

NHÓM 4: Lớp MT_K12

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Bài tập trong giáo trình (LT): 2.19 2.23 3.2 (tính $E(X)$ $D(X)$ cho bài 2.4) 3.18

BÀI LÀM

Bài 2.19:

$$f(x, y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2 + 2xy + 5y^2)}$$

a) Hàm mật độ lẽ của (X, Y)

$$f_X(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2 + 2xy + 5y^2)} dy$$

$$\text{Đặt } A = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5\left(y + \frac{1}{5}x\right)^2 + \frac{4}{5}x^2\right]} dy$$

$$\text{Đặt } u = y + \frac{1}{5}x \Rightarrow du = dy$$

$$A = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5u^2 + \frac{4}{5}x^2\right]} du$$

$$\text{Đặt } B = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5u^2 + \frac{4}{5}x^2\right]} \cdot e^{-\frac{1}{2}\left[5v^2 + \frac{4}{5}x^2\right]} dudv$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5(u^2 + v^2) + \frac{8}{5}x^2\right]} dudv$$

Xét miền $D: u^2 + v^2 \leq R^2$

$$\text{Xét } C = \iint e^{-\frac{1}{2}\left[5(u^2 + v^2) + \frac{8}{5}x^2\right]} dudv$$

$$= \int_0^{2\pi} d\varphi \int_0^r e^{-\frac{1}{2}\left(5r^2 + \frac{8}{5}x^2\right)} r dr \quad (u = r\cos\varphi; v = r\sin\varphi)$$

$$= \frac{1}{5} \int_0^{2\pi} d\varphi \int_{\frac{4}{5}x^2}^{\frac{1}{2}(5r^2 + \frac{8}{5}x^2)} e^{-t} dt \quad (t = \frac{1}{2}(5r^2 + \frac{8}{5}x^2) \Rightarrow dt = 5r dr)$$

$$= \frac{2}{5} \pi \cdot \left[e^{-\frac{4}{5}x^2} - e^{-\frac{1}{2}(5r^2 + \frac{8}{5}x^2)} \right]$$

$$\text{Nên } B = \lim_{r \rightarrow +\infty} C = \frac{2}{5} \pi e^{-\frac{4}{5}x^2}$$

Cho ta: $A = \sqrt{B} = \sqrt{\frac{2\pi}{5}} e^{-\frac{2}{5}x^2}$

Vậy $f_X(x) = \sqrt{\frac{2}{5\pi}} e^{-\frac{2}{5}x^2}$

$$f_Y(y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+2xy+5y^2)} dx$$

$$\text{Đặt } D = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(x+y)^2+4y^2]} dx$$

$$\text{Đặt } u = x + y \Rightarrow du = dx$$

$$D = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[u^2+4y^2]} du$$

$$\begin{aligned} \text{Đặt } E &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[u^2+4y^2]} \cdot e^{-\frac{1}{2}[v^2+4y^2]} dudv \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(u^2+v^2+8y^2)} dudv \end{aligned}$$

$$\text{Xét miền } D: u^2 + v^2 \leq R^2$$

$$\begin{aligned} \text{Xét } F &= \iint e^{-\frac{1}{2}(u^2+v^2+8y^2)} dudv \\ &= \int_0^{2\pi} d\varphi \int_0^r e^{-\frac{1}{2}(r^2+8y^2)} r dr \quad (u = r\cos\varphi; v = r\sin\varphi) \\ &= \int_0^{2\pi} d\varphi \int_{4y^2}^{\frac{1}{2}(r^2+8y^2)} e^{-t} dt \quad (t = \frac{1}{2}(r^2 + 8y^2) \Rightarrow dt = r dr) \\ &= 2\pi \cdot \left[e^{-4y^2} - e^{-\frac{1}{2}(r^2+8y^2)} \right] \end{aligned}$$

$$\text{Nên } E = \lim_{x \rightarrow +\infty} F = 2\pi e^{-4y^2}$$

Cho ta: $D = \sqrt{E} = \sqrt{2\pi} e^{-2y^2}$

Vậy $f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-2y^2}$

b) Hàm mật độ điều kiện của (X,Y)

$$f_{X/Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{\pi} e^{-\frac{1}{2}(x^2+2xy+5y^2)}}{\sqrt{\frac{2}{\pi}} e^{-2y^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+y)^2}$$

$$f_{Y/X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\frac{1}{\pi} e^{-\frac{1}{2}(x^2+2xy+5y^2)}}{\sqrt{\frac{2}{5\pi}} e^{-\frac{2}{5}x^2}}$$

$$= \sqrt{\frac{5}{2\pi}} e^{-\frac{1}{10}(x+5y)^2}$$

Bài 2.23:

$$f_X(x) = \begin{cases} \frac{1}{2} & , x \in [1,3] \\ 0 & , x \notin [1,3] \end{cases}$$

$$Y = 3X + 2$$

$$F_Y(y) = P(Y < y) = P(3X + 2 < y)$$

$$= \int_{3x+2 < y} f_X(x) dx = \begin{cases} 0 & y \leq 5 \\ \int_1^{\frac{y-2}{3}} \frac{1}{2} dx & 5 < y < 11 \\ 1 & y \geq 11 \end{cases} = \begin{cases} 0 & y \leq 5 \\ \frac{y-5}{6} & 5 < y < 11 \\ 1 & y \geq 11 \end{cases}$$

