Graph Structure

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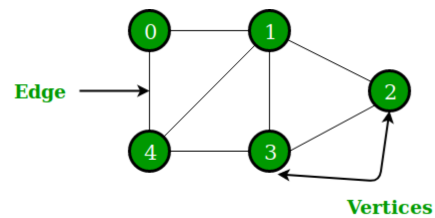
Graph

- A graph consists of **a finite set of vertices** (or nodes) and **set of edges** which connect a pair of nodes.
- G = {V, E}
  - V: set of vertices. V = {v_1, v_2,…, v_n}
  - E: set of edges. E = {e_1, e_2,…,e_m}
- Example:
  - V = {0, 1, 2, 3, 4}
  - E = {01, 04, 12, 13, 14, 23, 34}



Terminologies

Terminologies

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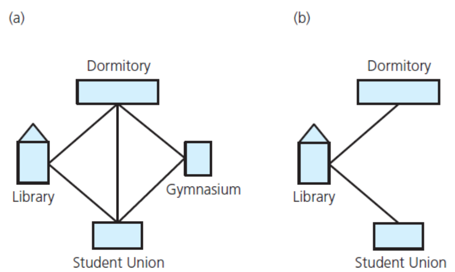
- A **subgraph** consists of a subset of a graph's vertices and a subset of its edges.
  - G'= {V', E'} is a subgraph of G = {V, E} if V' ⊆ V, E' ⊆ E



(a) A campus map as a graph;
(b) a subgraph

Terminologies

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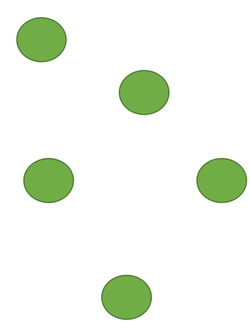
- **Vertex**: also called a **node**.
- **Edge**: connects two vertices.
- **Loop** (*self-edge*): An edge of the form (*v*, *v*).
- **Adjacent**: two vertices are **adjacent** if they are joined by an edge.

Terminologies

- **Path**: A sequence of edges that begins at one vertex and ends at another vertex.
  - If all vertices of a path is distinct, the path is **simple**.
- **Cycle**: A path that starts and ends at the same vertex and does not traverse the same edge more than once.
- **Acyclic** graph: A graph with no cycle.

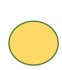
Terminologies

- **Null graph**: A graph having no edges



null graph

- **Trivial graph**: A graph with only one vertex.



trivial graph

Terminologies

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- **Path**: A sequence of edges that begins at one vertex and ends at another vertex.
  - If all vertices of a path is distinct, the path is simple.
- **Cycle**: A path that starts and ends at the same vertex and does not traverse the same edge more than once.
- **Acyclic graph**: A graph with no cycle.

Terminologies

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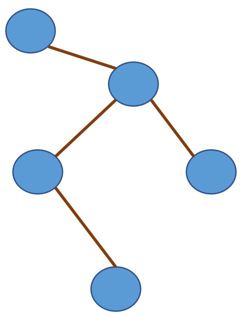
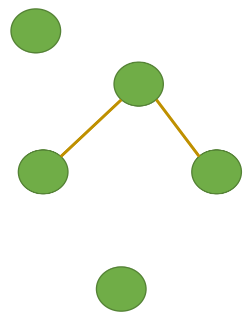
- **Connected graph:** A graph in which each pair of **distinct vertices** has a **path** between them.
- **Disconnected graph:** A graph does not contain at least two connected vertices.
- **Complete graph:** A graph in which each pairs of **distinct vertices** has an **edge** between them
- Graph cannot have duplicate edges between vertices.
  - **Multigraph**: does allow multiple edges

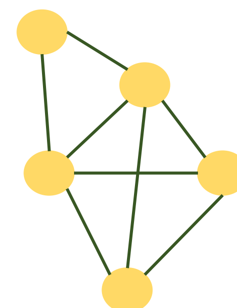
Terminologies



connected graph



disconnected graph



complete graph

Terminologies

- **Undirected graph**: the graph in which edges do not indicate a direction.
- **Directed graph, or digraph**: a graph in which each edge has a direction.
- **Weighted graph**: a graph with numbers (weights, costs) assigned to its edges.

