Chapter 2f
Program Verification

Mathematical Modeling (CO2011)

(Materials drawn from:
"Michael Huth and Mark Ryan. *Logic in Computer Science: Modelling and Reasoning about Systems*, 2nd Ed., Cambridge University Press, 2006.")

Nguyen An Khuong
Faculty of Computer Science and Engineering
University of Technology, VNU-HCM

Contents

Motivation

- One way of checking the correctness of programs is to explore the possible states that a computation system can reach during the execution of the program.
- Problems with this *model checking* approach:
  - Models become infinite.
  - Satisfaction/validity becomes undecidable.
- In this lecture, we cover a proof-based framework for program verification.

Characteristics of the Approach

Proof-based instead of model checking

Semi-automatic instead of automatic

Property-oriented not using full specification

Application domain fixed to sequential programs using integers

Interleaved with development rather than a-posteriori verification

Reasons for Program Verification

Documentation. Program properties formulated as theorems can serve as concise documentation

Time-to-market. Verification prevents/catches bugs and can reduce development time

Reuse. Clear specification provides basis for reuse

Certification. Verification is required in safety-critical domains such as nuclear power stations and aircraft cockpits

Framework for Software Verification

Convert informal description R of *requirements* for an application domain into formula ϕ_R.

Write program P that meets ϕ_R.

Prove that P satisfies ϕ_R.

Each step provides risks and opportunities.

Motivation of Core Language

- Real-world languages are quite large; many features and constructs
- Theoretical constructions such as Turing machines or lambda calculus are too far from actual applications; too low-level
- Idea: use subset of Pascal/C/C++/Java
- Benefit: we can study useful "realistic" examples

Expressions in Core Language

Expressions come as arithmetic expressions E:

$$E ::= n \mid x \mid (-E) \mid (E + E) \mid (E - E) \mid (E * E)$$

and boolean expressions B:

$$B ::= \texttt{true} \mid \texttt{false} \mid (!B) \mid (B \& B) \mid (B \| B) \mid (E < E)$$

Where are the other comparisons, for example ==?

Commands in Core Language

Commands cover some common programming idioms. Expressions are components of commands.

$$C ::= x = E \mid C; C \mid \texttt{if}\ B\ \{C\}\ \texttt{else}\ \{C\} \mid \texttt{while}\ B\ \{C\}$$

Example

Consider the factorial function:

$$
\begin{aligned}
0! &\stackrel{\text{def}}{=} 1 \\
(n+1)! &\stackrel{\text{def}}{=} (n+1) \cdot n!
\end{aligned}
$$

We shall show that after the execution of the following Core program, we have $y = x!$.

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

1. Core Programming Language
2. **Hoare Triples; Partial and Total Correctness**
3. Proof Calculus for Partial Correctness
4. Practical Aspects of Correctness Proofs
5. Correctness of the Factorial Function
6. Proof Calculus for Total Correctness
7. Homeworks

Example

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

Example

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- We need to be able to say that at the end, y is x!

Example

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- We need to be able to say that at the end, `y` is `x!`
- That means we require a *post-condition* $y = x!$

Example

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- Do we need pre-conditions, too?

Example

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- Do we need pre-conditions, too?
  Yes, they specify what needs to be the case before execution.
  Example: $x > 0$

Example

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y = 1;
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  Yes, they specify what needs to be the case before execution.
  Example: $x > 0$
- Do we have to prove the postcondition in one go?

Example

```
y = 1;
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while (z != x) { z = z + 1; y = y * z; }
```

- Do we need pre-conditions, too?
  Yes, they specify what needs to be the case before execution.
  Example: $x > 0$
- Do we have to prove the postcondition in one go?
  No, the postcondition of one line can be the pre-condition of the next!

Assertions on Programs

Shape of assertions

$$(\!|\phi|\!)\ P\ (\!|\psi|\!)$$

Informal meaning

If the program P is run in a state that satisfies ϕ, then the state resulting from P's execution will satisfy ψ.

(Slightly Trivial) Example

Informal specification

Given a positive number x, the program P calculates a number y whose square is less than x.

