CHAPTER 13

Multiple Integrals

OVERVIEW The problems we can solve by integrating functions of two and three variables are similar to the problems solved by single-variable integration, but more general. As in the previous chapter, we can perform the necessary calculations by drawing on our experience with functions of a single variable.

13.1 Double Integrals

We now show how to integrate a continuous function $f(x, y)$ over a bounded region in the xy-plane. There are many similarities between the "double" integrals we define here and the "single" integrals we defined in Chapter 4 for functions of a single variable. Every double integral can be evaluated in stages, using the single-integration methods already at our command.

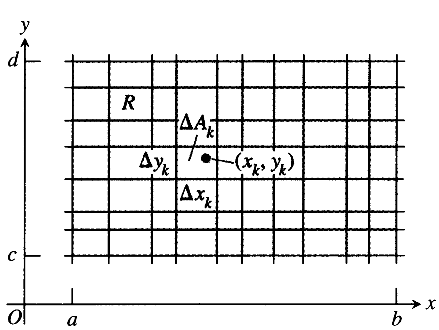
Double Integrals over Rectangles

Suppose that $f(x, y)$ is defined on a rectangular region R given by

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

We imagine R to be covered by a network of lines parallel to the x- and y-axes (Fig. 13.1). These lines divide R into small pieces of area $\Delta A = \Delta x \Delta y$. We number these in some order $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, choose a point (x_k, y_k) in each piece ΔA_k, and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k. \tag{1}$$



13.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \Delta y_k$.

If f is continuous throughout R, then, as we refine the mesh width to make both Δx and Δy go to zero, the sums in (1) approach a limit called the **double integral** of f over R. The notation for it is

$$\iint_R f(x, y)\,dA \qquad \text{or} \qquad \iint_R f(x, y)\,dx\,dy.$$

Thus,

$$\iint_R f(x, y)\,dA = \lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k. \tag{2}$$

As with functions of a single variable, the sums approach this limit no matter how the intervals $[a, b]$ and $[c, d]$ that determine R are partitioned, as long as the norms of the partitions both go to zero. The limit in (2) is also independent of the order in which the areas ΔA_k are numbered and independent of the choice of the point (x_k, y_k) within each ΔA_k. The values of the individual approximating sums S_n depend on these choices, but the sums approach the same limit in the end. The proof of the existence and uniqueness of this limit for a continuous function f is given in more advanced texts. The continuity of f is a sufficient condition for the existence of the double integral, but not a necessary one. The limit in question exists for many discontinuous functions as well.

Properties of Double Integrals

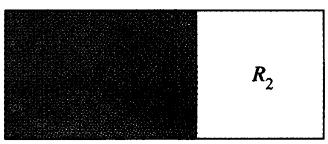
Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

1. $$\iint_R kf(x, y)\,dA = k\iint_R f(x, y)\,dA \qquad \text{(any number } k\text{)}$$

2. $$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$

3. $$\iint_R f(x, y)\,dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$$

4. $$\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$$



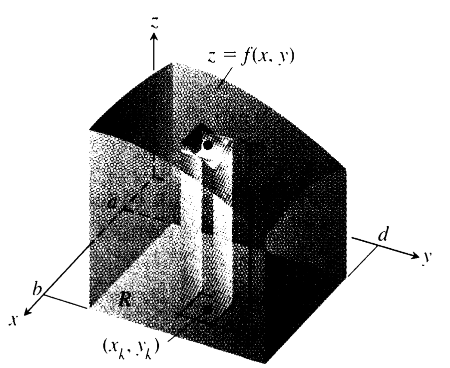
$$\iint_{R_1 \cup R_2} f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$$

13.2 Double integrals have the same kind of domain additivity property that single integrals have.

These are like the single-integral properties in Section 4.5. There is also an additivity property:

5. $$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA.$$

It holds when R is the union of two nonoverlapping rectangles R_1 and R_2 (Fig. 13.2). Again, we omit the proof.



13.3 Approximating solids with rectangular prisms leads us to define the volumes of more general prisms as double integrals. The volume of the prism shown here is the double integral of $f(x, y)$ over the base region R.

Double Integrals as Volumes

When $f(x, y)$ is positive, we may interpret the double integral of f over a rectangular region R as the volume of the solid prism bounded below by R and above by the surface $z = f(x, y)$ (Fig. 13.3). Each term $f(x_k, y_k)\,\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\,\Delta A_k$ is the volume of a vertical rectangular prism that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid. We *define* this volume to be

$$\text{Volume} = \lim S_n = \iint_R f(x, y)\,dA. \tag{3}$$

As you might expect, this more general method of calculating volume agrees with the methods in Chapter 5, but we will not prove this here.

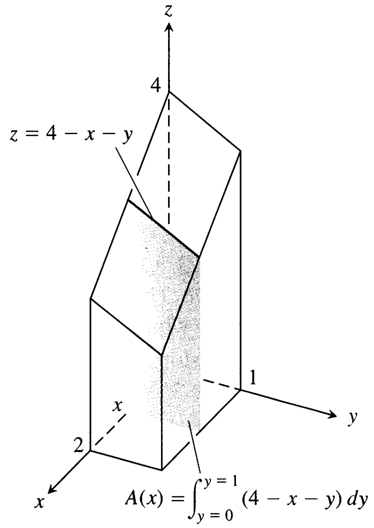
Fubini's Theorem for Calculating Double Integrals

Suppose we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R: $0 \le x \le 2,\ 0 \le y \le 1$ in the xy-plane. If we apply the method of slicing from Section 5.2, with slices perpendicular to the x-axis (Fig. 13.4), then the volume is

$$\int_{x=0}^{x=2} A(x)\ dx, \tag{4}$$

where $A(x)$ is the cross-section area at x. For each value of x we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,dy, \tag{5}$$



13.4 To obtain the cross-section area $A(x)$, we hold x fixed and integrate with respect to y.

which is the area under the curve $z = 4 - x - y$ in the plane of the cross section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining (4) and (5), we see that the volume of the entire solid is

$$\begin{aligned}\text{Volume} &= \int_{x=0}^{x=2} A(x)\,dx = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4 - x - y)\,dy\right)dx \\ &= \int_{x=0}^{x=2}\left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2}\left(\frac{7}{2} - x\right)dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = 5.\end{aligned} \tag{6}$$

If we had just wanted to write instructions for calculating the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx.$$

The expression on the right, called an **iterated** or **repeated integral,** says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then integrating the resulting expression in x with respect to x from $x = 0$ to $x = 2$.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis (Fig. 13.5)? As a function of y, the typical cross-section area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y. \tag{7}$$

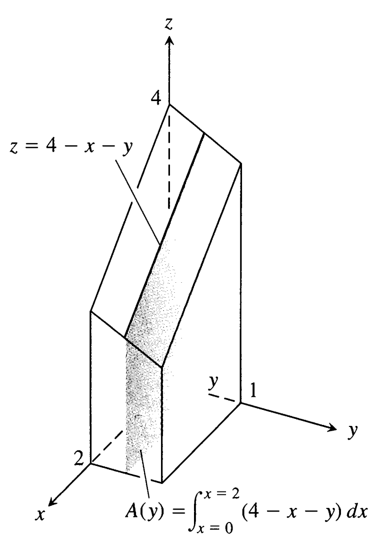
The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\,dy = \int_{y=0}^{y=1} (6 - 2y)\,dy = \left[6y - y^2\right]_0^1 = 5,$$

in agreement with our earlier calculation.

Again, we may give instructions for calculating the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\,dx\,dy.$$



13.5 To obtain the cross-section area $A(y)$, we hold y fixed and integrate with respect to x.

The expression on the right says we can find the volume by integrating $4 - x - y$ with respect to x from $x = 0$ to $x = 2$ (as in Eq. 7) and integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral the order of integration is first x and then y, the reverse of the order in Eq. (6).

What do these two volume calculations with iterated integrals have to do with the double integral

$$\iint_R (4 - x - y)\ dA$$

over the rectangle $R: 0 \le x \le 2,\ 0 \le y \le 1$? The answer is that they both give the value of the double integral. A theorem published in 1907 by Guido Fubini (1879–1943) says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration. (Fubini proved his theorem in greater generality, but this is how it translates into what we're doing now.)

Theorem 1

Fubini's Theorem (First Form)

If $f(x, y)$ is continuous on the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Fubini's theorem says that double integrals over rectangles can be calculated as iterated integrals. This means we can evaluate a double integral by integrating with respect to one variable at a time.

Fubini's theorem also says that we may calculate the double integral by integrating in *either* order, a genuine convenience, as we will see in Example 3. In particular, when we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

EXAMPLE 1 Calculate $\iint_R f(x, y)\ dA$ for

$$f(x, y) = 1 - 6x^2y \qquad \text{and} \qquad R:\quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's theorem,

$$\begin{aligned}\iint_R f(x, y)\, dA &= \int_{-1}^{1} \int_0^2 (1 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \Big[x - 2x^3y\Big]_{x=0}^{x=2} dy \\ &= \int_{-1}^{1} (2 - 16y)\, dy = \Big[2y - 8y^2\Big]_{-1}^{1} = 4.\end{aligned}$$

Reversing the order of integration gives the same answer:

$$\begin{aligned}\int_0^2 \int_{-1}^{1} (1 - 6x^2y)\, dy\, dx &= \int_0^2 \Big[y - 3x^2y^2\Big]_{y=-1}^{y=1} dx \\ &= \int_0^2 \Big[(1 - 3x^2) - (-1 - 3x^2)\Big]\, dx = \int_0^2 2\, dx = 4.\end{aligned}$$

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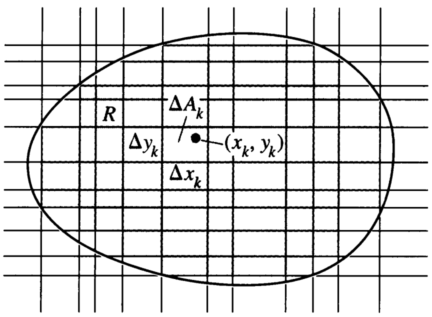
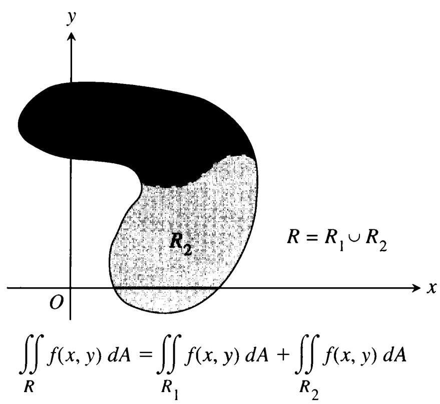
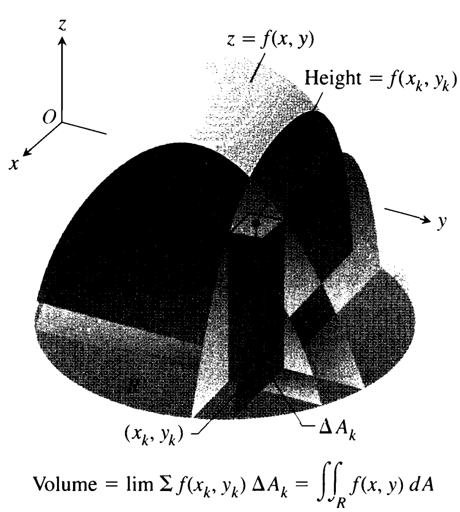


13.6 A rectangular grid partitioning a bounded nonrectangular region into cells.



13.7 The additivity property for rectangular regions holds for regions bounded by continuous curves.



13.8 We define the volumes of solids with curved bases the same way we define the volumes of solids with rectangular bases.

Technology *Multiple Integration* Most Computer Algebra Systems can calcuate both multiple and iterated integrals. The typical procedure is to apply the CAS integrate command in nested iterations according to the order of integration you specify:

Integral	Typical CAS Formulation
$\int\int x^2 y\, dx\, dy$	int(int(x ^ 2 * y, x), y);
$\int_{-\pi/3}^{\pi/4}\int_0^1 x \cos y\, dx\, dy$	int(int(x* cos(y), x = 0 . . 1), y = −Pi/3 . . Pi/4);

If a CAS cannot produce an exact value for a definite integral, it can usually find an approximate value numerically.

Double Integrals over Bounded Nonrectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded nonrectangular region, like the one shown in Fig. 13.6, we again imagine R to be covered by a rectangular grid, but we include in the partial sum only the small pieces of area $\Delta A = \Delta x \Delta y$ that lie entirely within the region (shaded in the figure). We number the pieces in some order, choose an arbitrary point (x_k, y_k) in each ΔA_k, and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

The only difference between this sum and the one in Eq. (1) for rectangular regions is that now the areas ΔA_k may not cover all of R. But as the mesh becomes increasingly fine and the number of terms in S_n increases, more and more of R is included. If f is continuous and the boundary of R is made from the graphs of a finite number of continuous functions of x and/or continuous functions of y joined end to end, then the sums S_n will have a limit as the norms of the partitions that define the rectangular grid independently approach zero. We call the limit the **double integral** of f over R:

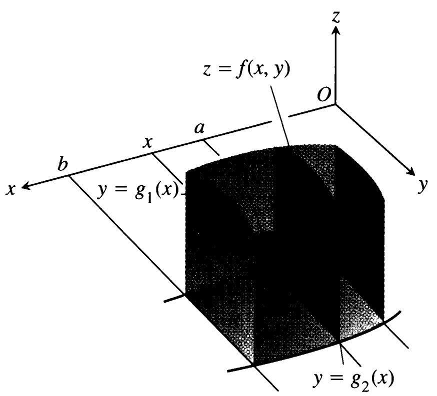
$$\iint_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum f(x_k, y_k)\, \Delta A_k.$$

This limit may also exist under less restrictive circumstances.

Double integrals of continuous functions over nonrectangular regions have the same algebraic properties as integrals over rectangular regions. The domain additivity property corresponding to property 5 says that if R is decomposed into nonoverlapping regions R_1 and R_2 with boundaries that are again made of a finite number of line segments or smooth curves (see Fig. 13.7 for an example), then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA.$$

If $f(x, y)$ is positive and continuous over R (Fig. 13.8), we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\, dA$, as before.



13.9 The area of the vertical slice shown here is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$.

If R is a region like the one shown in the xy-plane in Fig. 13.9, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-section area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

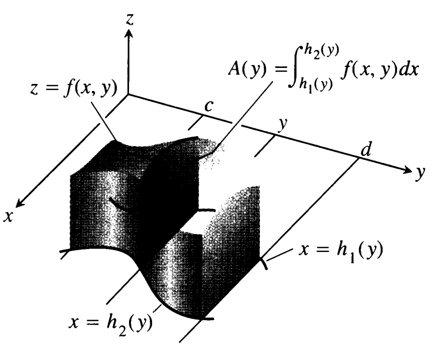
and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \tag{8}$$

Similarly, if R is a region like the one shown in Fig. 13.10, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \tag{9}$$

The fact that the iterated integrals in Eqs. (8) and (9) both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's theorem.



13.10 The volume of the solid shown here is

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Theorem 2

Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

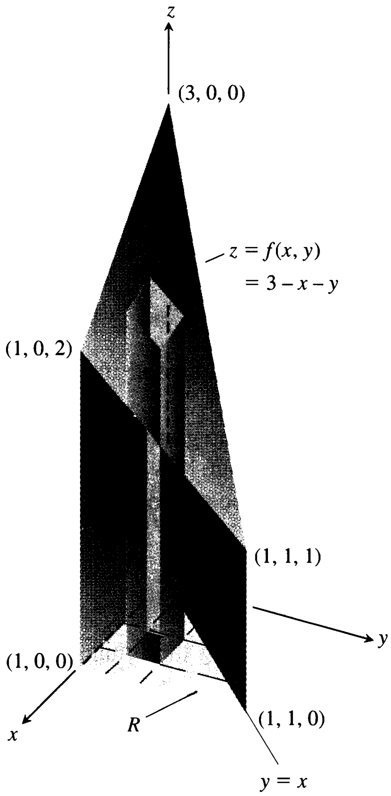
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 2 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

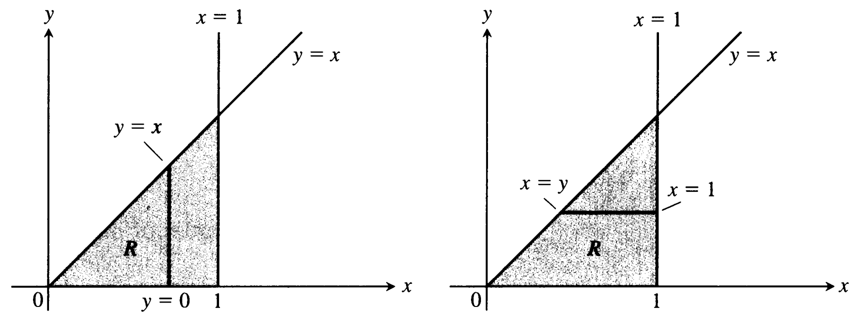
$$z = f(x, y) = 3 - x - y.$$

Solution See Fig. 13.11. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Fig. 13.11b). Hence,

$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1. \end{aligned}$$



(a)



(b) (c)

13.11 (a) Prism with a triangular base in the *xy*-plane. The volume of this prism is defined as a double integral over *R*. To evaluate it as an iterated integral, we may integrate first with respect to *y* and then with respect to *x*, or the other way around (Example 2). (b) Integration limits of

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x,y)\,dy\,dx.$$

If we integrate first with respect to *y*, we integrate along a vertical line through *R* and then integrate from left to right to include all the vertical lines in *R*.
(c) Integration limits of

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} f(x,y)\,dx\,dy.$$

If we integrate first with respect to *x*, we integrate along a horizontal line through *R* and then integrate from bottom to top to include all the horizontal lines in *R*.

When the order of integration is reversed (Fig. 13.11c), the integral for the volume is

$$\begin{aligned}
V &= \int_0^1\int_y^1 (3-x-y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.
\end{aligned}$$

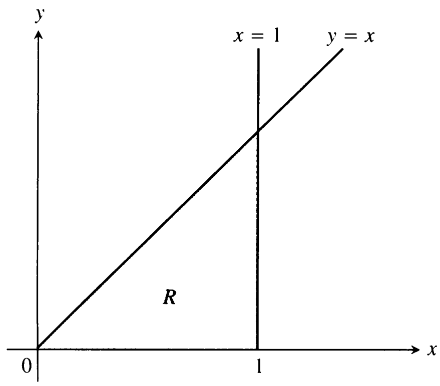
The two integrals are equal, as they should be. ❑

While Fubini's theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

EXAMPLE 3 Calculate

$$\iint_R \frac{\sin x}{x}\,dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.



13.12 The region of integration in Example 3.

Solution The region of integration is shown in Fig. 13.12. If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(y \frac{\sin x}{x} \bigg]_{y=0}^{y=x} \right) dx = \int_0^1 \sin x\, dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

we are stopped by the fact that $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions.

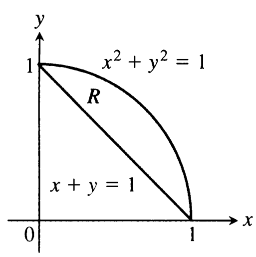
There is no general rule for predicting which order of integration will be the good one in circumstances like these, so don't worry about how to start your integrations. Just forge ahead and if the order you first choose doesn't work, try the other. ❑

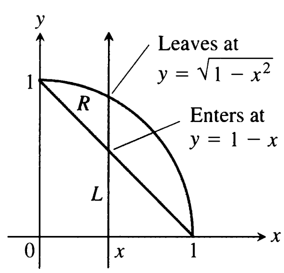
Finding the Limits of Integration

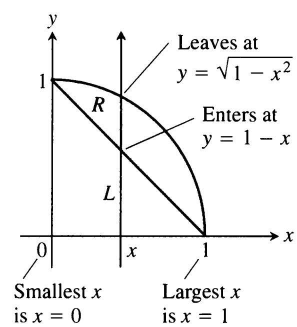
The hardest part of evaluating a double integral can be finding the limits of integration. Fortunately, there is a good procedure to follow.

Procedure for Finding Limits of Integration

A. To evaluate $\iint_R f(x, y)\, dA$ over a region R, integrating first with respect to y and then with respect to x, take the following steps:







1. *A sketch.* Sketch the region of integration and label the bounding curves.

2. *The y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration.

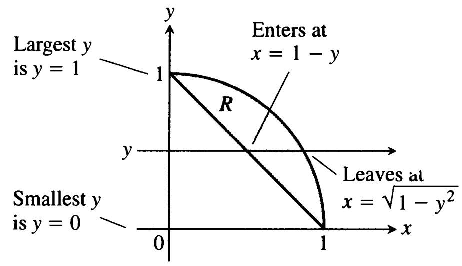
3. *The x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral is

$$\iint_R f(x, y)\, dA =$$

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

B. To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines. The integral is

$$\iint_R f(x, y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\,dx\,dy.$$



EXAMPLE 4 Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\,dy\,dx$$

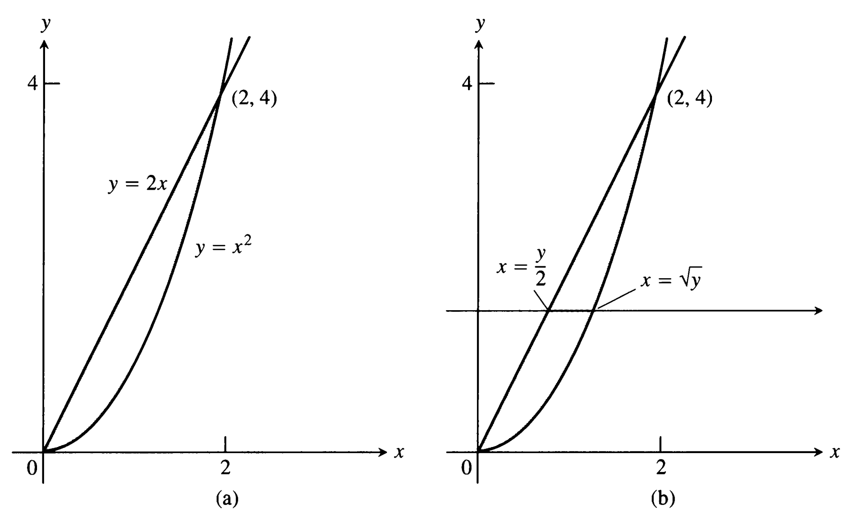
and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Fig. 13.13a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Fig. 13.13b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\,dx\,dy.$$

The common value of these integrals is 8.



13.13 Figure for Example 4. ❑

Exercises 13.1

Finding Regions of Integration and Double Integrals

In Exercises 1–10, sketch the region of integration and evaluate the integral.

1. $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$

2. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy)\, dy\, dx$

3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy$

4. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y)\, dx\, dy$

5. $\int_0^{\pi} \int_0^x x \sin y\, dy\, dx$

6. $\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$

7. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx\, dy$

8. $\int_1^2 \int_y^{y^2} dx\, dy$

9. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

10. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$

In Exercises 11–16, integrate f over the given region.

11. $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$

12. $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

13. $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

14. $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

15. $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

16. $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 17–20 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

17. $\int_{-2}^0 \int_v^{-v} 2\, dp\, dv$ (the pv-plane)

18. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\, dt\, ds$ (the st-plane)

19. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t\, du\, dt$ (the tu-plane)

20. $\int_0^3 \int_{-2}^{4-2u} \frac{4 - 2u}{v^2}\, dv\, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 21–30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\int_0^1 \int_2^{4-2x} dy\, dx$

22. $\int_0^2 \int_{y-2}^0 dx\, dy$

23. $\int_0^1 \int_y^{\sqrt{y}} dx\, dy$

24. $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

25. $\int_0^1 \int_1^{e^x} dy\, dx$

26. $\int_0^{\ln 2} \int_{e^y}^2 dx\, dy$

27. $\int_0^{3/2} \int_0^{9-4x^2} 16x\, dy\, dx$

28. $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$

29. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$

30. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\, dy\, dx$

Evaluating Double Integrals

In Exercises 31–40, sketch the region of integration, determine the order of integration, and evaluate the integral.

31. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$

32. $\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx$

33. $\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$

34. $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\, dy\, dx$

35. $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy$

36. $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$

37. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dx\, dy$

38. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4 + 1}$

39. $\iint_R (y - 2x^2)\, dA$ where R is the region inside the square $|x| + |y| = 1$

40. $\iint_R xy\, dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

41. Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

42. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

43. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

44. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

45. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

46. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

47. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

48. Find the volume of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

49. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

50. Find the volume of the solid that is bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm \sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

Integrals over Unbounded Regions

Evaluate the improper integrals in Exercises 51–54 as iterated integrals.

51. $\displaystyle\int_1^{\infty}\int_{e^{-x}}^{1} \frac{1}{x^3 y}\,dy\,dx$

52. $\displaystyle\int_{-1}^{1}\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1)\,dy\,dx$

53. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy$

54. $\displaystyle\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\,dx\,dy$

Approximating Double Integrals

In Exercises 55 and 56, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\,dA \approx \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

55. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

56. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided it lies within R)

Theory and Examples

57. Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \le 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

58. Integrate $f(x, y) = 1/[(x^2 - x)(y - 1)^{2/3}]$ over the infinite rectangle $2 \le x < \infty$, $0 \le y \le 2$.

59. A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1\int_0^y (x^2 + y^2)\,dx\,dy + \int_1^2\int_0^{2-y} (x^2 + y^2)\,dx\,dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

60. Evaluate the integral

$$\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\ dx.$$

(*Hint:* Write the integrand as an integral.)

61. What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\ dA?$$

Give reasons for your answer.

62. What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\ dA?$$

Give reasons for your answer.

63. Is it all right to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

64. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices (0, 1), (2, 0), and (1, 2)? Give reasons for your answer.

65. Prove that $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{b\to\infty}\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dx\,dy$

$$= 4\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2.$$

66. Evaluate the improper integral $\displaystyle\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\,dy\,dx.$

Numerical Evaluation

Use a double-integral evaluator to estimate the values of the integrals in Exercises 67–70.

67. $\displaystyle\int_1^3\int_1^x \frac{1}{xy}\,dy\,dx$

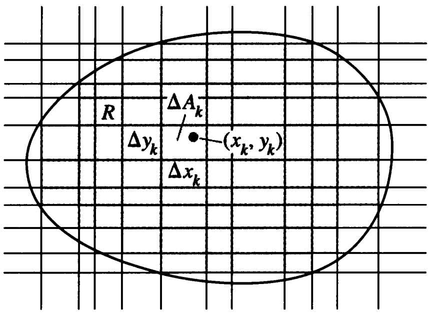
68. $\displaystyle\int_0^1\int_0^1 e^{-(x^2+y^2)}\,dy\,dx$

69. $\displaystyle\int_0^1\int_0^1 \tan^{-1} xy\,dy\,dx$

70. $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 3\sqrt{1 - x^2 - y^2}\,dy\,dx$

13.2 Areas, Moments, and Centers of Mass

In this section we show how to use double integrals to define and calculate the areas of bounded regions in the plane and the masses, moments, centers of mass, and radii of gyration of thin plates covering these regions. The calculations are similar to the ones in Chapter 5, but now we can handle a greater variety of shapes.



13.14 The first step in defining the area of a region is to partition the interior of the region into cells.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the partial sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \tag{1}$$

This approximates what we would like to call the area of R. As Δx and Δy approach zero, the coverage of R by the ΔA_k's (Fig. 13.14) becomes increasingly complete, and we define the area of R to be the limit

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \iint_R dA. \tag{2}$$

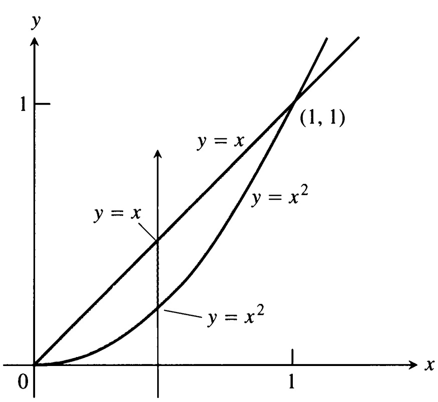
Definition

The **area** of a closed, bounded plane region R is

$$A = \iint_R dA. \tag{3}$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply.

To evaluate the integral in (3), we integrate the constant function $f(x, y) = 1$ over R.



13.15 The area of the region between the parabola and the line in Example 1 is

$$\int_0^1 \int_{x^2}^{x} dy\, dx.$$

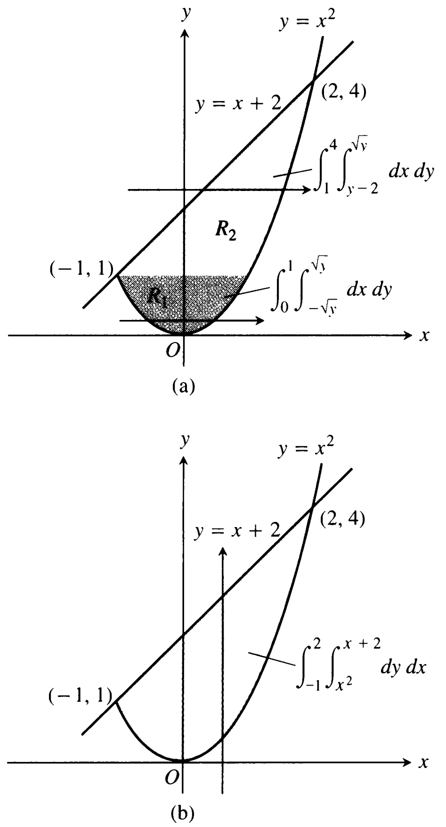
EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Fig. 13.15) and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 \Big[y\Big]_{x^2}^{x} dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}. \quad \square$$

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Fig. 13.16(a), we may



13.16 Calculating this area takes (a) two double integrals if the first integration is with respect to x, but (b) only one if the first integration is with respect to y (Example 2).

calculate the area as

$$A=\iint_{R_1} dA+\iint_{R_2} dA=\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy.$$

On the other hand, reversing the order of integration (Fig. 13.16b) gives

$$A=\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx.$$

This result is simpler and is the only one we would bother to write down in practice. The area is

$$A=\int_{-1}^{2}\Big[y\Big]_{x^2}^{x+2} dx=\int_{-1}^{2}(x+2-x^2)\,dx=\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}=\frac{9}{2}.$$ ❑

Average Value

The average value of an integrable function of a single variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a closed and bounded region that has a measurable area, the average value is the integral over the region divided by the area of the region. If f is the function and R the region, then

$$\textbf{Average value of } f \text{ over } R=\frac{1}{\text{area of } R}\iint_R f\,dA. \tag{4}$$

If f is the area density of a thin plate covering R, then the double integral of f over R divided by the area of R is the plate's average density in units of mass per unit area. If $f(x, y)$ is the distance from the point (x, y) to a fixed point P, then the average value of f over R is the average distance of points in R from P.

EXAMPLE 3 Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R\colon 0\le x\le\pi,\ 0\le y\le 1$.

Solution The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x\cos xy\,dy\,dx=\int_0^{\pi}\Big[\sin xy\Big]_{y=0}^{y=1}dx$$
$$=\int_0^{\pi}(\sin x-0)\,dx=-\cos x\Big]_0^{\pi}=1+1=2.$$

The area of R is π. The average value of f over R is $2/\pi$. ❑

Global warming

The "global warming" controversy deals with whether the average air temperature over the surface of the earth is increasing.

First and Second Moments and Centers of Mass

To find the moments and centers of mass of thin sheets and plates, we use formulas similar to those in Chapter 5. The main difference is that now, with double integrals, we can accommodate a greater variety of shapes and density functions. The formulas are given in Table 13.1, on the following page. The examples that follow show how the formulas are used.

The mathematical difference between the **first moments** M_x and M_y and the **moments of inertia,** or **second moments,** I_x and I_y is that the second moments use the *squares* of the "lever-arm" distances x and y.

Table 13.1 Mass and moment formulas for thin plates covering regions in the *xy*-plane

Density: $\delta(x, y)$

Mass: $M = \iint \delta(x, y)\, dA$ **First moments:** $M_x = \iint y\delta(x, y)\, dA, \quad M_y = \iint x\delta(x, y)\, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

Moments of inertia (second moments):

About the x-axis: $I_x = \iint y^2\, \delta(x, y)\, dA$ About the origin (polar moment): $I_0 = \iint (x^2 + y^2)\, \delta(x, y)\, dA = I_x + I_y$

About the y-axis: $I_y = \iint x^2\, \delta(x, y)\, dA$

About a line L: $I_L = \iint r^2(x, y)\delta(x, y)\, dA$, where $r(x, y) =$ distance from (x, y) to L

Radii of gyration: About the x-axis: $R_x = \sqrt{I_x/M}$

About the y-axis: $R_y = \sqrt{I_y/M}$

About the origin: $R_0 = \sqrt{I_0/M}$

The moment I_0 is also called the **polar moment** of inertia about the origin. It is calculated by integrating the density $\delta(x, y)$ (mass per unit area) times $r^2 = x^2 + y^2$, the square of the distance from a representative point (x, y) to the origin. Notice that $I_0 = I_x + I_y$; once we find two, we get the third automatically. (The moment I_0 is sometimes called I_z, for moment of inertia about the z-axis. The identity $I_z = I_x + I_y$ is then called the **Perpendicular Axis Theorem.**)

The **radius of gyration** R_x is defined by the equation

$$I_x = MR_x{}^2.$$

It tells how far from the x-axis the entire mass of the plate might be concentrated to give the same I_x. The radius of gyration gives a convenient way to express the moment of inertia in terms of a mass and a length. The radii R_y and R_0 are defined in a similar way, with

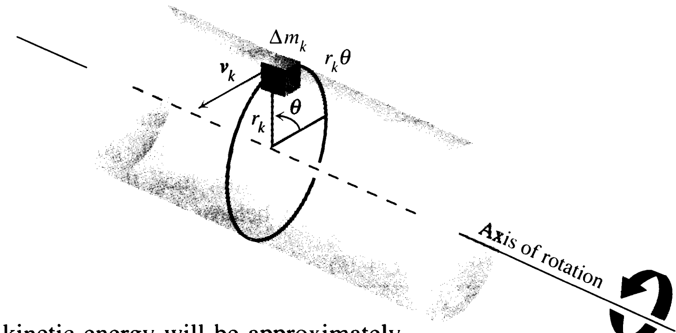
$$I_y = MR_y{}^2 \qquad \text{and} \qquad I_0 = MR_0{}^2.$$

We take square roots to get the formulas in Table 13.1.

Why the interest in moments of inertia? A body's first moments tell us about balance and about the torque the body exerts about different axes in a gravitational field. But if the body is a rotating shaft, we are more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity. This is where the second moment or moment of inertia comes in.

Think of partitioning the shaft into small blocks of mass Δm_k and let r_k denote the distance from the kth block's center of mass to the axis of rotation (Fig. 13.17). If the shaft rotates at an angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of mass will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}(r_k\theta) = r_k\frac{d\theta}{dt} = r_k\,\omega. \tag{5}$$



13.17 To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

The block's kinetic energy will be approximately

$$\frac{1}{2}\Delta m_k {v_k}^2 = \frac{1}{2}\Delta m_k (r_k\,\omega)^2 = \frac{1}{2}\,\omega^2 {r_k}^2\,\Delta m_k. \tag{6}$$

The kinetic energy of the shaft will be approximately

$$\sum \frac{1}{2}\,\omega^2 {r_k}^2 \Delta m_k. \tag{7}$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$\text{KE}_{\text{shaft}} = \int \frac{1}{2}\omega^2 r^2\,dm = \frac{1}{2}\omega^2 \int r^2\,dm. \tag{8}$$

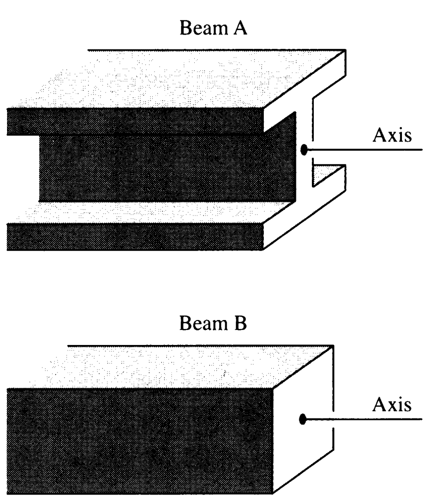
The factor

$$I = \int r^2\,dm \tag{9}$$

is the moment of inertia of the shaft about its axis of rotation, and we see from Eq. (8) that the shaft's kinetic energy is

$$\text{KE}_{\text{shaft}} = \frac{1}{2} I\omega^2. \tag{10}$$

To start a shaft of inertial moment I rotating at an angular velocity ω, we need to provide a kinetic energy of $\text{KE} = (1/2)I\omega^2$. To stop the shaft, we have to take this amount of energy back out. To start a locomotive with mass m moving at a linear velocity v, we need to provide a kinetic energy of $\text{KE} = (1/2)\,mv^2$. To stop the locomotive, we have to remove this amount of energy. The shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia takes into account not only the mass but also its distribution.

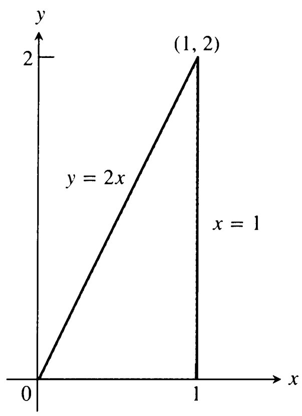


13.18 The greater the polar moment of inertia of the cross section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-section area, but A is stiffer.

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times I, the polar moment of inertia of a typical cross section of the beam perpendicular to the beam's longitudinal axis. The greater the value of I, the stiffer the beam and the less it will bend under a given load. That is why we use I beams instead of beams whose cross sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to maximize the value of I (Fig. 13.18).

If you want to see the moment of inertia at work, try the following experiment. Tape two coins to the ends of a pencil and twiddle the pencil about the center of mass. The moment of inertia accounts for the resistance you feel each time you change the direction of motion. Now move the coins an equal distance toward the center of mass and twiddle the pencil again. The system has the same mass and the same center of mass but now offers less resistance to the changes in motion. The moment of inertia has been reduced. The moment of inertia is what gives a baseball bat, golf club, or tennis racket its "feel." Tennis rackets that weigh the same, look the same, and have identical centers of mass will feel different and behave differently if their masses are not distributed the same way.

First moments are "balancing" moments. Second moments are "turning" moments.



13.19 The triangular region covered by the plate in Example 4.

EXAMPLE 4 A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's mass, first moments, center of mass, moments of inertia, and radii of gyration about the coordinate axes.

Solution We sketch the plate and put in enough detail to determine the limits of integration for the integrals we have to evaluate (Fig. 13.19).

