which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

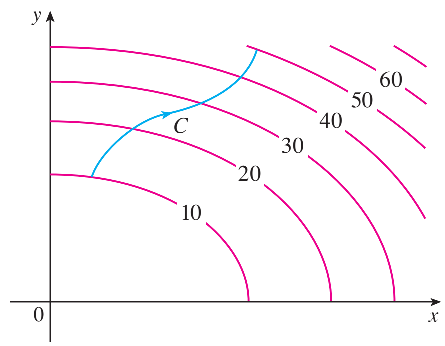
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 EXERCISES

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \qquad y = t^3 + t \qquad 0 \leqslant t \leqslant 1$$

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (2x - 3y)\,\mathbf{i} + (-3x + 4y - 8)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$

6. $\mathbf{F}(x, y) = (3x^2 - 2y^2)\,\mathbf{i} + (4xy + 3)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (xy\cos xy + \sin xy)\,\mathbf{i} + (x^2 \cos xy)\,\mathbf{j}$

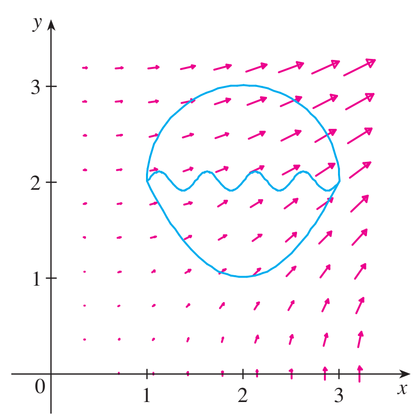
9. $\mathbf{F}(x, y) = (\ln y + 2xy^3)\,\mathbf{i} + (3x^2y^2 + x/y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (xy\cosh xy + \sinh xy)\,\mathbf{i} + (x^2 \cosh xy)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.

(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?



12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

12. $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$,
C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

13. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t + \sin \frac{1}{2}\pi t,\ t + \cos \frac{1}{2}\pi t \rangle, \quad 0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y) = \dfrac{y^2}{1 + x^2}\,\mathbf{i} + 2y \arctan x\,\mathbf{j}$,
C: $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

15. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

16. $\mathbf{F}(x, y, z) = (2xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (x^2 + 3z^2)\,\mathbf{k}$,
C: $x = t^2, \quad y = t + 1, \quad z = 2t - 1, \quad 0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = y^2 \cos z\,\mathbf{i} + 2xy \cos z\,\mathbf{j} - xy^2 \sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t^2\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi$

18. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + xe^y\,\mathbf{j} + (z + 1)e^z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

19–20 Show that the line integral is independent of path and evaluate the integral.

19. $\int_C \tan y\,dx + x \sec^2 y\,dy$,
C is any path from $(1, 0)$ to $(2, \pi/4)$

20. $\int_C (1 - ye^{-x})\,dx + e^{-x}\,dy$,
C is any path from $(0, 1)$ to $(1, 2)$

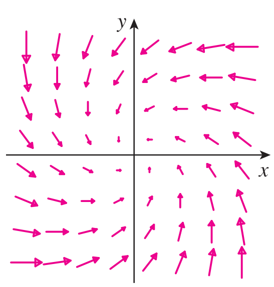
21–22 Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

21. $\mathbf{F}(x, y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}; \quad P(1, 1),\ Q(2, 4)$

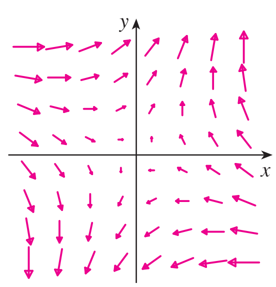
22. $\mathbf{F}(x, y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}; \quad P(0, 1),\ Q(2, 0)$

23–24 Is the vector field shown in the figure conservative? Explain.

23.



24.



CAS **25.** If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x \cos y)\,\mathbf{j}$, use a plot to guess whether $\mathbf{F}$ is conservative. Then determine whether your guess is correct.

26. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

27. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

28. Use Exercise 27 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

29–32 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

29. $\{(x, y) \mid x > 0,\ y > 0\}$

30 $\{(x, y) \mid x \neq 0\}$

31. $\{(x, y) \mid 1 < x^2 + y^2 < 4\}$

32. $\{(x, y) \mid x^2 + y^2 \leqslant 1 \text{ or } 4 \leqslant x^2 + y^2 \leqslant 9\}$

33. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

34. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 4 in Section 16.1. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m²/kg².)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 5 in Section 16.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\varepsilon = 8.985 \times 10^9$.)