Chapter 6

Impedance Matching Networks

6.1 Impedance Matching for Maximum Power Transfer

In this section we review the motivation for impedance matching and introduce important concepts which will be used in later chapters. Let us first illustrate the basic principles of impedance matching for maximum power transfer. In Figure 6.1 a source with impedance $Z_S = R_S + j\,X_S$ is connected to a load $Z_L = R_L + j\,X_L$. The peak voltage of the source (assumed to be sinusoidal) is V_S:

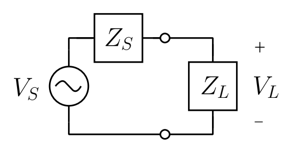


Figure 6.1: A voltage source driving an arbitrary load

The time-averaged real power delivered to the load can be written as

$$P_L = \frac{1}{2}\mathrm{Re}\{V_L I_L^*\} \tag{6.1}$$

where V_L and I_L are the voltage across and current through the load impedance, respectively. Note that the factor $\frac{1}{2}$ is present because V_L and I_L are "peak" phasors,i.e., phasors whose magnitude is the peak value of the sinusoidal time function. The current through the load is

$$I_L = \frac{V_L}{Z_L} \tag{6.2}$$

so the time-averaged real power is

$$\begin{aligned} P_L &= \frac{1}{2}|V_L|^2 Re\{\frac{1}{{Z_L}^*}\} \qquad (6.3) \\ &= \frac{1}{2}|V_L|^2 \frac{R_L}{|Z_L|^2} \end{aligned}$$

This can be written in terms of the source voltage V_S as follows:

$$\begin{aligned} P_L &= \frac{1}{2}|V_S|^2 \frac{R_L}{|Z_L + Z_S|^2} \\ &= \frac{1}{2}|V_S|^2 \frac{R_L}{(R_L + R_S)^2 + (X_L + X_S)^2} \end{aligned} \tag{6.4}$$

By studying Equation 6.4 we can determine what conditions are required to maximize the power delivered to the load. First we will assume that the source voltage and source impedance are set at the outset, i.e., they are not under our control. This will usually be the case. The problem is then to choose R_L and X_L to maximize P_L. We will restrict the solution to values of R_L that are greater than or equal to 0 because values of R_L that are < 0 would require that the load contain an active device, whereas we are considering only passive load terminations. The solution to this simple exercise is

$$Z_L = Z_S^* \tag{6.5}$$

This result states that the time-average real power delivered to a load is maximized when the load impedance is equal to the complex conjugate of the source impedance. Using Equation 6.5 in Equation 6.4, we obtain an expression for the maximum power that the source can deliver to an external passive load. This power is referred to as the power available from the source, P_{avs}:

$$\begin{aligned} P_{avs} &\equiv P_L|_{Z_L = Z_S^*} \\ &= \frac{|V_S|^2}{8R_S} \end{aligned} \tag{6.6}$$

The concept of *available power* is often used in the radio frequency literature. For example, the output power level of RF signal generators is usually specified by stating the available power in *dBm*, i.e., "decibels referred to 1 mW." A power level of 6 dBm indicates a power 6 dB higher than 1 mW, or approximately 4 mW. It is important to realize that when a signal generator is configured for a given output power level, that power level is the power available from the generator, P_{avs}, which is the power that the generator will deliver to a conjugately matched load. Signal generators commonly have source impedances of $50\,\Omega$ (sometimes $75\,\Omega$), i.e., the generator will deliver the rated power only to a $50\,\Omega$ load. With a different load impedance the power delivered to the load will be less than the rated value.

6.1.1 Mismatch Factor

The degree to which the actual power delivered to an arbitrary load is smaller than the available power can be quantified in terms of a *mismatch factor*, MF, a quantity that depends on the degree of impedance mismatch between the source and load. For an arbitrary load impedance Z_L the mismatch factor is defined as the ratio of actual delivered power to available power:

$$
\begin{aligned}
MF &= \frac{P_L}{P_{avs}} \\
&= \frac{4R_S R_L}{(R_S+R_L)^2+(X_S+X_L)^2} \qquad (6.7) \\
&= \frac{4R_S R_L}{|Z_S+Z_L|^2}
\end{aligned}
$$

Mismatch factor is a real number and $0 \leq MF \leq 1$. For given source and load impedances, the mismatch factor tells us what fraction of the available power will be delivered to the load. Since the mismatch factor is a ratio of two power levels, it makes sense to express the quantity in decibels. The *mismatch loss*, ML, in decibels is defined as

$$ML = -10 \log MF. \qquad (6.8)$$

6.1.1.1 Example - Mismatch Factor and Mismatch Loss

Suppose a $50\,\Omega$ signal generator has available power of 1 mW. The generator is to drive a load impedance of $250 + j100\,\Omega$. What is the power delivered to the load?

The problem could be solved by computing the power delivered to the load using Equation 6.1. Another approach is to compute the mismatch factor from Equation 6.7. This gives a mismatch factor $MF = 0.5$, so $P_L = P_{avs}MF = (1\,\mathrm{mW})0.5 = 0.5\,\mathrm{mW}$. Alternatively, we can express the powers in dBm and use the mismatch loss in dB. The mismatch loss $ML = -10\log(0.5) = 3\,\mathrm{dB}$. The power available from the source is 1 mW, or $10\log\frac{1\,\mathrm{mW}}{1\,\mathrm{mW}} = 0\,\mathrm{dBm}$. The power delivered to the load is $P_L = P_{avs} - ML = 0\,\mathrm{dBm} - 3\,\mathrm{dB} = -3\,\mathrm{dBm}$. Note that a power level of $-3\,\mathrm{dBm}$ is equal to 0.5 mW.

6.1.2 Properties of Lossless Impedance Matching Networks

We have illustrated how impedance matching (or mis-matching) influences the transfer of power from a source to a load. In many applications it is desirable to maximize the transfer of power from the source to the load. This can be achieved by using a lossless 2-port network inserted between the source and the load. The purpose of the *matching network* is to transform the load impedance, Z_L, into Z_S^* at the input terminals, thereby permitting the source to deliver all of its available power to the network. If a matching network is lossless, then all of the power that is delivered to the network must be delivered to the load. This simple concept has implications that may not be obvious at first glance.

Consider a source and load connected through a lossless matching network as shown in Figure 6.2. As already noted, the matching network transforms the load impedance Z_L into Z_S^* at the input terminals of the network. Since the source looks into the conjugate of its impedance, it delivers all of its available power P_{avs} to the network. Because no power is dissipated in a lossless network, all of this power is delivered to the load. Now the power available at the output of the lossless matching network must be the same as the power available from the source, and we have already stated that all of this available power is delivered to the load. Therefore the load must be conjugately matched to the output of the matching network, which means that the impedance at the output of the network (as seen by the load) is Z_L^*. Thus a lossless matching network has the property of providing a simultaneous conjugate match at the input and output ports.

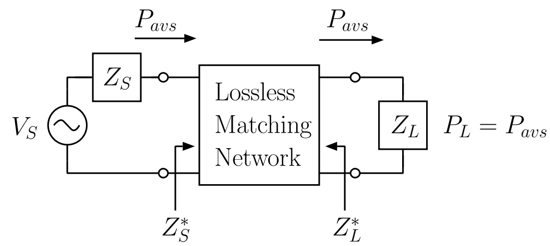


Figure 6.2: Source and load coupled by a lossless matching network

Another way to look at this is to note that the original source plus the matching network can be viewed as a new source with available power P_{avs} and source impedance Z_L^*. Adding a lossless network to the output of a source does not change the available power from the source; it only changes the source impedance. Thus, we can think of the matching network as (i) a network that transforms the load impedance into Z_S^* or (ii) a network that transforms the source impedance into Z_L^* without changing the available power of the original source. The network can be designed either way.

Most lumped-element matching networks are versions of ladder networks as shown in Figure 6.3. The ladder-type matching network in Figure 6.3 is assumed to be lossless; the

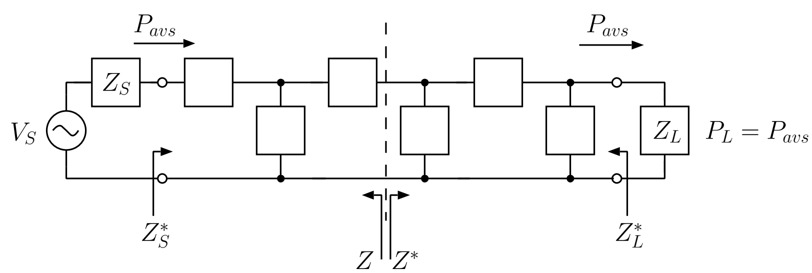


Figure 6.3: Ladder-type matching network

blank boxes represent purely reactive circuit elements. Now suppose that the network is "broken" at the dashed line. The part of the network to the left of the line can be thought of as a source. It will have the same available power as that of the original source, P_{avs}, and some source impedance, Z. The part of the network to the right of the dashed line is a load for this source and must receive all of the available power (since that power is ultimately transferred to Z_L). Thus, a conjugate match must exist at the junction defined by the dashed line. Clearly, the dashed line could have been drawn anywhere within the lossless network and the same argument would hold. **The conclusion is: A circuit consisting of a source connected to a load through a lossless matching network can be broken at any point between the source and the load and a conjugate match must exist between the two sides of the circuit**.

6.2 Impedance Matching with Lossless L-networks

6.2.1 Resistive Terminations

Figure 6.4 shows two resistances to be matched with a lossless L-network. The goal is to transform R_1 to R_2 at one frequency. The unknown reactances X_s and X_p are easily found with the use of a parallel-to-series transformation as in Figure 6.5.