The plate's mass is

$$\begin{aligned} M &= \int_0^1 \int_0^{2x} \delta(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} (6x + 6y + 6)\, dy\, dx \\ &= \int_0^1 \left[6xy + 3y^2 + 6y \right]_{y=0}^{y=2x} dx \\ &= \int_0^1 (24x^2 + 12x)\, dx = \left[8x^3 + 6x^2 \right]_0^1 = 14. \end{aligned}$$

The first moment about the x-axis is

$$\begin{aligned} M_x &= \int_0^1 \int_0^{2x} y\delta(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)\, dy\, dx \\ &= \int_0^1 \left[3xy^2 + 2y^3 + 3y^2 \right]_{y=0}^{y=2x} dx = \int_0^1 (28x^3 + 12x^2)\, dx \\ &= \left[7x^4 + 4x^3 \right]_0^1 = 11. \end{aligned}$$

A similar calculation gives

$$M_y = \int_0^1 \int_0^{2x} x\delta(x, y)\, dy\, dx = 10.$$

The coordinates of the center of mass are therefore

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}, \qquad \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

The moment of inertia about the x-axis is

$$\begin{aligned} I_x &= \int_0^1 \int_0^{2x} y^2\delta(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\, dy\, dx \\ &= \int_0^1 \left[2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_{y=0}^{y=2x} dx = \int_0^1 (40x^4 + 16x^3)\, dx \\ &= \left[8x^5 + 4x^4 \right]_0^1 = 12. \end{aligned}$$

Similarly, the moment of inertia about the y-axis is

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y)\, dy\, dx = \frac{39}{5}.$$

Since we know I_x and I_y, we do not need to evaluate an integral to find I_0; we can use the equation $I_0 = I_x + I_y$ instead:

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

The three radii of gyration are

$$R_x = \sqrt{I_x/M} = \sqrt{12/14} = \sqrt{6/7}$$

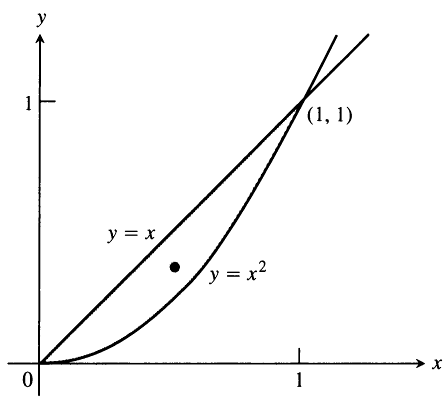
$$R_y = \sqrt{I_y/M} = \sqrt{\left(\frac{39}{5}\right)/14} = \sqrt{39/70}$$

$$R_0 = \sqrt{I_0/M} = \sqrt{\left(\frac{99}{5}\right)/14} = \sqrt{99/70}.$$

❑

Centroids of Geometric Figures

When the density of an object is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. As far as $\bar{x}$ and $\bar{y}$ are concerned, δ might as well be 1. Thus, when δ is constant, the location of the center of mass becomes a feature of the object's shape and not of the material of which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape. To find a centroid, we set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing first moments by masses.



13.20 Example 5 finds the centroid of the region shown here.

EXAMPLE 5 Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

Solution We sketch the region and include enough detail to determine the limits of integration (Fig. 13.20). We then set δ equal to 1 and evaluate the appropriate formulas from Table 13.1:

$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 \Big[y \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 \Big[xy \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

From these values of M, M_x, and M_y, we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $\left(\frac{1}{2}, \frac{2}{5}\right)$.

❑

Exercises 13.2

Area by Double Integration

In Exercises 1–8, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

The integrals and sums of integrals in Exercises 9–14 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

9. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\, dy$
10. $\displaystyle\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$
11. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$
12. $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy$
13. $\displaystyle\int_{-1}^0 \int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-x/2}^{1-x} dy\, dx$
14. $\displaystyle\int_0^2 \int_{x^2-4}^0 dy\, dx + \int_0^4 \int_0^{\sqrt{x}} dy\, dx$

Average Values

15. Find the average value of $f(x, y) = \sin(x + y)$ over
    a) the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$,
    b) the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$.
16. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \le x \le 1$, $0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.
17. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.
18. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2 \ln 2$, $\ln 2 \le y \le 2 \ln 2$.

Constant Density

19. Find the center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.
20. Find the moments of inertia and radii of gyration about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.
21. Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.
22. Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.
23. Find the centroid of the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$.
24. The area of the region in the first quadrant bounded by the parabola $y = 6x - x^2$ and the line $y = x$ is 125/6 square units. Find the centroid.
25. Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.
26. Find the moment of inertia about the x-axis of a thin plate of density $\delta = 1$ bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.
27. Find the centroid of the region between the x-axis and the arch $y = \sin x$, $0 \le x \le \pi$.
28. Find the moment of inertia with respect to the y-axis of a thin sheet of constant density $\delta = 1$ bounded by the curve $y = (\sin^2 x)/x^2$ and the interval $\pi \le x \le 2\pi$ of the x-axis.
29. *The centroid of an infinite region.* Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. (Use improper integrals in the mass-moment formulas.)
30. *The first moment of an infinite plate.* Find the first moment about the y-axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

Variable Density

31. Find the moment of inertia and radius of gyration about the x-axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.
32. Find the mass of a thin plate occupying the smaller region cut from the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ if $\delta(x, y) = 5x$.
33. Find the center of mass of a thin triangular plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.
34. Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.
35. Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$.

36. Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.

37. Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the x-axis, the lines $x = \pm 1$, and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.

38. Find the center of mass and the moment of inertia and radius of gyration about the x-axis of a thin rectangular plate bounded by the lines $x = 0$, $x = 20$, $y = -1$, and $y = 1$ if $\delta(x, y) = 1 + (x/20)$.

39. Find the center of mass, the moments of inertia and radii of gyration about the coordinate axes, and the polar moment of inertia and radius of gyration of a thin triangular plate bounded by the lines $y = x$, $y = -x$, and $y = 1$ if $\delta(x, y) = y + 1$.

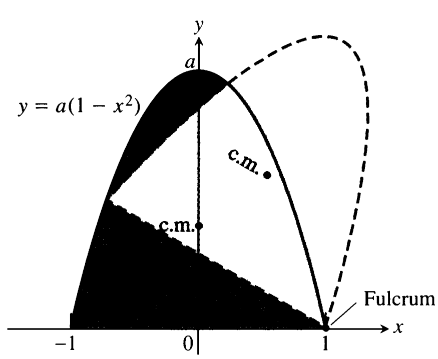
40. Repeat Exercise 39 for $\delta(x, y) = 3x^2 + 1$.

Theory and Examples

41. If $f(x, y) = (10{,}000\, e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacteria on the xy-plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \le x \le 5$ and $-2 \le y \le 0$.

42. If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

43. *Appliance design.* When we design an appliance, one of the concerns is how hard the appliance will be to tip over. When tipped, it will right itself as long as its center of mass lies on the correct side of the *fulcrum,* the point on which the appliance is riding as it tips. Suppose the profile of an appliance of approximately constant density is parabolic, like an old-fashioned radio. It fills the region $0 \le y \le a(1 - x^2)$, $-1 \le x \le 1$, in the xy-plane (Fig. 13.21). What values of a will guarantee that the appliance will have to be tipped more than 45° to fall over?



13.21 The profile of the appliance in Exercise 43.

44. *Minimizing a moment of inertia.* A rectangular plate of constant density $\delta(x, y) = 1$ occupies the region bounded by the lines $x = 4$ and $y = 2$ in the first quadrant. The moment of inertia I_a of the rectangle about the line $y = a$ is given by the integral

$$I_a = \int_0^4 \int_0^2 (y - a)^2 \, dy \, dx.$$

Find the value of a that minimizes I_a.

45. Find the centroid of the infinite region in the xy-plane bounded by the curves $y = 1/\sqrt{1 - x^2}$, $y = -1/\sqrt{1 - x^2}$, and the lines $x = 0$, $x = 1$.

46. Find the radius of gyration of a slender rod of constant linear density δ gm/cm and length L cm with respect to an axis

a) through the rod's center of mass perpendicular to the rod's axis;

b) perpendicular to the rod's axis at one end of the rod.

47. A thin plate of constant density δ occupies the region R in the xy-plane bounded by the curves $x = y^2$ and $x = 2y - y^2$ (see Exercise 34).

a) Find δ such that the plate has the same mass as the plate in Exercise 34.

b) Compare the value of δ found in part (a) with the average value of $\delta(x, y) = y + 1$ over R.

48. According to the *Texas Almanac,* Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0 each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation to the average temperature in Texas at time t_0. Your answer should involve information that is readily available in the *Texas Almanac.*

The Parallel Axis Theorem

Let $L_{\text{c.m.}}$ be a line in the xy-plane that runs through the center of mass of a thin plate of mass m covering a region in the plane. Let L be a line in the plane parallel to and h units away from $L_{\text{c.m.}}$. The **Parallel Axis Theorem** says that under these conditions the moments of inertia I_L and $I_{\text{c.m.}}$ of the plate about L and $L_{\text{c.m.}}$ satisfy the equation

$$I_L = I_{\text{c.m.}} + mh^2. \tag{1}$$

This equation gives a quick way to calculate one moment when the other moment and the mass are known.

49. *Proof of the Parallel Axis Theorem*

a) Show that the first moment of a thin flat plate about any line in the plane of the plate through the plate's center of mass is zero. (*Hint:* Place the center of mass at the origin with the line along the y-axis. What does the formula $\bar{x} = M_y/M$ then tell you?)

b) Use the result in (a) to derive the Parallel Axis Theorem. Assume that the plane is coordinatized in a way that makes $L_{\text{c.m.}}$ the y-axis and L the line $x = h$. Then expand the integrand of the integral for I_L to rewrite the integral as the sum of integrals whose values you recognize.

50. a) Use the Parallel Axis Theorem and the results of Example 4 to find the moments of inertia of the plate in Example 4 about the vertical and horizontal lines through the plate's center of mass.

b) Use the results in (a) to find the plate's moments of inertia about the lines $x = 1$ and $y = 2$.

Pappus's Formula

In addition to stating the centroid theorems in Section 5.10, Pappus knew that the centroid of the union of two nonoverlapping plane regions lies on the line segment joining their individual centroids. More specifically, suppose that m_1 and m_2 are the masses of thin plates P_1 and P_2 that cover nonoverlapping regions in the xy-plane. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be the vectors from the origin to the respective centers of mass of P_1 and P_2. Then the center of mass of the union $P_1 \cup P_2$ of the two plates is determined by the vector

$$\mathbf{c} = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1 + m_2}. \tag{2}$$

Equation (2) is known as **Pappus's formula.** For more than two nonoverlapping plates, as long as their number is finite, the formula generalizes to

$$\mathbf{c} = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + \cdots + m_n\mathbf{c}_n}{m_1 + m_2 + \cdots + m_n}. \tag{3}$$

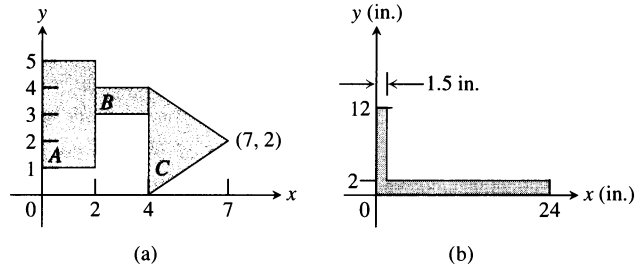
This formula is especially useful for finding the centroid of a plate of irregular shape that is made up of pieces of constant density whose centroids we know from geometry. We find the centroid of each piece and apply Eq. (3) to find the centroid of the plate.

51. Derive Pappus's formula (Eq. 2). (*Hint:* Sketch the plates as regions in the first quadrant and label their centers of mass as $(\overline{x}_1, \overline{y}_1)$ and $(\overline{x}_2, \overline{y}_2)$. What are the moments of $P_1 \cup P_2$ about the coordinate axes?)

52. Use Eq. (2) and mathematical induction to show that Eq. (3) holds for any positive integer $n > 2$.

53. Let A, B, and C be the shapes indicated in Fig. 13.22(a). Use Pappus's formula to find the centroid of

a) $A \cup B$ **b)** $A \cup C$ **c)** $B \cup C$
d) $A \cup B \cup C$



13.22 The figures for Exercises 53 and 54.

54. Locate the center of mass of the carpenter's square in Fig. 13.22(b).

55. An isosceles triangle T has base $2a$ and altitude h. The base lies along the diameter of a semicircular disk D of radius a so that the two together make a shape resembling an ice cream cone. What relation must hold between a and h to place the centroid of $T \cup D$ on the common boundary of T and D? inside T?

56. An isosceles triangle T of altitude h has as its base one side of a square Q whose edges have length s. (The square and triangle do not overlap.) What relation must hold between h and s to place the centroid of $T \cup Q$ on the base of the triangle? Compare your answer with the answer to Exercise 55.

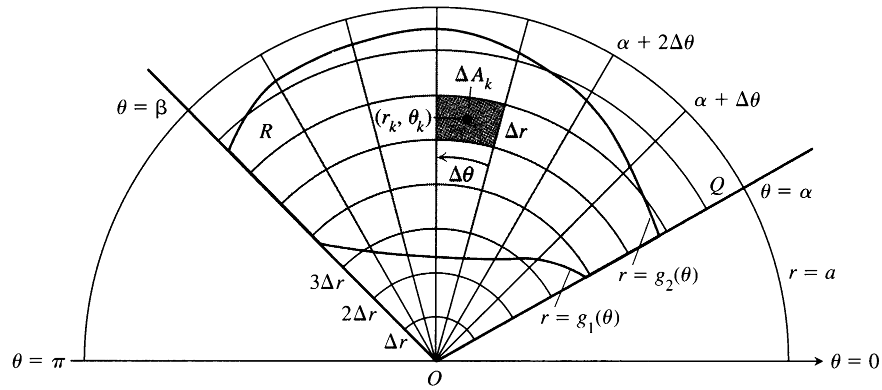
13.3 Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values.

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$. See Fig. 13.23.



13.23 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$ is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of Q by circular arcs and rays induces a partition of R.

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \ldots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \quad \theta = \alpha + \Delta\theta, \quad \theta = \alpha + 2\Delta\theta, \quad \ldots, \quad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles."

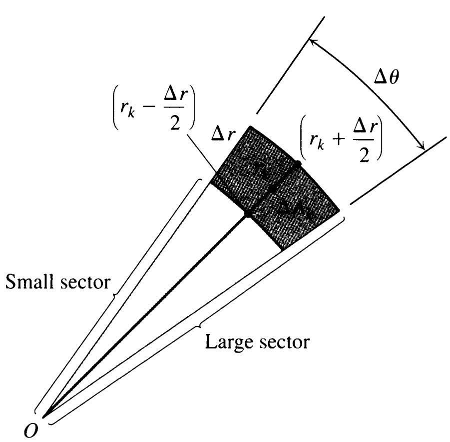
We number the polar rectangles that lie inside R (the order does not matter), calling their areas $\Delta A_1, \ \Delta A_2, \ \ldots, \ \Delta A_n$.

We let (r_k, θ_k) be the center of the polar rectangle whose area is ΔA_k. By "center" we mean the point that lies halfway between the circular arcs on the ray that bisects the arcs. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k. \tag{1}$$

If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R. In symbols,

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, dA.$$



13.24 The observation that

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

leads to the formula $\Delta A_k = r_k \Delta r \Delta\theta$. The text explains why.

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta\theta$. The radius of the inner arc bounding ΔA_k is $r_k - (\Delta r/2)$ (Fig. 13.24). The radius of the outer arc is $r_k + (\Delta r/2)$. The areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius:} \quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta \qquad \text{Outer radius:} \quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta. \tag{2}$$

Therefore,

$$\begin{aligned} \Delta A_k &= \text{Area of large sector} - \text{Area of small sector} \\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\Delta r) = r_k\Delta r\Delta\theta. \end{aligned}$$

Combining this result with Eq. (1) gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta\theta. \tag{3}$$

A version of Fubini's theorem now says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta. \tag{4}$$

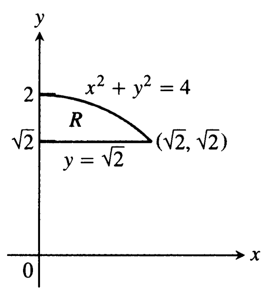
Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates.

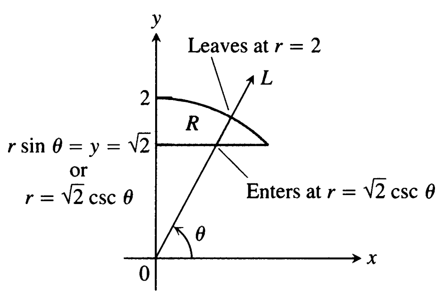
How to Integrate in Polar Coordinates

To evaluate $\iint_R f(r,\theta)\,dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

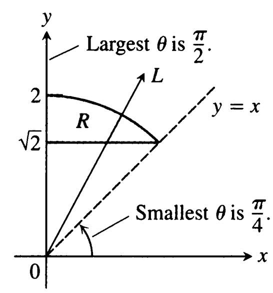
1. *A sketch.* Sketch the region and label the bounding curves.



2. *The r-limits of integration.* Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.



3. *The θ-limits of integration.* Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.



The integral is

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\,r\,dr\,d\theta.$$

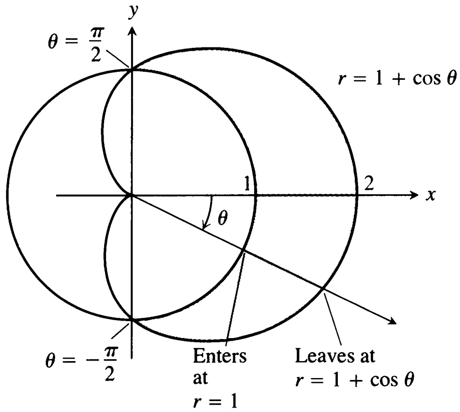
EXAMPLE 1 Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

Step 1: *A sketch.* We sketch the region and label the bounding curves (Fig. 13.25).

Step 2: *The r-limits of integration.* A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

Step 3: *The θ-limits of integration.* The rays from the origin that intersect R run from



13.25 The sketch for Example 1.

$\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\ r\ dr\ d\theta. \qquad \square$$

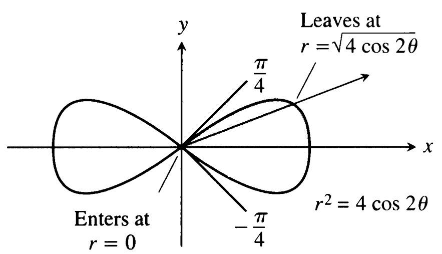
If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta. \qquad (5)$$

As you might expect, this formula for area is consistent with all earlier formulas, although we will not prove the fact.



13.26 To integrate over the shaded region, we run r from 0 to $\sqrt{4\cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Fig. 13.26) and see that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4 \sin 2\theta \Big]_0^{\pi/4} = 4. \qquad \square$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps.

Step 1: Substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral.

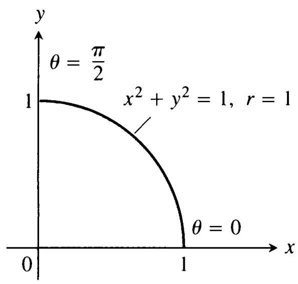
Step 2: Supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta, \qquad (6)$$

where G denotes the region of integration in polar coordinates. This is like the substitution method in Chapter 4 except that there are now two variables to substitute for instead of one. Notice that $dx\, dy$ is not replaced by $dr\, d\theta$ but by $r\, dr\, d\theta$. We will see why in Section 13.7.

EXAMPLE 3 Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.



13.27 In polar coordinates, this region is described by simple inequalities:

$0 \le r \le 1$ and $0 \le \theta \le \pi/2$

(Example 3).

Solution We sketch the plate to determine the limits of integration (Fig. 13.27).

In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Integration with respect to y gives

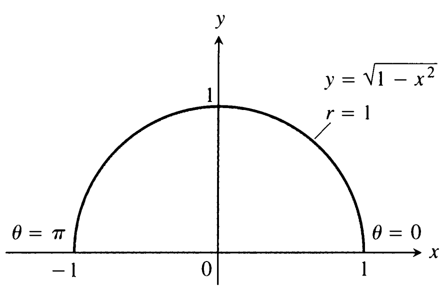
$$\int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. Substituting $x = r\cos\theta$, $y = r\sin\theta$, and replacing $dx\,dy$ by $r\,dr\,d\theta$, we get

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx &= \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.\end{aligned}$$

Why is the polar coordinate transformation so effective? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants. ❑



13.28 The semicircular region in Example 4 is the region

$0 \le r \le 1,\quad 0 \le \theta \le \pi.$

EXAMPLE 4 Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ (Fig. 13.28).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we must find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta$, $y = r\sin\theta$, and replacing $dy\,dx$ by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\begin{aligned}\iint_R e^{x^2+y^2}\, dy\, dx &= \int_0^{\pi} \int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi} \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta \\ &= \int_0^{\pi} \frac{1}{2}(e-1)\, d\theta = \frac{\pi}{2}(e-1).\end{aligned}$$

The r in the $r\,dr\,d\theta$ was just what we needed to integrate e^{r^2}. Without it we would have been stuck, as we were at the beginning. ❑

Exercises 13.3

Evaluating Polar Integrals

In Exercises 1–16, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

1. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy\, dx$

2. $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$

3. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

4. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

5. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

6. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$

7. $\int_0^6 \int_0^y x\,dx\,dy$

8. $\int_0^2 \int_0^x y\,dy\,dx$

9. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$

10. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^0 \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dx\,dy$

11. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$

12. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$

13. $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx$

14. $\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2\,dx\,dy$

15. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$

16. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx$

Finding Area in Polar Coordinates

17. Find the area of the region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2}$.

18. Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

19. Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

20. Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3,\ 0 \le \theta \le 2\pi$. The region looks like a snail shell.

21. Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

22. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Masses and Moments

23. Find the first moment about the x-axis of a thin plate of constant density $\delta(x, y) = 3$, bounded below by the x-axis and above by the cardioid $r = 1 - \cos\theta$.

24. Find the moment of inertia about the x-axis and the polar moment of inertia about the origin of a thin disk bounded by the circle $x^2 + y^2 = a^2$ if the disk's density at the point (x, y) is $\delta(x, y) = k(x^2 + y^2)$, k a constant.

25. Find the mass of a thin plate covering the region outside the circle $r = 3$ and inside the circle $r = 6\sin\theta$ if the plate's density function is $\delta(x, y) = 1/r$.

26. Find the polar moment of inertia about the origin of a thin plate covering the region that lies inside the cardioid $r = 1 - \cos\theta$ and outside the circle $r = 1$ if the plate's density function is $\delta(x, y) = 1/r^2$.

27. Find the centroid of the region enclosed by the cardioid $r = 1 + \cos\theta$.

28. Find the polar moment of inertia about the origin of a thin plate enclosed by the cardioid $r = 1 + \cos\theta$ if the plate's density function is $\delta(x, y) = 1$.

Average Values

29. Find the average height of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane.

30. Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane.

31. Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \le a^2$ to the origin.

32. Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \le 1$ to the boundary point $A(1, 0)$.

Theory and Examples

33. Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \le x^2 + y^2 \le e$.

34. Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \le x^2 + y^2 \le e^2$.

35. The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

36. The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

37. a) The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2}\,dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

b) (*Continuation of Section 7.6, Exercise 92.*) Evaluate

$$\lim_{x\to\infty} \operatorname{erf}(x) = \lim_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

38. Evaluate the integral
$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$$

39. Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \le 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \le 1$ exist? Give reasons for your answer.

40. Use the double integral in polar coordinates to derive the formula
$$A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$$
for the area of the fan-shaped region between the origin and polar curve $r = f(\theta), \alpha \le \theta \le \beta$.

41. Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0. Find the average value of d^2 over the region enclosed by the circle. (*Hint:* Simplify your work by placing the center of the circle at the origin and P_0 on the x-axis.)

42. Suppose that the area of a region in the polar coordinate plane is
$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta.$$

a) Sketch the region and find its area.

b) Use one of Pappus's theorems together with the centroid information in Exercise 26 of Section 5.10 to find the volume of the solid generated by revolvlng the region about the x-axis.

CAS Explorations and Projects

In Exercises 43–46, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

a) Plot the Cartesian region of integration in the xy-plane.

b) Change each boundary curve of the Cartesian region in (a) to its polar representation by solving its Cartesian equation for r and θ.

c) Using the results in (b), plot the polar region of integration in the $r\theta$-plane.

d) Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in (c) and evaluate the polar integral using the CAS integration utility.

43. $\displaystyle\int_0^1 \int_x^1 \frac{y}{x^2+y^2}\,dy\,dx$

44. $\displaystyle\int_0^1 \int_0^{x/2} \frac{x}{x^2+y^2}\,dy\,dx$

45. $\displaystyle\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2+y^2}}\,dx\,dy$

46. $\displaystyle\int_0^1 \int_y^{2-y} \sqrt{x+y}\,dx\,dy$

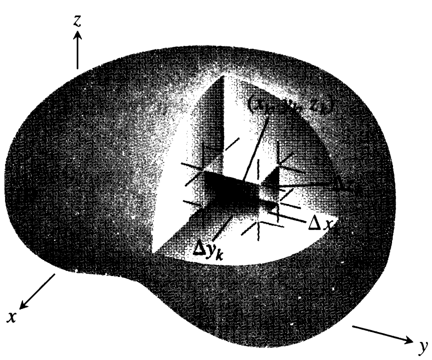
13.4 Triple Integrals in Rectangular Coordinates

We use triple integrals to find the volumes of three-dimensional shapes, the masses and moments of solids, and the average values of functions of three variables. In Chapter 14, we will also see how these integrals arise in the studies of vector fields and fluid flow.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space—the region occupied by a solid ball, for example, or a lump of clay—then the integral of F over D may be defined in the following way. We partition a rectangular region containing D into rectangular cells by planes parallel to the coordinate planes (Fig. 13.29). We number the cells that lie inside D from 1 to n in some order, a typical cell having dimensions Δx_k by Δy_k by Δz_k and volume ΔV_k. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k. \tag{1}$$



13.29 Partitioning a solid with rectangular cells of volume ΔV_k.

If F is continuous and the bounding surface of D is made of smooth surfaces joined along continuous curves, then as Δx_k, Δy_k, and Δz_k approach zero independently the sums S_n approach a limit

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\,dV. \tag{2}$$

We call this limit the **triple integral of *F* over *D*.** The limit also exists for some discontinuous functions.

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals. If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. $$\iiint_D kF\,dV = k\iiint_D F\,dV \qquad \text{(any number } k\text{)}$$

2. $$\iiint_D (F \pm G)\,dV = \iiint_D F\,dV \pm \iiint_D G\,dV$$

3. $$\iiint_D F\,dV \ge 0 \quad \text{if} \quad F \ge 0 \text{ on } D$$

4. $$\iiint_D F\,dV \ge \iiint_D G\,dV \quad \text{if} \quad F \ge G \text{ on } D.$$

Triple integrals also have an additivity property, used in physics and engineering as well as in mathematics. If the domain D of a continuous function F is partitioned by smooth surfaces into a finite number of nonoverlapping cells $D_1, D_2, \dots, D_n$, then

5. $$\iiint_D F\,dV = \iiint_{D_1} F\,dV + \iiint_{D_2} F\,dV + \cdots + \iiint_{D_n} F\,dV.$$

Volume of a Region in Space

If F is the constant function whose value is 1, then the sums in Eq. (1) reduce to

$$S_n = \sum F(x_k, y_k, z_k)\Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k. \tag{3}$$

As Δx, Δy, and Δz approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

Definition

The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV. \tag{4}$$

As we will see in a moment, this integral enables us to calculate the volumes of solids enclosed by curved surfaces.

Evaluation

We seldom evaluate a triple integral from its definition as a limit. Instead, we apply a three-dimensional version of Fubini's theorem to evaluate it by repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration.

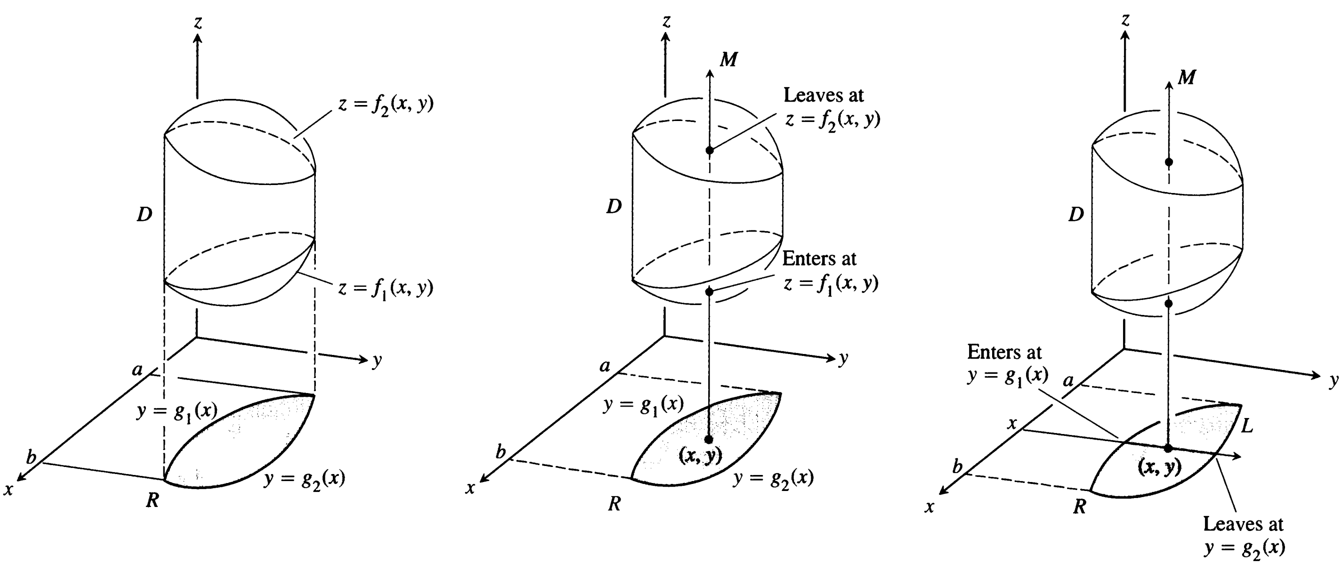
How to Find Limits of Integration in Triple Integrals

To evaluate

$$\iiint_D F(x, y, z)\, dV$$

over a region D, integrating first with respect to z, then with respect to y, finally with x, take the following steps.

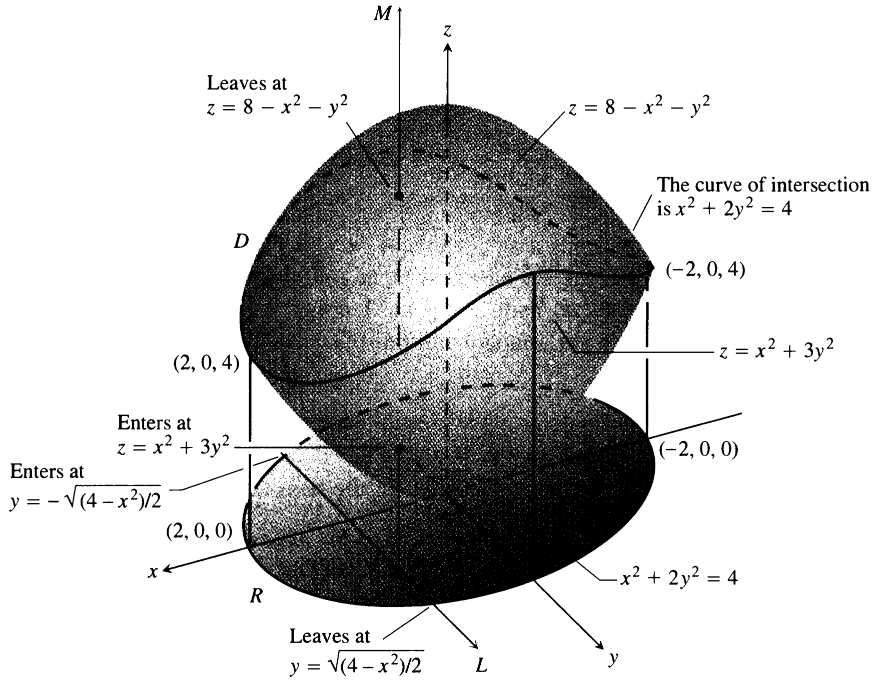
1. *A sketch.* Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.
2. *The z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.
3. *The y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *The x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.



13.30 The volume of the region enclosed by these two paraboloids is calculated in Example 1.

EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint\limits_D dz\,dy\,dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we take these steps.

Step 1: *A sketch.* The surfaces (Fig. 13.30) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

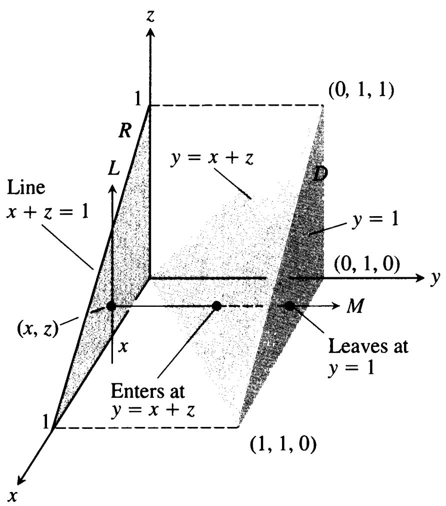
Step 2: *The z-limits of integration.* The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Step 3: *The y-limits of integration.* The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Step 4: *The x-limits of integration.* As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$V = \iiint\limits_D dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$



13.31 The tetrahedron in Example 2.

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u$$ ❑

In the next example, we project D onto the xz-plane instead of the xy-plane.

EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1).

Solution

Step 1: *A sketch.* We sketch D along with its "shadow" R in the xz-plane (Fig. 13.31). The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

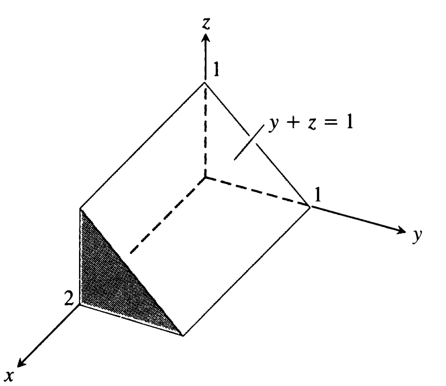
Step 2: *The y-limits of integration.* The line through a typical point (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

Step 3: *The z-limits of integration.* The line L through (x, z) parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Step 4: *The x-limits of integration.* As L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dy\, dz\, dx.$$ ❑

As we know, there are sometimes (but not always) two different orders in which the single integrations for evaluating a double integral may be worked. For triple integrals, there could be as many as *six*.



13.32 Example 3 gives six different iterated triple integrals for the volume of this prism.

EXAMPLE 3 Each of the following integrals gives the volume of the solid shown in Fig. 13.32.

a) $\displaystyle\int_0^1 \int_0^{1-z} \int_0^2 dx\, dy\, dz$ **b)** $\displaystyle\int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy$

c) $\displaystyle\int_0^1 \int_0^2 \int_0^{1-z} dy\, dx\, dz$ **d)** $\displaystyle\int_0^2 \int_0^1 \int_0^{1-z} dy\, dz\, dx$

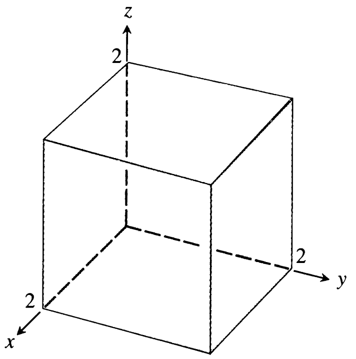
e) $\displaystyle\int_0^1 \int_0^2 \int_0^{1-y} dz\, dx\, dy$ **f)** $\displaystyle\int_0^2 \int_0^1 \int_0^{1-y} dz\, dy\, dx$ ❑

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV. \tag{5}$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of F over D is the average distance of points in D from the origin. If $F(x, y, z)$ is the density of a solid that occupies a region D in space, then the average value of F over D is the average density of the solid in units of mass per unit volume.



13.33 The region of integration in Example 4.

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Fig. 13.33). We then use Eq. (5) to calculate the average value of F over the cube.

The volume of the cube is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz = \int_0^2 \int_0^2 \left[\frac{x^2}{2} yz\right]_{x=0}^{x=2} dy\, dz = \int_0^2 \int_0^2 2yz\, dy\, dz$$

$$= \int_0^2 \left[y^2 z\right]_{y=0}^{y=2} dz = \int_0^2 4z\, dz = \left[2z^2\right]_0^2 = 8.$$

With these values, Eq. (5) gives

$$\begin{matrix}\text{Average value of} \\ xyz \text{ over the cube}\end{matrix} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz\, dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order dx, dy, dz, but any of the other five possible orders would have done as well. ❑

Exercises 13.4

Evaluating Triple Integrals in Different Iterations

1. Find the common value of the integrals in Example 3.
2. Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.
3. Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.
4. Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.
5. Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.
6. Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\, dx\, dy$ and $dz\, dy\, dx$ that give the volume of D. Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$

8. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$ **9.** $\displaystyle\int_1^e\int_1^e\int_1^e \frac{1}{xyz}\,dx\,dy\,dz$

10. $\displaystyle\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx$

11. $\displaystyle\int_0^1\int_0^{\pi}\int_0^{\pi} y\sin z\,dx\,dy\,dz$

12. $\displaystyle\int_{-1}^1\int_{-1}^1\int_{-1}^1 (x+y+z)\,dy\,dx\,dz$

13. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

14. $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

15. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$

16. $\displaystyle\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(u+v+w)\,du\,dv\,dw$ (uvw-space)

18. $\displaystyle\int_1^e\int_1^e\int_1^e \ln r\ln s\ln t\,dt\,dr\,ds$ (rst-space)

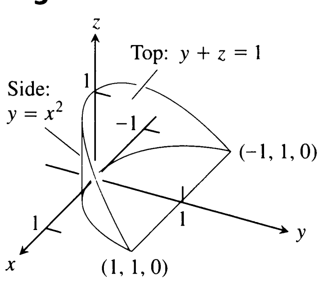
19. $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv$ (tvx-space)

20. $\displaystyle\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr$ (pqr-space)

Volumes Using Triple Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^1\int_{x^2}^1\int_0^{1-y} dz\,dy\,dx.$$

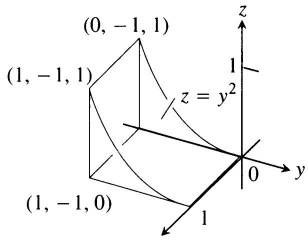


Rewrite the integral as an equivalent iterated integral in the order

a) $dy\,dz\,dx$ **b)** $dy\,dx\,dz$
c) $dx\,dy\,dz$ **d)** $dx\,dz\,dy$
e) $dz\,dx\,dy$

22. Here is the region of integration of the integral

$$\int_0^1\int_{-1}^0\int_0^{y^2} dz\,dy\,dx.$$

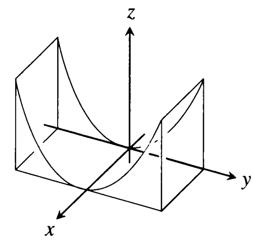


Rewrite the integral as an equivalent iterated integral in the order

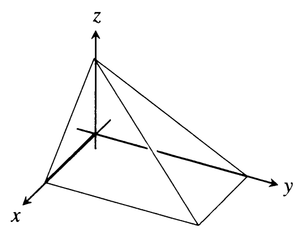
a) $dy\,dz\,dx$ **b)** $dy\,dx\,dz$
c) $dx\,dy\,dz$ **d)** $dx\,dz\,dy$
e) $dz\,dx\,dy$

Find the volumes of the regions in Exercises 23–36.

