Chương 6: DẠNG TOÀN PHƯƠNG

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n +$$

$$+ a_{21}x_2x_1 + a_{22}x_2^2 + \ldots + a_{2n}x_2x_n +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn}x_n^2$$

$$= X^T A X$$

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ . & \ldots & . \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ . \\ x_n \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & . & x_n \end{pmatrix}$$

A là ma trận đối xứng,
hạng của A là hạng của dạng toàn phương

VD1:

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = a_{11}x_1^2 + a_{12}x_1x_2 +$$

$$+ a_{21}x_2x_1 + a_{22}x_2^2$$

VD2:

$$f = 2x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_2 - x_1x_3 + 4x_2x_3$$

$$\to \begin{cases} a_{11} = 2,\ a_{22} = -2,\ a_{33} = 1,\ a_{12} = a_{21} = 1 \\ a_{13} = a_{31} = -1/2,\ a_{23} = a_{32} = 2 \end{cases}$$

$$\to f = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1/2 \\ 1 & -2 & 2 \\ -1/2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

VD3: Biểu diễn dạng toàn phương dạng X^TAX

a) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

b) $f(x_1, x_2, x_3) = x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3$

c) $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 4x_1x_3 + 4x_2x_3$

d) $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_3 - 2x_2x_3$

DẠNG TOÀN PHƯƠNG CHÍNH TẮC

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j \quad \to \quad f(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} b_{ii} y_i^2$$

1) Phép biến đổi trực giao

$$f(x_1, x_2, \ldots, x_n)$$

$$\downarrow$$

$$A \to P \to P_e$$

$$\to P_e^{-1} \left(or\ P_e^T = P_e^{-1}\right) \to D$$

$$\to \begin{cases} f(y_1, y_2, \ldots, y_n) = Y^T D Y \\ X = P_e Y \to Y = P_e^{-1} = P_e^T X \end{cases}$$

2) Phép biến đổi Lagrange

$f(x_1, x_2, \ldots, x_n) \to$ **Nhóm các số hạng để có dạng chính tắc**

$\exists\ a_{ii} \neq 0$

(có thể chỉ 1)

$a_{ii} = 0$

$i = \overline{1, n}$

VD4:

PHÉP BIẾN ĐỔI TRỰC GIAO

$$f(x_1,x_2,x_3) = 2x_1^2+2x_2^2+2x_3^2+ +2x_1x_2+2x_2x_3+2x_3x_1$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{l} \lambda = 4 \to u_1 = (1,1,1) \\ \lambda = 1 \to u_2 = (1,-1,0)\ \&\ u_3 = (1,1,-2) \end{array}\right.$$

$$S = \{u_1 = (1,1,1), u_2 = (1,-1,0), u_3 = (1,1,-2)\}$$

$$S_e = \left\{ e_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), e_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), e_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \right\}$$

$$P_e = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$P_e^{-1} = P_e^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$D = P_e^{-1}AP_e = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f(y_1,y_2,y_3) = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} D \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 4y_1^2 + y_2^2 + y_3^2$$

$$X = P_eY \to Y = P_e^{-1}X = P_e^TX$$

VD5: ***(tiếp VD4)***

$$f\left(x_1,x_2,x_3\right)=$$
$$2x_1^2+2x_2^2+2x_3^2+$$
$$+2x_1x_2+2x_2x_3+2x_3x_1$$

$$\Downarrow$$

$$f\left(y_1,y_2,y_3\right)=$$
$$4y_1^2+y_2^2+y_3^2$$

$$P_e=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$X=P_eY$$

Tính:

a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=? \rightarrow \begin{cases} f\left(x_1,x_2,x_3\right)=? \\ f\left(y_1,y_2,y_3\right)=? \end{cases}$

b) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} (0,2) \\ (1,-1) \\ (3,-3) \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=? \rightarrow \begin{cases} f\left(x_1,x_2,x_3\right)=? \\ f\left(y_1,y_2,y_3\right)=? \end{cases}$

c) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} (0,2,0) \\ (1,-1,0) \\ (-1,1,0) \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=? \rightarrow \begin{cases} f\left(x_1,x_2,x_3\right)=? \\ f\left(y_1,y_2,y_3\right)=? \end{cases}$

VD6:

a) Bằng phép biến đổi trực giao đưa dạng toàn phương về dạng chính tắc

$$f\left(x_1, x_2\right) = 5x_1^2 + 8x_2^2 - 4x_1x_2$$

b) Tính:

1) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = ? \rightarrow \begin{cases} f\left(x_1, x_2\right) = ? \\ f\left(y_1, y_2\right) = ? \end{cases}$

2) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-1,1) \\ (0,2) \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = ? \rightarrow \begin{cases} f\left(x_1, x_2\right) = ? \\ f\left(y_1, y_2\right) = ? \end{cases}$

3) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-1,0,1) \\ (0,1,0) \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = ? \rightarrow \begin{cases} f\left(x_1, x_2\right) = ? \\ f\left(y_1, y_2\right) = ? \end{cases}$

PHÉP BIẾN ĐỔI LAGRANGE

(tồn tại ít nhất một $a_{ii} \neq 0$)

→ Nhóm lần lượt các biến để được tổng các số hạng dạng bình phương

VD7:

$$f(x_1,x_2) = x_1^2 + 3x_1x_2 = \left(x_1^2 + 2.x_1.\frac{3}{2}x_2 + \frac{9}{4}x_2^2\right) - \frac{9}{4}x_2^2$$

$$= \left(x_1 + \frac{3}{2}x_2\right)^2 - \frac{9}{4}x_2^2$$

$$\to f(y_1,y_2) = y_1^2 - \frac{9}{4}y_2^2, \begin{cases} y_1 = x_1 + \dfrac{3}{2}x_2 \\ y_2 = x_2 \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\to \beta = \left\{(1,0), \left(-\frac{3}{2}, 1\right)\right\}$$