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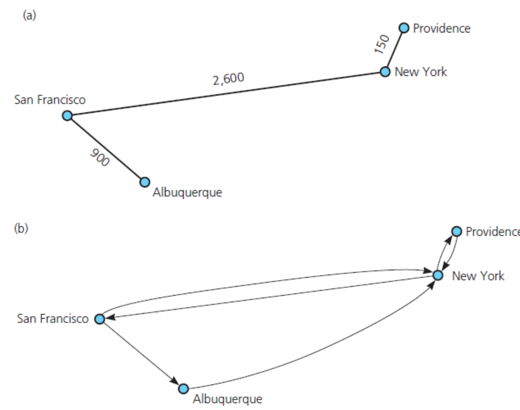
Terminologies

(a): undirected graph

(b): directed graph



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Graph Representation

Graph Representation

- Adjacency Matrix

- Adjacency List

Adjacency Matrix

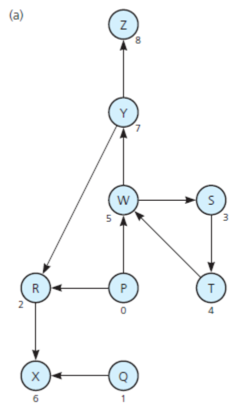
A[n][n] with n is the number of vertices.

- $A[i][j] = \begin{cases} 1 \; if \; there \; is \; an \; edge(i,j) \\ 0 \; if \; there \; is \; no \; edge \; (i,j) \end{cases}$

- $A[i][j] = \begin{cases} w \quad with \; w \; is \; the \; weight \; of \; edge(i,j) \\ \infty \qquad if \; there \; is \; no \; edge \; (i,j) \end{cases}$

Adjacency Matrix

(a)

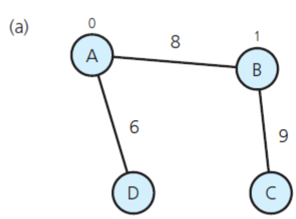


(b)

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

Adjacency Matrix

(a)



(b)

		0	1	2	3
		A	B	C	D
0	A	∞	8	∞	6
1	B	8	∞	9	∞
2	C	∞	9	∞	∞
3	D	6	∞	∞	∞

Adjacency List

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- A graph with n vertices has n linked chains.
- The i^{th} linked chain has a node for vertex j if and only if having edge (i,j).

Adjacency List

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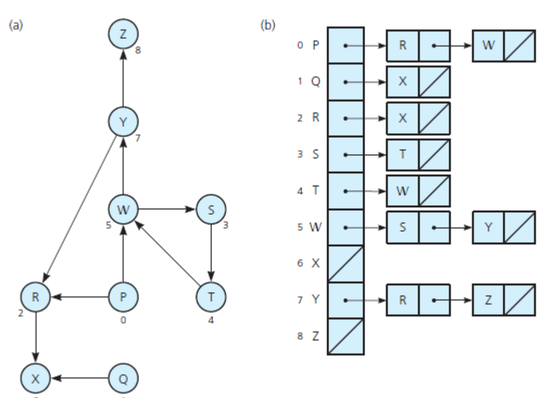
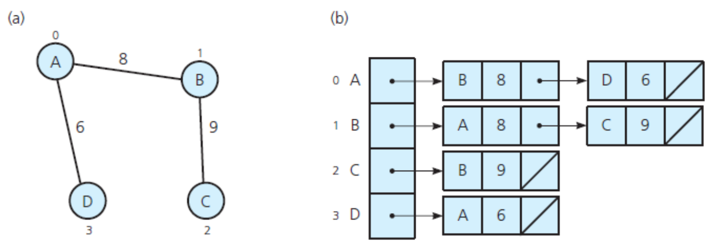


Adjacency List



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Graph Traversal

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Graph Traversal

- Visits (all) the vertices that it can reach.
- **Connected component** is subset of vertices visited during traversal that begins at given vertex.