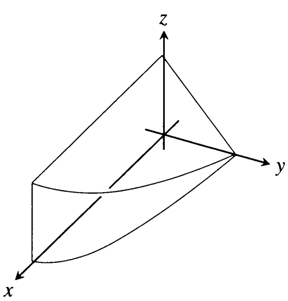
23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$



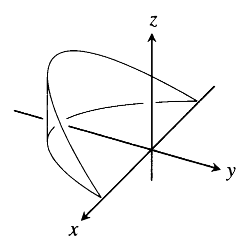
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



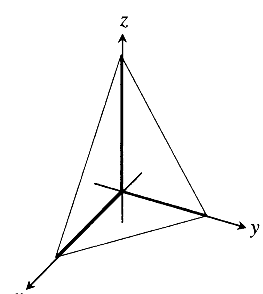
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



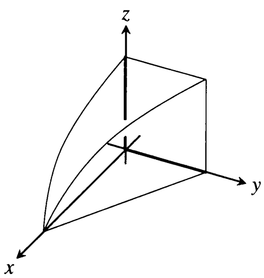
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$



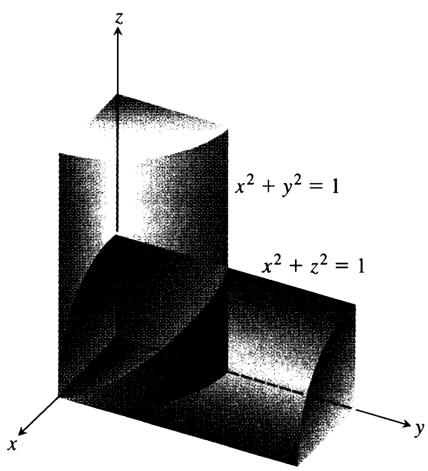
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y/2 + z/3 = 1$



28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$

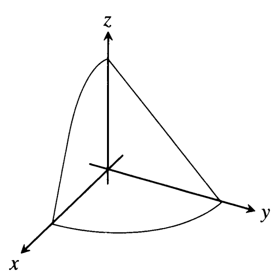


29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ (Fig. 13.34)

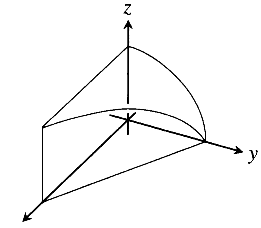


13.34 One-eighth of the region common to the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ in Exercise 29.

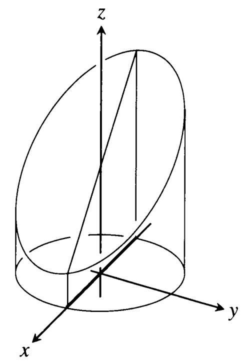
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant

34. The finite region bounded by the planes $z = x, x + z = 8$, $z = y, y = 8$, and $z = 0$.

35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane and the plane $z = x + 2$

36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

42. $\displaystyle\int_0^1 \int_0^1 \int_{x^2}^1 12xz\, e^{zy^2}\, dy\, dx\, dz$

43. $\displaystyle\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin \pi y^2}{y^2}\, dx\, dy\, dz$

44. $\displaystyle\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4 - z}\, dy\, dz\, dx$

Theory and Examples

45. Solve for a:

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}.$$

46. For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π?

47. What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4)\,dV?$$

Give reasons for your answer.

48. What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2)\,dV?$$

Give reasons for your answer.

CAS Explorations and Projects

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

49. $F(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.

50. $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$.

51. $F(x, y, z) = \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$.

52. $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \le 1$.

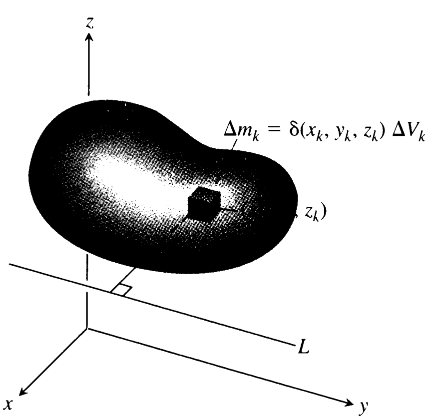
13.5 Masses and Moments in Three Dimensions

This section shows how to calculate the masses and moments of three-dimensional objects in Cartesian coordinates. The formulas are similar to those for two-dimensional objects. For calculations in spherical and cylindrical coordinates, see Section 13.6.

Masses and Moments

If $\delta(x, y, z)$ is the density of an object occupying a region D in space (mass per unit volume), the integral of δ over D gives the mass of the object. To see why, imagine partitioning the object into n mass elements like the one in Fig. 13.35. The object's mass is the limit

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\Delta V_k = \iiint_D \delta(x, y, z)\,dV. \quad (1)$$



13.35 To define an object's mass and moment of inertia about a line, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

If $r(x, y, z)$ is the distance from the point (x, y, z) in D to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k)\Delta V_k$ about the line L (shown in Fig. 13.35) is approximately $\Delta I_k = r^2(x_k, y_k, z_k)\Delta m_k$. The moment of inertia of the entire object about L is

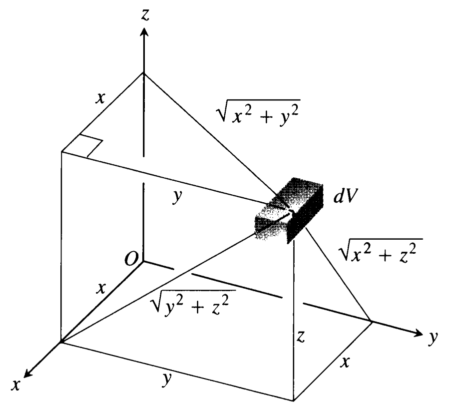
$$I_L = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\,\delta(x_k, y_k, z_k)\Delta V_k = \iiint_D r^2\delta\,dV.$$

If L is the x-axis, then $r^2 = y^2 + z^2$ (Fig. 13.36) and

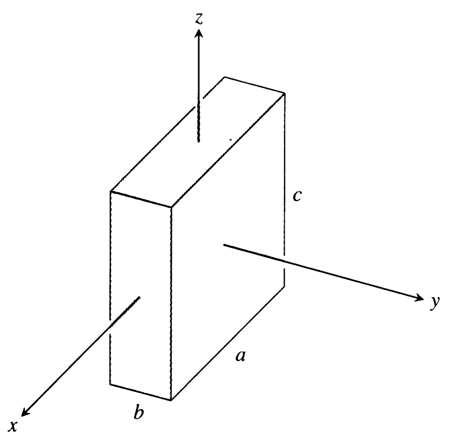
$$I_x = \iiint_D (y^2 + z^2)\,\delta\,dV.$$

Similarly,

$$I_y = \iiint_D (x^2 + z^2)\,\delta\,dV \qquad \text{and} \qquad I_z = \iiint_D (x^2 + y^2)\,\delta\,dV.$$



13.36 Distances from dV to the coordinate planes and axes.



13.37 Example 1 calculates I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block.

These and other useful formulas are summarized in Table 13.2.

Table 13.2 Mass and moment formulas for objects in space

Mass: $M = \iiint_D \delta\, dV \qquad (\delta = \text{density})$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments):

$$I_x = \iiint (y^2+z^2)\,\delta\, dV$$
$$I_y = \iiint (x^2+z^2)\,\delta\, dV$$
$$I_z = \iiint (x^2+y^2)\,\delta\, dV$$

Moment of inertia about a line L:

$I_L = \iiint r^2\,\delta\, dV \qquad (r(x,y,z) = \text{distance from points } (x,y,z) \text{ to line } L)$

Radius of gyration about a line L:

$$R_L = \sqrt{I_L/M}$$

EXAMPLE 1 Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Fig. 13.37.

Solution The preceding formula for I_x gives

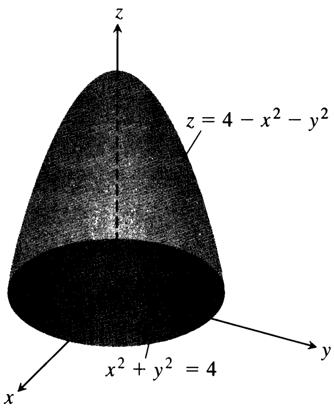
$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2+z^2)\,\delta\, dx\, dy\, dz. \tag{2}$$

We can avoid some of the work of integration by observing that $(y^2+z^2)\,\delta$ is an even function of x, y, and z and therefore

$$I_x = 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2+z^2)\,\delta\, dx\, dy\, dz = 4a\delta \int_0^{c/2}\int_0^{b/2} (y^2+z^2)\, dy\, dz$$
$$= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz$$
$$= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$
$$= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2+c^2) = \frac{M}{12}(b^2+c^2).$$

Similarly,

$$I_y = \frac{M}{12}(a^2+c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2+b^2).$$ ❑



13.38 Example 2 finds the center of mass of this solid.

EXAMPLE 2 Find the center of mass of a solid of constant density δ bounded below by the disk R: $x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Fig. 13.38).

Solution By symmetry, $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$M_{xy} = \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2\,r\,dr\,d\theta \qquad \text{Polar coordinates}$$

$$= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives

$$M = \iint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\,\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$, and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$. ❑

When the density of a solid object is constant (as in Examples 1 and 2), the center of mass is called the **centroid** of the object (as was the case for two-dimensional shapes in Section 13.2).

Exercises 13.5

Constant Density

The solids in Exercises 1–12 all have constant density $\delta = 1$.

1. Evaluate the integral for I_x in Eq. (2) directly to show that the shortcut in Example 1 gives the same answer. Use the results in Example 1 to find the radius of gyration of the rectangular solid about each coordinate axis.

2. The coordinate axes in the figure to the right run through the centroid of a solid wedge parallel to the labeled edges. Find I_x, I_y, and I_z if $a = b = 6$ and $c = 4$.

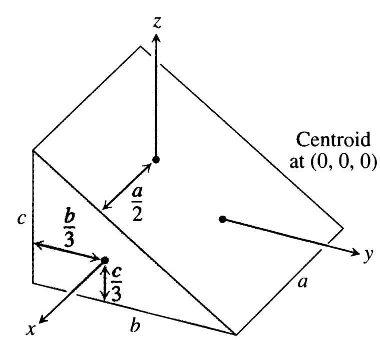
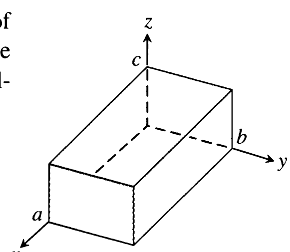


Figure for Exercise 2

3. Find the moments of inertia of the rectangular solid shown here with respect to its edges by calculating I_x, I_y, and I_z.



4. a) Find the centroid and the moments of inertia I_x, I_y, and I_z of the tetrahedron whose vertices are the points (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1).
   b) Find the radius of gyration of the tetrahedron about the x-axis. Compare it with the distance from the centroid to the x-axis.

5. A solid "trough" of constant density is bounded below by the surface $z = 4y^2$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -1$. Find the center of mass and the moments of inertia with respect to the three axes.

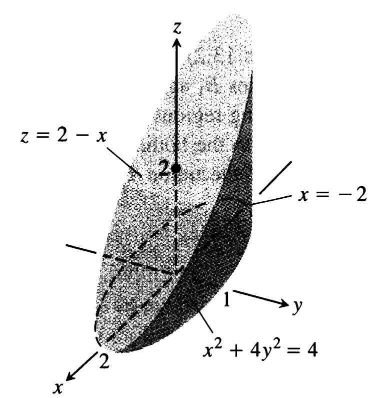
6. A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptic cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$ (see the figure).

a) Find $\bar{x}$ and $\bar{y}$.

b) Evaluate the integral

$$M_{xy} = \int_{-2}^{2} \int_{-(1/2)\sqrt{4-x^2}}^{(1/2)\sqrt{4-x^2}} \int_{0}^{2-x} z\,dz\,dy\,dx,$$

using integral tables to carry out the final integration with respect to x. Then divide M_{xy} by M to verify that $\bar{z} = 5/4$.



7. a) Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.

b) Find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.

8. A solid cube, 2 units on a side, is bounded by the planes $x = \pm 1$, $z = \pm 1$, $y = 3$, and $y = 5$. Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes.

9. A wedge like the one in Exercise 2 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line L: $z = 0$, $y = 6$ is $r^2 = (y - 6)^2 + z^2$. Then calculate the moment of inertia and radius of gyration of the wedge about L.

10. A wedge like the one in Exercise 2 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line L: $x = 4$, $y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then calculate the moment of inertia and radius of gyration of the wedge about L.

11. A solid like the one in Exercise 3 has $a = 4$, $b = 2$, and $c = 1$. Make a quick sketch to check for yourself that the square of the distance between a typical point (x, y, z) of the solid and the line L: $y = 2$, $z = 0$ is $r^2 = (y - 2)^2 + z^2$. Then find the moment of inertia and radius of gyration of the solid about L.

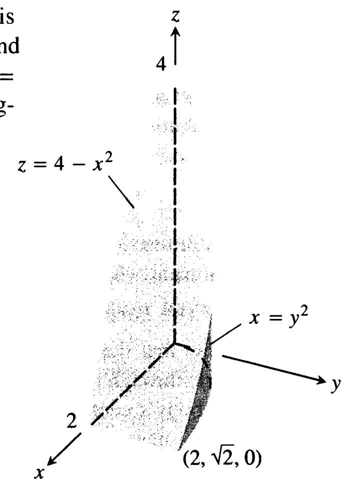
12. A solid like the one in Exercise 3 has $a = 4$, $b = 2$, and $c = 1$. Make a quick sketch to check for yourself that the square of the distance between a typical point (x, y, z) of the solid and the line L: $x = 4$, $y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then find the moment of inertia and radius of gyration of the solid about L.

Variable Density

In Exercises 13 and 14, find (a) the mass of the solid and (b) the center of mass.

13. A solid region in the first octant is bounded by the coordinate planes and the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$.

14. A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the figure). Its density function is $\delta(x, y, z) = kxy$.



In Exercises 15 and 16, find

a) the mass of the solid

b) the center of mass

c) the moments of inertia about the coordinate axes

d) the radii of gyration about the coordinate axes.

15. A solid cube in the first octant is bounded by the coordinate planes and by the planes $x = 1$, $y = 1$, and $z = 1$. The density of the cube is $\delta(x, y, z) = x + y + z + 1$.

16. A wedge like the one in Exercise 2 has dimensions $a = 2$, $b = 6$, and $c = 3$. The density is $\delta(x, y, z) = x + 1$. Notice that if the density is constant, the center of mass will be $(0, 0, 0)$.

17. Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density of the solid is $\delta(x, y, z) = 2y + 5$.

18. Find the mass of the solid region bounded by the parabolic surfaces $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Work

In Exercises 19 and 20, calculate the following.

a) The amount of work done by (constant) gravity g in moving the liquid filled in the container to the xy-plane (*Hint:* Partition the liquid in the container into small volume elements ΔV_i and find the work done (approximately) by gravity on each element.

Summation and passage to the limit gives a triple integral to evaluate.)

b) The work done by gravity in moving the center of mass down to the xy-plane

19. The container is a cubical box in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$. The density of the liquid filling the box is $\delta(x, y, z) = x + y + z + 1$ (refer to Exercise 15).

20. The container is in the shape of the region bounded by $y = 0$, $z = 0$, $z = 4 - x^2$, and $x = y^2$. The density of the liquid filling the region is $\delta(x, y, z) = kxy$ (see Exercise 14).

The Parallel Axis Theorem

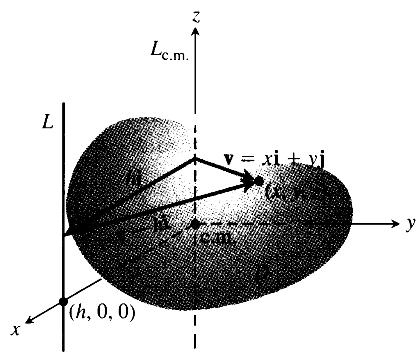
The Parallel Axis Theorem (Exercises 13.2) holds in three dimensions as well as in two. Let $L_{\text{c.m.}}$ be a line through the center of mass of a body of mass m and let L be a parallel line h units away from $L_{\text{c.m.}}$. The **Parallel Axis Theorem** says that the moments of inertia $I_{\text{c.m.}}$ and I_L of the body about $L_{\text{c.m.}}$ and L satisfy the equation

$$I_L = I_{\text{c.m.}} + mh^2. \tag{1}$$

As in the two-dimensional case, the theorem gives a quick way to calculate one moment when the other moment and the mass are known.

21. *Proof of the Parallel Axis Theorem*

a) Show that the first moment of a body in space about any plane through the body's center of mass is zero. (*Hint:* Place the body's center of mass at the origin and let the plane be the yz-plane. What does the formula $\bar{x} = M_{yz}/M$ then tell you?)



b) To prove the Parallel Axis Theorem, place the body with its center of mass at the origin, with the line $L_{\text{c.m.}}$ along the z-axis and the line L perpendicular to the xy-plane at the point $(h, 0, 0)$. Let D be the region of space occupied by the body. Then, in the notation of the figure,

$$I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2 \, dm. \tag{2}$$

Expand the integrand in this integral and complete the proof.

22. The moment of inertia about a diameter of a solid sphere of constant density and radius a is $(2/5)ma^2$, where m is the mass of the sphere. Find the moment of inertia about a line tangent to the sphere.

23. The moment of inertia of the solid in Exercise 3 about the z-axis is $I_z = abc(a^2 + b^2)/3$.

a) Use Eq. (1) to find the moment of inertia and radius of gyration of the solid about the line parallel to the z-axis through the solid's center of mass.

b) Use Eq. (1) and the result in (a) to find the moment of inertia and radius of gyration of the solid about the line $x = 0$, $y = 2b$.

24. If $a = b = 6$ and $c = 4$, the moment of inertia of the solid wedge in Exercise 2 about the x-axis is $I_x = 208$. Find the moment of inertia of the wedge about the line $y = 4$, $z = -4/3$ (the edge of the wedge's narrow end).

Pappus's Formula

Pappus's formula (Exercises 13.2) holds in three dimensions as well as in two. Suppose that bodies B_1 and B_2 of mass m_1 and m_2, respectively, occupy nonoverlapping regions in space and that $\mathbf{c}_1$ and $\mathbf{c}_2$ are the vectors from the origin to the bodies' respective centers of mass. Then the center of mass of the union $B_1 \cup B_2$ of the two bodies is determined by the vector

$$\mathbf{c} = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1 + m_2}. \tag{3}$$

As before, this formula is called **Pappus's formula.** As in the two-dimensional case, the formula generalizes to

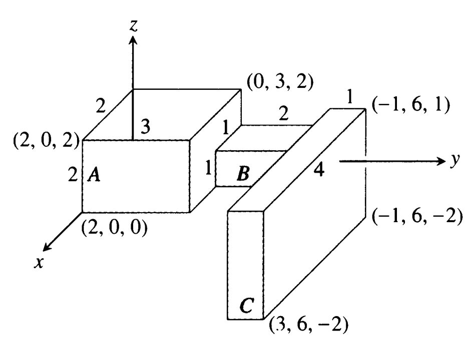
$$\mathbf{c} = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + \cdots + m_n\mathbf{c}_n}{m_1 + m_2 + \cdots + m_n} \tag{4}$$

for n bodies.

25. Derive Pappus's formula (Eq. 3). (*Hint:* Sketch B_1 and B_2 as nonoverlapping regions in the first octant and label their centers of mass $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$. Express the moments of $B_1 \cup B_2$ about the coordinate planes in terms of the masses m_1 and m_2 and the coordinates of these centers.)

26. The figure below shows a solid made from three rectangular solids of constant density $\delta = 1$. Use Pappus's formula to find the center of mass of

a) $A \cup B$ **b)** $A \cup C$
c) $B \cup C$ **d)** $A \cup B \cup C$.



27. a) Suppose that a solid right circular cone C of base radius a and altitude h is constructed on the circular base of a solid hemisphere S of radius a so that the union of the two solids resembles an ice cream cone. The centroid of a solid cone lies one-fourth of the way from the base toward the vertex. The centroid of a solid hemisphere lies three-eighths of the way from the base to the top. What relation must hold between h and a to place the centroid of $C \cup S$ in the common base of the two solids?

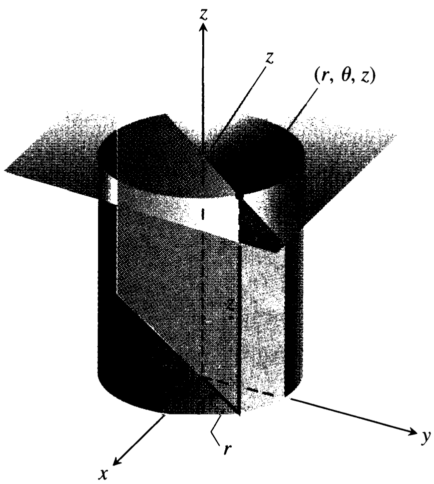
b) If you have not already done so, answer the analogous question about a triangle and a semicircle (Section 13.2, Exercise 55). The answers are not the same.

28. A solid pyramid P with height h and four congruent sides is built with its base as one face of a solid cube C whose edges have length s. The centroid of a solid pyramid lies one-fourth of the way from the base toward the vertex. What relation must hold between h and s to place the centroid of $P \cup C$ in the base of the pyramid? Compare your answer with the answer to Exercise 27. Also compare it to the answer to Exercise 56 in Section 13.2.

13.6 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates.

Cylindrical Coordinates



13.39 Cylindrical coordinates and typical surfaces of constant coordinate value.

Cylindrical coordinates (Fig. 13.39) are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. As we saw in Section 10.7, surfaces like these have equations of constant coordinate value:

$$r = 4 \qquad \text{Cylinder, radius 4, axis the } z\text{-axis}$$

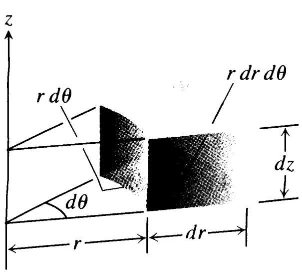
$$\theta = \frac{\pi}{3} \qquad \text{Plane containing the } z\text{-axis}$$

$$z = 2 \qquad \text{Plane perpendicular to the } z\text{-axis}$$

The volume element for subdividing a region in space with cylindrical coordinates is

$$dV = dz\, r\, dr\, d\theta \tag{1}$$

(Fig. 13.40). Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.



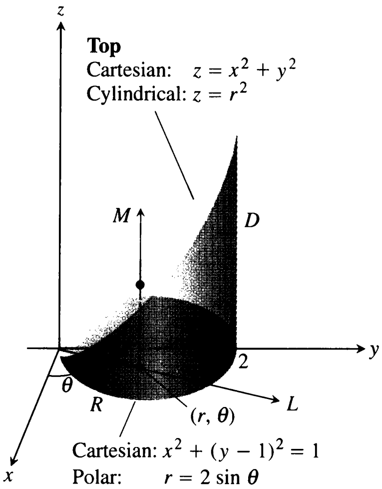
13.40 The volume element in cylindrical coordinates is $dV = dz\, r\, dr\, d\theta$.

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution

Step 1: *A sketch* (Fig. 13.41). The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r \sin\theta &= 0 \\ r &= 2 \sin\theta. \end{aligned}$$



13.41 The figure for Example 1.

Step 2: *The z-limits of integration.* A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

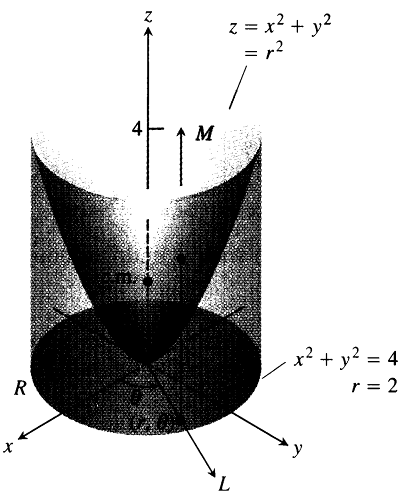
Step 3: *The r-limits of integration.* A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Step 4: *The θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example 1 illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized in the box on the following page.

EXAMPLE 2 Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$ and below by the xy-plane.



13.42 Example 2 shows how to find the centroid of this solid.

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Fig. 13.42). Its base R is the disk $|r| \leq 2$ in the xy-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed step 1 with our initial sketch. The remaining steps give the limits of integration.

Step 2: *The z-limits.* A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

Step 3: *The r-limits.* A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

Step 4: *The θ-limits.* As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^5}{2}\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3}\, d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi} \int_0^2 \int_0^{r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \Big[z\Big]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies outside the solid.

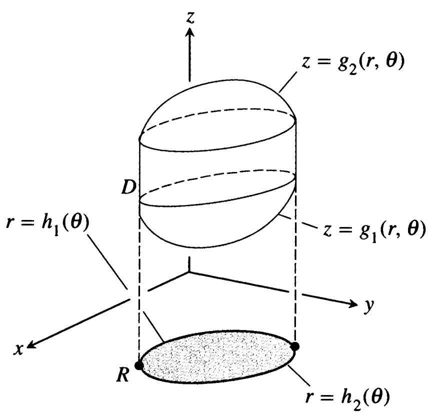
How to Integrate in Cylindrical Coordinates

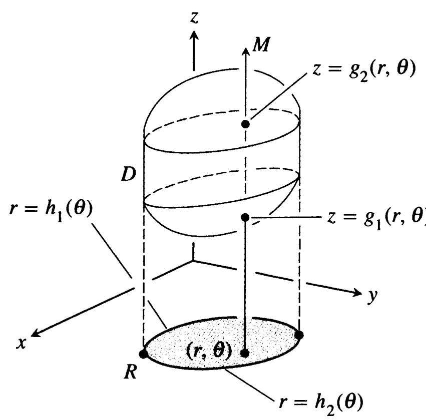
To evaluate

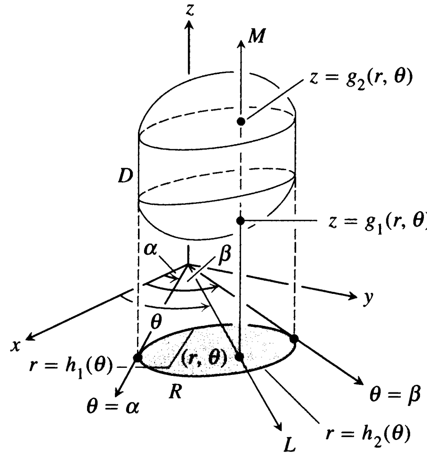
$$\iiint_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

1. *A sketch.* Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.
2. *The z-limits of integration.* Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.
3. *The r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.







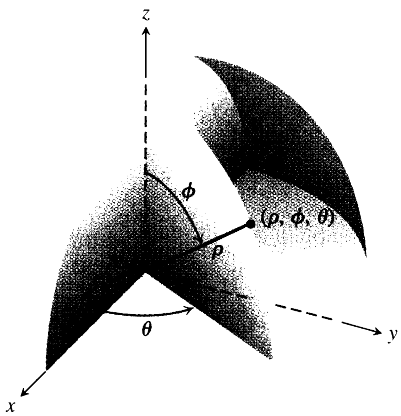
4. *The θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \tag{2}$$

Spherical Coordinates

Spherical coordinates (Fig. 13.43, on the following page) are good for describing spheres centered at the origin, half-planes hinged along the z-axis, and single-napped cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$\rho = 4$	Sphere, radius 4, center at origin
$\phi = \dfrac{\pi}{3}$	Cone opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis
$\theta = \dfrac{\pi}{3}$	Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with the positive x-axis



13.43 Spherical coordinates are measured with a distance and two angles.

The volume element in spherical coordinates is the volume of a **spherical wedge** defined by the differentials $d\rho$, $d\phi$, and $d\theta$ (Fig. 13.44). The wedge is approximately a rectangular box with one side a circular arc of length $\rho\, d\phi$, another side a circular arc of length $\rho \sin\phi\, d\theta$, and thickness $d\rho$. Therefore the volume element in spherical coordinates is

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta, \tag{3}$$

and triple integrals take the form

$$\iiint F(\rho, \phi, \theta)\, dV = \iiint F(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta. \tag{4}$$

To evaluate these integrals, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is shown in the following box. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

How to Integrate in Spherical Coordinates

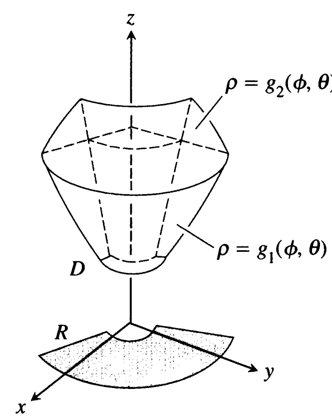
To evaluate

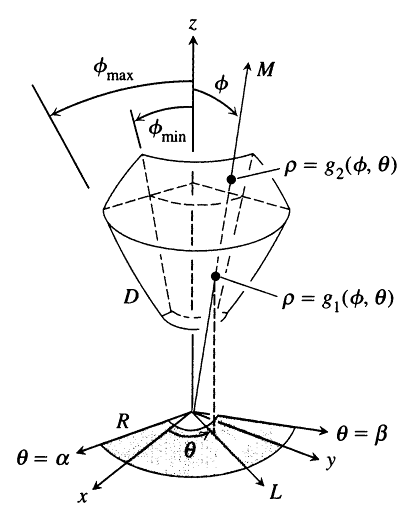
$$\iiint_D f(\rho, \phi, \theta)\ dV$$

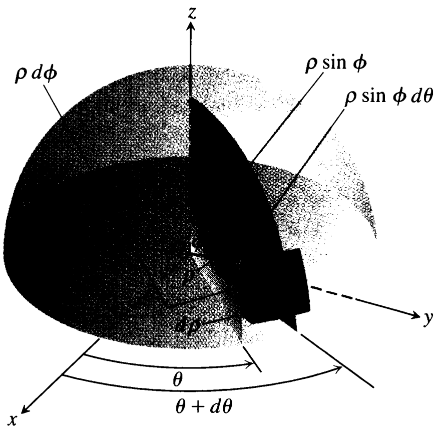
over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *A sketch.* Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.
2. *The ρ-limits of integration.* Draw a ray M from the origin making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.
3. *The ϕ-limits of integration.* For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ-limits of integration.
4. *The θ-limits of integration.* The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta. \tag{5}$$

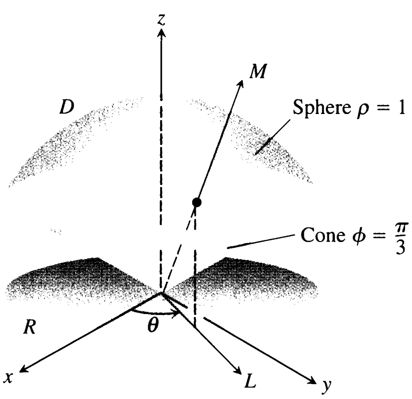






13.44 The volume element in spherical coordinates is

$$dV = d\rho \cdot \rho\, d\phi \cdot \rho \sin\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$



13.45 The solid in Example 3.

EXAMPLE 3 Find the volume of the upper region D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we take the following steps.

Step 1: *A sketch.* We sketch D and its projection R on the xy-plane (Fig. 13.45).

Step 2: *The ρ-limits of integration.* We draw a ray M from the origin making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

Step 3: *The ϕ-limits of integration.* The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

Step 4: *The θ-limits of integration.* The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$
\begin{aligned}
V &= \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.
\end{aligned}
$$

❑

EXAMPLE 4 A solid of constant density $\delta = 1$ occupies the region D in Example 3. Find the solid's moment of inertia about the z-axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \iiint \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta.$$

For the region in Example 3, this becomes

$$
\begin{aligned}
I_z &= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi\, d\phi\, d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\left(-\frac{1}{2}+1+\frac{1}{24}-\frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi}\frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.
\end{aligned}
$$

❑

Coordinate Conversion Formulas (from Section 10.8)

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding volume elements

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Exercises 13.6

Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$

2. $\displaystyle\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta$

3. $\displaystyle\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\,r\,dr\,d\theta$

4. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$

5. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$

6. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2} (r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta$

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

7. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$

8. $\displaystyle\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz$

9. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi} (r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$

10. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi} (r\sin\theta + 1)\,r\,d\theta\,dz\,dr$

11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a) $dz\,dr\,d\theta$
    b) $dr\,dz\,d\theta$
    c) $d\theta\,dz\,dr$

12. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a) $dz\,dr\,d\theta$
    b) $dr\,dz\,d\theta$
    c) $d\theta\,dz\,dr$

13. Give the limits of integration for evaluating the integral

$$\iiint f(r,\theta,z)\ dz\,r\,dr\,d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

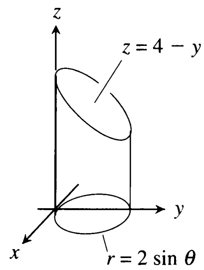
14. Convert the integral

$$\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_0^x (x^2 + y^2)\ dz\,dx\,dy$$

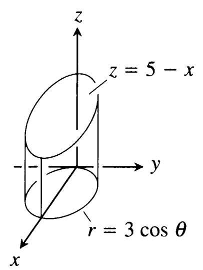
to an equivalent integral in cylindrical coordinates and evaluate the result.

In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,r\,dr\,d\theta$ over the given region D.

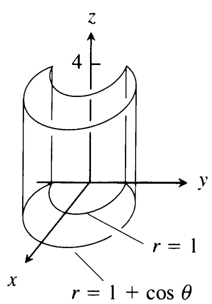
15. D is the right circular cylinder whose base is the circle $r = 2 \sin \theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



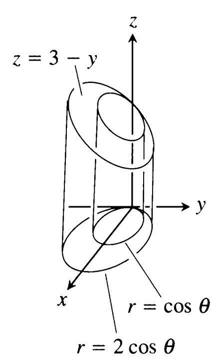
16. D is the right circular cylinder whose base is the circle $r = 3 \cos \theta$ and whose top lies in the plane $z = 5 - x$.



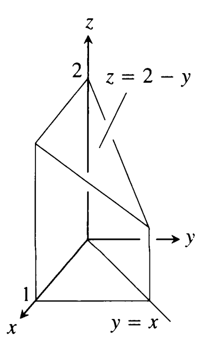
17. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



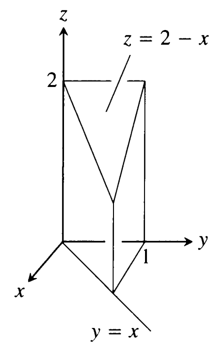
18. D is the solid right cylinder whose base is the region between the circles $r = \cos \theta$ and $r = 2 \cos \theta$, and whose top lies in the plane $z = 3 - y$.



19. D is the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



20. D is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

22. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

23. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

24. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

25. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

26. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} (\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders are possible and occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

28. $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi$

29. $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\pi/4} 12\rho \sin^3\phi \, d\phi \, d\theta \, d\rho$

30. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

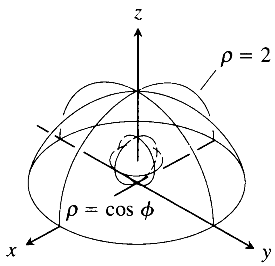
a) $d\rho\, d\phi\, d\theta$ **b)** $d\phi\, d\rho\, d\theta$

32. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

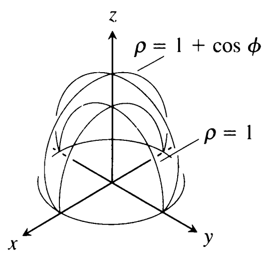
a) $d\rho\, d\phi\, d\theta$ **b)** $d\phi\, d\rho\, d\theta$

In Exercises 33–38, (a) find the spherical coordinate limits for the integral that calculates the volume of the given solid, and (b) then evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$



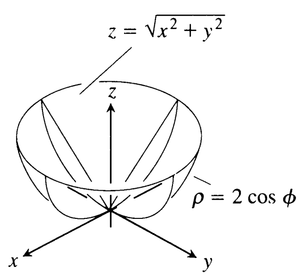
34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$



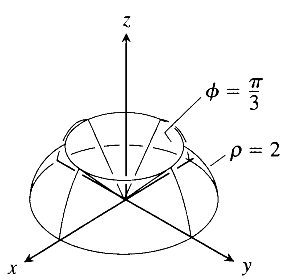
35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

36. The upper portion cut from the solid in Exercise 35 by the xy-plane

37. The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



38. The solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



Rectangular, Cylindrical, and Spherical Coordinates

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in (a) spherical, (b) cylindrical, and (c) rectangular coordinates.

40. Let D be the region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in (a) cylindrical and (b) spherical coordinates. Then (c) find V.

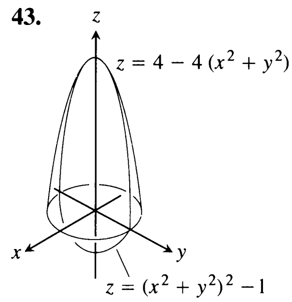
41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in (a) spherical, (b) cylindrical, and (c) rectangular coordinates. Then (d) find the volume by evaluating one of the three triple integrals.

42. Express the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$, as an iterated integral in (a) cylindrical and (b) spherical coordinates. Then (c) find I_z.

Volumes

Find the volumes of the solids in Exercises 43–48.

43.



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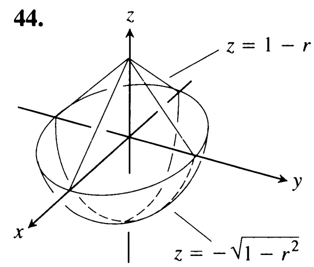

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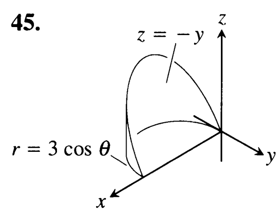

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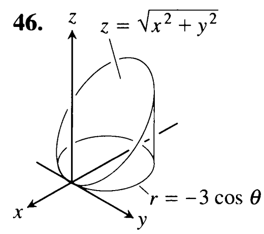

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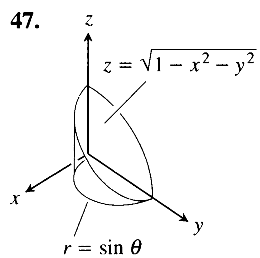
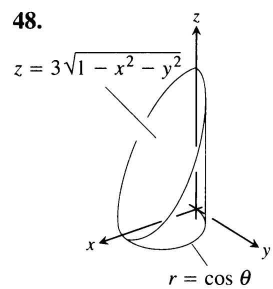


48.



49. Find the volume of the portion of the solid sphere $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

50. Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

51. Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.

52. Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

53. Find the volume of the region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

54. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

55. Find the volume of the solid cut from the thick-walled cylinder $1 \le x^2 + y^2 \le 2$ by the cones $z = \pm\sqrt{x^2 + y^2}$.

56. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

57. Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

58. Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.

59. Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

60. Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.

61. Find the volume of the region cut from the solid cylinder $x^2 + y^2 \le 1$ by the sphere $x^2 + y^2 + z^2 = 4$.

62. Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

63. Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

64. Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)

65. Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \le 1$.

66. Find the average value of the function $f(\rho, \phi, \theta) = \rho \cos\phi$ over the solid upper ball $\rho \le 1$, $0 \le \phi \le \pi/2$.

Masses, Moments, and Centroids

67. A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r$, $r \ge 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.

68. Find the centroid of the region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.

69. Find the centroid of the solid in Exercise 38.

70. Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.

71. Find the centroid of the region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy-plane.

72. Find the centroid of the region cut from the solid ball $r^2 + z^2 \le 1$ by the half-planes $\theta = -\pi/3$, $r \ge 0$, and $\theta = \pi/3$, $r \ge 0$.

73. Find the moment of inertia and radius of gyration about the z-axis of a thick-walled right circular cylinder bounded on the inside by the cylinder $r = 1$, on the outside by the cylinder $r = 2$, and on the top and bottom by the planes $z = 4$ and $z = 0$. (Take $\delta = 1$.)

74. Find the moment of inertia of a solid circular cylinder of radius 1 and height 2 (a) about the axis of the cylinder, (b) about a line through the centroid perpendicular to the axis of the cylinder. (Take $\delta = 1$.)

75. Find the moment of inertia of a right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base. (Take $\delta = 1$.)

76. Find the moment of inertia of a solid sphere of radius a about a diameter. (Take $\delta = 1$.)

77. Find the moment of inertia of a right circular cone of base radius a and height h about its axis. (*Hint:* Place the cone with its vertex at the origin and its axis along the z-axis.)