Assertion

$$(\!|x > 0|\!)\ P\ (\!|y \cdot y < x|\!)$$

Example for P

```
y = 0
```

Our first Hoare triple

$$(\!|x > 0|\!)\ \texttt{y = 0}\ (\!|y \cdot y < x|\!)$$

(Slightly Less Trivial) Example

Same assertion

$$(\!|x > 0|\!)\ P\ (\!|y \cdot y < x|\!)$$

Another example for P

```
y = 0;
while (y * y < x) {
y = y + 1;
}
y = y - 1;
```

Recall: Models in Predicate Logic

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a set $P^{\mathcal{M}} \subseteq A^n$.

Recall: Satisfaction Relation

The model $\mathcal{M}$ satisfies ϕ with respect to environment l, written $\mathcal{M} \models_l \phi$:

- in case ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, if the result $(a_1, a_2, \ldots, a_n)$ of evaluating $t_1, t_2, \ldots, t_n$ with respect to l is in $P^{\mathcal{M}}$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;
- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for some $a \in A$;

Recall: Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_l \psi$ does not hold;
- in case ϕ has the form $\psi_1 \vee \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds or $\mathcal{M} \models_l \psi_2$ holds;
- in case ϕ has the form $\psi_1 \wedge \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds and $\mathcal{M} \models_l \psi_2$ holds; and
- in case ϕ has the form $\psi_1 \rightarrow \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds whenever $\mathcal{M} \models_l \psi_2$ holds.

Hoare Triples

Definition

An assertion of the form $(\!|\phi|\!)\ P\ (\!|\psi|\!)$ is called a Hoare triple.

- ϕ is called the precondition, ψ is called the postcondition.
- A state of a Core program P is a function l that assigns each variable x in P to an integer $l(x)$.
- A state l satisfies ϕ if $\mathcal{M} \models_l \phi$, where $\mathcal{M}$ contains integers and gives the usual meaning to the arithmetic operations.
- Quantifiers in ϕ and ψ bind only variables that do *not* occur in the program P.

Example

Let $l(x) = -2$, $l(y) = 5$ and $l(z) = -1$. We have:

- $l \models \neg(x + y < z)$
- $l \not\models y = x \cdot z < z$
- $l \not\models \forall u(y < u \rightarrow y \cdot z < u \cdot z)$

Partial Correctness

Definition

We say that the triple $(\!|\phi|\!)\ P\ (\!|\psi|\!)$ is *satisfied under partial correctness* if, for all states which satisfy ϕ, the state resulting from P's execution satisfies ψ, provided that P terminates.

Notation

We write $\models_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$.

Extreme Example

$(\!|\phi|\!)$ `while true { x = 0; }` $(\!|\psi|\!)$

holds for all ϕ and ψ.

Total Correctness

Definition

We say that the triple $(\!|\phi|\!)\ P\ (\!|\psi|\!)$ is *satisfied under total correctness* if, for all states which satisfy ϕ, P is guaranteed to terminate and the resulting state satisfies ψ.

Notation

We write $\models_{\text{tot}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$.

Back to Factorial

Consider Fac1:

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

Back to Factorial

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- $\models_{\text{tot}} (\!| x \geq 0 |\!)$ Fac1 $(\!| y = x! |\!)$

Back to Factorial

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y = 1;
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- $\models_{\text{tot}} (\!| x \geq 0 |\!)$ `Fac1` $(\!| y = x! |\!)$
- $\not\models_{\text{tot}} (\!| \top |\!)$ `Fac1` $(\!| y = x! |\!)$

Back to Factorial

Consider Fac1:

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- $\models_{\text{tot}} (\!|x \geq 0|\!)$ Fac1 $(\!|y = x!|\!)$
- $\not\models_{\text{tot}} (\!|\top|\!)$ Fac1 $(\!|y = x!|\!)$
- $\models_{\text{par}} (\!|x \geq 0|\!)$ Fac1 $(\!|y = x!|\!)$

Back to Factorial

Consider `Fac1`:

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

- $\models_{\text{tot}} (\!| x \geq 0 |\!)$ `Fac1` $(\!| y = x! |\!)$
- $\not\models_{\text{tot}} (\!| \top |\!)$ `Fac1` $(\!| y = x! |\!)$
- $\models_{\text{par}} (\!| x \geq 0 |\!)$ `Fac1` $(\!| y = x! |\!)$
- $\models_{\text{par}} (\!| \top |\!)$ `Fac1` $(\!| y = x! |\!)$

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Strategy

We are looking for a proof calculus that allows us to establish

$$\vdash_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$$

where

- $\models_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$ holds whenever $\vdash_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$ (correctness), and
- $\vdash_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$ holds whenever $\models_{\text{par}} (\!|\phi|\!)\ P\ (\!|\psi|\!)$ (completeness).