Depth-First Search

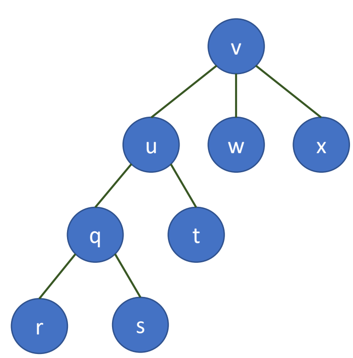
- Goes as far as possible from a vertex before backing up.

```
DFS(v: vertex)
{
    Mark v as visited
    for (each unvisited vertex u adjacent to v)
        DFS(u)
}
```

Depth-First Search

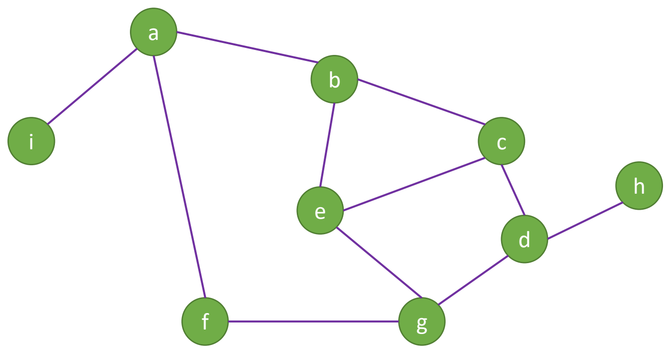
```
DFS(v: vertex)
      s = new empty stack
      s.push(v)
      Mark v as visited
      while (s is not empty)  {
            if (no unvisited vertices are adjacent to the vertex on
the top of the stack)
                  s.pop()
            else {
                  s.push(u)
                  Marked u as visited
            }
      }
```


Depth-First Search



v - u - q - r - s - t - w - x

Depth-First Search



DFS starts at ***a***:

DFS starts at ***e***:

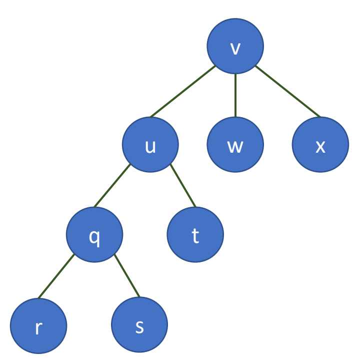
Breadth-First Search

- Visits all vertices adjacent to vertex before going forward.
- Breadth-first search uses a **queue**.

Breadth-First Search

```
BFS(v: Vertex)
    q = a new empty queue
    q.enqueue(v)
    Mark v as visited
    while (q is not empty){
        w = q.dequeue()
        for (each unvisited vertex u adjacent to w){
            Mark u as visited
            q.enqueue(u)
        }
    }
```


Breadth-First Search



v - u - w - x - q - t - r - s

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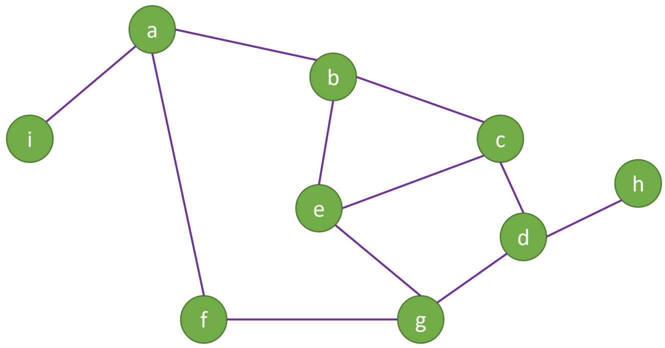
Breadth-First Search



BFS starts at ***a***:
BFS starts at ***e***:

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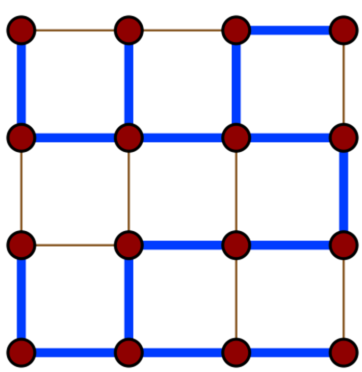
Minimum Spanning Tree

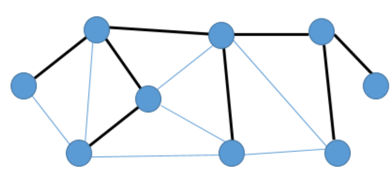
Spanning Tree

- A spanning tree
  - is a **subgraph** of undirected graph G
  - has **all** the vertices covered with **minimum** possible number of edges.

- does not have cycles
- cannot be disconnected.

Spanning Tree





Spanning Tree

- A connected graph G can have **more than one** spanning tree.
- All possible spanning trees of graph G, **have the same** number of **edges** and **vertices**.
- The spanning tree **does not have any cycle** (loops).
- The spanning tree is **minimally connected**.
- The spanning tree is **maximally acyclic**.