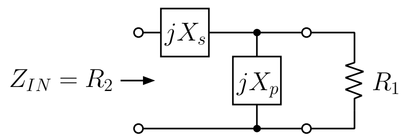


Figure 6.4: L-network which transforms R_1 into R_2

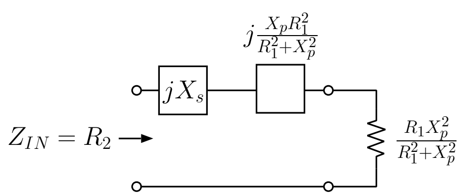


Figure 6.5: Parallel-to-series transformation

To make the input impedance equal to R_2, we choose

$$X_s = -\frac{X_p R_1^2}{R_1^2 + X_p^2} \tag{6.9}$$

$$R_2 = \frac{R_1 X_p^2}{R_1^2 + X_p^2}$$

Solving for X_p and X_s gives

$$X_p = \pm R_1 \sqrt{\frac{R_2}{R_1 - R_2}} \tag{6.10}$$

$$X_s = \mp\sqrt{R_2 R_1 - R_2^2} \tag{6.11}$$

The solutions yield real values for X_p and X_s (and hence purely reactive L-network components) only if $R_1 > R_2$. This leads to a rule for using a lossless L-network to match two resistances:

The shunt arm of the L-network is connected across the larger of the two resistances.

Also note that there are two possible solutions for the resistive matching problem corresponding to the upper and lower signs in Equation 6.10 and Equation 6.11. These solutions are referred to as "low-pass" and "high-pass" solutions. The circuit configurations for the two solutions are in shown Figures 6.6 and 6.7.

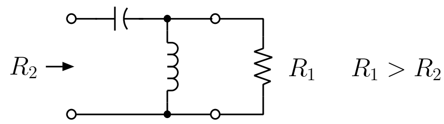


Figure 6.6: High-pass L-network

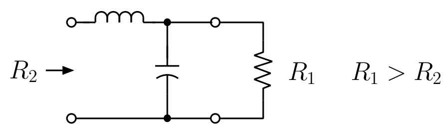


Figure 6.7: Low-pass L-network

The high-pass and low-pass designations refer to the behavior of the network transfer function at frequencies away from the design frequency.

6.2.2 "Q" of an L-network

Referring to Figure 6.5, it is apparent that the transformed L-network looks like a series-resonant circuit. At the design frequency the two reactances have equal magnitudes and opposite signs, so that the net series reactance is zero. Treating this circuit in the same manner as a resonant RLC network, the Q of the network is the magnitude of one of the series reactances divided by the series resistance:

$$Q = \left| \frac{X_p R_1^2/(R_1^2 + X_p^2)}{R_1 X_p^2/(R_1^2 + X_p^{2)}} \right| \tag{6.12}$$

$$\tag{6.13}$$

$$= \left| \frac{R_1}{X_p} \right| \tag{6.14}$$

$$\tag{6.15}$$

$$= \sqrt{\frac{R_1}{R_2} - 1} \tag{6.16}$$

This result shows that the Q of the L-network is determined by the terminating resistances R_1 and R_2, and, therefore, cannot be chosen independently. This suggests that the bandwidth of the matching network depends only on the ratio R_1/R_2.

A word of caution is in order. We have defined the Q of the L-network by analogy with the series RLC network. This analogy works well only when $R_1 \gg |X_p|$ or, equivalently, when $Q \gg 1$. In such cases the Q can be used to predict the 3 dB bandwidth of the network's voltage transfer function. For moderate or small values of Q the expression

$$Q = \frac{R_1}{|X_p|} = \sqrt{\frac{R_1}{R_2} - 1}$$

is still valid, but a simple relationship between Q and bandwidth does not exist, since the series reactance that results from the parallel-to-series transformation has a different frequency dependence from that of a simple inductor or capacitor. Thus an L-network does not behave exactly like a series RLC circuit. Only when the Q is very large will the equivalent series reactance behave approximately like a capacitor or inductor.

6.2.3 Summary: L-network design equations

The design of an L-network can be summarized as follows. If the terminating resistances are denoted by R_{big} and R_{small} (where $R_{big} > R_{small}$), then the design equations can be written in terms of the network Q,

$$Q = \sqrt{\frac{R_{big}}{R_{small}} - 1}, \tag{6.17}$$

as

$$X_p = \pm R_{big}/Q \qquad X_s = \mp R_{small} Q. \tag{6.18}$$

When the upper signs are chosen, the resulting network is of highpass type. The lower signs give a lowpass network. The L-network will be oriented such that the parallel arm is in shunt with R_{big}.

6.2.3.1 Example - Matching resistive source and load with a low-pass L-network

Match a 100 Ω source to a 25 Ω load with a lossless L-network having a low-pass topology. Since the series arm of the L-network connects to the smaller of the two resistances, the L-

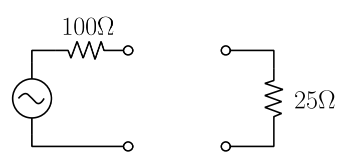


Figure 6.8: Example for L-net matching

network will be oriented as shown in Figure 6.9. A capacitor and inductor have been chosen for the shunt and series elements, respectively, because the problem statement specified a low-pass network.

$$\begin{aligned}
Q &= \sqrt{\tfrac{R_{big}}{R_{small}} - 1} &&= \sqrt{\tfrac{100}{25} - 1} &&= \sqrt{3} \\
X_p &= X_C &&= -\tfrac{R_{big}}{Q} &&= -\tfrac{100}{\sqrt{3}} &&= -57.7\,\Omega \\
X_s &= X_L &&= Q R_{small} &&= \sqrt{3}\,25 &&= 43.3\,\Omega
\end{aligned}$$

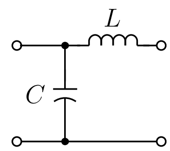


Figure 6.9: L-network orientation

The completed network is shown in Figure 6.10. It is interesting to compare the power

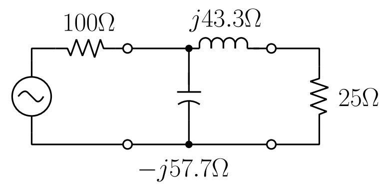


Figure 6.10: Completed lossless L-network with low-pass topology

delivered to the load with and without the matching network in place. Suppose the peak voltage of the source is 1 Volt. Then the power delivered to the load without the matching network would be

$$P_{out} = \frac{1}{2}\frac{|V_{out}|^2}{25} = \frac{1}{50}\left(\frac{1}{5}\right)^2 = 0.8\,\text{mW} \tag{6.19}$$

With the matching network the source "sees" 100 Ω; therefore, the power delivered to the matching network is

$$P_{in} = \frac{1}{2}\frac{(1/2)^2}{100} = \frac{1}{800} = 1.25\,\text{mW} \tag{6.20}$$

Since the matching network is lossless, all of this power will be delivered to the load. Thus, with the matching network, $P_{out} = 1.25\,\text{mW}$. The improvement gained is

$$10\log(1.25/.8) \simeq \underline{2\ \text{dB}} \tag{6.21}$$

6.2.4 Matching Complex Loads with a Lossless L-network

So far, only purely resistive source and load terminations have been considered. When complex source and loads are involved, there are two basic conceptual approaches that can be used:

1. Absorption - "absorb" the source or load reactance into the matching network.

2. Resonance - series or parallel resonate the source or load reactance at the frequency of interest.

These approaches will be illustrated by example in the following sections.

6.2.4.1 Example - Absorption

Absorption will be illustrated with an example. Suppose that it is necessary to match the source and load shown in Figure 6.11 at 100 MHz with a lossless L-network.

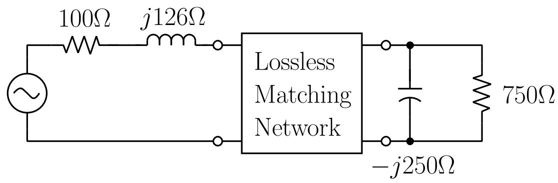


Figure 6.11: L-network with complex source and load

Absorption is applied by lumping the source and load reactances into the series and parallel reactances of the matching network, as shown in Figure 6.12. The lumped reactances

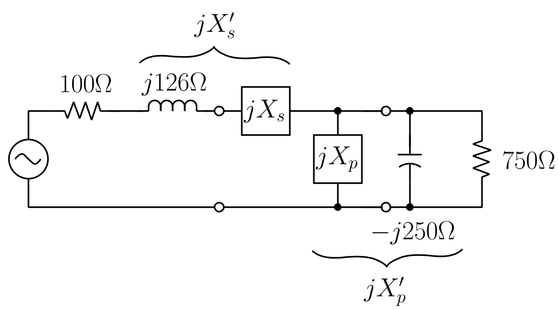


Figure 6.12: Absorbing the complex part of the load impedance into the network

X_s' and X_p' can be found by using the design equations for matching between two resistive terminations:

$$\begin{aligned} Q &= \sqrt{750/100 - 1} &&= 2.55 \\ X_s' &= \pm\,100\,(2.55) &&= \pm\,255 \\ X_p' &= \mp\,750/2.55 &&= \mp\,294.1 \end{aligned} \tag{6.22}$$

The lumped reactances can be written in terms of the reactances associated with the source and load, and the L-network reactances:

$$X_s' = X_s + 126, \tag{6.23}$$

$$X_p' = \frac{-250\,(X_p)}{-250 + X_p}. \tag{6.24}$$

Thus, values of X_s and X_p can be obtained:

$$\begin{aligned} X_s &= X_s' - 126 = \pm 255 - 126 = \begin{cases} 129\Omega \\ -381\Omega \end{cases} \\ X_p &= \frac{250X_p'}{250+X_p'} = \begin{cases} 1667.2\Omega \\ 135.1\Omega \end{cases} \end{aligned} \tag{6.25}$$

The two solutions found so far are shown in Figure 6.13.

Figure 6.13: Two L-network solutions.

