We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\, dt = \int_0^{2\pi} dt = 2\pi$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\, dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field.

16.4 EXERCISES

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C (x - y)\,dx + (x + y)\,dy$,
   C is the circle with center the origin and radius 2

2. $\oint_C xy\,dx + x^2\,dy$,
   C is the rectangle with vertices (0, 0), (3, 0), (3, 1), and (0, 1)

3. $\oint_C xy\,dx + x^2y^3\,dy$,
   C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x\,dx + y\,dy$, C consists of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0) and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1)

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2\,dx + 2x^2y\,dy$,
   C is the triangle with vertices (0, 0), (2, 2), and (2, 4)

6. $\int_C \cos y\,dx + x^2\sin y\,dy$,
   C is the rectangle with vertices (0, 0), (5, 0), (5, 2), and (0, 2)

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy$,
   C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

8. $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$,
   C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

9. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

10. $\int_C \sin y\,dx + x\cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \left\langle \sqrt{x} + y^3,\, x^2 + \sqrt{y} \right\rangle$,
    C consists of the arc of the curve $y = \sin x$ from (0, 0) to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to (0, 0)

12. $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$,
C is the triangle from (0, 0) to (2, 6) to (2, 0) to (0, 0)

13. $\mathbf{F}(x, y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$,
C is the circle $x^2 + y^2 = 25$ oriented clockwise

14. $\mathbf{F}(x, y) = \langle y - \ln(x^2 + y^2), 2 \tan^{-1}(y/x) \rangle$, C is the circle $(x - 2)^2 + (y - 3)^2 = 1$ oriented counterclockwise

CAS **15–16** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = y^2 e^x$, $Q(x, y) = x^2 e^y$,
C consists of the line segment from $(-1, 1)$ to $(1, 1)$ followed by the arc of the parabola $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

16. $P(x, y) = 2x - x^3 y^5$, $Q(x, y) = x^3 y^8$,
C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to (1, 0), then along the line segment to (0, 1), and then back to the origin along the y-axis.

18. A particle starts at the point $(-2, 0)$, moves along the x-axis to (2, 0), and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid,* with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$$

(c) Find the area of the pentagon with vertices (0, 0), (2, 1), (1, 3), (0, 2), and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a.

24. Use Exercise 22 to find the centroid of the triangle with vertices (0, 0), $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3\,dx \qquad I_y = \frac{\rho}{3} \oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 15.5.)

27. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

28. Complete the proof of the special case of Green's Theorem by proving Equation 3.

29. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\,dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

16.5 CURL AND DIVERGENCE

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.