78. A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z-axis if the density is (a) $\delta(r, \theta, z) = z$; (b) $\delta(r, \theta, z) = r$.

79. A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z-axis if the density is (a) $\delta(r, \theta, z) = z$; (b) $\delta(r, \theta, z) = z^2$.

80. A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia and radius of gyration about the z-axis if the density is

a) $\delta(\rho, \phi, \theta) = \rho^2$, **b)** $\delta(\rho, \phi, \theta) = r = \rho \sin\phi$.

81. Show that the centroid of the solid semi-ellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \le 1$, $z \ge 0$, lies on the z-axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.

82. Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)

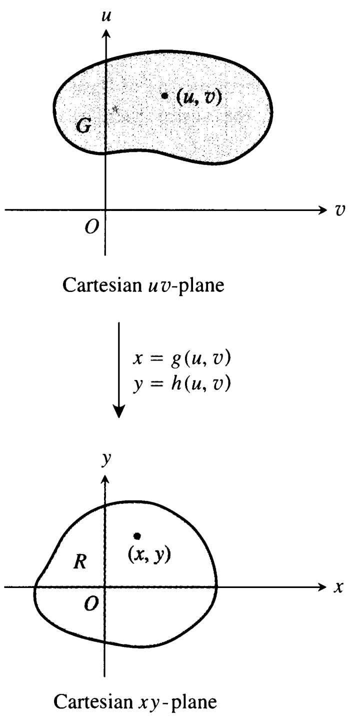
83. A solid right circular cylinder is bounded by the cylinder $r = a$ and the planes $z = 0$ and $z = h$, $h > 0$. Find the center of mass and the moment of inertia and radius of gyration about the z-axis if the density is $\delta(r, \theta, z) = z + 1$.

84. A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

85. A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center. What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

13.7 Substitutions in Multiple Integrals

This section shows how to evaluate multiple integrals by substitution. As in single integration, the goal of substitution is to replace complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both.



13.46 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane.

Substitutions in Double Integrals

The polar coordinate substitution of Section 13.3 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

as suggested in Fig. 13.46. We call R the **image** of G under the transformation, and G the **preimage** of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G?

The answer is: If g, h, and f have continuous partial derivatives and $J(u, v)$ (to be discussed in a moment) is zero only at isolated points, if at all, then

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(g(u, v), h(u, v))\,|J(u, v)|\,du\,dv. \tag{1}$$

The factor $J(u, v)$, whose absolute value appears in Eq. (1), is the *Jacobian* of the coordinate transformation, named after the mathematician Carl Jacobi.

Definition

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{2}$$

Notice the "Reversed" Order

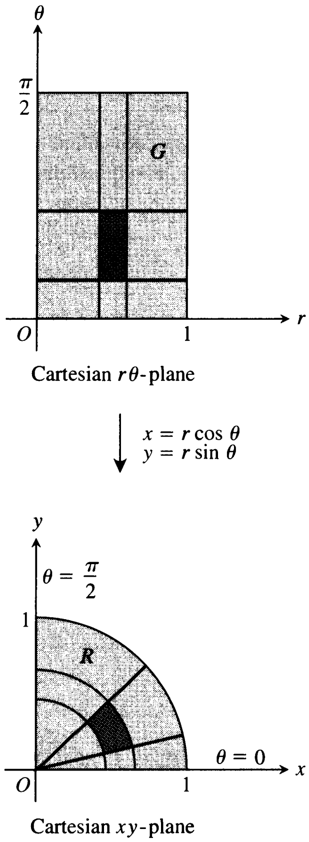
The transforming equations $x = g(u, v)$ and $y = h(u, v)$ go from G to R, but we use them to change an integral over R into an integral over G.

The Jacobian is also denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

to help remember how the determinant in Eq. (2) is constructed from the partial derivatives of x and y. The derivation of Eq. (1) is intricate and properly belongs to a course in advanced calculus. We will not give the derivation here.

For polar coordinates, we have r and θ in place of u and v. With $x = r\cos\theta$



13.47 The equations $x = r\cos\theta, y = r\sin\theta$ transform G into R.

and $y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Hence, Eq. (1) becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta,\ r\sin\theta)\,|r|\,dr\,d\theta$$

$$= \iint_G f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta, \qquad \text{If } r \geq 0 \qquad (3)$$

which is Eq. (6) in Section 13.3.

Figure 13.47 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle G: $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$ into the quarter circle R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy-plane.

Notice that the integral on the right-hand side of Eq. (3) is not the integral of $f(r\cos\theta,\ r\sin\theta)$ over a region in the polar coordinate plane. It is the integral of the product of $f(r\cos\theta,\ r\sin\theta)$ and r over a region G in the *Cartesian* $r\theta$-plane.

Here is an example of another substitution.

EXAMPLE 1 Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation

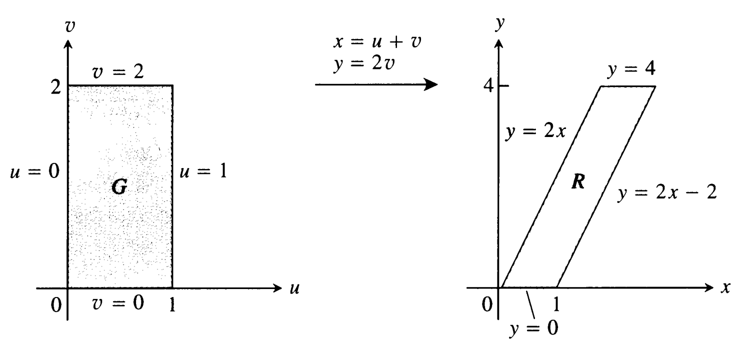
$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2} \qquad (4)$$

and integrating over an appropriate region in the uv-plane.

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Fig. 13.48).

To apply Eq. (1), we need to find the corresponding uv-region G and the Jacobian of the transformation. To find them, we first solve Eqs. (4) for x and y in terms of u and v. Routine algebra gives

$$x = u + v, \qquad y = 2v. \qquad (5)$$



13.48 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G. See Example 1.

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R (Fig. 13.48).

xy-equations for the boundary of R	**Corresponding uv-equations for the boundary of G**	**Simplified uv-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

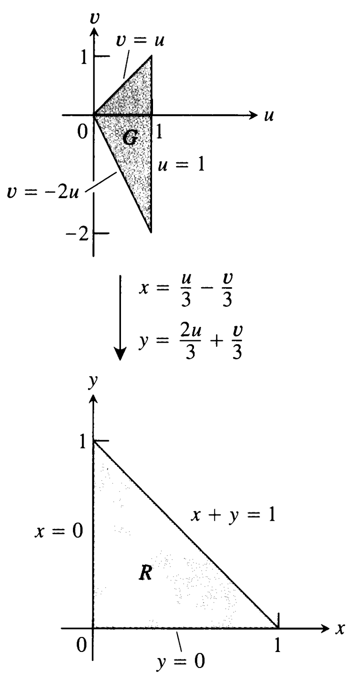
The Jacobian of the transformation (again from Eqs. 5) is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Eq. (1):

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u|J(u, v)|\, du\, dv$$
$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[u^2\Big]_0^1 dv = \int_0^2 dv = 2.$$

❑



13.49 The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform G into R. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms R into G. See Example 2.

EXAMPLE 2 Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x + y}\,(y - 2x)^2\, dy\, dx.$$

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Fig. 13.49). The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Routine algebra produces x and y as functions of u and v:

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}. \tag{6}$$

From Eqs. (6) we can find the boundaries of the uv-region G (Fig. 13.49).

xy-equations for the boundary of R	**Corresponding uv-equations for the boundary of G**	**Simplified uv-equations**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

The Jacobian of the transformation in Eq. (6) is

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\ \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Eq. (1), we evaluate the integral:

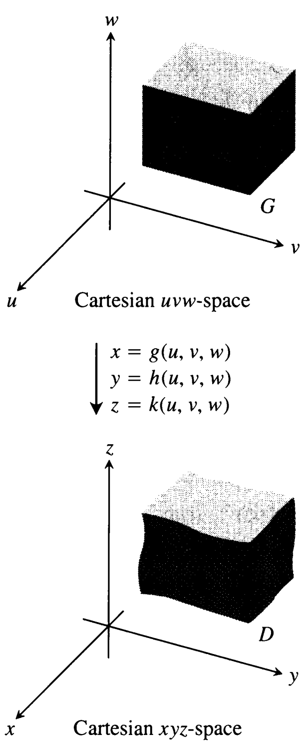
$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2}v^2\,|J(u,v)|\,dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2}v^2 \left(\frac{1}{3}\right) dv\,du = \frac{1}{3}\int_0^1 u^{1/2}\left(\frac{1}{3}v^3\right)\Bigg]_{v=-2u}^{v=u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}(u^3+8u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9}u^{9/2}\Bigg]_0^1 = \frac{2}{9}.$$ ❑

Substitutions in Triple Integrals

The cylindrical and spherical coordinate substitutions are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of three-dimensional regions. The method is like the method for double integrals except that now we work in three dimensions instead of two.



13.50 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz-space into an integral over a region G in Cartesian uvw-space.

Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u,v,w), \qquad y = h(u,v,w), \qquad z = k(u,v,w),$$

as suggested in Fig. 13.50. Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u,v,w), h(u,v,w), k(u,v,w)) = H(u,v,w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw. \tag{7}$$

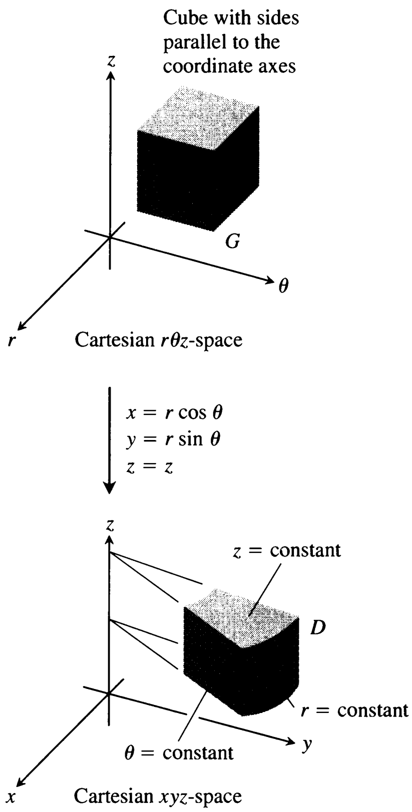
The factor $J(u, v, w)$, whose absolute value appears in this equation, is the **Jacobian determinant**

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}. \tag{8}$$

As in the two-dimensional case, the derivation of the change-of-variable formula in Eq. (7) is complicated and we will not go into it here.

For cylindrical coordinates, r, θ, and z take the place of u, v, and w. The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$



13.51 The equations $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ transform G into D.

(Fig. 13.51). The Jacobian of the transformation is

$$J(r, \theta, z) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Eq. (7) is

$$\iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(r, \theta, z)\,|r|\,dr\,d\theta\,dz. \tag{9}$$

We can drop the absolute value signs whenever $r \geq 0$.

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

(Fig. 13.52). The Jacobian of the transformation is

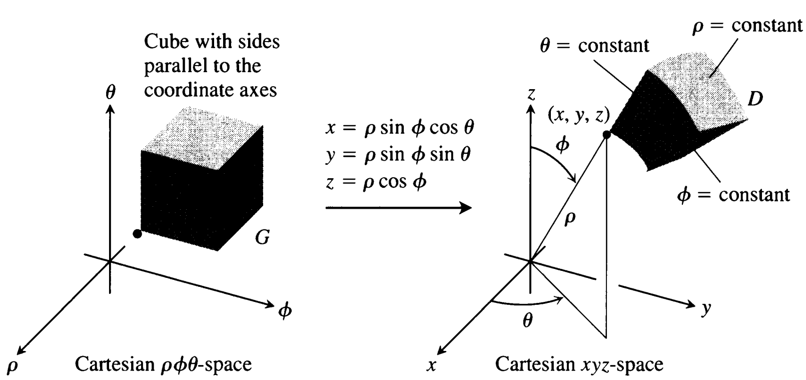
$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2\sin\phi \tag{10}$$

(Exercise 17). The corresponding version of Eq. (7) is

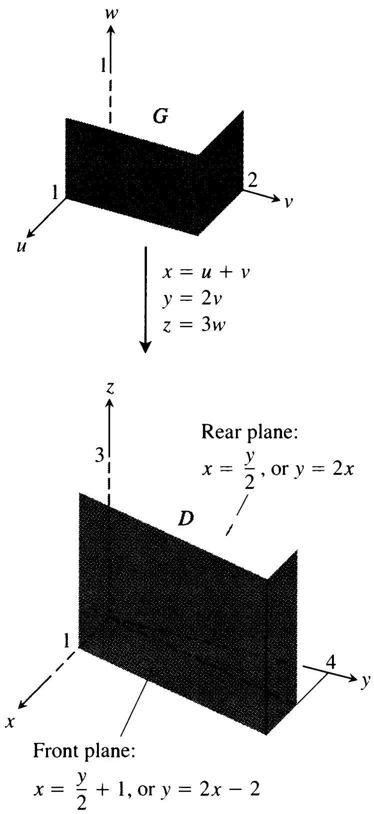
$$\iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(\rho, \phi, \theta)|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta. \tag{11}$$

We can drop the absolute value signs because $\sin\phi$ is never negative.

Here is an example of another substitution.



13.52 The equations $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ transform G into D.



13.53 The equations $x = u + v, y = 2v$, and $z = 3w$ transform G into D. Reversing the transformation by the equations $u = (2x - y)/2, v = y/2$, and $w = z/3$ transforms D into G. See Example 3.

EXAMPLE 3 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz$$

by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3 \tag{12}$$

and integrating over an appropriate region in uvw-space.

Solution We sketch the region D of integration in xyz-space and identify its boundaries (Fig. 13.53). In this case, the bounding surfaces are planes.

To apply Eq. (7), we need to find the corresponding uvw-region G and the Jacobian of the transformation. To find them, we first solve Eqs. (12) for x, y, and z in terms of u, v, and w. Routine algebra gives

$$x = u + v, \qquad y = 2v, \qquad z = 3w. \tag{13}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D:

xyz-equations for the boundary of D	Corresponding uvw-equations for the boundary of G	Simplified uvw-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Eqs. (13), is

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Eq. (7):

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + w)\,|J(u, v, w)|\,du\,dv\,dw$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\,du\,dv\,dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw\right]_0^1 dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w\right) dv\,dw = 6 \int_0^1 \left[\frac{v}{2} + vw\right]_0^2 dw = 6 \int_0^1 (1 + 2w)\,dw$$

$$= 6\Big[w + w^2\Big]_0^1 = 6(2) = 12.$$

❑

Carl Gustav Jacob Jacobi

Jacobi (1804–1851), one of nineteenth-century Germany's most accomplished scientists, developed the theory of determinants and transformations into a powerful tool for evaluating multiple integrals and solving differential equations. He also applied transformation methods to study nonelementary integrals like the ones that arise in the calculation of arc length. Like Euler, Jacobi was a prolific writer and an even more prolific calculator and worked in a variety of mathematical and applied fields.

Exercises 13.7

Transformations of Coordinates

1. a) Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the xy-plane. Sketch the transformed region in the uv-plane.

2. a) Solve the system

$$u = x + 2y, \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

3. a) Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

4. a) Solve the system

$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

5. Find the Jacobian $\partial(x, y)/\partial(u, v)$ for the transformation

a) $x = u \cos v, \quad y = u \sin v$

b) $x = u \sin v, \quad y = u \cos v.$

6. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a) $x = u \cos v, \quad y = u \sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4).$

Double Integrals

7. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

8. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

9. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -\frac{3}{2}x + 1$, $y = -\frac{3}{2}x + 3$, $y = -\frac{1}{4}x$, and $y = -\frac{1}{4}x + 1$.

10. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y)\, dx\, dy.$$

11. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

12. a) Find the Jacobian of the transformation $x = u$, $y = uv$, and sketch the region G: $1 \le u \le 2$, $1 \le uv \le 2$ in the uv-plane.

b) Then use Eq. (1) to transform the integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

into an integral over G, and evaluate both integrals.

13. A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy-plane. Find the first moment of the plate about the origin. (*Hint:* Use the transformation $x = ar \cos \theta$, $y = br \sin \theta$.)

14. The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

15. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3}\int_y^{2-2y}(x+2y)\,e^{(y-x)}\,dx\,dy$$

by first writing it as an integral over a region G in the uv-plane.

16. Use the transformation $x=u+(1/2)v$, $y=v$ to evaluate the integral

$$\int_0^2\int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

by first writing it as an integral over a region G in the uv-plane.

Triple Integrals

17. Evaluate the determinant in Eq. (10) to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is $\rho^2\sin\phi$.

18. Evaluate the integral in Example 3 by integrating with respect to x, y, and z.

19. Find the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(*Hint:* Let $x=au$, $y=bv$, and $z=cw$. Then find the volume of an appropriate region in uvw-space.)

20. Evaluate

$$\iiint |xyz|\,dx\,dy\,dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$$

(*Hint:* Let $x=au$, $y=bv$, and $z=cw$. Then integrate over an appropriate region in uvw-space.)

21. Let D be the region in xyz-space defined by the inequalities

$$1\le x\le 2,\qquad 0\le xy\le 2,\qquad 0\le z\le 1.$$

Evaluate

$$\iiint_D (x^2y+3xyz)\,dx\,dy\,dz$$

by applying the transformation

$$u=x,\qquad v=xy,\qquad w=3z$$

and integrating over an appropriate region G in uvw-space.

22. Assuming the result that the center of mass of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semi-ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)\le 1$, $z\ge 0$, lies on the z-axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)

Single Integrals

23. *Substitutions in single integrals.* How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Chapter 13 Questions to Guide Your Review

1. Define the double integral of a function of two variables over a bounded region in the coordinate plane.
2. How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.
3. How are double integrals used to calculate areas, average values, masses, moments, centers of mass, and radii of gyration? Give examples.
4. How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.
5. Define the triple integral of a function $f(x, y, z)$ over a bounded region in space.
6. How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.
7. How are triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, centers of mass, and radii of gyration? Give examples.
8. How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?
9. How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.
10. How are substitutions in double integrals pictured as transformations of two-dimensional regions? Give a sample calculation.
11. How are substitutions in triple integrals pictured as transformations of three-dimensional regions? Give a sample calculation.

CHAPTER 13 PRACTICE EXERCISES

Planar Regions of Integration

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy$

2. $\displaystyle\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx$

3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$

4. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} x\,y\,dx\,dy$

Reversing the Order of Integration

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\displaystyle\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy$

6. $\displaystyle\int_0^1\int_{x^2}^{x} \sqrt{x}\,dy\,dx$

7. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy$

8. $\displaystyle\int_0^2\int_0^{4-x^2} 2x\,dy\,dx$

Evaluating Double Integrals

Evaluate the integrals in Exercises 9–12.

9. $\displaystyle\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy$

10. $\displaystyle\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy$

11. $\displaystyle\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$

12. $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin\pi x^2}{x^2}\,dx\,dy$

Areas and Volumes

13. Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy-plane.

14. Find the area of the "triangular" region in the xy-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.

15. Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

16. Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy-plane.

Average Values

Find the average value of $f(x, y) = xy$ over the regions in Exercises 17 and 18.

17. The square bounded by the lines $x = 1$, $y = 1$ in the first quadrant

18. The quarter circle $x^2 + y^2 \le 1$ in the first quadrant

Masses and Moments

19. Find the centroid of the "triangular" region bounded by the lines $x = 2$, $y = 2$ and the hyperbola $xy = 2$ in the xy-plane.

20. Find the centroid of the region between the parabola $x + y^2 - 2y = 0$ and the line $x + 2y = 0$ in the xy-plane.

21. Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$, bounded by the y-axis and the lines $y = 2x$ and $y = 4$ in the xy-plane.

22. Find the polar moment of inertia about the center of a thin rectangular sheet of constant density $\delta = 1$ bounded by the lines

a) $x = \pm 2$, $y = \pm 1$ in the xy-plane
b) $x = \pm a$, $y = \pm b$ in the xy-plane.

(*Hint:* Find I_x. Then use the formula for I_x to find I_y and add the two to find I_0.)

23. Find the moment of inertia and radius of gyration about the x-axis of a thin plate of constant density δ covering the triangle with vertices (0, 0), (3, 0), and (3, 2) in the xy-plane.

24. Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes of a thin plate bounded by the line $y = x$ and the parabola $y = x^2$ in the xy-plane if the density is $\delta(x, y) = x + 1$.

25. Find the mass and first moments about the coordinate axes of a thin square plate bounded by the lines $x = \pm 1$, $y = \pm 1$ in the xy-plane if the density is $\delta(x, y) = x^2 + y^2 + 1/3$.

26. Find the moment of inertia and radius of gyration about the x-axis of a thin triangular plate of constant density δ whose base lies along the interval $[0, b]$ on the x-axis and whose vertex lies on the line $y = h$ above the x-axis. As you will see, it does not matter where on the line this vertex lies. All such triangles have the same moment of inertia and radius of gyration.

Polar Coordinates

Evaluate the integrals in Exercises 27 and 28 by changing to polar coordinates.

27. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{(1+x^2+y^2)^2}$

28. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$

29. Find the centroid of the region in the polar coordinate plane defined by the inequalities $0 \le r \le 3$ and $-\pi/3 \le \theta \le \pi/3$.

30. Find the centroid of the region in the first quadrant bounded by the rays $\theta = 0$ and $\theta = \pi/2$ and the circles $r = 1$ and $r = 3$.

31. a) Find the centroid of the region in the polar coordinate plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

b) CALCULATOR Sketch the region and show the centroid in your sketch.

32. a) Find the centroid of the plane region defined by the polar coordinate inequalities $0 \le r \le a$, $-\alpha \le \theta \le \alpha$ $(0 < \alpha \le \pi)$. How does the centroid move as $\alpha \to \pi^-$?

b) CALCULATOR Sketch the region for $\alpha = 5\pi/6$ and show the centroid in your sketch.

33. Integrate the function $f(x, y) = 1/(1 + x^2 + y^2)^2$ over the region enclosed by one loop of the lemniscate $(x^2 + y^2)^2 - (x^2 - y^2) = 0$.

34. Integrate $f(x, y) = 1/(1 + x^2 + y^2)^2$ over

a) the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, \sqrt{3})$;

b) the first quadrant of the xy-plane.

Triple Integrals in Cartesian Coordinates

Evaluate the integrals in Exercises 35–38.

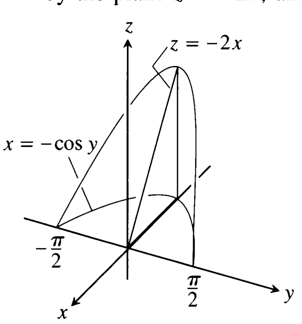
35. $$\int_0^\pi \int_0^\pi \int_0^\pi \cos(x + y + z)\, dx\, dy\, dz$$

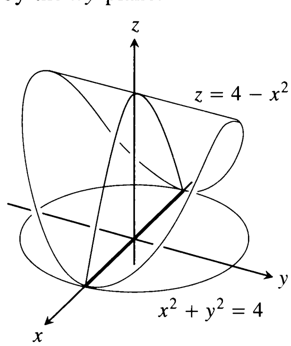
36. $$\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dz\, dy\, dx$$

37. $$\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z)\, dz\, dy\, dx$$

38. $$\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\, dy\, dz\, dx$$

39. Find the volume of the wedge-shaped region enclosed on the side by the cylinder $x = -\cos y$, $-\pi/2 \le y \le \pi/2$, on the top by the plane $z = -2x$, and below by the xy-plane.





40. Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy-plane.

41. Find the average value of $f(x, y, z) = 30xz\sqrt{x^2 + y}$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 3$, $z = 1$.

42. Find the average value of ρ over the solid sphere $\rho \le a$ (spherical coordinates).

Cylindrical and Spherical Coordinates

43. Convert

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\, dz\, r\, dr\, d\theta, \quad r \ge 0$$

to (a) rectangular coordinates with the order of integration $dz\, dx\, dy$, and (b) spherical coordinates. Then (c) evaluate one of the integrals.

44. (a) Convert to cylindrical coordinates. Then (b) evaluate the new integral.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{(x^2+y^2)} 21\, xy^2\, dz\, dy\, dx$$

45. (a) Convert to spherical coordinates. Then (b) evaluate the new integral.

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 dz\, dy\, dx$$

46. Write an iterated triple integral for the integral of $f(x, y, z) = 6 + 4y$ over the region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes in (a) rectangular coordinates, (b) cylindrical coordinates, (c) spherical coordinates. Then (d) find the integral of f by evaluating one of the triple integrals.

47. Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3 \sin\theta \cos\theta\, z^2\, dz\, dr\, d\theta.$$

Arrange the order of integration to be z first, then y, then x.

48. The volume of a solid is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx.$$

a) Describe the solid by giving equations for the surfaces that form its boundary.

b) Convert the integral to cylindrical coordinates but do not evaluate the integral.

49. Let D be the smaller spherical cap cut from a solid ball of radius 2 by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in (a) rectangular, (b) cylindrical, and (c) spherical coordinates. *Do not evaluate the integrals.*

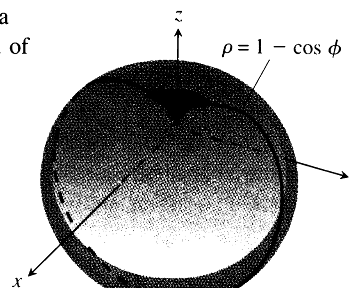
50. Express the moment of inertia I_z of the solid hemisphere bounded below by the plane $z = 0$ and above by the sphere $x^2 + y^2 + z^2 = 1$ as an iterated integral in (a) rectangular, (b) cylindrical, and (c) spherical coordinates. *Do not evaluate the integrals.*

51. *Spherical vs. cylindrical coordinates.* Triple integrals involving spherical shapes do not always require spherical coordinates for convenient evaluation. Some calculations may be accomplished more easily with cylindrical coordinates. As a case in point, find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the plane $z = 2$ by using (a) cylindrical coordinates, (b) spherical coordinates.

52. Find the moment of inertia about the z-axis of a solid of constant density $\delta = 1$ that is bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \pi/3$ (spherical coordinates).

53. Find the moment of inertia of a solid of constant density δ bounded by two concentric spheres of radii a and b $(a < b)$ about a diameter.

54. Find the moment of inertia about the z-axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$.



Substitutions

55. Show that if $u = x - y$ and $v = y$, then

$$\int_0^\infty \int_0^x e^{-sx} f(x-y, y)\, dy\, dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v)\, du\, dv.$$

56. What relationship must hold between the constants a, b, and c to make

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(ax^2+2bxy+cy^2)}\, dx\, dy = 1?$$

(*Hint:* Let $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$, where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$. Then $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)

CHAPTER 13 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Volumes

1. The base of a sand pile covers the region in the xy-plane that is bounded by the parabola $x^2 + y = 6$ and the line $y = x$. The height of the sand above the point (x, y) is x^2. Express the volume of sand as (a) a double integral, (b) a triple integral. Then (c) find the volume.

2. A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.

3. Find the volume of the portion of the solid cylinder $x^2 + y^2 \le 1$ that lies between the planes $z = 0$ and $x + y + z = 2$.

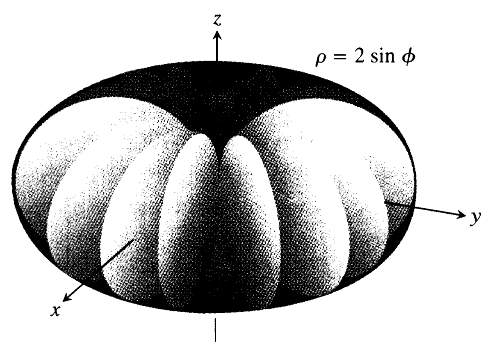
4. Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

5. Find the volume of the region bounded above by the paraboloid $z = 3 - x^2 - y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$.

6. Find the volume of the region enclosed by the spherical coordinate surface $\rho = 2\sin\phi$ (Fig. 13.54).

7. A circular cylindrical hole is bored through a solid sphere, the axis of the hole being a diameter of the sphere. The volume of the remaining solid is

$$V = 2\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta.$$



13.54 The surface in Exercise 6.

a) Find the radius of the hole and the radius of the sphere.
b) Evaluate the integral.

8. Find the volume of material cut from the solid sphere $r^2 + z^2 \le 9$ by the cylinder $r = 3\sin\theta$.

9. Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

10. Find the volume of the region in the first octant that lies between the cylinders $r = 1$ and $r = 2$ and that is bounded below by the xy-plane and above by the surface $z = xy$.

Changing the Order of Integration

In Exercises 11 and 12, sketch the region of integration and write an equivalent iterated integral with the order of integration reversed.

11. $\displaystyle\int_0^1 \int_{x^2}^{x} f(x, y)\, dy\, dx$ **12.** $\displaystyle\int_0^4 \int_y^{2\sqrt{y}} f(x, y)\, dx\, dy$

13. Evaluate the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx.$$

(*Hint:* Use the relation

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\, dy$$

to form a double integral and evaluate the integral by changing the order of integration.)

14. a) Show, by changing to polar coordinates, that

$$\int_0^{a \sin\beta} \int_{y \cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2 + y^2)\, dx\, dy = a^2\beta\left(\ln a - \frac{1}{2}\right),$$

where $a > 0$ and $0 < \beta < \pi/2$.

b) Rewrite the Cartesian integral with the order of integration reversed.

15. By changing the order of integration, show that the following double integral can be reduced to a single integral:

$$\int_0^x \int_0^u e^{m(x-t)} f(t)\, dt\, du = \int_0^x (x - t)e^{m(x-t)} f(t)\, dt.$$

Similarly, it can be shown that

$$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\, dt\, du\, dv = \int_0^x \frac{(x - t)^2}{2} e^{m(x-t)} f(t)\, dt.$$

16. Sometimes a multiple integral with variable limits can be changed into one with constant limits. By changing the order of integration, show that

$$\int_0^1 f(x)\left(\int_0^x g(x - y) f(y)\, dy\right) dx$$
$$= \int_0^1 f(y)\left(\int_y^1 g(x - y) f(x)\, dx\right) dy$$
$$= \frac{1}{2}\int_0^1 \int_0^1 g(|x - y|) f(x)\, f(y)\, dx\, dy.$$

Masses and Moments

17. A thin plate of constant density is to occupy the triangular region in the first quadrant of the xy-plane having vertices (0, 0), $(a, 0)$, and $(a, 1/a)$. What value of a will minimize the plate's polar moment of inertia about the origin?

18. Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y-axis and the lines $y = 2x$ and $y = 4$ in the xy-plane.

19. Find the centroid of the region in the polar coordinate plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

20. Find the centroid of the boomerang-shaped region between the parabolas $y^2 = -4(x - 1)$ and $y^2 = -2(x - 2)$ in the xy-plane.

21. The counterweight of a flywheel of constant density 1 has the form of the smaller segment cut from a circle of radius a by a chord at a distance b from the center ($b < a$). Find the mass of the counterweight and its polar moment of inertia about the center of the wheel.

22. Find the radii of gyration about the x- and y-axes of a thin plate of density $\delta = 1$ enclosed by one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

23. A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z-axis if the density is (a) $\delta(r, \theta, z) = z$; (b) $\delta(r, \theta, z) = r$.

24. A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z-axis if the density is (a) $\delta(r, \theta, z) = z$; (b) $\delta(r, \theta, z) = z^2$.

25. Use spherical coordinates to find the centroid of a solid hemisphere of radius a.

26. Find the moment of inertia and radius of gyration of a solid sphere of radius a and density $\delta = 1$ about a diameter of the sphere.

Theory and Applications

27. Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\, dy\, dx,$$

where a and b are positive numbers and

$$\max(b^2x^2, a^2y^2) = \begin{cases} b^2x^2 & \text{if} \quad b^2x^2 \ge a^2y^2 \\ a^2y^2 & \text{if} \quad b^2x^2 < a^2y^2. \end{cases}$$

28. Show that

$$\iint \frac{\partial^2 F(x, y)}{\partial x\, \partial y}\, dx\, dy$$

over the rectangle $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, is

$$F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).$$

29. Suppose that $f(x, y)$ can be written as a product $f(x, y) = F(x)G(y)$ of a function of x and a function of y. Then the integral of f over the rectangle R: $a \le x \le b$, $c \le y \le d$ can be evaluated as a product as well, by the formula

$$\iint_R f(x, y)\, dA = \left(\int_a^b F(x)\, dx\right)\left(\int_c^d G(y)\, dy\right). \tag{1}$$

The argument is that

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_a^b F(x) G(y)\, dx \right) dy \quad \text{(i)}$$

$$= \int_c^d \left(G(y) \int_a^b F(x)\, dx \right) dy \quad \text{(ii)}$$

$$= \int_c^d \left(\int_a^b F(x)\, dx \right) G(y)\, dy \quad \text{(iii)}$$

$$= \left(\int_a^b F(x)\, dx \right) \int_c^d G(y)\, dy. \quad \text{(iv)}$$

a) Give reasons for steps (i)–(iv).

When it applies, Eq. (1) can be a time saver. Use it to evaluate the following integrals.

b) $\displaystyle\int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y\, dy\, dx$

c) $\displaystyle\int_1^2 \int_{-1}^1 \frac{x}{y^2}\, dx\, dy$

30. Let $D_{\mathbf{u}} f$ denote the derivative of $f(x, y) = (x^2 + y^2)/2$ in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$.

a) Find the average value of $D_{\mathbf{u}} f$ over the triangular region cut from the first quadrant by the line $x + y = 1$.

b) Show in general that the average value of $D_{\mathbf{u}} f$ over a region in the xy-plane is the value of $D_{\mathbf{u}} f$ at the centroid of the region.

31. *The value of* $\Gamma(1/2)$. As we saw in Additional Exercises 49 and 50 in Chapter 7, the gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

extends the factorial function from the nonnegative integers to other real values. Of particular interest in the theory of differential equations is the number

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{(1/2)-1} e^{-t}\, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt. \quad (2)$$

a) If you have not yet done Exercise 37 in Section 13.3, do it now to show that

$$I = \int_0^\infty e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}.$$

b) Substitute $y = \sqrt{t}$ in Eq. (2) to show that $\Gamma(1/2) = 2I = \sqrt{\pi}$.

32. The electrical charge distribution on a circular plate of radius R meters is $\sigma(r, \theta) = kr(1 - \sin\theta)$ coulomb/m^2 (k a constant). Integrate σ over the plate to find the total charge Q.

33. *A parabolic rain gauge.* A bowl is in the shape of the graph of $z = x^2 + y^2$ from $z = 0$ to $z = 10$ in. You plan to calibrate the bowl to make it into a rain gauge. What height in the bowl would correspond to 1 in. of rain? 3 in. of rain?

34. *Water in a satellite dish.* A parabolic satellite dish is 2 m wide and 1/2 m deep. Its axis of symmetry is tilted 30 degrees from the vertical.

a) Set up, but do not evaluate, a triple integral in rectangular coordinates that gives the amount of water the satellite dish will hold. (*Hint:* Put your coordinate system so that the satellite dish is in "standard position" and the plane of the water level is slanted.) (*Caution:* The limits of integration are not "nice.")

b) What would be the smallest tilt of the satellite dish so that it holds no water?

35. *Cylindrical shells.* In Section 5.4, we learned how to find the volume of a solid of revolution using the shell method, namely if the region between the curve $y = f(x)$ and the x-axis from a to b $(0 < a < b)$ is revolved about the y-axis the volume of the resulting solid is $\int_a^b 2\pi x f(x)\, dx$. Prove that finding volumes by using triple integrals gives the same result. (*Hint:* Use cylindrical coordinates with the roles of y and z changed.)

36. *An infinite half-cylinder.* Let D be the interior of the infinite right circular half-cylinder of radius 1 with its single-end face suspended 1 unit above the origin and its axis the ray from $(0, 0, 1)$ to ∞. Use cylindrical coordinates to evaluate

$$\iiint_D z(r^2 + z^2)^{-5/2}\, dV.$$

37. *Hypervolume.* We have learned that $\int_a^b 1\, dx$ is the length of the interval $[a, b]$ on the number line (one-dimensional space), $\iint_R 1\, dA$ is the area of region R in the xy-plane (two-dimensional space), and $\iiint_D 1\, dV$ is the volume of the region D in three-dimensional space (xyz-space). We could continue: If Q is a region in 4-space ($xyzw$-space), then $\iiiint_Q 1\, dV$ is the "hypervolume" of Q. Use your generalizing abilities and a Cartesian coordinate system of 4-space to find the hypervolume inside the unit 4-sphere $x^2 + y^2 + z^2 + w^2 = 1$.

CHAPTER 14

Integration in Vector Fields

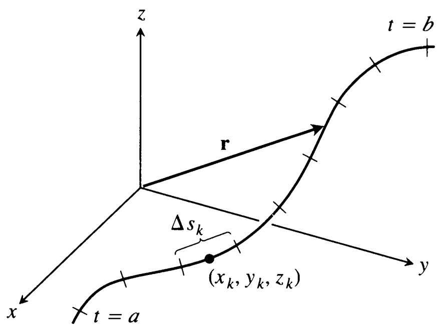
OVERVIEW This chapter treats integration in vector fields. The mathematics in this chapter is the mathematics that is used to describe the properties of electromagnetism, explain the flow of heat in stars, and calculate the work it takes to put a satellite in orbit.

14.1 Line Integrals

When a curve $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$, $a \le t \le b$, passes through the domain of a function $f(x, y, z)$ in space, the values of f along the curve are given by the composite function $f(g(t), h(t), k(t))$. If we integrate this composite with respect to arc length from $t = a$ to $t = b$, we calculate the so-called line integral of f along the curve. Despite the three-dimensional geometry, the line integral is an ordinary integral of a real-valued function over an interval of real numbers.

The importance of line integrals lies in their application. These are the integrals with which we calculate the work done by variable forces along paths in space and the rates at which fluids flow along curves and across boundaries.

Definitions and Notation



14.1 The curve $\mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$, partitioned into small arcs from $t = a$ to $t = b$. The length of a typical subarc is Δs_k.

Suppose that $f(x, y, z)$ is a function whose domain contains the curve $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$, $a \le t \le b$. We partition the curve into a finite number of subarcs (Fig. 14.1). The typical subarc has length Δs_k. In each subarc we choose a point (x_k, y_k, z_k) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k. \tag{1}$$

If f is continuous and the functions g, h, and k have continuous first derivatives, then the sums in (1) approach a limit as n increases, and the lengths Δs_k approach zero. We call this limit the **integral of f over the curve from a to b.** If the curve is denoted by a single letter, C for example, the notation for the integral is

$$\int_C f(x, y, z)\,ds \qquad \text{“The integral of } f \text{ over } C\text{”} \tag{2}$$

Evaluation for Smooth Curves

If $\mathbf{r}(t)$ is smooth for $a \leq t \leq b$ ($\mathbf{v} = d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$), we can use the equation

$$s(t) = \int_a^t |\mathbf{v}(\tau)|\, d\tau \qquad \text{Eq. (4) of Section 11.3, with } t_0 = a$$

to express ds in Eq. (2) as $ds = |\mathbf{v}(t)|\, dt$. A theorem from advanced calculus says that we can then evaluate the integral of f over C as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))|\mathbf{v}(t)|\, dt.$$

This formula will evaluate the integral correctly no matter what parametrization we use, as long as the parametrization is smooth.

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

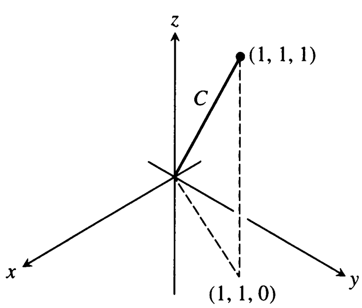
1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}, \qquad a \leq t \leq b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt. \tag{3}$$

Notice that if f has the constant value 1, then the integral of f over C gives the length of C.



