ROBOTICS

CHAPTER 6: DIFFERENTIAL KINEMATICS

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CONTENTS

6.1 DIFFERENTIAL KINEMATICS PROBLEM

- "Gives the relationship between the joint velocities and the corresponding end-effector linear and angular velocity. "
- This mapping is described by a matrix, termed geometric Jacobian, which depends on the manipulator configuration.
- Alternatively, if the end-effector pose is expressed with reference to a minimal representation in the operational space, it is possible to compute the Jacobian matrix via differentiation of the direct kinematics function with respect to the joint variables.
- The Jacobian is useful for:
  - Finding singularities,
  - Analyzing redundancy,
  - Determining inverse kinematics algorithms, describing the mapping between forces applied to the end-effector and resulting torques at the joints (statics) and, deriving dynamic equations of motion and designing operational space control schemes.
  - Finally, the kineto-statics duality concept is illustrated, which is at the basis of the definition of velocity and force manipulability ellipsoids.

6.2 GEOMETRIC JACOBIAN

❖ Consider an n-DOF manipulator. The direct kinematics equation can be written in the form:

$$T_e(q) = \begin{bmatrix} R_e(q) & p_e(q) \\ 0^T & 1 \end{bmatrix}$$

❖ The goal of the differential kinematics is to find the relationship between the joint velocities and the end-effector linear and angular velocities.

❖ The expression of the end-effector linear velocity $\dot{p}_e$ and angular velocity ω_e as a function of the joint velocities $\dot{q}$:

$$\dot{p}_e = J_P(q)\dot{q}$$
$$\omega_e = J_O(q)\dot{q}$$

Or in compact form: $v_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = J(q)\dot{q}$

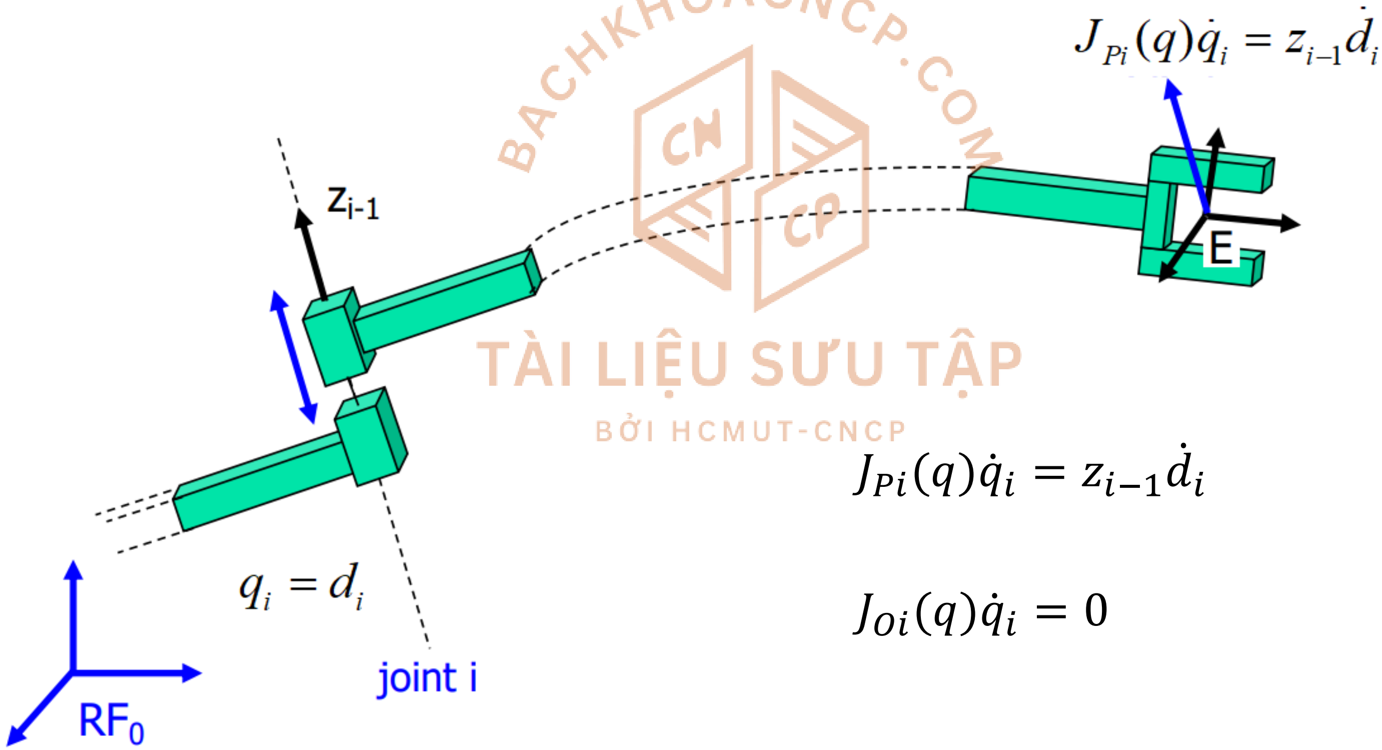
❖ Geometric Jacobian:

