ROBOTICS

CHAPTER 5: INVERSE KINEMATICS

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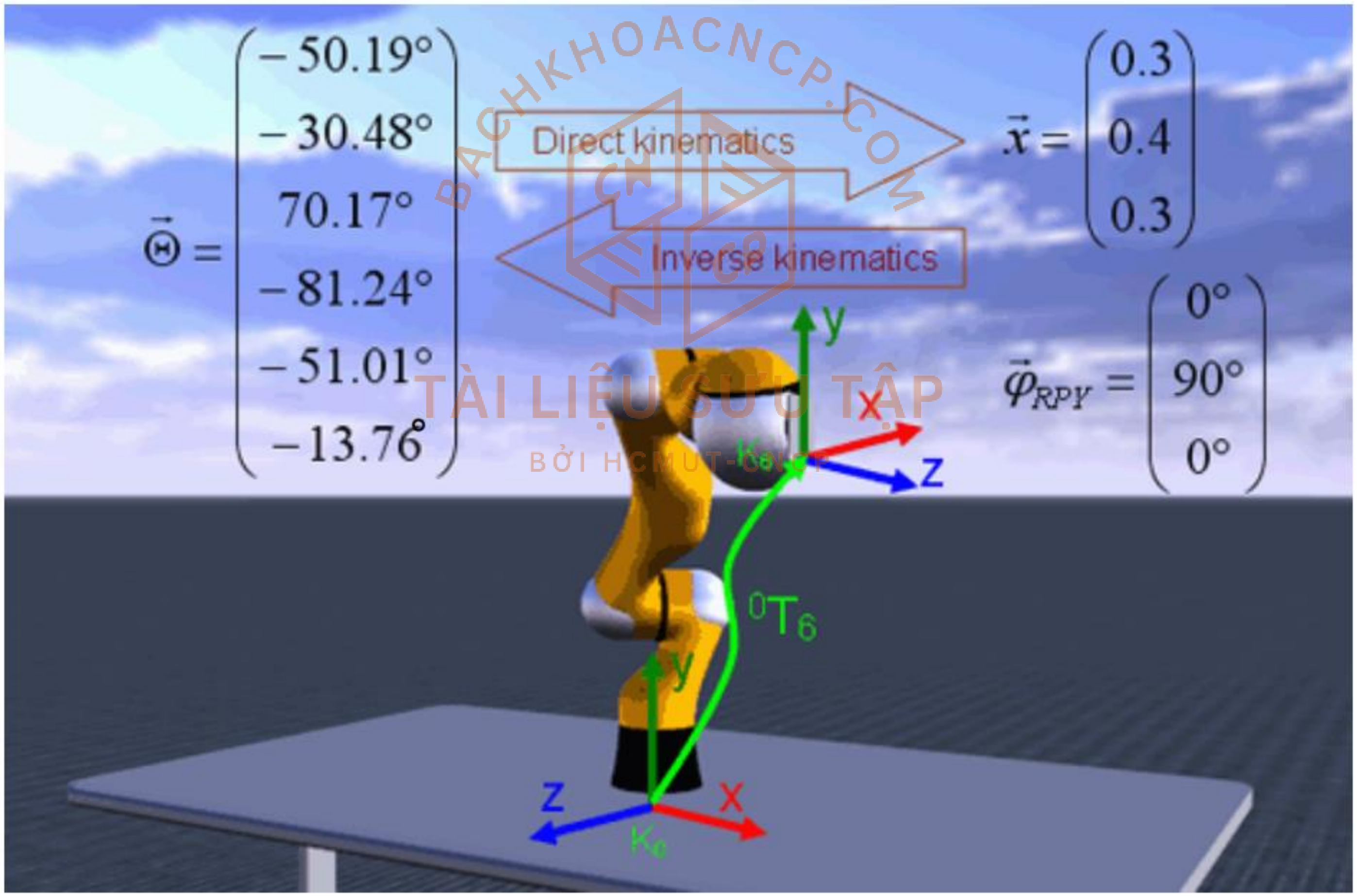
CONTENTS

5.1 INVERSE KINEMATICS PROBLEM

❖ "Given a desired end-effector pose (position + orientation), find the values of the joint variables that will realize it"



5.1 INVERSE KINEMATICS PROBLEM

❖ A synthesis problem, with input data in the form:

- Classical formulation: inverse kinematics for a given end-effector pose

$$T = \begin{bmatrix} R & p \\ 000 & 1 \end{bmatrix} = {}^{0}A_n(q)$$

- Generalized formulation: inverse kinematics for a given value of task variables

$$r = \begin{bmatrix} p \\ \theta \end{bmatrix} = f_r(q)$$

- The equations to solve are in general nonlinear, and thus it is not always possible to find a closed-form solution.
- Multiple solutions may exist.
- Infinite solutions may exist, e.g., in the case of a kinematically redundant manipulator.
- There might be no admissible solutions, in view of the manipulator kinematic structure.