Bài 3.18:

$$f(x,y) = \begin{cases} 8xy & \text{nếu } 0 \leq y \leq x \leq 1 \\ 0 & \text{nếu} \end{cases}$$

a)

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y) dx = \begin{cases} 0, y \notin [0,1] \\ \int_y^1 8xy dx, y \in [0,1] \end{cases}$$

$$= \begin{cases} 0 & , y \notin [0,1] \\ 4y(1-y^2) & , y \in [0,1] \end{cases}$$

Nên:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

$$\Leftrightarrow f_{X|Y=\frac{1}{2}}(x) = \frac{f\left(x, \frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)} = \begin{cases} \frac{8x}{3} dx & , x \in \left[\frac{1}{2}, 1\right] \\ 0 & , x \notin \left[\frac{1}{2}, 1\right] \end{cases}$$

b)

$$E\left(X \middle| Y = \frac{1}{2}\right) = \int_{\frac{1}{2}}^1 x \frac{8x}{3} dx = \frac{8}{3} \int_{\frac{1}{2}}^1 x^2 dx = \frac{8}{9} \left(1 - \frac{1}{8}\right) = \frac{7}{9}$$

c)

$$P\left(X < 2 \middle| Y < \frac{1}{2}\right) = \frac{P\left(X < \frac{1}{2} \text{ và } Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)}$$

$$P\left(X < \frac{1}{2} \text{ và } Y < \frac{1}{2}\right) = \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}} f(x,y) dx dy = \int_0^{\frac{1}{2}} dx \int_0^x 8xy dy = \int_0^{\frac{1}{2}} 8x dx \cdot \frac{x^2}{2} = \frac{1}{16}$$

$$P\left(Y < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f_Y(y) dy = \int_0^{\frac{1}{2}} 4y(1-y^2) dy$$

$$= \int_0^{\frac{1}{2}} (4y - 4y^3) dy = 2 \cdot \frac{1}{4} - \frac{1}{16} = \frac{7}{16}$$

Nên:

$$P\left(X < \frac{1}{2} \middle| Y < \frac{1}{2}\right) = \frac{1}{7}$$

$$P\left(x < \frac{1}{2} \middle| Y = \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\frac{1}{2}} f_{X|Y=\frac{1}{2}}(x) dx = 0$$

Bài 3.2 (tính $E(X)$, $D(X)$ của bài 2.4):

$$\begin{aligned}
 E(X) &= \sum_{n=1}^{\infty} x_i p_i \\
 &= 0.p + 1.qp + 2q^2p + \dots + nq^n p + \dots \\
 &= p \sum_{n=1}^{\infty} nq^n = pq \sum_{n=1}^{\infty} nq^{n-1} = pq \sum_{n=1}^{\infty} (q^n)' \\
 &= pq \left(\sum_{n=1}^{\infty} q^n \right)' = pq \left(\frac{q}{1-q} \right)' = pq \left(\frac{1}{1-q^2} \right)' = \frac{q}{p}
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum_{n=1}^{\infty} x_i^2 p_i = 0.p + 1^2 qp + \dots + n^2 q^n p + \dots \\
 &= p \sum_{n=1}^{\infty} n^2 q^n = pq \sum_{n=1}^{\infty} n^2 q^{n-1} = pq \sum_{n=1}^{\infty} (nq^n)' \\
 &= pq \left(q \sum_{n=1}^{\infty} nq^{n-1} \right)' = pq \left(q \sum_{n=1}^{\infty} (q^n)' \right)' \\
 &= pq \left(q \left(\sum_{n=1}^{\infty} q^n \right)' \right)' = pq \left(q \cdot \left(\frac{q}{1-q} \right)' \right)' \\
 &= pq \left(\frac{q}{(1-q)^2} \right)' = pq \frac{(1-q^2)}{(1-q)^4} = pq \frac{(1+q)}{p^3} = \frac{q(1+q)}{p^2}
 \end{aligned}$$

Vậy:

$$D(X) = E(X^2) - (E(X))^2 = \frac{q + q^2}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2}$$