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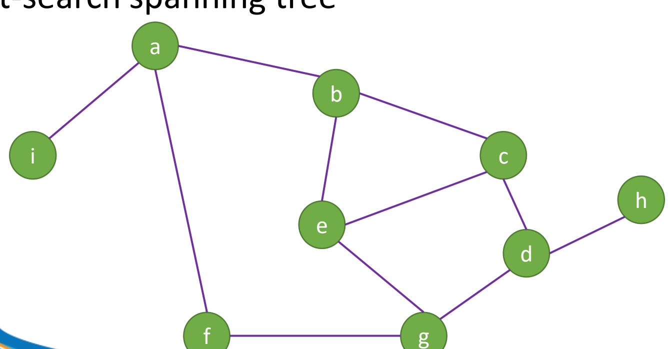
Spanning Tree

- Depth-first-search spanning tree
- Breadth-first-search spanning tree



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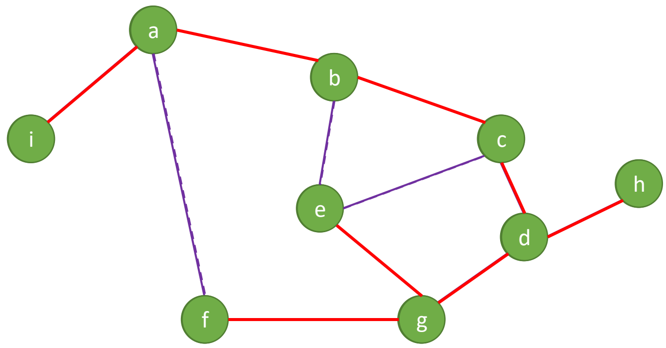
Spanning Tree



DFS spanning tree

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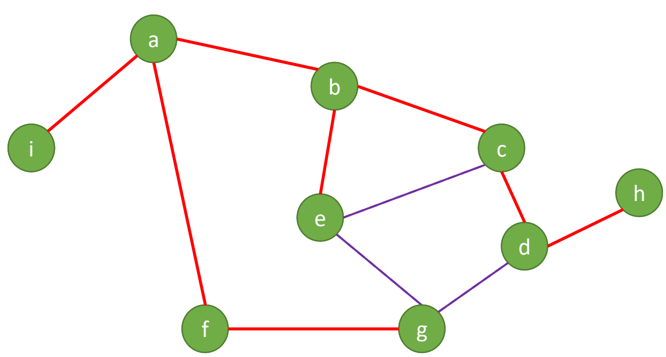
Spanning Tree



BFS spanning tree

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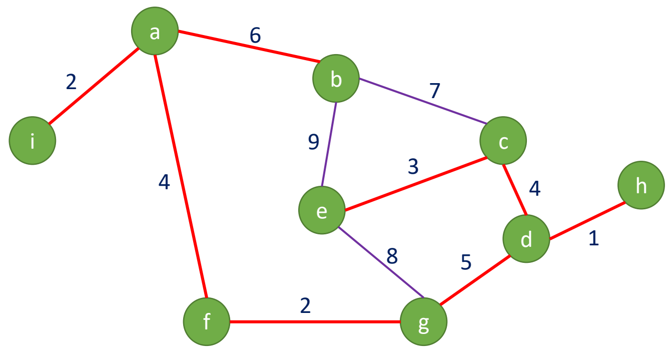
Minimum Spanning Tree

- A minimum spanning tree is a spanning tree that has **minimum weight** than all other spanning trees of the same graph.



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Prim’s Minimum Spanning Tree

- Begins with any vertex.
- Initially, the tree T contains only the starting vertex.
- At each stage,
  - Select the least cost edge *e*(v, u) with *v* in T and *u* not in T.
  - Add *u* and *e* to T