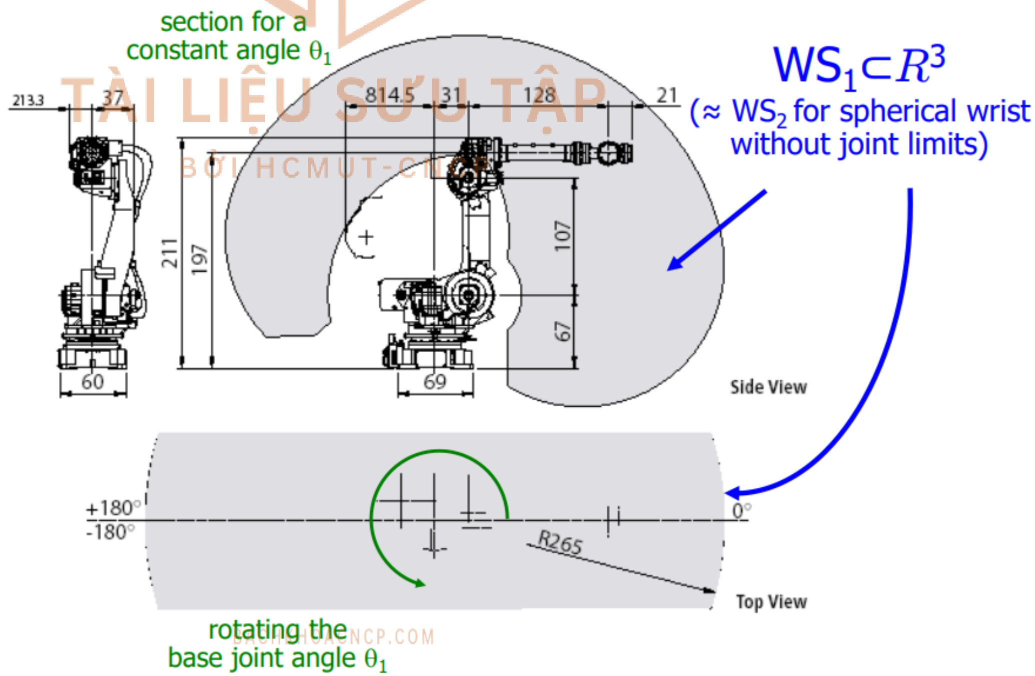
5.1 INVERSE KINEMATICS PROBLEM

- Primary workspace WS1: set of all positions **p** that can be reached with at least one orientation ($\boldsymbol{\theta}$ or **R**)
  - Out of WS1 there is no solution to the problem
  - When p ∈ WS1, there is a suitable $\boldsymbol{\theta}$ (or **R**) for which a solution exists

- Secondary (or dexterous) workspace WS2: set of positions **p** that can be reached with any orientation (among those feasible for the robot direct kinematics)

  When **p** ∈ WS2, there exists a solution for any feasible $\boldsymbol{\theta}$ (or **R**)

- WS2 ⊆ WS1



5.1 INVERSE KINEMATICS PROBLEM

❖ SOLUTION METHODS

ANALYTICAL solution (in closed form)

- Preferred, if it can be found*
- Use ad-hoc geometric inspection
- Algebraic methods (solution of polynomial equations)
- systematic ways for generating a reduced set of equations to be solved

* Sufficient conditions for 6-dof arms

- 3 consecutive rotational joint axes are incident (e.g., spherical wrist), or
- 3 consecutive rotational joint axes are parallel

NUMERICAL solution (in iterative form)