$$J = \begin{bmatrix} J_P \\ J_O \end{bmatrix}$$

6.2 GEOMETRIC JACOBIAN

❖ **PRISMATIC JOINT:**

Note: orientation of Frame i with respect to Frame i − 1 does not vary by moving Joint i



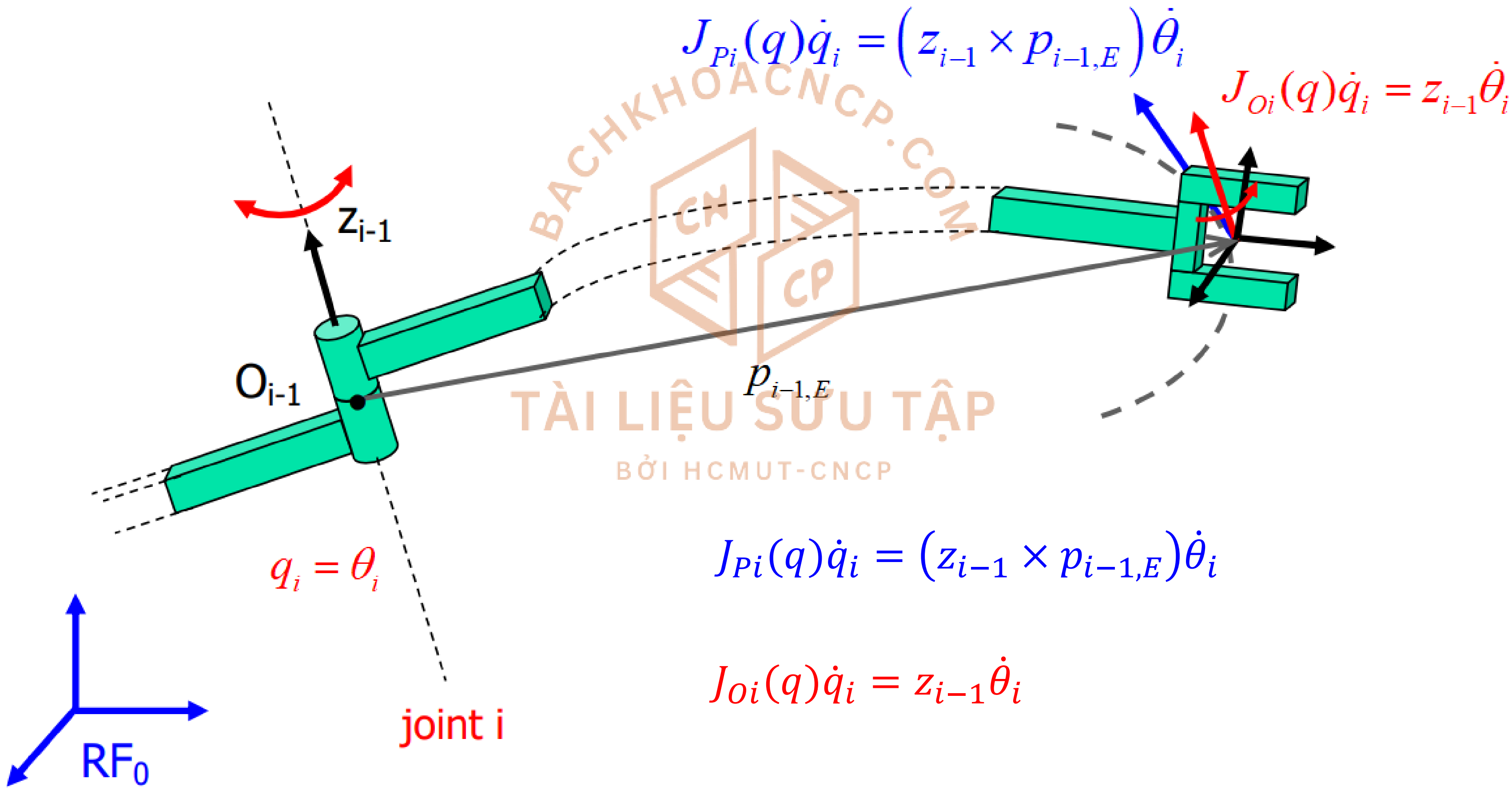
$$J_{Pi}(q)\dot{q}_i = z_{i-1}\dot{d}_i$$

