Single queues

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2020-2021/Semester 1

Outline

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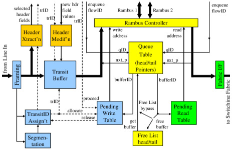
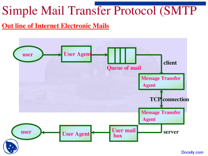
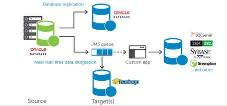
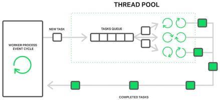
Queues in real world









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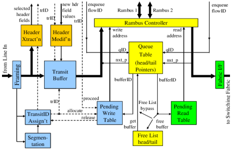
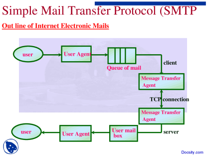
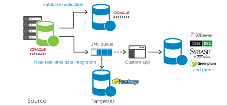
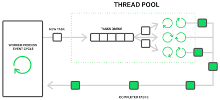


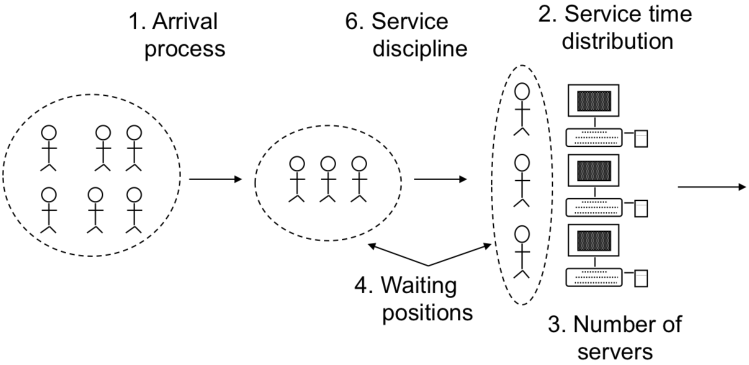






Are there any common structures of queues?

Basic components of a queue



- Customers can be people, parts, vehicles, machines, jobs,...
- Queue might not be a physical line.

Kendall Notation System

- $A/S/m/B/K/SD$
  - A: Arrival process,
  - S: Service time distribution,
  - m: Number of servers,
  - B: Number of buffers (system capacity),
  - K: Population size,
  - SD: Service discipline.

- Arrival times: $t_1, t_2, \ldots, t_j$.
- Interarrival times: $\tau_j = t_j - t_{j-1}$.
- τ_j form a sequence of Independent and Identically Distributed (IID) random variables.
- Exponential + IID $\rightarrow$ Poisson.
- Notation:
  - M = Memoryless = Poisson,
  - E = Erlang,
  - H = Hyper-exponential,
  - G = General $\rightarrow$ Results valid for all distributions.

- Time each student spends at the terminal.
- Service times are IID.
- Distribution: M, E, H, or G.
- Device $=$ Service center $=$ Queue.
- Buffer $=$ Waiting positions.

- First-Come-First-Served (FCFS);
- Last-Come-First-Served (LCFS);
- Last-Come-First-Served with Preempt and Resume (LCFS-PR);
- Round-Robin (RR) with a fixed quantum.
- Small Quantum $\rightarrow$ Processor Sharing (PS);
- Infinite Server: (IS) $=$ fixed delay;
- Shortest Processing Time first (SPT);
- Shortest Remaining Processing Time first (SRPT);
- Shortest Expected Processing Time first (SEPT);
- Shortest Expected Remaining Processing Time first (SERPT).
- Biggest-In-First-Served (BIFS);
- Loudest-Voice-First-Served (LVFS).

- M: Exponential,
- E_k: Erlang with parameter k,
- H_k: Hyper-exponential with parameter k,
- D: Deterministic $\rightarrow$ constant,
- G: General $\rightarrow$ All.
- Memoryless:
  - Expected time to the next arrival is always $1/\lambda$ regardless of the time since the last arrival,
  - Remembering the past history does not help.