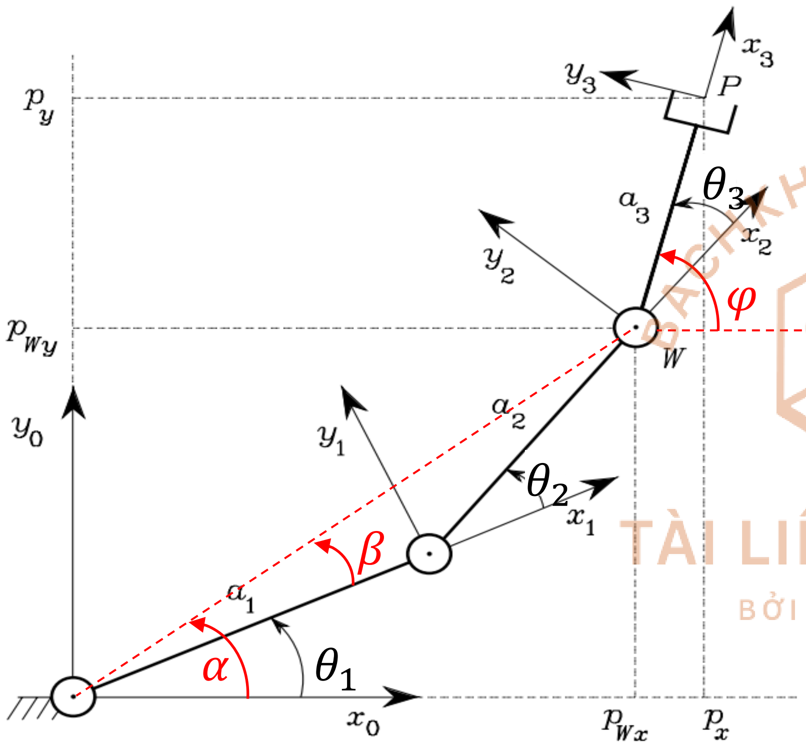
- Certainly needed if n>m (redundant case), or at/close to singularities
- Slower, but easier to be set up
- In its basic form, it uses the (analytical) Jacobian matrix of the direct kinematics map

$$J_r(q) = \frac{\partial f_r(q)}{\partial q}$$

- Newton method, Gradient method, and so on…

5.2 INVERSE KINEMATICS SOLUTIONS: THREE-LINK PLANAR ARM

❖ ALGEBRAIC SOLUTION



$$
{}^{0}T_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

From Direct Kinematics:

$$\varphi = \theta_1 + \theta_2 + \theta_3 \qquad (1)$$

$$p_{Wx} = p_x - a_3c\varphi = a_1c_1 + a_2c_{12} \qquad (2)$$

$$p_{Wy} = p_y - a_3s\varphi = a_1c_1 + a_2s_{12} \qquad (3)$$

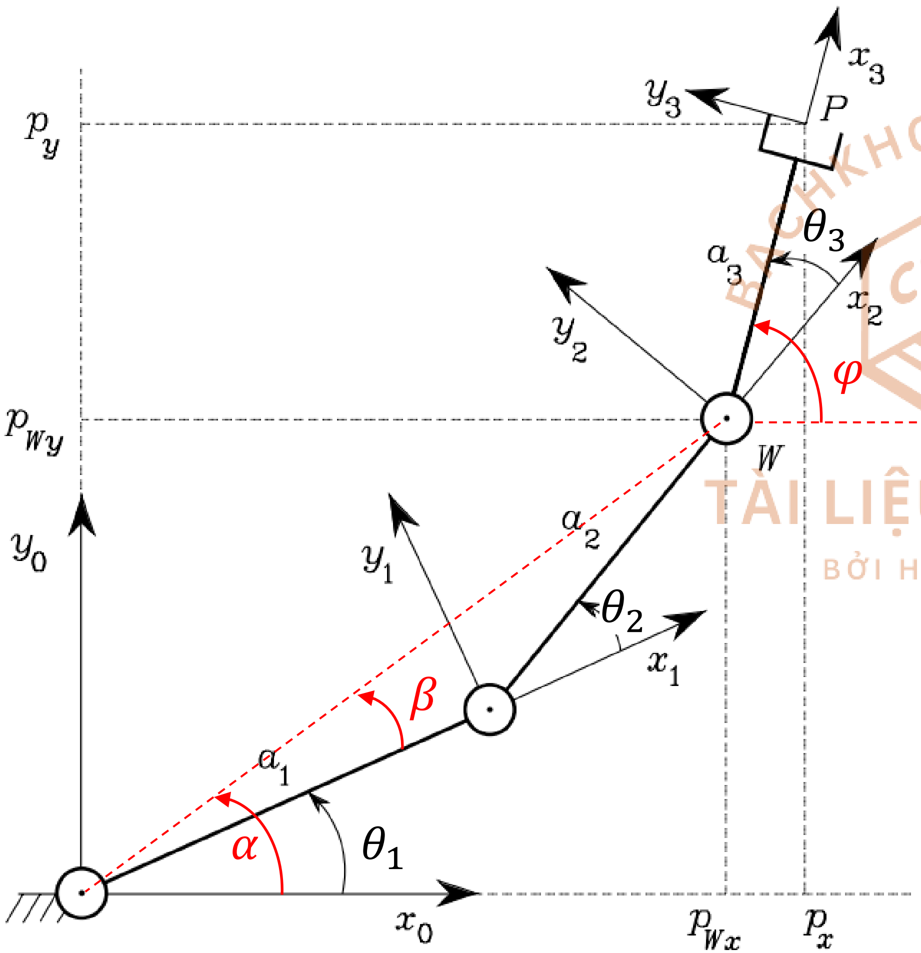
$$\Rightarrow p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2a_1a_2c_2$$