Prim’s Minimum Spanning Tree

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```
primAlgorithm(v: Vertex)
   Mark v as visited and include it in the minimum spanning tree
   while (there are unvisited vertices)
   {
         Find the least-cost edge e(v, u) from a visited vertex v to some unvisited vertex u
         Mark u as visited
         Add the vertex u and the edge e(v, u) to the minimum spanning tree
   }
```


Prim’s Minimum Spanning Tree

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```
PrimSpanningTree(matrix[N][N], source)
{
   for v = 0 to N-1 {
         length[v] = matrix[source][v]
         parent[v] = source }
   Mark source //Add source to the spanning tree
   for step = 1 to N-1 {
         Find the vertex v such that length[v] is smallest and v is not in spanning tree
         Mark v
         for all vertices u not in vertexSet
               if (length[u] > matrix[v][u]){
                     length[u] = matrix[v][u]
                     parent[u] = v }
   }
}
```

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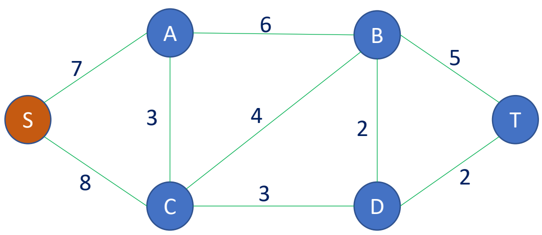
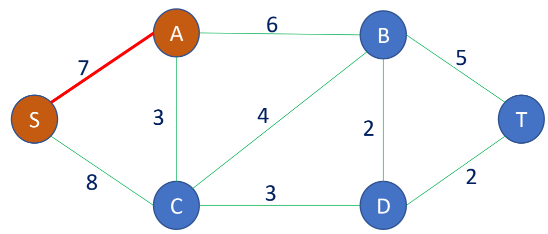
Prim's Minimum Spanning Tree





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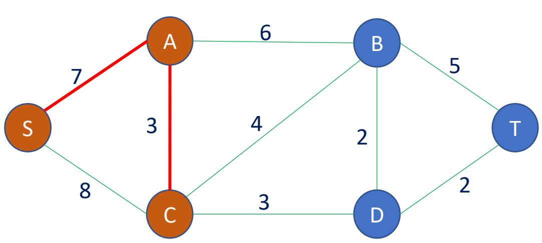
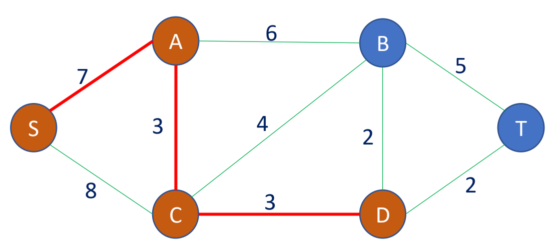
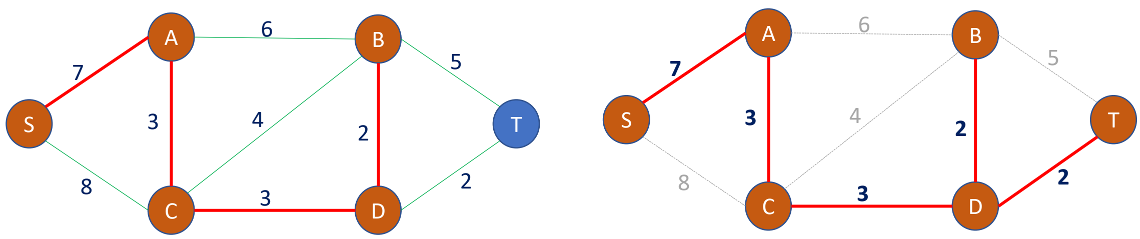
Prim's Minimum Spanning Tree





Prim's Minimum Spanning Tree



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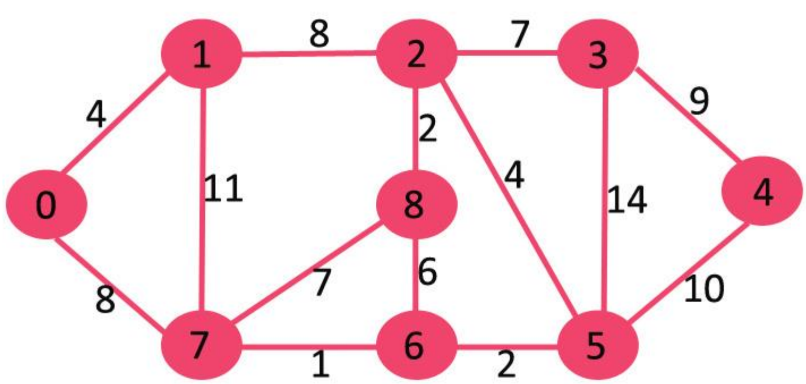
Example