$$J_{Oi}(q)\dot{q}_i = 0$$

(Ref: [3.1.1], [3.1.2] Robotics Modelling, Planning and Control)

6.2 GEOMETRIC JACOBIAN

❖ **REVOLUTE JOINT:**



$$J_{Pi}(q)\dot{q}_i = \left(z_{i-1} \times p_{i-1,E}\right)\dot{\theta}_i$$

$$J_{Oi}(q)\dot{q}_i = z_{i-1}\dot{\theta}_i$$

(Ref: [3.1.1], [3.1.2] Robotics Modelling, Planning and Control)

6.2 GEOMETRIC JACOBIAN

❖ EXPRESSION OF GEOMETRIC JACOBIAN

$$v_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} J_P(q) \\ J_O(q) \end{bmatrix} \dot{q} = \begin{bmatrix} J_{P1}(q) & \dots & J_{Pn}(q) \\ J_{O1}(q) & \dots & J_{On}(q) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dots \\ \dot{q}_n \end{bmatrix}$$

- Prismatic i-th joint:

$$J_{Pi}(q) = z_{i-1}$$

$$J_{Oi}(q) = 0$$

- Revolute i-th joint:

$$J_{Pi}(q) = z_{i-1} \times p_{i-1,E} = z_{i-1} \times (p_E - p_{i-1})$$

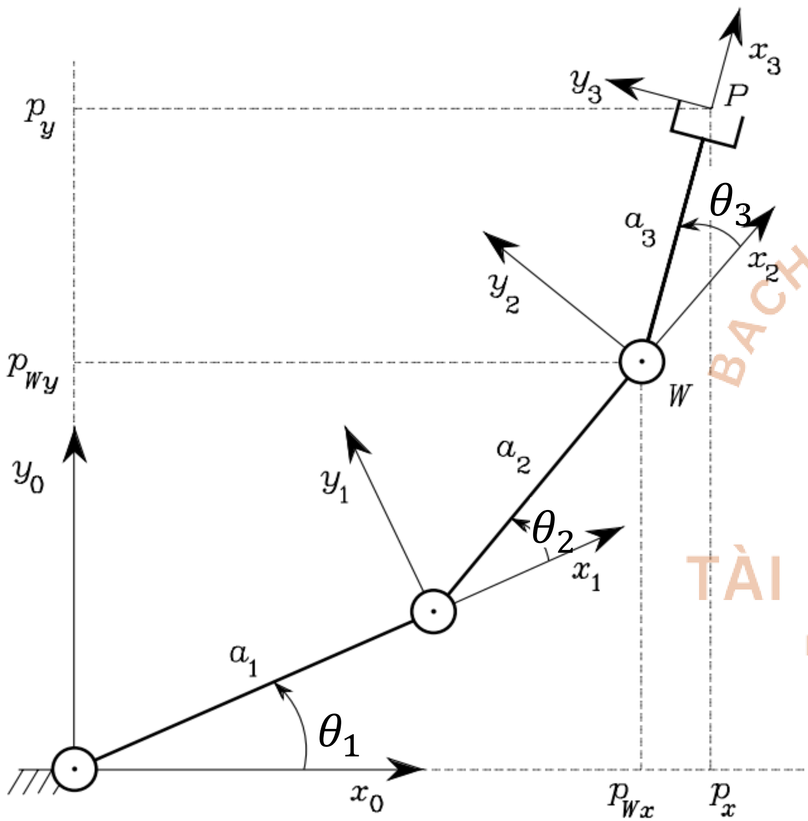
$$J_{Oi}(q) = z_{i-1}$$

$$\boldsymbol{u} \times \boldsymbol{p} = \boldsymbol{S}(\boldsymbol{u})\boldsymbol{p} = \begin{bmatrix} -u_z p_y + u_y p_z \\ u_z p_x - u_x p_z \\ -u_y p_x + u_x p_y \end{bmatrix}$$

