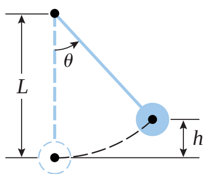
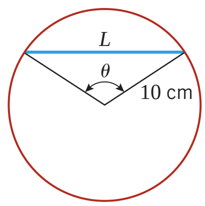
(d) Does y have a minimum value? A maximum value? If so, find them.

25. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

26. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).



▲ **Figure Ex-25**



▲ **Figure Ex-26**

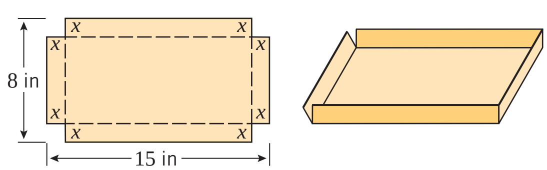
27–28 Express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.] ■

27. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$

28. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

29. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 in by 15 in, by cutting out squares with sides of length x from each corner and bending up the sides.

(a) Express the volume V as a function of x.
(b) Find the domain of V.
(c) Plot the graph of the function V obtained in part (a) and estimate the range of this function.
(d) In words, describe how the volume V varies with x, and discuss how one might construct boxes of maximum volume.



▲ **Figure Ex-29**

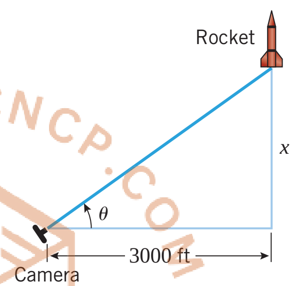
30. Repeat Exercise 29 assuming the box is constructed in the same fashion from a 6-inch-square sheet of metal.

31. A construction company has adjoined a 1000 ft^2 rectangular enclosure to its office building. Three sides of the enclosure are fenced in. The side of the building adjacent to the enclosure is 100 ft long and a portion of this side is used as the fourth side of the enclosure. Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions.

(a) Find a formula for L in terms of x and y.
(b) Find a formula that expresses L as a function of x alone.
(c) What is the domain of the function in part (b)?
(d) Plot the function in part (b) and estimate the dimensions of the enclosure that minimize the amount of fencing required.

32. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The rocket rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.

(a) Express the height x as a function of the elevation angle θ.
(b) Find the domain of the function in part (a).
(c) Plot the graph of the function in part (a) and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height.



◀ **Figure Ex-32**

33. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.

(a) Estimate the radius r and the height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]
(b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.
(c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

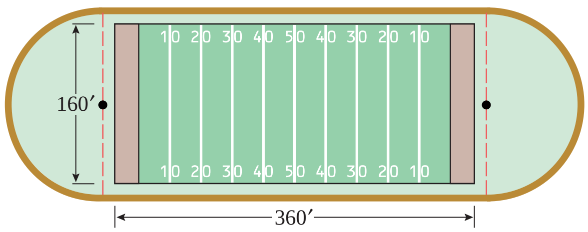
34. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure on the next page. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.

(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]
(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the football field and a straightaway. Make a graph of L versus x.

(cont.)

(c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.

(d) Use the graph to estimate the length of the longest possible straightaways, and then find that length exactly.



▲ **Figure Ex-34**

35–36 (i) Explain why the function f has one or more holes in its graph, and state the x-values at which those holes occur. (ii) Find a function g whose graph is identical to that of f, but without the holes. ■

35. $f(x) = \dfrac{(x+2)(x^2-1)}{(x+2)(x-1)}$ **36.** $f(x) = \dfrac{x^2 + |x|}{|x|}$

37. In 2001 the National Weather Service introduced a new wind chill temperature (WCT) index. For a given outside temperature T and wind speed v, the wind chill temperature index is the equivalent temperature that exposed skin would feel with a wind speed of v mi/h. Based on a more accurate model of cooling due to wind, the new formula is

$$\text{WCT} = \begin{cases} T, & 0 \le v \le 3 \\ 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}, & 3 < v \end{cases}$$

where T is the temperature in °F, v is the wind speed in mi/h, and WCT is the equivalent temperature in °F. Find the WCT to the nearest degree if $T = 25$°F and
(a) $v = 3$ mi/h (b) $v = 15$ mi/h (c) $v = 46$ mi/h.

Source: Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.

38–40 Use the formula for the wind chill temperature index described in Exercise 37. ■

38. Find the air temperature to the nearest degree if the WCT is reported as -60°F with a wind speed of 48 mi/h.

39. Find the air temperature to the nearest degree if the WCT is reported as -10°F with a wind speed of 48 mi/h.

40. Find the wind speed to the nearest mile per hour if the WCT is reported as 5°F with an air temperature of 20°F.

✔ QUICK CHECK ANSWERS 0.1

1. (a) $[-1, +\infty)$ (b) 6 (c) $|t| + 4$ (d) 8 (e) $[4, +\infty)$ **2.** (a) M (b) I **3.** (a) $[-3, 3)$ (b) $[-2, 2]$ (c) -1 (d) 1 (e) $-\frac{3}{4}$; $-\frac{3}{2}$ **4.** (a) yes; domain: $\{65, 70, 71, 73, 75\}$; range: $\{48, 50, 52, 56\}$ (b) no **5.** (a) $l = 2w$ (b) $A = l^2/2$ (c) $w = \sqrt{A/2}$

0.2 NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

■ ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f + g$, $f - g$, fg, and f/g. For example, $f + g$ is defined by the formula

$$(f + g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f + g$ is obtained by adding the values of f and g. Equation (1) provides a formula for $f + g$ but does not say anything about the domain of $f + g$. However, for the right side of this equation to be defined, x must lie in the domains of both f and g, so we define the domain of $f + g$ to be the intersection of these two domains. More generally, we make the following definition.

0.2.1 DEFINITION Given functions f and g, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

For the functions $f+g$, $f-g$, and fg we define the domain to be the intersection of the domains of f and g, and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

If f is a constant function, that is, $f(x) = c$ for all x, then the product of f and g is cg, so multiplying a function by a constant is a special case of multiplying two functions.

Example 1 Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find the domains and formulas for the functions $f+g$, $f-g$, fg, f/g, and $7f$.

Solution. First, we will find the formulas and then the domains. The formulas are

$$(f+g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}\,) + (x-3) = x - 2 + \sqrt{x-2} \tag{2}$$
$$(f-g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}\,) - (x-3) = 4 - x + \sqrt{x-2} \tag{3}$$
$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2}\,)(x-3) \tag{4}$$
$$(f/g)(x) = f(x)/g(x) = \frac{1+\sqrt{x-2}}{x-3} \tag{5}$$
$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2} \tag{6}$$

The domains of f and g are $[2, +\infty)$ and $(-\infty, +\infty)$, respectively (their natural domains). Thus, it follows from Definition 0.2.1 that the domains of $f+g$, $f-g$, and fg are the intersection of these two domains, namely,

$$[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty) \tag{7}$$

Moreover, since $g(x) = 0$ if $x = 3$, the domain of f/g is (7) with $x = 3$ removed, namely,

$$[2, 3) \cup (3, +\infty)$$

Finally, the domain of $7f$ is the same as the domain of f. ◀

We saw in the last example that the domains of the functions $f+g$, $f-g$, fg, and f/g were the natural domains resulting from the formulas obtained for these functions. The following example shows that this will not always be the case.

Example 2 Show that if $f(x) = \sqrt{x}$, $g(x) = \sqrt{x}$, and $h(x) = x$, then the domain of fg is not the same as the natural domain of h.

Solution. The natural domain of $h(x) = x$ is $(-\infty, +\infty)$. Note that

$$(fg)(x) = \sqrt{x}\sqrt{x} = x = h(x)$$

on the domain of fg. The domains of both f and g are $[0, +\infty)$, so the domain of fg is

$$[0, +\infty) \cap [0, +\infty) = [0, +\infty)$$

by Definition 0.2.1. Since the domains of fg and h are different, it would be misleading to write $(fg)(x) = x$ without including the restriction that this formula holds only for $x \geq 0$. ◀

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function

$$f(g(x)) = (g(x))^2 = (x+1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x+1)^2$$

In general, we make the following definition.

Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, and one needs $g(x)$ in the domain of f to compute $f(g(x))$.

0.2.2 DEFINITION Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

Example 3 Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of $g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◀

Note that the functions $f \circ g$ and $g \circ f$ in Example 3 are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

Example 4 Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x}, \quad g(x) = 1/x, \quad h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = \sqrt{1/x^3} = 1/x^{3/2}$$ ◀

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x+1)^2$$

To evaluate $h(x)$ for a given value of x, we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \quad \text{and} \quad f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x+1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x, trying to break the evaluation into two steps performed in succession.
- The first operation in the evaluation will determine a function g and the second a function f.
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

Example 5 Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x)) \qquad \boxed{g(x) = x^3 \text{ and } f(x) = \sin x}$$ ◀

Table 0.2.1 gives some more examples of decomposing functions into compositions.

Table 0.2.1

COMPOSING FUNCTIONS

FUNCTION	$g(x)$ INSIDE	$f(x)$ OUTSIDE	COMPOSITION
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5) = f(g(x))$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x} = f(g(x))$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x} = f(g(x))$
$\dfrac{1}{x+1}$	$x+1$	$\dfrac{1}{x}$	$\dfrac{1}{x+1} = f(g(x))$

REMARK There is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2+1)^{10}$ as a composition that differ from that in Table 0.2.1:

$$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^2 \text{ and } f(x) = x^5}$$

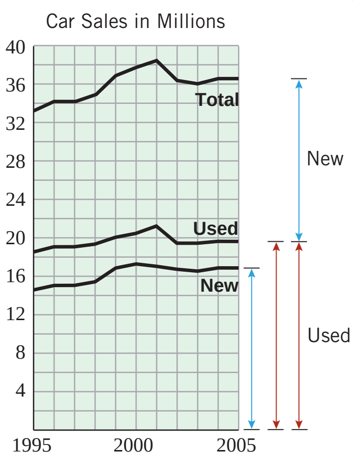
$$(x^2+1)^{10} = [(x^2+1)^3]^{10/3} = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^3 \text{ and } f(x) = x^{10/3}}$$

NEW FUNCTIONS FROM OLD

The remainder of this section will be devoted to considering the geometric effect of performing basic operations on functions. This will enable us to use known graphs of functions to visualize or sketch graphs of related functions. For example, Figure 0.2.1 shows the graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ over a certain time period. Those graphs can be used to construct the graph of the total car sales

$$T(t) = N(t) + U(t)$$

by adding the values of $N(t)$ and $U(t)$ for each value of t. In general, the graph of $y = f(x) + g(x)$ can be constructed from the graphs of $y = f(x)$ and $y = g(x)$ by adding corresponding y-values for each x.



Source: NADA.

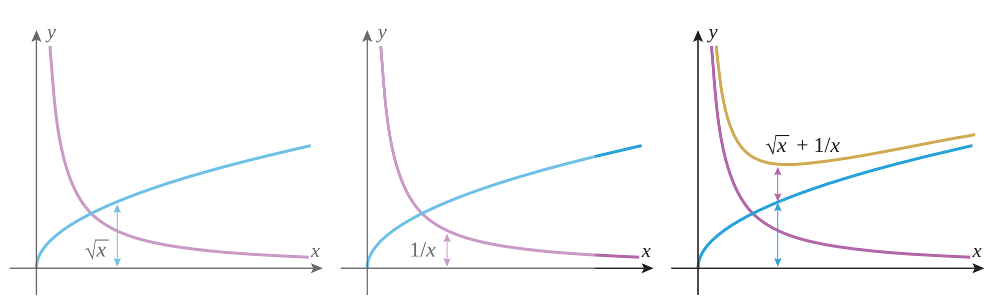
▲ **Figure 0.2.1**

▶ **Example 6** Referring to Figure 0.1.4 for the graphs of $y = \sqrt{x}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{x} + 1/x$ for $x \geq 0$.

Solution. To add the corresponding y-values of $y = \sqrt{x}$ and $y = 1/x$ graphically, just imagine them to be "stacked" on top of one another. This yields the sketch in Figure 0.2.2. ◀

Use the technique in Example 6 to sketch the graph of the function

$$\sqrt{x} - \frac{1}{x}$$



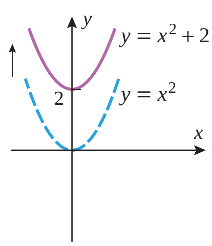
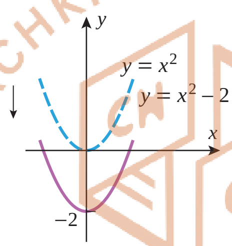
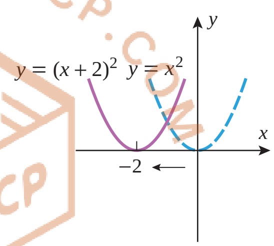
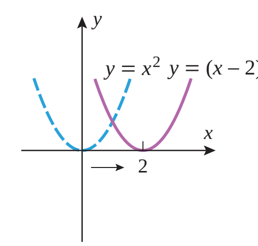
▶ **Figure 0.2.2**

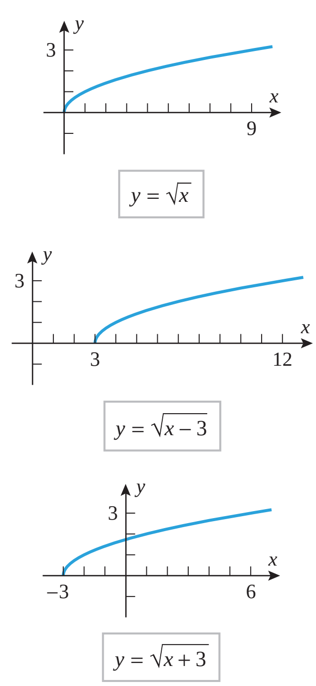
Add the y-coordinates of $\sqrt{x}$ and $1/x$ to obtain the y-coordinate of $\sqrt{x} + 1/x$.

TRANSLATIONS

Table 0.2.2 illustrates the geometric effect on the graph of $y = f(x)$ of adding or subtracting a *positive* constant c to f or to its independent variable x. For example, the first result in the table illustrates that adding a positive constant c to a function f adds c to each y-coordinate of its graph, thereby shifting the graph of f up by c units. Similarly, subtracting c from f shifts the graph down by c units. On the other hand, if a positive constant c is added to x, then the value of $y = f(x + c)$ at $x - c$ is $f(x)$; and since the point $x - c$ is c units to the left of x on the x-axis, the graph of $y = f(x + c)$ must be the graph of $y = f(x)$ shifted left by c units. Similarly, subtracting c from x shifts the graph of $y = f(x)$ right by c units.

Table 0.2.2
TRANSLATION PRINCIPLES

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				



▲ **Figure 0.2.3**

Before proceeding to the next examples, it will be helpful to review the graphs in Figures 0.1.4 and 0.1.9.

▶ **Example 7** Sketch the graph of

$$\text{(a)}\ \ y = \sqrt{x-3} \qquad \text{(b)}\ \ y = \sqrt{x+3}$$

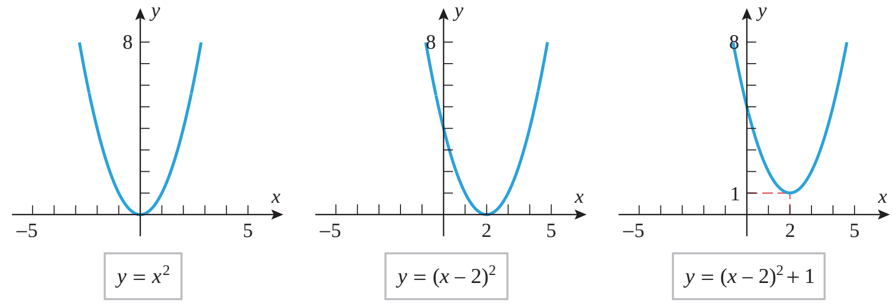
Solution. Using the translation principles given in Table 0.2.2, the graph of the equation $y = \sqrt{x-3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units. The graph of $y = \sqrt{x+3}$ can be obtained by translating the graph of $y = \sqrt{x}$ left 3 units (Figure 0.2.3). ◀

▶ **Example 8** Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Web Appendix H for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure 0.2.4). ◀

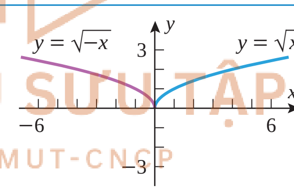
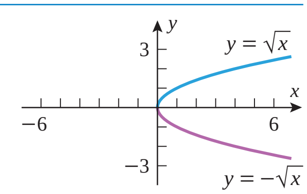


▶ Figure 0.2.4

REFLECTIONS

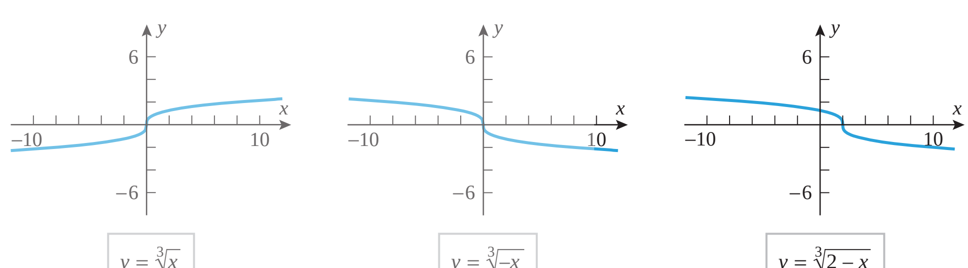
The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent to $-y = f(x)$]. This is summarized in Table 0.2.3.

Table 0.2.3
REFLECTION PRINCIPLES

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

▶ **Example 9** Sketch the graph of $y = \sqrt[3]{2 - x}$.

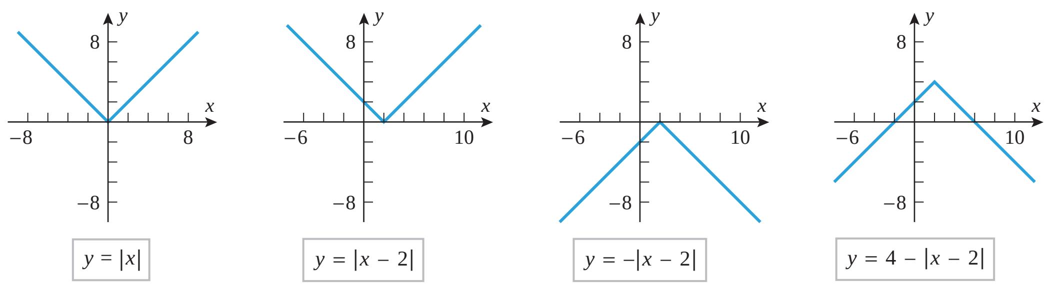
Solution. Using the translation and reflection principles in Tables 0.2.2 and 0.2.3, we can obtain the graph by a reflection followed by a translation as follows: First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x - 2)} = \sqrt[3]{2 - x}$ (Figure 0.2.5). ◀



▶ Figure 0.2.5

▶ **Example 10** Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: First translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure 0.2.6). ◀



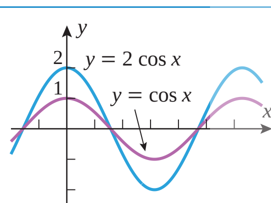
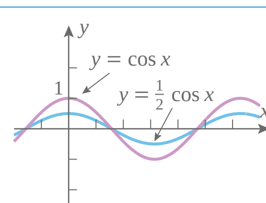
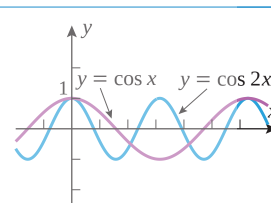
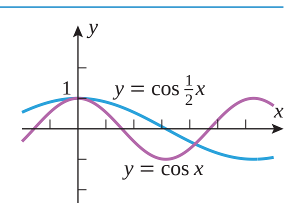
▲ **Figure 0.2.6**

STRETCHES AND COMPRESSIONS

Multiplying $f(x)$ by a *positive* constant c has the geometric effect of stretching the graph of $y = f(x)$ in the y-direction by a factor of c if $c > 1$ and compressing it in the y-direction by a factor of $1/c$ if $0 < c < 1$. For example, multiplying $f(x)$ by 2 doubles each y-coordinate, thereby stretching the graph vertically by a factor of 2, and multiplying by $\frac{1}{2}$ cuts each y-coordinate in half, thereby compressing the graph vertically by a factor of 2. Similarly, multiplying x by a *positive* constant c has the geometric effect of compressing the graph of $y = f(x)$ by a factor of c in the x-direction if $c > 1$ and stretching it by a factor of $1/c$ if $0 < c < 1$. [If this seems backwards to you, then think of it this way: The value of $2x$ changes twice as fast as x, so a point moving along the x-axis from the origin will only have to move half as far for $y = f(2x)$ to have the same value as $y = f(x)$, thereby creating a horizontal compression of the graph.] All of this is summarized in Table 0.2.4.

Describe the geometric effect of multiplying a function f by a *negative* constant in terms of reflection and stretching or compressing. What is the geometric effect of multiplying the independent variable of a function f by a *negative* constant?