Example: $M/M/3/20/1500/FCFS$

- Time between successive arrivals is exponentially distributed.
- Service times are exponentially distributed.
- Three servers,
- 20 Buffers = 3 service + 17 waiting,
  After 20, all arriving jobs are lost,
- Total of 1500 jobs that can be serviced.
- Service discipline is FCFS.
- Defaults:
  - Infinite buffer capacity,
  - Infinite population size,
  - FCFS service discipline.
- $G/G/1 = G/G/1/1/1/FCFS$.

- Bulk arrivals/service.
- $M^{[x]}$: x represents the group size.
- $G^{[x]}$: a bulk arrival or service process with general inter-group times.
- Example:
  - $M^{[x]}/M/1$: Single server queue with bulk Poisson arrivals and exponential service times;
  - $M/G^{[x]}/m$: Poisson arrival process, bulk service with general service time distribution, and m servers.

Outline

Typical performance questions

What is the ...

- average number of customers in the system?
- average time a customer spends in the system?
- probability a customer is rejected?
- fraction of time a server is idle?

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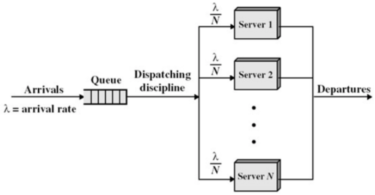

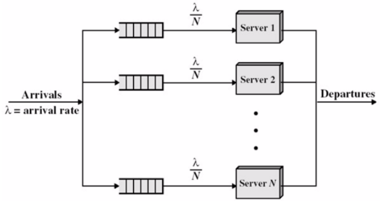

- What is average time waiting in the queue?
- What is variability of time in the queue?

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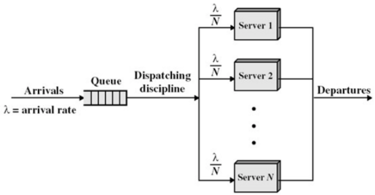
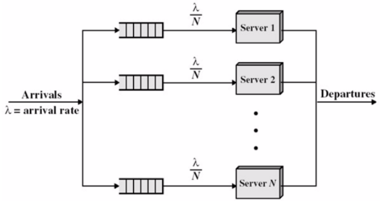
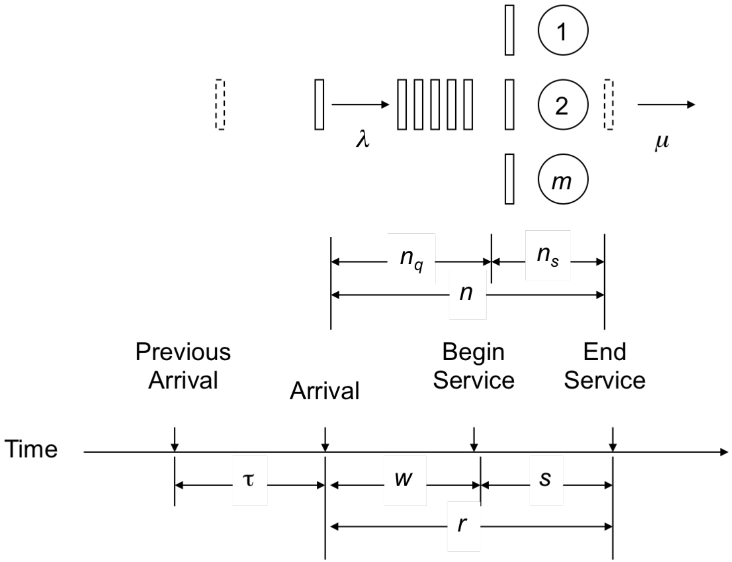




- What is average time waiting in the queue?
- What is variability of time in the queue?

None of done incorrectly outcomes are investigated to produce metrics.

Key variables (1)



Key variables (2)

- τ: Inter-arrival time = time between two successive arrivals.
- λ: Average arrival rate = $1/E[\tau]$
  - May be a function of the state of the system
    E.g., number of jobs already in the system.
- s: Service time per job.
- μ: Average service rate per server = $1/E[s]$.
- Total service rate for m servers is $m\mu$.
- n: Number of jobs in the system.
  Note: n includes jobs currently receiving service as well as those waiting in the queue.

Key variables (3)

- n_q: Number of jobs waiting.
- n_s: Number of jobs receiving service.
- r: Response time or the time in the system (system time)
  - time waiting + time receiving service.
- w: Waiting time
  - Time between arrival and beginning of service.