$$\Rightarrow c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2c_2}; s_2 = \pm\sqrt{1-c_2^2}$$

$$\boxed{\Rightarrow \theta_2 = ATAN2(s_2, c_2)}$$

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

❖ ALGEBRAIC SOLUTION



From (2), (3):

$$\begin{bmatrix} a_1 + a_2c_2 & -a_2s_2 \\ a_2s_2 & a_1 + a_2c_2 \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_{Wx} \\ p_{Wy} \end{bmatrix}$$

$$\Rightarrow s_1 = \frac{(a_1 + a_2c_2)p_{Wy} - a_2s_2p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2c_2)p_{Wx} + a_2s_2p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}$$

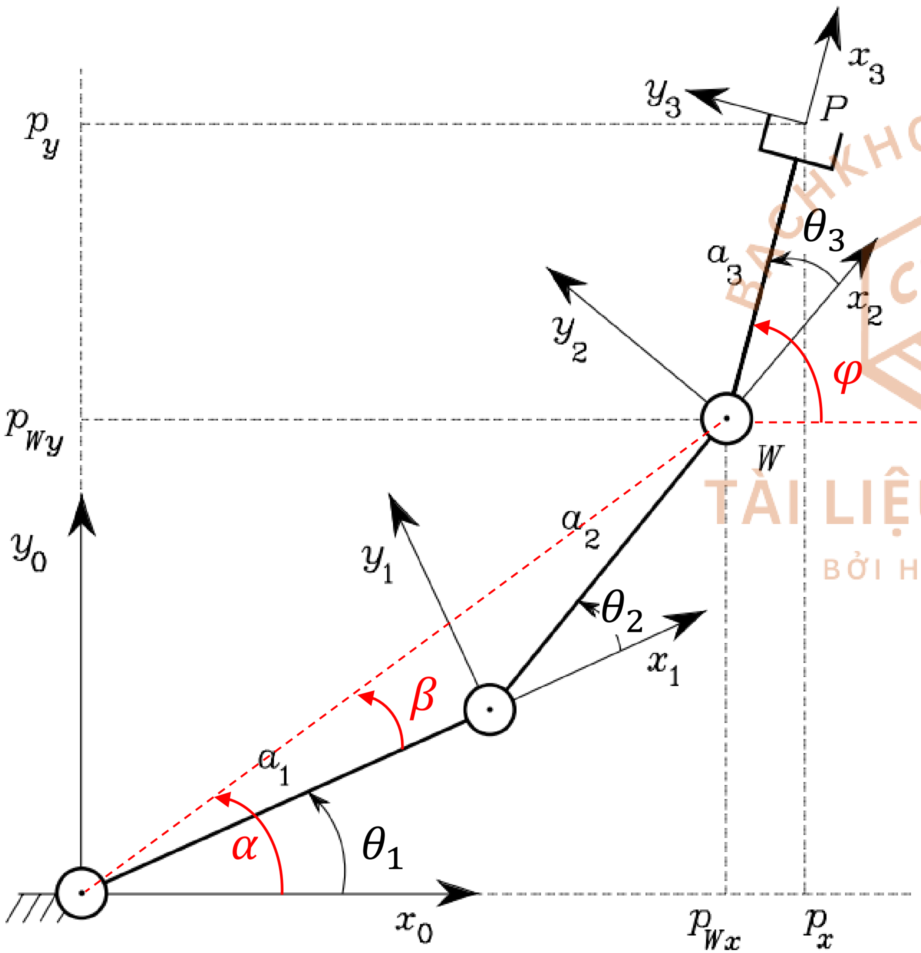
$$\boxed{\Rightarrow \theta_1 = ATAN2(s_1, c_1)}$$

$$\boxed{\theta_3 = \varphi - \theta_1 - \theta_2}$$

If φ is not specified, then the arm is redundant and there exist infinite solutions to the inverse kinematics problem.