Rules for Partial Correctness

$$\frac{(\!|\phi|\!)\ C_1\ (\!|\eta|\!) \qquad (\!|\eta|\!)\ C_2\ (\!|\psi|\!)}{(\!|\phi|\!)\ C_1; C_2\ (\!|\psi|\!)}\ [\text{Composition}]$$

Rules for Partial Correctness (continued)

$$\frac{}{(\!|[x \to E]\psi|\!)\ x = E\ (\!|\psi|\!)}\ [\text{Assignment}]$$

Examples

Let P be the program `x = 2`.
Using

$$\frac{}{(\!|[x \to E]\psi|\!)\ x = E\ (\!|\psi|\!)}\ [\text{Assignment}]$$

we can prove:

- $(\!|2 = 2|\!)\ P\ (\!|x = 2|\!)$
- $(\!|2 = 4|\!)\ P\ (\!|x = 4|\!)$
- $(\!|2 = y|\!)\ P\ (\!|x = y|\!)$
- $(\!|2 > 0|\!)\ P\ (\!|x > 0|\!)$

More Examples

Let P be the program `x = x + 1`.
Using

$$\frac{}{(\![x \to E]\psi]\!)\ x = E\ (\!|\psi|\!)}\ [\text{Assignment}]$$

we can prove:

- $(\!|x + 1 = 2|\!)\ P\ (\!|x = 2|\!)$
- $(\!|x + 1 = y|\!)\ P\ (\!|x = y|\!)$

Rules for Partial Correctness (continued)

$$\frac{(\!|\phi \wedge B|\!)\ C_1\ (\!|\psi|\!) \qquad (\!|\phi \wedge \neg B|\!)\ C_2\ (\!|\psi|\!)}{(\!|\phi|\!)\ \texttt{if}\ B\ \texttt{\{}\ C_1\ \texttt{\}}\ \texttt{else}\ \texttt{\{}\ C_2\ \texttt{\}}\ (\!|\psi|\!)}\ [\text{If-statement}]$$

$$\frac{(\!|\psi \wedge B|\!)\ C\ (\!|\psi|\!)}{(\!|\psi|\!)\ \texttt{while}\ B\ \texttt{\{}\ C\ \texttt{\}}\ (\!|\psi \wedge \neg B|\!)}\ [\text{Partial-while}]$$

Rules for Partial Correctness (continued)

$$\frac{\vdash_{AR} \phi' \to \phi \qquad (\!|\phi|\!)\ C\ (\!|\psi|\!) \qquad \vdash_{AR} \psi \to \psi'}{(\!|\phi'|\!)\ C\ (\!|\psi'|\!)}\ \text{[Implied]}$$

Proof Tableaux

Proofs have tree shape

All rules have the structure

$$\frac{\text{something}}{\text{something else}}$$

As a result, all proofs can be written as a tree.

Practical concern

These trees tend to be very wide when written out on paper. Thus we are using a linear format, called *proof tableaux*.

Interleave Formulas with Code

$$\frac{(\!|\phi|\!)\ C_1\ (\!|\eta|\!) \qquad (\!|\eta|\!)\ C_2\ (\!|\psi|\!)}{(\!|\phi|\!)\ C_1; C_2\ (\!|\psi|\!)}\ [\text{Composition}]$$

Shape of rule suggests format for proof of $C_1; C_2; \ldots; C_n$:

$(\!|\phi_0|\!)$
$C_1;$
$(\!|\phi_1|\!)$ justification
$C_2;$
$\vdots$
$(\!|\phi_{n-1}|\!)$ justification
$C_n;$
$(\!|\phi_n|\!)$ justification

Working Backwards

Overall goal

Find a proof that at the end of executing a program P, some condition ψ holds.