Table 0.2.4
STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c ($c > 1$)	Multiply $f(x)$ by c ($0 < c < 1$)	Multiply x by c ($c > 1$)	Multiply x by c ($0 < c < 1$)
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

SYMMETRY

Figure 0.2.7 illustrates three types of symmetries: ***symmetry about the x-axis***, ***symmetry about the y-axis***, and ***symmetry about the origin***. As illustrated in the figure, a curve is symmetric about the x-axis if for each point (x, y) on the graph the point $(x, -y)$ is also on the graph, and it is symmetric about the y-axis if for each point (x, y) on the graph the point $(-x, y)$ is also on the graph. A curve is symmetric about the origin if for each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. (Equivalently, a graph is symmetric about the origin if rotating the graph 180° about the origin leaves it unchanged.) This suggests the following symmetry tests.

> Explain why the graph of a nonzero function cannot be symmetric about the x-axis.

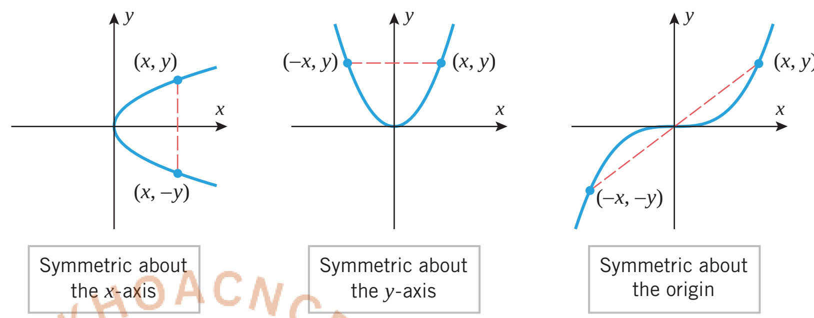


Figure 0.2.7

0.2.3 THEOREM ***(Symmetry Tests)***

(*a*) *A plane curve is symmetric about the y-axis if and only if replacing x by $-x$ in its equation produces an equivalent equation.*

(*b*) *A plane curve is symmetric about the x-axis if and only if replacing y by $-y$ in its equation produces an equivalent equation.*

(*c*) *A plane curve is symmetric about the origin if and only if replacing both x by $-x$ and y by $-y$ in its equation produces an equivalent equation.*

Example 11 Use Theorem 0.2.3 to identify symmetries in the graph of $x = y^2$.

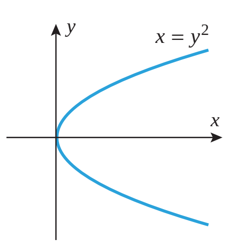


Figure 0.2.8

Solution. Replacing y by $-y$ yields $x = (-y)^2$, which simplifies to the original equation $x = y^2$. Thus, the graph is symmetric about the x-axis. The graph is not symmetric about the y-axis because replacing x by $-x$ yields $-x = y^2$, which is not equivalent to the original equation $x = y^2$. Similarly, the graph is not symmetric about the origin because replacing x by $-x$ and y by $-y$ yields $-x = (-y)^2$, which simplifies to $-x = y^2$, and this is again not equivalent to the original equation. These results are consistent with the graph of $x = y^2$ shown in Figure 0.2.8. ◀

EVEN AND ODD FUNCTIONS

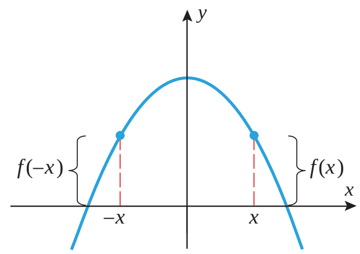
A function f is said to be an ***even function*** if

$$f(-x) = f(x) \tag{8}$$

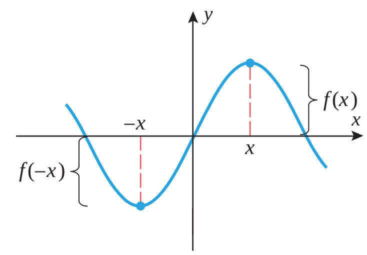
and is said to be an ***odd function*** if

$$f(-x) = -f(x) \tag{9}$$

Geometrically, the graphs of even functions are symmetric about the y-axis because replacing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$, which is equivalent to the original equation $y = f(x)$ by (8) (see Figure 0.2.9). Similarly, it follows from (9) that graphs of odd functions are symmetric about the origin (see Figure 0.2.10). Some examples of even functions are x^2, x^4, x^6, and $\cos x$; and some examples of odd functions are x^3, x^5, x^7, and $\sin x$.



▲ **Figure 0.2.9** This is the graph of an even function since $f(-x) = f(x)$.



▲ **Figure 0.2.10** This is the graph of an odd function since $f(-x) = -f(x)$.

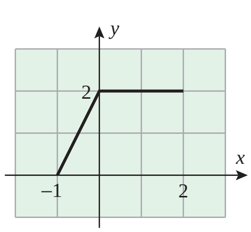
✔ QUICK CHECK EXERCISES 0.2 *(See page 27 for answers.)*

1. Let $f(x) = 3\sqrt{x} - 2$ and $g(x) = |x|$. In each part, give the formula for the function and state the corresponding domain.
   (a) $f + g$: ________ Domain: ________
   (b) $f - g$: ________ Domain: ________
   (c) fg: ________ Domain: ________
   (d) f/g: ________ Domain: ________
2. Let $f(x) = 2 - x^2$ and $g(x) = \sqrt{x}$. In each part, give the formula for the composition and state the corresponding domain.
   (a) $f \circ g$: ________ Domain: ________
   (b) $g \circ f$: ________ Domain: ________
3. The graph of $y = 1 + (x - 2)^2$ may be obtained by shifting the graph of $y = x^2$ ________ (left/right) by ________ unit(s) and then shifting this new graph ________ (up/down) by ________ unit(s).
4. Let
$$f(x) = \begin{cases} |x + 1|, & -2 \le x \le 0 \\ |x - 1|, & 0 < x \le 2 \end{cases}$$
   (a) The letter of the alphabet that most resembles the graph of f is ________.
   (b) Is f an even function?

EXERCISE SET 0.2 Graphing Utility

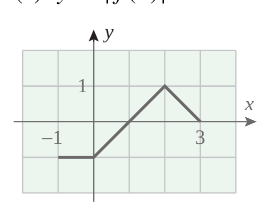
FOCUS ON CONCEPTS

1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
   (a) $y = f(x) - 1$ (b) $y = f(x - 1)$
   (c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$



◀ **Figure Ex-1**

2. Use the graph in Exercise 1 to sketch the graphs of the following equations.
   (a) $y = -f(-x)$ (b) $y = f(2 - x)$
   (c) $y = 1 - f(2 - x)$ (d) $y = \frac{1}{2}f(2x)$
3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
   (a) $y = f(x + 1)$ (b) $y = f(2x)$
   (c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$



◀ **Figure Ex-3**

4. Use the graph in Exercise 3 to sketch the graph of the equation $y = f(|x|)$.

5–24 Sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = x^2$, $y = \sqrt{x}$, $y = 1/x$, $y = |x|$, or $y = \sqrt[3]{x}$ appropriately. Then use a graphing utility to confirm that your sketch is correct. ■

5. $y = -2(x+1)^2 - 3$

6. $y = \frac{1}{2}(x-3)^2 + 2$

7. $y = x^2 + 6x$

8. $y = \frac{1}{2}(x^2 - 2x + 3)$

9. $y = 3 - \sqrt{x+1}$

10. $y = 1 + \sqrt{x-4}$

11. $y = \frac{1}{2}\sqrt{x} + 1$

12. $y = -\sqrt{3x}$

13. $y = \dfrac{1}{x-3}$

14. $y = \dfrac{1}{1-x}$

15. $y = 2 - \dfrac{1}{x+1}$

16. $y = \dfrac{x-1}{x}$

17. $y = |x+2| - 2$

18. $y = 1 - |x-3|$

19. $y = |2x-1| + 1$

20. $y = \sqrt{x^2 - 4x + 4}$

21. $y = 1 - 2\sqrt[3]{x}$

22. $y = \sqrt[3]{x-2} - 3$

23. $y = 2 + \sqrt[3]{x+1}$

24. $y + \sqrt[3]{x-2} = 0$

25. (a) Sketch the graph of $y = x + |x|$ by adding the corresponding y-coordinates on the graphs of $y = x$ and $y = |x|$.
(b) Express the equation $y = x + |x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

26. Sketch the graph of $y = x + (1/x)$ by adding corresponding y-coordinates on the graphs of $y = x$ and $y = 1/x$. Use a graphing utility to confirm that your sketch is correct.

27–28 Find formulas for $f+g$, $f-g$, fg, and f/g, and state the domains of the functions.

27. $f(x) = 2\sqrt{x-1},\ g(x) = \sqrt{x-1}$

28. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

29. Let $f(x) = \sqrt{x}$ and $g(x) = x^3 + 1$. Find
(a) $f(g(2))$ (b) $g(f(4))$ (c) $f(f(16))$
(d) $g(g(0))$ (e) $f(2+h)$ (f) $g(3+h)$.

30. Let $g(x) = \sqrt{x}$. Find
(a) $g(5s+2)$ (b) $g(\sqrt{x}+2)$ (c) $3g(5x)$
(d) $\dfrac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2 - g(x^2)$
(g) $g(1/\sqrt{x}\,)$ (h) $g((x-1)^2)$ (i) $g(x+h)$.

31–34 Find formulas for $f \circ g$ and $g \circ f$, and state the domains of the compositions.

31. $f(x) = x^2,\ g(x) = \sqrt{1-x}$

32. $f(x) = \sqrt{x-3},\ g(x) = \sqrt{x^2+3}$

33. $f(x) = \dfrac{1+x}{1-x},\ g(x) = \dfrac{x}{1-x}$

34. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

35–36 Find a formula for $f \circ g \circ h$.

35. $f(x) = x^2 + 1,\ g(x) = \dfrac{1}{x},\ h(x) = x^3$

36. $f(x) = \dfrac{1}{1+x},\ g(x) = \sqrt[3]{x},\ h(x) = \dfrac{1}{x^3}$

37–42 Express f as a composition of two functions; that is, find g and h such that $f = g \circ h$. [*Note:* Each exercise has more than one solution.]

37. (a) $f(x) = \sqrt{x+2}$ (b) $f(x) = |x^2 - 3x + 5|$

38. (a) $f(x) = x^2 + 1$ (b) $f(x) = \dfrac{1}{x-3}$

39. (a) $f(x) = \sin^2 x$ (b) $f(x) = \dfrac{3}{5 + \cos x}$

40. (a) $f(x) = 3\sin(x^2)$ (b) $f(x) = 3\sin^2 x + 4\sin x$

41. (a) $f(x) = \left(1 + \sin(x^2)\right)^3$ (b) $f(x) = \sqrt{1 - \sqrt[3]{x}}$

42. (a) $f(x) = \dfrac{1}{1-x^2}$ (b) $f(x) = |5 + 2x|$

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. The domain of $f+g$ is the intersection of the domains of f and g.

44. The domain of $f \circ g$ consists of all values of x in the domain of g for which $g(x) \neq 0$.

45. The graph of an even function is symmetric about the y-axis.

46. The graph of $y = f(x+2) + 3$ is obtained by translating the graph of $y = f(x)$ right 2 units and up 3 units.

FOCUS ON CONCEPTS

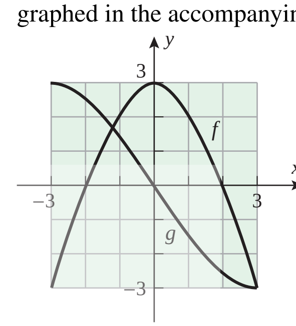
47. Use the data in the accompanying table to make a plot of $y = f(g(x))$.

x	−3	−2	−1	0	1	2	3
$f(x)$	−4	−3	−2	−1	0	1	2
$g(x)$	−1	0	1	2	3	−2	−3

▲ **Table Ex-47**

48. Find the domain of $g \circ f$ for the functions f and g in Exercise 47.

49. Sketch the graph of $y = f(g(x))$ for the functions graphed in the accompanying figure.



◀ **Figure Ex-49**

50. Sketch the graph of $y = g(f(x))$ for the functions graphed in Exercise 49.

51. Use the graphs of f and g in Exercise 49 to estimate the solutions of the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

52. Use the table given in Exercise 47 to solve the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

53–56 Find

$$\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \frac{f(w)-f(x)}{w-x}$$

Simplify as much as possible. ■

53. $f(x)=3x^2-5$ **54.** $f(x)=x^2+6x$

55. $f(x)=1/x$ **56.** $f(x)=1/x^2$

57. Classify the functions whose values are given in the accompanying table as even, odd, or neither.

x	−3	−2	−1	0	1	2	3
$f(x)$	5	3	2	3	1	−3	5
$g(x)$	4	1	−2	0	2	−1	−4
$h(x)$	2	−5	8	−2	8	−5	2

▲ **Table Ex-57**

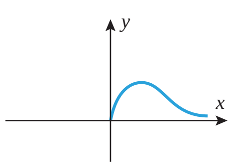
58. Complete the accompanying table so that the graph of $y=f(x)$ is symmetric about
(a) the y-axis (b) the origin.

x	−3	−2	−1	0	1	2	3
$f(x)$	1		−1	0		−5	

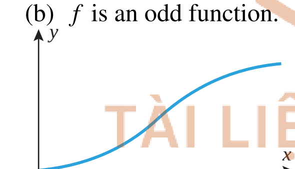
▲ **Table Ex-58**

59. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
(a) the x-axis (b) the y-axis (c) the origin.

60. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that
(a) f is an even function (b) f is an odd function.



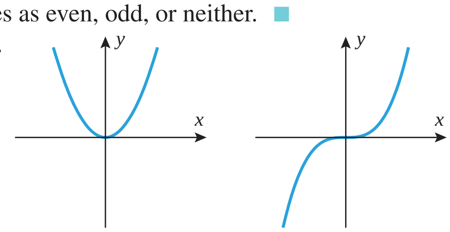
▲ **Figure Ex-59**



▲ **Figure Ex-60**

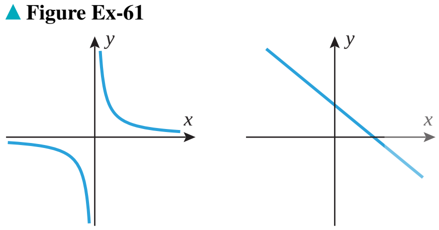
61–62 Classify the functions graphed in the accompanying figures as even, odd, or neither. ■

61.



▲ **Figure Ex-61**

62.



▲ **Figure Ex-62**

63. In each part, classify the function as even, odd, or neither.
(a) $f(x)=x^2$ (b) $f(x)=x^3$
(c) $f(x)=|x|$ (d) $f(x)=x+1$
(e) $f(x)=\dfrac{x^5-x}{1+x^2}$ (f) $f(x)=2$

64. Suppose that the function f has domain all real numbers. Determine whether each function can be classified as even or odd. Explain.
(a) $g(x)=\dfrac{f(x)+f(-x)}{2}$ (b) $h(x)=\dfrac{f(x)-f(-x)}{2}$

65. Suppose that the function f has domain all real numbers. Show that f can be written as the sum of an even function and an odd function. [*Hint:* See Exercise 64.]

66–67 Use Theorem 0.2.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin. ■

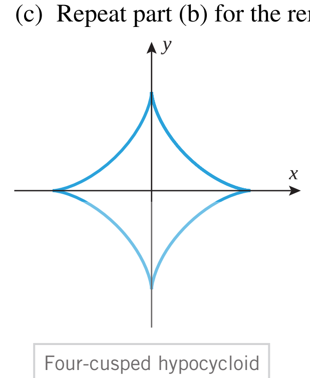
66. (a) $x=5y^2+9$ (b) $x^2-2y^2=3$
(c) $xy=5$

67. (a) $x^4=2y^3+y$ (b) $y=\dfrac{x}{3+x^2}$
(c) $y^2=|x|-5$

68–69 (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants. ■

68. $9x^2+4y^2=36$ **69.** $4x^2+16y^2=16$

70. The graph of the equation $x^{2/3}+y^{2/3}=1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.
(a) Use Theorem 0.2.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin.
(b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
(c) Repeat part (b) for the remaining three quadrants.



◀ **Figure Ex-70**

71. The equation $y=|f(x)|$ can be written as

$$y=\begin{cases} f(x), & f(x)\ge 0 \\ -f(x), & f(x)<0 \end{cases}$$

which shows that the graph of $y=|f(x)|$ can be obtained from the graph of $y=f(x)$ by retaining the portion that lies on or above the x-axis and reflecting about the x-axis the portion that lies below the x-axis. Use this method to obtain the graph of $y = |2x - 3|$ from the graph of $y = 2x - 3$.

72–73 Use the method described in Exercise 71. ■

72. Sketch the graph of $y = |1 - x^2|$.

73. Sketch the graph of
(a) $f(x) = |\cos x|$ (b) $f(x) = \cos x + |\cos x|$.

74. The ***greatest integer function***, $\lfloor x \rfloor$, is defined to be the greatest integer that is less than or equal to x. For example, $\lfloor 2.7 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$, and $\lfloor 4 \rfloor = 4$. In each part, sketch the graph of $y = f(x)$.
(a) $f(x) = \lfloor x \rfloor$ (b) $f(x) = \lfloor x^2 \rfloor$
(c) $f(x) = \lfloor x \rfloor^2$ (d) $f(x) = \lfloor \sin x \rfloor$

75. Is it ever true that $f \circ g = g \circ f$ if f and g are nonconstant functions? If not, prove it; if so, give some examples for which it is true.

✔ QUICK CHECK ANSWERS 0.2

1. (a) $(f+g)(x) = 3\sqrt{x} - 2 + x;\ x \geq 0$ (b) $(f-g)(x) = 3\sqrt{x} - 2 - x;\ x \geq 0$ (c) $(fg)(x) = 3x^{3/2} - 2x;\ x \geq 0$ (d) $(f/g)(x) = \dfrac{3\sqrt{x} - 2}{x};\ x > 0$ **2.** (a) $(f \circ g)(x) = 2 - x;\ x \geq 0$ (b) $(g \circ f)(x) = \sqrt{2 - x^2};\ -\sqrt{2} \leq x \leq \sqrt{2}$ **3.** right; 2; up; 1 **4.** (a) W (b) yes

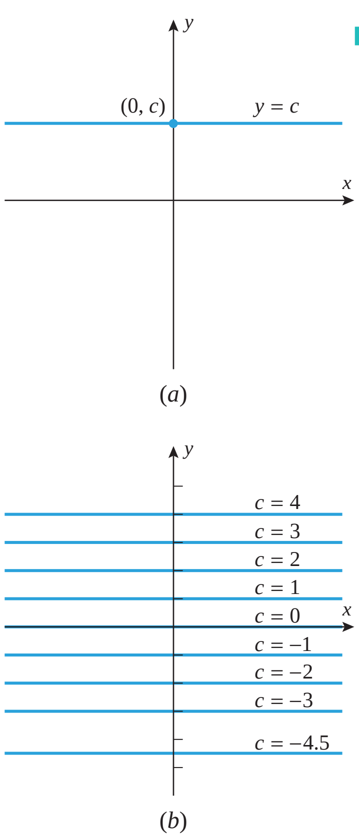
0.3 FAMILIES OF FUNCTIONS

Functions are often grouped into families according to the form of their defining formulas or other common characteristics. In this section we will discuss some of the most basic families of functions.

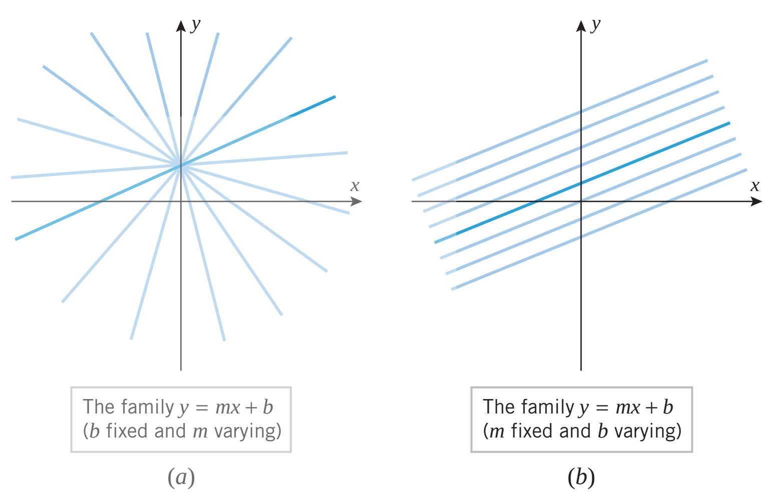
FAMILIES OF CURVES

The graph of a constant function $f(x) = c$ is the graph of the equation $y = c$, which is the horizontal line shown in Figure 0.3.1a. If we vary c, then we obtain a set or ***family*** of horizontal lines such as those in Figure 0.3.1b.