14.2 The integration path in Example 1.

EXAMPLE 1 Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin and the point (1, 1, 1) (Fig. 14.2).

Solution We choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}, \qquad 0 \leq t \leq 1.$$

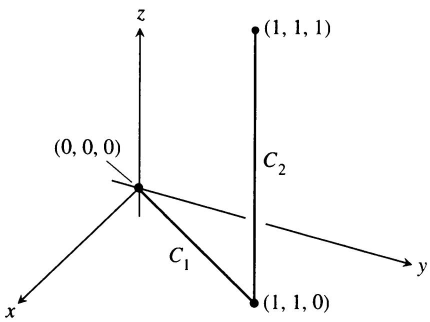
The components have continuous first derivatives and $|\mathbf{v}(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of f over C is

$$\begin{aligned}
\int_C f(x, y, z)\, ds &= \int_0^1 f(t, t, t)\left(\sqrt{3}\right) dt \qquad \text{Eq. (3)} \\
&= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt \\
&= \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0.
\end{aligned}$$

❑

Additivity

Line integrals have the useful property that if a curve C is made by joining a finite number of curves $C_1, C_2, \ldots, C_n$ end to end, then the integral of a function over



14.3 The path of integration in Example 2.

C is the sum of the integrals over the curves that make it up:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds. \tag{4}$$

EXAMPLE 2 Figure 14.3 shows another path from the origin to (1, 1, 1), the union of line segments C_1 and C_2. Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

Solution We choose the simplest parametrizations for C_1 and C_2 we can think of, checking the lengths of the velocity vectors as we go along:

$$C_1:\quad \mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j},\quad 0 \le t \le 1;\quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2:\quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 1;\quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

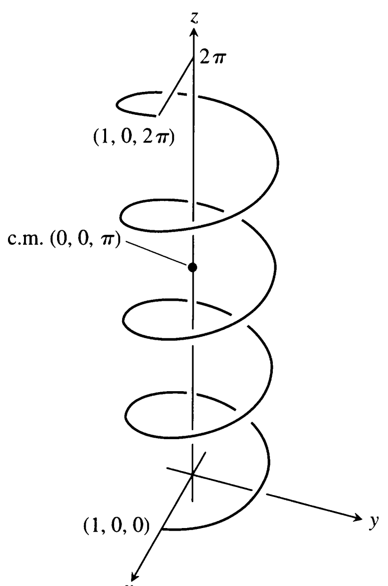
With these parametrizations we find that

$$\begin{aligned}
\int_{C_1 \cup C_2} f(x, y, z)\,ds &= \int_{C_1} f(x, y, z)\,ds + \int_{C_2} f(x, y, z)\,ds && \text{Eq. (4)}\\
&= \int_0^1 f(t, t, 0)\,\sqrt{2}\,dt + \int_0^1 f(1, 1, t)(1)\,dt && \text{Eq. (3)}\\
&= \int_0^1 (t - 3t^2 + 0)\,\sqrt{2}\,dt + \int_0^1 (1 - 3 + t)(1)\,dt\\
&= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}.
\end{aligned}$$ ❑

Notice three things about the integrations in Examples 1 and 2. First, as soon as the components of the appropriate curve were substituted into the formula for f, the integration became a standard integration with respect to t. Second, the integral of f over $C_1 \cup C_2$ was obtained by integrating f over each section of the path and adding the results. Third, the integrals of f over C and $C_1 \cup C_2$ had different values. For most functions, the value of the integral along a path joining two points changes if you change the path between them. For some functions, however, the value remains the same, as we will see in Section 14.3.

Mass and Moment Calculations

We treat coil springs and wires like masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ (mass per unit length). The spring's or wire's mass, center-of-mass, and moments are then calculated with the formulas in Table 14.1, on the following page. The formulas also apply to thin rods.



14.4 The helical spring in Example 3.

EXAMPLE 3 A coil spring lies along the helix

$$\mathbf{r}(t) = (\cos 4t)\,\mathbf{i} + (\sin 4t)\,\mathbf{j} + t\,\mathbf{k},\qquad 0 \le t \le 2\pi.$$

The spring's density is a constant, $\delta = 1$. Find the spring's mass and center of mass, and its moment of inertia and radius of gyration about the z-axis.

Solution We sketch the spring (Fig. 14.4). Because of the symmetries involved, the center of mass lies at the point $(0, 0, \pi)$ on the z-axis.

Table 14.1 Mass and moment formulas for coil springs, thin rods, and wires lying along a smooth curve C in space

Mass: $M = \int_C \delta(x, y, z)\, ds$

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\,\delta\, ds$$

$$I_z = \int_C (x^2 + y^2)\,\delta\, ds, \qquad I_L = \int_C r^2\,\delta\, ds$$

$r(x, y, z) =$ distance from point (x, y, z) to line L

Radius of gyration about a line L: $R_L = \sqrt{I_L/M}$

For the remaining calculations, we first find $|\mathbf{v}(t)|$:

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{(-4 \sin 4t)^2 + (4 \cos 4t)^2 + 1} = \sqrt{17}.$$

We then evaluate the formulas from Table 14.1 using Eq. (3):

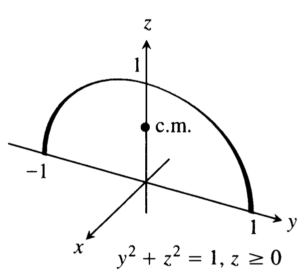
$$M = \int_{\text{Helix}} \delta\, ds = \int_0^{2\pi} (1)\sqrt{17}\, dt = 2\pi\sqrt{17}$$

$$I_z = \int_{\text{Helix}} (x^2 + y^2)\delta\, ds = \int_0^{2\pi} (\cos^2 4t + \sin^2 4t)(1)\sqrt{17}\, dt$$

$$= \int_0^{2\pi} \sqrt{17}\, dt = 2\pi\sqrt{17}$$

$$R_z = \sqrt{I_z/M} = \sqrt{2\pi\sqrt{17}/(2\pi\sqrt{17})} = 1.$$

Notice that the radius of gyration about the z-axis is the radius of the cylinder around which the helix winds. ❑



14.5 Example 4 shows how to find the center of mass of a circular arch of variable density.

EXAMPLE 4 A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1$, $z \geq 0$, in the yz-plane (Fig. 14.5). Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the yz-plane with its mass distributed symmetrically about the z-axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\,\mathbf{j} + (\sin t)\,\mathbf{k}, \qquad 0 \leq t \leq \pi.$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1.$$

The formulas in Table 14.1 then give

$$M = \int_C \delta\, ds = \int_C (2 - z)\, ds = \int_0^{\pi} (2 - \sin t)\, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\,\delta\, ds = \int_C z(2 - z)\, ds = \int_0^{\pi} (\sin t)(2 - \sin t)\, dt$$

$$= \int_0^{\pi} (2 \sin t - \sin^2 t)\, dt = \frac{8 - \pi}{2}$$

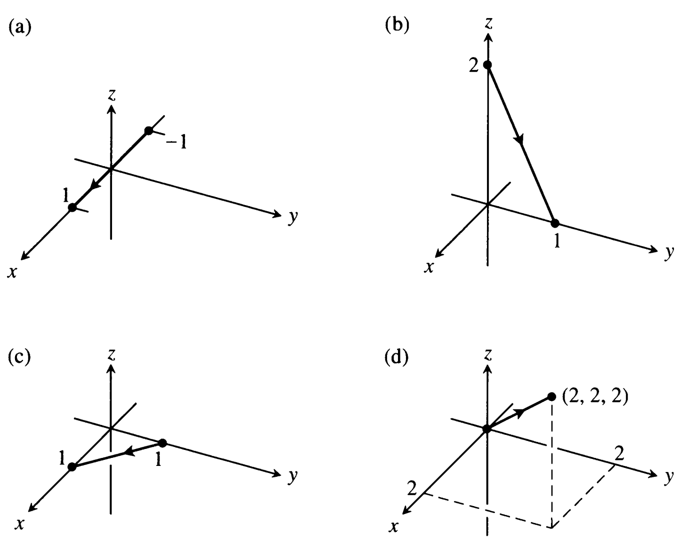
$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

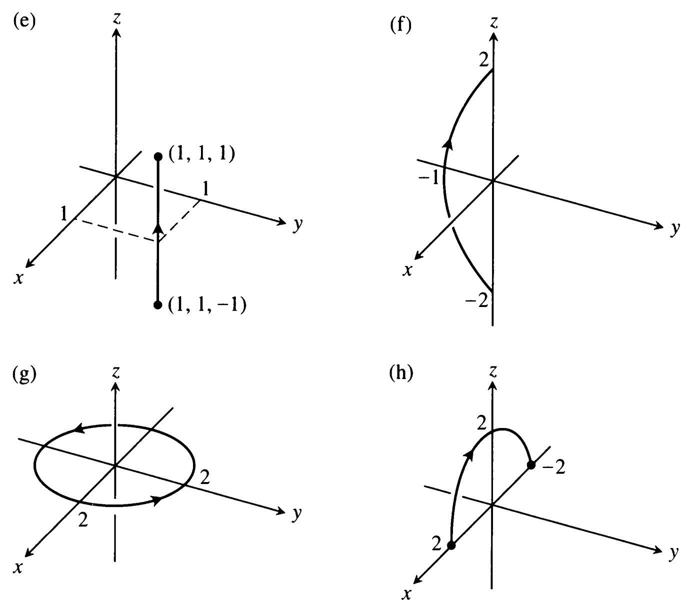
With $\bar{z}$ to the nearest hundredth, the center of mass is (0, 0, 0.57). ❑

Exercises 14.1

Graphs of Vector Equations

Match the vector equations in Exercises 1–8 with the graphs in Fig. 14.6.





14.6 The graphs for Exercises 1–8.

1. $\mathbf{r}(t) = t\,\mathbf{i} + (1 - t)\,\mathbf{j}, \quad 0 \le t \le 1$

2. $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\,\mathbf{k}, \quad -1 \le t \le 1$

3. $\mathbf{r}(t) = (2 \cos t)\,\mathbf{i} + (2 \sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi$

4. $\mathbf{r}(t) = t\,\mathbf{i}, \quad -1 \le t \le 1$

5. $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 2$

6. $\mathbf{r}(t) = t\,\mathbf{j} + (2 - 2t)\,\mathbf{k}, \quad 0 \le t \le 1$

7. $\mathbf{r}(t) = (t^2 - 1)\,\mathbf{j} + 2t\,\mathbf{k}, \quad -1 \le t \le 1$

8. $\mathbf{r}(t) = (2 \cos t)\,\mathbf{i} + (2 \sin t)\,\mathbf{k}, \quad 0 \le t \le \pi$

Evaluating Line Integrals over Space Curves

9. Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t$, $y=(1-t)$, $z=0$, from $(0,1,0)$ to $(1,0,0)$.

10. Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t$, $y=(1-t)$, $z=1$, from $(0,1,1)$ to $(1,0,1)$.

11. Evaluate $\int_C (xy+y+z)\,ds$ along the curve $\mathbf{r}(t)=2t\,\mathbf{i}+t\,\mathbf{j}+(2-2t)\,\mathbf{k}$, $0\le t\le 1$.

12. Evaluate $\int_C \sqrt{x^2+y^2}\,ds$ along the curve $\mathbf{r}(t)=(4\cos t)\,\mathbf{i}+(4\sin t)\,\mathbf{j}+3t\,\mathbf{k}$, $-2\pi\le t\le 2\pi$.

13. Find the line integral of $f(x,y,z)=x+y+z$ over the straight-line segment from $(1,2,3)$ to $(0,-1,1)$.

14. Find the line integral of $f(x,y,z)=\sqrt{3}/(x^2+y^2+z^2)$ over the curve $\mathbf{r}(t)=t\,\mathbf{i}+t\,\mathbf{j}+t\,\mathbf{k}$, $1\le t\le\infty$.

15. Integrate $f(x,y,z)=x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ (Fig. 14.7a) given by

C_1: $\mathbf{r}(t)=t\,\mathbf{i}+t^2\,\mathbf{j}$, $0\le t\le 1$

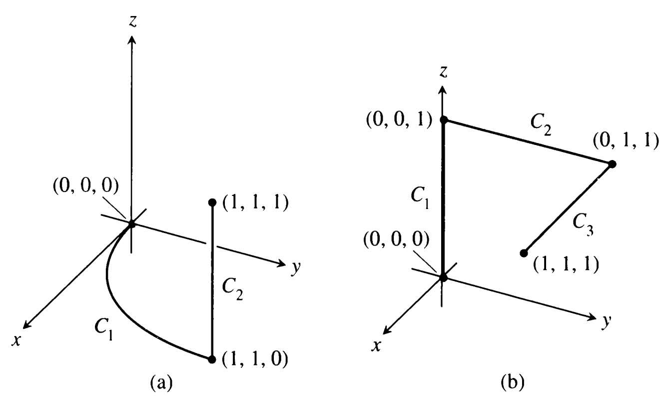
C_2: $\mathbf{r}(t)=\mathbf{i}+\mathbf{j}+t\,\mathbf{k}$, $0\le t\le 1$

16. Integrate $f(x,y,z)=x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ (Fig. 14.7b) given by

C_1: $\mathbf{r}(t)=t\,\mathbf{k}$, $0\le t\le 1$

C_2: $\mathbf{r}(t)=t\,\mathbf{j}+\mathbf{k}$, $0\le t\le 1$

C_3: $\mathbf{r}(t)=t\,\mathbf{i}+\mathbf{j}+\mathbf{k}$, $0\le t\le 1$



14.7 The paths of integration for Exercises 15 and 16.

17. Integrate $f(x,y,z)=(x+y+z)/(x^2+y^2+z^2)$ over the path $\mathbf{r}(t)=t\,\mathbf{i}+t\,\mathbf{j}+t\,\mathbf{k}$, $0<a\le t\le b$.

18. Integrate $f(x,y,z)=-\sqrt{x^2+z^2}$ over the circle

$$\mathbf{r}(t)=(a\cos t)\,\mathbf{j}+(a\sin t)\,\mathbf{k},\ 0\le t\le 2\pi.$$

Line Integrals over Plane Curves

In Exercises 19–22, integrate f over the given curve.

19. $f(x,y)=x^3/y$, C: $y=x^2/2$, $0\le x\le 2$

20. $f(x,y)=(x+y^2)/\sqrt{1+x^2}$, C: $y=x^2/2$ from $(1,1/2)$ to $(0,0)$

21. $f(x,y)=x+y$, C: $x^2+y^2=4$ in the first quadrant from $(2,0)$ to $(0,2)$

22. $f(x,y)=x^2-y$, C: $x^2+y^2=4$ in the first quadrant from $(0,2)$ to $(\sqrt{2},\sqrt{2})$

Mass and Moments

23. Find the mass of a wire that lies along the curve $\mathbf{r}(t)=(t^2-1)\,\mathbf{j}+2t\,\mathbf{k}$, $0\le t\le 1$, if the density is $\delta=(3/2)\,t$.

24. A wire of density $\delta(x,y,z)=15\sqrt{y+2}$ lies along the curve $\mathbf{r}(t)=(t^2-1)\,\mathbf{j}+2t\,\mathbf{k}$, $-1\le t\le 1$. Find its center of mass. Then sketch the curve and center of mass together.

25. Find the mass of a thin wire lying along the curve $\mathbf{r}(t)=\sqrt{2}\,t\,\mathbf{i}+\sqrt{2}\,t\,\mathbf{j}+(4-t^2)\,\mathbf{k}$, $0\le t\le 1$, if the density is (a) $\delta=3t$, (b) $\delta=1$.

26. Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t)=t\,\mathbf{i}+2t\,\mathbf{j}+(2/3)t^{3/2}\,\mathbf{k}$, $0\le t\le 2$, if the density is $\delta=3\sqrt{5+t}$.

27. A circular wire hoop of constant density δ lies along the circle $x^2+y^2=a^2$ in the xy-plane. Find the hoop's moment of inertia and radius of gyration about the z-axis.

28. A slender rod of constant density lies along the line segment $\mathbf{r}(t)=t\,\mathbf{j}+(2-2t)\,\mathbf{k}$, $0\le t\le 1$, in the yz-plane. Find the moments of inertia and radii of gyration of the rod about the three coordinate axes.

29. A spring of constant density δ lies along the helix

$$\mathbf{r}(t)=(\cos t)\,\mathbf{i}+(\sin t)\,\mathbf{j}+t\,\mathbf{k},\ 0\le t\le 2\pi.$$

a) Find I_z and R_z.

b) Suppose you have another spring of constant density δ that is twice as long as the spring in (a) and lies along the helix for $0\le t\le 4\pi$. Do you expect I_z and R_z for the longer spring to be the same as those for the shorter one, or should they be different? Check your predictions by calculating I_z and R_z for the longer spring.

30. A wire of constant density $\delta=1$ lies along the curve

$$\mathbf{r}(t)=(t\cos t)\,\mathbf{i}+(t\sin t)\,\mathbf{j}+(2\sqrt{2}/3)\,t^{3/2}\,\mathbf{k},\quad 0\le t\le 1.$$

Find $\bar{z}$, I_z, and R_z.

31. Find I_x and R_x for the arch in Example 4.

32. Find the center of mass, and the moments of inertia and radii of gyration about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t)=t\,\mathbf{i}+\frac{2\sqrt{2}}{3}t^{3/2}\,\mathbf{j}+\frac{t^2}{2}\,\mathbf{k},\quad 0\le t\le 2,$$

if the density is $\delta=1/(t+1)$.

CAS Explorations and Projects

In Exercises 33–36, use a CAS to perform the following steps to evaluate the line integrals:

a) Find $ds=|\mathbf{v}(t)|\,dt$ for the path $\mathbf{r}(t)=g(t)\,\mathbf{i}+h(t)\,\mathbf{j}+k(t)\,\mathbf{k}$.

b) Express the integrand $f(g(t),h(t),k(t))\,|\mathbf{v}(t)|$ as a function of the parameter t.

c) Evaluate $\int_C f\,ds$ using Eq. (3) in the text.

33. $f(x, y, z) = \sqrt{1 + 30x^2 + 10y}$; $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t^2\,\mathbf{k}$, $0 \le t \le 2$

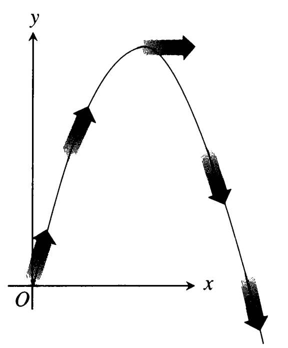
34. $f(x, y, z) = \sqrt{1 + x^3 + 5y^3}$; $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{3}t^2\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 2$

35. $f(x, y, z) = x\sqrt{y} - 3z^2$; $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + 5t\,\mathbf{k}$, $0 \le t \le 2\pi$

36. $f(x, y, z) = \left(1 + \frac{9}{4}z^{1/3}\right)^{1/4}$; $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t^{5/2}\,\mathbf{k}$, $0 \le t \le 2\pi$

14.2 Vector Fields, Work, Circulation, and Flux

When we study physical phenomena that are represented by vectors, we replace integrals over closed intervals by integrals over paths through vector fields. We use such integrals to find the work done in moving an object along a path against a variable force (a vehicle sent into space against Earth's gravitational field) or to find the work done by a vector field in moving an object along a path through the field (the work done by an accelerator in raising the energy of a particle). We also use line integrals to find the rates at which fluids flow along and across curves.



14.8 The velocity vectors $\mathbf{v}(t)$ of a projectile's motion make a vector field along the trajectory.

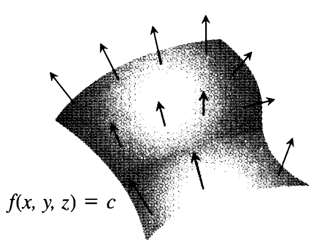
Vector Fields

A **vector field** on a domain in the plane or in space is a function that assigns a vector to each point in the domain. A field of three-dimensional vectors might have a formula like

$$\mathbf{F}(x, y, z) = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}.$$

The field is **continuous** if the **component functions** M, N, and P are continuous, **differentiable** if M, N, and P are differentiable, and so on. A field of two-dimensional vectors might have a formula like

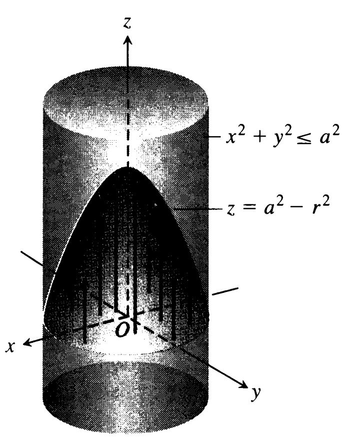
$$\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}.$$



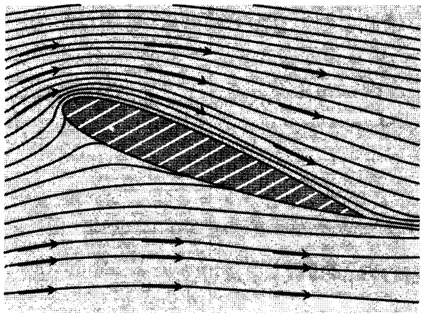
14.9 The field of gradient vectors ∇f on a surface $f(x, y, z) = c$.

If we attach a projectile's velocity vector to each point of the projectile's trajectory in the plane of motion, we have a two-dimensional field defined along the trajectory. If we attach the gradient vector of a scalar function to each point of a level surface of the function, we have a three-dimensional field on the surface. If we attach the velocity vector to each point of a flowing fluid, we have a three-dimensional field defined on a region in space. These and other fields are illustrated in Figs. 14.8–14.16. Some of the illustrations give formulas for the fields as well.

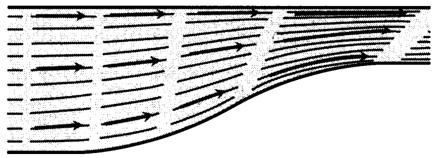
To sketch the fields that had formulas, we picked a representative selection of domain points and sketched the vectors attached to them. Notice the convention that the arrows representing the vectors are drawn with their tails, not their heads, at the points where the vector functions are evaluated. This is different from the way we drew the position vectors of the planets and projectiles in Chapter 11, with their tails at the origin and their heads at the planet's and projectile's locations.



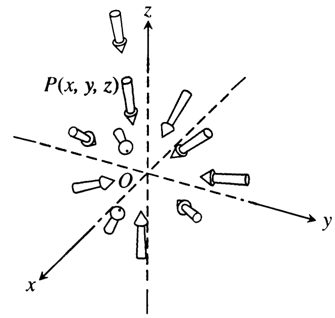
14.10 The flow of fluid in a long cylindrical pipe. The vectors $\mathbf{v} = (a^2 - r^2)\,\mathbf{k}$ inside the cylinder that have their bases in the xy-plane have their tips on the paraboloid $z = a^2 - r^2$.



14.11 Velocity vectors of a flow around an airfoil in a wind tunnel. The streamlines were made visible by kerosene smoke. (Adapted from *NCFMF Book of Film Notes,* 1974, MIT Press with Education Development Center, Inc., Newton, Massachusetts.)

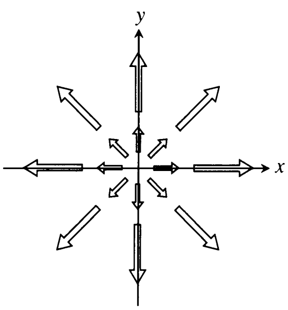


14.12 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length. (Adapted from *NCFMF Book of Film Notes,* 1974, MIT Press with Education Development Center, Inc., Newton, Massachusetts.)

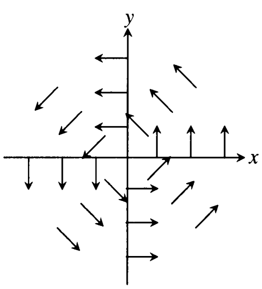


14.13 Vectors in the gravitational field

$$\mathbf{F} = -\frac{GM(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}.$$



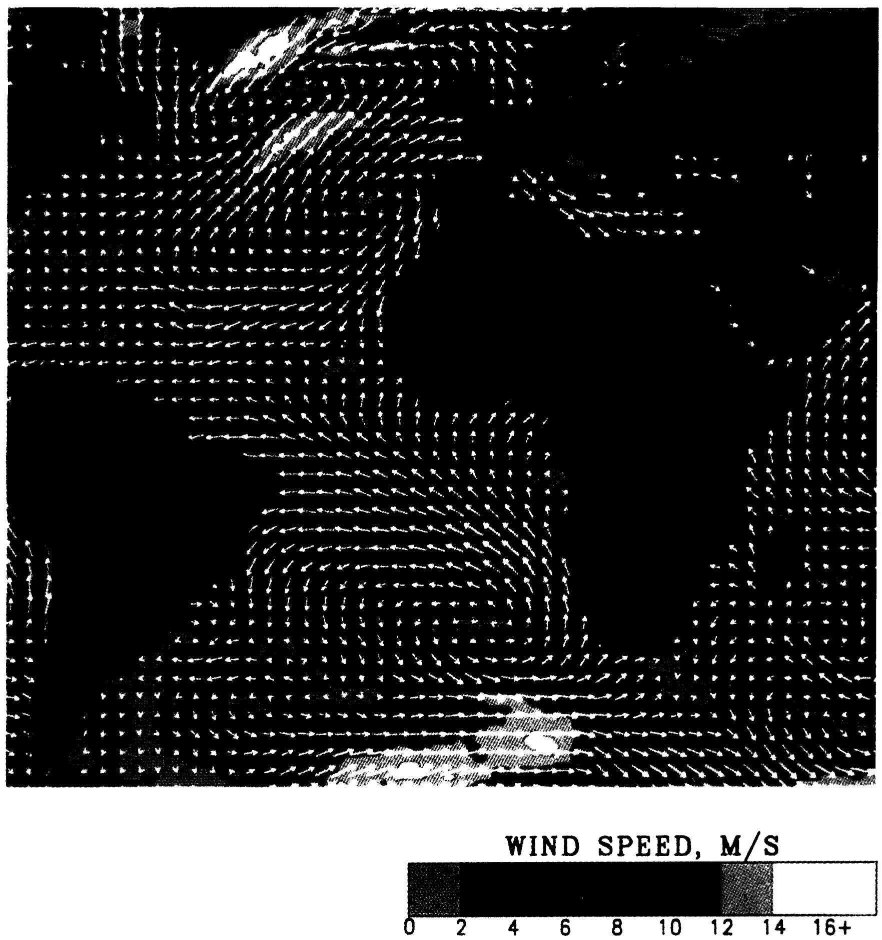
14.14 The radial field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.



14.15 The circumferential or "spin" field of unit vectors

$$\mathbf{F} = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.



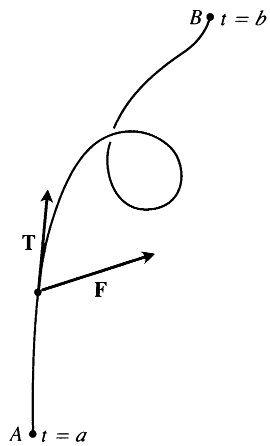
14.16 NASA's *Seasat* used radar during a 3-day period in September 1978 to take 350,000 wind measurements over the world's oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm south of Greenland.

Gradient Fields

Definition

The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$



14.17 The work done by a continuous field **F** over a smooth path $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ from A to B is the integral of $\mathbf{F} \cdot \mathbf{T}$ over the path from $t = a$ to $t = b$.

EXAMPLE 1 Find the gradient field of $f(x, y, z) = xyz$.

Solution The gradient field of f is the field $\mathbf{F} = \nabla f = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. ❑

As we will see in Section 14.3, gradient fields are of special importance in engineering, mathematics, and physics.

The Work Done by a Force over a Curve in Space

Suppose that the vector field $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind) and that

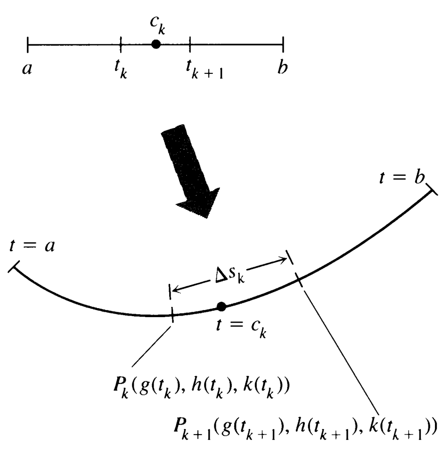
$$\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}, \quad a \le t \le b,$$

is a smooth curve in the region. Then the integral of $\mathbf{F} \cdot \mathbf{T}$, the scalar component of **F** in the direction of the curve's unit tangent vector, over the curve is called the work done by **F** over the curve from a to b (Fig. 14.17).

Definition

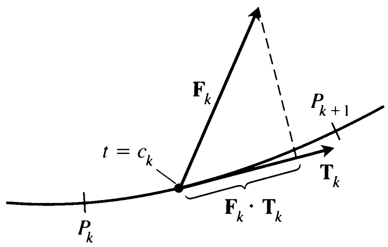
The **work** done by a force $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ over a smooth curve $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$ from $t = a$ to $t = b$ is

$$W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T}\, ds. \tag{1}$$



14.18 Each partition of a parameter interval $a \le t \le b$ induces a partition of the curve $\mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$.

We motivate Eq. (1) with the same kind of reasoning we used in Section 5.8 to derive the formula $W = \int_a^b F(x)\,dx$ for the work done by a continuous force of magnitude $F(x)$ directed along an interval of the x-axis. We divide the curve into short segments, apply the constant-force-times-distance formula for work to approximate the work over each curved segment, add the results to approximate the work over the entire curve, and calculate the work as the limit of the approximating sums as the segments become shorter and more numerous. To find exactly what the limiting integral should be, we partition the parameter interval $I = [a, b]$ in the usual way and choose a point c_k in each subinterval $[t_k, t_{k+1}]$. The partition of I determines ("induces," we say) a partition of the curve, with the point P_k being the tip of the position vector **r** at $t = t_k$ and Δs_k being the length of the curve segment $P_k P_{k+1}$ (Fig. 14.18). If $\mathbf{F}_k$ denotes the value of **F** at the point on the curve corresponding to $t = c_k$, and $\mathbf{T}_k$ denotes the curve's tangent vector at this point, then $\mathbf{F}_k \cdot \mathbf{T}_k$ is the scalar component of **F** in the direction of **T** at $t = c_k$



14.19 An enlarged view of the curve segment P_kP_{k+1} in Fig. 14.18, showing the force vector and unit tangent vector at the point on the curve where $t = c_k$.

(Fig. 14.19). The work done by **F** along the curve segment P_kP_{k+1} will be approximately

$$\begin{pmatrix}\text{force component in}\\ \text{direction of motion}\end{pmatrix} \times \begin{pmatrix}\text{distance}\\ \text{applied}\end{pmatrix} = \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k.$$

The work done by **F** along the curve from $t = a$ to $t = b$ will be approximately

$$\sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k.$$

As the norm of the partition of $[a, b]$ approaches zero, the norm of the induced partition of the curve approaches zero and these sums approach the line integral

$$\int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T}\, ds.$$

The sign of the number we calculate with this integral depends on the direction in which the curve is traversed as t increases. If we reverse the direction of motion, we reverse the direction of **T** and change the sign of $\mathbf{F} \cdot \mathbf{T}$ and its integral.

Notation and Evaluation

Table 14.2 shows six ways to write the work integral in Eq. (1).

Table 14.2 Different ways to write the work integral

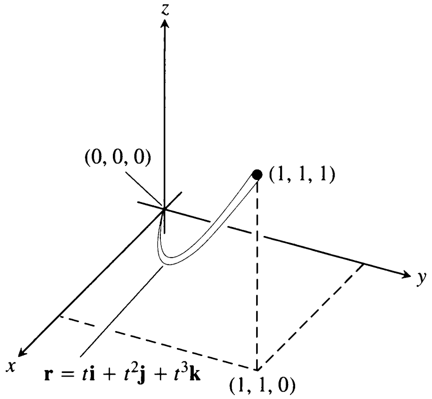
$\mathbf{W} = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_{t=a}^{t=b} \mathbf{F} \cdot d\mathbf{r}$	Compact differential form
$= \int_{a}^{b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$	Expanded to include dt; emphasizes the parameter t and velocity vector $d\mathbf{r}/dt$
$= \int_{a}^{b} \left(M\frac{dg}{dt} + N\frac{dh}{dt} + P\frac{dk}{dt}\right) dt$	Emphasizes the component functions
$= \int_{a}^{b} \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt$	Abbreviates the components of **r**
$= \int_{a}^{b} M dx + N dy + P dz$	dt's canceled; the most common form

Despite their variety, the formulas in Table 14.2 are all evaluated the same way.

How to Evaluate a Work Integral

To evaluate the work integral, take these steps:

1. Evaluate **F** on the curve as a function of the parameter t.
2. Find $d\mathbf{r}/dt$.
3. Dot **F** with $d\mathbf{r}/dt$.
4. Integrate from $t = a$ to $t = b$.



14.20 The curve in Example 2.

EXAMPLE 2 Find the work done by $\mathbf{F} = (y - x^2)\,\mathbf{i} + (z - y^2)\,\mathbf{j} + (x - z^2)\,\mathbf{k}$ over the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \le t \le 1$, from (0, 0, 0) to (1, 1, 1) (Fig. 14.20).

Solution

Step 1: *Evaluate* $\mathbf{F}$ *on the curve.*

$$\begin{aligned}\mathbf{F} &= (y - x^2)\,\mathbf{i} + (z - y^2)\,\mathbf{j} + (x - z^2)\,\mathbf{k}\\ &= \underbrace{(t^2 - t^2)}_{0}\,\mathbf{i} + (t^3 - t^4)\,\mathbf{j} + (t - t^6)\,\mathbf{k}\end{aligned}$$

Step 2: *Find* $d\mathbf{r}/dt$.

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

Step 3: *Dot* $\mathbf{F}$ *with* $d\mathbf{r}/dt$.

$$\begin{aligned}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} &= [(t^3 - t^4)\,\mathbf{j} + (t - t^6)\,\mathbf{k}]\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8\end{aligned}$$

Step 4: *Integrate from* $t = 0$ *to* $t = 1$.

$$\begin{aligned}\text{Work} &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt\\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}\end{aligned}$$

❑

Flow Integrals and Circulation

Instead of being a force field, suppose that $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of $\mathbf{F}\cdot\mathbf{T}$ along a curve in the region gives the fluid's flow along the curve.

Definitions

If $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$, $a \le t \le b$, is a smooth curve in the domain of a continuous velocity field $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$, the **flow** along the curve from $t = a$ to $t = b$ is the integral of $\mathbf{F}\cdot\mathbf{T}$ over the curve from a to b:

$$\text{Flow} = \int_a^b \mathbf{F}\cdot\mathbf{T}\,ds. \tag{2}$$

The integral in this case is called a **flow integral.** If the curve is a closed loop, the flow is called the **circulation** around the curve.

We evaluate flow integrals the same way we evaluate work integrals.

EXAMPLE 3 A fluid's velocity field is $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le \pi/2$.

Solution

Step 1: *Evaluate* $\mathbf{F}$ *on the curve.*

$$\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k} = (\cos t)\,\mathbf{i} + t\,\mathbf{j} + (\sin t)\,\mathbf{k}$$

Step 2: *Find* $d\mathbf{r}/dt$.

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}$$

Step 3: *Find* $\mathbf{F} \cdot (d\mathbf{r}/dt)$.

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1) \\ &= -\sin t \cos t + t \cos t + \sin t\end{aligned}$$

Step 4: *Integrate from* $t = a$ *to* $t = b$.

$$\begin{aligned}\text{Flow} &= \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\,dt \\ &= \left[\frac{\cos^2 t}{2} + t \sin t\right]_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2}\end{aligned}$$

❑

EXAMPLE 4 Find the circulation of the field $\mathbf{F} = (x - y)\,\mathbf{i} + x\,\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$, $0 \le t \le 2\pi$.

Solution

1. On the circle, $\mathbf{F} = (x - y)\,\mathbf{i} + x\,\mathbf{j} = (\cos t - \sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}$.
2. $\dfrac{d\mathbf{r}}{dt} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}$
3. $\mathbf{F} \cdot \dfrac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$
4. $$\begin{aligned}\text{Circulation} &= \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} (1 - \sin t \cos t)\,dt \\ &= \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi} = 2\pi.\end{aligned}$$

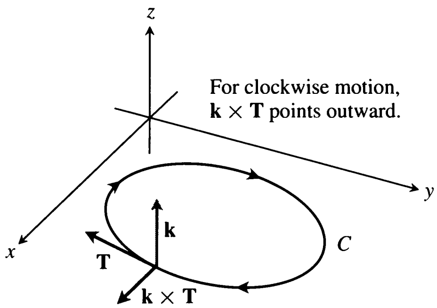
❑

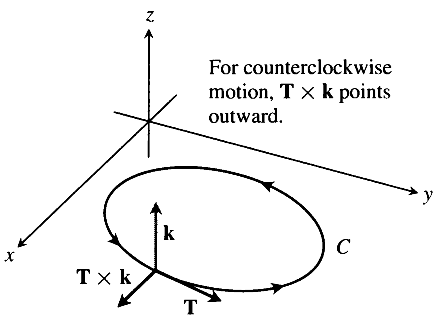
Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy-plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. The value of this integral is called the flux of $\mathbf{F}$ across C. *Flux* is Latin for *flow,* but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ would still be called the flux of the field across C.

Definition

If C is a smooth closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit





14.21 To find an outward unit normal vector for a smooth curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$.

normal vector on C, the **flux** of $\mathbf{F}$ across C is given by the following line integral:

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{3}$$

Notice the difference between flux and circulation. The flux of $\mathbf{F}$ across C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of $\mathbf{F}$ in the direction of the outward normal. The circulation of $\mathbf{F}$ around C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{T}$, the scalar component of $\mathbf{F}$ in the direction of the unit tangent vector. Flux is the integral of the normal component of $\mathbf{F}$; circulation is the integral of the tangential component of $\mathbf{F}$.

To evaluate the integral in (3), we begin with a parametrization

$$x = g(t), \qquad y = h(t), \qquad a \le t \le b,$$

that traces the curve C exactly once as t increases from a to b. We can find the outward unit normal vector $\mathbf{n}$ by crossing the curve's unit tangent vector $\mathbf{T}$ with the vector $\mathbf{k}$. But which order do we choose, $\mathbf{T} \times \mathbf{k}$ or $\mathbf{k} \times \mathbf{T}$? Which one points outward? It depends on which way C is traversed as the parameter t increases. If the motion is clockwise, then $\mathbf{k} \times \mathbf{T}$ points outward; if the motion is counterclockwise, then $\mathbf{T} \times \mathbf{k}$ points outward (Fig. 14.21). The usual choice is $\mathbf{n} = \mathbf{T} \times \mathbf{k}$, the choice that assumes counterclockwise motion. Thus, while the value of the arc length integral in the definition of flux in Eq. (3) does not depend on which way C is traversed, the formulas we are about to derive for evaluating the integral in Eq. (3) will assume counterclockwise motion.

In terms of components,

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

If $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$, then

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence,

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx.$$

We put a directed circle $\circlearrowleft$ on the last integral as a reminder that the integration around the closed curve C is to be in the counterclockwise direction. To evaluate this integral, we express M, dy, N, and dx in terms of t and integrate from $t = a$ to $t = b$. We do not need to know either $\mathbf{n}$ or ds to find the flux.

The Formula for Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} \text{ across } C) = \oint_C M\, dy - N\, dx \tag{4}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t), a \le t \le b$, that traces C counterclockwise exactly once.

