NHÓM 4: Lớp MT_K12

……………….

Bài tập trong giáo trình (LT): 2.19 2.23 3.2 (tính E(X) D(X) cho bài 2.4) 3.18

BÀI LÀM

Bài 2.19:

$$f(x,y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2+2xy+5y^2)}$$

a) Hàm mật độ lề của (X,Y)

$$f_X(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+2xy+5y^2)} dy$$

$$Đặt\ A = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5\left(y+\frac{1}{5}x\right)^2+\frac{4}{5}x^2\right]} dy$$

$$Đặt\ u = y + \frac{1}{5}x \Rightarrow du = dy$$

$$A = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5u^2+\frac{4}{5}x^2\right]} du$$

$$Đặt\ B = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5u^2+\frac{4}{5}x^2\right]} . e^{-\frac{1}{2}\left[5v^2+\frac{4}{5}x^2\right]} dudv$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[5(u^2+v^2)+\frac{8}{5}x^2\right]} dudv$$

$$Xét\ miền\ D: u^2+v^2 \le R^2$$

$$Xét\ C = \iint e^{-\frac{1}{2}\left[5(u^2+v^2)+\frac{8}{5}x^2\right]} dudv$$

$$= \int_0^{2\pi} d\varphi \int_0^{r} e^{-\frac{1}{2}\left(5r^2+\frac{8}{5}x^2\right)} rdr \qquad (u = r\cos\varphi; v = r\sin\varphi)$$

$$= \frac{1}{5}\int_0^{2\pi} d\varphi \int_{\frac{4}{5}x^2}^{\frac{1}{2}\left(5r^2+\frac{8}{5}x^2\right)} e^{-t} dt \qquad \left(t = \frac{1}{2}\left(5r^2+\frac{8}{5}x^2\right) \Rightarrow dt = 5rdr\right)$$

$$= \frac{2}{5}\pi.\left[e^{-\frac{4}{5}x^2} - e^{-\frac{1}{2}\left(5r^2+\frac{8}{5}x^2\right)}\right]$$

$$Nên\ \ B = \lim_{r\to+\infty} C = \frac{2}{5}\pi e^{-\frac{4}{5}x^2}$$

$$Cho\ ta:\ A = \sqrt{B} = \sqrt{\frac{2\pi}{5}}e^{-\frac{2}{5}x^2}$$

$$V\hat{a}y\ f_X(x) = \sqrt{\frac{2}{5\pi}}e^{-\frac{2}{5}x^2}$$

$$f_Y(y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+2xy+5y^2)}dx$$

$$Đặt\ D = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(x+y)^2+4y^2]}dx$$

$$Đặt\ u = x + y\ \Rightarrow du = dx$$

$$D = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[u^2+4y^2]}du$$

$$Đặt\ E = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[u^2+4y^2]}.e^{-\frac{1}{2}[v^2+4y^2]}dudv$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(u^2+v^2+8y^2)}\,dudv$$

$$Xét\ miền\ D: u^{2+}v^2 \leq R^2$$

$$Xét\ F = \iint e^{-\frac{1}{2}(u^2+v^2+8y^2)}\,dudv$$

$$= \int_0^{2\pi} d\varphi \int_0^r e^{-\frac{1}{2}(r^2+8y^2)}rdr \qquad (u = rcos\varphi; v = rsin\varphi)$$

$$= \int_0^{2\pi} d\varphi \int_{4y^2}^{\frac{1}{2}(r^2+8y^2)} e^{-t}dt \qquad (t = \frac{1}{2}(r^2 + 8y^2) \Rightarrow dt = rdr$$

$$= 2\pi.\left[e^{-4y^2} - e^{-\frac{1}{2}(r^2+8y^2)}\right]$$

$$Nên\ E = \lim_{x\to+\infty} F = 2\pi e^{-4y^2}$$

$$Cho\ ta:\ D = \sqrt{E} = \sqrt{2\pi}e^{-2y^2}$$

$$V\hat{a}y\ f_Y(y) = \sqrt{\frac{2}{\pi}}e^{-2y^2}$$

b) Hàm mật độ điều kiện của (X,Y)

$$f_{X/Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{\pi}e^{-\frac{1}{2}(x^2+2xy+5y^2)}}{\sqrt{\frac{2}{\pi}}e^{-2y^2}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x+y)^2}$$

$$f_{Y/X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\frac{1}{\pi}e^{-\frac{1}{2}(x^2+2xy+5y^2)}}{\sqrt{\frac{2}{5\pi}}e^{-\frac{2}{5}x^2}}$$

$$= \sqrt{\frac{5}{2\pi}}e^{-\frac{1}{10}(x+5y)^2}$$

Bài 2.23:

$$f_X(x) = \begin{cases} \frac{1}{2} & , x \in [1,3] \\ 0 & , x \notin [1,3] \end{cases}$$

$$Y = 3X + 2$$

$$F_Y(y) = P(Y < y) = P(3X + 2 < y)$$

$$= \int_{3x+2<y} f_X(x)dx = \begin{cases} 0 & y \le 5 \\ \int_1^{\frac{y-2}{3}} \frac{1}{2}dx & 5 < y < 11 \\ 1 & y \ge 11 \end{cases} = \begin{cases} 0 & y \le 5 \\ \frac{y-5}{6} & 5 < y < 11 \\ 1 & y \ge 11 \end{cases}$$