Constants that are varied to produce families of curves are called ***parameters***. For example, recall that an equation of the form $y = mx + b$ represents a line of slope m and y-intercept b. If we keep b fixed and treat m as a parameter, then we obtain a family of lines whose members all have y-intercept b (Figure 0.3.2a), and if we keep m fixed and treat b as a parameter, we obtain a family of parallel lines whose members all have slope m (Figure 0.3.2b).



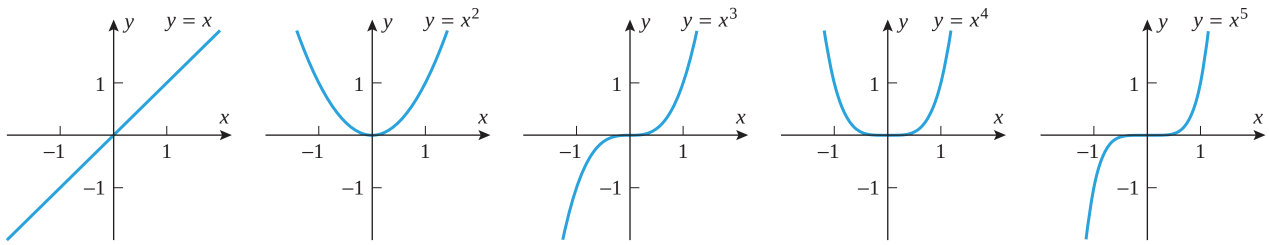
▲ **Figure 0.3.1**



► **Figure 0.3.2**

POWER FUNCTIONS; THE FAMILY $y = x^n$

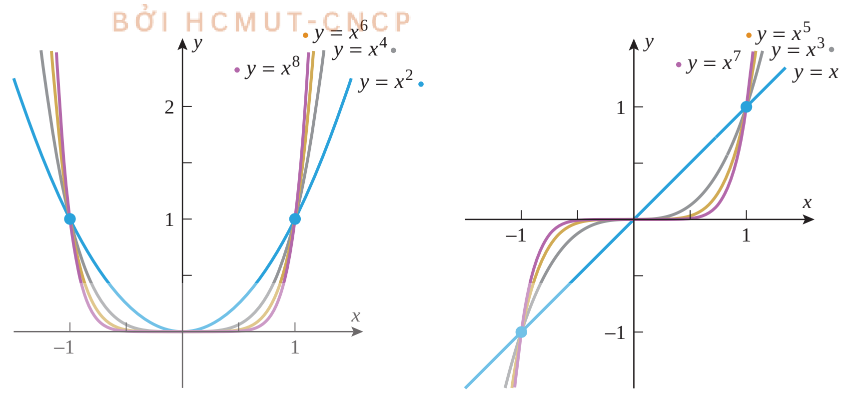
A function of the form $f(x) = x^p$, where p is constant, is called a ***power function***. For the moment, let us consider the case where p is a positive integer, say $p = n$. The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 0.3.3. The first graph is the line with slope 1 that passes through the origin, and the second is a parabola that opens up and has its vertex at the origin (see Web Appendix H).



▲ **Figure 0.3.3**

For $n \geq 2$ the shape of the curve $y = x^n$ depends on whether n is even or odd (Figure 0.3.4):

- For even values of n, the functions $f(x) = x^n$ are even, so their graphs are symmetric about the y-axis. The graphs all have the general shape of the graph of $y = x^2$, and each graph passes through the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.
- For odd values of n, the functions $f(x) = x^n$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $y = x^3$, and each graph passes through the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.



The family $y = x^n$ (n even)

The family $y = x^n$ (n odd)

▶ **Figure 0.3.4**

REMARK The flattening and steepening effects can be understood by considering what happens when a number x is raised to higher and higher powers: If $-1 < x < 1$, then the absolute value of x^n *decreases* as n increases, thereby causing the graphs to become flatter on this interval as n increases (try raising $\frac{1}{2}$ or $-\frac{1}{2}$ to higher and higher powers). On the other hand, if $x > 1$ or $x < -1$, then the absolute value of x^n *increases* as n increases, thereby causing the graphs to become steeper on these intervals as n increases (try raising 2 or -2 to higher and higher powers).

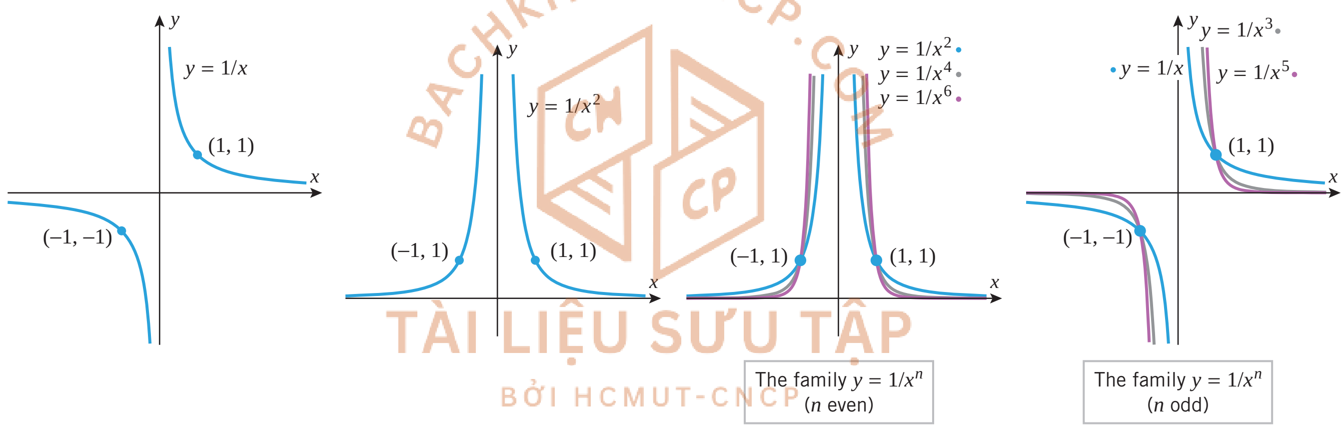
THE FAMILY $y = x^{-n}$

If p is a negative integer, say $p = -n$, then the power functions $f(x) = x^p$ have the form $f(x) = x^{-n} = 1/x^n$. Figure 0.3.5 shows the graphs of $y = 1/x$ and $y = 1/x^2$. The graph of $y = 1/x$ is called an ***equilateral hyperbola*** (for reasons to be discussed later).

As illustrated in Figure 0.3.5, the shape of the curve $y = 1/x^n$ depends on whether n is even or odd:

- For even values of n, the functions $f(x) = 1/x^n$ are even, so their graphs are symmetric about the y-axis. The graphs all have the general shape of the curve $y = 1/x^2$, and each graph passes through the points $(-1, 1)$ and $(1, 1)$. As n increases, the graphs become steeper over the intervals $-1 < x < 0$ and $0 < x < 1$ and become flatter over the intervals $x > 1$ and $x < -1$.
- For odd values of n, the functions $f(x) = 1/x^n$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $y = 1/x$, and each graph passes through the points $(1, 1)$ and $(-1, -1)$. As n increases, the graphs become steeper over the intervals $-1 < x < 0$ and $0 < x < 1$ and become flatter over the intervals $x > 1$ and $x < -1$.
- For both even and odd values of n the graph $y = 1/x^n$ has a break at the origin (called a ***discontinuity***), which occurs because division by zero is undefined.

By considering the value of $1/x^n$ for a fixed x as n increases, explain why the graphs become flatter or steeper as described here for increasing values of n.



▲ **Figure 0.3.5**

INVERSE PROPORTIONS

Recall that a variable y is said to be ***inversely proportional to a variable x*** if there is a positive constant k, called the ***constant of proportionality***, such that

$$y = \frac{k}{x} \tag{1}$$

Since k is assumed to be positive, the graph of (1) has the same shape as $y = 1/x$ but is compressed or stretched in the y-direction. Also, it should be evident from (1) that doubling x multiplies y by $\frac{1}{2}$, tripling x multiplies y by $\frac{1}{3}$, and so forth.

Equation (1) can be expressed as $xy = k$, which tells us that the product of inversely proportional variables is a positive constant. This is a useful form for identifying inverse proportionality in experimental data.

Table 0.3.1

x	0.8	1	2.5	4	6.25	10
y	6.25	5	2	1.25	0.8	0.5

▶ **Example 1** Table 0.3.1 shows some experimental data.

(a) Explain why the data suggest that y is inversely proportional to x.

(b) Express y as a function of x.

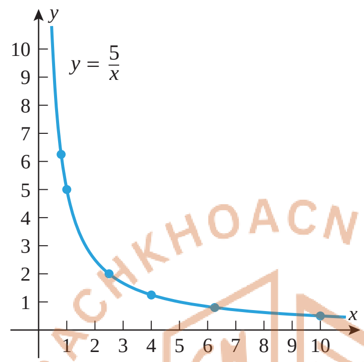
(c) Graph your function and the data together for $x > 0$.

Solution. For every data point we have $xy = 5$, so y is inversely proportional to x and $y = 5/x$. The graph of this equation with the data points is shown in Figure 0.3.6. ◂

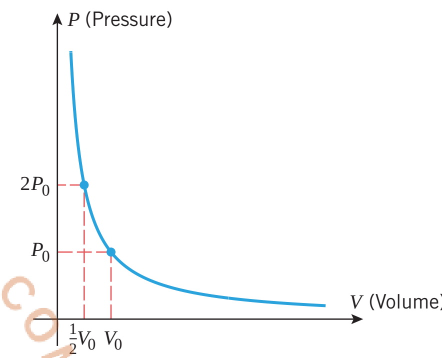
Inverse proportions arise in various laws of physics. For example, ***Boyle's law*** in physics states that *if a fixed amount of an ideal gas is held at a constant temperature, then the product of the pressure P exerted by the gas and the volume V that it occupies is constant*; that is,

$$PV = k$$

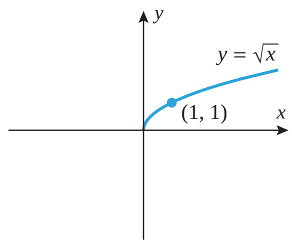
This implies that the variables P and V are inversely proportional to one another. Figure 0.3.7 shows a typical graph of volume versus pressure under the conditions of Boyle's law. Note how doubling the pressure corresponds to halving the volume, as expected.



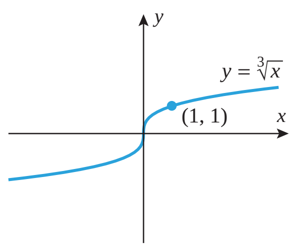
▲ **Figure 0.3.6**



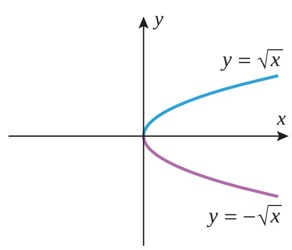
▲ **Figure 0.3.7** Doubling pressure corresponds to halving volume



(*a*)



(*b*)



(*c*)

▲ **Figure 0.3.8**

POWER FUNCTIONS WITH NONINTEGER EXPONENTS

If $p = 1/n$, where n is a positive integer, then the power functions $f(x) = x^p$ have the form

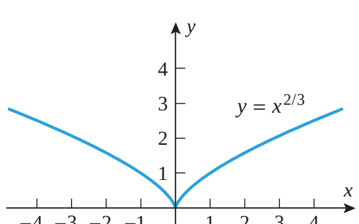
$$f(x) = x^{1/n} = \sqrt[n]{x}$$

In particular, if $n = 2$, then $f(x) = \sqrt{x}$, and if $n = 3$, then $f(x) = \sqrt[3]{x}$. The graphs of these functions are shown in parts (*a*) and (*b*) of Figure 0.3.8.

Since every real number has a real cube root, the domain of the function $f(x) = \sqrt[3]{x}$ is $(-\infty, +\infty)$, and hence the graph of $y = \sqrt[3]{x}$ extends over the entire x-axis. In contrast, the graph of $y = \sqrt{x}$ extends only over the interval $[0, +\infty)$ because $\sqrt{x}$ is imaginary for negative x. As illustrated in Figure 0.3.8*c*, the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ form the upper and lower halves of the parabola $x = y^2$. In general, the graph of $y = \sqrt[n]{x}$ extends over the entire x-axis if n is odd, but extends only over the interval $[0, +\infty)$ if n is even.

Power functions can have other fractional exponents. Some examples are

$$f(x) = x^{2/3}, \quad f(x) = \sqrt[5]{x^3}, \quad f(x) = x^{-7/8} \tag{2}$$



▲ **Figure 0.3.9**

The graph of $f(x) = x^{2/3}$ is shown in Figure 0.3.9. We will discuss expressions involving irrational exponents later.

TECHNOLOGY MASTERY

Graphing utililties sometimes omit portions of the graph of a function involving fractional exponents (or radicals). If $f(x) = x^{p/q}$, where p/q is a positive fraction in *lowest terms*, then you can circumvent this problem as follows:

- If p is even and q is odd, then graph $g(x) = |x|^{p/q}$ instead of $f(x)$.
- If p is odd and q is odd, then graph $g(x) = (|x|/x)|x|^{p/q}$ instead of $f(x)$.

Use a graphing utility to generate graphs of $f(x) = \sqrt[5]{x^3}$ and $f(x) = x^{-7/8}$ that show all of their significant features.

POLYNOMIALS

A ***polynomial in x*** is a function that is expressible as a sum of finitely many terms of the form cx^n, where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$$2x+1, \quad 3x^2+5x-\sqrt{2}, \quad x^3, \quad 4\,(=4x^0), \quad 5x^7-x^4+3$$

The function $(x^2-4)^3$ is also a polynomial because it can be expanded by the binomial formula (see the inside front cover) and expressed as a sum of terms of the form cx^n:

$$(x^2-4)^3 = (x^2)^3 - 3(x^2)^2(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64 \tag{3}$$

A more detailed review of polynomials appears in Appendix C.

A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order:

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$
$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants $c_0, c_1, \ldots, c_n$ are called the ***coefficients*** of the polynomial. When a polynomial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the ***degree*** of the polynomial. Nonzero constant polynomials are considered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as ***linear***, ***quadratic***, ***cubic***, ***quartic***, and ***quintic***, respectively. For example,

$3+5x$	x^2-3x+1	$2x^3-7$
Has degree 1 (linear)	Has degree 2 (quadratic)	Has degree 3 (cubic)
$8x^4-9x^3+5x-3$	$\sqrt{3}+x^3+x^5$	$(x^2-4)^3$
Has degree 4 (quartic)	Has degree 5 (quintic)	Has degree 6 [see (3)]

The constant 0 is a polynomial called the ***zero polynomial***. In this text we will take the degree of the zero polynomial to be undefined. Other texts may use different conventions for the degree of the zero polynomial.

The natural domain of a polynomial in x is $(-\infty, +\infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We already know that the graphs of polynomials of degree 0 and 1 are lines and that the graphs of polynomials of degree 2 are parabolas. Figure 0.3.10 shows the graphs of some typical polynomials of higher degree. Later, we will discuss polynomial graphs in detail, but for now it suffices to observe that graphs of polynomials are very well behaved in the sense that they have no discontinuities or sharp corners. As illustrated in Figure 0.3.10, the graphs of polynomials wander up and down for awhile in a roller-coaster fashion, but eventually that behavior stops and the graphs steadily rise or fall indefinitely as one travels along the curve in either the positive or negative direction. We will see later that the number of peaks and valleys is less than the degree of the polynomial.

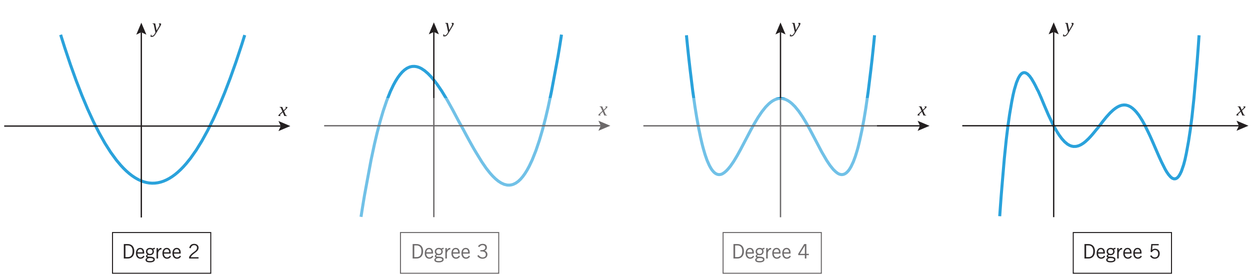


Figure 0.3.10

RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a ***rational function***. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

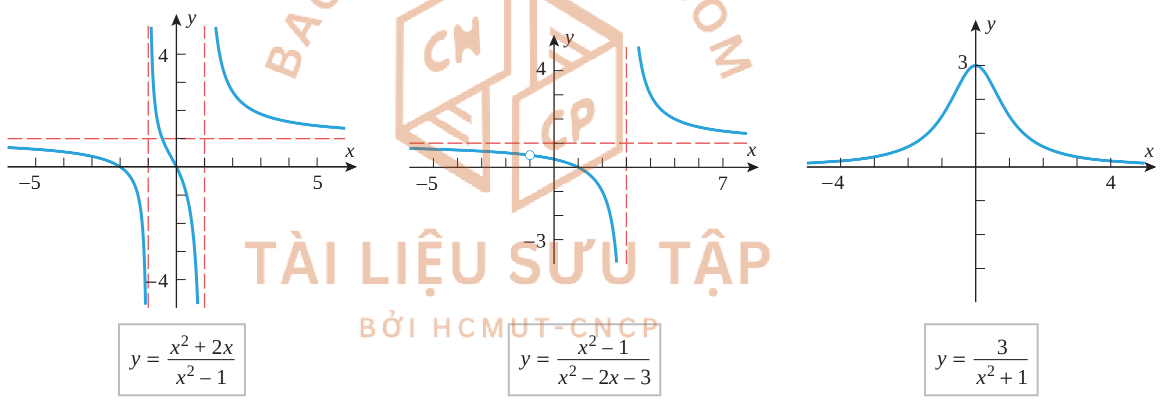
consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the rational function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

consists of all values of x, except $x = 1$ and $x = -1$. Its graph is shown in Figure 0.3.11 along with the graphs of two other typical rational functions.

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

- Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero.
- Unlike polynomials, rational functions may have numbers at which they are not defined. Near such points, many rational functions have graphs that closely approximate a vertical line, called a ***vertical asymptote***. These are represented by the dashed vertical lines in Figure 0.3.11.
- Unlike the graphs of nonconstant polynomials, which eventually rise or fall indefinitely, the graphs of many rational functions eventually get closer and closer to some horizontal line, called a ***horizontal asymptote***, as one traverses the curve in either the positive or negative direction. The horizontal asymptotes are represented by the dashed horizontal lines in the first two parts of Figure 0.3.11. In the third part of the figure the x-axis is a horizontal asymptote.



▲ **Figure 0.3.11**

ALGEBRAIC FUNCTIONS

Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, multiplication, division, and root extraction) are called ***algebraic functions***. Some examples are

$$f(x) = \sqrt{x^2 - 4}, \quad f(x) = 3\sqrt[3]{x}(2 + x), \quad f(x) = x^{2/3}(x + 2)^2$$

As illustrated in Figure 0.3.12, the graphs of algebraic functions vary widely, so it is difficult to make general statements about them. Later in this text we will develop general calculus methods for analyzing such functions.

THE FAMILIES $y = A \sin Bx$ AND $y = A \cos Bx$

In this text we will assume that the independent variable of a trigonometric function is in radians unless otherwise stated. A review of trigonometric functions can be found in Appendix B.

Many important applications lead to trigonometric functions of the form

$$f(x) = A\sin(Bx - C) \quad \text{and} \quad g(x) = A\cos(Bx - C) \tag{4}$$

where A, B, and C are nonzero constants. The graphs of such functions can be obtained by stretching, compressing, translating, and reflecting the graphs of $y = \sin x$ and $y = \cos x$

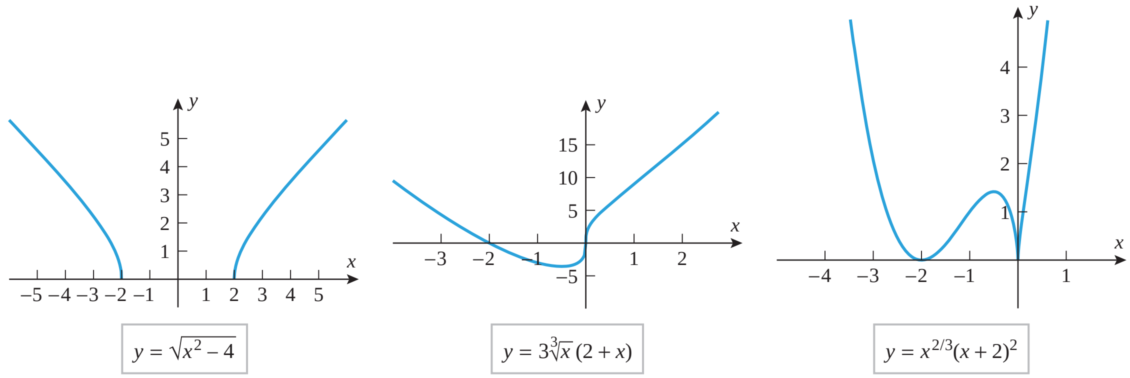


Figure 0.3.12

appropriately. To see why this is so, let us start with the case where $C = 0$ and consider how the graphs of the equations

$$y = A \sin Bx \quad \text{and} \quad y = A \cos Bx$$

relate to the graphs of $y = \sin x$ and $y = \cos x$. If A and B are positive, then the effect of the constant A is to stretch or compress the graphs of $y = \sin x$ and $y = \cos x$ vertically and the effect of the constant B is to compress or stretch the graphs of $\sin x$ and $\cos x$ horizontally. For example, the graph of $y = 2 \sin 4x$ can be obtained by stretching the graph of $y = \sin x$ vertically by a factor of 2 and compressing it horizontally by a factor of 4. (Recall from Section 0.2 that the multiplier of x *stretches* when it is less than 1 and *compresses* when it is greater than 1.) Thus, as shown in Figure 0.3.13, the graph of $y = 2 \sin 4x$ varies between -2 and 2, and repeats every $2\pi/4 = \pi/2$ units.