5.2 INVERSE KINEMATICS SOLUTIONS: THREE-LINK PLANAR ARM

❖ GEOMETRIC SOLUTION



The application of the cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points W and O gives

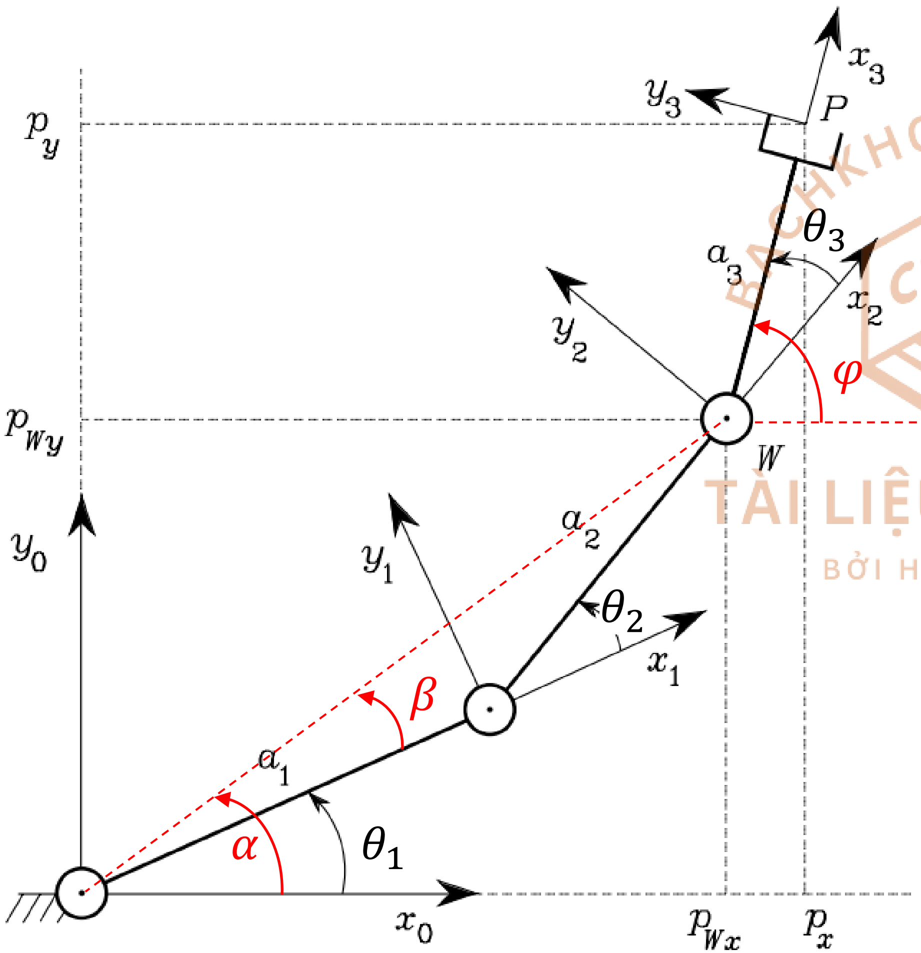
$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2)$$

$$\Rightarrow c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2c_2}; s_2 = \pm\sqrt{1-c_2^2}$$

$$\boxed{\Rightarrow \theta_2 = ATAN2(s_2, c_2)}$$

5.2 INVERSE KINEMATICS SOLUTIONS: THREE-LINK PLANAR ARM

❖ GEOMETRIC SOLUTION



Notice that the determination of α depends on the sign of p_{Wx} and p_{Wy}

$$\alpha = ATAN2(p_{Wy}, p_{Wx})$$

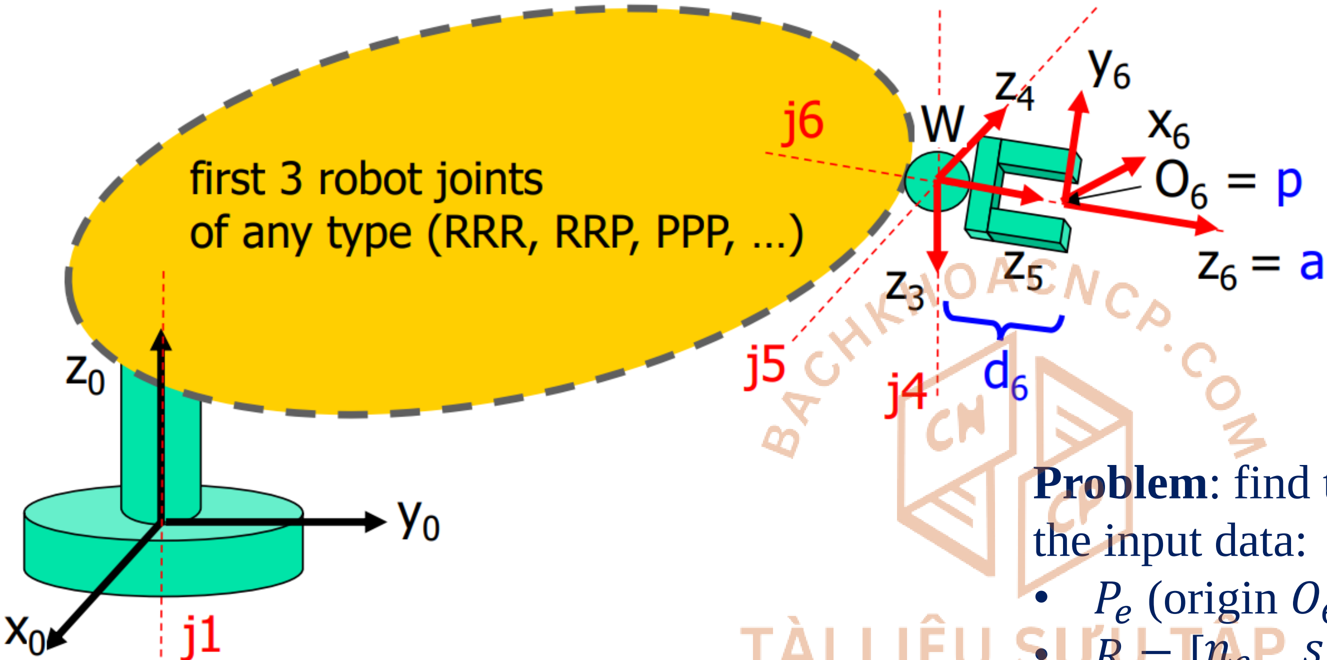
To compute β, applying again the cosine theorem yields