The element values (in μH and pF) were obtained from the calculated reactances using the design frequency of 100 MHz. It should be noted that the two solutions found so far are not the only possibilities for this particular source and load. This becomes apparent if series-to-parallel transformations are applied to the source and load. After transformation, the source and load look like Figure 6.14.

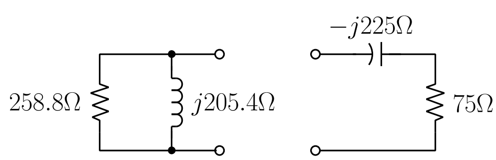


Figure 6.14: Source and load after application of series-to-parallel transformation

For simplicity, the Norton equivalent current source has been omitted from the source representation. Note that the smaller of the two resistances is now on the load side and hence it is possible to match the source and load using L-networks oriented as in Figure 6.15.

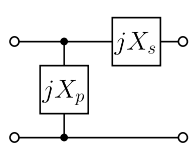


Figure 6.15: L-network reoriented for smaller resistance on load side

The values of X_s and X_p could be found using absorption. Alternatively, the resonance concept can be used. For illustration, the resonance concept will be used to find the remaining L-network solutions.

6.2.4.2 Example - Resonance

Continuing with the example from the previous section, the resonance concept will be employed to find two more L-network solutions for the source and load shown in Figure 6.11. After transforming the source from series to parallel form, and the load from parallel to series form, the source and load are represented as in Figure 6.14. To apply the resonance concept, the source and load are augmented with reactances that resonate with the source and load reactances as shown in Figure 6.16.

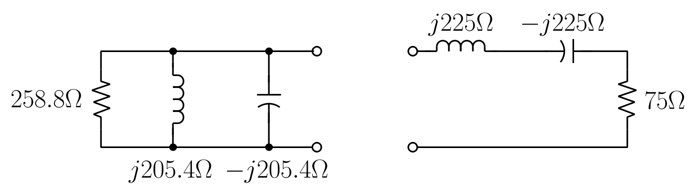


Figure 6.16: Transformed source and load augmented with resonating reactances

After resonating the source and load in this manner, the new source and load impedances are purely real (resistive) at the design frequency. An L-network is then designed to match these two resistances, i.e.,

$$\begin{aligned} Q &= \sqrt{\frac{258.76}{75} - 1} &&= 1.57 \\ X_s' &= \pm 75\,(1.57) &&= \pm 117.8 \\ X_p' &= \mp 258.76\,/1.57 &&= \mp 164.8 \end{aligned} \tag{6.26}$$

Now the circuit can be drawn as shown in Figure 6.17. To complete the design, the resonating

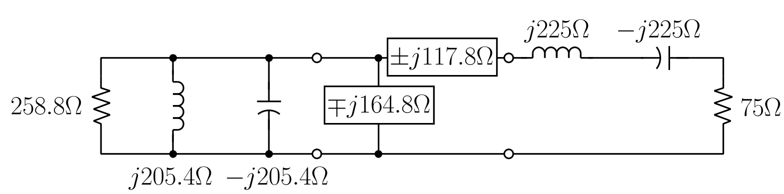


Figure 6.17: L-net with resonating reactances

reactances must be incorporated into the matching network. For example, when the upper signs are chosen, the net parallel reactance will be $-j205.4\Omega || -j164.8\Omega = -j91.4\Omega$. The net series reactance will be $j117.8\Omega + j225\Omega = j342.8\Omega$. The final solutions are shown in Figure 6.18.

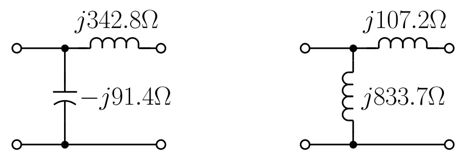


Figure 6.18: Resonating reactances incorporated into matching network

Thus, we have found four possible solutions that can be obtained using a lossless L-network.

The example considered here allowed 4 possible L-network solutions because transforming the source and load caused R_{big} and R_{small} to swap positions. This will not always be the case. Thus, for some source and load combinations, there will be only two L-network solutions, and in other cases there will be four solutions.

6.3 Harmonic Attenuation in Lossless Matching Networks Using Traps

In certain applications it is necessary to provide a match at some frequency ω_o and to attenuate one or more other frequencies. Typically the undesired frequencies are harmonics of ω_o, i.e., $2\omega_o$, $3\omega_o$, etc.; however, they could also be subharmonic or even unrelated to ω_o.

We will concentrate on the case where the undesired frequencies are harmonics. Generalization to other cases is straightforward. The basic idea is to first design a lossless matching network to provide a match at the desired frequency (ω_o). This matching network might consist of an L-, T-, or Pi-network. Next, one or more series elements of the network are replaced with parallel L-C-networks. Alternatively, or in addition, the shunt elements of the network are replaced with series L-C-networks. The replacement (series or parallel L-C) elements are designed to have the same reactance as the original elements at ω_o; however, they are designed to be resonant at the undesired harmonic frequency. As an example, consider Figure 6.19 which shows a matching network with a match at $f_o = 10^7/2\pi$ Hz.

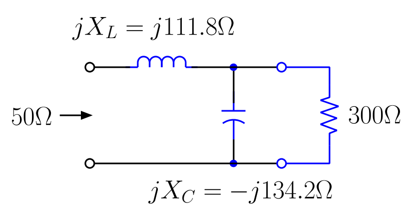


Figure 6.19: $L = 11.18\,\mu$H, $C = 745.2\,$pF

Suppose it is desired to "trap" the second harmonic $2f_o$ in the shunt arm. The shunt arm would be replaced with a series L-C that has reactance $-134.2\,\Omega$ at f_o and is series resonant (looks like a short circuit) at $2f_o$. The design equations are

$$2\,\omega_o = 1/\sqrt{LC} \tag{6.27}$$

$$\omega_o\,L - \frac{1}{\omega_o C} = -134.2 \tag{6.28}$$

The solution is

$$\begin{aligned} L &= 4.47\,\mu\text{H} \\ C &= 558.9\,\text{pF} \end{aligned} \tag{6.29}$$

The new network looks like Figure 6.20.

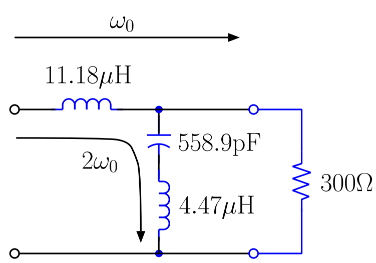


Figure 6.20: Network with “trapped” second harmonic

The series trap acts to shunt the component at frequency $2f_o$ to ground.

The third (or any other) harmonic could be trapped in the series arm. The inductor would be replaced with a parallel LC whose element values are found from

$$3\omega_o = \frac{1}{\sqrt{LC}} \tag{6.30}$$

$$-\frac{1}{\omega_o\,L} + \omega_o\,C = -\frac{1}{111.8} \tag{6.31}$$

The solution is

$$\begin{aligned} C &= 111.8\,\text{pF} \\ L &= 9.94\,\mu\text{H} \end{aligned} \tag{6.32}$$

Figure 6.21 shows the network with both traps installed. The network will look like an open circuit to the $3f_o$ component.

It should be noted that this approach will work only when applied to *capacitive* shunt elements and *inductive* series elements if frequencies *higher* than ω_o are to be trapped. Thus, the original network must have a low-pass topology. The student should convince himself or herself that such is the case. On the other hand, if frequencies smaller than ω_o are to be trapped, then the approach can be employed only with inductive shunt elements and capacitive series elements.

The trapping approach is also useful for reducing the feed-through of nonharmonic frequency components. If the frequencies of the undesired components are very close to the desired frequency, it may not be possible to build an effective trap, because the Q of the series or parallel L-C circuits will be too high to be realized.

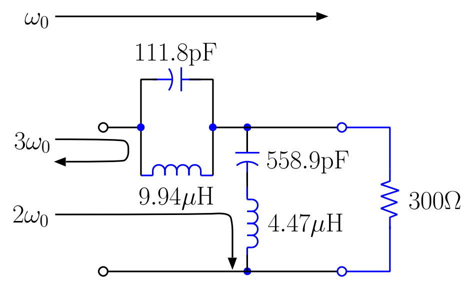


Figure 6.21: Network with second and third harmonics trapped

6.4 Three-element Matching Networks

The L-network does not give the designer freedom to choose the Q (bandwidth) or phase shift of the matching network. The addition of a third matching element makes it possible to design for a match and a specified phase shift or Q. The following section describes a procedure for designing 3 element matching networks with specified Q. Then, a general procedure allowing for specified attenuation and phase shift will be discussed.

6.4.1 Design of Pi- and T-networks for Specified Bandwidth (Q)

First consider the Pi-network with a low-pass topology. In addition, we assume that absorption has been used so that the shunt reactance of the source or load impedance is included in the Pi-network. The problem is then reduced to matching between two resistive terminations as shown in Figure 6.22 (where it is assumed that $R_2 > R_1$).

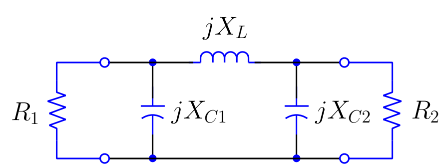


Figure 6.22: Matching two resistive terminations where $R_2 > R_1$

The Pi-network can be thought of as two back-to-back L-networks that act to match both R_1 and R_2 to a "virtual resistance" R_v as shown in Figure 6.23.