Outline

Little's Law

Named after Little (1961).

Little's Law

For any queuing system that has a steady state and has an average rate of λ,

$$E[n] = \lambda E[r]$$

If the average system time is 2 hours, and customers arrive at a rate of 3 per hour then on average, there are 6 customers in the system.

Discussion on Little's Law

Based on a black-box view of the system.

Little's law can be used for a system or any part of the system.

- Average number in queue = arrival rate × average waiting time
- Average number in service = arrival rate × average service time

There is no drop in number

Little's law requires no assumptions about arrival or service time distribution, the size of population, or limits on the system.

Example on Little's Law

- Consider problem
  - A monitor on a disk server showed that the average time to satisfy an I/O request was 100 milliseconds.
  - The I/O rate was about 100 requests per second.
  - What was the average number of requests at the disk server?

- Using Little's law, average number in the disk server

$$
\begin{aligned}
&= \text{Arrival rate} \times \text{system time} \\
&= 100 \text{ (requests/second)} \times (0.1 \text{ seconds}) \\
&= 10 \text{ requests.}
\end{aligned}
$$

Outline

(Recall) Stochastic processes

- Process: sequence (family) of random variables that are functions of time.
  E.g., $n(t)$: number of jobs waiting for CPU of a computer system at time t.
- Each $n(t)$ is random variable which can be defined by a probability distribution. $\Rightarrow$ Stochastic process.

(Recall) Stochastic processes

- Process: sequence (family) of random variables that are functions of time.
  E.g., $n(t)$: number of jobs waiting for CPU of a computer system at time t.
- Each $n(t)$ is random variable which can be defined by a probability distribution. $\Rightarrow$ Stochastic process.
- Types of stochastic processes
  - Discrete or Continuous State Processes
    - $n(t)$ is a discrete-state process
    - $w(t)$ is a continuous-state process
  - Markov Processes
  - Birth-death Processes
  - Poisson Processes

Markov processes

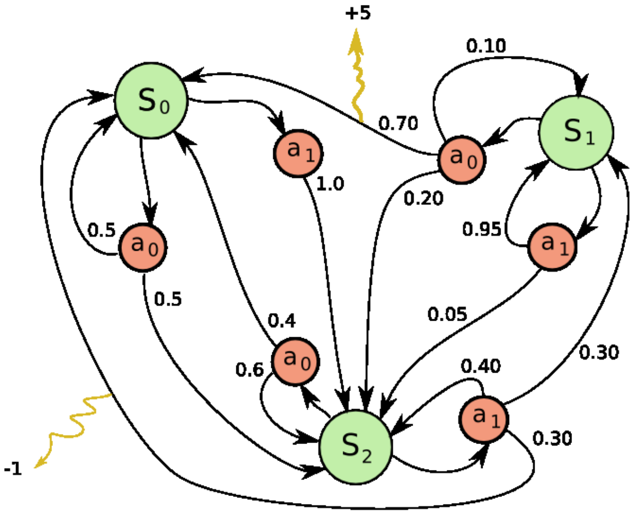
- A **Markov process** = A stochastic process in which future states are independent of the past and depend only on the present. ⇒ easier to analyze, not have to remember past trajectory.
- A **Markov chain** = A discrete-state Markov process
- Some remarks:
  - Knowing current (present) state is sufficient
  - Not necessary to know how long the process has been in the current state. ⇒State time has a memoryless distribution.

E.g., exponential distribution

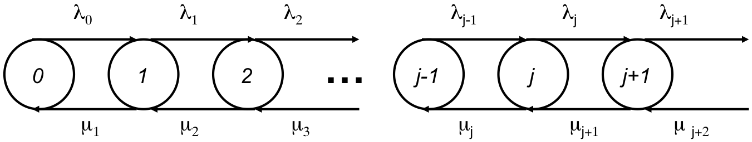
$$
\begin{aligned}
Pr\{X > s+t \mid X > t\} &= \frac{Pr\{X>s+t \text{ and } X>t\}}{Pr\{X>t\}} \\
&= \frac{Pr\{X>s+t\}}{Pr\{X>s\}} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\
&= e^{-\lambda s}
\end{aligned}
$$

The time spent by a job in such a queue is a Markov process and the number of jobs in the queue is a Markov chain.