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2 c_2$$

$$\Rightarrow \beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\boxed{\theta_1 = \alpha \pm \beta} \quad \boxed{\theta_3 = \varphi - \theta_1 - \theta_2}$$

5.2 INVERSE KINEMATICS SOLUTION OF MANUPULATOR WITH SPHERICAL WRIST



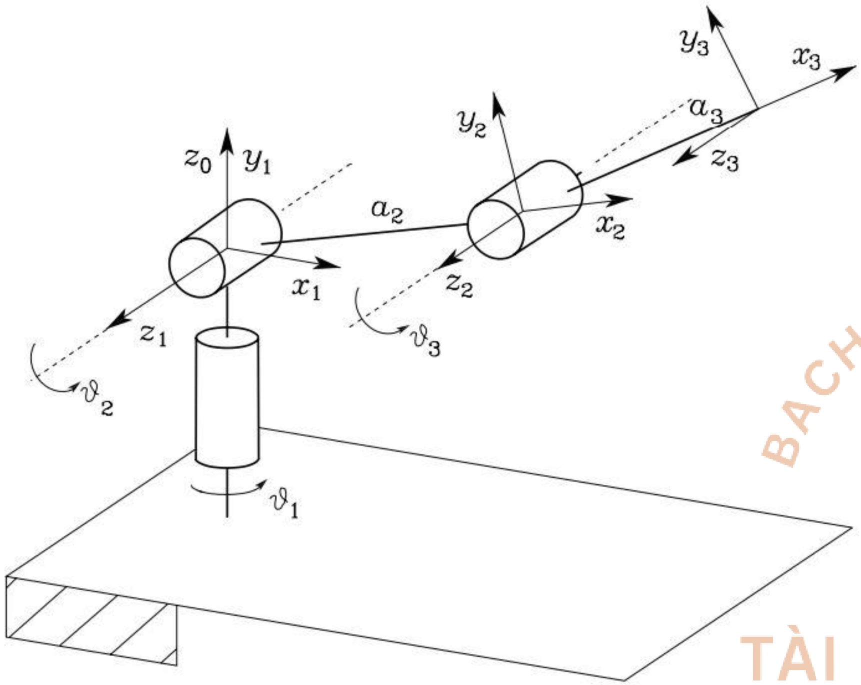
Problem: find the joint variables q_1, q_2,…, q_6 from the input data:

- P_e (origin O_6)
- $R = [n_e \quad s_e \quad a_e]$ (orientation of RF_6)

Solution:

- Compute wrist position $P_{W=}P_e - d_6 a_e$
- Solve inverse kinematics for (q_1, q_2, q_3)
- Compute ${}^0R_3(q_1, q_2, q_3)$
- Compute ${}^3R_6(q_4, q_5, q_6) = {}^0R_3^T R$
- Solve inverse kinematics for the spherical wrist (q_4, q_5, q_6)

5.2 INVERSE KINEMATICS SOLUTIONS: ARTICALUATED ARM



$$^0T_3 = {}^0A_1\,{}^1A_2\,{}^2A_3 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & c_1(a_2c_2 + a_3c_{23}) \\ s_1c_{23} & -s_1s_{23} & -c_1 & s_1(a_2c_2 + a_3c_{23}) \\ s_{23} & c_{23} & 0 & a_2s_2 + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From Direct Kinematics:

$$p_{Wx} = c_1(a_2c_2 + a_3c_{23})$$

$$p_{Wy} = s_1(a_2c_2 + a_3c_{23})$$

$$p_{Wz} = a_2s_2 + a_3s_{23}$$

Link	a_i	α_i	d_i	θ_i
1	0	$\pi/2$	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