Because the series arms of both L-networks are connected to R_v, it is clear that R_v is smaller than R_1 and R_2. Define the Q's of the two L-networks to be Q_1 and Q_2 where

$$Q_1 = \sqrt{\frac{R_1}{R_v} - 1}, \quad \text{and} \quad Q_2 = \sqrt{\frac{R_2}{R_v} - 1} \tag{6.33}$$

We are assuming that $R_2 > R_1$, and therefore Q_2 will be larger than Q_1. For most practical purposes the Q of the Pi-network can be approximated by Q_2. This is especially true if $R_2 \gg R_1$. If R_2 is only slightly larger than R_1, then the overall Q of the network will be somewhat larger than Q_2. The design procedure follows:

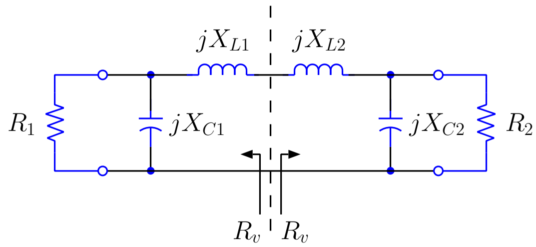


Figure 6.23: The Pi-network as 2 back-to-back L-networks

1. Determine the required Q of the matching network by considering the required bandwidth, BW, and center frequency, f_o ($Q = f_o/BW$). This Q is taken to be equal to Q_2, and thus the virtual resistance, R_v, is determined. **Note that R_v must be smaller than R_1, and therefore the Pi-network can only be used to obtain a larger Q than would have been provided by the simpler L-network**. Also note that the relationship $BW = f_o/Q$ is only exactly true for a simple parallel or series RLC circuit. Thus, the actual bandwidth of your circuit may be different from the specified design value. If a particular design requires that the bandwidth be precisely determined, it is a good idea to simulate the performance of the matching network using a computer-aided design program in order to verify that the performance will be satisfactory.

2. Once R_v is found, the values of X_{C1}, X_{L1}, X_{C2}, and X_{L2} can be calculated using the previously derived formulas for L-network matching.

The design procedure can be summarized by Equations 6.34, where Q_2 is determined by the desired bandwidth. You should verify that these equations result from steps (1) and (2):

$$X_{C2} = -\frac{R_2}{Q_2} \tag{6.34}$$

$$X_{C1} = -\sqrt{\frac{R_1 R_2}{(Q_2^2+1) - \frac{R_2}{R_1}}} \tag{6.35}$$

$$\tag{6.36}$$

$$\tag{6.37}$$

$$X_L = \frac{R_2 Q_2 + R_2\sqrt{\frac{R_1}{R_2}(Q_2^2+1) - 1}}{Q_2^2+1} \tag{6.38}$$

The Pi-network is most useful for matching when the values of R_1 and R_2 are not too small. If R_1 and R_2 are small, the virtual resistance will be even smaller, and the capacitor values will turn out to be impractically large. If either terminating resistance is significantly less than 50 Ω, the T-network will usually be a more practical choice. One possible T-network is the band-pass case shown in Figure 6.24.

As before, we can think of this network as two back-to-back L-networks as shown in Figure 6.25.

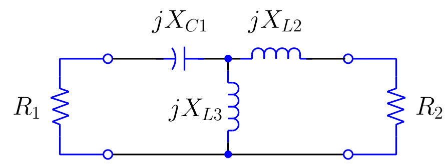


Figure 6.24: Band-pass T-network

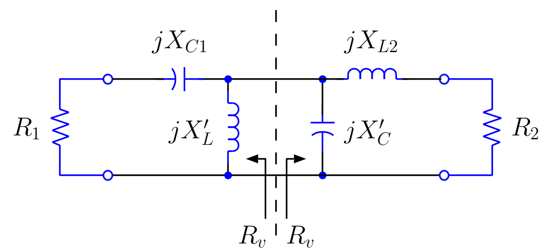


Figure 6.25: Band-pass T-network as 2 back-to-back L-networks

The Q's of the two L-networks are

$$Q_1 = \sqrt{\frac{R_v}{R_1} - 1} \quad \text{and} \quad Q_2 = \sqrt{\frac{R_v}{R_2} - 1} \tag{6.39}$$

Note that since $R_v > R_1$ and $R_v > R_2$, this network will have a larger Q than a single L-network that matches R_1 to R_2. The overall Q of the network is set by Q_1 since we assume $R_2 > R_1$. The design formulas are

$$X_{C1} = -R_1 Q_1 \tag{6.40}$$

$$X_{L2} = R_2 \sqrt{\frac{R_1}{R_2}(Q_1^2 + 1) - 1} \tag{6.41}$$

$$\tag{6.42}$$

$$X_{L3} = \frac{R_1(Q_1^2 + 1)}{Q_1 - \sqrt{\frac{R_1}{R_2}(Q_1^2 + 1) - 1}} \tag{6.43}$$

Note carefully that the two elements X'_L and X'_C that appear in the back-to-back L-networks are combined into a single element, X_{L3}, in the T-network. In practice this is not necessary and, in fact, may not be desirable. Whether or not the elements are combined makes no difference at the design frequency, but it may make a significant difference at frequencies well removed from the design frequency. The different possibilities are best examined using a CAD program.

You should be able to derive similar formulas for the other possible topologies, e.g., band-pass Pi, low-pass T, etc.

6.4.2 Matching Two Resistive Terminations with Specified Attenuation and Phase Shift

In this section we will consider a more general type of matching network than was considered in section 6.2. Specifically, we allow the matching element impedances to be complex in the initial development of our solution. This will allow for a solutions with specified attenuation and phase shift. Then we will specialize to the cases where the elements are purely reactive (lossless networks) and purely resistive (lossy networks or attenuators). Note that the terminating impedances are assumed to be real (resistive).

6.4.2.1 Pi-network with specified attenuation and phase shift

Referring to Figure 6.26 we make the following assumptions:

Y_1, Y_2 are assumed to be <u>real</u>.

Y_A, Y_B, Y_C may be complex.

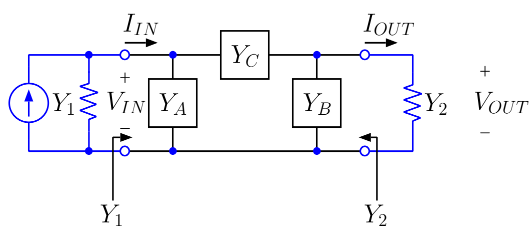


Figure 6.26: Matching two resistive terminations with a Pi-network

The conditions required for a match can be summarized as follows:

1. Must see Y_1 looking in from the left when terminated with Y_2 on the right. Thus,

$$Y_1 = Y_A + \frac{(Y_2 + Y_B)Y_C}{Y_2 + Y_B + Y_C} \tag{6.44}$$

2. Must see Y_2 looking in from the right when terminated with Y_1 on the left. Thus,

$$Y_2 = Y_B + \frac{(Y_1 + Y_A)Y_C}{Y_1 + Y_A + Y_C} \tag{6.45}$$

So far, we have 2 equations but 3 unknowns. The third equation can be used to specify either the Q of the network (and therefore, the bandwidth) or the phase shift and attenuation. For now, we consider the latter. The third equation can be written in the form:

$$e^{\alpha + j\beta} = \sqrt{\frac{V_{IN}\, I_{IN}}{V_{OUT}\, I_{OUT}}} \tag{6.46}$$

Equation 6.46 can be interpreted as follows. Since Y_1 and Y_2 are assumed to be real, the input voltage and current will be in phase and so will the output voltage and current.

Denoting the phase of the input voltage (current) by θ_{IN} and the phase of the output voltage (current) by θ_{OUT}

$$\begin{aligned} e^{\alpha+j\beta} &= \sqrt{\frac{|V_{IN}|\,|I_{IN}|}{|V_{OUT}|\,|I_{OUT}|}}\; e^{j(\theta_{IN}-\theta_{OUT})} \qquad (6.47) \\ &= \sqrt{\frac{P_{IN}}{P_{OUT}}}\; e^{j(\theta_{IN}-\theta_{OUT})} \end{aligned}$$

Thus

$$\beta = \theta_{IN} - \theta_{OUT} \qquad (6.48)$$

and

$$\alpha = \frac{1}{2} \ln \frac{P_{IN}}{P_{OUT}} \qquad (6.49)$$

Therefore, β is the phase shift of the network and α is the power attenuation expressed in nepers.

Equations 6.44, 6.45 and 6.46 must be solved for the unknowns Y_A, Y_B and Y_C. Note that Equation 6.46 can be written in terms of the unknowns as shown below. Define $\theta = \alpha + j\beta$, then

$$\begin{aligned} e^{\theta} &= \sqrt{\frac{V_{IN}\,I_{IN}}{V_{OUT}\,I_{OUT}}} = \frac{V_{IN}}{V_{OUT}}\sqrt{\frac{I_{IN}/V_{IN}}{I_{OUT/V_{OUT}}}} \qquad (6.50) \\ & \qquad (6.51) \\ &= \frac{V_{IN}}{V_{OUT}}\sqrt{\frac{Y_1}{Y_2}} \qquad (6.52) \\ & \qquad (6.53) \\ e^{\theta} &= (1 + Y_C^{-1}(Y_2 + Y_B))\sqrt{\frac{Y_1}{Y_2}} \qquad (6.54) \end{aligned}$$

Equations 6.44, 6.45 and 6.46 can now be solved. The result is

$$\begin{aligned} Y_C &= \frac{\sqrt{Y_1 Y_2}}{\sinh\theta} \qquad (6.55) \\ Y_B &= \frac{Y_2}{\tanh\theta} - Y_C \qquad (6.56) \\ Y_A &= \frac{Y_1}{\tanh\theta} - Y_C \qquad (6.57) \end{aligned}$$

6.4.2.2 T-network with specified attenuation and phase shift

Similar considerations apply to using a T-network to match two resistive terminations as shown in Figure 6.27.