Markov decision process in learning
Vision



Birth-death processes



- A **Birth-death process** = A discrete-state Markov process in which the transitions are restricted to neighboring states.
  $\Rightarrow$ Process in state n can change only to state $n+1$ or $n-1$.

Number of jobs in a queue with a single server and individual arrivals (not bulk arrivals).

- An arrival (birth): state changed by $+1$.
- A departure (death): state changed by -1.

Birth-death processes (2)

Theorem

The steady-state probability p_n of a birth-death process being in state n is given by

$$p_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} p_0, n = 1, 2, \dots, \infty$$

Here, p_0 is the probability of being in the zero state.

Poisson processes (1)

- Inter-arrival time $\tau =$ IID (identically and independently distributed) and exponential
  - number of arrivals n over a given interval $(t, t + x)$ has a Poisson distribution $P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}$
  - Arrival process is a **Poisson process** or **Poisson stream**.

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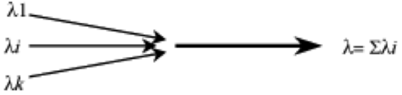
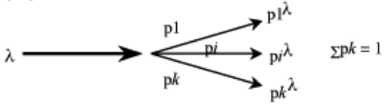
- Properties
  1. Merging: $\lambda = \sum_{i=1}^{k} \lambda_i$.

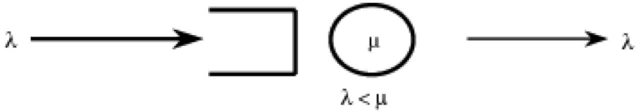


  2. Splitting: If the probability of a job going to i^{th} substream is p_i, each substream is also Poisson with a mean rate of $p_i\lambda$.

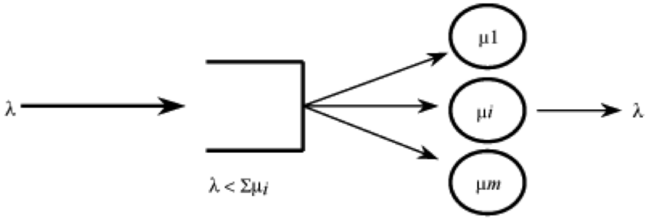


Poisson processes (2)

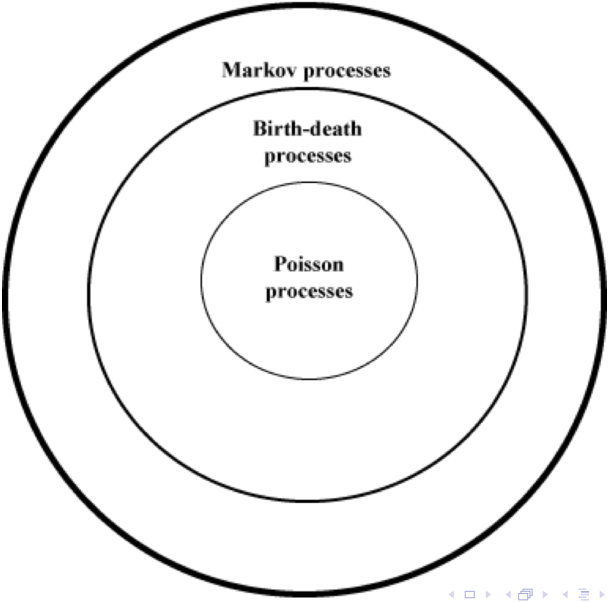
- Properties (cont.)
  (3) If the arrivals to a single server with exponential service time are Poisson with mean rate λ, the departures are also Poisson with the same rate λ provided $\lambda < \mu$.



- Properties (cont.)
  (4) If the arrivals to a service facility with m service centers are Poisson with a mean rate λ, the departures also constitute a Poisson stream with the same rate λ, provided $\lambda < \sum_i \mu_i$.
    - Here, the servers are assumed to have exponentially distributed service times.



Relationship among stochastics processes



Outline

Rules for All Queues: apply to G/G/m queues (1)

(1) Stability Condition: $\lambda < m\mu$

- Finite-population and infinite-buffer systems are always stable.

