EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^{1} = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

15.3 EXERCISES

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_0^4 \int_0^{\sqrt{y}} xy^2\, dx\, dy$

2. $\displaystyle\int_0^1 \int_{2x}^{2} (x - y)\, dy\, dx$

3. $\displaystyle\int_0^1 \int_{x^2}^{x} (1 + 2y)\, dy\, dx$

4. $\displaystyle\int_0^2 \int_y^{2y} xy\, dx\, dy$

5. $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

6. $\displaystyle\int_0^1 \int_0^{v} \sqrt{1 - v^2}\, du\, dv$

7–18 Evaluate the double integral.

7. $\displaystyle\iint_D y^2\, dA, \quad D = \{(x, y) \mid -1 \leqslant y \leqslant 1,\ -y - 2 \leqslant x \leqslant y\}$

8. $\displaystyle\iint_D \frac{y}{x^5 + 1}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$

9. $\displaystyle\iint_D x\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \sin x\}$

10. $\displaystyle\iint_D x^3\, dA, \quad D = \{(x, y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$

11. $\displaystyle\iint_D y^2 e^{xy}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 4,\ 0 \leqslant x \leqslant y\}$

12. $\displaystyle\iint_D x\sqrt{y^2 - x^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

13. $\displaystyle\iint_D x \cos y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

14. $\displaystyle\iint_D (x + y)\, dA$, D is bounded by $y = \sqrt{x}$ and $y = x^2$

15. $\displaystyle\iint_D y^3\, dA$,
D is the triangular region with vertices $(0, 2)$, $(1, 1)$, $(3, 2)$

16. $\displaystyle\iint_D xy^2\, dA$, D is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

17. $\displaystyle\iint_D (2x - y)\, dA$,
D is bounded by the circle with center the origin and radius 2

18. $\displaystyle\iint_D 2xy\, dA$, D is the triangular region with vertices $(0, 0)$, $(1, 2)$, and $(0, 3)$

19–28 Find the volume of the given solid.

19. Under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$

20. Under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$

21. Under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$, and $(1, 2)$

22. Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, $z = 0$

23. Bounded by the coordinate planes and the plane $3x + 2y + z = 6$

24. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

25. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

26. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

27. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

28. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

29. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

30. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

31–32 Find the volume of the solid by subtracting two volumes.

31. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

32. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

33–34 Sketch the solid whose volume is given by the iterated integral.

33. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$ **34.** $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

CAS **35–38** Use a computer algebra system to find the exact volume of the solid.

35. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

36. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

37. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

38. Enclosed by $z = x^2 + y^2$ and $z = 2y$

39–44 Sketch the region of integration and change the order of integration.

39. $\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$ **40.** $\int_0^1 \int_{4x}^4 f(x, y)\, dy\, dx$

41. $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\, dx\, dy$ **42.** $\int_0^3 \int_0^{\sqrt{9-y}} f(x, y)\, dx\, dy$

43. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$ **44.** $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

45–50 Evaluate the integral by reversing the order of integration.

45. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ **46.** $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$

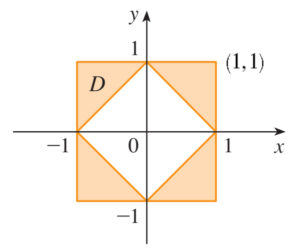
47. $\int_0^4 \int_{\sqrt{x}}^2 \dfrac{1}{y^3 + 1}\, dy\, dx$ **48.** $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$

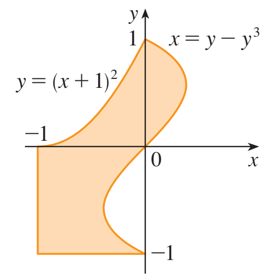
49. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

50. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

51–52 Express D as a union of regions of type I or type II and evaluate the integral.

51. $\iint_D x^2\, dA$ **52.** $\iint_D y\, dA$





53–54 Use Property 11 to estimate the value of the integral.

53. $\iint_Q e^{-(x^2+y^2)^2}\, dA$, Q is the quarter-circle with center the origin and radius $\frac{1}{2}$ in the first quadrant

54. $\iint_T \sin^4(x + y)\, dA$, T is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

55–56 Find the average value of f over region D.

55. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

56. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

57. Prove Property 11.

58. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

59. Evaluate $\iint_D (x^2 \tan x + y^3 + 4)\, dA$, where $D = \{(x, y) \mid x^2 + y^2 \leqslant 2\}$. [*Hint:* Exploit the fact that D is symmetric with respect to both axes.]

60. Use symmetry to evaluate $\iint_D (2 - 3x + 4y)\, dA$, where D is the region bounded by the square with vertices $(\pm 5, 0)$ and $(0, \pm 5)$.

61. Compute $\iint_D \sqrt{1 - x^2 - y^2}\, dA$, where D is the disk $x^2 + y^2 \leqslant 1$, by first identifying the integral as the volume of a solid.

CAS **62.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)