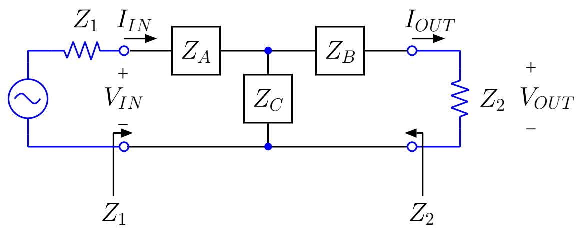


Figure 6.27: Matching two resistive terminations with a T-network

The equations that ensure a simultaneous conjugate match at both ports are

$$Z_1 = Z_A + \frac{Z_C(Z_B + Z_2)}{Z_C + Z_B + Z_2} \tag{6.58}$$

$$Z_2 = Z_B + \frac{Z_C(Z_A + Z_1)}{Z_C + Z_A + Z_1} \tag{6.59}$$

The third equation is the same as before,

$$e^{\alpha + j\beta} = e^{\theta} = \sqrt{\frac{V_{IN}\, I_{IN}}{V_{OUT}\, I_{OUT}}} \tag{6.60}$$

which can be written

$$e^{\theta} = \frac{I_{IN}}{I_{OUT}}\sqrt{\frac{Z_1}{Z_2}} \tag{6.61}$$

The solutions for Z_A, Z_B and Z_C are

$$Z_C = \frac{\sqrt{Z_1\, Z_2}}{\sinh\theta} \tag{6.62}$$

$$Z_B = \frac{Z_2}{\tanh\theta} - Z_C \tag{6.63}$$

$$Z_A = \frac{Z_1}{\tanh\theta} - Z_C \tag{6.64}$$

The interpretation of $\theta = \alpha + j\beta$ is the same for the T-network as it was for the Pi-network, that is,

$$\begin{aligned} \beta &= \theta_{IN} - \theta_{OUT} \quad \Rightarrow \quad \text{voltage (current) phase shift in radians} \\ \alpha &= \tfrac{1}{2}\ln\tfrac{P_{IN}}{P_{OUT}} \quad \Rightarrow \quad \text{power attenuation in nepers} \end{aligned} \tag{6.65}$$

Remember that these discussions have assumed that Y_1 and Y_2 (or Z_1 and Z_2) are real (resistive). Complex loads are handled by incorporating the reactive part of the termination into the matching network using either resonance or absorption as discussed in section 6.2.

6.4.3 Design of Lossless Pi- and T- Matching Networks with Specified Phase Shift

The solutions found so far allow the designer to specify both phase shift and attenuation of the network. In practice one is usually concerned with one of the special cases: (i) Y_A, Y_B, Y_C or Z_A, Z_B, Z_C are purely reactive — this corresponds to the lossless matching network; (ii) Y_A, Y_B, Y_C orZ_A, Z_B, Z_C are purely resistive — this corresponds to a lossy network. Networks of type (ii) are often used to provide specified amounts of attenuation and/or isolation between circuits. In this section we will consider the lossless matching networks.

When Y_A, Y_B, Y_C or Z_A, Z_B, Z_C are purely reactive, then the network is lossless and $\alpha = 0$. Hence $\theta = j\beta$ and the design equations reduce to

Pi:

$$Y_C = \frac{\sqrt{Y_1 Y_2}}{j \sin\beta} \tag{6.66}$$

$$Y_B = \frac{Y_2}{j \tan\beta} - Y_C \tag{6.67}$$

$$Y_A = \frac{Y_1}{j \tan\beta} - Y_C \tag{6.68}$$

T:

$$Z_C = \frac{\sqrt{Z_1 Z_2}}{j \sin\beta} \tag{6.69}$$

$$Z_A = \frac{Z_1}{j \tan\beta} - Z_C \tag{6.70}$$

$$Z_B = \frac{Z_2}{j \tan\beta} - Z_C \tag{6.71}$$

A word of caution on interpreting β when source and/or load admittances are complex. We have seen that when the source and load are resistive, β can be interpreted as the voltage or current phase shift, since voltage and current are in phase at both the input and output. If the source and/or load are complex, then the voltage and current are not in phase. In this situation β has a different interpretation for the Pi- and T-networks. This can be seen by considering Figure 6.28.

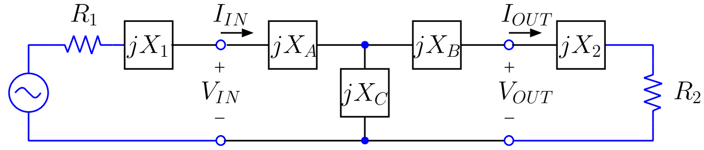


Figure 6.28: T-network with complex source and load

The network would be designed by incorporating X_1 and X_2 into the matching network as shown in Figure 6.29.

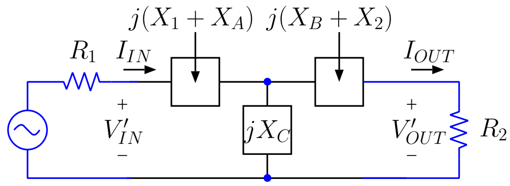


Figure 6.29: Network designed with X_1 and X_2 incorporated into the matching network

The currents I_{IN} and I_{OUT} in Figure 6.29 are the same as those shown in Figure 6.28, but the voltages are not. In fact, V'_{IN} and V'_{OUT} are related to the V_{IN} and V_{OUT} of Figure 6.28 by

$$V'_{OUT} = V_{OUT}\,\frac{R_2}{R_2 + j\,X_2} \tag{6.72}$$

$$V_{IN} = V'_{IN}\,\frac{R_1 - jX_1}{R_1} \tag{6.73}$$

Now, β is the phase shift between I_{IN} and I_{OUT} or, equivalently, V'_{IN} and V'_{OUT}. It is not the phase shift between V_{IN} and V_{OUT}, however. The voltage phase shift $(\theta_{V_{IN}} - \theta_{V_{OUT}})$ can be determined as follows:

$$\frac{V_{IN}}{V_{OUT}} = \frac{V'_{IN}}{V'_{OUT}}\,\frac{R_1 - jX_1}{R_2 + jX_2}\,\frac{R_2}{R_1} \tag{6.74}$$

Define

$$\theta_{Z_1} = \tan^{-1}\frac{X_1}{R_1} \tag{6.75}$$

$$\theta_{Z_2} = \tan^{-1}\frac{X_2}{R_2} \tag{6.76}$$

Then

$$\begin{aligned}\theta_{V_{IN}} - \theta_{V_{OUT}} &= \theta'_{V_{IN}} - \theta'_{V_{OUT}} - \theta_{Z_1} - \theta_{Z_2} \\ &= \beta - \theta_{Z_1} - \theta_{Z_2}\end{aligned} \tag{6.77}$$

Thus for the lossless T-network β gives the current phase shift, and the voltage phase shift can be found from Equation 6.77.

Similar considerations lead to the conclusion that β gives the voltage phase shift for the lossless Pi-network. In this case Equation 6.78 yields the current phase shift:

$$\theta_{I_{IN}} - \theta_{I_{OUT}} = \beta - \theta_{Y_1} - \theta_{Y_2} \tag{6.78}$$

6.4.3.1 Example - Design of a lossless Pi-network with specified phase shift

Design a lossless Pi-network to match a load of 300 Ω to a 50 Ω source with V_{OUT} leading V_{IN} by 45° at $\omega = 10^7$ rad/s as in Figure 6.30. We want $\theta_{V_{OUT}} - \theta_{V_{IN}} = 45°$. Hence

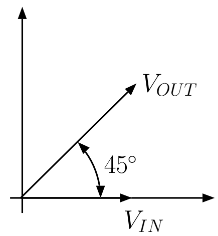


Figure 6.30: Phasor diagram showing V_{OUT} leading V_{IN} by 45°.

$\beta = -45°$

$$\begin{aligned}
Y_1 &= \frac{1}{50} &&= 20\,\mathrm{mS} \\
Y_2 &= \frac{1}{300} &&= 3.3\,\mathrm{mS} \\
Y_C &= \frac{\sqrt{20(3.33)}}{j\ \sin(-\pi/4)} &&= +j\,11.54\,\mathrm{mS} \\
Y_B &= \frac{3.33\,\mathrm{mS}}{j\ \tan(-\pi/4)} - j1154\,\mathrm{mS} &&= -j\,8.21\,\mathrm{mS} \\
Y_A &= \frac{20\,\mathrm{mS}}{j\ \tan(-\pi/4)} - j1154\,\mathrm{mS} &&= +j\,8.46\,\mathrm{mS}
\end{aligned} \tag{6.79}$$

The final solution is shown in Figure 6.31. The network will have a band-pass transfer

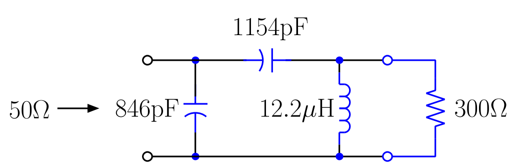


Figure 6.31: Pi-network solution

function characteristic, as the series capacitance and shunt inductor guarantee that the network transfer function will have zero response at DC, whereas the shunt capacitor will provide high frequency attenuation which increases at 6 dB per octave for frequencies well above the design frequency.

Complex source or load impedances can be handled by incorporating the load reactances into the network, as illustrated in the following example.

6.4.3.2 Example - Matching complex load with a specified current phase shift.

At $\omega = 10^7$ rad/s design a lossless matching network to match a load impedance of 150 - j 75 Ω to a generator having an internal impedance of 50 Ω. The output current is to be in phase with the input current as in Figure 6.32. Start by trying to find a T-network solution:

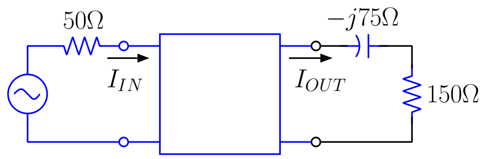


Figure 6.32: Example

$$\begin{aligned} \beta &= \theta_{I_{IN}} - \theta_{I_{OUT}} = 0 \\ \sin\beta &= 0 \\ \tan\beta &= 0 \end{aligned} \tag{6.80}$$

Thus, Z_A, Z_B, $Z_C \Rightarrow \infty$. This illustrates that 0 current phase shift cannot be obtained with a T-network!