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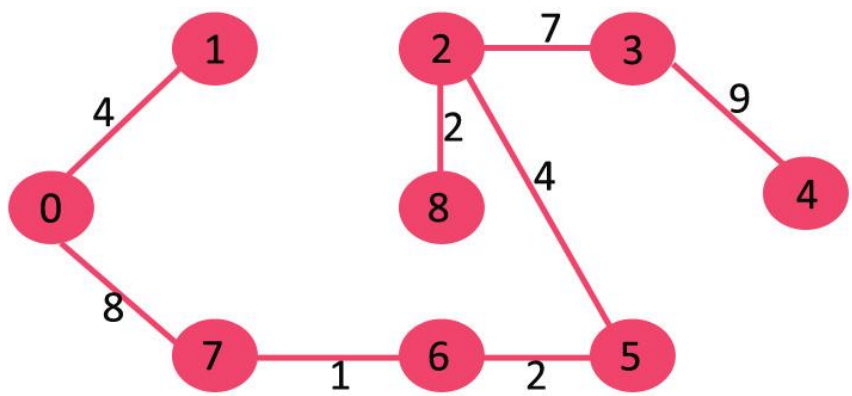
Example



Shortest Path

Dijkstra's Shortest Path Algorithm

- Given a graph and a source vertex in the graph, find shortest paths from source to all vertices in the given graph.
- **Dijkstra's** algorithm is very **similar** to **Prim's** algorithm for minimum spanning tree.
- This algorithm is applicable to graphs with **non-negative weights** only.

Dijkstra's Shortest Path Algorithm

```
shortestPath(matrix[N][N], source, length[])
```

Input:

```
    matrix[N][N]: adjacency matrix of Graph G
with N vertices
    source: the source vertex
```

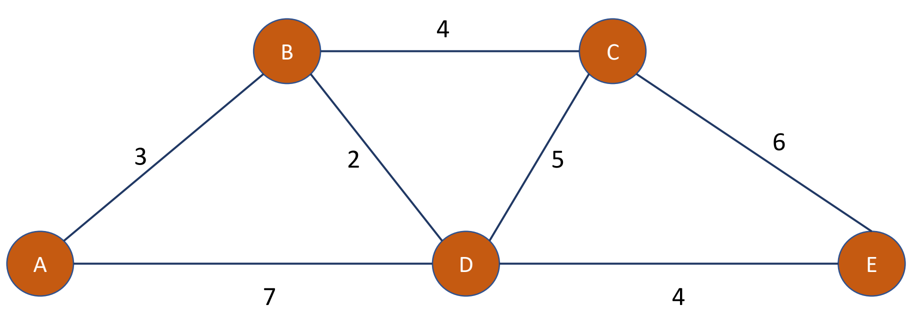
Output:

```
    length[]: the length of the shortest path
from source to all vertices in G.
```

Dijkstra's Shortest Path Algorithm

```
shortestPath(matrix[N][N], source, length[])
{
   for v = 0 to N-1
         length[v] = matrix[source][v]
   length[source] = 0 //why?
   for step = 1 to N {
         Find the vertex v such that length[v] is smallest
and v is not in vertexSet
         Add v to vertexSet
         for all vertices u not in vertexSet
               if (length[u] > length[v] + matrix[v][u]){
                     length[u] = length[v] + matrix[v][u]
                     parent[u] = v }
   }
}
```


Example



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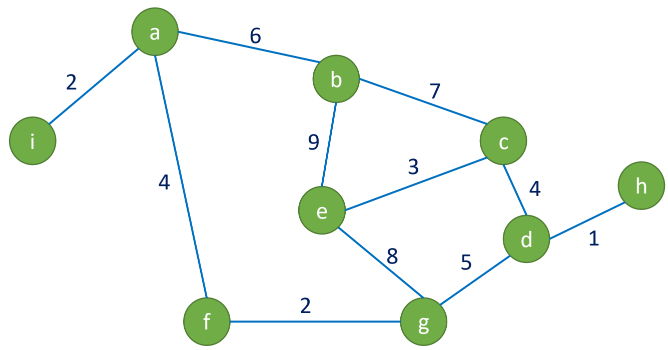
Example



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Questions and Answers