Probability, Statistics and Random Processes

Homework 10

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1. Population standard deviation $\sigma = 20$.

a) We have the margin of error for the 95% confidence interval: $E = 20$

Given 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

$$\rightarrow \ z_{\alpha/2} = z_{0.025} = 1.96$$

The sample size must be:

$$n = \left(\frac{z_{\alpha/2} * \sigma}{E}\right)^2 = \left(\frac{1.96 * 20}{20}\right)^2 \approx 4$$

b) We have the margin of error for the 99% confidence interval: $E = 20$

Given 99% confidence interval → Significance level $\alpha = 1 - \frac{99}{100} = 0.01$.

$$\rightarrow \ z_{\alpha/2} = z_{0.005} = 2.58$$

The sample size must be:

$$n = \left(\frac{z_{\alpha/2} * \sigma}{E}\right)^2 = \left(\frac{2.58 * 20}{20}\right)^2 \approx 7$$

2. The sample mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{91.6 + 88.75 + 90.8 + 89.95 + 91.3}{5} = 90.48$$

Given a 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

$$\rightarrow \ z_{\alpha/2} = z_{0.025} = 1.96$$

The 95% two-sided confidence interval for the mean yield:

$$\bar{x} - z_{\frac{\alpha}{2}} * \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\frac{\alpha}{2}} * \frac{\sigma}{\sqrt{n}}$$

$$\rightarrow 90.48 - 1.96 * \frac{3}{\sqrt{5}} \leq \mu \leq 90.48 + 1.96 * \frac{3}{\sqrt{5}}$$

$$\rightarrow 87.85 \leq \mu \leq 93.11$$

$$\rightarrow (87.85, 93.11)$$

3. Given the sample size $n = 12$.

a) The sample mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = 2.08$$

The sample standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} = 0.156$$

Given a 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

$$\rightarrow z_{\alpha/2} = z_{0.025} = 1.96$$

The 95% two-sided confidence interval for the mean number of CATs performed:

$$\bar{x} - t_{\frac{\alpha}{2}} * \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}} * \frac{s}{\sqrt{n}}$$

$$\rightarrow 2.08 - 1.96 * \frac{0.156}{\sqrt{12}} \leq \mu \leq 2.08 + 1.96 * \frac{0.156}{\sqrt{12}}$$

$$\rightarrow 1.99 \leq \mu \leq 2.17$$

$$\rightarrow (1.99, 2.17)$$

b) The mean performed by all clinics 1.95 is lower than the lower bound of the 95% confidence interval. Therefore, this particular clinic performs more CAT scans than average.

4. Given the sample size $n = 39$.

The sample mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = 4.7$$

The sample standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} = 0.63$$

Given a 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

Given degree of freedom $df = n - 1 = 39 - 1 = 38$:

$$\rightarrow \chi_{\alpha/2} = \chi_{0.025} = 56.9$$

$$\rightarrow \chi_{1-\alpha/2} = \chi_{0.975} = 22.88$$

The 95% two-sided confidence interval for the standard deviation:

$$\sqrt{\frac{(n-1)s^2}{{\chi_{\alpha/2}}^2}} \leq \sigma \leq \sqrt{\frac{(n-1)s^2}{{\chi_{1-\alpha/2}}^2}}$$

$$\rightarrow \sqrt{\frac{(39-1) * 0.63^2}{56.9^2}} \leq \sigma \leq \sqrt{\frac{(39-1) * 0.63^2}{22.88^2}}$$

$$\rightarrow 0.068 \leq \mu \leq 0.170$$

$$\rightarrow (0.068, 0.170)$$

5. The sample proportion: $\hat{p} = \frac{412}{768} = 0.536$, with sample size $n = 768$.

a) Given a 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

$$\rightarrow \; z_{\alpha/2} = z_{0.025} = 1.96$$

The 95% two-sided confidence interval for proportion:

$$\hat{p} - \; z_{\frac{\alpha}{2}} * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + \; z_{\frac{\alpha}{2}} * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\rightarrow 0.536 - 1.96 * \sqrt{\frac{0.536 * (1 - 0.536)}{768}} \leq p \leq 0.536 + 1.96 * \sqrt{\frac{0.536 * (1 - 0.536)}{768}}$$

$$\rightarrow 0.5 \leq p \leq 0.57$$

$$\rightarrow (0.50, 0.57)$$

b) Given a 95% confidence interval → Significance level $\alpha = 1 - \frac{95}{100} = 0.05$.

$$\rightarrow \; z_{\alpha/2} = z_{0.025} = 1.96$$

The 95% lower confidence bound for proportion:

$$\hat{p} - \; z_{\frac{\alpha}{2}} * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p$$

$$\rightarrow 0.536 - 1.96 * \sqrt{\frac{0.536 * (1 - 0.536)}{768}} \leq p$$

$$\rightarrow 0.5 \leq p$$