## 16.7 EXERCISES

**1.** Let $S$ be the boundary surface of the box enclosed by the planes $x = 0$, $x = 2$, $y = 0$, $y = 4$, $z = 0$, and $z = 6$. Approximate $\iint_S e^{-0.1(x+y+z)}\, dS$ by using a Riemann sum as in Definition 1, taking the patches $S_{ij}$ to be the rectangles that are the faces of the box $S$ and the points $P_{ij}^*$ to be the centers of the rectangles.

**2.** A surface $S$ consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that $f$ is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4$$

Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches $S_{ij}$ to be four quarter-cylinders and the top and bottom disks.

**3.** Let $H$ be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose $f$ is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing $H$ into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

**4.** Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where $g$ is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where $S$ is the sphere $x^2 + y^2 + z^2 = 4$.

**5–18** Evaluate the surface integral.

**5.** $\iint_S x^2yz\, dS$,
$S$ is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

**6.** $\iint_S xy\, dS$,
$S$ is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

**7.** $\iint_S yz\, dS$,
$S$ is the part of the plane $x + y + z = 1$ that lies in the first octant

**8.** $\iint_S y\, dS$,
$S$ is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

**9.** $\iint_S yz\, dS$,
$S$ is the surface with parametric equations $x = u^2$, $y = u \sin v$, $z = u \cos v$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

**10.** $\iint_S \sqrt{1 + x^2 + y^2}\, dS$,
$S$ is the helicoid with vector equation $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

**11.** $\iint_S x^2z^2\, dS$,
$S$ is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$

**12.** $\iint_S z\, dS$,
$S$ is the surface $x = y + 2z^2$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$

**13.** $\iint_S y\, dS$,
$S$ is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$

**14.** $\iint_S y^2\, dS$,
$S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$-plane

**15.** $\iint_S (x^2z + y^2z)\, dS$,
$S$ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

**16.** $\iint_S xz\, dS$,
$S$ is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$

**17.** $\iint_S (z + x^2y)\, dS$,
$S$ is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant

**18.** $\iint_S (x^2 + y^2 + z^2)\, dS$,
$S$ is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

**19–30** Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface $S$. In other words, find the flux of $\mathbf{F}$ across $S$. For closed surfaces, use the positive (outward) orientation.

**19.** $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

**20.** $\mathbf{F}(x, y, z) = y\, \mathbf{i} + x\, \mathbf{j} + z^2\, \mathbf{k}$,
$S$ is the helicoid of Exercise 10 with upward orientation

**21.** $\mathbf{F}(x, y, z) = xze^y\, \mathbf{i} - xze^y\, \mathbf{j} + z\, \mathbf{k}$,
$S$ is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation

**22.** $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + z^4\, \mathbf{k}$,
$S$ is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation

**23.** $\mathbf{F}(x, y, z) = x\, \mathbf{i} - z\, \mathbf{j} + y\, \mathbf{k}$,
$S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation toward the origin

**24.** $\mathbf{F}(x, y, z) = xz\, \mathbf{i} + x\, \mathbf{j} + y\, \mathbf{k}$,
$S$ is the hemisphere $x^2 + y^2 + z^2 = 25$, $y \geqslant 0$, oriented in the direction of the positive $y$-axis

**25.** $\mathbf{F}(x, y, z) = y\, \mathbf{j} - z\, \mathbf{k}$,
$S$ consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

**26.** $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + 4x^2\, \mathbf{j} + yz\, \mathbf{k}$, $S$ is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with upward orientation

27. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
$S$ is the cube with vertices $(\pm1, \pm1, \pm1)$

28. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, $S$ is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

29. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, $S$ is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

30. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
$S$ is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

CAS 31. Evaluate $\iint_S xyz\, dS$ correct to four decimal places, where $S$ is the surface $z = xy$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

CAS 32. Find the exact value of $\iint_S x^2yz\, dS$, where $S$ is the surface in Exercise 31.

CAS 33. Find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where $S$ is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the $xy$-plane.

CAS 34. Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the $xy$-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

35. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where $S$ is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

36. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where $S$ is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

37. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

38. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

39. (a) Give an integral expression for the moment of inertia $I_z$ about the $z$-axis of a thin sheet in the shape of a surface $S$ if the density function is $\rho$.
(b) Find the moment of inertia about the $z$-axis of the funnel in Exercise 38.

40. Let $S$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If $S$ has constant density $k$, find (a) the center of mass and (b) the moment of inertia about the $z$-axis.

41. A fluid has density 870 kg/m$^3$ and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where $x$, $y$, and $z$ are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leqslant z \leqslant 1$.

42. Seawater has density 1025 kg/m$^3$ and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where $x$, $y$, and $z$ are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

43. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

44. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm1, \pm1, \pm1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

45. The temperature at the point $(x, y, z)$ in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

46. The temperature at a point in a ball with conductivity $K$ is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a$ with center at the center of the ball.

47. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(r) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant $c$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere $S$ with center the origin is independent of the radius of $S$.

## 16.8 STOKES' THEOREM

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region $D$ to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface $S$ to a line integral around the boundary curve of $S$ (which is a space curve). Figure 1 shows