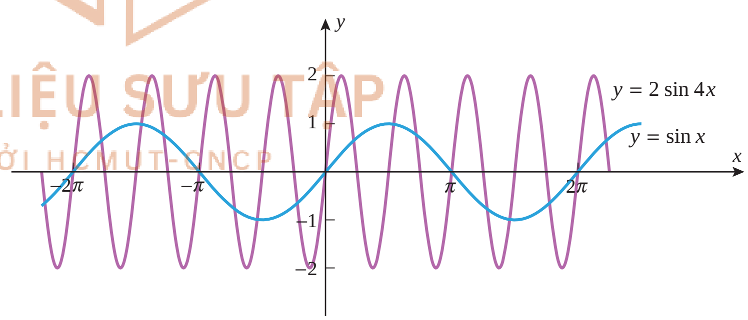


Figure 0.3.13

In general, if A and B are positive numbers, then the graphs of

$$y = A \sin Bx \quad \text{and} \quad y = A \cos Bx$$

oscillate between $-A$ and A and repeat every $2\pi/B$ units, so we say that these functions have ***amplitude*** A and ***period*** $2\pi/B$. In addition, we define the ***frequency*** of these functions to be the reciprocal of the period, that is, the frequency is $B/2\pi$. If A or B is negative, then these constants cause reflections of the graphs about the axes as well as compressing or stretching them; and in this case the amplitude, period, and frequency are given by

$$\text{amplitude} = |A|, \quad \text{period} = \frac{2\pi}{|B|}, \quad \text{frequency} = \frac{|B|}{2\pi}$$

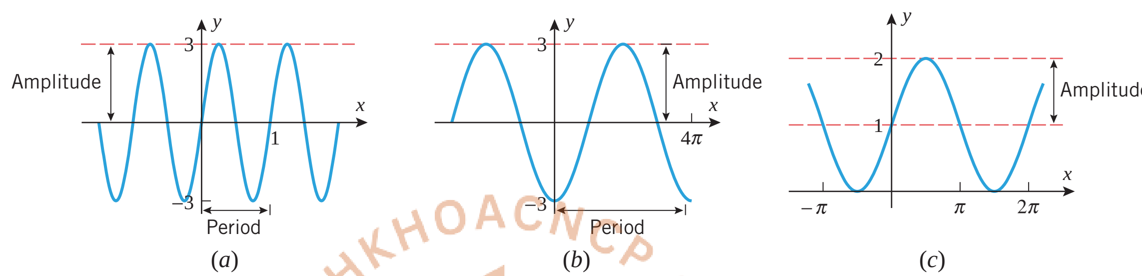
Example 2 Make sketches of the following graphs that show the period and amplitude.

(a) $y = 3 \sin 2\pi x$ (b) $y = -3 \cos 0.5x$ (c) $y = 1 + \sin x$

Solution (a). The equation is of the form $y = A \sin Bx$ with $A = 3$ and $B = 2\pi$, so the graph has the shape of a sine function, but it has an amplitude of $A = 3$ and a period of $2\pi/B = 2\pi/2\pi = 1$ (Figure 0.3.14*a*).

Solution (b). The equation is of the form $y = A \cos Bx$ with $A = -3$ and $B = 0.5$, so the graph has the shape of a cosine curve that has been reflected about the x-axis (because $A = -3$ is negative), but with amplitude $|A| = 3$ and period $2\pi/B = 2\pi/0.5 = 4\pi$ (Figure 0.3.14*b*).

Solution (c). The graph has the shape of a sine curve that has been translated up 1 unit (Figure 0.3.14*c*). ◀



▲ **Figure 0.3.14**

THE FAMILIES $y = A\sin(Bx - C)$ AND $y = A\cos(Bx - C)$

To investigate the graphs of the more general families

$$y = A\sin(Bx - C) \quad \text{and} \quad y = A\cos(Bx - C)$$

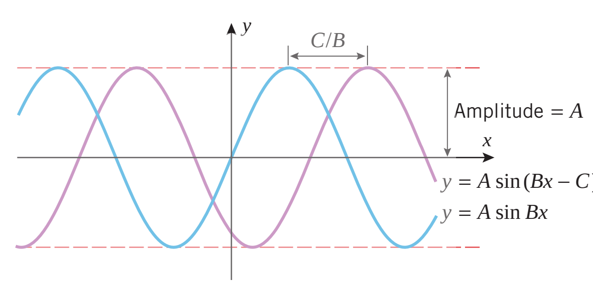
it will be helpful to rewrite these equations as

$$y = A\sin\left[B\left(x - \frac{C}{B}\right)\right] \quad \text{and} \quad y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$$

In this form we see that the graphs of these equations can be obtained by translating the graphs of $y = A\sin Bx$ and $y = A\cos Bx$ to the left or right, depending on the sign of C/B. For example, if $C/B > 0$, then the graph of

$$y = A\sin[B(x - C/B)] = A\sin(Bx - C)$$

can be obtained by translating the graph of $y = A\sin Bx$ to the right by C/B units (Figure 0.3.15). If $C/B < 0$, the graph of $y = A\sin(Bx - C)$ is obtained by translating the graph of $y = A\sin Bx$ to the left by $|C/B|$ units.

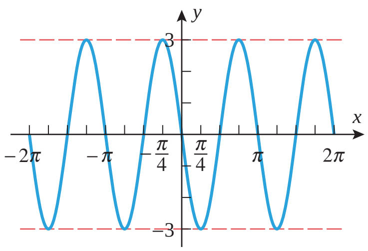


▶ **Figure 0.3.15**

▶ **Example 3** Find the amplitude and period of

$$y = 3\cos\left(2x + \frac{\pi}{2}\right)$$

and determine how the graph of $y = 3\cos 2x$ should be translated to produce the graph of this equation. Confirm your results by graphing the equation on a calculator or computer.



▲ **Figure 0.3.16**

Solution. The equation can be rewritten as

$$y = 3\cos\left[2x - \left(-\frac{\pi}{2}\right)\right] = 3\cos\left[2\left(x - \left(-\frac{\pi}{4}\right)\right)\right]$$

which is of the form

$$y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$$

with $A = 3$, $B = 2$, and $C/B = -\pi/4$. It follows that the amplitude is $A = 3$, the period is $2\pi/B = \pi$, and the graph is obtained by translating the graph of $y = 3\cos 2x$ left by $|C/B| = \pi/4$ units (Figure 0.3.16). ◀

✔ QUICK CHECK EXERCISES 0.3 *(See page 38 for answers.)*

1. Consider the family of functions $y = x^n$, where n is an integer. The graphs of $y = x^n$ are symmetric with respect to the y-axis if n is ________. These graphs are symmetric with respect to the origin if n is ________. The y-axis is a vertical asymptote for these graphs if n is ________.

2. What is the natural domain of a polynomial?

3. Consider the family of functions $y = x^{1/n}$, where n is a nonzero integer. Find the natural domain of these functions if n is
(a) positive and even (b) positive and odd
(c) negative and even (d) negative and odd.

4. Classify each equation as a polynomial, rational, algebraic, or not an algebraic function.
(a) $y = \sqrt{x} + 2$ (b) $y = \sqrt{3}x^4 - x + 1$
(c) $y = 5x^3 + \cos 4x$ (d) $y = \dfrac{x^2 + 5}{2x - 7}$
(e) $y = 3x^2 + 4x^{-2}$

5. The graph of $y = A\sin Bx$ has amplitude ________ and is periodic with period ________.

EXERCISE SET 0.3 Graphing Utility

1. (a) Find an equation for the family of lines whose members have slope $m = 3$.
(b) Find an equation for the member of the family that passes through $(-1, 3)$.
(c) Sketch some members of the family, and label them with their equations. Include the line in part (b).

2. Find an equation for the family of lines whose members are perpendicular to those in Exercise 1.

3. (a) Find an equation for the family of lines with y-intercept $b = 2$.
(b) Find an equation for the member of the family whose angle of inclination is $135°$.
(c) Sketch some members of the family, and label them with their equations. Include the line in part (b).

4. Find an equation for
(a) the family of lines that pass through the origin
(b) the family of lines with x-intercept $a = 1$
(c) the family of lines that pass through the point $(1, -2)$
(d) the family of lines parallel to $2x + 4y = 1$.

5. Find an equation for the family of lines tangent to the circle with center at the origin and radius 3.

6. Find an equation for the family of lines that pass through the intersection of $5x - 3y + 11 = 0$ and $2x - 9y + 7 = 0$.

7. The U.S. Internal Revenue Service uses a 10-year linear depreciation schedule to determine the value of various business items. This means that an item is assumed to have a value of zero at the end of the tenth year and that at intermediate times the value is a linear function of the elapsed time. Sketch some typical depreciation lines, and explain the practical significance of the y-intercepts.

8. Find all lines through $(6, -1)$ for which the product of the x- and y-intercepts is 3.

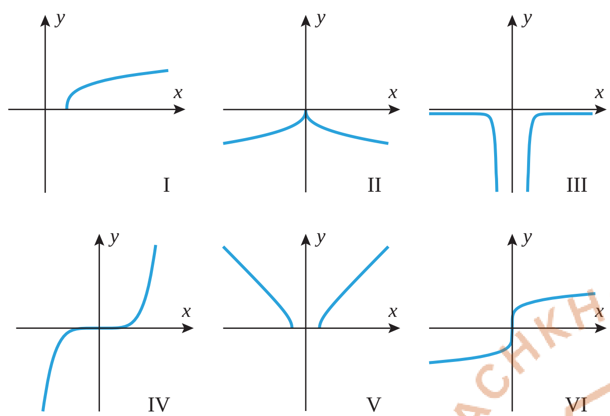
FOCUS ON CONCEPTS

9–10 State a geometric property common to all lines in the family, and sketch five of the lines. ■

9. (a) The family $y = -x + b$
(b) The family $y = mx - 1$
(c) The family $y = m(x + 4) + 2$
(d) The family $x - ky = 1$

10. (a) The family $y = b$
(b) The family $Ax + 2y + 1 = 0$
(c) The family $2x + By + 1 = 0$
(d) The family $y - 1 = m(x + 1)$

11. In each part, match the equation with one of the accompanying graphs.
(a) $y = \sqrt[5]{x}$ (b) $y = 2x^5$
(c) $y = -1/x^8$ (d) $y = \sqrt{x^2 - 1}$
(e) $y = \sqrt[4]{x - 2}$ (f) $y = -\sqrt[5]{x^2}$



▲ **Figure Ex-11**

12. The accompanying table gives approximate values of three functions: one of the form kx^2, one of the form kx^{-3}, and one of the form $kx^{3/2}$. Identify which is which, and estimate k in each case.

x	0.25	0.37	2.1	4.0	5.8	6.2	7.9	9.3
$f(x)$	640	197	1.08	0.156	0.0513	0.0420	0.0203	0.0124
$g(x)$	0.0312	0.0684	2.20	8.00	16.8	19.2	31.2	43.2
$h(x)$	0.250	0.450	6.09	16.0	27.9	30.9	44.4	56.7

▲ **Table Ex-12**

13–14 Sketch the graph of the equation for $n = 1, 3,$ and 5 in one coordinate system and for $n = 2, 4,$ and 6 in another coordinate system. If you have a graphing utility, use it to check your work. ■

13. (a) $y = -x^n$ (b) $y = 2x^{-n}$ (c) $y = (x - 1)^{1/n}$

14. (a) $y = 2x^n$ (b) $y = -x^{-n}$
(c) $y = -3(x + 2)^{1/n}$

15. (a) Sketch the graph of $y = ax^2$ for $a = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(b) Sketch the graph of $y = x^2 + b$ for $b = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = ax^2 + b$.

16. (a) Sketch the graph of $y = a\sqrt{x}$ for $a = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(b) Sketch the graph of $y = \sqrt{x} + b$ for $b = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = a\sqrt{x} + b$.

17–18 Sketch the graph of the equation by making appropriate transformations to the graph of a basic power function. If you have a graphing utility, use it to check your work. ■

17. (a) $y = 2(x + 1)^2$ (b) $y = -3(x - 2)^3$
(c) $y = \dfrac{-3}{(x + 1)^2}$ (d) $y = \dfrac{1}{(x - 3)^5}$

18. (a) $y = 1 - \sqrt{x + 2}$ (b) $y = 1 - \sqrt[3]{x + 2}$
(c) $y = \dfrac{5}{(1 - x)^3}$ (d) $y = \dfrac{2}{(4 + x)^4}$

19. Sketch the graph of $y = x^2 + 2x$ by completing the square and making appropriate transformations to the graph of $y = x^2$.

20. (a) Use the graph of $y = \sqrt{x}$ to help sketch the graph of $y = \sqrt{|x|}$.
(b) Use the graph of $y = \sqrt[3]{x}$ to help sketch the graph of $y = \sqrt[3]{|x|}$.

21. As discussed in this section, Boyle's law states that at a constant temperature the pressure P exerted by a gas is related to the volume V by the equation $PV = k$.
(a) Find the appropriate units for the constant k if pressure (which is force per unit area) is in newtons per square meter (N/m^2) and volume is in cubic meters (m^3).
(b) Find k if the gas exerts a pressure of 20,000 N/m^2 when the volume is 1 liter (0.001 m^3).
(c) Make a table that shows the pressures for volumes of 0.25, 0.5, 1.0, 1.5, and 2.0 liters.
(d) Make a graph of P versus V.

22. A manufacturer of cardboard drink containers wants to construct a closed rectangular container that has a square base and will hold $\frac{1}{10}$ liter (100 cm^3). Estimate the dimensions of the container that will require the least amount of material for its manufacture.

23–24 A variable y is said to be ***inversely proportional to the square of a variable*** x if y is related to x by an equation of the form $y = k/x^2$, where k is a nonzero constant, called the ***constant of proportionality***. This terminology is used in these exercises. ■

23. According to ***Coulomb's law***, the force F of attraction between positive and negative point charges is inversely proportional to the square of the distance x between them.
(a) Assuming that the force of attraction between two point charges is 0.0005 newton when the distance between them is 0.3 meter, find the constant of proportionality (with proper units).
(b) Find the force of attraction between the point charges when they are 3 meters apart.
(c) Make a graph of force versus distance for the two charges.

(cont.)

(d) What happens to the force as the particles get closer and closer together? What happens as they get farther and farther apart?

24. It follows from Newton's Law of Universal Gravitation that the weight W of an object (relative to the Earth) is inversely proportional to the square of the distance x between the object and the center of the Earth, that is, $W = C/x^2$.
(a) Assuming that a weather satellite weighs 2000 pounds on the surface of the Earth and that the Earth is a sphere of radius 4000 miles, find the constant C.
(b) Find the weight of the satellite when it is 1000 miles above the surface of the Earth.
(c) Make a graph of the satellite's weight versus its distance from the center of the Earth.
(d) Is there any distance from the center of the Earth at which the weight of the satellite is zero? Explain your reasoning.

25–28 True–False Determine whether the statement is true or false. Explain your answer.

25. Each curve in the family $y = 2x + b$ is parallel to the line $y = 2x$.

26. Each curve in the family $y = x^2 + bx + c$ is a translation of the graph of $y = x^2$.

27. If a curve passes through the point $(2, 6)$ and y is inversely proportional to x, then the constant of proportionality is 3.

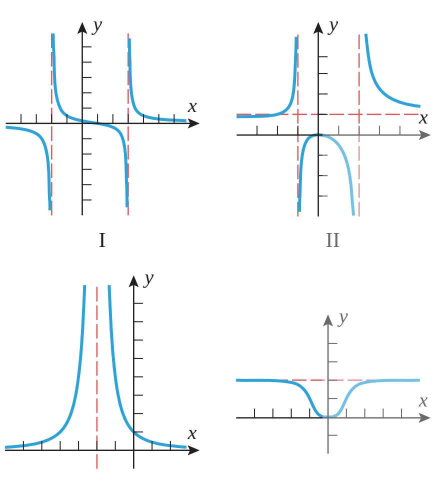
28. Curves in the family $y = -5\sin(A\pi x)$ have amplitude 5 and period $2/|A|$.

FOCUS ON CONCEPTS

29. In each part, match the equation with one of the accompanying graphs, and give the equations for the horizontal and vertical asymptotes.

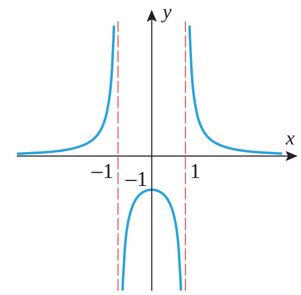
(a) $y = \dfrac{x^2}{x^2 - x - 2}$ (b) $y = \dfrac{x-1}{x^2 - x - 6}$

(c) $y = \dfrac{2x^4}{x^4 + 1}$ (d) $y = \dfrac{4}{(x+2)^2}$



▲ **Figure Ex-29**

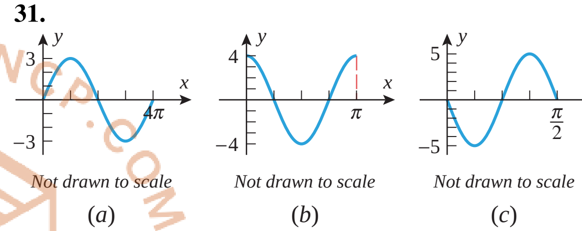
30. Find an equation of the form $y = k/(x^2 + bx + c)$ whose graph is a reasonable match to that in the accompanying figure. If you have a graphing utility, use it to check your work.



◀ **Figure Ex-30**

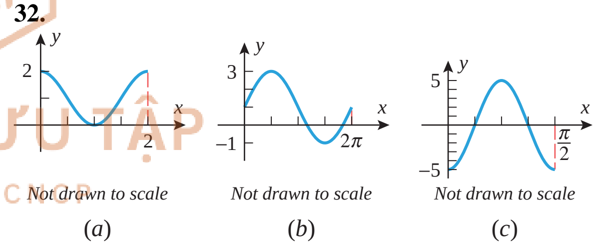
31–32 Find an equation of the form $y = D + A\sin Bx$ or $y = D + A\cos Bx$ for each graph.

31.



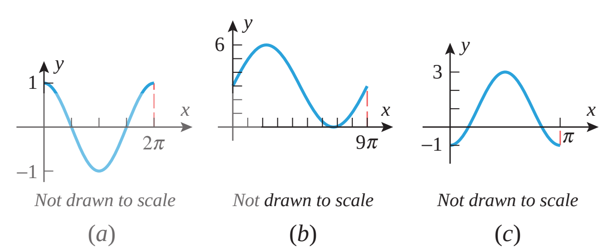
▲ **Figure Ex-31**

32.



▲ **Figure Ex-32**

33. In each part, find an equation for the graph that has the form $y = y_0 + A\sin(Bx - C)$.



▲ **Figure Ex-33**

34. In the United States, a standard electrical outlet supplies sinusoidal electrical current with a maximum voltage of $V = 120\sqrt{2}$ volts (V) at a frequency of 60 hertz (Hz). Write an equation that expresses V as a function of the time t, assuming that $V = 0$ if $t = 0$. [*Note:* 1 Hz = 1 cycle per second.]

35–36 Find the amplitude and period, and sketch at least two periods of the graph by hand. If you have a graphing utility, use it to check your work.

35. (a) $y = 3\sin 4x$ (b) $y = -2\cos \pi x$
(c) $y = 2 + \cos\left(\dfrac{x}{2}\right)$

36. (a) $y = -1 - 4\sin 2x$ (b) $y = \frac{1}{2}\cos(3x - \pi)$
(c) $y = -4\sin\left(\dfrac{x}{3} + 2\pi\right)$

37. Equations of the form

$$x = A_1 \sin \omega t + A_2 \cos \omega t$$

arise in the study of vibrations and other periodic motion. Express the equation

$$x = 5\sqrt{3}\sin 2\pi t + \tfrac{5}{2}\cos 2\pi t$$

in the form $x = A\sin(\omega t + \theta)$, and use a graphing utility to confirm that both equations have the same graph.

