

The simplification of the first integral follows from the symmetry of the standard normal density about zero, which leads to the relationship

$$\Phi(z) = 1 - \Phi(-z) \tag{6}$$

For the second integral, we use u - substitution and some calculus, with $u = \frac{z^2}{2}$ and $du = zdz$, we have

$$\begin{aligned}\sigma \int_{\frac{-\mu}{\sigma}}^{\infty} z\, \phi(z) dz \; &= \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{-\mu}{\sigma}}^{\infty} z\, exp \left(\frac{-z^2}{2}\right) dz \\ &= \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{\mu^2}{2\sigma^2}}^{\infty} e^{-u} du \\ &= -\frac{\sigma}{\sqrt{2\pi}} e^{-u} \Big|_{\frac{\mu^2}{2\sigma^2}}^{\infty} \\ &= \frac{\sigma}{\sqrt{2\pi}} \left(e^{-\infty} + e^{-\frac{\mu^2}{2\sigma^2}}\right) \\ &= \sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} \\ &= \sigma \phi \left(\frac{\mu^2}{2\sigma^2}\right)\end{aligned}$$

It is worth pointing out, as the author does, that, if $\mu = 0$, we obtain

$$\mathrm{EE}(\mu = 0, \sigma) = \frac{\sigma}{\sqrt{2\pi}} \approx 0.4\sigma. \tag{7}$$

That is, the expected exposure is approximately 40% of the standard deviation of the value.

1.3 Expected Positive Exposure (EPE)

This formula is given as Appendix 8A(iii) in the book. We prove the formula here. The EPE is the average of the EE over time. We assume that the mean exposure is 0 and allow the variance of the exposure to scale linearly with time, yielding

$$V(t) = \sigma\sqrt{t}Z$$

where σ represents the annualized standard deviation. Using the result for EE above, for any point in time t, we have

$$\mathrm{EE}(\mu = 0, \sigma, t) = \frac{\sigma\sqrt{t}}{\sqrt{2\pi}}.$$

Integrating this exposure over the life of the position and dividing by the length of the time interval yields the following for EPE:

$$\begin{aligned}
\mathrm{EPE} &= \frac{\sigma}{T\sqrt{2\pi}}\int_0^T \sqrt{t}dt \\
&= \frac{\sigma}{T\sqrt{2\pi}}\frac{2}{3}\, t^{\frac{3}{2}}\Big|_0^T \\
&= \frac{2\sigma}{3T\sqrt{2\pi}}T^{\frac{3}{2}} \\
&= \frac{2\sigma}{3\sqrt{2\pi}}\sqrt{T} \\
&\approx .27\sigma\sqrt{T} \qquad (8)
\end{aligned}$$

If we let $T = 1$ year, we obtain

$$\mathrm{EPE}(T = 0...1) \approx .27\sigma\sqrt{T} \qquad (9)$$

In comparison, the expected exposure at the 1 year mark is

$$\mathrm{EE}(T = 1) \approx .40\sigma\sqrt{T} \qquad (10)$$

This makes sense, of course. Exposure is increasing for the entire year in this case, like it would for a forward contract or an option. The average exposure is thus less than the final exposure.

2 Additional formulas from Chapter 8

2.1 Appendix 8B: Exposure of a forward contract

Suppose the exposure evolves with a constant drift and standard deviation:

$$dV_t = \mu dt + \sigma dW.$$

If we assume that $V_0 \equiv 0$, we have

$$V_t = \mu t + \sigma W_t$$

and

$$V_t \sim N(\mu t, \sigma^2 t).$$

Using the formulas derived above under the assumption of normally distributed exposure values, we obtain

$$PFE(\alpha, t) = \mu t + \sigma\sqrt{t}\Phi^{-1}(\alpha). \tag{11}$$

and

$$EE(t) = \mu t \Phi\left(\frac{\mu}{\sigma\sqrt{t}}\right) + \sigma\sqrt{t}\phi\left(\frac{\mu}{\sigma\sqrt{t}}\right). \tag{12}$$

2.2 Appendix 8C: Exposure of a swap

Suppose the exposure evolves with 0 drift and normal *variance* that scales directly with time into the future but inversely with the square of time to expiry. (We defer discussion of why it should behave this way for now. As the author notes, the term $(T-t)$ corresponds to the approximate duration of the swap of maturity T at time t. This holds, for example, when the expected future value is 0 at all future dates, which would imply a flat yield curve.)

$$V_t \sim N(0, \sigma^2 t(T-t)^2). \tag{13}$$

(Note: The normal variance is given here; the book gives the standard deviation.) Denoting the standard deviation of this distribution by $\Sigma(t, T)$ and differentiating with respect to t, we have

$$\Sigma(t,T) = \sigma\sqrt{t}(T-t)$$
$$\frac{d}{dt}\Sigma(t,T) = \frac{\sigma T}{2\sqrt{t}} - \frac{3\sigma\sqrt{t}}{2}$$

We set the derivative equal to 0 to find the maximum value of the uncertainty

$$\frac{\sigma T}{2\sqrt{t}} - \frac{3\sigma\sqrt{t}}{2} = 0$$
$$\frac{\sigma T}{2\sqrt{t}} = \frac{3\sigma\sqrt{t}}{2}$$
$$t = \frac{T}{3}$$

Looking at the formula for $\Sigma(t, T)$, we see that it is 0 for $t = 0$, positive for $t < T$, and 0 again for $t = T$, which means that the critical value must be a maximum. The fact that this approximation works well can be seen by examing the swap considered in Spreadsheet 9.2.

