

Chapter 8

Discrete Probability

Discrete Structures for Computing

TÀI LIỆU SƯU TẬP

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen Ngoc Le
Faculty of Computer Science and Engineering
University of Technology - VNUHCM
{htnguyen;trtanh}@hcmut.edu.vn

Discrete Probability

Huynh Tuong Nguyen,
Tran Tuan Anh, Nguyen
Ngoc Le



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Contents

1 Introduction

Randomness

2 Probability

3 Probability Rules

4 Random variables

5 Probability Models

Geometric Model

Binomial Model

Discrete Probability

Huynh Tuong Nguyen,
Tran Tuan Anh, Nguyen
Ngoc Le



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

BACHKHOACNCP.COM

Course outcomes

Course learning outcomes

L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic
	L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing
	L.O.2.2 – Use proving methods: direct, contrapositive, induction
	L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory
	L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures
	L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Discrete Probability

Huynh Tuong Nguyen,
Tran Tuan Anh, Nguyen
Ngoc Le



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Motivations

- Gambling



- Real life problems



- **Computer Science:** cryptology – deals with encrypting codes or the design of error correcting codes



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models



Which of these are random phenomena?

- The number you receive when rolling a fair dice
- The sequence for lottery special prize (by law!)
- Your blood type (No!)
- You met the red light on the way to school
 - The traffic light is not random. It has timer.
 - The pattern of your riding is random.

So what is special about randomness?

In the long run, they are predictable and have relative frequency (fraction of times that the event occurs over and over and over).



- **Experiment** (*thí nghiệm*): a procedure that yields one of a given set of possible outcomes.
 - Tossing a coin to see the face
- **Sample space** (*không gian mẫu*): set of possible **outcomes**
 - {Head, Tail}
- **Event** (*sự kiện*): a subset of sample space.
 - You see Head after an experiment. {Head} is an event.



Example

Example (1)

Experiment: Rolling a die. What is the sample space?

Answer: $\{1, 2, 3, 4, 5, 6\}$

Example (2)

Experiment: Rolling two dice. What is the sample space?

Answer: It depends on what we're going to ask!

- The total number?
 $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- The number of each die?
 $\{(1,1), (1,2), (1,3), \dots, (6,6)\}$

Which is better?

The latter one, because they are **equally likely outcomes**



The Law of Large Numbers

Definition

The Law of Large Numbers (*Luật số lớn*) states that the **long-run relative frequency** of repeated independent events gets closer and closer to the **true** relative frequency as the number of trials increases.

Example

Do you believe that the true relative frequency of **Head** when you toss a coin is 50%?

Let's try!



Be Careful!

Don't misunderstand the Law of Large Numbers (LLN). It can lead to money lost and poor business decisions.

Example

I had 8 children, all of them are girls. Thanks to LLN (!?), there are high possibility that the next one will be a boy.
(Overpopulation!!!)

Example

I'm playing *Bầu cua tôm cá*, the fish has not appeared in recent 5 games, it will be more likely to be fish next game. Thus, I bet all my money in fish. **(Sorry, you lose!)**





Definition

The **probability** (xác suất) of an event E of a finite nonempty sample space of **equally likely outcomes** S is:

$$p(E) = \frac{|E|}{|S|}.$$

- Note that $E \subseteq S$ so $0 \leq |E| \leq |S|$
- $0 \leq p(E) \leq 1$
 - 0 indicates impossibility
 - 1 indicates certainty

People often say: "It has a **20%** probability"

Examples

Example (1)

What is the probability of getting a Head when tossing a coin?

Answer:

- There are $|S| = 2$ possible outcomes
- Getting a Head is $|E| = 1$ outcome, so $p(E) = 1/2 = 0.5 = 50\%$

Example (2)

What is the probability of getting a 7 by rolling two dice?

Answer:

- **Product rule:** There are a total of 36 equally likely possible outcomes
- There are six successful outcomes: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)
- Thus, $|E| = 6$, $|S| = 36$, $p(E) = 6/36 = 1/6$



Examples

Example (3)

We toss a coin 6 times. What is probability of H in 6th toss, if all the previous 5 are T?

Answer:

Don't be silly! Still $1/2$.