38. Determine the number of solutions of $x = 2\sin x$, and use a graphing or calculating utility to estimate them.

✔ QUICK CHECK ANSWERS 0.3

1. even; odd; negative **2.** $(-\infty, +\infty)$ **3.** (a) $[0, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(0, +\infty)$ (d) $(-\infty, 0) \cup (0, +\infty)$ **4.** (a) algebraic (b) polynomial (c) not algebraic (d) rational (e) rational **5.** $|A|$; $2\pi/|B|$

0.4 INVERSE FUNCTIONS; INVERSE TRIGONOMETRIC FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers, and in music a melodic inversion reverses an ascending interval to the corresponding descending interval. In mathematics the term ***inverse*** *is used to describe functions that reverse one another in the sense that each undoes the effect of the other. In this section we discuss this fundamental mathematical idea. In particular, we introduce inverse trigonometric functions to address the problem of recovering an angle that could produce a given trigonometric function value.*

INVERSE FUNCTIONS

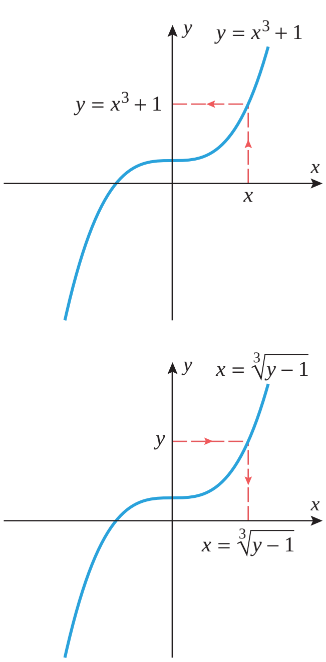
The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y-1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 0.4.1).



▲ **Figure 0.4.1**

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y-1}$ discussed above. When these functions are composed in either order, they cancel out the effect of one another in the sense that

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3+1) - 1} = x \\ f(g(y)) &= [g(y)]^3 + 1 = \left(\sqrt[3]{y-1}\right)^3 + 1 = y \end{aligned} \tag{1}$$

Pairs of functions with these two properties are so important that there is special terminology for them.

0.4.1 DEFINITION If the functions f and g satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
$$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$

then we say that ***f is an inverse of g*** and ***g is an inverse of f*** or that ***f and g are inverse functions***.

WARNING

If f is a function, then the -1 in the symbol f^{-1} always denotes an inverse and *never* an exponent. That is,

$f^{-1}(x)$ *never means* $\dfrac{1}{f(x)}$

It can be shown (Exercise 62) that if a function f has an inverse, then that inverse is unique. Thus, if a function f has an inverse, then we are entitled to talk about "the" inverse of f, in which case we denote it by the symbol f^{-1}.

Example 1 The computations in (1) show that $g(y) = \sqrt[3]{y-1}$ is the inverse of $f(x) = x^3 + 1$. Thus, we can express g in inverse notation as

$$f^{-1}(y) = \sqrt[3]{y-1}$$

and we can express the equations in Definition 0.4.1 as

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(y)) &= y \quad \text{for every } y \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

We will call these the ***cancellation equations*** for f and f^{-1}.

CHANGING THE INDEPENDENT VARIABLE

The formulas in (2) use x as the independent variable for f and y as the independent variable for f^{-1}. Although it is often convenient to use different independent variables for f and f^{-1}, there will be occasions on which it is desirable to use the same independent variable for both. For example, if we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we would want to use x as the independent variable and y as the dependent variable for both functions. Thus, to graph the functions $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y-1}$ of Example 1 in the same xy-coordinate system, we would change the independent variable y to x, use y as the dependent variable for both functions, and graph the equations

$$y = x^3 + 1 \quad \text{and} \quad y = \sqrt[3]{x-1}$$

We will talk more about graphs of inverse functions later in this section, but for reference we give the following reformulation of the cancellation equations in (2) using x as the independent variable for both f and f^{-1}:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{3}$$

Example 2 Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x \quad ◀$$

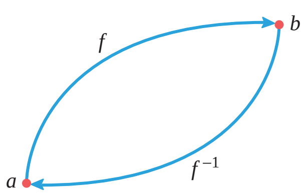
In general, if a function f has an inverse and $f(a) = b$, then the procedure in Example 3 shows that $a = f^{-1}(b)$; that is, f^{-1} maps each output of f back into the corresponding input (Figure 0.4.2).

▶ **Example 3** Given that the function f has an inverse and that $f(3) = 5$, find $f^{-1}(5)$.

Solution. Apply f^{-1} to both sides of the equation $f(3) = 5$ to obtain

$$f^{-1}(f(3)) = f^{-1}(5)$$

and now apply the first equation in (3) to conclude that $f^{-1}(5) = 3$. ◀



▲ **Figure 0.4.2** If f maps a to b, then f^{-1} maps b back to a.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (3) imply the following relationships between the domains and ranges of f and f^{-1}:

$$\begin{aligned} &\text{domain of } f^{-1} = \text{range of } f \\ &\text{range of } f^{-1} = \text{domain of } f \end{aligned} \tag{4}$$

One way to show that two sets are the same is to show that each is a subset of the other. Thus we can establish the first equality in (4) by showing that the domain of f^{-1} is a subset of the range of f and that the range of f is a subset of the domain of f^{-1}. We do this as follows: The first equation in (3) implies that f^{-1} is defined at $f(x)$ for all values of x in the domain of f, and this implies that the range of f is a subset of the domain of f^{-1}. Conversely, if x is in the domain of f^{-1}, then the second equation in (3) implies that x is in the range of f because it is the image of $f^{-1}(x)$. Thus, the domain of f^{-1} is a subset of the range of f. We leave the proof of the second equation in (4) as an exercise.

A METHOD FOR FINDING INVERSE FUNCTIONS

At the beginning of this section we observed that solving $y = f(x) = x^3 + 1$ for x as a function of y produces $x = f^{-1}(y) = \sqrt[3]{y-1}$. The following theorem shows that this is not accidental.

0.4.2 THEOREM *If an equation $y = f(x)$ can be solved for x as a function of y, say $x = g(y)$, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$.*

PROOF Substituting $y = f(x)$ into $x = g(y)$ yields $x = g(f(x))$, which confirms the first equation in Definition 0.4.1, and substituting $x = g(y)$ into $y = f(x)$ yields $y = f(g(y))$, which confirms the second equation in Definition 0.4.1. ■

Theorem 0.4.2 provides us with the following procedure for finding the inverse of a function.

An alternative way to obtain a formula for $f^{-1}(x)$ with x as the independent variable is to reverse the roles of x and y at the outset and solve the equation $x = f(y)$ for y as a function of x.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

▶ **Example 4** Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution. Following the procedure stated above, we first write

$$y = \sqrt{3x-2}$$

Then we solve this equation for x as a function of y:

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

which tells us that

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2) \tag{5}$$

Since we want x to be the independent variable, we reverse x and y in (5) to produce the formula

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2) \tag{6}$$

We know from (4) that the domain of f^{-1} is the range of f. In general, this need not be the same as the natural domain of the fomula for f^{-1}. Indeed, in this example the natural domain of (6) is $(-\infty, +\infty)$, whereas the range of $f(x) = \sqrt{3x-2}$ is $[0, +\infty)$. Thus, if we want to make the domain of f^{-1} clear, we must express it explicitly by rewriting (6) as

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2), \quad x \geq 0 \ ◀$$

EXISTENCE OF INVERSE FUNCTIONS

The procedure we gave above for finding the inverse of a function f was based on solving the equation $y = f(x)$ for x as a function of y. This procedure can fail for two reasons—the function f may not have an inverse, or it may have an inverse but the equation $y = f(x)$ cannot be solved explicitly for x as a function of y. Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

If a function f has an inverse, then it must assign distinct outputs to distinct inputs. For example, the function $f(x) = x^2$ cannot have an inverse because it assigns the same value to $x = 2$ and $x = -2$, namely,

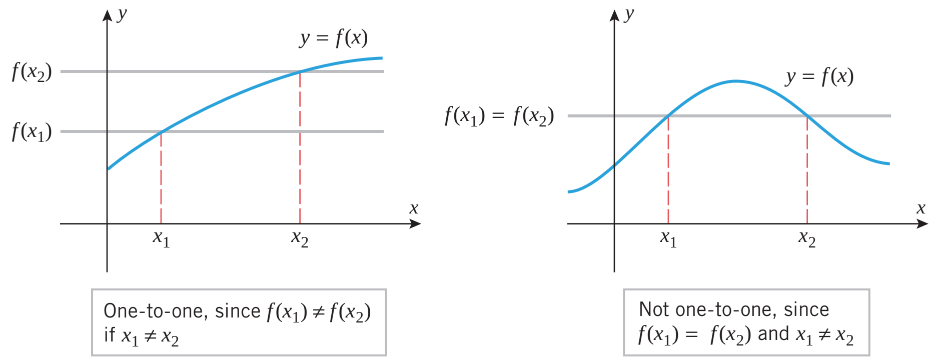
$$f(2) = f(-2) = 4$$

Thus, if $f(x) = x^2$ were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. But this is impossible because $f^{-1}(4)$ cannot have two different values. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x as a function of y. We run into trouble immediately because the resulting equation $x = \pm\sqrt{y}$ does not express x as a *single* function of y.

A function that assigns distinct outputs to distinct inputs is said to be ***one-to-one*** or ***invertible***, so we know from the preceding discussion that if a function f has an inverse, then it must be one-to-one. The converse is also true, thereby establishing the following theorem.

0.4.3 THEOREM *A function has an inverse if and only if it is one-to-one.*

Stated algebraically, a function f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; stated geometrically, a function f is one-to-one if and only if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 0.4.3). The latter statement together with Theorem 0.4.3 provides the following geometric test for determining whether a function has an inverse.

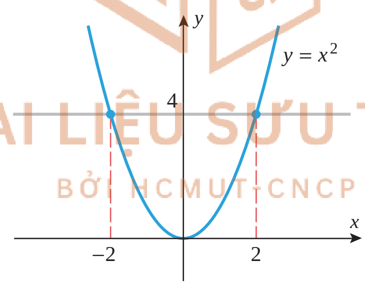


▶ **Figure 0.4.3**

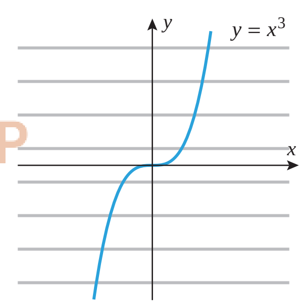
> **0.4.4 THEOREM (*The Horizontal Line Test*)** *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

▶ **Example 5** Use the horizontal line test to show that $f(x) = x^2$ has no inverse but that $f(x) = x^3$ does.

Solution. Figure 0.4.4 shows a horizontal line that cuts the graph of $y = x^2$ more than once, so $f(x) = x^2$ is not invertible. Figure 0.4.5 shows that the graph of $y = x^3$ is cut at most once by any horizontal line, so $f(x) = x^3$ is invertible. [Recall from Example 2 that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.] ◀



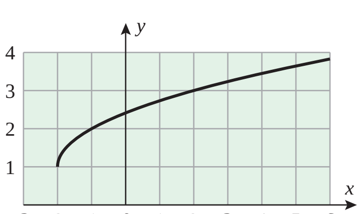
▲ **Figure 0.4.4**



▲ **Figure 0.4.5**

▶ **Example 6** Explain why the function f that is graphed in Figure 0.4.6 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀



▲ **Figure 0.4.6**

> The function $f(x) = x^3$ in Figure 0.4.5 is an example of an increasing function. Give an example of a decreasing function and compute its inverse.

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

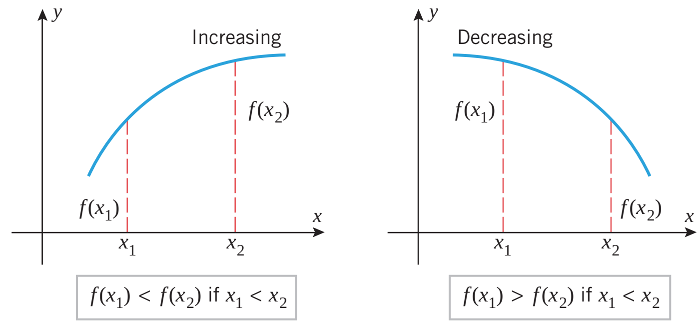
A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

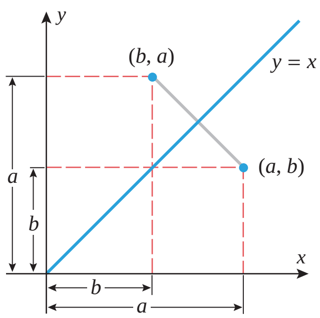
and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$

(Figure 0.4.7). It is evident geometrically that increasing and decreasing functions pass the horizontal line test and hence are invertible.

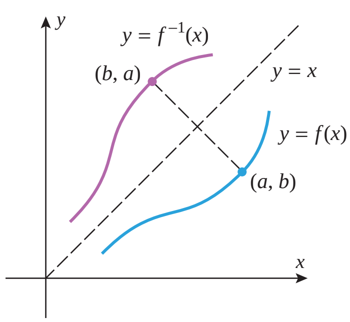


▶ **Figure 0.4.7**



The points (a, b) and (b, a) are reflections about $y = x$.

▲ **Figure 0.4.8**



The graphs of f and f^{-1} are reflections about $y = x$.

▲ **Figure 0.4.9**

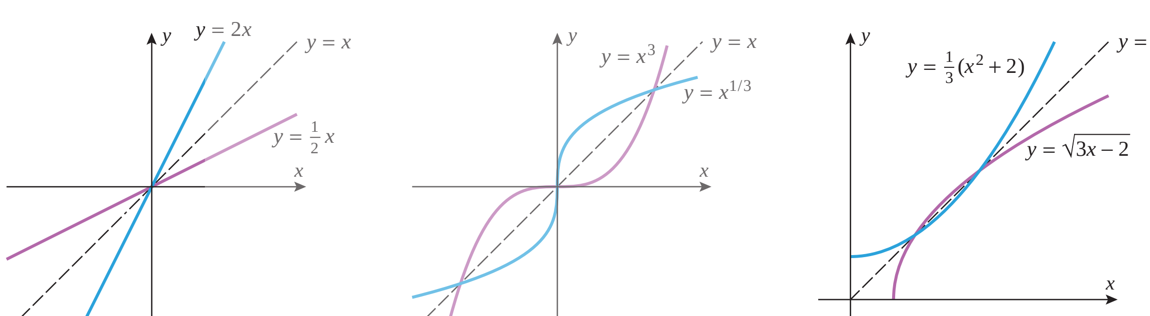
GRAPHS OF INVERSE FUNCTIONS

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions so we can compare the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 0.4.8), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 0.4.9). In summary, we have the following result.

0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*

▶ **Example 7** Figure 0.4.10 shows the graphs of the inverse functions discussed in Examples 2 and 4. ◀



▲ **Figure 0.4.10**

RESTRICTING DOMAINS FOR INVERTIBILITY

If a function g is obtained from a function f by placing restrictions on the domain of f, then g is called a ***restriction*** of f. Thus, for example, the function

$$g(x) = x^3, \quad x \geq 0$$

is a restriction of the function $f(x) = x^3$. More precisely, it is called the restriction of x^3 to the interval $[0, +\infty)$.

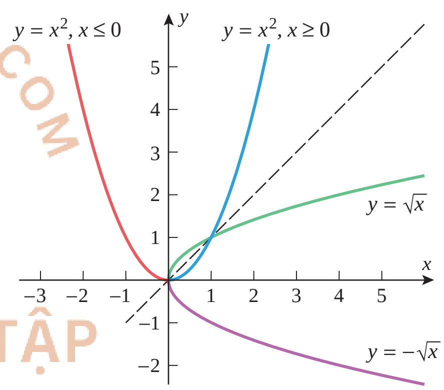
Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately. For example, we showed earlier that $f(x) = x^2$ is not invertible. However, consider the restricted functions

$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

the union of whose graphs is the complete graph of $f(x) = x^2$ (Figure 0.4.11). These restricted functions are each one-to-one (hence invertible), since their graphs pass the horizontal line test. As illustrated in Figure 0.4.12, their inverses are

$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$

▲ Figure 0.4.11



▲ Figure 0.4.12

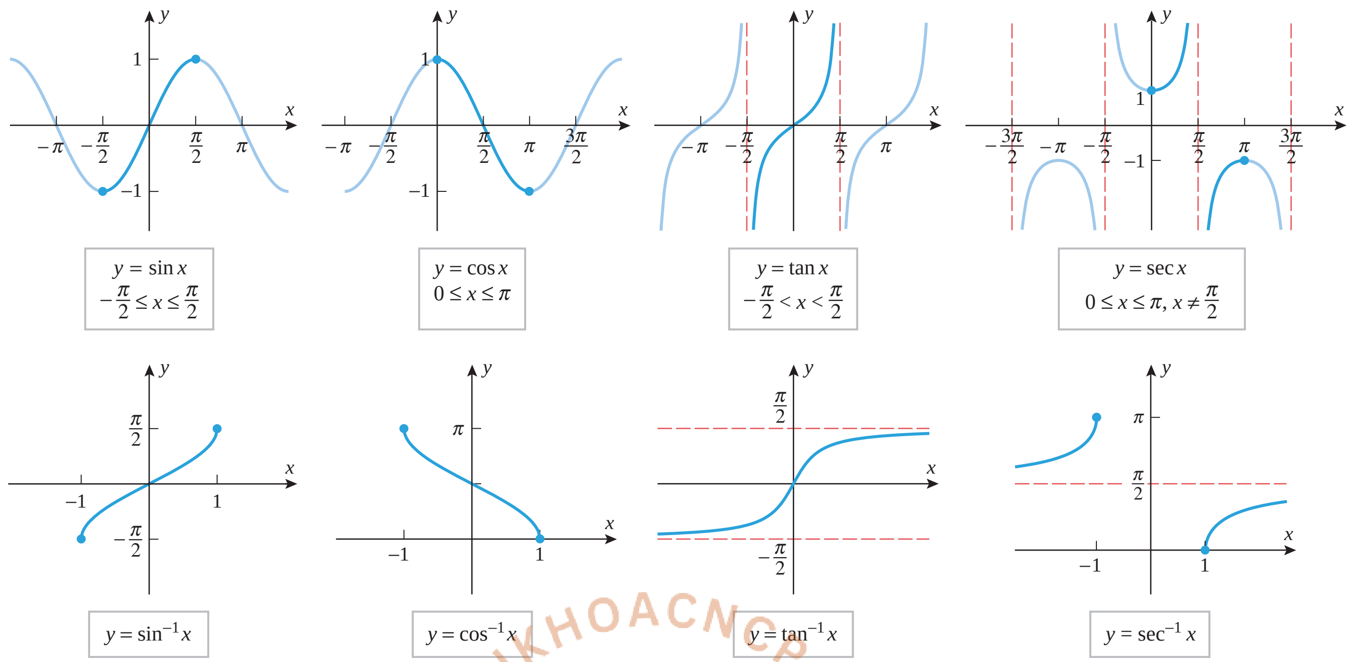
INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle x using a known value of $\sin x$, $\cos x$, or some other trigonometric function. Recall that problems of this type involve the computation of "arc functions" such as $\arcsin x$, $\arccos x$, and so forth. We will conclude this section by studying these arc functions from the viewpoint of general inverse functions.

The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test. To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "inverse trigonometric functions" to be the inverses of these restricted functions. The top part of Figure 0.4.13 shows geometrically how these restrictions are made for $\sin x$, $\cos x$, $\tan x$, and $\sec x$, and the bottom part of the figure shows the graphs of the corresponding inverse functions

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

(also denoted by $\arcsin x$, $\arccos x$, $\arctan x$, and $\operatorname{arcsec} x$). Inverses of $\cot x$ and $\csc x$ are of lesser importance and will be considered in the exercises.



▲ **Figure 0.4.13**

If you have trouble visualizing the correspondence between the top and bottom parts of Figure 0.4.13, keep in mind that a reflection about $y = x$ converts vertical lines into horizontal lines, and vice versa; and it converts x-intercepts into y-intercepts, and vice versa.

The following formal definitions summarize the preceding discussion.

0.4.6 DEFINITION The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$

0.4.7 DEFINITION The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

0.4.8 DEFINITION The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

0.4.9 DEFINITION* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

WARNING

The notations $\sin^{-1} x, \cos^{-1} x, \ldots$ are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonometric functions. If we want to express the reciprocal $1/\sin x$ using an exponent, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x < \pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in some earlier editions of this text. Each definition has advantages and disadvantages, but we will use the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Sage*.

Table 0.4.1 summarizes the basic properties of the inverse trigonometric functions we have considered. You should confirm that the domains and ranges listed in this table are consistent with the graphs shown in Figure 0.4.13.