All vectors should be expressed in the same reference frame (here, the base frame RF_0)

6.3 JACOBIAN OF TYPICAL MANIPULATOR STRUCTURES

❖ THREE-LINK PLANAR ARM



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$z_0 = [0 \quad 0 \quad 1]^T$$
$$p_0 = [0 \quad 0 \quad 0]^T$$

$$^0A_1 = \begin{array}{cccc} & & z_1 & p_1 \\ \end{array}$$
$$= \begin{bmatrix} c_1 & -s_1 & \boxed{0} & \boxed{a_1c_1} \\ s_1 & c_1 & \boxed{0} & \boxed{a_1s_1} \\ 0 & 0 & \boxed{1} & \boxed{0} \\ 0 & 0 & \boxed{0} & \boxed{1} \end{bmatrix}$$

$$^0A_2 = \begin{array}{cccc} & & z_2 & p_2 \\ \end{array}$$
$$= \begin{bmatrix} c_{12} & -s_{12} & \boxed{0} & \boxed{a_1c_1 + a_2c_{12}} \\ s_{12} & c_{12} & \boxed{0} & \boxed{a_1s_1 + a_2s_{12}} \\ 0 & 0 & \boxed{1} & \boxed{0} \\ 0 & 0 & \boxed{0} & \boxed{1} \end{bmatrix}$$

$$^0T_3 = \begin{bmatrix} c_{123} & -s_{123} & \boxed{0} & \boxed{a_1c_1 + a_2c_{12} + a_3c_{123}} \\ s_{123} & c_{123} & \boxed{0} & \boxed{a_1s_1 + a_2s_{12} + a_3s_{123}} \\ 0 & 0 & \boxed{1} & \boxed{0} \\ 0 & 0 & \boxed{0} & \boxed{1} \end{bmatrix}$$
$$\begin{array}{cccc} & & z_3 & p_E \end{array}$$

6.3 JACOBIAN OF TYPICAL MANIPULATOR STRUCTURES

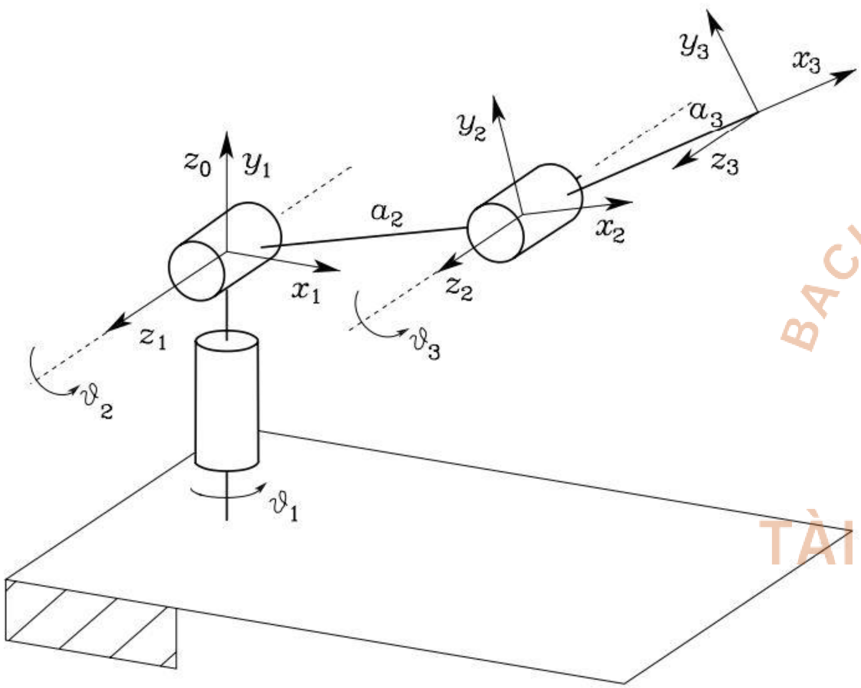
❖ THREE-LINK PLANAR ARM

$$J(q) = \begin{bmatrix} z_0 \times (p_3 - p_0) & z_1 \times (p_3 - p_1) & z_2 \times (p_3 - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix}$$