VD8:

$$f(x_1,x_2) = 3x_1^2 + x_1x_2 - x_2^2$$

$$= \left((\sqrt{3}x_1)^2 + 2.\sqrt{3}x_1.\frac{1}{2\sqrt{3}}x_2 + \frac{1}{12}x_2^2\right) - \frac{13}{12}x_2^2$$

$$= \left(\sqrt{3}x_1 + \frac{1}{2\sqrt{3}}x_2\right)^2 - \frac{13}{12}x_2^2 = 3\left(x_1 + \frac{1}{6}x_2\right)^2 - \frac{13}{12}x_2^2$$

$$\to f(y_1,y_2) = 3y_1^2 - \frac{13}{12}y_2^2, \begin{cases} y_1 = x_1 + \dfrac{1}{6}x_2 \\ y_2 = x_2 \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1/6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1/6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\to \beta = \left\{(1,0), \left(-\frac{1}{6}, 1\right)\right\}$$

VD9:

$$\beta=\left\{(1,0,0),\left(-\frac{3}{4},1,0\right),\left(\frac{40}{31},-\frac{12}{31},1\right)\right\}$$

$$f(x_1,x_2,x_3)=2x_1^2+5x_2^2-4x_3^2-3x_1x_2-4x_1x_3$$

$$=\left((\sqrt{2}x_1)^2-2.\sqrt{2}x_1.\frac{1}{2\sqrt{2}}(3x_2-4x_3)+\frac{1}{8}(3x_2-4x_3)^2\right)$$

$$-\frac{1}{8}(3x_2-4x_3)^2+5x_2^2-4x_3^2$$

$$=\left(\sqrt{2}x_1+\frac{1}{2\sqrt{2}}(3x_2-4x_3)\right)^2+\frac{31}{8}x_2^2+3x_2x_3-6x_3^2$$

$$=2\left(x_1+\frac{3}{4}x_2-x_3\right)^2+\left(\sqrt{\frac{31}{8}}x_2+2.\sqrt{\frac{31}{8}}x_2.\frac{3\sqrt{8}}{2\sqrt{31}}x_3+\frac{18}{31}x_3^2\right)-\frac{204}{31}x_3^2$$

$$=2\left(x_1+\frac{3}{4}x_2-x_3\right)^2+\left(\sqrt{\frac{31}{8}}x_2+\frac{3\sqrt{8}}{2\sqrt{31}}x_3\right)^2-\frac{204}{31}x_3^2$$

$$=2\left(x_1+\frac{3}{4}x_2-x_3\right)^2+\frac{31}{8}\left(x_2+\frac{12}{31}x_3\right)^2-\frac{204}{31}x_3^2 \Rightarrow f(y_1,y_2,y_3)=2y_1^2+\frac{31}{8}y_2^2-\frac{204}{31}y_3^2,$$

$$\begin{cases} y_1=x_1+\frac{3}{4}x_2-x_3 \\ y_2=x_2+\frac{12}{31}x_3 \\ y_3=x_3 \end{cases}$$

$$\Uparrow$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=\begin{pmatrix} 1 & 3/4 & -1 \\ 0 & 1 & 12/31 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Uparrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 & -3/4 & 40/31 \\ 0 & 1 & -12/31 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\Uparrow$$

(leads to β above)

PHÉP BIẾN ĐỔI LAGRANGE

(tất cả các a_{ii} = 0)

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_k = y_k\,,\ k = \overline{3,n} \end{cases}$$

Đưa về dạng Lagrange ở trên, và thực hiện các bước tính tương tự

VD10:

$$f(X) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3 \end{cases} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{aligned} f(Y) &= 2(y_1 + y_2)(y_1 - y_2) + 2(y_1 + y_2)y_3 + 2(y_1 - y_2) \\ &= 2y_1^2 - 2y_2^2 + 4y_1y_3 = 2\left[y_1^2 + 2y_1.y_3\right] - 2y_2^2 \\ &= 2\left[y_1^2 + 2y_1.y_3 + y_3^2\right] - 2y_3^2 - 2y_2^2 \\ &= 2(y_1 + y_3)^2 - 2y_2^2 - 2y_3^2 \end{aligned}$$

$$f(Z) = 2z_1^2 - 2z_1^2 - 2z_1^2, \quad \begin{cases} z_1 = y_1 + y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\beta = \{(1,1,0),(1,-1,0),(-1,-1,1)\}$$

VD11: Sử dụng phép biến đổi trực giao đưa f về dạng toàn phương chính tắc

1) $f(x_1, x_2) = 5x_1^2 + 8x_2^2 - 4x_1x_2$

2) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

4) $f(x_1, x_2, x_3) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$

5) $f(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

6) $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_3 - 2x_2x_3$

7) $f(x_1, x_2, x_3) = -2x_1^2 - 5x_2^2 - 5x_3^2 - 4x_1x_2 + 4x_1x_3 + 8x_2x_3$

8) $f(x_1, x_2, x_3) = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

9) $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 4x_1x_3 + 4x_2x_3$

10) $f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

VD12: **Sử dụng phép biến đổi Lagrange đưa f về dạng toàn phương chính tắc**

a). $f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

b). $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

c). $f(x_1, x_2, x_3) = x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3$

d). $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 4x_1x_3 + 4x_2x_3$

e). $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_3 - 2x_2x_3$