Table 0.4.1
PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle x in radian measure such that

$$\sin x = \tfrac{1}{2} \tag{7}$$

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to solve the equation

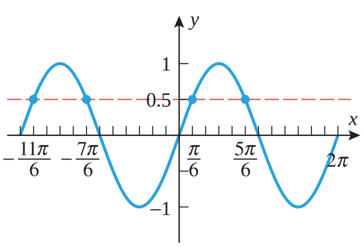
$$\sin x = y \tag{8}$$

Because $\sin x$ repeats periodically, this equation has infinitely many solutions for x; however, if we solve this equation as

$$x = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 0.4.14 shows four solutions of Equation (7), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \pi/6 \tag{9}$$



▲ **Figure 0.4.14**

TECHNOLOGY MASTERY

Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (9) numerically by showing that

$$\sin^{-1}(0.5) \approx 0.523598775598\ldots \approx \pi/6$$

In general, if we view $x = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \le x \le \pi/2$ imposes the geometric requirement that the angle x in standard position terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.

▶ **Example 8** Find exact values of

$$\text{(a)}\ \sin^{-1}(1/\sqrt{2}) \qquad \text{(b)}\ \sin^{-1}(-1)$$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $x = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$. ◀

If $x = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can x lie? Answer the same question for

$$x = \tan^{-1} y \quad \text{and} \quad x = \sec^{-1} y$$

TECHNOLOGY MASTERY

Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{10}$$

is useful (Exercise 50). Use this formula to show that

$$\sec^{-1}(2.25) \approx 1.11 \quad \text{and} \quad \sec^{-1}(-2.25) \approx 2.03$$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 0.4.15*a*). Moreover, the unlabeled acute angle in Figure 0.4.15*a* is $\cos^{-1} x$, since the cosine of that angle is x, and the unlabeled side in that figure has length $\sqrt{1-x^2}$ by the Theorem of Pythagoras (Figure 0.4.15*b*). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \le x \le 1$; for example,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \tag{11}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2} \tag{12}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2} \tag{13}$$

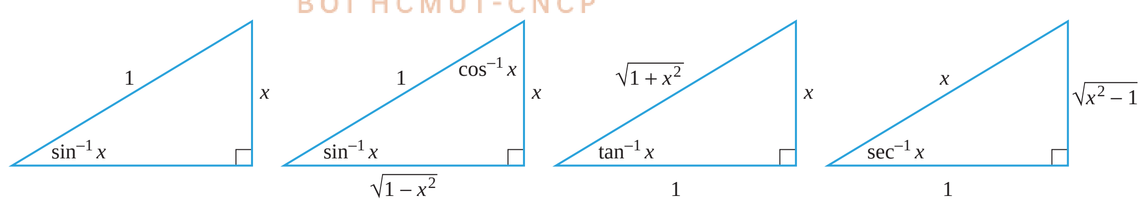
$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \tag{14}$$

There is little to be gained by memorizing these identities. What is important is the mastery of the *method* used to obtain them.

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 0.4.15*c* and 0.4.15*d* (verify). Those triangles reveal additional useful identities; for example,

$$\sec(\tan^{-1} x) = \sqrt{1+x^2} \tag{15}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \quad (x \ge 1) \tag{16}$$



(*a*) (*b*) (*c*) (*d*)

Figure 0.4.15

REMARK

The triangle technique does not always produce the most general form of an identity. For example, in Exercise 61 we will ask you to derive the following extension of Formula (16) that is valid for $x \le -1$ as well as $x \ge 1$:

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{|x|} \quad (|x| \ge 1) \tag{17}$$

Referring to Figure 0.4.13, observe that the inverse sine and inverse tangent are odd functions; that is,

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x) \tag{18–19}$$

Example 9 Figure 0.4.16 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

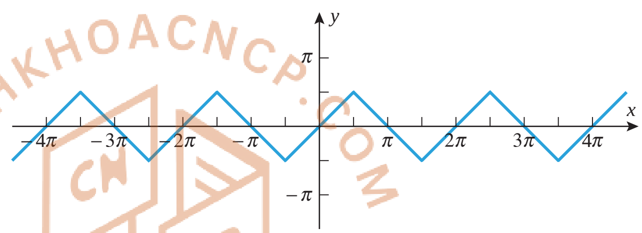
Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \le x \le \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 0.4.16). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if the quantity x lies in the interval $\pi/2 \le x \le 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \le x \le \pi/2$, so

$$\sin^{-1}[\sin(x - \pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x - \pi)] = -\sin^{-1}[\sin(x - \pi)] = -(x - \pi)$$

This shows that on the interval $\pi/2 \le x \le 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 0.4.16. ◀



▶ **Figure 0.4.16**

✔ QUICK CHECK EXERCISES 0.4 *(See page 52 for answers.)*

1. In each part, determine whether the function f is one-to-one.
(a) $f(t)$ is the number of people in line at a movie theater at time t.
(b) $f(x)$ is the measured high temperature (rounded to the nearest °F) in a city on the xth day of the year.
(c) $f(v)$ is the weight of v cubic inches of lead.

2. A student enters a number on a calculator, doubles it, adds 8 to the result, divides the sum by 2, subtracts 3 from the quotient, and then cubes the difference. If the resulting number is x, then ________ was the student's original number.

3. If $(3, -2)$ is a point on the graph of an odd invertible function f, then ________ and ________ are points on the graph of f^{-1}.

4. In each part, determine the exact value without using a calculating utility.
(a) $\sin^{-1}(-1) =$ ________
(b) $\tan^{-1}(1) =$ ________
(c) $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) =$ ________
(d) $\cos^{-1}\left(\frac{1}{2}\right) =$ ________
(e) $\sec^{-1}(-2) =$ ________

5. In each part, determine the exact value without using a calculating utility.
(a) $\sin^{-1}(\sin \pi/7) =$ ________
(b) $\sin^{-1}(\sin 5\pi/7) =$ ________
(c) $\tan^{-1}(\tan 13\pi/6) =$ ________
(d) $\cos^{-1}(\cos 12\pi/7) =$ ________

EXERCISE SET 0.4 Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions.
(a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$
(b) $f(x) = 3x + 1,\ g(x) = 3x - 1$
(c) $f(x) = \sqrt[3]{x - 2},\ g(x) = x^3 + 2$
(d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.
(a) $f(x) = 3x + 2$ (b) $f(x) = \sqrt{x - 1}$
(c) $f(x) = |x|$ (d) $f(x) = x^3$
(e) $f(x) = x^2 - 2x + 2$ (f) $f(x) = \sin x$

4. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
(a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = x^3 - 3x^2 + 3x - 1$

FOCUS ON CONCEPTS

5. In each part, determine whether the function f defined by the table is one-to-one.

(a)

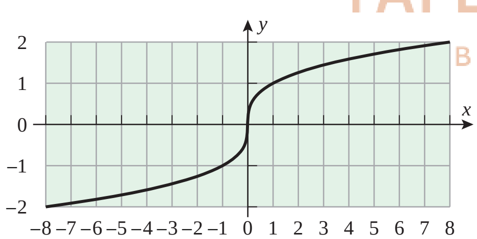
x	1	2	3	4	5	6
$f(x)$	−2	−1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	−7	6	−3	1	4

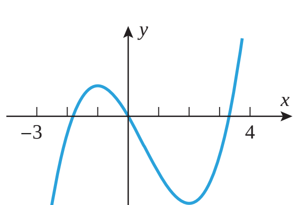
6. A face of a broken clock lies in the xy-plane with the center of the clock at the origin and 3:00 in the direction of the positive x-axis. When the clock broke, the tip of the hour hand stopped on the graph of $y = f(x)$, where f is a function that satisfies $f(0) = 0$.
(a) Are there any times of the day that cannot appear in such a configuration? Explain.
(b) How does your answer to part (a) change if f must be an invertible function?
(c) How do your answers to parts (a) and (b) change if it was the tip of the minute hand that stopped on the graph of f?

7. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.
(b) Find the domain and range of f^{-1}.
(c) Sketch the graph of f^{-1}.



▲ **Figure Ex-7**

8. (a) Explain why the function f graphed in the accompanying figure has no inverse function on its domain $-3 \le x \le 4$.
(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.



◀ **Figure Ex-8**

9–16 Find a formula for $f^{-1}(x)$. ■

9. $f(x) = 7x - 6$

10. $f(x) = \dfrac{x+1}{x-1}$

11. $f(x) = 3x^3 - 5$

12. $f(x) = \sqrt[5]{4x+2}$

13. $f(x) = 3/x^2, \quad x < 0$

14. $f(x) = 5/(x^2+1), \quad x \ge 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$

16. $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}. ■

17. $f(x) = (x+2)^4, \quad x \ge 0$

18. $f(x) = \sqrt{x+3}$

19. $f(x) = -\sqrt{3-2x}$

20. $f(x) = x - 5x^2, \quad x \ge 1$

21. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to
(a) $x \ge -b/(2a)$ (b) $x \le -b/(2a)$.

FOCUS ON CONCEPTS

22. The formula $F = \frac{9}{5}C + 32$, where $C \ge -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.
(a) Find a formula for the inverse function.
(b) In words, what does the inverse function tell you?
(c) Find the domain and range of the inverse function.

23. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length y in meters as a function of the same length x in miles.
(b) Find a formula for the inverse of f.
(c) Describe what the formula $x = f^{-1}(y)$ tells you in practical terms.

24. Let $f(x) = x^2$, $x > 1$, and $g(x) = \sqrt{x}$.
(a) Show that $f(g(x)) = x$, $x > 1$, and $g(f(x)) = x$, $x > 1$.
(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.
(c) Do parts (a) and (b) contradict one another? Explain.

25. (a) Show that $f(x) = (3-x)/(1-x)$ is its own inverse.
(b) What does the result in part (a) tell you about the graph of f?

26. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

27. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

28. Let $f(x) = \dfrac{x^3}{x^2+1}$. Find x if $f^{-1}(x) = 2$.

29. Prove that if $a^2 + bc \neq 0$, then the graph of

$$f(x) = \frac{ax + b}{cx - a}$$

is symmetric about the line $y = x$.

30. (a) Prove: If f and g are one-to-one, then so is the composition $f \circ g$.
(b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

31–34 True–False Determine whether the statement is true or false. Explain your answer. ■

31. If f is an invertible function such that $f(2) = 2$, then $f^{-1}(2) = \frac{1}{2}$.

32. If f and g are inverse functions, then f and g have the same domain.

33. A one-to-one function is invertible.

34. The range of the inverse tangent function is the interval $-\pi/2 \leq y \leq \pi/2$.

35. Given that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

36. Given that $\theta = \sec^{-1} 2.6$, find the exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\csc\theta$.

37. For which values of x is it true that
(a) $\cos^{-1}(\cos x) = x$
(b) $\cos(\cos^{-1} x) = x$
(c) $\tan^{-1}(\tan x) = x$
(d) $\tan(\tan^{-1} x) = x$?

38–39 Find the exact value of the given quantity. ■

38. $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$

39. $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

40–41 Complete the identities using the triangle method (Figure 0.4.15). ■

40. (a) $\sin(\cos^{-1} x) = ?$
(b) $\tan(\cos^{-1} x) = ?$
(c) $\csc(\tan^{-1} x) = ?$
(d) $\sin(\tan^{-1} x) = ?$

41. (a) $\cos(\tan^{-1} x) = ?$
(b) $\tan(\cos^{-1} x) = ?$
(c) $\sin(\sec^{-1} x) = ?$
(d) $\cot(\sec^{-1} x) = ?$

42. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.
(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Confirm that your sketches agree with those in Figure 0.4.13.
(c) Use your graphing utility to graph $y = \sin^{-1} x$ and $y = \cos^{-1} x$; confirm that the graphs agree with those in Figure 0.4.13.

43. In each part, sketch the graph and check your work with a graphing utility.
(a) $y = \sin^{-1} 2x$
(b) $y = \tan^{-1} \frac{1}{2}x$

44. The ***law of cosines*** states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

45–46 Use a calculating utility to approximate the solution of each equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.] ■

45. (a) $\sin x = 0.37$, $\pi/2 < x < \pi$
(b) $\sin\theta = -0.61$, $180° < \theta < 270°$

46. (a) $\cos x = -0.85$, $\pi < x < 3\pi/2$
(b) $\cos\theta = 0.23$, $-90° < \theta < 0°$

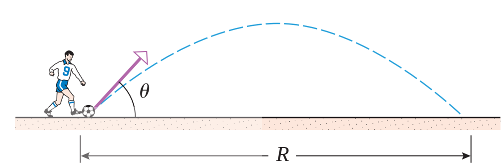
FOCUS ON CONCEPTS

47. (a) Use a calculating utility to evaluate the expressions $\sin^{-1}(\sin^{-1} 0.25)$ and $\sin^{-1}(\sin^{-1} 0.9)$, and explain what you think is happening in the second calculation.
(b) For what values of x in the interval $-1 \leq x \leq 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1} x)$?

48. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accompanying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by

$$R = \frac{v^2}{g}\sin 2\theta$$

where v is the initial speed of the ball and g is the acceleration due to gravity. Using $g = 9.8\ \text{m/s}^2$, approximate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?



▲ **Figure Ex-48**

49–50 The function $\cot^{-1} x$ is defined to be the inverse of the restricted cotangent function

$$\cot x, \quad 0 < x < \pi$$

and the function $\csc^{-1} x$ is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad -\pi/2 < x < \pi/2, \quad x \neq 0$$

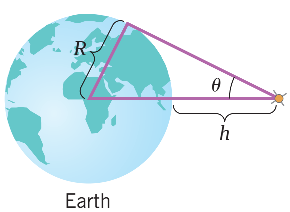
Use these definitions in these and in all subsequent exercises that involve these functions. ■

49. (a) Sketch the graphs of $\cot^{-1} x$ and $\csc^{-1} x$.
(b) Find the domain and range of $\cot^{-1} x$ and $\csc^{-1} x$.

50. Show that
(a) $\cot^{-1} x = \begin{cases} \tan^{-1}(1/x), & \text{if } x > 0 \\ \pi + \tan^{-1}(1/x), & \text{if } x < 0 \end{cases}$
(b) $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}, \quad \text{if } |x| \geq 1$
(c) $\csc^{-1} x = \sin^{-1} \dfrac{1}{x}, \quad \text{if } |x| \geq 1.$

51. Most scientific calculators have keys for the values of only $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The formulas in Exercise 50 show how a calculator can be used to obtain values of $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Express your answers in degrees, rounded to the nearest tenth of a degree.
(a) $\cot^{-1} 0.7$ (b) $\sec^{-1} 1.2$ (c) $\csc^{-1} 2.3$

52. An Earth-observing satellite has horizon sensors that can measure the angle θ shown in the accompanying figure. Let R be the radius of the Earth (assumed spherical) and h the distance between the satellite and the Earth's surface.
(a) Show that $\sin\theta = \dfrac{R}{R+h}$.
(b) Find θ, to the nearest degree, for a satellite that is 10,000 km from the Earth's surface (use $R = 6378$ km).



◀ **Figure Ex-52**

53. The number of hours of daylight on a given day at a given point on the Earth's surface depends on the latitude λ of the point, the angle γ through which the Earth has moved in its orbital plane during the time period from the vernal equinox (March 21), and the angle of inclination ϕ of the Earth's axis of rotation measured from ecliptic north ($\phi \approx 23.45°$). The number of hours of daylight h can be approximated by the formula

$$h = \begin{cases} 24, & D \geq 1 \\ 12 + \frac{2}{15}\sin^{-1} D, & |D| < 1 \\ 0, & D \leq -1 \end{cases}$$

where

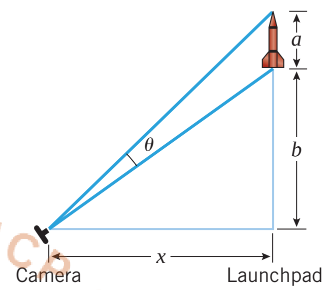
$$D = \frac{\sin\phi \sin\gamma \tan\lambda}{\sqrt{1 - \sin^2\phi \sin^2\gamma}}$$

and $\sin^{-1} D$ is in degree measure. Given that Fairbanks, Alaska, is located at a latitude of $\lambda = 65°$ N and also that $\gamma = 90°$ on June 20 and $\gamma = 270°$ on December 20, approximate
(a) the maximum number of daylight hours at Fairbanks to one decimal place
(b) the minimum number of daylight hours at Fairbanks to one decimal place.

Source: This problem was adapted from *TEAM, A Path to Applied Mathematics*, The Mathematical Association of America, Washington, D.C., 1985.

54. A camera is positioned x feet from the base of a missile launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{b}$$

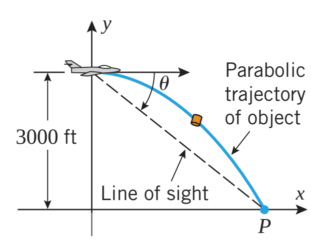


◀ **Figure Ex-54**

55. An airplane is flying at a constant height of 3000 ft above water at a speed of 400 ft/s. The pilot is to release a survival package so that it lands in the water at a sighted point P. If air resistance is neglected, then the package will follow a parabolic trajectory whose equation relative to the coordinate system in the accompanying figure is

$$y = 3000 - \frac{g}{2v^2}x^2$$

where g is the acceleration due to gravity and v is the speed of the airplane. Using $g = 32$ ft/s^2, find the "line of sight" angle θ, to the nearest degree, that will result in the package hitting the target point.



◀ **Figure Ex-55**

56. Prove:
(a) $\sin^{-1}(-x) = -\sin^{-1} x$
(b) $\tan^{-1}(-x) = -\tan^{-1} x$.

57. Prove:
(a) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
(b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

58. Prove:
(a) $\sin^{-1} x = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}} \quad (|x| < 1)$
(b) $\cos^{-1} x = \dfrac{\pi}{2} - \tan^{-1} \dfrac{x}{\sqrt{1-x^2}} \quad (|x| < 1).$

59. Prove:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

provided $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$. [*Hint:* Use an identity for $\tan(\alpha+\beta)$.]

60. Use the result in Exercise 59 to show that
(a) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \pi/4$
(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \pi/4$.

61. Use identities (10) and (13) to obtain identity (17).

62. Prove: A one-to-one function f cannot have two different inverses.

✔ QUICK CHECK ANSWERS 0.4

1. (a) not one-to-one (b) not one-to-one (c) one-to-one **2.** $\sqrt[3]{x} - 1$ **3.** $(-2, 3)$; $(2, -3)$ **4.** (a) $-\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/3$ (e) $2\pi/3$ **5.** (a) $\pi/7$ (b) $2\pi/7$ (c) $\pi/6$ (d) $2\pi/7$

0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have replaced logarithm tables for numerical calculations, the logarithmic functions have wide-ranging applications in mathematics and science. In this section we will review some properties of exponents and logarithms and then use our work on inverse functions to develop results about exponential and logarithmic functions.

IRRATIONAL EXPONENTS

Recall from algebra that if b is a nonzero real number, then nonzero *integer* powers of b are defined by

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}} \quad \text{and} \quad b^{-n} = \frac{1}{b^n}$$

and if $n = 0$, then $b^0 = 1$. Also, if p/q is a positive *rational* number expressed in lowest terms, then

$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p \quad \text{and} \quad b^{-p/q} = \frac{1}{b^{p/q}}$$

If b is negative, then some fractional powers of b will have imaginary values—the quantity $(-2)^{1/2} = \sqrt{-2}$, for example. To avoid this complication, we will assume throughout this section that $b > 0$, even if it is not stated explicitly.

There are various methods for defining *irrational* powers such as

$$2^{\pi}, \quad 3^{\sqrt{2}}, \quad \pi^{-\sqrt{7}}$$

One approach is to define irrational powers of b via successive approximations using rational powers of b. For example, to define 2^{π} consider the decimal representation of π:

$$3.1415926\ldots$$

From this decimal we can form a sequence of rational numbers that gets closer and closer to π, namely,

$$3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159$$

and from these we can form a sequence of *rational* powers of 2:

$$2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}$$

Since the exponents of the terms in this sequence get successively closer to π, it seems plausible that the terms themselves will get successively closer to some number. It is that number that we *define* to be 2^{π}. This is illustrated in Table 0.5.1, which we generated using

Table 0.5.1

x	2^x
3	8.000000
3.1	8.574188
3.14	8.815241
3.141	8.821353
3.1415	8.824411
3.14159	8.824962
3.141592	8.824974
3.1415926	8.824977

a calculator. The table suggests that to four decimal places the value of 2^π is

$$2^\pi \approx 8.8250 \qquad (1)$$

TECHNOLOGY MASTERY

Use a calculating utility to verify the results in Table 0.5.1, and then verify (1) by using the utility to compute 2^π directly.

With this notion for irrational powers, we remark without proof that the following familiar laws of exponents hold for all real values of p and q:

$$b^p b^q = b^{p+q}, \qquad \frac{b^p}{b^q} = b^{p-q}, \qquad (b^p)^q = b^{pq}$$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \qquad f(x) = \left(\tfrac{1}{2}\right)^x, \qquad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^\pi$ would *not* be classified as exponential functions, since they have a variable base and a constant exponent.

Figure 0.5.1 illustrates that the graph of $y = b^x$ has one of three general forms, depending on the value of b. The graph of $y = b^x$ has the following properties:

- The graph passes through (0, 1) because $b^0 = 1$.
- If $b > 1$, the value of b^x increases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely. If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x.
- If $0 < b < 1$, the value of b^x decreases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x. If you traverse the graph from right to left, the values of b^x increase indefinitely.
- If $b = 1$, then the value of b^x is constant.