Common situation

If P has the shape $C_1; \ldots ; C_n$, we need to find the weakest formula ψ' such that

$$(\!|\psi'|\!)\ C_n\ (\!|\psi|\!)$$

Terminology

The weakest formula ψ' is called *weakest precondition*.

Example

$(\!| y < 3 |\!)$

$(\!| y + 1 < 4 |\!)$ Implied

```
y = y + 1;
```

$(\!| y < 4 |\!)$ Assignment

Another Example

Can we claim $u = x + y$ after `z = x; z = z + y; u = z;` ?

$(\!\|\top\|\!)$	
$(\!\|x + y = x + y\|\!)$	Implied
`z = x;`	
$(\!\|z + y = x + y\|\!)$	Assignment
`z = z + y;`	
$(\!\|z = x + y\|\!)$	Assignment
`u = z;`	
$(\!\|u = x + y\|\!)$	Assignment

An Alternative Rule for If

We have:

$$\frac{(\!|\phi \wedge B|\!)\ C_1\ (\!|\psi|\!) \qquad (\!|\phi \wedge \neg B|\!)\ C_2\ (\!|\psi|\!)}{(\!|\phi|\!)\ \texttt{if}\ B\ \texttt{\{}\ C_1\ \texttt{\}}\ \texttt{else}\ \texttt{\{}\ C_2\ \texttt{\}}\ (\!|\psi|\!)}\ [\text{If-statement}]$$

Sometimes, the following *derived rule* is more suitable:

$$\frac{(\!|\phi_1|\!)\ C_1\ (\!|\psi|\!) \qquad (\!|\phi_2|\!)\ C_2\ (\!|\psi|\!)}{(\!|(B \to \phi_1) \wedge (\neg B \to \phi_2)|\!)\ \texttt{if}\ B\ \texttt{\{}\ C_1\ \texttt{\}}\ \texttt{else}\ \texttt{\{}\ C_2\ \texttt{\}}\ (\!|\psi|\!)}\ [\text{If-stmt 2}]$$

Example

Consider this implementation of Succ:

```
a = x + 1;
if (a - 1 == 0) {
y = 1;
} else {
y = a;
}
```

Can we prove $(\!|\top|\!)$ Succ $(\!|y = x + 1|\!)$?

Another Example

```
⋮
if ( a - 1 == 0 ) {
```

$(\!|1 = x + 1|\!)$ If-Statement 2

```
  y = 1;
```

$(\!|y = x + 1|\!)$ Assignment

```
} else {
```

$(\!|a = x + 1|\!)$ If-Statement 2

```
  y = a;
```

$(\!|y = x + 1|\!)$ Assignment

```
}
```

$(\!|y = x + 1|\!)$ If-Statement 2

Another Example

$(\!|\top|\!)$

$(\!|(x + 1 - 1 = 0 \to 1 = x + 1) \wedge$
$(\neg(x + 1 - 1 = 0) \to x + 1 = x + 1)|\!)$ Implied

```
a = x + 1;
```

$(\!|(a - 1 = 0 \to 1 = x + 1) \wedge$
$(\neg(a - 1 = 0) \to a = x + 1)|\!)$ Assignment

```
if ( a - 1 == 0 ) {
```

$(\!|1 = x + 1|\!)$ If-Statement 2

```
   y = 1;
```

$(\!|y = x + 1|\!)$ Assignment

```
} else {
```

$(\!|a = x + 1|\!)$ If-Statement 2

```
   y = a;
```

$(\!|y = x + 1|\!)$ Assignment

Recall: Partial-while Rule

$$\frac{(\!|\psi \land B|\!)\ C\ (\!|\psi|\!)}{(\!|\psi|\!)\ \texttt{while}\ B\ \texttt{\{}\ C\ \texttt{\}}\ (\!|\psi \land \neg B|\!)}\ [\text{Partial-while}]$$

Factorial Example

We shall show that the following Core program Fac1 meets this specification:

