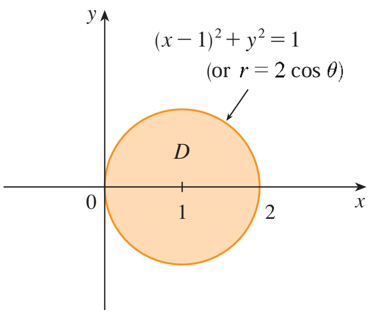


FIGURE 9

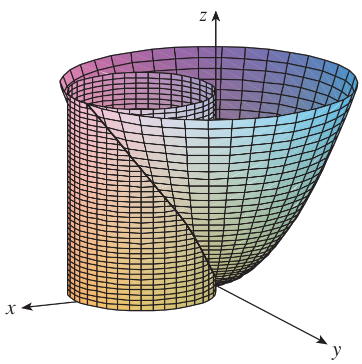


FIGURE 10

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x-1)^2 + y^2 = 1$$

(See Figures 9 and 10.) In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r,\theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta = 8\int_0^{\pi/2}\cos^4\theta\, d\theta = 8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

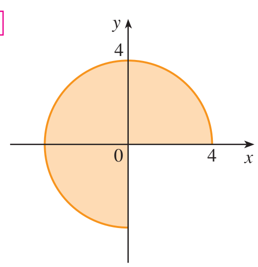
$$= 2\int_0^{\pi/2}\left[1 + 2\cos 2\theta + \tfrac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

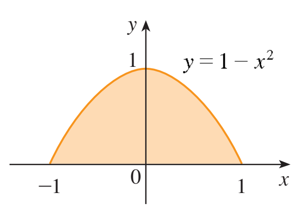
15.4 EXERCISES

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

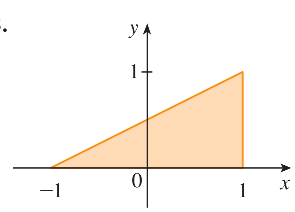
1.



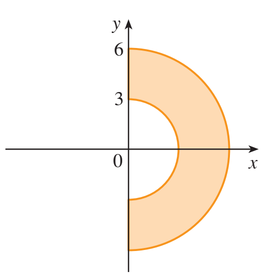
2.



3.



4.



5–6 Sketch the region whose area is given by the integral and evaluate the integral.

5. $\displaystyle\int_\pi^{2\pi}\int_4^7 r\, dr\, d\theta$

6. $\displaystyle\int_0^{\pi/2}\int_0^{4\cos\theta} r\, dr\, d\theta$

7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D xy\, dA$,
where D is the disk with center the origin and radius 3

8. $\iint_R (x + y)\, dA$, where R is the region that lies to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

9. $\iint_R \cos(x^2 + y^2)\, dA$, where R is the region that lies above the x-axis within the circle $x^2 + y^2 = 9$

10. $\iint_R \sqrt{4 - x^2 - y^2}\, dA$,
where $R = \{(x, y) \mid x^2 + y^2 \leqslant 4,\ x \geqslant 0\}$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

13. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$

14. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

15–18 Use a double integral to find the area of the region.

15. One loop of the rose $r = \cos 3\theta$

16. The region enclosed by the curve $r = 4 + 3\cos\theta$

17. The region within both of the circles $r = \cos\theta$ and $r = \sin\theta$

18. The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

19–27 Use polar coordinates to find the volume of the given solid.

19. Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leqslant 4$

20. Below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane

21. Enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$

22. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

23. A sphere of radius a

24. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

25. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

26. Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

27. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

28. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

29–32 Evaluate the iterated integral by converting to polar coordinates.

29. $\displaystyle\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(x^2 + y^2)\, dy\, dx$

30. $\displaystyle\int_{0}^{a} \int_{-\sqrt{a^2-y^2}}^{0} x^2 y\, dx\, dy$

31. $\displaystyle\int_{0}^{1} \int_{y}^{\sqrt{2-y^2}} (x + y)\, dx\, dy$

32. $\displaystyle\int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

33. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

34. An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler.
(a) If $0 < R \leqslant 100$, what is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?
(b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R.

35. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy\, dy\, dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy\, dy\, dx$$

into one double integral. Then evaluate the double integral.

36. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$
$$= \lim_{a \to \infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a \to \infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\, x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

37. Use the result of Exercise 36 part (c) to evaluate the following integrals.

(a) $\displaystyle\int_{0}^{\infty} x^2 e^{-x^2}\, dx$ (b) $\displaystyle\int_{0}^{\infty} \sqrt{x}\, e^{-x}\, dx$