Let us consider a Pi-network. For the Pi-network, β is the voltage phase shift. As noted earlier, β can be written in terms of the current phase shift and the phase angles of the terminating impedances:

$$\beta = \theta_{I_{IN}} - \theta_{I_{OUT}} - \theta_{Z_1} - \theta_{Z_2} = 0 - 0 - 0 - \theta_{Z_2} = -\theta_{Z_2}$$

$$\theta_{Z_2} = \tan^{-1}\frac{-75}{150} = -26.6° \tag{6.81}$$

Thus $\beta = +26.6°$. To proceed, the load is transformed so that absorption can be used as in Figure 6.33. Then

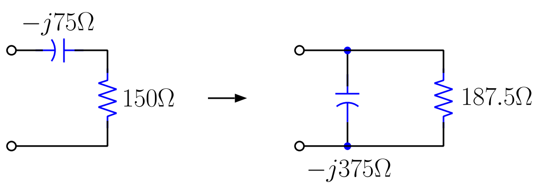


Figure 6.33: Transform load

$$\begin{aligned} Y_C &= \frac{\sqrt{(1/50)(1/187.5)}}{j\ \sin\ 26.6°} &&= -j\,23.07\,\mathrm{mS} \\ Y_A &= \frac{1/50}{j\ \tan\ 26.6°} + j\,23.07\,\mathrm{mS} &&= -j\,16.9\,\mathrm{mS} \end{aligned} \tag{6.82}$$

$$\frac{-1}{j\,375} + Y_B = \frac{1/187.5}{j\ \tan 26.6°} + j\,23.07\,\mathrm{mS}$$

$$Y_B = j\,9.73\,\mathrm{mS} \tag{6.83}$$

The final result is shown in Figure 6.34.

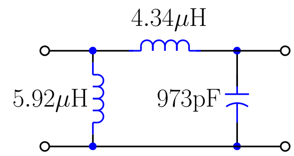


Figure 6.34: Final Result.

This example illustrated how to properly account for phase shifts in complex load impedance. A similar approach would be necessary if the source impedance was complex.

We now turn from the purely reactive 3-element matching networks to a discussion of purely resistive networks.

6.4.4 Resistive Three-element Matching Networks

The purely resistive matching networks are also known as "attenuators," "attenuating pads," or just "pads." Here we assume Y_A, Y_B, Y_C (or Z_A, Z_B, and Z_C) are real (resistances). The resistive matching network can be employed to provide a broadband match between two resistive terminations. Of course, some power loss must be accepted. This type of network is used to build attenuators with specified attenuation. In addition, as we will see, the resistive network can be used to ensure that a particular device sees a well-defined impedance, even when the input of the following device has a variable or unknown impedance.

In this application Y_A, Y_B, Y_C are real (resistive). The terminating impedances are also assumed to be resistive. There will be no phase shift and hence

$$\theta = \alpha \ \text{(real) nepers} \tag{6.84}$$

As noted previously

$$2\alpha = \ln \frac{P_{IN}}{P_{OUT}} \tag{6.85}$$

Thus, α determines the attenuation of the network. It is useful to relate α to the attenuation in dB:

$$\begin{aligned} \text{Attenuation in dB} &= 10 \log \frac{P_{IN}}{P_{OUT}} \\ &= 20 \log e^{\alpha} \\ &= 8.686\alpha \end{aligned} \tag{6.86}$$

If a network that provides 10 dB of attenuation is required, α would be $10/8.686 = 1.1513$ nepers.

6.4.4.1 Example - Design a 10 dB Pi-type resistive attenuator

Design a 10 dB Pi-type attenuator for use in a system with 75 Ω impedance ($R_1 = R_2 = 75\,\Omega$) as in Figure 6.35.

$$
\begin{aligned}
\alpha &= 10/8.686 &&= 1.1513 \\
Y_1 &= Y_2 &&= \tfrac{1}{75} &&= 13.33 \text{ mS} \\
Y_C &= \tfrac{\sqrt{Y_1 Y_2}}{\sinh\ \alpha} &&= \tfrac{13.33 \text{ mS}}{1.423} &&= 9.370 \text{ mS} \\
Y_A &= Y_B &&= \tfrac{13.33 \text{ mS}}{\tanh\ \alpha} - Y_C &&= 6.922 \text{ mS} \\
Z_A &= Z_B &&= 144.5\,\Omega &&\simeq 145\,\Omega \\
Z_C &= 106.7\,\Omega &&\simeq 107\,\Omega
\end{aligned}
$$

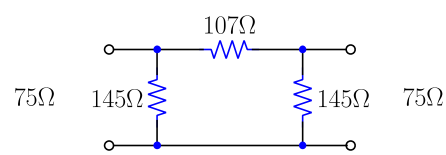


Figure 6.35: 10 dB Pi-type attenuator with 75 Ω impedance

The resistive matching networks/attenuators have the useful characteristic of isolating the impedance found at the input from the impedance that terminates the output. This can be seen by computing the input impedance for the extreme cases of shorted and open-circuit output terminations in the example considered above:

$$
\begin{aligned}
\text{shorted output :} \quad Z_{IN} &= 61.6\,\Omega \\
\text{open output :} \quad Z_{IN} &= 92.0\,\Omega
\end{aligned}
$$

Clearly, for any resistive termination, the input impedance will be in the range 61.6 to 92.0 Ω. This property can be used to advantage when the input impedance of a particular stage is not well known or is subject to variation. If this stage follows a stage that requires a certain load impedance in order to operate correctly, an attenuator can sometimes be an effective "isolation" stage. For example, for proper operation of certain types of frequency mixers, it is necessary that all of the frequency components at the output of the mixer "see" a well-defined impedance (usually 50 Ω). Typically the frequencies at the output of a mixer may span a very broad range, while the following stage is often a narrow-band IF amplifier. The IF amplifier would often have a well defined input impedance only within its passband. A resistive matching network is sometimes employed between the mixer and the IF amplifier in order to ensure proper termination of the mixer output port.

It is important to note that the attenuation of a resistive matching network will only be equal to the design value if the network is operated between the impedances for which it was designed. It is important to remember this, since the resistive networks are often found in applications where the terminating impedances are substantially different from the "correct" values. The example cited in the previous paragraph is one such case.

6.4.4.2 Minimum-loss Resistive Matching Networks

In some cases, especially when a broadband impedance match is required, one would like to design a resistive network that has the smallest possible attenuation (α). The problem is then to find the solution that yields the smallest α subject to the constraint that all resistive elements have values greater than or equal to zero. This constraint ensures that the solution corresponds to a passive network.

Consider the T-network in Figure 6.36.

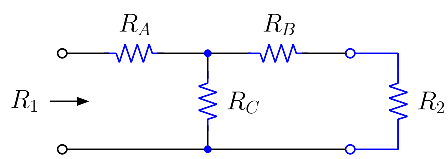


Figure 6.36: T-network

$$R_C = \frac{\sqrt{R_1 R_2}}{\sinh\ \alpha} \tag{6.87}$$

$$R_B = \frac{R_2}{\tanh\ \alpha} - \frac{\sqrt{R_1 R_2}}{\sinh\ \alpha} \tag{6.88}$$

$$R_A = \frac{R_1}{\tanh\ \alpha} - \frac{\sqrt{R_1 R_2}}{\sinh\ \alpha} \tag{6.89}$$

The goal is to find the smallest α that gives values for R_A, R_B, R_C which are all ≥ 0. Note from Equation 6.87 that

$$\sinh\ \alpha = \frac{\sqrt{R_1 R_2}}{R_C} \tag{6.90}$$

Since sinh α increases monotonically as a function of α, as shown in Figure 6.37, minimizing α is equivalent to minimizing sinh α. Clearly then, we should use the largest value of R_C

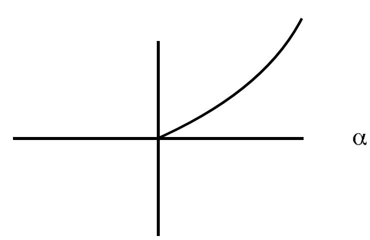


Figure 6.37: sinh α as a function of α

for which $R_B \geq 0$ and $R_A \geq 0$ (see Equation 6.90). Using

$$\cosh\,\alpha = \sqrt{1 + \sinh^2\,\alpha} \tag{6.91}$$

$$= \sqrt{1 + R_1\, R_2/{R_C}^2} \tag{6.92}$$

we find from Equation 6.88 and Equation 6.89

$$R_B = R_2\sqrt{\frac{R_C^2}{R_1\, R_2} + 1} - R_C \tag{6.93}$$

$$R_A = R_1\sqrt{\frac{R_C^2}{R_1\, R_2} + 1} - R_C \tag{6.94}$$

For the moment, assume $R_2 > R_1$. Then, as R_C is increased in order to decrease $\sinh(\alpha)$, R_A will decrease and will become equal to zero when (see Equation 6.94)

$$R_C = \frac{R_1}{\sqrt{1 - R_1/R_2}} \tag{6.95}$$

Be aware that R_B will always be greater than zero if $R_2 > R_1$. So, if $R_2 > R_1$, the minimum attenuation is obtained when R_C is given by Equation 6.95. The corresponding value for α (denoted by α_{min}) is obtained from Equation 6.92 and is

$$\alpha_{min} = \cosh^{-1}\sqrt{R_2/R_1} \tag{6.96}$$

The value for R_B is obtained by using Equation 6.95 in Equation 6.93 to yield

$$R_B = R_2\sqrt{1 - R_1\,/R_2} \tag{6.97}$$

Notice that since $R_A = 0$, the minimum-loss resistive network reduces to an L-network with the series arm (R_B) connected to the larger of the two terminating resistances.