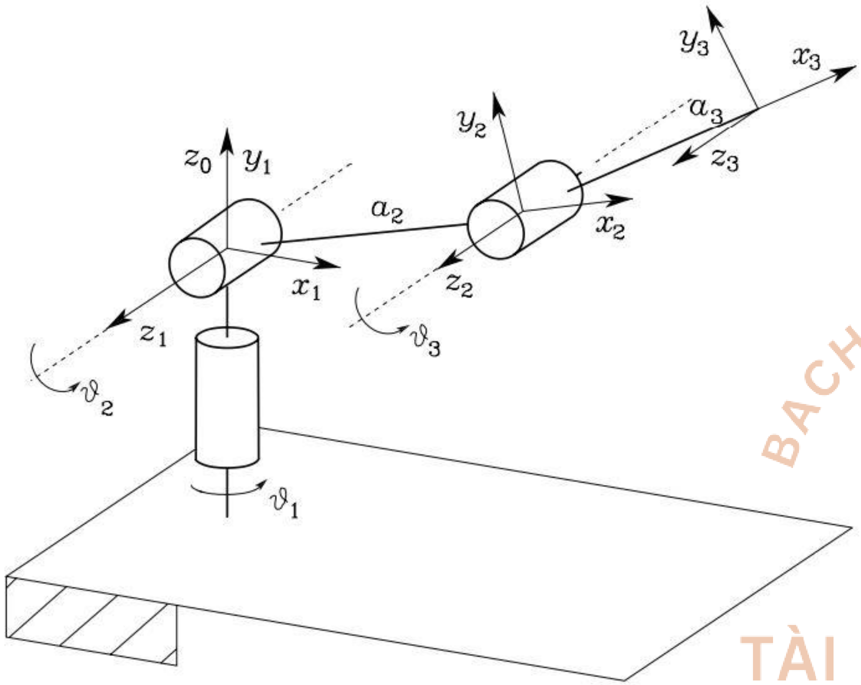
$$\Rightarrow p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3$$

$$\Rightarrow c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$\boxed{\Rightarrow \theta_3 = ATAN2(s_3, c_3)}$$

$$\left(Condition: |a_2 - a_3| \le \sqrt{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2} \le a_2 + a_3\right)$$

5.2 INVERSE KINEMATICS SOLUTIONS: ARTICALUATED ARM



From Direct Kinematics:

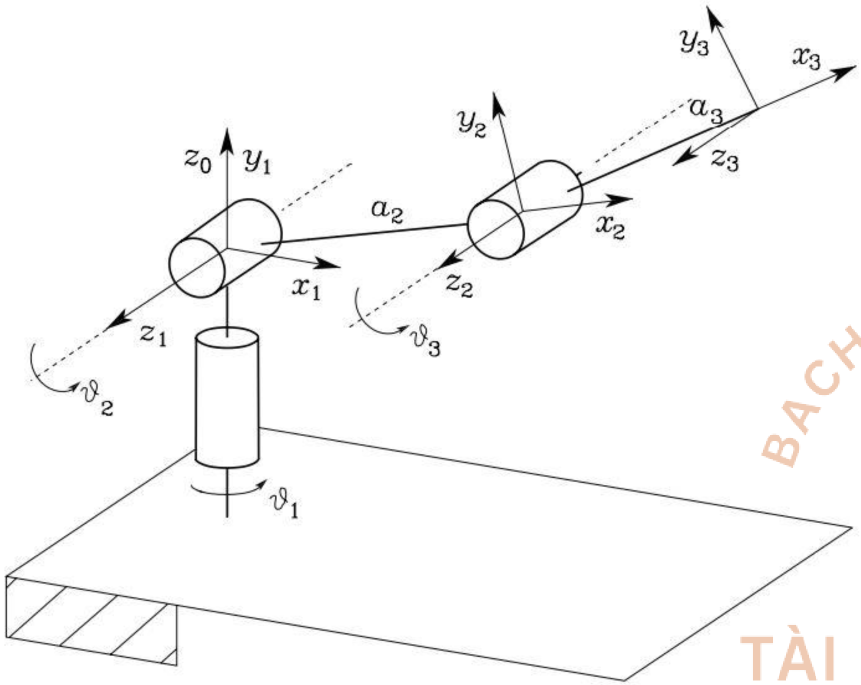
$$p_{Wx}^2 + p_{Wy}^2 = \left(a_2 c_2 + a_3 c_{23}\right)^2$$