2.3 Appendix 8D: Exposure of a cross-currency swap

For this example, we decompose the cross-currency swap into an FX forward and an interest rate swap.

$$\text{CCS} = \text{FXF} + \text{IRS}$$

This enables us to use the formulas from Appendix 8B and Appendix 8C to compute the exposure of the cross-currency swap. For the FX forward, Appendix 8B gives us

$$V_t \sim N(0, \sigma_{FX}^2\, t).$$

For the interest rate swap, Appendix 8C gives us

$$V_t \sim N(0, \sigma_{IR}^2\, t\,(T-t)^2).$$

We also need the formula for the variance of the sum of 2 random variables:

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$$

or

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\,\sigma_X\,\sigma_Y$$

Applying this to the decomposition

$$\text{CCS} = \text{FXF} + \text{IRS}$$

we obtain

$$\sigma_{CCS}^2(t) = \sigma_{FX}^2\, t + \sigma_{IR}^2\, t(T-t)^2 + 2\rho\,\sigma_{FX}\,\sigma_{IR}\, t(T-t),$$

so that

$$V_{CCS}(t) \sim N\left(0, \sqrt{\sigma_{FX}^2\, t + \sigma_{IR}^2\, t(T-t)^2 + 2\rho\,\sigma_{FX}\,\sigma_{IR}\, t(T-t)}\right) \tag{14}$$

2.4 Appendix 8E: Simple netting calculation

From Appendix 8A, under the assumption of normally distributed exposure values, the Expected Exposure is given by

$$\mathrm{EE}_i = \mu_i\, \Phi\left(\frac{\mu_i}{\sigma_i}\right) + \sigma_i\, \phi\left(\frac{\mu_i}{\sigma_i}\right) \tag{15}$$

The formula for the variance of the sum of X and Y that we developed above can be extended to n random variables:

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\, \sigma_i \sigma_j \\
&= \sum_{i=1}^{n} \sigma_i^2 + \sum_{j \neq i}^{n} \rho_{ij}\, \sigma_i \sigma_j \\
&= \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \rho_{ij}\, \sigma_i \sigma_j
\end{aligned}$$

We simplify the remaining analysis by assuming that the random variable all have mean $\mu = 0$ and the same variance σ^2. Further, we assume that the correlations are identically equal to a constant that we denote by $\bar{\rho}$. Our formula for the portfolio variance simplifies to

$$\begin{aligned}
\sum_{i=1}^{n} \sigma^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \bar{\rho}\,\sigma^2 \;&=\; n\sigma^2 + n(n-1)\bar{\rho}\,\sigma^2 \\
&= \sigma^2(n + n(n-1)\bar{\rho})
\end{aligned} \tag{16}$$

We are assuming that $\mu = 0$. Applying the formula for Expected Exposure,

$$\mathrm{EE} = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right),$$

and using

$$\phi(0) = \frac{1}{\sqrt{2\pi}} \approx 0.40,$$

we obtain

$$\mathrm{EE}_{NS} \approx 0.40\,\sigma\sqrt{n + n(n-1)\bar{\rho}} \tag{17}$$

If there is not netting, we simply sum the variances to obtain

$$\mathrm{EE}_{NN} \approx 0.40\,\sigma n \tag{18}$$

The ratio of these two quantities is the netting benefit:

$$\frac{\text{EE}_{NS}}{\text{EE}_{NN}} = \frac{\sqrt{n + n(n-1)\bar{\rho}}}{n} \tag{19}$$

For perfect correlation, $\rho = 100\%$, we get no benefit:

$$\frac{\text{EE}_{NS}}{\text{EE}_{NN}} = \frac{\sqrt{n + n(n-1)}}{n} = \frac{\sqrt{n^2}}{n} = 1.$$

For uncorrelated exposures, $\rho = 0$, we get

$$\frac{\text{EE}_{NS}}{\text{EE}_{NN}} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

This is close to optimal, as the square root means, when using a constant correlations for many random variables, they cannot all be negative numbers far from 0. Here,

$$\begin{aligned}
\text{EE}_{NS} &= \sqrt{n + n(n-1)\bar{\rho}} \\
&\Rightarrow n + n(n-1)\bar{\rho} > 0 \text{ (since the radicand must be positive)} \\
&\Rightarrow \bar{\rho} \geq \frac{-1}{n-1} \approx 0
\end{aligned}$$

for n large. Of course, the assumption that ρ is a constant is an oversimplification. Nonetheless, the idea of assuming a constant average correlation over many exposures is useful in many contexts.