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

Thus, to show:

$$(\!|\top|\!)\ \texttt{Fac1}\ (\!|y = x!|\!)$$

Partial Correctness of `Fac1`

$\vdots$

$(\!|\, y = z! \,|\!)$

`while ( z != x ) {`

$(\!\|\, y = z! \wedge z \neq x \,\|\!)$	Invariant
$(\!\|\, y \cdot (z+1) = (z+1)! \,\|\!)$	Implied
`z = z + 1;`	
$(\!\|\, y \cdot z = z! \,\|\!)$	Assignment
`y = y * z;`	
$(\!\|\, y = z! \,\|\!)$	Assignment
`}`	
$(\!\|\, y = z! \wedge \neg(z \neq x) \,\|\!)$	Partial-while
$(\!\|\, y = x! \,\|\!)$	Implied

Partial Correctness of `Fac1`

$(\!|\top|\!)$	
$(\!|(1 = 0!)|\!)$	Implied
`y = 1;`	
$(\!|y = 0!|\!)$	Assignment
`z = 0;`	
$(\!|y = z!|\!)$	Assignment
`while ( z != x ) {`	
$\vdots$	
`}`	
$(\!|y = z! \land \neg(z \neq x)|\!)$	Partial-while
$(\!|y = x!|\!)$	Implied

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Ideas for Total Correctness

- The only source of non-termination is the `while` command.
- If we can show that the value of an integer expression decreases in each iteration, but never becomes negative, we have proven termination.
  Why? Well-foundedness of natural numbers
- We shall include this argument in a new version of the `while` rule.

Rules for Partial Correctness (continued)

$$\frac{(\!|\psi \wedge B|\!)\ C\ (\!|\psi|\!)}{(\!|\psi|\!)\ \texttt{while}\ B\ \{\ C\ \}\ (\!|\psi \wedge \neg B|\!)}\ [\text{Partial-while}]$$

$$\frac{(\!|\psi \wedge B \wedge 0 \leq E = E_0|\!)\ C\ (\!|\psi \wedge 0 \leq E < E_0|\!)}{(\!|\psi \wedge 0 \leq E|\!)\ \texttt{while}\ B\ \{\ C\ \}\ (\!|\psi \wedge \neg B|\!)}\ [\text{Total-while}]$$

Factorial Example (Again!)

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

What could be a good variant E?

Factorial Example (Again!)

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

What could be a good variant E?

E must strictly decrease in the loop, but not become negative.

Factorial Example (Again!)

```
y = 1;
z = 0;
while (z != x) { z = z + 1; y = y * z; }
```

What could be a good variant E?

E must strictly decrease in the loop, but not become negative.

Answer:

$$x - z$$

Total Correctness of `Fac1`

$$\vdots$$

$(\!|y = z! \wedge 0 \le x - z|\!)$	
`while ( z != x ) {`	
$\quad(\!|y = z! \wedge z \neq x \wedge 0 \le x - z = E_0|\!)$	Invariant
$\quad(\!|y \cdot (z+1) = (z+1)! \wedge 0 \le x - (z+1) < E_0|\!)$	Implied
`    z = z + 1;`	
$\quad(\!|y \cdot z = z! \wedge 0 \le x - z < E_0|\!)$	Assignment
`    y = y * z;`	
$\quad(\!|y = z! \wedge 0 \le x - z < E_0|\!)$	Assignment
`}`	
$(\!|y = z! \wedge \neg(z \neq x)|\!)$	Total-while
$(\!|y = x!|\!)$	Implied

Total Correctness of `Fac1`

$(\!|x \leq 0|\!)$
$(\!|(1 = 0! \wedge 0 \leq x - 0|\!)$ Implied

```
y = 1;
```

$(\!|y = 0! \wedge 0 \leq x - 0|\!)$ Assignment

```
z = 0;
```

$(\!|y = z! \wedge 0 \leq x - z|\!)$ Assignment

```
while ( z != x ) {
    ⋮
}
```

$(\!|y = z! \wedge \neg(z \neq x)|\!)$ Total-while
$(\!|y = x!|\!)$ Implied

HW

Do as much as possible (at least ALL designated) problems given in Section 4.6 in [2]