EXAMPLE 5 Find the flux of $\mathbf{F} = (x - y)\,\mathbf{i} + x\,\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane.

Solution The parametrization $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j},\ 0 \le t \le 2\pi$, traces the circle counterclockwise exactly once. We can therefore use this parametrization in Eq. (4). With

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\,dt$$
$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\,dt,$$

we find

$$\text{Flux} = \int_C M\,dy - N\,dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t)\,dt \qquad \text{Eq. (4)}$$
$$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2}\,dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi.$$

The flux of $\mathbf{F}$ across the circle is π. Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux. ❑

Exercises 14.2

Vector and Gradient Fields

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

2. $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$

3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$

4. $g(x, y, z) = xy + yz + xz$

5. Give a formula $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at (0, 0).)

6. Give a formula $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at (0, 0) and that at any other point (a, b), $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Work

In Exercises 7–12, find the work done by force $\mathbf{F}$ from (0, 0, 0) to (1, 1, 1) over each of the following paths (Fig. 14.22):

a) The straight-line path C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 1$

b) The curved path C_2: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k},\quad 0 \le t \le 1$

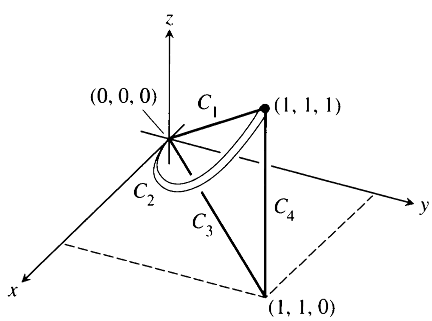
c) The path $C_3 \cup C_4$ consisting of the line segment from (0, 0, 0) to (1, 1, 0) followed by the segment from (1, 1, 0) to (1, 1, 1)

7. $\mathbf{F} = 3y\,\mathbf{i} + 2x\,\mathbf{j} + 4z\,\mathbf{k}$ **8.** $\mathbf{F} = [1/(x^2 + 1)]\,\mathbf{j}$

9. $\mathbf{F} = \sqrt{z}\,\mathbf{i} - 2x\,\mathbf{j} + \sqrt{y}\,\mathbf{k}$ **10.** $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$

11. $\mathbf{F} = (3x^2 - 3x)\,\mathbf{i} + 3z\,\mathbf{j} + \mathbf{k}$

12. $\mathbf{F} = (y + z)\,\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$



14.22 The paths from (0, 0, 0) to (1, 1, 1).

In Exercises 13–16, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

13. $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} - yz\,\mathbf{k}$
$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 1$

14. $\mathbf{F} = 2y\,\mathbf{i} + 3x\,\mathbf{j} + (x + y)\,\mathbf{k}$
$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + (t/6)\,\mathbf{k},\quad 0 \le t \le 2\pi$

15. $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 2\pi$

16. $\mathbf{F} = 6z\,\mathbf{i} + y^2\,\mathbf{j} + 12x\,\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + (t/6)\,\mathbf{k},\quad 0 \le t \le 2\pi$

Line Integrals and Vector Fields in the Plane

17. Evaluate $\int_C xy\,dx + (x + y)\,dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

18. Evaluate $\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices (0, 0), (1, 0), and (0, 1).

19. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ for the vector field $\mathbf{F} = x^2\,\mathbf{i} - y\,\mathbf{j}$ along the curve $x = y^2$ from (4, 2) to (1, −1).

20. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1).

21. Find the work done by the force $\mathbf{F} = xy\,\mathbf{i} + (y-x)\,\mathbf{j}$ over the straight line from (1, 1) to (2, 3).

22. Find the work done by the gradient of $f(x,y) = (x+y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from (2, 0) to itself.

23. Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\,\mathbf{i} + y\,\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\,\mathbf{i} + x\,\mathbf{j}$$

around and across each of the following curves.

a) The circle $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j},\quad 0 \le t \le 2\pi$

b) The ellipse $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j},\quad 0 \le t \le 2\pi$

24. Find the flux of the fields

$$\mathbf{F}_1 = 2x\,\mathbf{i} - 3y\,\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\,\mathbf{i} + (x-y)\,\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j},\quad 0 \le t \le 2\pi.$$

In Exercises 25–28, find the circulation and flux of the field **F** around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j}$, $0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = t\,\mathbf{i},\ -a \le t \le a$.

25. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$

26. $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

27. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$

28. $\mathbf{F} = -y^2\,\mathbf{i} + x^2\,\mathbf{j}$

29. Evaluate the flow integral of the velocity field $\mathbf{F} = (x+y)\,\mathbf{i} - (x^2+y^2)\,\mathbf{j}$ along each of the following paths from (1, 0) to $(-1, 0)$ in the xy-plane.

a) The upper half of the circle $x^2 + y^2 = 1$

b) The line segment from (1, 0) to $(-1, 0)$

c) The line segment from (1, 0) to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$

30. Find the flux of the field **F** in Exercise 29 outward across the triangle with vertices (1, 0), (0, 1), $(-1, 0)$.

Sketching and Finding Fields in the Plane

31. Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{j}$$

(see Fig. 14.15) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.

32. Draw the radial field

$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$$

(see Fig. 14.14) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.

33. a) Find a field $\mathbf{G} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ in the xy-plane with the property that at any point $(a,b) \ne (0,0)$, **G** is a vector of magnitude $\sqrt{a^2+b^2}$ tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the counterclockwise direction. (The field is undefined at (0, 0).)

b) How is **G** related to the spin field **F** in Fig. 14.15?

34. a) Find a field $\mathbf{G} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ in the xy-plane with the property that at any point $(a,b) \ne (0,0)$, **G** is a unit vector tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the clockwise direction.

b) How is **G** related to the spin field **F** in Fig. 14.15?

35. Find a field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ in the xy-plane with the property that at each point $(x,y) \ne (0,0)$, **F** is a unit vector pointing toward the origin. (The field is undefined at (0, 0).)

36. Find a field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ in the xy-plane with the property that at each point $(x,y) \ne (0,0)$, **F** points toward the origin and $|\mathbf{F}|$ is (a) the distance from (x,y) to the origin, (b) inversely proportional to the distance from (x,y) to the origin. (The field is undefined at (0, 0).)

Flow Integrals in Space

In Exercises 37–40, **F** is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t.

37. $\mathbf{F} = -4xy\,\mathbf{i} + 8y\,\mathbf{j} + 2\,\mathbf{k}$
$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \mathbf{k},\quad 0 \le t \le 2$

38. $\mathbf{F} = x^2\,\mathbf{i} + yz\,\mathbf{j} + y^2\,\mathbf{k}$
$\mathbf{r}(t) = 3t\,\mathbf{j} + 4t\,\mathbf{k},\quad 0 \le t \le 1$

39. $\mathbf{F} = (x-z)\,\mathbf{i} + x\,\mathbf{k}$
$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{k},\quad 0 \le t \le \pi$

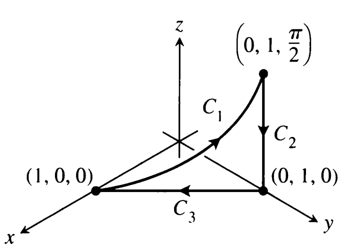
40. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + 2\,\mathbf{k}$
$\mathbf{r}(t) = (-2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + 2t\,\mathbf{k},\quad 0 \le t \le 2\pi$

41. Find the circulation of $\mathbf{F} = 2x\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t:

C_1: $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le \pi/2$

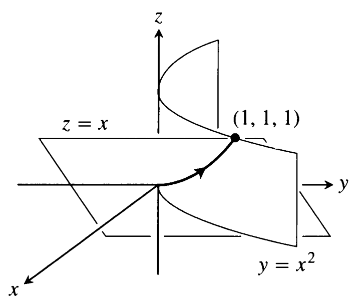
C_2: $\mathbf{r}(t) = \mathbf{j} + (\pi/2)(1-t)\,\mathbf{k},\quad 0 \le t \le 1$

C_3: $\mathbf{r}(t) = t\,\mathbf{i} + (1-t)\,\mathbf{j},\quad 0 \le t \le 1$



42. Let C be the ellipse in which the plane $2x + 3y - z = 0$ meets the cylinder $x^2 + y^2 = 12$. Show, without evaluating either line integral directly, that the circulation of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around C in either direction is zero.

43. The field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ is the velocity field of a flow in space. Find the flow from (0, 0, 0) to (1, 1, 1) along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (*Hint:* Use $t = x$ as the parameter.)



44. Find the flow of the field $\mathbf{F} = \nabla(xy^2z^3)$

a) once around the curve C in Exercise 42, clockwise as viewed from above.

b) along the line segment from (1, 1, 1) to (2, 1, −1).

Theory and Examples

45. Suppose $f(t)$ is differentiable and positive for $a \le t \le b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \le t \le b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t-axis, the graph of f, and the lines $t = a$ and $t = b$? Give reasons for your answer.

46. A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = k\left[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}\right].$$

CAS Explorations and Projects

In Exercises 47–52, use a CAS to perform the following steps for finding the work done by force $\mathbf{F}$ over the given path:

a) Find $d\mathbf{r}$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.

b) Evaluate the force $\mathbf{F}$ along the path.

c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

47. $\mathbf{F} = xy^6\mathbf{i} + 3x(xy^5 + 2)\mathbf{j}$; $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$

48. $\mathbf{F} = \dfrac{3}{1 + x^2}\mathbf{i} + \dfrac{2}{1 + y^2}\mathbf{j}$; $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi$

49. $\mathbf{F} = (y + yz\cos xyz)\mathbf{i} + (x^2 + xz\cos xyz)\mathbf{j} + (z + xy\cos xyz)\mathbf{k}$; $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$

50. $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + ze^x\mathbf{k}$; $\mathbf{r}(t) = -t\mathbf{i} + \sqrt{t}\mathbf{j} + 3t\mathbf{k}$, $1 \le t \le 4$

51. $\mathbf{F} = (2y + \sin x)\mathbf{i} + (z^2 + (1/3)\cos y)\mathbf{j} + x^4\mathbf{k}$; $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin 2t\,\mathbf{k}$, $-\pi/2 \le t \le \pi/2$

52. $\mathbf{F} = (x^2y)\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$; $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2\sin^2(t) - 1)\mathbf{k}$, $0 \le t \le 2\pi$

14.3 Path Independence, Potential Functions, and Conservative Fields

In gravitational and electric fields, the amount of work it takes to move a mass or a charge from one point to another depends only on the object's initial and final positions and not on the path taken in between. This section discusses the notion of path independence of work integrals and describes the remarkable properties of fields in which work integrals are path independent.

Path Independence

If A and B are two points in an open region D in space, the work $\int \mathbf{F} \cdot d\mathbf{r}$ done in moving a particle from A to B by a field $\mathbf{F}$ defined on D usually depends on the path taken. For some special fields, however, the integral's value is the same for all paths from A to B. If this is true for all points A and B in D, we say that the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent in D and that $\mathbf{F}$ is conservative on D.

Definitions

Let $\mathbf{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B in D the work $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ done in moving from A to B is the same over all paths from A to B. Then the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent in *D*** and the field $\mathbf{F}$ is **conservative on *D*.**

The word *conservative* comes from physics, where it refers to fields in which the principle of conservation of energy holds (it does, in conservative fields).

Under conditions normally met in practice, a field $\mathbf{F}$ is conservative if and only if it is the gradient field of a scalar function f; that is, if and only if $\mathbf{F} = \nabla f$ for some f. The function f is then called a potential function for $\mathbf{F}$.

Definition

If $\mathbf{F}$ is a field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function** for $\mathbf{F}$.

An electric potential is a scalar function whose gradient field is an electric field. A gravitational potential is a scalar function whose gradient field is a gravitational field, and so on. As we will see, once we have found a potential function f for a field $\mathbf{F}$, we can evaluate all the work integrals in the domain of $\mathbf{F}$ by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A). \tag{1}$$

If you think of ∇f for functions of several variables as being something like the derivative f' for functions of a single variable, then you see that Eq. (1) is the vector calculus analogue of the Fundamental Theorem of Calculus formula

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Conservative fields have other remarkable properties we will study as we go along. For example, saying that $\mathbf{F}$ is conservative on D is equivalent to saying that the integral of $\mathbf{F}$ around every closed path in D is zero. Naturally, we need to impose conditions on the curves, fields, and domains to make Eq. (1) and its implications hold.

We assume that all curves are **piecewise smooth,** i.e., made up of finitely many smooth pieces connected end to end, as discussed in Section 11.1. We also assume that the components of $\mathbf{F}$ have continuous first partial derivatives. When $\mathbf{F} = \nabla f$, this continuity requirement guarantees that the mixed second derivatives of the potential function f are equal, a result we will find revealing in studying conservative fields $\mathbf{F}$.

We assume D to be an *open* region in space. This means that every point in D is the center of a ball that lies entirely in D. We also assume D to be **connected,** which in an open region means that every point can be connected to every other point by a smooth curve that lies in the region.

Line Integrals in Conservative Fields

The following result provides a convenient way to evaluate a line integral in a conservative field. The result establishes that the value of the integral depends only on the endpoints and not on the specific path joining them.

Theorem 1
The Fundamental Theorem of Line Integrals

1. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Proof That $\mathbf{F} = \nabla f$ Implies Path Independence of the Integral Suppose that A and B are two points in D and that C: $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$, $a \le t \le b$, is a smooth curve in D joining A and B. Along the curve, f is a differentiable function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \qquad \text{Chain Rule}$$

$$= \nabla f \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}. \qquad \text{Because } \mathbf{F} = \nabla f \tag{2}$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_a^b \frac{df}{dt}\,dt \qquad \text{Eq. (2)}$$

$$= f(g(t), h(t), k(t))\Big]_a^b = f(B) - f(A).$$

Thus, the value of the work integral depends only on the values of f at A and B and not on the path in between. This proves Part 2 as well as the forward implication in Part 1. We omit the more technical proof of the reverse implication. ❑

EXAMPLE 1 Find the work done by the conservative field

$$\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} = \nabla(xyz)$$

along any smooth curve C joining the point $(-1, 3, 9)$ to $(1, 6, -4)$.

Solution With $f(x, y, z) = xyz$, we have

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} \qquad \mathbf{F} = \nabla f$$

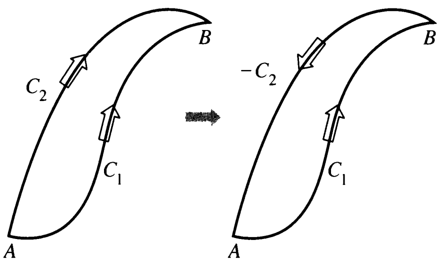
$$= f(B) - f(A) \qquad \text{Fundamental Theorem, Part 2}$$

$$= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)}$$

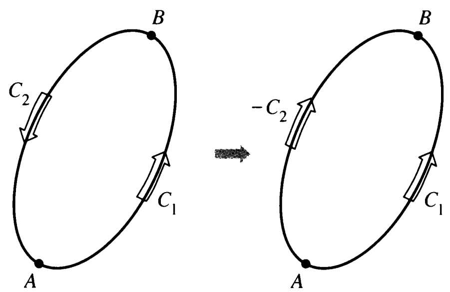
$$= (1)(6)(-4) - (-1)(3)(9)$$

$$= -24 + 27 = 3.$$

❑



14.23 If we have two paths from A to B, one of them can be reversed to make a loop.



14.24 If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B.

Theorem 2

The following statements are equivalent:

1. $\int \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop in D.
2. The field $\mathbf{F}$ is conservative on D.

***Proof That* (1) ⇒ (2)** We want to show that for any two points A and B in D the integral of $\mathbf{F}\cdot d\mathbf{r}$ has the same value over any two paths C_1 and C_2 from A to B. We reverse the direction on C_2 to make a path $-C_2$ from B to A (Fig. 14.23). Together, C_1 and $-C_2$ make a closed loop C, and

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C}\mathbf{F}\cdot d\mathbf{r} = 0.$$

Thus the integrals over C_1 and C_2 give the same value.

***Proof That* (2) ⇒ (1)** We want to show that the integral of $\mathbf{F}\cdot d\mathbf{r}$ is zero over any closed loop C. We pick two points A and B on C and use them to break C into two pieces: C_1 from A and B followed by C_2 from B back to A (Fig. 14.24). Then

$$\oint_{C}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{A}^{B}\mathbf{F}\cdot d\mathbf{r} - \int_{A}^{B}\mathbf{F}\cdot d\mathbf{r} = 0. \qquad \square$$

The following diagram summarizes the results of Theorems 1 and 2.

	Theorem 1		Theorem 2	
$\mathbf{F} = \nabla f$ on D	⇔	$\mathbf{F}$ conservative on D	⇔	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ over any closed path in D

Now that we see how convenient it is to evaluate line integrals in conservative fields, two questions remain:

1. How do we know when a given field $\mathbf{F}$ is conservative?
2. If $\mathbf{F}$ is in fact conservative, how do we find a potential function f (so that $\mathbf{F} = \nabla f$)?

Finding Potentials for Conservative Fields

The test for being conservative is this:

The Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ be a field whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{3}$$

Proof We show that Eqs. (3) must hold if $\mathbf{F}$ is conservative. There is a potential function f such that

$$\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y\,\partial z} \\ &= \frac{\partial^2 f}{\partial z\,\partial y} \\ &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}. \end{aligned}$$

Continuity implies that the mixed partial derivatives are equal.

The other two equations in (3) are proved similarly.

The second half of the proof, that Eqs. (3) imply that $\mathbf{F}$ is conservative, is a consequence of Stokes's theorem, taken up in Section 14.7. ❑

When we know that $\mathbf{F}$ is conservative, we usually want to find a potential function for $\mathbf{F}$. This requires solving the equation $\nabla f = \mathbf{F}$ or

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$$

for f. We accomplish this by integrating the three equations

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P.$$

EXAMPLE 2 Show that $\mathbf{F} = (e^x \cos y + yz)\,\mathbf{i} + (xz - e^x \sin y)\,\mathbf{j} + (xy + z)\,\mathbf{k}$ is conservative and find a potential function for it.

Solution We apply the test in Eqs. (3) to

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

Together, these equalities tell us that there is a function f with $\nabla f = \mathbf{F}$.

We find f by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z. \tag{4}$$

We integrate the first equation with respect to x, holding y and z fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

We write the constant of integration as a function of y and z because its value may change if y and z change. We then calculate $\partial f/\partial y$ from this equation and match it with the expression for $\partial f/\partial y$ in Eq. (4). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g/\partial y = 0$. Therefore, g is a function of z alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f/\partial z$ from this equation and match it to the formula for $\partial f/\partial z$ in Eq. (4). This gives

$$xy + \frac{dh}{dz} = xy + z, \qquad \text{or} \qquad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We have found infinitely many potential functions for **F**, one for each value of C. ❑

EXAMPLE 3 Show that $\mathbf{F} = (2x - 3)\,\mathbf{i} - z\,\mathbf{j} + (\cos z)\,\mathbf{k}$ is not conservative.

Solution We apply the component test in Eqs. (3) and find right away that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\cos z) = 0, \qquad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(-z) = -1.$$

The two are unequal, so **F** is not conservative. No further testing is required. ❑

Exact Differential Forms

As we will see in the next section and again later on, it is often convenient to express work and circulation integrals in the "differential" form

$$\int_A^B M\,dx + N\,dy + P\,dz$$

mentioned in Section 14.2. Such integrals are relatively easy to evaluate if $M\,dx + N\,dy + P\,dz$ is the differential of a function f. For then

$$\begin{aligned}\int_A^B M\,dx + N\,dy + P\,dz &= \int_A^B \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \int_A^B \nabla f \cdot d\mathbf{r} \\ &= f(B) - f(A). \qquad \text{Theorem 1}\end{aligned}$$

Thus

$$\int_A^B df = f(B) - f(A),$$

just as with differentiable functions of a single variable.

Definitions

The form $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is called a **differential form.** A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some (scalar) function f throughout D.

Notice that if $M\,dx + N\,dy + P\,dz = df$ on D, then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the gradient field of f on D. Conversely, if $\mathbf{F} = \nabla f$, then the form $M\,dx + N\,dy + P\,dz$ is exact. The test for the form's being exact is therefore the same as the test for $\mathbf{F}$'s being conservative.

The Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{5}$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

EXAMPLE 4 Show that $y\,dx + x\,dy + 4\,dz$ is exact, and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over the line segment from $(1, 1, 1)$ to $(2, 3, -1)$.

Solution We let $M = y$, $N = x$, $P = 4$ and apply the test of Eq. (5):

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

These equalities tell us that $y\,dx + x\,dy + 4\,dz$ is exact, so

$$y\,dx + x\,dy + 4\,dz = df$$

for some function f, and the integral's value is $f(2, 3, -1) - f(1, 1, 1)$.

We find f up to a constant by integrating the equations

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 4. \tag{6}$$

From the first equation we get

$$f(x, y, z) = xy + g(y, z).$$

The second equation tells us that

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \qquad \text{or} \qquad \frac{\partial g}{\partial y} = 0.$$

Hence, g is a function of z alone, and

$$f(x, y, z) = xy + h(z).$$

The third of Eqs. (6) tells us that

$$\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \qquad \text{or} \qquad h(z) = 4z + C.$$

Therefore, $$f(x, y, z) = xy + 4z + C.$$

The value of the integral is

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3.$$

❑

Exercises 14.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$
2. $\mathbf{F} = (y \sin z)\,\mathbf{i} + (x \sin z)\,\mathbf{j} + (xy \cos z)\,\mathbf{k}$
3. $\mathbf{F} = y\,\mathbf{i} + (x + z)\,\mathbf{j} - y\,\mathbf{k}$
4. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$
5. $\mathbf{F} = (z + y)\,\mathbf{i} + z\,\mathbf{j} + (y + x)\,\mathbf{k}$
6. $\mathbf{F} = (e^x \cos y)\,\mathbf{i} - (e^x \sin y)\,\mathbf{j} + z\,\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\,\mathbf{i} + 3y\,\mathbf{j} + 4z\,\mathbf{k}$
8. $\mathbf{F} = (y + z)\,\mathbf{i} + (x + z)\,\mathbf{j} + (x + y)\,\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}\,(\mathbf{i} + x\,\mathbf{j} + 2x\,\mathbf{k})$
10. $\mathbf{F} = (y \sin z)\,\mathbf{i} + (x \sin z)\,\mathbf{j} + (xy \cos z)\,\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x + y))\,\mathbf{i} + \left(\sec^2(x + y) + \dfrac{y}{y^2 + z^2}\right)\mathbf{j} + \dfrac{z}{y^2 + z^2}\,\mathbf{k}$
12. $\mathbf{F} = \dfrac{y}{1 + x^2y^2}\,\mathbf{i} + \left(\dfrac{x}{1 + x^2y^2} + \dfrac{z}{\sqrt{1 - y^2z^2}}\right)\mathbf{j} + \left(\dfrac{y}{\sqrt{1 - y^2z^2}} + \dfrac{1}{z}\right)\mathbf{k}$

Evaluating Line Integrals

In Exercises 13–22, show that the differential forms in the integrals are exact. Then evaluate the integrals.

13. $\displaystyle\int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$
14. $\displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$
15. $\displaystyle\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$
16. $\displaystyle\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1 + z^2}\,dz$
17. $\displaystyle\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz$
18. $\displaystyle\int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y\,dx + \left(\frac{1}{y} - 2x \sin y\right)dy + \frac{1}{z}\,dz$
19. $\displaystyle\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z \ln y\,dz$
20. $\displaystyle\int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$
21. $\displaystyle\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz$
22. $\displaystyle\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$
23. Evaluate the integral
$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$
from Example 4 by finding parametric equations for the line segment from $(1, 1, 1)$ to $(2, 3, -1)$ and evaluating the line integral of $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + 4\,\mathbf{k}$ along the segment. Since $\mathbf{F}$ is conservative, the integral is independent of the path.
24. Evaluate $\displaystyle\int_C x^2\,dx + yz\,dy + (y^2/2)\,dz$
along the line segment C joining $(0, 0, 0)$ to $(0, 3, 4)$.

Theory, Applications, and Examples

Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B.

25. $\displaystyle\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$
26. $\displaystyle\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}}$

In Exercises 27 and 28, express $\mathbf{F}$ in the form ∇f.

27. $\mathbf{F} = \dfrac{2x}{y}\,\mathbf{i} + \left(\dfrac{1 - x^2}{y^2}\right)\mathbf{j}$
28. $\mathbf{F} = (e^x \ln y)\,\mathbf{i} + \left(\dfrac{e^x}{y} + \sin z\right)\mathbf{j} + (y \cos z)\,\mathbf{k}$
29. Find the work done by $\mathbf{F} = (x^2 + y)\,\mathbf{i} + (y^2 + x)\,\mathbf{j} + ze^z\,\mathbf{k}$ over the following paths from $(1, 0, 0)$ to $(1, 0, 1)$.
    - **a)** The line segment $x = 1, y = 0, 0 \le z \le 1$
    - **b)** The helix $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + (t/2\pi)\,\mathbf{k}, 0 \le t \le 2\pi$
    - **c)** The x-axis from $(1, 0, 0)$ to $(0, 0, 0)$ followed by the parabola $z = x^2, y = 0$ from $(0, 0, 0)$ to $(1, 0, 1)$
30. Find the work done by $\mathbf{F} = e^{yz}\,\mathbf{i} + (xz\,e^{yz} + z \cos y)\,\mathbf{j} + (xy\,e^{yz} + \sin y)\,\mathbf{k}$ over the following paths from $(1, 0, 1)$ to $(1, \pi/2, 0)$.
    - **a)** The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \le t \le 1$
    - **b)** The line segment from $(1, 0, 1)$ to the origin followed by the line segment from the origin to $(1, \pi/2, 0)$
    - **c)** The line segment from $(1, 0, 1)$ to $(1, 0, 0)$, followed by the x-axis from $(1, 0, 0)$ to the origin, followed by the parabola $y = \pi x^2/2, z = 0$

31. Let $\mathbf{F} = \nabla(x^3 y^2)$ and let C be the path in the xy-plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways:

a) Find parametrizations for the segments that make up C and evaluate the integral;

b) Use the fact that $f(x, y) = x^3 y^2$ is a potential function for $\mathbf{F}$.

32. Evaluate $\int_C 2x \cos y\, dx - x^2 \sin y\, dy$ along the following paths C in the xy-plane.

a) The parabola $y = (x-1)^2$ from $(1, 0)$ to $(0, 1)$

b) The line segment from $(-1, \pi)$ to $(1, 0)$

c) The x-axis from $(-1, 0)$ to $(1, 0)$

d) The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 \le t \le 2\pi$, counterclockwise from $(1, 0)$ back to $(1, 0)$

33. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \qquad (G, m, \text{ and } M \text{ are constants}).$$

34. (*Continuation of Exercise 33.*) Let P_1 and P_2 be points at distances s_1 and s_2 from the origin. Show that the work done by the gravitational field in Exercise 33 in moving a particle from P_1 to P_2 is the quantity

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

35. a) How are the constants a, b, and c related if the following differential form is exact?

$$(ay^2 + 2czx)\, dx + y(bx + cz)\, dy + (ay^2 + cx^2)\, dz$$

b) For what values of b and c will

$$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$

be a gradient field?

36. Suppose that $\mathbf{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.$$

Show that $\nabla g = \mathbf{F}$.

37. You have been asked to find the path along which a force field $\mathbf{F}$ will perform the least work in moving a particle between two locations. A quick calculation on your part shows $\mathbf{F}$ to be conservative. How should you respond? Give reasons for your answer.

38. By experiment, you find that a force field $\mathbf{F}$ performs only half as much work in moving an object along path C_1 from A to B as it does in moving the object along path C_2 from A to B. What can you conclude about $\mathbf{F}$? Give reasons for your answer.

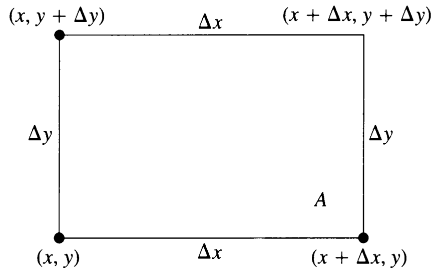
14.4 Green's Theorem in the Plane

We now come to a theorem that can be used to describe the relationship between the way an incompressible fluid flows along or across the boundary of a plane region and the way it moves inside the region. The connection between the fluid's boundary behavior and its internal behavior is made possible by the notions of divergence and curl. The divergence of a fluid's velocity field measures the rate at which fluid is being piped into or out of the region at any given point. The curl measures the fluid's rate of rotation at each point.

Green's theorem states that, under conditions usually met in practice, the outward flux of a vector field across the boundary of a plane region equals the double integral of the divergence of the field over the interior of the region. In another form, it states that the counterclockwise circulation of a field around the boundary of a region equals the double integral of the curl of the field over the region.

Green's theorem is one of the great theorems of calculus. It is deep and surprising and has far-reaching consequences. In pure mathematics, it ranks in importance with the Fundamental Theorem of Calculus. In applied mathematics, the generalizations of Green's theorem to three dimensions provide the foundation for theorems about electricity, magnetism, and fluid flow.

We talk in terms of velocity fields of fluid flows because fluid flows are easy to picture. We would like you to be aware, however, that Green's theorem applies to any vector field satisfying certain mathematical conditions. It does not depend for its validity on the field's having a particular physical interpretation.



14.25 The rectangle for defining the flux density (divergence) of a vector field at a point (x, y).

Flux Density at a Point: Divergence

We need two new ideas for Green's theorem. The first is the idea of the flux density of a vector field at a point, which in mathematics is called the *divergence* of the vector field. We obtain it in the following way.

Suppose that $\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ is the velocity field of a fluid flow in the plane and that the first partial derivatives of M and N are continuous at each point of a region R. Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R (Fig. 14.25). The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy. The rate at which fluid leaves the rectangle across the bottom edge is approximately

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta x = -N(x, y)\Delta x. \tag{1}$$

This is the scalar component of the velocity at (x, y) in the direction of the outward normal times the length of the segment. If the velocity is in meters per second, for example, the exit rate will be in meters per second times meters or square meters per second. The rates at which the fluid crosses the other three sides in the directions of their outward normals can be estimated in a similar way. All told, we have

$$\begin{aligned}
&\text{Top:} && \mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\,\Delta x = N(x, y + \Delta y)\Delta x \\
&\text{Bottom:} && \mathbf{F}(x, y) \cdot (-\mathbf{j})\,\Delta x = -N(x, y)\Delta x \\
&\text{Right:} && \mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\,\Delta y = M(x + \Delta x, y)\,\Delta y \\
&\text{Left:} && \mathbf{F}(x, y) \cdot (-\mathbf{i})\,\Delta y = -M(x, y)\,\Delta y.
\end{aligned} \tag{2}$$

Combining opposite pairs gives

$$\text{Top and bottom:} \quad (N(x, y + \Delta y) - N(x, y))\,\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \tag{3}$$

$$\text{Right and left:} \quad (M(x + \Delta x, y) - M(x, y))\,\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y. \tag{4}$$

Adding (3) and (4) gives

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y.$$

We now divide by $\Delta x\,\Delta y$ to estimate the total flux per unit area or flux density for the rectangle:

$$\frac{\text{Flux across rectangle boundary}}{\text{Rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

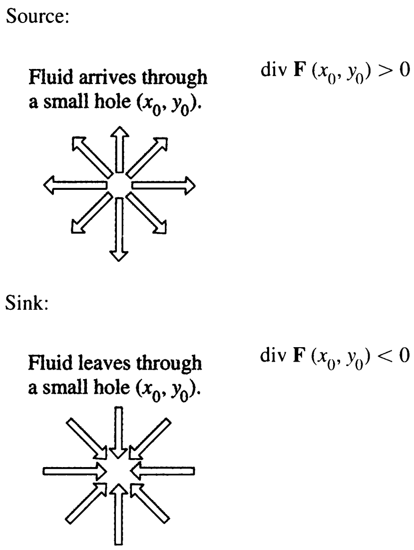
Finally, we let Δx and Δy approach zero to define what we call the *flux density* of $\mathbf{F}$ at the point (x, y).

In mathematics, we call the flux density the *divergence* of $\mathbf{F}$. The symbol for it is div $\mathbf{F}$, pronounced "divergence of $\mathbf{F}$" or "div $\mathbf{F}$."

Definition

The **flux density** or **divergence** of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{5}$$



14.26 In the flow of an incompressible fluid across a plane region, the divergence is positive at a "source," a point where fluid enters the system, and negative at a "sink," a point where the fluid leaves the system.

Intuitively, if water were flowing into a region through a small hole at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since water would be flowing out of a small rectangle about (x_0, y_0), the divergence of $\mathbf{F}$ at (x_0, y_0) would be positive. If the water were draining out instead of flowing in, the divergence would be negative. See Fig. 14.26.

EXAMPLE 1 Find the divergence of $\mathbf{F}(x, y) = (x^2 - y)\,\mathbf{i} + (xy - y^2)\,\mathbf{j}$.

Solution We use the formula in Eq. (5):

$$\begin{aligned}\text{div}\,\mathbf{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x}(x^2 - y) + \frac{\partial}{\partial y}(xy - y^2) \\ &= 2x + x - 2y = 3x - 2y.\end{aligned}$$

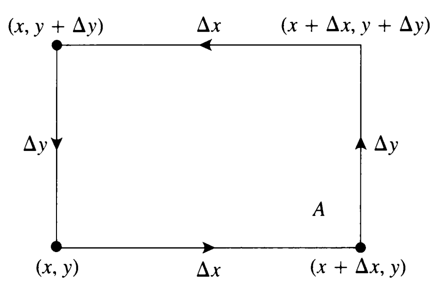
❑

Circulation Density at a Point: The Curl

The second of the two new ideas we need for Green's theorem is the idea of circulation density of a vector field $\mathbf{F}$ at a point, which in mathematics is called the *curl* of $\mathbf{F}$. To obtain it, we return to the velocity field

$$\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$$

and the rectangle A. The rectangle is redrawn here as Fig. 14.27.



14.27 The rectangle for defining the circulation density (curl) of a vector field at a point (x, y).

The counterclockwise circulation of $\mathbf{F}$ around the boundary of A is the sum of flow rates along the sides. For the bottom edge, the flow rate is approximately

$$\mathbf{F}(x, y) \cdot \mathbf{i}\,\Delta x = M(x, y)\,\Delta x. \tag{6}$$

This is the scalar component of the velocity $\mathbf{F}(x, y)$ in the direction of the tangent vector $\mathbf{i}$ times the length of the segment. The rates of flow along the other sides in the counterclockwise direction are expressed in a similar way. In all, we have

$$\begin{aligned}
&\text{Top:} && \mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i})\,\Delta x = -M(x, y + \Delta y)\,\Delta x \\
&\text{Bottom:} && \mathbf{F}(x, y) \cdot \mathbf{i}\,\Delta x = M(x, y)\,\Delta x \\
&\text{Right:} && \mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\,\Delta y = N(x + \Delta x, y)\,\Delta y \\
&\text{Left:} && \mathbf{F}(x, y) \cdot (-\mathbf{j})\,\Delta y = -N(x, y)\,\Delta y.
\end{aligned} \tag{7}$$

We add opposite pairs to get

Top and bottom:

$$-(M(x, y + \Delta y) - M(x, y))\,\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \tag{8}$$

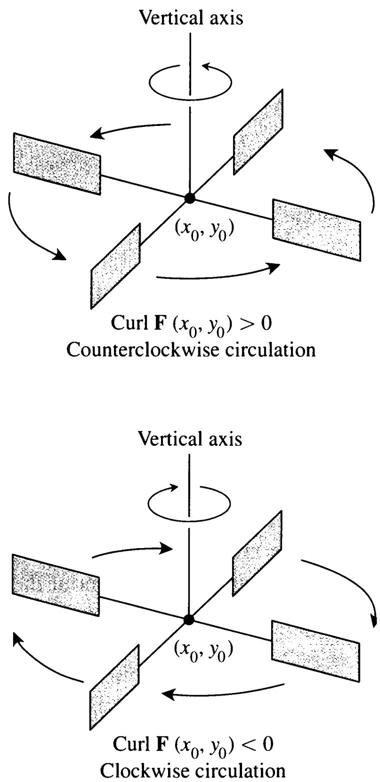
Right and left:

$$(N(x + \Delta x, y) - N(x, y))\,\Delta y \approx \left(\frac{\partial N}{\partial x}\,\Delta x\right)\Delta y. \tag{9}$$

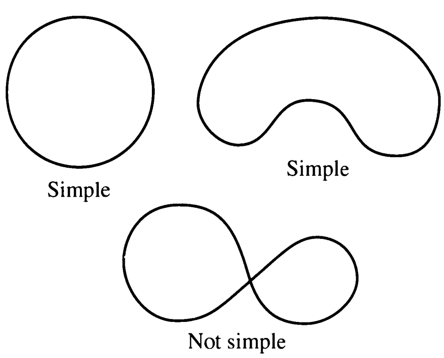
Adding (8) and (9) and dividing by $\Delta x \Delta y$ gives an estimate of the circulation density for the rectangle:

$$\frac{\text{Circulation around rectangle}}{\text{Rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Finally, we let Δx and Δy approach zero to define what we call the *circulation density* of $\mathbf{F}$ at the point (x, y).



14.28 In the flow of an incompressible fluid over a plane region, the curl measures the rate of the fluid's rotation at a point. The curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.



14.29 In proving Green's theorem, we distinguish between two kinds of closed curves, simple and not simple. Simple curves do not cross themselves. A circle is simple but a figure 8 is not.

Definition

The **circulation density** or **curl** of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ at the point (x, y) is

$$\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{10}$$

If water is moving about a region in the xy-plane in a thin layer, then the circulation, or curl, at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel will spin if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane (Fig. 14.28).

EXAMPLE 2 Find the curl of the vector field

$$\mathbf{F}(x, y) = (x^2 - y)\,\mathbf{i} + (xy - y^2)\,\mathbf{j}.$$

Solution We use the formula in Eq. (10):

$$\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - y) = y + 1.$$ ❑

Green's Theorem in the Plane

In one form, Green's theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane (Fig. 14.29) equals the double integral of the divergence of the field over the region enclosed by the curve. Recall the formulas for flux in Eqs. (3) and (4) in Section 14.2.

Theorem 3
Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ across a simple closed curve C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{divergence integral}} \tag{11}$$

In another form, Green's theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the curl of the field over the region enclosed by the curve.

Theorem 4
Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of curl $\mathbf{F}$ over the

(continued)

For a two-dimensional field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, the integral in Eq. (2), Section 14.2, for circulation takes the equivalent form

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C M\,dx + N\,dy.$$

region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{T}\,ds}_{\text{counterclockwise circulation}} = \oint_C M\,dx + N\,dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{curl integral}} \tag{12}$$

The two forms of Green's theorem are equivalent. Applying Eq. (11) to the field $\mathbf{G}_1 = N\,\mathbf{i} - M\,\mathbf{j}$ gives Eq. (12), and applying Eq. (12) to $\mathbf{G}_2 = -N\,\mathbf{i} + M\,\mathbf{j}$ gives Eq. (11).

We need two kinds of assumptions for Green's theorem to hold. First, we need conditions on M and N to ensure the existence of the integrals. The usual assumptions are that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R. Second, we need geometric conditions on the curve C. It must be simple, closed, and made up of pieces along which we can integrate M and N. The usual assumptions are that C is piecewise smooth. The proof we give for Green's theorem, however, assumes things about the shape of R as well. You can find proofs that are less restrictive in more advanced texts. First let's look at some examples.