Bài 3.18:

$$f(x,y) = \begin{cases} 8xy & n\acute{\hat{e}}u\ 0 \le y \le x \le 1 \\ 0 & n\acute{\hat{e}}u \end{cases}$$

$a)$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \begin{cases} 0 & , y \notin [0,1] \\ \int_y^1 8xydx & , y \in [0,1] \end{cases}$$

$$= \begin{cases} 0 & , y \notin [0,1] \\ 4y(1-y^2) & , y \in [0,1] \end{cases}$$

$Nên:$

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

$$\Leftrightarrow \ f_{X|Y=\frac{1}{2}}(x) = \frac{f\left(x, \frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)} = \begin{cases} \frac{8x}{3} dx & , x \in \left[\frac{1}{2}, 1\right] \\ 0 & , x \notin \left[\frac{1}{2}, 1\right] \end{cases}$$

$b)$

$$E\left(X \middle| Y = \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} x \frac{8x}{3} dx = \frac{8}{3} \int_{\frac{1}{2}}^{1} x^2 dx = \frac{8}{9}\left(1 - \frac{1}{8}\right) = \frac{7}{9}$$

$c)$

$$P\left(X < 2 \middle| Y < \frac{1}{2}\right) = \frac{P\left(X < \frac{1}{2} \text{ và } Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)}$$

$$P\left(X < \frac{1}{2} \text{ và } Y < \frac{1}{2}\right) = \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}} f(x,y) dx dy = \int_0^{\frac{1}{2}} dx \int_0^x 8xy dy = \int_0^{\frac{1}{2}} 8x dx \cdot \frac{x^2}{2} = \frac{1}{16}$$

$$P\left(Y < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f_Y(y) dy = \int_0^{\frac{1}{2}} 4y(1-y^2) dy$$

$$= \int_0^{\frac{1}{2}} (4y - 4y^3) dy = 2.\frac{1}{4} - \frac{1}{16} = \frac{7}{16}$$

$Nên:$

$$P\left(X < \frac{1}{2} \middle| Y < \frac{1}{2}\right) = \frac{1}{7}$$

$$P\left(x < \frac{1}{2} \middle| Y = \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\frac{1}{2}} f_{X|Y=\frac{1}{2}}(x) dx = 0$$

<u>Bài 3.2</u> (tính E(X), D(X) của bài 2.4):

$$E(X) = \sum_{n=1}^{\infty} x_i\, p_i$$

$$= 0.p + 1.qp + 2q^2p + \ \dots + nq^np + \ \dots$$

$$= p\sum_{n=1}^{\infty} nq^n \ = pq\sum_{n=1}^{\infty} nq^{n-1} = pq\sum_{n=1}^{\infty} (q^n)'$$

$$= \ pq\left(\sum_{n=1}^{\infty} q^n\right)' = pq\left(\frac{q}{1-q}\right)' = pq\left(\frac{1}{1-q^2}\right) = \frac{q}{p}$$

$$E(X^2) = \sum_{n=1}^{\infty} {x_i}^2\, p_i = \ 0.p + 1^2qp + \ \dots + n^2q^np + \ \dots$$

$$= \ p\sum_{n=1}^{\infty} n^2\, q^n = pq\sum_{n=1}^{\infty} n^2q^{n-1} = pq\sum_{n=1}^{\infty} (nq^n)'$$

$$= \ pq\left(q\sum_{n=1}^{\infty} nq^{n-1}\right)' = pq\left(q\sum_{n=1}^{\infty} (q^n)'\right)'$$

$$= pq\left(q\left(\sum_{n=1}^{\infty} q^n\right)'\right)' = pq\left(q.\left(\frac{q}{1-q}\right)'\right)'$$

$$= pq\left(\frac{q}{(1-q)^2}\right)' = pq\,\frac{(1-q^2)}{(1-q)^4} = pq\,\frac{(1+q)}{p^3} = \frac{q(1+q)}{p^2}$$

Vậy:

$$D(X) = E(X^2) - \ \big(E(X)\big)^2 = \frac{q+q^2}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2}$$