$$\Rightarrow J(q) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

6.3 JACOBIAN OF TYPICAL MANIPULATOR STRUCTURES

❖ **ARTICULATED ARM**



Link	a_i	α_i	d_i	θ_i
1	0	$\pi/2$	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$z_0 = [0 \quad 0 \quad 1]^T$$
$$p_0 = [0 \quad 0 \quad 0]^T$$

$$\begin{array}{cc} & \begin{array}{cccc} & & z_1 & p_1 \end{array} \\ {}^0A_1 = & \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} & & z_2 & p_2 \end{array} \\ {}^0A_2 = & \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & a_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & a_2s_1c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} {}^0A_3 = & \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & c_1(a_2c_2 + a_3c_{23}) \\ s_1c_{23} & -s_1s_{23} & -c_1 & s_1(a_2c_2 + a_3c_{23}) \\ s_{23} & c_{23} & 0 & a_2s_2 + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & \begin{array}{cccc} \phantom{c_1c_{23}} & \phantom{-c_1s_{23}} & z_3 & p_E \end{array} \end{array}$$

6.3 JACOBIAN OF TYPICAL MANIPULATOR STRUCTURES

❖ ARTICULATED ARM

$$J(q) = \begin{bmatrix} z_0 \times (p_3 - p_0) & z_1 \times (p_3 - p_1) & z_2 \times (p_3 - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix}$$

$$\Rightarrow J(q) = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

6.4 KINEMATIC SINGULARITIES

- Configurations where the Jacobian loses rank => Kinematic singularities

- Reasons to find the singularities of a manipulator:
  - Singularities represent configurations at which mobility of the structure is reduced, i.e., it is not possible to impose an arbitrary motion to the end-effector.
  - When the structure is at a singularity, infinite solutions to the inverse kinematics problem may exist.
  - In the neighbourhood of a singularity, small velocities in the operational space may cause large velocities in the joint space.

6.4 KINEMATIC SINGULARITIES

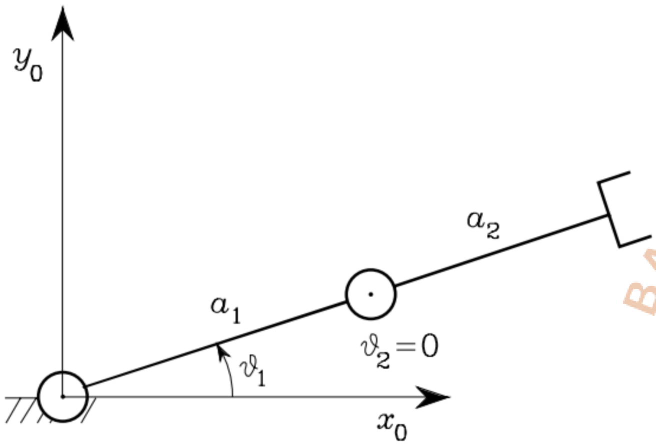
❖ Singularities can be classified into:

- Boundary singularities that occur when the manipulator is either outstretched or retracted.
- Internal singularities that occur inside the reachable workspace and are generally caused by the alignment of two or more axes of motion, or else by the attainment of particular end-effector configurations. These singularities can be encountered anywhere in the reachable workspace for a planned path in the operational space.

❖ There are a number of methods that can be used to determine the singularities of the Jacobian. In this chapter, we will exploit the fact that a square matrix is singular when its determinant is equal to zero.

6.4 KINEMATIC SINGULARITIES

❖ SINGULARITIES OF PLANAR 2R ARM



- Jacobian:

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\det(J) = a_1 a_2 s_2$$

- Singularities:

$$\det(J) = 0 \Leftrightarrow \theta_2 = 0 \text{ or } \theta_2 = \pi$$

6.4 SINGULARITY DECOUPLING

- **For manipulators having a spherical wrist: Split the problem of singularity computation into two separate problems:**
  - Computation of arm singularities resulting from the motion of the first 3 or more links.
  - Computation of wrist singularities resulting from the motion of the wrist joints.