EXAMPLE 3 Verify both forms of Green's theorem for the field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution We first express all functions, derivatives, and differentials in terms of t:

$$M = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\,dt,$$

$$N = \cos t, \qquad dy = d(\sin t) = \cos t\,dt,$$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

The two sides of Eq. (11):

$$\oint_C M\,dy - N\,dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1 + 0)\,dx\,dy$$

$$= \iint_R dx\,dy = \text{area of unit circle} = \pi.$$

The two sides of Eq. (12):

$$\oint_C M\,dx + N\,dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$$

$$= \int_0^{2\pi} (-\sin t \cos t + 1)\,dt = 2\pi$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\,dx\,dy = 2\iint_R dx\,dy = 2\pi.$$

❑

Using Green's Theorem to Evaluate Line Integrals

If we construct a closed curve C by piecing a number of different curves end to end, the process of evaluating a line integral over C can be lengthy because there are so many different integrals to evaluate. However, if C bounds a region R to which Green's theorem applies, we can use Green's theorem to change the line integral around C into one double integral over R.

EXAMPLE 4 Evaluate the integral

$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's theorem to change the line integral into a double integral over the square.

1. *With Eq.* (11): Taking $M = xy$, $N = y^2$, and C and R as the square's boundary and interior gives

$$\oint_C xy\,dy - y^2\,dx = \iint_R (y + 2y)\,dx\,dy = \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \int_0^1 \left[3xy\right]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\bigg]_0^1 = \frac{3}{2}.$$

2. *With Eq.* (12): Taking $M = -y^2$ and $N = xy$ gives the same result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R (y - (-2y))\,dx\,dy = \frac{3}{2}.$$

❑

The Green of Green's Theorem

The Green of Green's theorem was George Green (1793–1841), a self-taught scientist in Nottingham, England. Green's work on the mathematical foundations of gravitation, electricity, and magnetism was published privately in 1828 in a short book entitled *An Essay on the Application of Mathematical Analysis to Electricity and Magnetism.* The book sold all of fifty-two copies (fewer than one hundred were printed), the copies going mostly to Green's patrons and personal friends. A few weeks before Green's death in 1841, William Thomson noticed a reference to Green's book and in 1845 was finally able to locate a copy. Excited by what he read, Thomson shared Green's ideas with other scientists and had the book republished in a series of journal articles. Green's mathematics provided the foundation on which Thomson, Stokes, Rayleigh, and Maxwell built the present-day theory of electromagnetism.

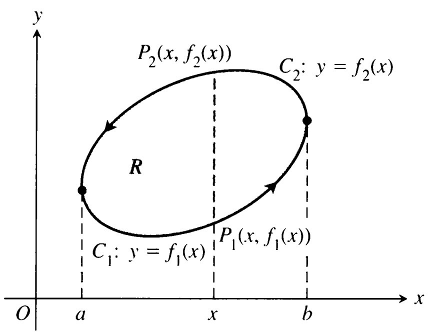
EXAMPLE 5 Calculate the outward flux of the field $\mathbf{F}(x, y) = x\,\mathbf{i} + y^2\,\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's theorem, we can change the line integral to one double integral. With $M = x$, $N = y^2$, C the square, and R the square's interior, we have

$$\begin{aligned}
\text{Flux} &= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy \qquad \text{Green's theorem} \\
&= \int_{-1}^{1} \int_{-1}^{1} (1 + 2y)\, dx\, dy = \int_{-1}^{1} \left[x + 2xy \right]_{x=-1}^{x=1} dy \\
&= \int_{-1}^{1} (2 + 4y)\, dy = \left[2y + 2y^2 \right]_{-1}^{1} = 4.
\end{aligned}$$

❑

A Proof of Green's Theorem (Special Regions)



14.30 The boundary curve C is made up of C_1, the graph of $y = f_1(x)$, and C_2, the graph of $y = f_2(x)$.

Let C be a smooth simple closed curve in the xy-plane with the property that lines parallel to the axes cut it in no more than two points. Let R be the region enclosed by C and suppose that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R. We want to prove the circulation-curl form of Green's theorem,

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy. \tag{13}$$

Figure 14.30 shows C made up of two directed parts:

$$C_1:\quad y = f_1(x), \quad a \le x \le b, \qquad C_2:\quad y = f_2(x), \quad b \ge x \ge a.$$

For any x between a and b, we can integrate $\partial M/\partial y$ with respect to y from $y = f_1(x)$ to $y = f_2(x)$ and obtain

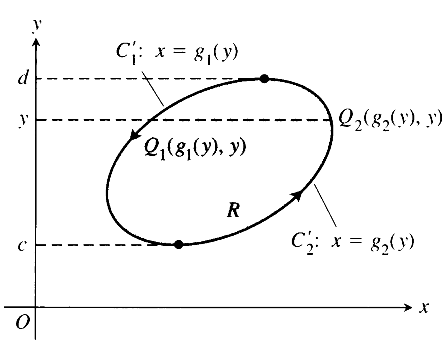
$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)). \tag{14}$$

We can then integrate this with respect to x from a to b:

$$\begin{aligned}
\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx \\
&= -\int_b^a M(x, f_2(x))\, dx - \int_a^b M(x, f_1(x))\, dx \\
&= -\int_{C_2} M\, dx - \int_{C_1} M\, dx \\
&= -\oint_C M\, dx.
\end{aligned}$$

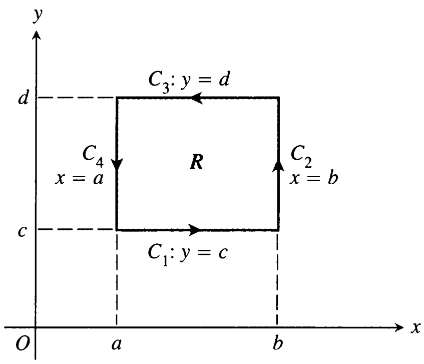
Therefore

$$\oint_C M\, dx = \iint_R \left(-\frac{\partial M}{\partial y} \right) dx\, dy. \tag{15}$$



14.31 The boundary curve C is made up of C_1', the graph of $x = g_1(y)$, and C_2', the graph of $x = g_2(y)$.

Equation (15) is half the result we need for Eq. (13). We derive the other half by integrating $\partial N/\partial x$ first with respect to x and then with respect to y, as suggested by Fig. 14.31. This shows the curve C of Fig. 14.30 decomposed into the two



14.32 To prove Green's theorem for a rectangle, we divide the boundary into four directed line segments.

directed parts C_1': $x = g_1(y)$, $d \geq y \geq c$ and C_2': $x = g_2(y)$, $c \leq y \leq d$. The result of this double integration is

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy. \tag{16}$$

Combining Eqs. (15) and (16) gives Eq. (13). This concludes the proof. ❑

Extending the Proof to Other Regions

The argument we just gave does not apply directly to the rectangular region in Fig. 14.32 because the lines $x = a$, $x = b$, $y = c$, and $y = d$ meet the region's boundary in more than two points. However, if we divide the boundary C into four directed line segments,

$$C_1:\quad y = c,\quad a \leq x \leq b, \qquad C_2:\quad x = b,\quad c \leq y \leq d,$$
$$C_3:\quad y = d,\quad b \geq x \geq a, \qquad C_4:\quad x = a,\quad d \geq y \geq c,$$

we can modify the argument in the following way.

Proceeding as in the proof of Eq. (16), we have

$$\begin{aligned}\int_c^d\int_a^b \frac{\partial N}{\partial x}\,dx\,dy &= \int_c^d (N(b, y) - N(a, y))\,dy \\ &= \int_c^d N(b, y)\,dy + \int_d^c N(a, y)\,dy \\ &= \int_{C_2} N\,dy + \int_{C_4} N\,dy.\end{aligned} \tag{17}$$

Because y is constant along C_1 and C_3, $\int_{C_1} N\,dy = \int_{C_3} N\,dy = 0$, so we can add $\int_{C_1} N\,dy + \int_{C_3} N\,dy$ to the right-hand side of Eq. (17) without changing the equality. Doing so, we have

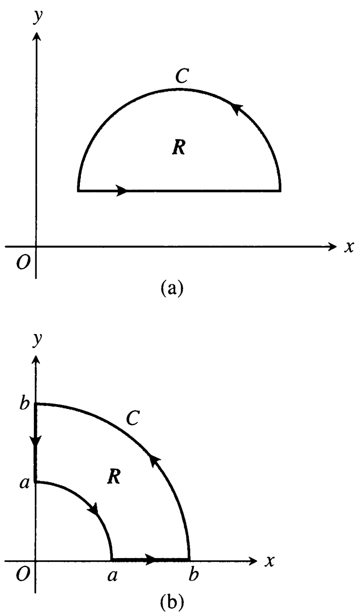
$$\int_c^d\int_a^b \frac{\partial N}{\partial x}\,dx\,dy = \oint_C N\,dy. \tag{18}$$

Similarly, we can show that

$$\int_a^b\int_c^d \frac{\partial M}{\partial y}\,dy\,dx = -\oint_C M\,dx. \tag{19}$$

Subtracting Eq. (19) from Eq. (18), we again arrive at

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$$

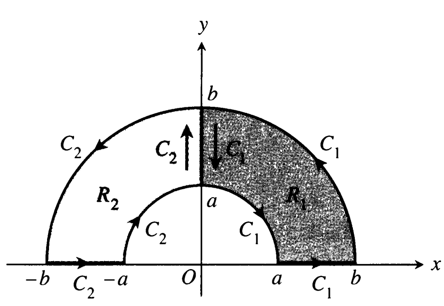


14.33 Other regions to which Green's theorem applies.

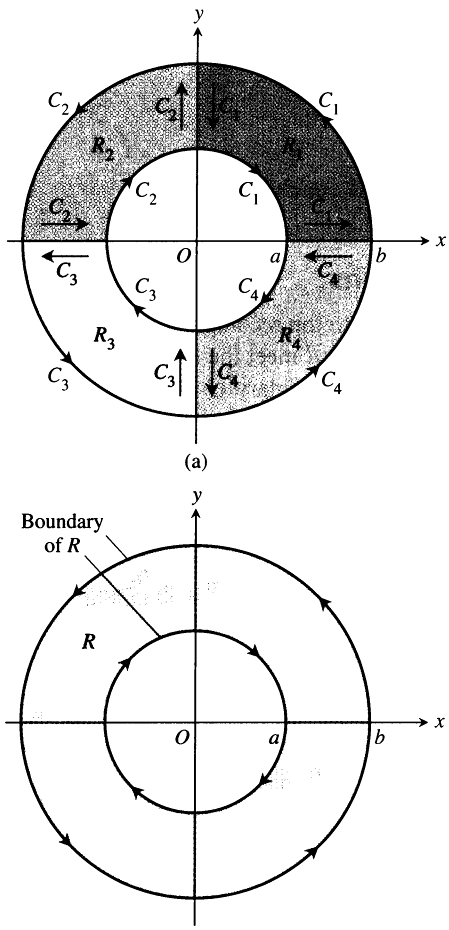
Regions like those in Fig. 14.33 can be handled with no greater difficulty. Equation (13) still applies. It also applies to the horseshoe-shaped region R shown in Fig. 14.34, as we see by putting together the regions R_1 and R_2 and their boundaries. Green's theorem applies to C_1, R_1 and to C_2, R_2, yielding

$$\int_{C_1} M\,dx + N\,dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

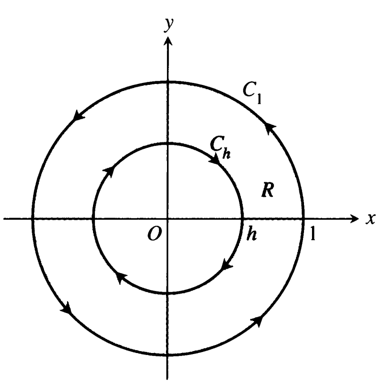
$$\int_{C_2} M\,dx + N\,dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$$



14.34 A region R that combines regions R_1 and R_2.



14.35 The annular region R combines four smaller regions. In polar coordinates, $r = a$ for the inner circle, $r = b$ for the outer circle, and $a \le r \le b$ for the region itself.



14.36 Green's theorem may be applied to the annular region R by integrating along the boundaries as shown (Example 6).

When we add these two equations, the line integral along the y-axis from b to a for C_1 cancels the integral over the same segment but in the opposite direction for C_2. Hence

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

where C consists of the two segments of the x-axis from $-b$ to $-a$ and from a to b and of the two semicircles, and where R is the region inside C.

The device of adding line integrals over separate boundaries to build up an integral over a single boundary can be extended to any finite number of subregions. In Fig. 14.35(a), let C_1 be the boundary, oriented counterclockwise, of the region R_1 in the first quadrant. Similarly for the other three quadrants: C_i is the boundary of the region R_i, $i = 1, 2, 3, 4$. By Green's theorem,

$$\oint_{C_i} M\,dx + N\,dy = \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{20}$$

We add Eqs. (20) for $i = 1, 2, 3, 4$, and get (Fig. 14.35b):

$$\oint_{r=b} (M\,dx + N\,dy) + \oint_{r=a} (M\,dx + N\,dy) = \iint_{a \le r \le b} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{21}$$

Equation (21) says that the double integral of $(\partial N/\partial x) - (\partial M/\partial y)$ over the annular ring R equals the line integral of $M\,dx + N\,dy$ over the complete boundary of R in the direction that keeps R on our left as we progress (Fig. 14.35b).

EXAMPLE 6 Verify the circulation form of Green's theorem (Eq. 12) on the annular ring R: $h^2 \le x^2 + y^2 \le 1$, $0 < h < 1$ (Fig. 14.36), if

$$M = \frac{-y}{x^2 + y^2}, \qquad N = \frac{x}{x^2 + y^2}.$$

Solution The boundary of R consists of the circle

$$C_1: \quad x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi,$$

traversed counterclockwise as t increases, and the circle

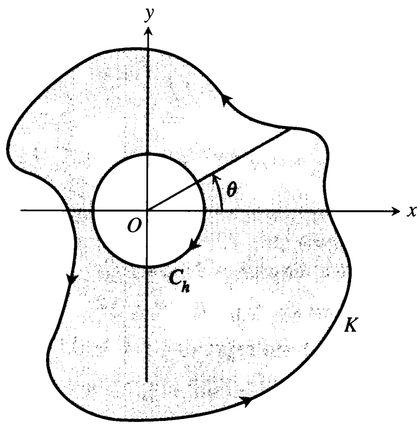
$$C_h: \quad x = h\cos\theta, \quad y = -h\sin\theta, \quad 0 \le \theta \le 2\pi,$$

traversed clockwise as θ increases. The functions M and N and their partial derivatives are continuous throughout R. Moreover,

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x},\end{aligned}$$

so

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0.$$



14.37 The region bounded by the circle C_h and the curve K.

The integral of $M\,dx + N\,dy$ over the boundary of R is

$$\int_C M\,dx + N\,dy = \oint_{C_1} \frac{x\,dy - y\,dx}{x^2 + y^2} + \oint_{C_h} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt - \int_0^{2\pi} \frac{h^2(\cos^2\theta + \sin^2\theta)}{h^2}\,d\theta$$

$$= 2\pi - 2\pi = 0. \qquad \square$$

The functions M and N in Example 6 are discontinuous at (0, 0), so we cannot apply Green's theorem to the circle C_1 and the region inside it. We must exclude the origin. We do so by excluding the points inside C_h.

We could replace the circle C_1 in Example 6 by an ellipse or any other simple closed curve K surrounding C_h (Fig. 14.37). The result would still be

$$\oint_K (M\,dx + N\,dy) + \oint_{C_h} (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = 0,$$

which leads to the surprising conclusion that

$$\oint_K (M\,dx + N\,dy) = 2\pi$$

for any such curve K. We can explain this result by changing to polar coordinates. With

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dx = -r\sin\theta\,d\theta + \cos\theta\,dr, \qquad dy = r\cos\theta\,d\theta + \sin\theta\,dr,$$

we have

$$\frac{x\,dy - y\,dx}{x^2+y^2} = \frac{r^2(\cos^2\theta + \sin^2\theta)\,d\theta}{r^2} = d\theta,$$

and θ increases by 2π as we traverse K once counterclockwise.

Exercises 14.4

Verifying Green's Theorem

In Exercises 1–4, verify Green's theorem by evaluating both sides of Eqs. (11) and (12) for the field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. Take the domains of integration in each case to be the disk R: $x^2 + y^2 \le a^2$ and its bounding circle C: $\mathbf{r} = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j}$, $0 \le t \le 2\pi$.

1. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$

2. $\mathbf{F} = y\,\mathbf{i}$

3. $\mathbf{F} = 2x\,\mathbf{i} - 3y\,\mathbf{j}$

4. $\mathbf{F} = -x^2y\,\mathbf{i} + xy^2\,\mathbf{j}$

Counterclockwise Circulation and Outward Flux

In Exercises 5–10, use Green's theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

5. $\mathbf{F} = (x - y)\,\mathbf{i} + (y - x)\,\mathbf{j}$
C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

6. $\mathbf{F} = (x^2 + 4y)\,\mathbf{i} + (x + y^2)\,\mathbf{j}$
C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

7. $\mathbf{F} = (y^2 - x^2)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$
C: The triangle bounded by $y = 0, x = 3$, and $y = x$

8. $\mathbf{F} = (x + y)\,\mathbf{i} - (x^2 + y^2)\,\mathbf{j}$
C: The triangle bounded by $y = 0, x = 1$, and $y = x$

9. $\mathbf{F} = (x + e^x \sin y)\,\mathbf{i} + (x + e^x \cos y)\,\mathbf{j}$
C: The right-hand loop of the lemniscate $r^2 = \cos 2\theta$

10. $\mathbf{F} = \left(\tan^{-1}\dfrac{y}{x}\right)\mathbf{i} + \ln(x^2 + y^2)\,\mathbf{j}$
C: The boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2, 0 \le \theta \le \pi$

11. Find the counterclockwise circulation and outward flux of the field $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

12. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = (-\sin y)\,\mathbf{i} + (x \cos y)\,\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

13. Find the outward flux of the field

$$\mathbf{F} = \left(3xy - \frac{x}{1+y^2}\right)\mathbf{i} + (e^x + \tan^{-1} y)\,\mathbf{j}$$

across the cardioid $r = a(1 + \cos\theta)$, $a > 0$.

14. Find the counterclockwise circulation of $\mathbf{F} = (y + e^x \ln y)\,\mathbf{i} + (e^x/y)\,\mathbf{j}$ around the boundary of the region that is bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$.

Work

In Exercises 15 and 16, find the work done by $\mathbf{F}$ in moving a particle once counterclockwise around the given curve.

15. $\mathbf{F} = 2xy^3\,\mathbf{i} + 4x^2y^2\,\mathbf{j}$
 C: The boundary of the "triangular" region in the first quadrant enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$

16. $\mathbf{F} = (4x - 2y)\,\mathbf{i} + (2x - 4y)\,\mathbf{j}$
 C: The circle $(x-2)^2 + (y-2)^2 = 4$

Evaluating Line Integrals in the Plane

Apply Green's theorem to evaluate the integrals in Exercises 17–20.

17. $\displaystyle\oint_C (y^2\,dx + x^2\,dy)$
 C: The triangle bounded by $x = 0, x + y = 1, y = 0$

18. $\displaystyle\oint_C (3y\,dx + 2x\,dy)$
 C: The boundary of $0 \le x \le \pi, 0 \le y \le \sin x$

19. $\displaystyle\oint_C (6y + x)\,dx + (y + 2x)\,dy$
 C: The circle $(x-2)^2 + (y-3)^2 = 4$

20. $\displaystyle\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$
 C: Any simple closed curve in the plane for which Green's theorem holds

Calculating Area with Green's Theorem

If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's theorem, the area of R is given by:

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx \qquad (22)$$

The reason is that by Eq. (11), run backward,

$$\text{Area of } R = \iint_R dy\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx$$
$$= \oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx.$$

Use the Green's theorem area formula (Eq. 22) to find the areas of the regions enclosed by the curves in Exercises 21–24.

21. The circle $\mathbf{r}(t) = (a \cos t)\,\mathbf{i} + (a \sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi$

22. The ellipse $\mathbf{r}(t) = (a \cos t)\,\mathbf{i} + (b \sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi$

23. The astroid (Fig. 9.42) $\mathbf{r}(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le 2\pi$

24. The curve (Fig. 9.75) $\mathbf{r}(t) = t^2\,\mathbf{i} + ((t^3/3) - t)\,\mathbf{j}$, $-\sqrt{3} \le t \le \sqrt{3}$

Theory and Examples

25. Let C be the boundary of a region on which Green's theorem holds. Use Green's theorem to calculate

 a) $\displaystyle\oint_C f(x)\,dx + g(y)\,dy$,

 b) $\displaystyle\oint_C ky\,dx + hx\,dy$ (k and h constants).

26. Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

27. What is special about the integral

$$\oint_C 4x^3y\,dx + x^4\,dy?$$

Give reasons for your answer.

28. What is special about the integral

$$\oint_C -y^3\,dx + x^3\,dy?$$

Give reasons for your answer.

29. Show that if R is a region in the plane bounded by a piecewise smooth simple closed curve C, then

$$\text{Area of } R = \oint_C x\,dy = -\oint_C y\,dx.$$

30. Suppose that a nonnegative function $y = f(x)$ has a continuous first derivative on $[a, b]$. Let C be the boundary of the region in the xy-plane that is bounded below by the x-axis, above by the graph of f, and on the sides by the lines $x = a$ and $x = b$. Show

that

$$\int_a^b f(x)\,dx = -\oint_C y\,dx.$$

31. Let A be the area and $\bar{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise smooth simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\bar{x}.$$

32. Let I_y be the moment of inertia about the y-axis of the region in Exercise 31. Show that

$$\frac{1}{3}\oint_C x^3\,dy = -\oint_C x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx = I_y.$$

33. *Green's theorem and Laplace's equation.* Assuming that all the necessary derivatives exist and are continuous, show that if $f(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

then

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0$$

for all closed curves C to which Green's theorem applies. (The converse is also true: If the line integral is always zero, then f satisfies the Laplace equation.)

34. Among all smooth simple closed curves in the plane, oriented counterclockwise, find the one along which the work done by

$$\mathbf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3\right)\mathbf{i} + x\,\mathbf{j}$$

is greatest. (*Hint:* Where is curl $\mathbf{F}$ positive?)

35. Green's theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (Fig. 14.38).

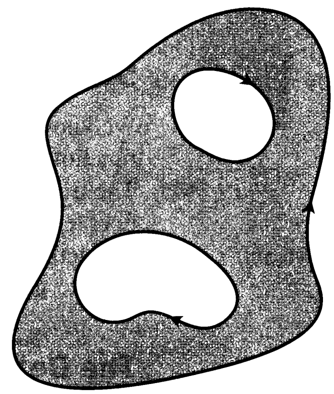
a) Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n}\,ds.$$

b) Let K be an arbitrary smooth simple closed curve in the plane that does not pass through $(0, 0)$. Use Green's theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n}\,ds$$

has two possible values, depending on whether $(0, 0)$ lies inside K or outside K.



14.38 Green's theorem holds for regions with more than one hole (Exercise 35).

36. *Bendixson's criterion.* The **streamlines** of a planar fluid flow are the smooth curves traced by the fluid's individual particles. The vectors $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ of the flow's velocity field are the tangent vectors of the streamlines. Show that if the flow takes place over a *simply connected* region R (no holes or missing points) and that if $M_x + N_y \neq 0$ throughout R, then none of the streamlines in R is closed. In other words, no particle of fluid ever has a closed trajectory in R. The criterion $M_x + N_y \neq 0$ is called **Bendixson's criterion** for the nonexistence of closed trajectories.

37. Establish Eq. (16) to finish the proof of the special case of Green's theorem.

38. Establish Eq. (19) to complete the argument for the extension of Green's theorem.

39. Can anything be said about the curl of a conservative two-dimensional vector field? Give reasons for your answer.

40. Does Green's theorem give any information about the circulation of a conservative field? Does this agree with anything else you know? Give reasons for your answer.

CAS Explorations and Projects

In Exercises 41–44, use a CAS and Green's theorem to find the counterclockwise circulation of the field $\mathbf{F}$ around the simple closed curve C. Perform the following CAS steps:

a) Plot C in the xy-plane.

b) Determine the integrand $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ for the curl form of Green's theorem.

c) Determine the (double integral) limits of integration from your plot in (a) and evaluate the curl integral for the circulation.

41. $\mathbf{F} = (2x - y)\,\mathbf{i} + (x + 3y)\,\mathbf{j}$, C: The ellipse $x^2 + 4y^2 = 4$

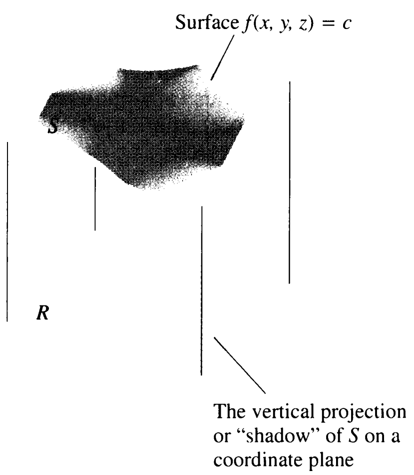
42. $\mathbf{F} = (2x^3 - y^3)\,\mathbf{i} + (x^3 + y^3)\,\mathbf{j}$, C: The ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

43. $\mathbf{F} = x^{-1}e^y\,\mathbf{i} + (e^y \ln x + 2x)\,\mathbf{j}$,
C: The boundary of the region defined by $y = 1 + x^4$ (below) and $y = 2$ (above)

44. $\mathbf{F} = x\,e^y\,\mathbf{i} + 4x^2 \ln y\,\mathbf{j}$, C: The triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$

14.5 Surface Area and Surface Integrals

We know how to integrate a function over a flat region in a plane, but what if the function is defined over a curved surface? How do we calculate its integral then? The trick to evaluating one of these so-called surface integrals is to rewrite it as a double integral over a region in a coordinate plane beneath the surface (Fig. 14.39). In Sections 14.7 and 14.8 we will see how surface integrals provide just what we need to generalize the two forms of Green's theorem to three dimensions.



14.39 As we will soon see, the integral of a function $g(x, y, z)$ over a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane.

The Definition of Surface Area

Figure 14.40 shows a surface S lying above its "shadow" region R in a plane beneath it. The surface is defined by the equation $f(x, y, z) = c$. If the surface is **smooth** (∇f is continuous and never vanishes on S), we can define and calculate its area as a double integral over R.

The first step in defining the area of S is to partition the region R into small rectangles ΔA_k of the kind we would use if we were defining an integral over R. Directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we may approximate with a portion ΔP_k of the tangent plane. To be specific, we suppose that ΔP_k is a portion of the plane that is tangent to the surface at the point $T_k(x_k, y_k, z_k)$ directly above the back corner C_k of ΔA_k. If the tangent plane is parallel to R, then ΔP_k will be congruent to ΔA_k. Otherwise, it will be a parallelogram whose area is somewhat larger than the area of ΔA_k.

Figure 14.41 gives a magnified view of $\Delta\sigma_k$ and ΔP_k, showing the gradient vector $\nabla f(x_k, y_k, z_k)$ at T_k and a unit vector $\mathbf{p}$ that is normal to R. The figure also shows the angle γ_k between ∇f and $\mathbf{p}$. The other vectors in the picture, $\mathbf{u}_k$ and $\mathbf{v}_k$, lie along the edges of the patch ΔP_k in the tangent plane. Thus, both $\mathbf{u}_k \times \mathbf{v}_k$ and ∇f are normal to the tangent plane.

We now need the fact from advanced vector geometry that $|(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}|$ is the area of the projection of the parallelogram determined by $\mathbf{u}_k$ and $\mathbf{v}_k$ onto any plane whose normal is $\mathbf{p}$. In our case, this translates into the statement

$$|(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}| = \Delta A_k. \tag{1}$$

Now, $|\mathbf{u}_k \times \mathbf{v}_k|$ itself is the area ΔP_k (standard fact about cross products) so Eq. (1) becomes

$$\underbrace{|\mathbf{u}_k \times \mathbf{v}_k|}_{\Delta P_k}\ \underbrace{|\mathbf{p}|}_{1}\ \underbrace{|\cos\,(\text{angle between } \mathbf{u}_k \times \mathbf{v}_k \text{ and } \mathbf{p})|}_{\substack{\text{same as } |\cos\gamma_k| \text{ because} \\ \nabla f \text{ and } \mathbf{u}_k \times \mathbf{v}_k \text{ are both} \\ \text{normal to the tangent plane}}} = \Delta A_k \tag{2}$$

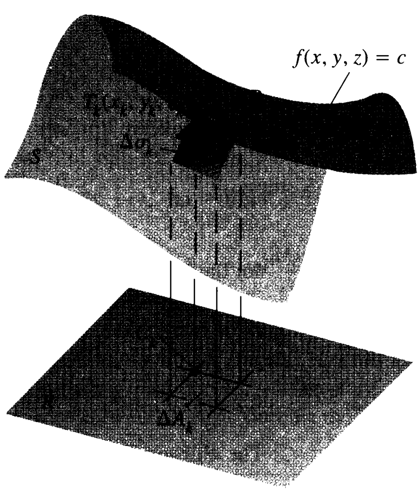
or

$$\Delta P_k |\cos\gamma_k| = \Delta A_k$$

or

$$\Delta P_k = \frac{\Delta A_k}{|\cos\gamma_k|},$$

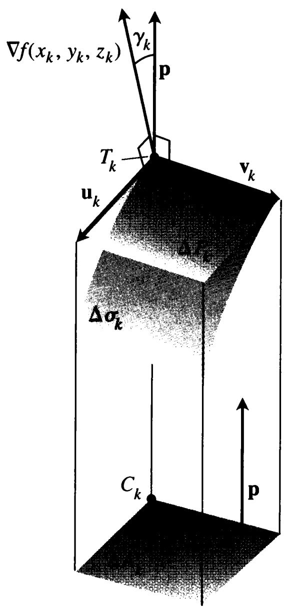
provided $\cos\gamma_k \neq 0$. We will have $\cos\gamma_k \neq 0$ as long as ∇f is not parallel to the ground plane and $\nabla f \cdot \mathbf{p} \neq 0$.



14.40 A surface S and its vertical projection onto a plane beneath it. You can think of R as the shadow of S on the plane. The tangent plate ΔP_k approximates the surface patch $\Delta\sigma_k$ above ΔA_k.

Since the patches ΔP_k approximate the surface patches $\Delta\sigma_k$ that fit together to make S, the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos\gamma_k|} \tag{3}$$



14.41 Magnified view from the preceding figure. The vector $\mathbf{u}_k \times \mathbf{v}_k$ (not shown) is parallel to the vector ∇f because both vectors are normal to the plane of ΔP_k.

looks like an approximation of what we might like to call the surface area of S. It also looks as if the approximation would improve if we refined the partition of R. In fact, the sums on the right-hand side of Eq. (3) are approximating sums for the double integral

$$\iint_R \frac{1}{|\cos\gamma|}\, dA. \tag{4}$$

We therefore define the **area** of S to be the value of this integral whenever it exists.

A Practical Formula

For any surface $f(x, y, z) = c$, we have $|\nabla f \cdot \mathbf{p}| = |\nabla f||\mathbf{p}||\cos\gamma|$, so

$$\frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}.$$

This combines with Eq. (4) to give a practical formula for area.

The Formula for Surface Area

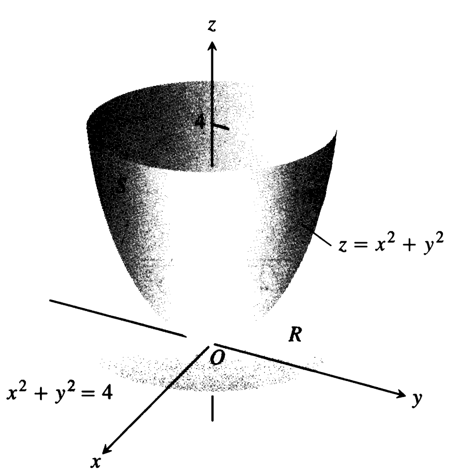
The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA, \tag{5}$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Thus, the area is the double integral over R of the magnitude of ∇f divided by the magnitude of the scalar component of ∇f normal to R.

We reached Eq. (5) under the assumption that $\nabla f \cdot \mathbf{p} \neq 0$ throughout R and that ∇f is continuous. Whenever the integral exists, however, we define its value to be the area of the portion of the surface $f(x, y, z) = c$ that lies over R.



14.42 The area of this parabolic surface is calculated in Example 1.

EXAMPLE 1 Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution We sketch the surface S and the region R below it in the xy-plane (Fig. 14.42). The surface S is part of the level surface $f(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal to the plane of R, we can take $\mathbf{p} = \mathbf{k}$.

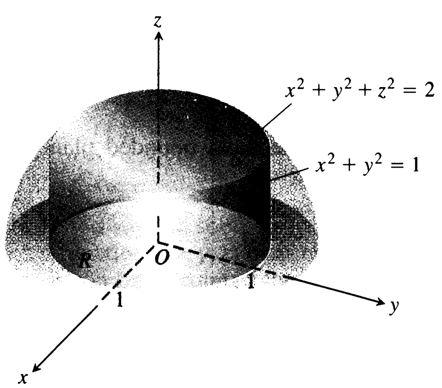
At any point (x, y, z) on the surface, we have

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 - z \\ \nabla f &= 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} \\ |\nabla f| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

In the region $R, dA = dx\,dy$. Therefore,

$$\begin{aligned}
\text{Surface area} &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA && \text{Eq. (5)}\\
&= \iint_{x^2+y^2\le 4} \sqrt{4x^2+4y^2+1}\, dx\, dy\\
&= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\, r\, dr\, d\theta && \text{Polar coordinates}\\
&= \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_0^2 d\theta\\
&= \int_0^{2\pi}\frac{1}{12}(17^{3/2}-1)\, d\theta = \frac{\pi}{6}(17\sqrt{17}-1).
\end{aligned}$$

❑



14.43 The cap cut from the hemisphere by the cylinder projects vertically onto the disk R: $x^2+y^2 \le 1$ (Example 2).

EXAMPLE 2 Find the area of the cap cut from the hemisphere $x^2+y^2+z^2 = 2, z \ge 0$, by the cylinder $x^2+y^2=1$ (Fig. 14.43).

Solution The cap S is part of the level surface $f(x,y,z) = x^2+y^2+z^2 = 2$. It projects one-to-one onto the disk R: $x^2+y^2 \le 1$ in the xy-plane. The vector $\mathbf{p} = \mathbf{k}$ is normal to the plane of R.

At any point on the surface,

$$\begin{aligned}
f(x,y,z) &= x^2+y^2+z^2\\
\nabla f &= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}\\
|\nabla f| &= 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2} && \text{Because } x^2+y^2+z^2=2 \text{ at points of } S\\
|\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |2z| = 2z.
\end{aligned}$$

Therefore,

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2}\iint_R \frac{dA}{z}. \tag{6}$$

What do we do about the z?

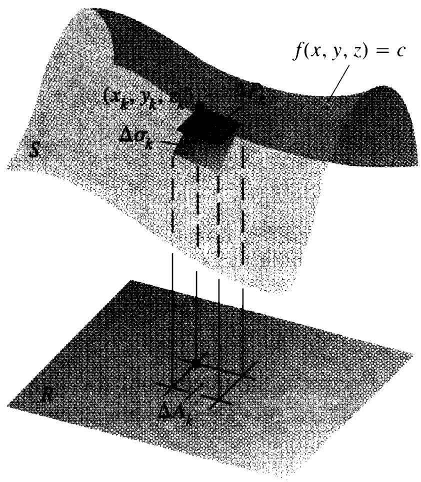
Since z is the z-coordinate of a point on the sphere, we can express it in terms of x and y as

$$z = \sqrt{2-x^2-y^2}.$$

We continue the work of Eq. (6) with this substitution:

$$\begin{aligned}
\text{Surface area} &= \sqrt{2}\iint_R \frac{dA}{z} = \sqrt{2}\iint_{x^2+y^2\le 1}\frac{dA}{\sqrt{2-x^2-y^2}}\\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r\, dr\, d\theta}{\sqrt{2-r^2}} && \text{Polar coordinates}\\
&= \sqrt{2}\int_0^{2\pi}\left[-(2-r^2)^{1/2}\right]_{r=0}^{r=1} d\theta\\
&= \sqrt{2}\int_0^{2\pi}(\sqrt{2}-1)\, d\theta = 2\pi(2-\sqrt{2}).
\end{aligned}$$

❑



14.44 If we know how an electrical charge is distributed over a surface, we can find the total charge with a suitably modified surface integral.

Surface Integrals

We now show how to integrate a function over a surface, using the ideas just developed for calculating surface area.

Suppose, for example, that we have an electrical charge distributed over a surface $f(x, y, z) = c$ like the one shown in Fig. 14.44 and that the function $g(x, y, z)$ gives the charge per unit area (charge density) at each point on S. Then we may calculate the total charge on S as an integral in the following way.

We partition the shadow region R on the ground plane beneath the surface into small rectangles of the kind we would use if we were defining the surface area of S. Then directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we approximate with a parallelogram-shaped portion of tangent plane, ΔP_k.

Up to this point the construction proceeds as in the definition of surface area, but now we take one additional step: We evaluate g at (x_k, y_k, z_k) and then approximate the total charge on the surface patch $\Delta\sigma_k$ by the product $g(x_k, y_k, z_k)\Delta P_k$. The rationale is that when the partition of R is sufficiently fine, the value of g throughout $\Delta\sigma_k$ is nearly constant and ΔP_k is nearly the same as $\Delta\sigma_k$. The total charge over S is then approximated by the sum

$$\text{Total charge} \approx \sum g(x_k, y_k, z_k)\,\Delta P_k = \sum g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma_k|}. \tag{7}$$

If f, the function defining the surface S, and its first partial derivatives are continuous, and if g is continuous over S, then the sums on the right-hand side of Eq. (7) approach the limit

$$\iint_R g(x, y, z)\frac{dA}{|\cos\gamma|} = \iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\,dA \tag{8}$$

as the partition of R is refined in the usual way. This limit is called the integral of g over the surface S and is calculated as a double integral over R. The value of the integral is the total charge on the surface S.

As you might expect, the formula in Eq. (8) defines the integral of *any* function g over the surface S as long as the integral exists.

Definitions

If R is the shadow region of a surface S defined by the equation $f(x, y, z) = c$, and g is a continuous function defined at the points of S, then the **integral of g over S** is the integral

$$\iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\,dA, \tag{9}$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$. The integral itself is called a **surface integral.**

The integral in (9) takes on different meanings in different applications. If g has the constant value 1, the integral gives the area of S. If g gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell.

Algebraic Properties: The Surface Area Differential

We can abbreviate the integral in (9) by writing $d\sigma$ for $(|\nabla f|/|\nabla f \cdot \mathbf{p}|)\,dA$.

The Surface Area Differential and the Differential Form for Surface Integrals

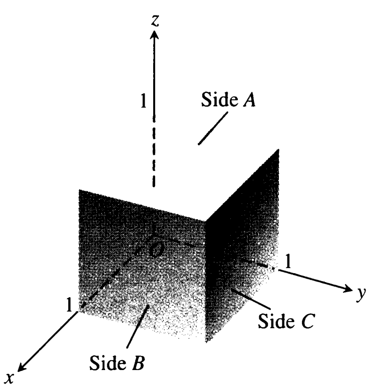
$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA \qquad \iint\limits_S g\, d\sigma \qquad (10)$$

surface area differential — differential formula for surface integrals

Surface integrals behave like other double integrals, the integral of the sum of two functions being the sum of their integrals and so on. The domain additivity property takes the form

$$\iint\limits_S g\, d\sigma = \iint\limits_{S_1} g\, d\sigma + \iint\limits_{S_2} g\, d\sigma + \cdots + \iint\limits_{S_n} g\, d\sigma.$$

The idea is that if S is partitioned by smooth curves into a finite number of nonoverlapping smooth patches (i.e., if S is **piecewise smooth**), then the integral over S is the sum of the integrals over the patches. Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube. We integrate over a turtle shell of welded plates by integrating one plate at a time and adding the results.