Some typical members of the family of exponential functions are graphed in Figure 0.5.2. This figure illustrates that the graph of $y = (1/b)^x$ is the reflection of the graph of $y = b^x$ about the y-axis. This is because replacing x by $-x$ in the equation $y = b^x$ yields

$$y = b^{-x} = (1/b)^x$$

The figure also conveys that for $b > 1$, the larger the base b, the more rapidly the function $f(x) = b^x$ increases for $x > 0$.

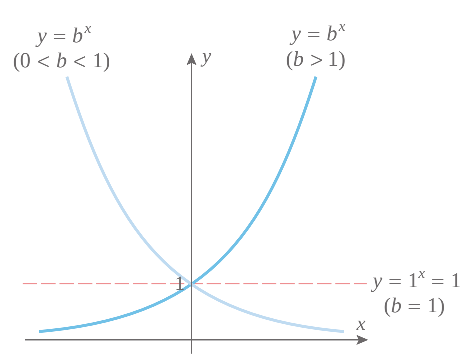


Figure 0.5.1

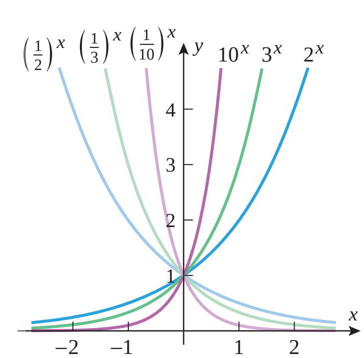


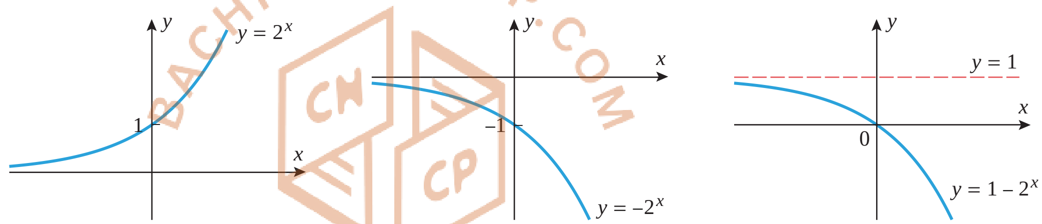
Figure 0.5.2 The family $y = b^x$ $(b > 0)$

The domain and range of the exponential function $f(x) = b^x$ can also be found by examining Figure 0.5.1:

- If $b > 0$, then $f(x) = b^x$ is defined and has a real value for every real value of x, so the natural domain of every exponential function is $(-\infty, +\infty)$.
- If $b > 0$ and $b \neq 1$, then as noted earlier the graph of $y = b^x$ increases indefinitely as it is traversed in one direction and decreases toward zero but never reaches zero as it is traversed in the other direction. This implies that the range of $f(x) = b^x$ is $(0, +\infty)$.*

▶ Example 1 Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

Solution. Start with a graph of $y = 2^x$. Reflect this graph across the x-axis to obtain the graph of $y = -2^x$, then translate that graph upward by 1 unit to obtain the graph of $y = 1 - 2^x$ (Figure 0.5.3). The dashed line in the third part of Figure 0.5.3 is a horizontal asymptote for the graph. You should be able to see from the graph that the domain of f is $(-\infty, +\infty)$ and the range is $(-\infty, 1)$. ◀



▲ **Figure 0.5.3**

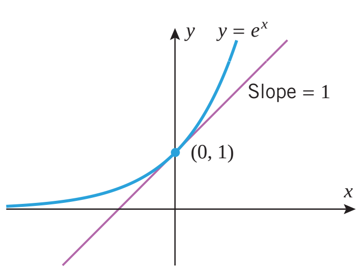
THE NATURAL EXPONENTIAL FUNCTION

The use of the letter e is in honor of the Swiss mathematician Leonhard Euler (biography on p. 3) who is credited with recognizing the mathematical importance of this constant.

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value to six decimal places is

$$e \approx 2.718282 \tag{2}$$

This base is important in calculus because, as we will prove later, $b = e$ is the only base for which the slope of the tangent line** to the curve $y = b^x$ at any point P on the curve is equal to the y-coordinate at P. Thus, for example, the tangent line to $y = e^x$ at $(0, 1)$ has slope 1 (Figure 0.5.4).



▲ **Figure 0.5.4** The tangent line to the graph of $y = e^x$ at $(0, 1)$ has slope 1.

The function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typography, the natural exponential function is sometimes written as $\exp(x)$, in which case the relationship $e^{x_1+x_2} = e^{x_1}e^{x_2}$ would be expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

*We are assuming without proof that the graph of $y = b^x$ is a curve without breaks, gaps, or holes.
**The precise definition of a tangent line will be discussed later. For now your intuition will suffice.

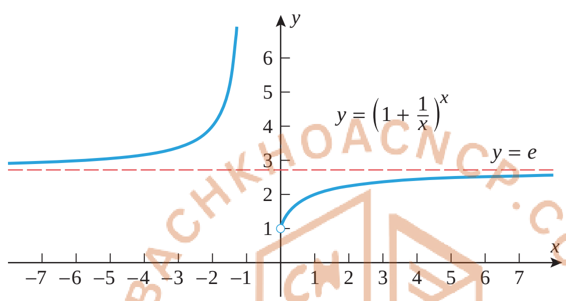
TECHNOLOGY MASTERY

Your technology utility should have keys or commands for approximating e and for graphing the natural exponential function. Read your documentation on how to do this and use your utility to confirm (2) and to generate the graphs in Figures 0.5.2 and 0.5.4.

The constant e also arises in the context of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x \tag{3}$$

As shown in Figure 0.5.5, $y = e$ is a horizontal asymptote of this graph. As a result, the value of e can be approximated to any degree of accuracy by evaluating (3) for x sufficiently large in absolute value (Table 0.5.2).



▲ **Figure 0.5.5**

Table 0.5.2

APPROXIMATIONS OF e BY $(1 + 1/x)^x$ FOR INCREASING VALUES OF x

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

LOGARITHMIC FUNCTIONS

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. Thus, for example,

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

$$\boxed{10^2 = 100} \quad \boxed{10^{-3} = 1/1000} \quad \boxed{2^4 = 16} \quad \boxed{b^0 = 1} \quad \boxed{b^1 = b}$$

Logarithms with base 10 are called ***common logarithms*** and are often written without explicit reference to the base. Thus, the symbol $\log x$ generally denotes $\log_{10} x$.

We call the function $f(x) = \log_b x$ the ***logarithmic function with base b***.

Logarithmic functions can also be viewed as inverses of exponential functions. To see why this is so, observe from Figure 0.5.1 that if $b > 0$ and $b \neq 1$, then the graph of $f(x) = b^x$ passes the horizontal line test, so b^x has an inverse. We can find a formula for this inverse with x as the independent variable by solving the equation

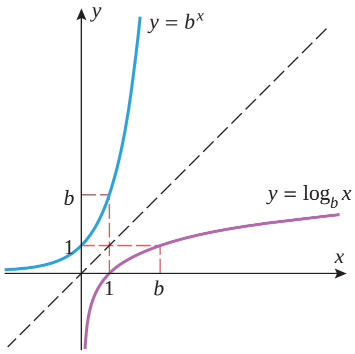
$$x = b^y$$

for y as a function of x. But this equation states that y is the logarithm to the base b of x, so it can be rewritten as

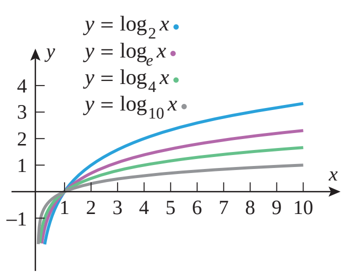
$$y = \log_b x$$

Thus, we have established the following result.

0.5.1 THEOREM *If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.*



▲ **Figure 0.5.6**



▲ **Figure 0.5.7** The family $y = \log_b x$ $(b > 1)$

TECHNOLOGY MASTERY

Use your graphing utility to generate the graphs of $y = \ln x$ and $y = \log x$.

It follows from this theorem that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (see Figure 0.5.6 for the case where $b > 1$). Figure 0.5.7 shows the graphs of $y = \log_b x$ for various values of b. Observe that they all pass through the point $(1, 0)$.

The most important logarithms in applications are those with base e. These are called ***natural logarithms*** because the function $\log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. For example,

$$\underbrace{\ln 1 = 0}_{\text{Since } e^0 = 1}, \quad \underbrace{\ln e = 1}_{\text{Since } e^1 = e}, \quad \underbrace{\ln 1/e = -1}_{\text{Since } e^{-1} = 1/e}, \quad \underbrace{\ln(e^2) = 2}_{\text{Since } e^2 = e^2}$$

In general,

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$

As shown in Table 0.5.3, the inverse relationship between b^x and $\log_b x$ produces a correspondence between some basic properties of those functions.

Table 0.5.3

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

It also follows from the cancellation properties of inverse functions [see (3) in Section 0.4] that

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for all real values of } x \\ b^{\log_b x} &= x \quad \text{for } x > 0 \end{aligned} \tag{4}$$

In the special case where $b = e$, these equations become

$$\begin{aligned} \ln(e^x) &= x \quad \text{for all real values of } x \\ e^{\ln x} &= x \quad \text{for } x > 0 \end{aligned} \tag{5}$$

In words, the functions b^x and $\log_b x$ cancel out the effect of one another when composed in either order; for example,

$$\log 10^x = x, \quad 10^{\log x} = x, \quad \ln e^x = x, \quad e^{\ln x} = x, \quad \ln e^5 = 5, \quad e^{\ln \pi} = \pi$$

SOLVING EQUATIONS INVOLVING EXPONENTIALS AND LOGARITHMS

You should be familiar with the following properties of logarithms from your earlier studies.

> **0.5.2 THEOREM** ***(Algebraic Properties of Logarithms)*** *If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then:*
>
> (a) $\log_b(ac) = \log_b a + \log_b c$ — Product property
>
> (b) $\log_b(a/c) = \log_b a - \log_b c$ — Quotient property
>
> (c) $\log_b(a^r) = r \log_b a$ — Power property
>
> (d) $\log_b(1/c) = -\log_b c$ — Reciprocal property

WARNING

Expressions of the form $\log_b(u + v)$ and $\log_b(u - v)$ have no useful simplifications. In particular,

$$\log_b(u + v) \neq \log_b(u) + \log_b(v)$$
$$\log_b(u - v) \neq \log_b(u) - \log_b(v)$$

These properties are often used to expand a single logarithm into sums, differences, and multiples of other logarithms and, conversely, to condense sums, differences, and multiples of logarithms into a single logarithm. For example,

$$\log \frac{xy^5}{\sqrt{z}} = \log xy^5 - \log \sqrt{z} = \log x + \log y^5 - \log z^{1/2} = \log x + 5\log y - \tfrac{1}{2}\log z$$

$$5\log 2 + \log 3 - \log 8 = \log 32 + \log 3 - \log 8 = \log \frac{32 \cdot 3}{8} = \log 12$$

$$\tfrac{1}{3}\ln x - \ln(x^2 - 1) + 2\ln(x + 3) = \ln x^{1/3} - \ln(x^2 - 1) + \ln(x + 3)^2 = \ln \frac{\sqrt[3]{x}(x + 3)^2}{x^2 - 1}$$

An equation of the form $\log_b x = k$ can be solved for x by rewriting it in the exponential form $x = b^k$, and an equation of the form $b^x = k$ can be solved by rewriting it in the logarithm form $x = \log_b k$. Alternatively, the equation $b^x = k$ can be solved by taking *any* logarithm of both sides (but usually log or ln) and applying part (*c*) of Theorem 0.5.2. These ideas are illustrated in the following example.

▶ **Example 2** Find x such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x + 1) = 5$ (c) $5^x = 7$

Solution (a). Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

Solution (b). Converting the equation to exponential form yields

$$x + 1 = e^5 \quad \text{or} \quad x = e^5 - 1 \approx 147.41$$

Solution (c). Converting the equation to logarithmic form yields

$$x = \log_5 7 \approx 1.21$$

Alternatively, taking the natural logarithm of both sides and using the power property of logarithms yields

$$x \ln 5 = \ln 7 \quad \text{or} \quad x = \frac{\ln 7}{\ln 5} \approx 1.21$$ ◀

Erik Simonsen/Getty Images

Power to satellites can be supplied by batteries, fuel cells, solar cells, or radioisotope devices.

Example 3 A satellite that requires 7 watts of power to operate at full capacity is equipped with a radioisotope power supply whose power output P in watts is given by the equation

$$P = 75e^{-t/125}$$

where t is the time in days that the supply is used. How long can the satellite operate at full capacity?

Solution. The power P will fall to 7 watts when

$$7 = 75e^{-t/125}$$

The solution for t is as follows:

$$\begin{aligned} 7/75 &= e^{-t/125} \\ \ln(7/75) &= \ln(e^{-t/125}) \\ \ln(7/75) &= -t/125 \\ t &= -125\ln(7/75) \approx 296.4 \end{aligned}$$

so the satellite can operate at full capacity for about 296 days. ◀

Here is a more complicated example.

Example 4 Solve $\dfrac{e^x - e^{-x}}{2} = 1$ for x.

Solution. Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2$$

or equivalently,

$$e^x - \frac{1}{e^x} = 2$$

Multiplying through by e^x yields

$$e^{2x} - 1 = 2e^x \quad \text{or} \quad e^{2x} - 2e^x - 1 = 0$$

This is really a quadratic equation in disguise, as can be seen by rewriting it in the form

$$(e^x)^2 - 2e^x - 1 = 0$$

and letting $u = e^x$ to obtain

$$u^2 - 2u - 1 = 0$$

Solving for u by the quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

or, since $u = e^x$,

$$e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative, so we discard the negative value $1 - \sqrt{2}$; thus,

$$\begin{aligned} e^x &= 1 + \sqrt{2} \\ \ln e^x &= \ln(1 + \sqrt{2}) \\ x &= \ln(1 + \sqrt{2}) \approx 0.881 \end{aligned}$$ ◀

CHANGE OF BASE FORMULA FOR LOGARITHMS

Scientific calculators generally have no keys for evaluating logarithms with bases other than 10 or e. However, this is not a serious deficiency because it is possible to express a logarithm with any base in terms of logarithms with any other base (see Exercise 42). For example, the following formula expresses a logarithm with base b in terms of natural logarithms:

$$\log_b x = \frac{\ln x}{\ln b} \tag{6}$$

We can derive this result by letting $y = \log_b x$, from which it follows that $b^y = x$. Taking the natural logarithm of both sides of this equation we obtain $y \ln b = \ln x$, from which (6) follows.

Example 5 Use a calculating utility to evaluate $\log_2 5$ by expressing this logarithm in terms of natural logarithms.

Solution. From (6) we obtain

$$\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.321928$$

LOGARITHMIC SCALES IN SCIENCE AND ENGINEERING

Logarithms are used in science and engineering to deal with quantities whose units vary over an excessively wide range of values. For example, the "loudness" of a sound can be measured by its ***intensity*** I (in watts per square meter), which is related to the energy transmitted by the sound wave—the greater the intensity, the greater the transmitted energy, and the louder the sound is perceived by the human ear. However, intensity units are unwieldy because they vary over an enormous range. For example, a sound at the threshold of human hearing has an intensity of about 10^{-12} W/m^2, a close whisper has an intensity that is about 100 times the hearing threshold, and a jet engine at 50 meters has an intensity that is about $10{,}000{,}000{,}000{,}000 = 10^{13}$ times the hearing threshold. To see how logarithms can be used to reduce this wide spread, observe that if

$$y = \log x$$

then increasing x by a *factor* of 10 *adds* 1 unit to y since

$$\log 10x = \log 10 + \log x = 1 + y$$

Physicists and engineers take advantage of this property by measuring loudness in terms of the ***sound level*** β, which is defined by

$$\beta = 10 \log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2 is a reference intensity close to the threshold of human hearing. The units of β are ***decibels*** (dB), named in honor of the telephone inventor Alexander Graham Bell. With this scale of measurement, *multiplying* the intensity I by a factor of 10 *adds* 10 dB to the sound level β (verify). This results in a more tractable scale than intensity for measuring sound loudness (Table 0.5.4). Some other familiar logarithmic scales are the ***Richter scale*** used to measure earthquake intensity and the **pH *scale*** used to measure acidity in chemistry, both of which are discussed in the exercises.

Table 0.5.4

β (dB)	I/I_0
0	$10^0 = 1$
10	$10^1 = 10$
20	$10^2 = 100$
30	$10^3 = 1000$
40	$10^4 = 10{,}000$
50	$10^5 = 100{,}000$
⋮	⋮
120	$10^{12} = 1{,}000{,}000{,}000{,}000$

Regina Mitchell-Ryall, Tony Gray/NASA/Getty Images

The roar of a space shuttle near the launchpad would damage your hearing without ear protection.

▶ **Example 6** A space shuttle taking off generates a sound level of 150 dB near the launchpad. A person exposed to this level of sound would experience severe physical injury. By comparison, a car horn at one meter has a sound level of 110 dB, near the threshold of pain for many people. What is the ratio of sound intensity of a space shuttle takeoff to that of a car horn?

Solution. Let I_1 and β_1 (= 150 dB) denote the sound intensity and sound level of the space shuttle taking off, and let I_2 and β_2 (= 110 dB) denote the sound intensity and sound level of a car horn. Then

$$\begin{aligned}
I_1/I_2 &= (I_1/I_0)/(I_2/I_0) \\
\log(I_1/I_2) &= \log(I_1/I_0) - \log(I_2/I_0) \\
10\log(I_1/I_2) &= 10\log(I_1/I_0) - 10\log(I_2/I_0) = \beta_1 - \beta_2 \\
10\log(I_1/I_2) &= 150 - 100 = 40 \\
\log(I_1/I_2) &= 4
\end{aligned}$$

Thus, $I_1/I_2 = 10^4$, which tells us that the sound intensity of the space shuttle taking off is 10,000 times greater than a car horn! ◀

EXPONENTIAL AND LOGARITHMIC GROWTH

Table 0.5.5

x	e^x	$\ln x$
1	2.72	0.00
2	7.39	0.69
3	20.09	1.10
4	54.60	1.39
5	148.41	1.61
6	403.43	1.79
7	1096.63	1.95
8	2980.96	2.08
9	8103.08	2.20
10	22026.47	2.30
100	2.69×10^{43}	4.61
1000	1.97×10^{434}	6.91

The growth patterns of e^x and $\ln x$ illustrated in Table 0.5.5 are worth noting. Both functions increase as x increases, but they increase in dramatically different ways—the value of e^x increases extremely rapidly and that of $\ln x$ increases extremely slowly. For example, the value of e^x at $x = 10$ is over 22,000, but at $x = 1000$ the value of $\ln x$ has not even reached 7.

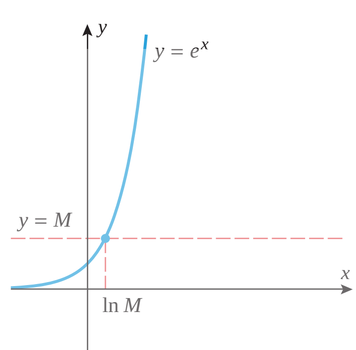
A function f is said to ***increase without bound*** as x increases if the values of $f(x)$ eventually exceed any specified positive number M (no matter how large) as x increases indefinitely. Table 0.5.5 strongly suggests that $f(x) = e^x$ increases without bound, which is consistent with the fact that the range of this function is $(0, +\infty)$. Indeed, if we choose any positive number M, then we will have $e^x = M$ when $x = \ln M$, and since the values of e^x increase as x increases, we will have

$$e^x > M \quad \text{if} \quad x > \ln M$$

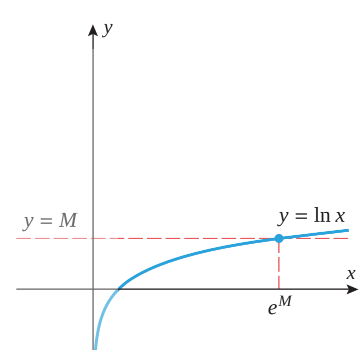
(Figure 0.5.8). It is not clear from Table 0.5.5 whether $\ln x$ increases without bound as x increases because the values grow so slowly, but we know this to be so since the range of this function is $(-\infty, +\infty)$. To see this algebraically, let M be any positive number. We will have $\ln x = M$ when $x = e^M$, and since the values of $\ln x$ increase as x increases, we will have

$$\ln x > M \quad \text{if} \quad x > e^M$$

(Figure 0.5.9).



▲ **Figure 0.5.8** The value of $y = e^x$ will exceed an arbitrary positive value of M when $x > \ln M$.



▲ **Figure 0.5.9** The value of $y = \ln x$ will exceed an arbitrary positive value of M when $x > e^M$.

✔ QUICK CHECK EXERCISES 0.5

(See page 63 for answers.)