Example (4)

Which is more likely:

- Rolling an 8 when 2 dice are rolled?
- Rolling an 8 when 3 dice are rolled?

Answer:

Two dice: $5/36 \approx 0.139$

Three dice: $21/216 \approx 0.097$



Formal Probability

Rule 1

A probability is a number **between 0 and 1**.

$$0 \leq p(E) \leq 1$$

Rule 2: Something has to happen rule

The probability of the set of all possible outcomes of a trial **must be 1**.

$$p(S) = 1$$

Rule 3: Compliment Rule

The probability of an event occurring is 1 minus the probability that it doesn't occur.

$$p(E) = 1 - p(\overline{E})$$

Discrete Probability

Huynh Tuong Nguyen,
Tran Tuan Anh, Nguyen
Ngoc Le



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Examples

Example

What is the probability of **NOT** drawing a heart card from 52 deck cards?

Answer:

Let E be the event of **getting** a heart from 52 deck cards. We have:

$$p(E) = 13/52 = 1/4$$

By the compliment rule, the probability of NOT getting a heart card is:

$$p(\overline{E}) = 1 - p(E) = 3/4$$





Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

General Addition Rule

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

- If $E_1 \cap E_2 = \emptyset$: They are **disjoint**, which means they can't occur together
- then, $p(E_1 \cup E_2) = p(E_1) + p(E_2)$

Example

Example (1)

If you choose a number between 1 and 100, what is the probability that it is divisible by either 2 or 5?

Short Answer:

$$\frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}$$

Example (2)

There are a survey that about 45% of VN population has Type O blood, 40% type A, 11% type B and the rest type AB. What is the probability that a blood donor has Type A or Type B?

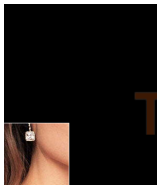
Short Answer:

$$40\% + 11\% = 51\%$$



Conditional Probability (Xác suất có điều kiện)

- “Knowledge” changes probabilities



Conditional Probability

Definition

$p(E | F)$ = Probability of event E given that event F **has occurred**.

General Multiplication Rule

$$\begin{aligned} p(E \cap F) &= p(E) \times p(F | E) \\ &= p(F) \times p(E | F) \end{aligned}$$



Example

Example

What is the probability of drawing a red card and then another red card **without replacement** (không hoàn lại)?

Solution

E : the event of drawing the first red card

F : the event of drawing the second red card

$$p(E) = 26/52 = 1/2$$

$$p(F|E) = 25/51$$

So the event of drawing a red card and then another red card is

$$p(E \cap F) = p(E) \times p(F|E) = 1/2 \times 25/51 = 25/102$$



Independence

Definition

Events E and F are independent (*độc lập*) whenever

$$p(E | F) = p(E)$$

- The outcome of one event does not influence the probability of the other.
- Example: $p(\text{"Head"} | \text{"It's raining outside"}) = p(\text{"Head"})$
- If E and F are independent

$$p(E \cap F) = p(E) \times p(F)$$

Disjoint \neq Independence

Disjoint events cannot be independent. They have no outcomes in common, so knowing that one occurred means the other did not.



Bayes's Theorem



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Example

If we know that the probability that a person has tuberculosis (TB) is $p(\text{TB}) = 0.0005$.

We also know $p(+|\text{TB}) = 0.999$ and $p(-|\overline{\text{TB}}) = 0.99$.

What is $p(\text{TB}|+)$ and $p(\overline{\text{TB}}|-)$?

Theorem (Bayes's Theorem)

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

Expected Value: Center

An **insurance** company charges \$50 a year. Can company make a profit? Assuming that it made a research on 1000 people and have following table:

Outcome	Payroll	Probability
	x	$p(X = x)$
Death	10,000	$\frac{1}{1000}$
Disability	5000	$\frac{2}{1000}$
Neither	0	$\frac{997}{1000}$

- X is a **discrete random variable** (*biến ngẫu nhiên rời rạc*)

The company **expects** that they have to pay each customer:

$$E(X) = \$10,000\left(\frac{1}{1000}\right) + \$5000\left(\frac{2}{1000}\right) + \$0\left(\frac{997}{1000}\right) = \$20$$

Expected value (giá trị kỳ vọng)

$$E(X) = \sum x \cdot p(X = x)$$



Variance: The Spread

- Of course, the expected value \$20 will not happen in reality
- There will be **variability**. Let's calculate!