$$\Rightarrow a_2 c_2 + a_3 c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}$$

$$\Rightarrow \begin{cases} c_2 = \dfrac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}\left(a_2 + a_3 c_3\right) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \\ s_2 = \dfrac{\mp\sqrt{p_{Wx}^2 + p_{Wy}^2}\, a_3 s_3 + p_{Wz}\left(a_2 + a_3 c_3\right)}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \end{cases}$$

$$\boxed{\Rightarrow \theta_2 = ATAN2(s_2, c_2)}$$

5.2 INVERSE KINEMATICS SOLUTIONS: ARTICALUATED ARM



From Direct Kinematics:

$$p_{Wx} = \pm c_1 \sqrt{p_{Wx}^2 + p_{Wy}^2}$$

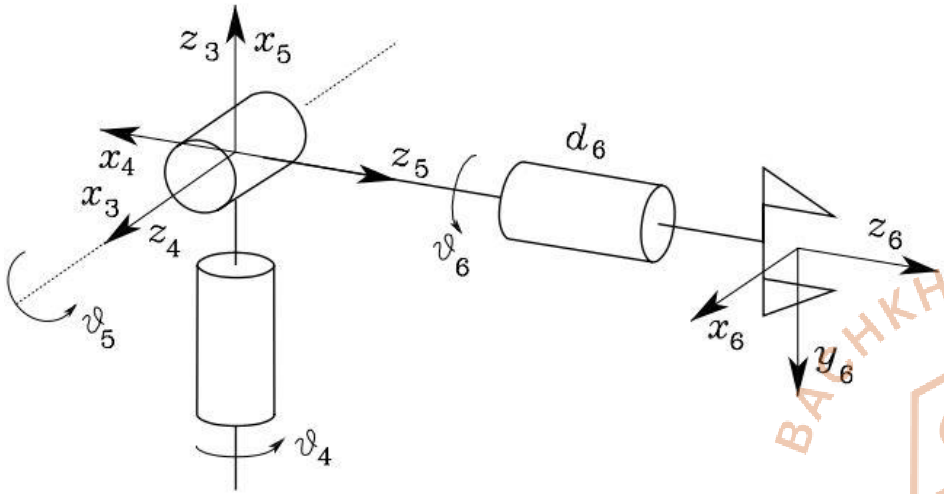
$$p_{Wy} = \pm s_1 \sqrt{p_{Wx}^2 + p_{Wy}^2}$$

$$\Rightarrow \theta_1 = ATAN2\left(p_{Wy}, p_{Wx}\right)$$

$$or\ \theta_1 = ATAN2\left(-p_{Wy}, -p_{Wx}\right)$$

- There exist four solutions according to the values of $\theta_1, \theta_2, \theta_3$
- It is possible to find the solutions only if at least $p_{Wx} \neq 0\ or\ p_{Wy} \neq 0$
- In case $p_{Wx} = p_{Wy} = 0$, an infinity of solutions is obtained => the arm in such configuration is kinematically singular

5.2 INVERSE KINEMATICS SOLUTIONS: SPHERICAL WRIST



Problem: find the joint variables θ_4, θ_5, θ_6 corresponding to a given end-effector orientation ${}^3T_6$

$$ {}^3T_6 = \begin{bmatrix} {}^3n_x & {}^3s_x & {}^3a_x \\ {}^3n_y & {}^3s_y & {}^3a_y \\ {}^3n_z & {}^3s_z & {}^3a_z \end{bmatrix} $$

Link	a_i	α_i	d_i	θ_i
4	0	$-\pi/2$	0	θ_4
5	0	$\pi/2$	0	θ_5
6	0	0	d_6	θ_6

$$ {}^3T_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \Rightarrow \begin{cases} \theta_4 = ATAN2\left({}^3a_y, {}^3a_x\right) \\ \theta_5 = ATAN2\left(\sqrt{\left({}^3a_x\right)^2 + \left({}^3a_y\right)^2}, {}^3a_z\right), \text{ for } \theta_5 \in (0,\pi) \\ \theta_6 = ATAN2\left({}^3s_z, -{}^3n_z\right) \end{cases} $$

$$ \textit{and} \begin{cases} \theta_4 = ATAN2\left(-{}^3a_y, -{}^3a_x\right) \\ \theta_5 = ATAN2\left(-\sqrt{\left({}^3a_x\right)^2 + \left({}^3a_y\right)^2}, {}^3a_z\right), \text{ for } \theta_5 \in (-\pi,0) \\ \theta_6 = ATAN2\left(-{}^3s_z, {}^3n_z\right) \end{cases} $$