1. The function $y = \left(\frac{1}{2}\right)^x$ has domain ______ and range ______.
2. The function $y = \ln(1 - x)$ has domain ______ and range ______.
3. Express as a power of 4:
   (a) 1 (b) 2 (c) $\frac{1}{16}$ (d) $\sqrt{8}$ (e) 5.
4. Solve each equation for x.
   (a) $e^x = \frac{1}{2}$ (b) $10^{3x} = 1{,}000{,}000$
   (c) $7e^{3x} = 56$
5. Solve each equation for x.
   (a) $\ln x = 3$ (b) $\log(x - 1) = 2$
   (c) $2\log x - \log(x + 1) = \log 4 - \log 3$

EXERCISE SET 0.5

Graphing Utility

1–2 Simplify the expression without using a calculating utility.

1. (a) $-8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$
2. (a) 2^{-4} (b) $4^{1.5}$ (c) $9^{-0.5}$

3–4 Use a calculating utility to approximate the expression. Round your answer to four decimal places.

3. (a) $2^{1.57}$ (b) $5^{-2.1}$
4. (a) $\sqrt[5]{24}$ (b) $\sqrt[8]{0.6}$

5–6 Find the exact value of the expression without using a calculating utility.

5. (a) $\log_2 16$ (b) $\log_2\left(\frac{1}{32}\right)$
   (c) $\log_4 4$ (d) $\log_9 3$
6. (a) $\log_{10}(0.001)$ (b) $\log_{10}(10^4)$
   (c) $\ln(e^3)$ (d) $\ln(\sqrt{e})$

7–8 Use a calculating utility to approximate the expression. Round your answer to four decimal places.

7. (a) $\log 23.2$ (b) $\ln 0.74$
8. (a) $\log 0.3$ (b) $\ln \pi$

9–10 Use the logarithm properties in Theorem 0.5.2 to rewrite the expression in terms of r, s, and t, where $r = \ln a$, $s = \ln b$, and $t = \ln c$.

9. (a) $\ln a^2\sqrt{bc}$ (b) $\ln \dfrac{b}{a^3 c}$
10. (a) $\ln \dfrac{\sqrt[3]{c}}{ab}$ (b) $\ln \sqrt{\dfrac{ab^3}{c^2}}$

11–12 Expand the logarithm in terms of sums, differences, and multiples of simpler logarithms.

11. (a) $\log(10x\sqrt{x - 3})$ (b) $\ln \dfrac{x^2 \sin^3 x}{\sqrt{x^2 + 1}}$
12. (a) $\log \dfrac{\sqrt[3]{x + 2}}{\cos 5x}$ (b) $\ln \sqrt{\dfrac{x^2 + 1}{x^3 + 5}}$

13–15 Rewrite the expression as a single logarithm.

13. $4\log 2 - \log 3 + \log 16$
14. $\frac{1}{2}\log x - 3\log(\sin 2x) + 2$
15. $2\ln(x + 1) + \frac{1}{3}\ln x - \ln(\cos x)$

16–23 Solve for x without using a calculating utility.

16. $\log_{10}(1 + x) = 3$
17. $\log_{10}(\sqrt{x}) = -1$
18. $\ln(x^2) = 4$
19. $\ln(1/x) = -2$
20. $\log_3(3^x) = 7$
21. $\log_5(5^{2x}) = 8$
22. $\ln 4x - 3\ln(x^2) = \ln 2$
23. $\ln(1/x) + \ln(2x^3) = \ln 3$

24–29 Solve for x without using a calculating utility. Use the natural logarithm anywhere that logarithms are needed.

24. $3^x = 2$
25. $5^{-2x} = 3$
26. $3e^{-2x} = 5$
27. $2e^{3x} = 7$
28. $e^x - 2xe^x = 0$
29. $xe^{-x} + 2e^{-x} = 0$
30. Solve $e^{-2x} - 3e^{-x} = -2$ for x without using a calculating utility. [*Hint:* Rewrite the equation as a quadratic equation in $u = e^{-x}$.]

FOCUS ON CONCEPTS

31–34 In each part, identify the domain and range of the function, and then sketch the graph of the function without using a graphing utility.

31. (a) $f(x) = \left(\frac{1}{2}\right)^{x-1} - 1$ (b) $g(x) = \ln|x|$
32. (a) $f(x) = 1 + \ln(x - 2)$ (b) $g(x) = 3 + e^{x-2}$
33. (a) $f(x) = \ln(x^2)$ (b) $g(x) = e^{-x^2}$
34. (a) $f(x) = 1 - e^{-x+1}$ (b) $g(x) = 3\ln\sqrt[3]{x - 1}$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The function $y = x^3$ is an exponential function.
36. The graph of the exponential function with base b passes through the point $(0, 1)$.
37. The natural logarithm function is the logarithmic function with base e.

38. The domain of a logarithmic function is the interval $x > 1$.

39. Use a calculating utility and the change of base formula (6) to find the values of $\log_2 7.35$ and $\log_5 0.6$, rounded to four decimal places.

40–41 Graph the functions on the same screen of a graphing utility. [Use the change of base formula (6), where needed.]

40. $\ln x,\ e^x,\ \log x,\ 10^x$

41. $\log_2 x,\ \ln x,\ \log_5 x,\ \log x$

42. (a) Derive the general change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

(b) Use the result in part (a) to find the exact value of $(\log_2 81)(\log_3 32)$ without using a calculating utility.

43. Use a graphing utility to estimate the two points of intersection of the graphs of $y = 1.3^x$ and $y = \log_{1.3} x$.

44. Use a graphing utility to estimate the two points of intersection of the graphs of $y = 0.6^{(x^2)}$ and $y = \log_{0.6}(x^2)$.

FOCUS ON CONCEPTS

45. (a) Is the curve in the accompanying figure the graph of an exponential function? Explain your reasoning.
(b) Find the equation of an exponential function that passes through the point $(4, 2)$.
(c) Find the equation of an exponential function that passes through the point $\left(2, \frac{1}{4}\right)$.
(d) Use a graphing utility to generate the graph of an exponential function that passes through the point $(2, 5)$.

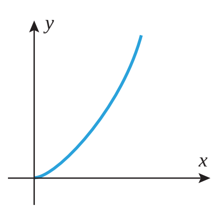


Figure Ex-45

46. (a) Make a conjecture about the general shape of the graph of $y = \log(\log x)$, and sketch the graph of this equation and $y = \log x$ in the same coordinate system.
(b) Check your work in part (a) with a graphing utility.

47. Find the fallacy in the following "proof" that $\frac{1}{8} > \frac{1}{4}$. Multiply both sides of the inequality $3 > 2$ by $\log \frac{1}{2}$ to get

$$3 \log \tfrac{1}{2} > 2 \log \tfrac{1}{2}$$
$$\log \left(\tfrac{1}{2}\right)^3 > \log \left(\tfrac{1}{2}\right)^2$$
$$\log \tfrac{1}{8} > \log \tfrac{1}{4}$$
$$\tfrac{1}{8} > \tfrac{1}{4}$$

48. Prove the four algebraic properties of logarithms in Theorem 0.5.2.

49. If equipment in the satellite of Example 3 requires 15 watts to operate correctly, what is the operational lifetime of the power supply?

50. The equation $Q = 12e^{-0.055t}$ gives the mass Q in grams of radioactive potassium-42 that will remain from some initial quantity after t hours of radioactive decay.
(a) How many grams were there initially?
(b) How many grams remain after 4 hours?
(c) How long will it take to reduce the amount of radioactive potassium-42 to half of the initial amount?

51. The acidity of a substance is measured by its pH value, which is defined by the formula

$$\text{pH} = -\log[H^+]$$

where the symbol $[H^+]$ denotes the concentration of hydrogen ions measured in moles per liter. Distilled water has a pH of 7; a substance is called *acidic* if it has pH < 7 and *basic* if it has pH > 7. Find the pH of each of the following substances and state whether it is acidic or basic.

	SUBSTANCE	$[H^+]$
(a)	Arterial blood	3.9×10^{-8} mol/L
(b)	Tomatoes	6.3×10^{-5} mol/L
(c)	Milk	4.0×10^{-7} mol/L
(d)	Coffee	1.2×10^{-6} mol/L

52. Use the definition of pH in Exercise 51 to find $[H^+]$ in a solution having a pH equal to
(a) 2.44 (b) 8.06.

53. The perceived loudness β of a sound in decibels (dB) is related to its intensity I in watts per square meter (W/m^2) by the equation

$$\beta = 10 \log(I/I_0)$$

where $I_0 = 10^{-12}\ \text{W/m}^2$. Damage to the average ear occurs at 90 dB or greater. Find the decibel level of each of the following sounds and state whether it will cause ear damage.

	SOUND	I
(a)	Jet aircraft (from 50 ft)	$1.0 \times 10^2\ \text{W/m}^2$
(b)	Amplified rock music	$1.0\ \text{W/m}^2$
(c)	Garbage disposal	$1.0 \times 10^{-4}\ \text{W/m}^2$
(d)	TV (mid volume from 10 ft)	$3.2 \times 10^{-5}\ \text{W/m}^2$

54–56 Use the definition of the decibel level of a sound (see Exercise 53).

54. If one sound is three times as intense as another, how much greater is its decibel level?

55. According to one source, the noise inside a moving automobile is about 70 dB, whereas an electric blender generates 93 dB. Find the ratio of the intensity of the noise of the blender to that of the automobile.

56. Suppose that the intensity level of an echo is $\frac{2}{3}$ the intensity level of the original sound. If each echo results in another echo, how many echoes will be heard from a 120 dB sound given that the average human ear can hear a sound as low as 10 dB?

57. On the ***Richter scale***, the magnitude M of an earthquake is related to the released energy E in joules (J) by the equation

$$\log E = 4.4 + 1.5M$$

(a) Find the energy E of the 1906 San Francisco earthquake that registered $M = 8.2$ on the Richter scale.
(b) If the released energy of one earthquake is 10 times that of another, how much greater is its magnitude on the Richter scale?

58. Suppose that the magnitudes of two earthquakes differ by 1 on the Richter scale. Find the ratio of the released energy of the larger earthquake to that of the smaller earthquake. [*Note:* See Exercise 57 for terminology.]

✔ QUICK CHECK ANSWERS 0.5

1. $(-\infty, +\infty)$; $(0, +\infty)$ **2.** $(-\infty, 1)$; $(-\infty, +\infty)$ **3.** (a) 4^0 (b) $4^{1/2}$ (c) 4^{-2} (d) $4^{3/4}$ (e) $4^{\log_4 5}$ **4.** (a) $\ln\frac{1}{2} = -\ln 2$ (b) 2 (c) $\ln 2$ **5.** (a) e^3 (b) 101 (c) 2

CHAPTER 0 REVIEW EXERCISES

Graphing Utility

1. Sketch the graph of the function

$$f(x) = \begin{cases} -1, & x \le -5 \\ \sqrt{25 - x^2}, & -5 < x < 5 \\ x - 5, & x \ge 5 \end{cases}$$

2. Use the graphs of the functions f and g in the accompanying figure to solve the following problems.
(a) Find the values of $f(-2)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) For what values of x is $f(x) < 2$?
(d) What are the domain and range of f?
(e) What are the domain and range of g?
(f) Find the zeros of f and g.

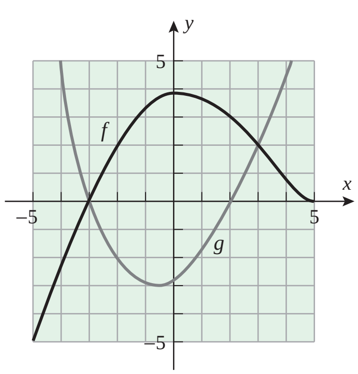


Figure Ex-2

3. A glass filled with water that has a temperature of 40°F is placed in a room in which the temperature is a constant 70°F. Sketch a rough graph that reasonably describes the temperature of the water in the glass as a function of the elapsed time.

4. You want to paint the top of a circular table. Find a formula that expresses the amount of paint required as a function of the radius, and discuss all of the assumptions you have made in finding the formula.

5. A rectangular storage container with an open top and a square base has a volume of 8 cubic meters. Material for the base costs \$5 per square meter and material for the sides \$2 per square meter.
(a) Find a formula that expresses the total cost of materials as a function of the length of a side of the base.
(b) What is the domain of the cost function obtained in part (a)?

6. A ball of radius 3 inches is coated uniformly with plastic.
(a) Express the volume of the plastic as a function of its thickness.
(b) What is the domain of the volume function obtained in part (a)?

7. A box with a closed top is to be made from a 6 ft by 10 ft piece of cardboard by cutting out four squares of equal size (see the accompanying figure), folding along the dashed lines, and tucking the two extra flaps inside.
(a) Find a formula that expresses the volume of the box as a function of the length of the sides of the cut-out squares.
(b) Find an inequality that specifies the domain of the function in part (a).
(c) Use the graph of the volume function to estimate the dimensions of the box of largest volume.

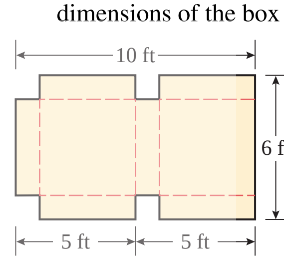


Figure Ex-7

8. Let C denote the graph of $y = 1/x$, $x > 0$.
(a) Express the distance between the point $P(1, 0)$ and a point Q on C as a function of the x-coordinate of Q.
(b) What is the domain of the distance function obtained in part (a)?

(cont.)

(c) Use the graph of the distance function obtained in part (a) to estimate the point Q on C that is closest to the point P.

9. Sketch the graph of the equation $x^2 - 4y^2 = 0$.

10. Generate the graph of $f(x) = x^4 - 24x^3 - 25x^2$ in two different viewing windows, each of which illustrates a different property of f. Identify each viewing window and a characteristic of the graph of f that is illustrated well in the window.

11. Complete the following table.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	0	−1	2	1	3	−2	−3	4	−4
$g(x)$	3	2	1	−3	−1	−4	4	−2	0
$(f \circ g)(x)$									
$(g \circ f)(x)$									

▲ **Table Ex-11**

12. Let $f(x) = -x^2$ and $g(x) = 1/\sqrt{x}$. Find formulas for $f \circ g$ and $g \circ f$ and state the domain of each composition.

13. Given that $f(x) = x^2 + 1$ and $g(x) = 3x + 2$, find all values of x such that $f(g(x)) = g(f(x))$.

14. Let $f(x) = (2x - 1)/(x + 1)$ and $g(x) = 1/(x - 1)$.
(a) Find $f(g(x))$.
(b) Is the natural domain of the function $h(x) = (3 - x)/x$ the same as the domain of $f \circ g$? Explain.

15. Given that

$$f(x) = \frac{x}{x-1}, \quad g(x) = \frac{1}{x}, \quad h(x) = x^2 - 1$$

find a formula for $f \circ g \circ h$ and state the domain of this composition.

16. Given that $f(x) = 2x + 1$ and $h(x) = 2x^2 + 4x + 1$, find a function g such that $f(g(x)) = h(x)$.

17. In each part, classify the function as even, odd, or neither.
(a) $x^2 \sin x$ (b) $\sin^2 x$ (c) $x + x^2$ (d) $\sin x \tan x$

18. (a) Write an equation for the graph that is obtained by reflecting the graph of $y = |x - 1|$ about the y-axis, then stretching that graph vertically by a factor of 2, then translating that graph down 3 units, and then reflecting that graph about the x-axis.
(b) Sketch the original graph and the final graph.

19. In each part, describe the family of curves.
(a) $(x - a)^2 + (y - a^2)^2 = 1$
(b) $y = a + (x - 2a)^2$

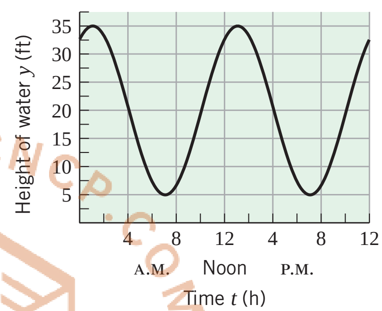
20. Find an equation for a parabola that passes through the points $(2, 0)$, $(8, 18)$, and $(-8, 18)$.

21. Suppose that the expected low temperature in Anchorage, Alaska (in °F), is modeled by the equation

$$T = 50 \sin \frac{2\pi}{365}(t - 101) + 25$$

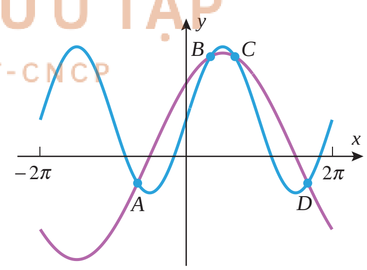
where t is in days and $t = 0$ corresponds to January 1.
(a) Sketch the graph of T versus t for $0 \le t \le 365$.
(b) Use the model to predict when the coldest day of the year will occur.
(c) Based on this model, how many days during the year would you expect the temperature to be below 0°F?

22. The accompanying figure shows a model for the tide variation in an inlet to San Francisco Bay during a 24-hour period. Find an equation of the form $y = y_0 + y_1 \sin(at + b)$ for the model, assuming that $t = 0$ corresponds to midnight.



◀ **Figure Ex-22**

23. The accompanying figure shows the graphs of the equations $y = 1 + 2\sin x$ and $y = 2\sin(x/2) + 2\cos(x/2)$ for $-2\pi \le x \le 2\pi$. Without the aid of a calculator, label each curve by its equation, and find the coordinates of the points A, B, C, and D. Explain your reasoning.



◀ **Figure Ex-23**

24. The electrical resistance R in ohms (Ω) for a pure metal wire is related to its temperature T in °C by the formula

$$R = R_0(1 + kT)$$

in which R_0 and k are positive constants.
(a) Make a hand-drawn sketch of the graph of R versus T, and explain the geometric significance of R_0 and k for your graph.
(b) In theory, the resistance R of a pure metal wire drops to zero when the temperature reaches absolute zero ($T = -273$°C). What information does this give you about k?
(c) A tungsten bulb filament has a resistance of 1.1 Ω at a temperature of 20°C. What information does this give you about R_0 for the filament?

(cont.)

(d) At what temperature will the tungsten filament have a resistance of 1.5 Ω?

25. (a) State conditions under which two functions, f and g, will be inverses, and give several examples of such functions.
(b) In words, what is the relationship between the graphs of $y = f(x)$ and $y = g(x)$ when f and g are inverse functions?
(c) What is the relationship between the domains and ranges of inverse functions f and g?
(d) What condition must be satisfied for a function f to have an inverse? Give some examples of functions that do not have inverses.

26. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.
(b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

27. In each part, find $f^{-1}(x)$ if the inverse exists.
(a) $f(x) = 8x^3 - 1$
(b) $f(x) = x^2 - 2x + 1$
(c) $f(x) = (e^x)^2 + 1$
(d) $f(x) = (x + 2)/(x - 1)$
(e) $f(x) = \sin\left(\dfrac{1-2x}{x}\right), \quad \dfrac{2}{4+\pi} \le x \le \dfrac{2}{4-\pi}$
(f) $f(x) = \dfrac{1}{1 + 3\tan^{-1} x}$

28. Let $f(x) = (ax + b)/(cx + d)$. What conditions on a, b, c, and d guarantee that f^{-1} exists? Find $f^{-1}(x)$.

29. In each part, find the exact numerical value of the given expression.
(a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
(b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

30. In each part, sketch the graph, and check your work with a graphing utility.
(a) $f(x) = 3\sin^{-1}(x/2)$
(b) $f(x) = \cos^{-1} x - \pi/2$
(c) $f(x) = 2\tan^{-1}(-3x)$
(d) $f(x) = \cos^{-1} x + \sin^{-1} x$

31. Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the x-axis, how many miles to the right of the origin will it have to travel?

32. Suppose that the graph of $y = 10^x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 100 mi above the x-axis, how many feet to the right of the origin will it have to travel?

33. Express the following function as a rational function of x:

$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

34. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

35. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
(b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

36. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity v (in feet per second) of the package t seconds after it is released is given by $v = 24.61(1 - e^{-1.3t})$.
(a) Graph v versus t.
(b) Show that the graph has a horizontal asymptote $v = c$.
(c) The constant c is called the ***terminal velocity***. Explain what the terminal velocity means in practical terms.
(d) Can the package actually reach its terminal velocity? Explain.
(e) How long does it take for the package to reach 98% of its terminal velocity?

37. A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, N, after t years will be given by the formula

$$N = \frac{220}{1 + 10(0.83^t)}$$

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.
(a) Graph N versus t.
(b) How many years must the state of Colorado maintain a program to care for the sheep?
(c) How many bighorn sheep can the environment in the protected area support? [*Hint:* Examine the graph of N versus t for large values of t.]

38. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature T (in °F) of the potato t minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F.
(a) During what interval of time would the potato be considered done?
(b) How long does it take for the difference between the potato and oven temperatures to be cut in half?

39. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
(b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

40. (a) Show that for $x > 0$ and $k \ne 0$ the equations

$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$

have the same solutions. *(cont.)*

(b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.

(c) Estimate the positive solution(s) of $x^8 = e^x$.

41. Consider $f(x) = x^2 \tan x + \ln x$, $0 < x < \pi/2$.

(a) Explain why f is one-to-one.

(b) Use a graphing utility to generate the graph of f. Then sketch the graphs of f and f^{-1} together. What are the asymptotes for each graph?