The preceding discussion was based on the T-network, although the final result was found to be an L-network. The same result would have been obtained if the starting point had been the Pi-network.

6.4.4.3 Summary of Resistive Minimum-loss Network

The resistive minimum-loss network reduces to an L-network with series arm connected to the larger of the terminating resistances as in Figure 6.38.

$$\text{Loss in dB} = 8.686\ \cosh^{-1}\sqrt{R_{big}/R_{small}}$$

$$R_{shunt} = \frac{R_{small}}{\sqrt{1 - R_{small}/R_{big}}} \tag{6.98}$$

$$R_{series} = R_{big}\sqrt{1 - R_{small}/R_{big}}$$

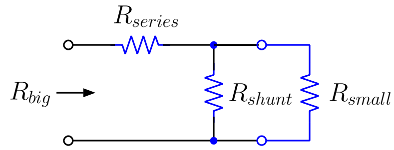


Figure 6.38: Resistive minimum-loss network

6.5 References

1. Smith, Jack, *Modern Communication Circuits*, McGraw-Hill, 1986.

2. Terman, Frederick Emmons, *Radio Engineers Handbook*, McGraw Hill, 1943.

6.6 Homework Problems

1. You are given a "black box" with two output terminals (a "1-port"). You play around with the box for awhile and make the following observations:

   (a) The output voltage from the box is sinusoidal.

   (b) The **peak magnitude** of the open circuit voltage at the output of the box is found to be 5 V.

   (c) You connect a 50 Ω resistor across the terminals and find that the peak magnitude of the voltage across the resistor is 2.795 V.

   (d) You short the output of the box and find the **peak magnitude** of the short circuit current is 100 mA.

   Find the power available from the source. Express your result in dBm.

2. Consider a source with impedance $Z_S = R_S + jX_S$ and suppose that the source drives a load that is purely resistive. Denote the load resistance by R_L.

   (a) What load resistance should be used if the goal is to maximize the power delivered to the load?

   (b) Find an expression for the power delivered to the load resistance found in part 2a. Express your result in terms of the source parameters only, i.e., P_{avs}, R_S and X_S.

3. Find the lossless network having the *minimum number of elements* to match a source impedance of 20-j60 Ω to a load impedance having an equivalent circuit of 100 Ω of resistance shunted by 50 Ω of capacitive reactance. Draw the network and label all parts.

4. Consider the design of a lossless L-network to match the source and load shown in Figure 6.39. All resistances and reactances are in ohms, and the current source magnitude is the peak value.

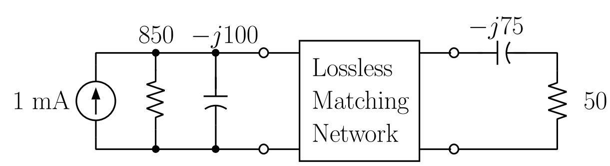


Figure 6.39: Complex source and load to be matched with a lossless L-network.

   (a) Find the power available from the source. Express your result in dBm.

   (b) How much power would be delivered to the load if a lossless matching network was not used, i.e., if the load is connected directly to the source? Express your result in dBm.

(c) There are four possible solutions for the matching network if an L-network is used. Find all four. Sketch all solutions and indicate whether the elements are inductors or capacitors.

(d) Verify two of your designs by plotting the path from the load to the source on a Smith Chart.

5. Consider a source with impedance $Z_S = R_S + jX_S$ and a load that is purely resistive. Denote the load resistance by R_L. Suppose that $R_S < R_L$. Under what condition(s) are there more than 2 lossless L-networks that conjugately match the source to the load? Derive a simple inequality that can be checked to determine whether there are 2 or 4 solutions.

6. The source and load shown in Figure 6.40 can be matched with an L-network that consists of two capacitors. All resistances and reactances are given in ohms.

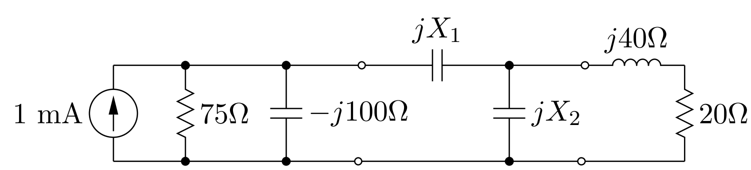


Figure 6.40: Source and load matched with 2-capacitor L-network.

Find X_1 and X_2 that will cause all of the available source power to be delivered to the load. Both X_1 and X_2 must be <0.

7. Design a matching network to match the source and load shown in Figure 6.41. Use a T-network such that I_{out} lags I_{in} by 60° at $\omega = 10^7$. What is the voltage phase shift?

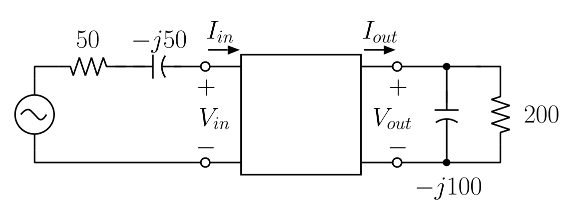


Figure 6.41: Source and load for lossless T-network.

8. The circuit in Figure 6.42 is matched at $\omega = 10^8\, rad\, s^{-1}$. Redesign for a match with the second harmonic attenuated in the shunt arm connected to the 100 Ω resistor and the third harmonic attenuated in the shunt arm connected to the 400 Ω resistor.

9. Consider the design of a band-pass T-type matching network with specified Q. This can be approached by thinking of the T-net as two back-to-back L-networks. One of the L-networks has a high-pass topology and the other has low-pass topology.

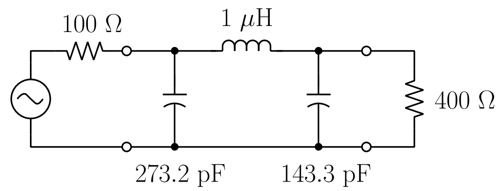


Figure 6.42: Circuit matched at $\omega = 10^8\ s^{-1}$.

Find a band-pass T-network that will match a 50 Ω load to a 10 Ω source impedance. Find the solution with a series inductor connected to the 10 Ω termination and a capacitor connected to the 50 Ω termination. Choose the overall Q of the network to be approximately equal to 10, i.e., choose the virtual resistance, R_v, such that the L-net with the highest Q has Q=10. Draw the T-network and label all components. Note: You should combine the two shunt elements into one single element for this problem.

10. Consider in Figure 6.43 the design of a T-network that transforms a 50 Ω load impedance to 5 Ω:

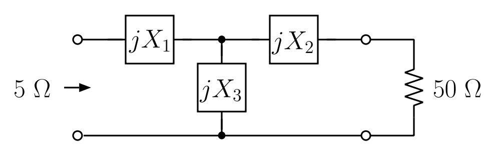


Figure 6.43: T-network transforming 50 Ω load impedance to 5 Ω.

Design a T-network with a Q of 6. Find the solution with $X_1 > 0$ and $X_2 < 0$. Specify X_1, X_2 and X_3 to the nearest ohm.

11. Consider the design of a T-network that transforms a 50 Ω load impedance to 10 Ω. Suppose that the reactance of the shunt element of the T-network is fixed at +j200 Ω as shown in Figure 6.44:

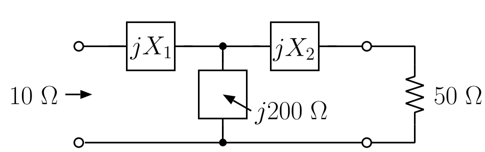


Figure 6.44: T-network transforming 50 Ω load impedance to 10 Ω.

Find two sets of values (X_1, X_2) which will give a match.

12. Design a T-network (as shown in Figure 6.45) that does an impedance inversion, i.e., that gives the following mapping between Z_L and Z_{IN} where R_0 is a positive real constant:

$$Z_{IN} = \frac{R_0^2}{Z_L} \tag{6.99}$$

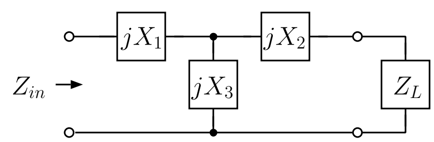


Figure 6.45: T-network providing impedance inversion.

Find expressions for X_1, X_2 and X_3. You may assume that $X_1 > 0$.

13. Consider the design of the matching network in Figure 6.46. Find the reactance X_s and the turns ratio of the ideal transformer, n. The ideal transformer is characterized by the relations:

$$V_2 = nV_1 \quad \text{and} \quad I_2 = -I_1/n \tag{6.100}$$

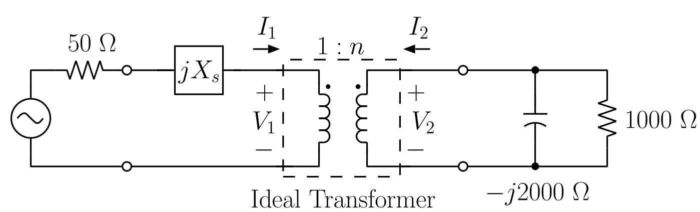


Figure 6.46: Matching network to find X_s and N

14. Suppose that you build an L-type matching network using a variable inductor and capacitor as shown in Figure 6.47. This network is to be used to match the impedance of a load, Z_L, to the output of a generator that has a 50 Ω impedance. At some frequency you know that the inductor's reactance can be adjusted to any value in the range from +j10 to +j100 Ω. At the same frequency the capacitor's reactance can be adjusted to any value in the range from -j10 to -j100 Ω. On a Smith Chart show the region corresponding to all possible values of load impedance, Z_L, that can be matched to 50 Ω using this network. Normalize your Smith Chart to 50 Ω. Shade or color the region of the chart that corresponds to the "matchable" load impedances.