- Variance (**phương sai**)

$$V(X) = \sum (x - E(X))^2 \cdot p(X = x)$$

$$\bullet V(X) = 9980^2 \left(\frac{1}{1000}\right) + 4980^2 \left(\frac{2}{1000}\right) + (-20)^2 \left(\frac{997}{1000}\right) = 149,600$$

- Standard deviation (**độ lệch chuẩn**)

$$SD(X) = \sqrt{V(X)}$$

$$\bullet SD(X) = \sqrt{149,600} \approx \$386.78$$

Comment

The company expects to pay out \$20, and make \$30. However, the standard deviation of \$386.78 indicates that it's no sure thing. That's pretty big spread (and risk) for an average profit of \$20.





Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Example

Some people madly drink Coca-Cola, hoping to find a ticket to see Big Bang. Let's call tearing a bottle's label **trial** (*phép thử*):

- There are only possible outcomes (**congrats** or **good luck**)
- The probability of success, p , is the same on every trial, say 0.06
- The trials are independent. Finding a ticket in the first bottle does not change what might happen in the second one.

• Bernoulli Trials

- Another examples: tossing a coin many times, results of testing TB on many patients, ...

Geometric Model (Mô hình hình học)



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Question: How long it will take us to achieve a success, given p , the probability of success?

Definition (Geometric probability model: Geom(p))

p = probability of success ($q = 1 - p$ = probability of failure)

X = number of trials until the first success occurs

$$p(X = x) = q^{x-1}p$$

Expected value: $\mu = \frac{1}{p}$

Standard deviation: $\sigma = \sqrt{\frac{q}{p^2}}$

Geometric Model: Example

Example

If the probability of finding a Sound Fest ticket is $p = 0.06$, how many bottles do you expect to open before you find a ticket? What is the probability that the first ticket is in one of the first four bottles?

Solution

Let X = number of trials until a ticket is found
We can model X with $\text{Geom}(0.06)$.

$$E(X) = \frac{1}{0.06} \approx 16.7$$

$$\begin{aligned} P(X \leq 4) &= P(X=1) + P(X=2) + P(X=3) + P(X=4) \\ &= (0.06) + (0.94)(0.06) + (0.94)^2(0.06) \\ &\quad + (0.94)^3(0.06) \\ &\approx 0.2193 \end{aligned}$$

Conclusion: We expect to open 16.7 bottles to find a ticket.
About 22% of time we'll find one within the first 4 bottles.



Contents

Introduction

Randomness

Probability

Probability Rules

Random variables

Probability Models

Geometric Model

Binomial Model

Binomial Model (Mô hình nhị thức)

Previous Question: How long it will take us to achieve a success, given p , the probability of success?

New Question: You buy 5 Coca-Cola. What's the probability you get **exactly** 2 Sound Fest tickets?

Definition (Binomial probability model: $\text{Binom}(n, p)$)

n = number of trials

p = probability of success ($q = 1 - p$ = probability of failure)

X = number of successes in n trials

$$p(X = x) = \binom{n}{x} p^x q^{n-x}$$

Expected value: $\mu = np$

Standard deviation: $\sigma = \sqrt{npq}$



Binomial Model: Example

Example

Suppose you buy 20 Coca-Cola bottles. What are the mean and standard deviation of the number of winning bottles among them? What is the probability that there are 2 or 3 tickets?

Solution

Let X = number of tickets among $n = 20$ bottles

We can model X with $\text{Binom}(20, 0.06)$.

$$E(X) = np = 20(0.06) = 1.2$$

$$SD(X) = \sqrt{npq} = \sqrt{20(0.06)(0.94)} \approx 1.06$$

$$\begin{aligned} P(X = 2 \text{ or } 3) &= P(X = 2) + P(X = 3) \\ &= \binom{20}{2} (0.06)^2 (0.94)^{18} + \binom{20}{3} (0.06)^3 (0.94)^{17} \\ &\approx 0.2246 + 0.0860 = 0.3106 \end{aligned}$$

Conclusion: In 20 bottles, we expect to find an average of 1.2 tickets, with a sd of 1.06. About 31% of the time we'll find 2 or 3 tickets among 20 bottles.