(2) Number in System versus Number in Queue:

- $n = n_q + n_s$, where n, n_q, and n_s are random variables;
- $E[n] = E[n_q] + E[n_s]$;
- If the service rate is independent of the number in the queue, $Cov(n_q, n_s) = 0$

$$\mathrm{Var}[n] = \mathrm{Var}[n_q] + \mathrm{Var}[n_s].$$

Rules for All Queues: apply to G/G/m queues (2)

(3) Number versus Time: if jobs are not lost due to insufficient buffers,
   - Mean number of jobs in the system = Arrival rate $\times$ Mean response time.

(4) Similarly,
   - Mean number of jobs in the queue = Arrival rate $\times$ Mean waiting time.
   - This is Little's law as mentioned later.

(5) Time in System versus Time in Queue $r = w + s$, where r, w, and s are random variables.
   - $E[r] = E[w] + E[s]$.

(6) If the service rate is independent of the number of jobs in the queue, $Cov(w, s) = 0$

$$\text{Var}[r] = \text{Var}[w] + \text{Var}[s].$$

Outline

M/M/1

Definition

- Interarrival times, service times are exponentially distributed.
- One server.
- No limitation on buffer and population.
- FCFS service discipline.

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- Interarrival times, service times are exponentially distributed.
- One server.
- No limitation on buffer and population.
- FCFS service discipline.

- It is the most commonly used type of queue.
- Need to know only arrival rate λ and service rate μ.

- A birth-death process with

$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots, \infty$$
$$\mu_n = \mu, \quad n = 1, 2, \ldots, \infty$$

- Probability of n jobs in the system

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, n = 1, 2, \ldots, \infty$$

- Traffic intensity $\rho = \lambda/\mu$.

$$p_n = \rho^n p_0$$

We have

$$\sum_{i=0}^{\infty} p_i = 1$$
$$p_0(1 + \rho^1 + \rho^2 + ...) = 1$$
$$p_0 = \frac{1}{1+\rho^1+\rho^2+\ldots} = 1 - \rho$$

$$\Rightarrow p_n = \rho^n(1 - \rho)$$

- Utilization of the server = Probability of having one or more jobs in the system

$$U = 1 - p_0 = \rho$$

- Mean number of jobs in the system

$$E[n] = \sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} n \rho^n (1-\rho) = \frac{\rho}{1-\rho}$$

- Variance of the number of jobs in the system

$$\begin{aligned} \mathrm{Var}[n] &= E[n^2] - (E[n])^2 \\ &= \textstyle\sum_{n=1}^{\infty} n^2 (1-\rho)\rho^n - (E[n])^2 \\ &= \tfrac{\rho}{(1-\rho)^2} \end{aligned}$$

- Probability of n or more jobs in the system

$$P(\geq n \text{ jobs in the system}) = \sum_{j=n}^{\infty} p_j = \rho^n$$

- By Little's Law

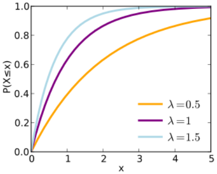
$$E[n] = \lambda E[r]$$

Then,

$$E[r] = \frac{E[n]}{\lambda} = \frac{1/\mu}{1-\rho}$$

- Cumulative distribution function of the response time

$$F(r) = 1 - e^{-r\mu(1-\rho)}.$$



$\Rightarrow$ The response time is exponentially distributed.

- q-percentile of the response time (i.e., $F(r) = q/100$)

$$1 - e^{-r_q\mu(1-\rho)} = \frac{q}{100}.$$

Hence,

$$r_q = \frac{1}{\mu(1-\rho)\ln\left(\frac{100}{100-q}\right)}.$$

- Cumulative distribution function of the waiting time

$$F(w) = 1 - \rho e^{-w\mu(1-\rho)}.$$

- Mean waiting time

$$E[w] = \rho \frac{1/\mu}{1-\rho} = E[r] - \frac{1}{\mu}$$

- This is a truncated exponential distribution. Its q-percentile is given by

$$w_q = \frac{1}{\mu(1-\rho)\ln\left(\frac{100\rho}{100-q}\right)}.$$

- The above formula applies only if q is greater than $100(1-\rho)$. All lower percentiles are zero.

$$w_q = \max\left\{0, \frac{E[w]}{\rho}\ln\left(\frac{100\rho}{100-q}\right)\right\}.$$

- Mean number of jobs in the queue:

$$E[n_q] = \sum_{n=1}^{\infty}(n-1)p_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}.$$
$$E[n_q] = E[n] - \rho$$

Note

All results for $M/M/1$ queues including some for the busy period are summarized in Box 31.1 in the book of R. Jain.