15. Consider the network shown in Figure 6.48. The resistor, capacitor, and inductor are variable with $25\,\Omega \leq R \leq 50\,\Omega$, $25\,\Omega \leq X_L \leq 50\,\Omega$, and $10\,\Omega \leq X_C \leq 40\,\Omega$. On a

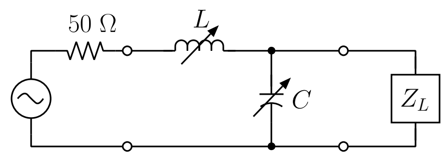


Figure 6.47: L-network with variable inductor and capacitor.

Smith Chart show the region corresponding to all possible values of input admittance, Y_{in}. Normalize your Smith Chart to 50 Ω.

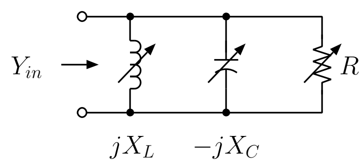


Figure 6.48: Network with variable resistor, capacitor and inductor.

16. Design a resistive T-type attenuator that results in 10 dB attenuation when used with 50 Ω source and load impedances.
    (a) Consider all possible resistive loads and find upper and lower bounds on the input resistance of the attenuator.
    (b) Now design a resistive Pi-type attenuator for the same parameters as the T-type attenuator.
    (c) Repeat part 16a for the Pi-type attenuator.
    (d) Suppose the Pi-type attenuator is used with a 600 Ω load. Find the attenuation in dB where attenuation is defined to be 10 log (P_{in}/P_{out}).
17. An 8 dB attenuator that is designed to match a 75 Ω source to a 300 Ω load is used between a 75 Ω source and a 10 Ω load. The source has available power of 10 dBm. Find the power delivered to the 10 Ω load. Express your result in dBm.
18. Consider a resistive attenuator/matching-network that is designed to work with 50 Ω source and load resistances, as in Figure 6.49. When connected to 50 Ω terminations, the attenuator is designed to give an attenuation of 10 dB, i.e., $P_{in}/P_{out} = 10$, and also to provide a 50 Ω match at both input and output. Now, suppose that the attenuator is used with an arbitrary load resistance, R_L (the source impedance is still 50 Ω). Denote the power available from the 50 Ω source by P_{avs} and find an expression for the power delivered to the load resistor, R_L. Your result should be in terms of P_{avs} and R_L only (and numerical constants).
19. Design a minimum-loss resistive matching network that will match a 300 Ω source to a 50 Ω load. What is the attenuation in dB?

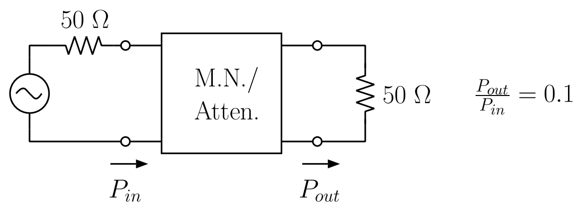


Figure 6.49: Resistive attenuator/matching-network.

20. Suppose that you have a lossless matching network of unknown type that transforms 50 Ω (resistive) to 450 Ω (resistive) and you need to couple a 50 Ω source to a resistive load, R_L, which may take on any positive, real, value. You have the choice of connecting the 50 Ω source directly to R_L, or using the matching network between the source and R_L (with the 50 Ω side of the matching network connected to the source). For what values of R_L would more power be delivered to R_L with the matching network in the system?

21. Design a lossless L-network that will match a 5 Ω source to a 200 Ω load. Use an L-network with high-pass topology.

    (a) Draw the network and indicate the reactances of the series and shunt elements.

    (b) Now, suppose that you build the network that you designed using a lossy inductor that has Q_L=32. Because the network contains a lossy element, the power that is actually delivered to the 200 Ω resistor will be smaller than the power available from the source. Find the loss of the network. The loss is defined as the difference (in decibels) between P_{avs} and the power that is actually delivered to the 200 Ω load.

22. Design the lossless L-network having a lowpass topology and with the series arm connected to the source, as shown in Figure 6.50.

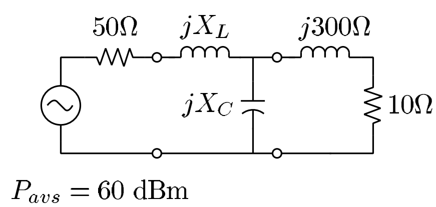


Figure 6.50: Lossless L-network with lowpass topology

    (a) Specify X_C and X_L.

    (b) Suppose the matching network you just designed is used and the imaginary part of the load impedance increases by 10%, i.e., suppose the load impedance changes to

$10 + j\ 330\ \Omega$. Determine the change in the power delivered to the load. Express your result in dB and be sure to correctly indicate the sign of the change.

23. A 50 Ω source has $P_{avs} = 4\,\text{mW}$. It is necessary to couple the source to a load with impedance $Z_L = 2 + j20\,\Omega$.

    (a) Find the power delivered to the load when the load is connected directly to the source. Express your answer in dBm.

    (b) There are 4 lossless LC L-networks that will match this source and load. Find the solution that has a capacitive series arm and the shunt arm connected across the load. You will receive full credit only if you specify the L-network having the specified topology.

24. Consider the source and load shown in the Figure.

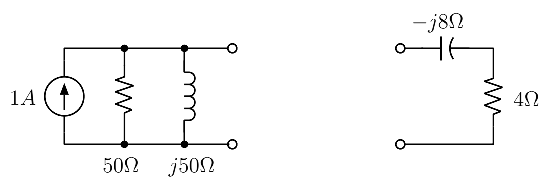


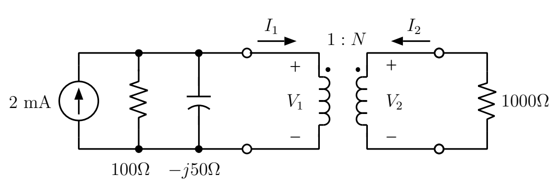
Figure 6.51: Source and load for problem 24.

    (a) Find the power available from the source. Express your answer in dBm.

    (b) Match this source and load using a lossless L-network. Any valid solution will be accepted.

    (c) Now suppose that you can only use a single lossless inductor or capacitor, either in series or in shunt, to couple the source to the load. If the goal is to maximize the power delivered to the load under this constraint, find the best possible solution.

25. A source is coupled to a resistive load, $R_L = 1000\,\Omega$, through an ideal transformer. The specified source current is the peak value. The ideal transformer is described by the relations: $V_2 = NV_1$ and $I_2 = -I_1/N$.



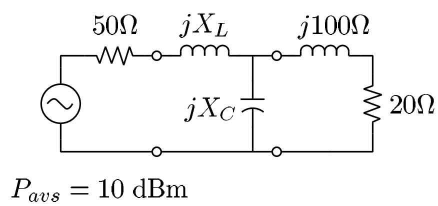
    (a) Find the power available from the source. Express your result in dBm.

    (b) Assuming that the turns ratio, N, is a positive real number find the value of N that maximizes the power delivered to the load.

(c) Assuming that N is set equal to the optimum value found in part (b.), determine the power delivered to the load. Express your result in dBm.

(d) Suppose the transformer is removed from the system, i.e. suppose that the $1000\,\Omega$ load is connected directly to the source. Find the power that would be delivered to the load. Express your result in dBm.

26. Design the lossless L-network with a low-pass topology having the series arm connected to the source, as shown below.



(a) Specify X_L and X_C.

(b) Determine the power that would be delivered to the $(20 + j100)\Omega$ load if the source was connected directly to the load. Express your result in dBm.

(c) Suppose the L-network designed in part a. is replaced with a new network consisting of a single capacitor. Sketch the single-capacitor network that will maximize the power delivered to the load. Specify the reactance of the capacitor and the power that would be delivered to the load (in dBm).

27. Consider the problem of coupling a $200\,\Omega$ source with $P_{avs} = 0$ dBm to the load $Z_L = 10 + j100\,\Omega$.

(a) Design a passive, lossless L-network that matches the source to the load and consists of 2 capacitors (no inductors may be used). Find the solution with the shunt element of the L-network connected across the load. Sketch the system, including the source, the matching network, and the load, and label each capacitor with its impedance.

(b) An ideal, lossless transformer with turns ratio N:1 is used between the given source and load. The turns ratio is a positive real number, and N>1. Find N that results in the largest possible power delivered to the load. Include a sketch showing the source, the transformer, and the load. Be sure to indicate the correct orientation of the transformer (i.e. specify which side of the transformer corresponds to the larger number of turns "N").

(c) Find the power delivered to the load if the transformer specified in part b. is used between the source and load. Express your result in dBm.

28. An ideal transformer can be used to reduce the mismatch loss (increase the mismatch factor) between an arbitrary source impedance $Z_S = R_S + jX_S$ ($R_S > 0$) and a pure resistance R_L as shown in the Figure. Suppose that the turns ratio, N, of the transformer can be adjusted to any positive real number ($N > 0$).

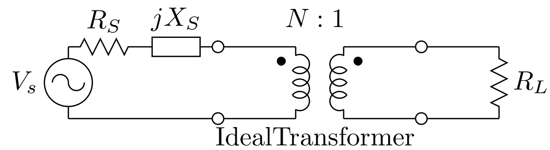


Figure 6.52: Ideal transformer matching complex source to real load.

(a) For given Z_S and R_L, there is an optimum turns ratio, N_{opt}, that maximizes the power delivered to the load. Find an expression for N_{opt}.

(b) Assuming that the optimum turns ratio is used, the resulting mismatch factor ($MF = \frac{P_{out}}{P_{avs}}$) depends only on the phase angle of the source impedance, i.e. if $Z_S = |Z_S|e^{j\theta}$, the mismatch factor can be written in terms of θ. Find such an expression for the mismatch factor.