$M/M/1$: exercise

Problem

On a network gateway, measurements show that the packets arrive at a mean rate of 125 packets per second (pps) and the gateway takes about two milliseconds to forward them.

- Using an $M/M/1$ model, analyze the gateway.
- What is the probability of buffer overflow if the gateway had only 13 buffers?
- How many buffers do we need to keep packet loss below one packet per million?

$M/M/1$: exercise(1)

- Arrival rate $\lambda = 125$ pps.
- Service rate $\mu = 1/.002 = 500$ pps.
- Gateway Utilization $\rho = \lambda/\mu = 0.25$.
- Probability of n packets in the gateway:
  $(1-\rho)\rho^n = 0.75 \times 0.25\rho^n$.
- Mean Number of packets in the gateway:
  $\rho/(1-\rho) = 0.25/0.75$.
- Mean time spent in the gateway:
  $(1/\mu)/(1-\rho) = (1/500)/(1-0.25) = 2.66$ milliseconds.
- Probability of buffer overflow:
  P(more than 13 packets in gateway)
  $$= \rho^{13} = 0.25^{13} = 14.9 \times 10^{-9}$$
  $$\approx 15 \text{ packets per billion packets.}$$

$M/M/1$: exercise(2)

- An airport runway for arrivals only
- Arriving aircraft join a single queue for the runway
- Exponentially distributed service time with a rate $\mu = 27$ arrivals/hour.
- Poisson arrivals with a rate $\lambda = 20$ arrivals/hour.
- Compute
  - Time in the airport runway system
  - Number of aircrafts in the runway system
  - Waiting time for the runway
  - Number of aircrafts waiting for the runway

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- Arriving aircraft join a single queue for the runway
- Exponentially distributed service time with a rate $\mu = 27$ arrivals/hour.
- Poisson arrivals with a rate $\lambda = 20$ arrivals/hour.

- Compute
  - Time in the airport runway system
    $E[r] = \frac{1}{\mu-\lambda} = \frac{1}{27-20} = \frac{1}{7}$hour.
  - Number of aircrafts in the runway system
    $E[n] = \lambda E[r] = \frac{20}{27-20} = 2.9$ aircrafts.
  - Waiting time for the runway
    $E[w] = E[r] - 1/\mu = \frac{1}{7} - \frac{1}{27} = 6.4$ min
  - Number of aircrafts waiting for the runway
    $E[n_q] = ...$

Outline

Definition

- Interarrival times, service times are exponentially distributed.
- n **servers.**
- No limitation on buffer and population.
- FCFS service discipline.

- A birth-death process with

$$\lambda_n = \lambda$$
$$\mu_n = \begin{cases} n\mu & n = 1, 2, \dots, m-1 \\ m\mu & n = m, m+1, \dots, +\infty \end{cases}$$

- Traffic intensity $\rho = \lambda/(m\mu)$
- Probability of zero job in the system

$$p_0 = \left[1 + \frac{(m\rho)^m}{m!(1-\rho} + \sum_{n=1}^{m-1} \frac{(m\rho)^n}{n!} \right]^{-1}$$

- Probability of n jobs in the system

$$\mu_n = \begin{cases} \frac{\lambda^n}{n!\mu^n} p_0 & n = 1, 2, \dots, m-1 \\ \frac{\lambda^n}{m!m^{n-m}\mu^n} p_0 & n = m, m+1, \dots, +\infty \end{cases}$$

Computer center

Computer center

Students arrive at a computer center in Poisson manner of rate 10 students/hour. Each student spends an average of 20 minutes at a terminal in exponential distribution. The center has 5 terminals. Let analyze the center usage.

- Traffic intensity $\rho = \lambda/(5\mu) = 0.167/(5 \times 0.05) = 0.67$
- Probability of all terminals being idle is $p_0 = ... = 0.0318$
- Probability of all terminals being busy is $\frac{(m\rho)^m}{m!(1-\rho)} p_0 = 0.33$.