14.45 To integrate a function over the surface of a cube, we integrate over each face and add the results (Example 3).

EXAMPLE 3 Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Fig. 14.45).

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint\limits_{\substack{\text{cube}\\\text{surface}}} xyz\, d\sigma = \iint\limits_{\text{side } A} xyz\, d\sigma + \iint\limits_{\text{side } B} xyz\, d\sigma + \iint\limits_{\text{side } C} xyz\, d\sigma.$$

Side A is the surface $f(x, y, z) = z = 1$ over the square region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane. For this surface and region,

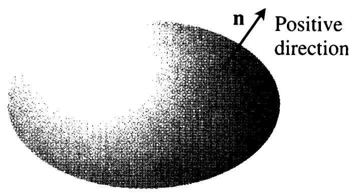
$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1,$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \frac{1}{1}\, dx\, dy = dx\, dy,$$

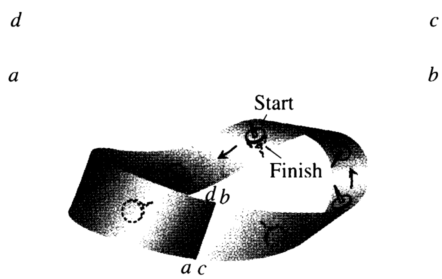
$$xyz = xy(1) = xy,$$

and

$$\iint\limits_{\text{side } A} xyz\, d\sigma = \iint\limits_{R_{xy}} xy\, dx\, dy = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2}\, dy = \frac{1}{4}.$$



14.46 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.



14.47 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

Symmetry tells us that the integrals of xyz over sides B and C are also 1/4. Hence,

$$\iint_{\substack{\text{cube}\\ \text{surface}}} xyz\, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

❑

Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\mathbf{n}$ of unit normal vectors on S that varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface.** The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Fig. 14.46).

The Möbius band in Fig. 14.47 is not orientable. No matter where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

The Surface Integral for Flux

Suppose that $\mathbf{F}$ is a continuous vector field defined over an oriented surface S and that $\mathbf{n}$ is the chosen unit normal field on the surface. We call the integral of $\mathbf{F}\cdot\mathbf{n}$ over S the flux across S in the positive direction. Thus, the flux is the integral over S of the scalar component of $\mathbf{F}$ in the direction of $\mathbf{n}$.

Definition

The **flux** of a three-dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is given by the formula

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma. \tag{11}$$

The definition is analogous to the flux of a two-dimensional field $\mathbf{F}$ across a plane curve C. In the plane (Section 14.2), the flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds,$$

the integral of the scalar component of $\mathbf{F}$ normal to the curve.

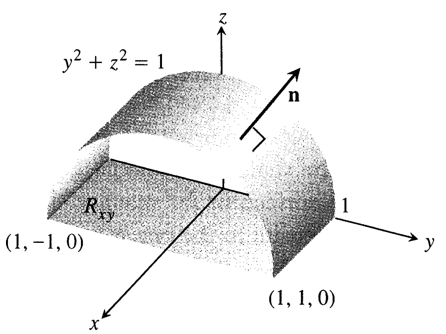
If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, the flux of $\mathbf{F}$ across S is the net rate at which fluid is crossing S in the chosen positive direction. We will discuss such flows in more detail in Section 14.7.

If $\mathbf{S}$ is part of a level surface $g(x, y, z) = c$, then $\mathbf{n}$ may be taken to be one of the two fields

$$\mathbf{n} = \pm\frac{\nabla g}{|\nabla g|}, \tag{12}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma && \text{Eq. (11)} \\ &= \iint_R \left(\mathbf{F}\cdot\frac{\pm\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}\,dA && \text{Eqs. (12) and (10)} \\ &= \iint_R \mathbf{F}\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{p}|}\,dA. && (13) \end{aligned}$$



14.48 Example 4 calculates the flux of a vector field outward through this surface. The area of the shadow region R_{xy} is 2.

EXAMPLE 4 Find the flux of $\mathbf{F} = yz\,\mathbf{j} + z^2\,\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \ge 0$, by the planes $x = 0$ and $x = 1$.

Solution The outward normal field on S (Fig. 14.48) may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\mathbf{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\,\mathbf{j} + 2z\,\mathbf{k}}{2\sqrt{1}} = y\,\mathbf{j} + z\,\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$, we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g\cdot\mathbf{k}|}\,dA = \frac{2}{|2z|}\,dA = \frac{1}{z}\,dA.$$

We can drop the absolute value bars because $z \ge 0$ on S.

The value of $\mathbf{F}\cdot\mathbf{n}$ on the surface is given by the formula

$$\begin{aligned} \mathbf{F}\cdot\mathbf{n} &= (yz\,\mathbf{j} + z^2\,\mathbf{k})\cdot(y\,\mathbf{j} + z\,\mathbf{k}) \\ &= y^2z + z^3 = z(y^2 + z^2) \\ &= z. && y^2 + z^2 = 1 \text{ on } S \end{aligned}$$

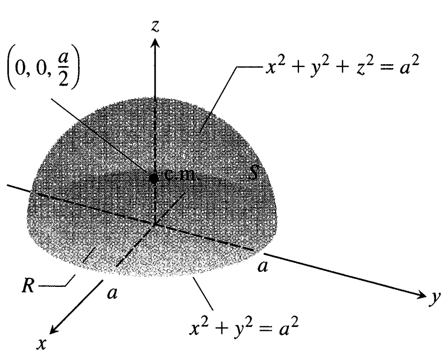
Therefore, the flux of $\mathbf{F}$ outward through S is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S (z)\left(\frac{1}{z}\,dA\right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2.$$

❑

Moments and Masses of Thin Shells

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated with the formulas in Table 14.3.



14.49 The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top (Example 5).

EXAMPLE 5 Find the center of mass of a thin hemispherical shell of radius a and constant density δ.

Solution We model the shell with the hemisphere

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \qquad z \ge 0$$

(Fig. 14.49). The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$ from the formula $\bar{z} = M_{xy}/M$.

The mass of the shell is

$$M = \iint_S \delta\,d\sigma = \delta\iint_S d\sigma = (\delta)(\text{area of } S) = 2\pi a^2\delta.$$

Table 14.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta(x, y, z)\, d\sigma$ $\quad$ ($\delta(x, y, z)$ = density at (x, y, z), mass per unit area)

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\delta\, d\sigma, \qquad M_{xz} = \iint_S y\delta\, d\sigma, \qquad M_{xy} = \iint_S z\delta\, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia:

$$I_x = \iint_S (y^2 + z^2)\,\delta\, d\sigma, \qquad I_y = \iint_S (x^2 + z^2)\,\delta\, d\sigma,$$

$$I_z = \iint_S (x^2 + y^2)\,\delta\, d\sigma, \qquad I_L = \iint_S r^2\,\delta\, d\sigma,$$

$r(x, y, z)$ = distance from point (x, y, z) to line L

Radius of gyration about a line L: $\quad R_L = \sqrt{I_L/M}$

To evaluate the integral for M_{xy}, we take $\mathbf{p} = \mathbf{k}$ and calculate

$$|\nabla f| = |2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}| = 2\sqrt{x^2 + y^2 + z^2} = 2a$$
$$|\nabla f \cdot \mathbf{p}| = |\nabla f \cdot \mathbf{k}| = |2z| = 2z$$
$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \frac{a}{z}\, dA.$$

Then

$$M_{xy} = \iint_S z\delta\, d\sigma = \delta \iint_R z\frac{a}{z}\, dA = \delta a \iint_R dA = \delta a(\pi a^2) = \delta\pi a^3$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The shell's center of mass is the point $(0, 0, a/2)$. ❑

Exercises 14.5

Surface Area

1. Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

2. Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.

3. Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

4. Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy-plane.

5. Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the xy-plane.

6. Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

7. Find the area of the ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.

8. Find the area of the upper portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$.

9. Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \le y^2 + z^2 \le 4$ in the yz-plane.

10. Find the area of the surface cut from the paraboloid $x^2 + y + z^2 = 2$ by the plane $y = 0$.

11. Find the area of the surface $x^2 - 2\ln x + \sqrt{15}y - z = 0$ above the square R: $1 \le x \le 2, 0 \le y \le 1$, in the xy-plane.

12. Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square R: $0 \le x \le 1, 0 \le y \le 1$, in the xy-plane.

Surface Integrals

13. Integrate $g(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = a, y = a, z = a$.

14. Integrate $g(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.

15. Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a$, $y = b$, and $z = c$.

16. Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$.

17. Integrate $g(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

18. Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0, x = 1$, and $z = 0$.

Flux Across a Surface

In Exercises 19 and 20, find the flux of the field $\mathbf{F}$ across the portion of the given surface in the specified direction.

19. $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
S: rectangular surface $z = 0$, $0 \le x \le 2$, $0 \le y \le 3$, direction $\mathbf{k}$

20. $\mathbf{F}(x, y, z) = yx^2\mathbf{i} - 2\mathbf{j} + xz\mathbf{k}$
S: rectangular surface $y = 0$, $-1 \le x \le 2$, $2 \le z \le 7$, direction $-\mathbf{j}$

In Exercises 21–26, find the flux of the field $\mathbf{F}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

21. $\mathbf{F}(x, y, z) = z\mathbf{k}$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

23. $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$

24. $\mathbf{F}(x, y, z) = zx\mathbf{i} + zy\mathbf{j} + z^2\mathbf{k}$

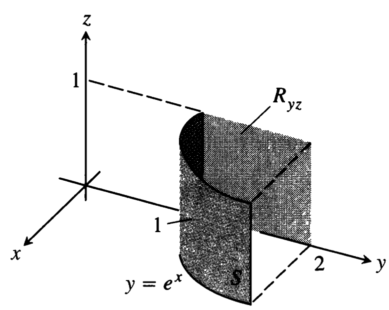
25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

26. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

27. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ upward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 1$, and $z = 0$.

28. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

29. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x-axis onto the rectangle $R_{yz}: 1 \le y \le 2, 0 \le z \le 1$ in the yz-plane (Fig. 14.50). Let $\mathbf{n}$ be the unit vector normal to S that points away from the yz-plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across S in the direction of $\mathbf{n}$.



14.50 The surface and region in Exercise 29.

30. Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y-axis onto the xz-plane is the rectangle R_{xz}: $1 \le x \le e, 0 \le z \le 1$. Let $\mathbf{n}$ be the unit vector normal to S that points away from the xz-plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through S in the direction of $\mathbf{n}$.

31. Find the outward flux of the field $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = a, y = a, z = a$.

32. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \le 25$ by the plane $z = 3$.

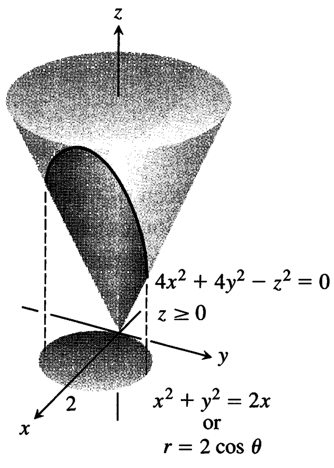
Moments and Masses

33. Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.

34. Find the centroid of the surface cut from the cylinder $y^2 + z^2 = 9$, $z \ge 0$, by the planes $x = 0$ and $x = 3$ (resembles the surface in Example 4).

35. Find the center of mass and the moment of inertia and radius of gyration about the z-axis of a thin shell of constant density δ cut from the cone $x^2+y^2-z^2=0$ by the planes $z=1$ and $z=2$.

36. Find the moment of inertia about the z-axis of a thin shell of constant density δ cut from the cone $4x^2+4y^2-z^2=0$, $z\geq 0$, by the circular cylinder $x^2+y^2=2x$ (Fig. 14.51).



14.51 The surface in Exercise 36.

37. a) Find the moment of inertia about a diameter of a thin spherical shell of radius a and constant density δ. (Work with a hemispherical shell and double the result.)

b) Use the Parallel Axis Theorem (Exercises 13.5) and the result in (a) to find the moment of inertia about a line tangent to the shell.

38. a) Find the centroid of the lateral surface of a solid cone of base radius a and height h (cone surface minus the base).

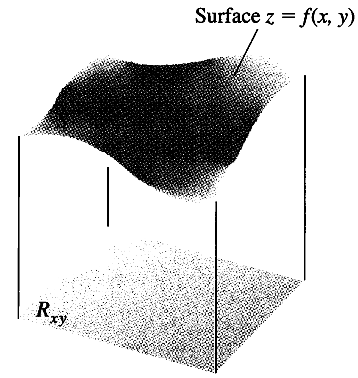
b) Use Pappus's formula (Exercises 13.5) and the result in (a) to find the centroid of the complete surface of a solid cone (side plus base).

c) A cone of radius a and height h is joined to a hemisphere of radius a to make a surface S that resembles an ice cream cone. Use Pappus's formula and the results in (a) and Example 5 to find the centroid of S. How high does the cone have to be to place the centroid in the plane shared by the bases of the hemisphere and cone?

Special Formulas for Surface Area

If S is the surface defined by a function $z=f(x,y)$ that has continuous first partial derivatives throughout a region R_{xy} in the xy-plane (Fig. 14.52), then S is also the level surface $F(x,y,z)=0$ of the function $F(x,y,z)=f(x,y)-z$. Taking the unit normal to R_{xy} to be $\mathbf{p}=\mathbf{k}$ then gives

$$|\nabla F| = |f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k}| = \sqrt{f_x{}^2+f_y{}^2+1},$$
$$|\nabla F\cdot\mathbf{p}| = |(f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k})\cdot\mathbf{k}| = |-1| = 1,$$



14.52 For a surface $z=f(x,y)$, the surface area formula in Eq. (5) takes the form
$$A=\iint_{R_{xy}}\sqrt{f_x{}^2+f_y{}^2+1}\,dx\,dy.$$

and

$$A=\iint\limits_{R_{xy}}\frac{|\nabla F|}{|\nabla F\cdot\mathbf{p}|}\,dA=\iint\limits_{R_{xy}}\sqrt{f_x{}^2+f_y{}^2+1}\,dx\,dy. \qquad (14)$$

Similarly, the area of a smooth surface $x=f(y,z)$ over a region R_{yz} in the yz-plane is

$$A=\iint\limits_{R_{yz}}\sqrt{f_y{}^2+f_z{}^2+1}\,dy\,dz, \qquad (15)$$

and the area of a smooth $y=f(x,z)$ over a region R_{xz} in the xz-plane is

$$A=\iint\limits_{R_{xz}}\sqrt{f_x{}^2+f_z{}^2+1}\,dx\,dz. \qquad (16)$$

Use Eqs. (14)–(16) to find the areas of the surfaces in Exercises 39–44.

39. The surface cut from the bottom of the paraboloid $z=x^2+y^2$ by the plane $z=3$

40. The surface cut from the "nose" of the paraboloid $x=1-y^2-z^2$ by the yz-plane

41. The portion of the cone $z=\sqrt{x^2+y^2}$ that lies over the region between the circle $x^2+y^2=1$ and the ellipse $9x^2+4y^2=36$ in the xy-plane. (*Hint:* Use formulas from geometry to find the area of the region.)

42. The triangle cut from the plane $2x+6y+3z=6$ by the bounding planes of the first octant. Calculate the area three ways, once with each area formula

43. The surface in the first octant cut from the cylinder $y=(2/3)z^{3/2}$ by the planes $x=1$ and $y=16/3$

44. The portion of the plane $y+z=4$ that lies above the region cut from the first quadrant of the xz-plane by the parabola $x=4-z^2$

14.6 Parametrized Surfaces

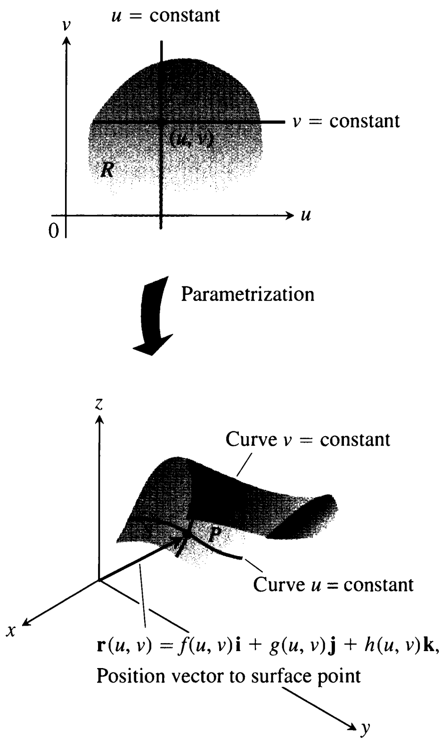
We have defined curves in the plane in three different ways:

Explicit form:	$y = f(x)$
Implicit form:	$F(x, y) = 0$
Parametric vector form:	$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad a \le t \le b.$

We have analogous definitions of surfaces in space:

Explicit form:	$z = f(x, y)$
Implicit form:	$F(x, y, z) = 0.$

There is also a parametric form that gives the position of a point on the surface as a vector function of two variables. The present section extends the investigation of surface area and surface integrals to surfaces described parametrically.



14.53 A parametrized surface.

Parametrizations of Surfaces

Let

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k} \tag{1}$$

be a continuous vector function that is defined on a region R in the uv-plane and one-to-one on the interior of R (Fig. 14.53). We call the range of $\mathbf{r}$ the **surface** S defined or traced by $\mathbf{r}$, and Eq. (1) together with the domain R constitute a **parametrization** of the surface. The variables u and v are the **parameters,** and R is the **parameter domain.** To simplify our discussion, we will take R to be a rectangle defined by inequalities of the form $a \le u \le b, c \le v \le d$. The requirement that $\mathbf{r}$ be one-to-one on the interior of R ensures that S does not cross itself. Notice that Eq. (1) is the vector equivalent of *three* parametric equations:

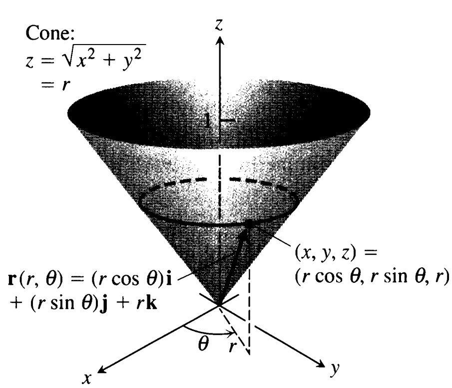
$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

EXAMPLE 1 Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$

Solution Here, cylindrical coordinates provide everything we need. A typical point (x, y, z) on the cone (Fig. 14.54) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Taking $u = r$ and $v = \theta$ in Eq. (1) gives the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi. \quad \square$$

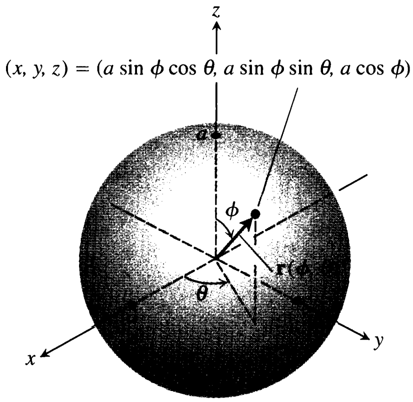


14.54 The cone in Example 1.

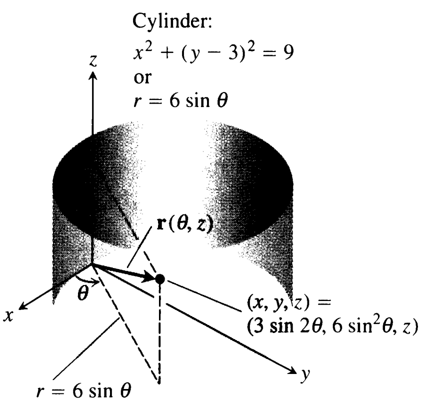
EXAMPLE 2 Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution Spherical coordinates provide what we need. A typical point (x, y, z) on the sphere (Fig. 14.55) has $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, and $z = a\cos\phi$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Taking $u = \phi$ and $v = \theta$ in Eq. (1) gives the parametrization

$$\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k},$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi. \quad \square$$



14.55 The sphere in Example 2.



14.56 The cylinder in Example 3.

EXAMPLE 3 Find a parametrization of the cylinder

$$x^2 + (y-3)^2 = 9, \qquad 0 \le z \le 5.$$

Solution In cylindrical coordinates, a point (x, y, z) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. For points on the cylinder $x^2 + (y-3)^2 = 9$ (Fig. 14.56), $r = 6\sin\theta$, $0 \le \theta \le \pi$ (Section 10.7, Example 5). A typical point on the cylinder therefore has

$$x = r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta$$

$$y = r\sin\theta = 6\sin^2\theta$$

$$z = z.$$

Taking $u = \theta$ and $v = z$ in Eq. (1) gives the parametrization

$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\,\mathbf{i} + (6\sin^2\theta)\,\mathbf{j} + z\,\mathbf{k}, \qquad 0 \le \theta \le \pi, \quad 0 \le z \le 5. \quad \square$$

Surface Area

Our goal is to find a double integral for calculating the area of a curved surface S based on the parametrization

$$\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d.$$

We need to assume that S is smooth enough for the construction we are about to carry out. The definition of smoothness involves the partial derivatives of $\mathbf{r}$ with respect to u and v:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

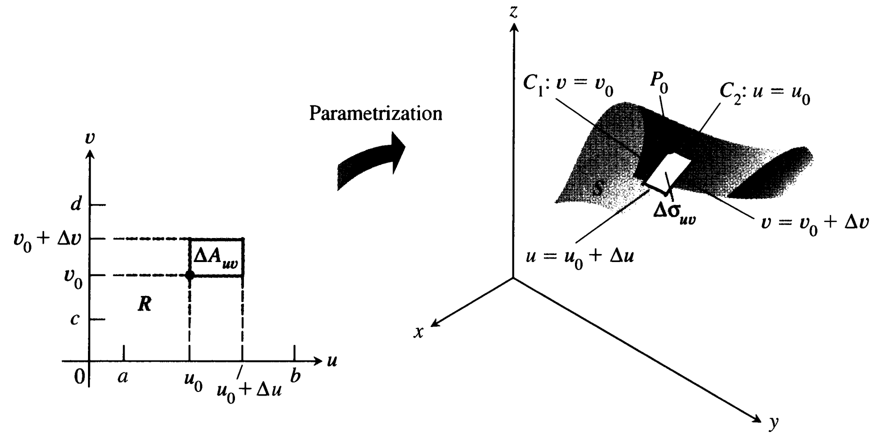
Definition

A parametrized surface $\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the parameter domain.

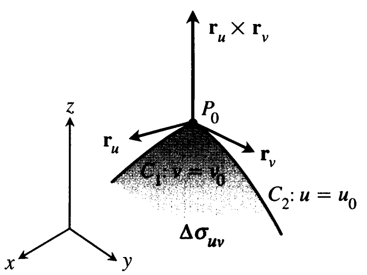
Now consider a small rectangle ΔA_{uv} in R with sides on the lines $u = u_0$, $u = u_0 + \Delta u$, $v = v_0$, and $v = v_0 + \Delta v$ (Fig. 14.57, on the following page). Each side of ΔA_{uv} maps to a curve on the surface S, and together these four curves bound a "curved area element" $\Delta\sigma_{uv}$. In the notation of the figure, the side $v = v_0$ maps to curve C_1, the side $u = u_0$ maps to C_2, and their common vertex (u_0, v_0) maps to P_0. Figure 14.58 (on the following page) shows an enlarged view of $\Delta\sigma_{uv}$. The vector $\mathbf{r}_u(u_0, v_0)$ is tangent to C_1 at P_0. Likewise, $\mathbf{r}_v(u_0, v_0)$ is tangent to C_2 at P_0. The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface at P_0. (Here is where we begin to use the assumption that S is smooth. We want to be sure that $\mathbf{r}_u \times \mathbf{r}_v \ne \mathbf{0}$.)

We next approximate the surface element $\Delta\sigma_{uv}$ by the parallelogram on the tangent plane whose sides are determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$ (Fig. 14.59, on the following page). The area of this parallelogram is

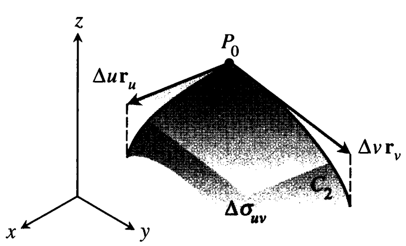
$$|\Delta u\,\mathbf{r}_u \times \Delta v\,\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v. \tag{2}$$



14.57 A rectangular area element ΔA_{uv} in the uv-plane maps onto a curved area element $\Delta\sigma_{uv}$ on S.



14.58 A magnified view of a surface area element $\Delta\sigma_{uv}$.



14.59 The parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$ approximates the surface area element $\Delta\sigma_{uv}$.

A partition of the region R in the uv-plane by rectangular regions ΔA_{uv} generates a partition of the surface S into surface area elements $\Delta\sigma_{uv}$. We approximate the area of each surface element $\Delta\sigma_{uv}$ by the parallelogram area in Eq. (2) and sum these areas together to obtain an approximation of the area of S:

$$\sum_u \sum_v |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v. \tag{3}$$

As Δu and Δv approach zero independently, the continuity of $\mathbf{r}_u$ and $\mathbf{r}_v$ guarantees that the sum in Eq. (3) approaches the double integral $\int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$. This double integral gives the area of the surface S.

Parametric Formula for the Area of a Smooth Surface

The area of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

As in Section 14.5, we can abbreviate the integral in (4) by writing $d\sigma$ for $|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$.

Surface Area Differential and the Differential Formula for Surface Area

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv \qquad\qquad \iint_S d\sigma \tag{5}$$

surface area differential — differential formula for surface area

EXAMPLE 4 Find the surface area of the cone in Example 1 (Fig. 14.54).

Solution In Example 1 we found the parametrization

$$\mathbf{r}(r,\theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + r\,\mathbf{k}, \qquad 0 \le r \le 1, \qquad 0 \le \theta \le 2\pi.$$

To apply Eq. (4) we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\,\mathbf{i} - (r\sin\theta)\,\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\,\mathbf{k}.$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\,r$. The area of the cone is

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta \qquad \text{Eq. (4) with } u = r, v = \theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\,d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2}.$$

❑

EXAMPLE 5 Find the surface area of a sphere of radius a.

Solution We use the parametrization from Example 2:

$$\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\,\mathbf{i} + (a\sin\phi\sin\theta)\,\mathbf{j} + (a\cos\phi)\,\mathbf{k},$$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

For $\mathbf{r}_\phi \times \mathbf{r}_\theta$ we get

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\,\mathbf{k}.$$

Thus,

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi(\sin^2\phi + \cos^2\phi)}$$

$$= a^2\sqrt{\sin^2\phi} = a^2\sin\phi,$$

since $\sin\phi \ge 0$ for $0 \le \phi \le \pi$. Therefore the area of the sphere is

$$A = \int_0^{2\pi}\int_0^\pi a^2\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\Big[-a^2\cos\phi\Big]_0^\pi\,d\theta = \int_0^{2\pi} 2a^2\,d\theta = 4\pi a^2.$$

❑

Surface Integrals

Having found the formula for calculating the area of a parametrized surface, we can now integrate a function over the surface using the parametrized form.

Definition

If S is a smooth surface defined parametrically as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k},\ a \le u \le b,\ c \le v \le d$, and $G(x, y, z)$ is a continuous function defined on S, then the **integral of G over S** is

$$\iint_S G(x, y, z)\, d\sigma = \int_c^d \int_a^b G(f(u, v), g(u, v), h(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

EXAMPLE 6 Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

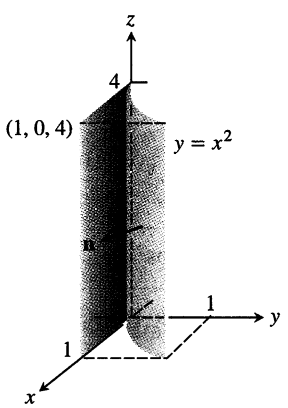
Solution Continuing the work in Examples 1 and 4, we have $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}\,r$ and

$$\iint_S x^2\, d\sigma = \int_0^{2\pi} \int_0^1 (r^2 \cos^2\theta)(\sqrt{2}\,r)\, dr\, d\theta \qquad x = r\cos\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3 \cos^2\theta\, dr\, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}.$$

❑



14.60 The parabolic surface in Example 7.

EXAMPLE 7 Find the flux of $\mathbf{F} = yz\,\mathbf{i} + x\,\mathbf{j} - z^2\,\mathbf{k}$ outward through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$ (Fig. 14.60).

Solution On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\,\mathbf{i} + x^2\,\mathbf{j} + z\,\mathbf{k},\ 0 \le x \le 1,\ 0 \le z \le 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\,\mathbf{i} - \mathbf{j}.$$

The unit normal pointing outward from the surface is

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\,\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field is

$$\mathbf{F} = yz\,\mathbf{i} + x\,\mathbf{j} - z^2\,\mathbf{k} = x^2 z\,\mathbf{i} + x\,\mathbf{j} - z^2\,\mathbf{k}.$$

Thus,

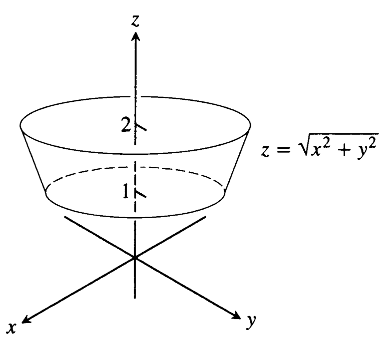
$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{4x^2 + 1}}\Big((x^2 z)(2x) + (x)(-1) + (-z^2)(0)\Big)$$

$$= \frac{2x^3 z - x}{\sqrt{4x^2 + 1}}.$$

The flux of $\mathbf{F}$ outward through the surface is

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\,|\mathbf{r}_x\times\mathbf{r}_z|\,dx\,dz \\
&= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\sqrt{4x^2+1}\,dx\,dz \\
&= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2\right]_{x=0}^{x=1} dz \\
&= \int_0^4 \frac{1}{2}(z-1)\,dz = \frac{1}{4}(z-1)^2\Big]_0^4 \\
&= \frac{1}{4}(9) - \frac{1}{4}(1) = 2.
\end{aligned}$$

❑



14.61 The cone frustum in Example 8.

EXAMPLE 8 Find the center of mass of a thin shell of constant density δ cut from the cone $z = \sqrt{x^2+y^2}$ by the planes $z = 1$ and $z = 2$ (Fig. 14.61).

Solution The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$. We find $\bar{z} = M_{xy}/M$. Working as in Examples 1 and 4 we have

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\,\mathbf{k}, \qquad 1\le r\le 2,\quad 0\le\theta\le 2\pi,$$

and

$$|\mathbf{r}_r\times\mathbf{r}_\theta| = \sqrt{2}\,r.$$

Therefore,

$$\begin{aligned}
M &= \iint_S \delta\,d\sigma = \int_0^{2\pi}\int_1^2 \delta\sqrt{2}\,r\,dr\,d\theta \\
&= \delta\sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2}\right]_1^2 d\theta = \delta\sqrt{2}\int_0^{2\pi}\left(2-\frac{1}{2}\right)d\theta \\
&= \delta\sqrt{2}\left[\frac{3\theta}{2}\right]_0^{2\pi} = 3\pi\,\delta\sqrt{2} \\
M_{xy} &= \iint_S \delta z\,d\sigma = \int_0^{2\pi}\int_1^2 \delta r\sqrt{2}\,r\,dr\,d\theta \\
&= \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta = \delta\sqrt{2}\int_0^{2\pi}\left[\frac{r^3}{3}\right]_1^2 d\theta \\
&= \delta\sqrt{2}\int_0^{2\pi}\frac{7}{3}\,d\theta = \frac{14}{3}\pi\,\delta\sqrt{2} \\
\bar{z} &= \frac{M_{xy}}{M} = \frac{14\pi\,\delta\sqrt{2}}{3(3\pi\,\delta\sqrt{2})} = \frac{14}{9}.
\end{aligned}$$

The shell's center of mass is the point $(0, 0, 14/9)$.

Exercises 14.6

Finding Parametrizations for Surfaces

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

1. The paraboloid $z = x^2 + y^2$, $z \le 4$
2. The paraboloid $z = 9 - x^2 - y^2$, $z \ge 0$
3. The first-octant portion of the cone $z = \sqrt{x^2 + y^2}/2$ between the planes $z = 0$ and $z = 3$
4. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$
5. The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$
6. The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$
7. The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$
8. The upper portion cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$
9. The surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 2$, and $z = 0$
10. The surface cut from the parabolic cylinder $y = x^2$ by the planes $z = 0, z = 3$, and $y = 2$
11. The portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 0$ and $x = 3$
12. The portion of the cylinder $x^2 + z^2 = 4$ above the xy-plane between the planes $y = -2$ and $y = 2$
13. The portion of the plane $x + y + z = 1$
    - **a)** inside the cylinder $x^2 + y^2 = 9$
    - **b)** inside the cylinder $y^2 + z^2 = 9$
14. The portion of the plane $x - y + 2z = 2$
    - **a)** inside the cylinder $x^2 + z^2 = 3$
    - **b)** inside the cylinder $y^2 + z^2 = 2$
15. The portion of the cylinder $(x - 2)^2 + z^2 = 4$ between the planes $y = 0$ and $y = 3$
16. The portion of the cylinder $y^2 + (z - 5)^2 = 25$ between the planes $x = 0$ and $x = 10$

Areas of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

17. The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$
18. The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$
19. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$
20. The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$
21. The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$
22. The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$
23. The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$
24. The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$
25. The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$
26. The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

Parametrized Surface Integrals

In Exercises 27–34, integrate the given function over the given surface.

27. $G(x, y, z) = x$, over the parabolic cylinder $y = x^2, 0 \le x \le 2$, $0 \le z \le 3$
28. $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4, z \ge 0$, $1 \le x \le 4$
29. $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
30. $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2, z \ge 0$
31. $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$, in the xy-plane
32. $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}, 0 \le z \le 1$
33. $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2, z \ge 0$
34. $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Flux Across Parametrized Surfaces

In Exercises 35–44, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the given direction.

35. $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 1$, and $z = 0$

36. $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \le x \le 1$, by the planes $z = 0$ and $z = 2$

37. $\mathbf{F} = z\,\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin

38. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin

39. $\mathbf{F} = 2xy\,\mathbf{i} + 2yz\,\mathbf{j} + 2xz\,\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \le x \le a$, $0 \le y \le a$, in the xy-plane

40. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$

41. $\mathbf{F} = xy\,\mathbf{i} - z\,\mathbf{k}$ outward (normal away from the z-axis) through the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$

42. $\mathbf{F} = y^2\,\mathbf{i} + xz\,\mathbf{j} - \mathbf{k}$ outward (normal away from the z-axis) through the cone $z = 2\sqrt{x^2 + y^2}$, $0 \le z \le 2$

43. $\mathbf{F} = -x\,\mathbf{i} - y\,\mathbf{j} + z^2\,\mathbf{k}$ outward (normal away from the z-axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$

44. $\mathbf{F} = 4x\,\mathbf{i} + 4y\,\mathbf{j} + 2\,\mathbf{k}$ outward (normal away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$

Moments and Masses

45. Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.

46. Find the center of mass and the moment of inertia and radius of gyration about the z-axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.

47. Find the moment of inertia about the z-axis of a thin spherical shell $x^2 + y^2 + z^2 = a^2$ of constant density δ.

48. Find the moment of inertia about the z-axis of a thin conical shell $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$, of constant density δ.

Tangent Planes to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, which is the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 49–52, find an equation for the plane that is tangent to the surface at the given point P_0. Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

49. The cone $\mathbf{r}(r, \theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + r\,\mathbf{k}$, $r \ge 0$, $0 \le \theta \le 2\pi$ at the point $P_0\left(\sqrt{2}, \sqrt{2}, 2\right)$ corresponding to $(r, \theta) = (2, \pi/4)$

50. The hemisphere surface

$$\mathbf{r}(\phi, \theta) = (4\sin\phi\cos\theta)\,\mathbf{i} + (4\sin\phi\sin\theta)\,\mathbf{j} + (4\cos\phi)\,\mathbf{k},$$

$0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$, at the point $P_0\left(\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right)$ corresponding to $(\phi, \theta) = (\pi/6, \pi/4)$

51. The circular cylinder $\mathbf{r}(\theta, z) = (3\sin 2\theta)\,\mathbf{i} + (6\sin^2\theta)\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le \pi$, at the point $P_0\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right)$ corresponding to $(\theta, z) = (\pi/3, 0)$ (See Example 3.)

52. The parabolic cylinder surface $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} - x^2\,\mathbf{k}$, $-\infty < x < \infty$, $-\infty < y < \infty$, at the point $P_0(1, 2, -1)$ corresponding to $(x, y) = (1, 2)$

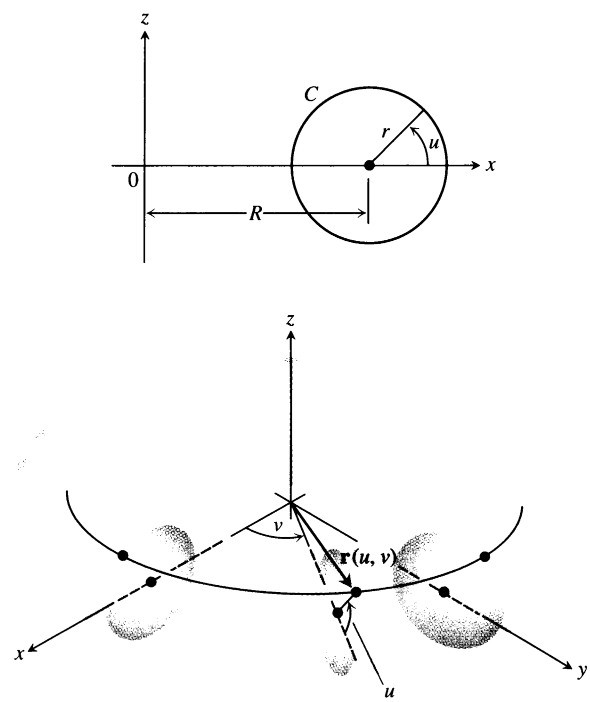
Further Examples of Parametrizations

53. a) A *torus of revolution* (doughnut) is the surface obtained by rotating a circle C in the xz-plane about the z-axis in space. If the radius of C is $r > 0$ and the center is $(R, 0, 0)$, show that a parametrization of the torus is

$$\mathbf{r}(u, v) = ((R + r\cos u)\cos v)\,\mathbf{i} + ((R + r\cos u)\sin v)\,\mathbf{j} + (r\sin u)\,\mathbf{k},$$

where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ are the angles in Fig. 14.62.

b) Show that the surface area of the torus is $A = 4\pi^2 R r$.



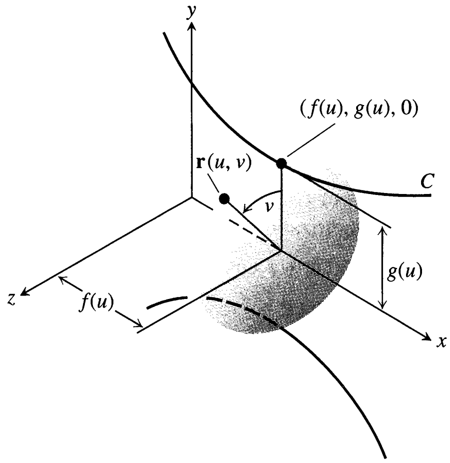
14.62 The torus surface in Exercise 53.

54. *Parametrization of a surface of revolution.* Suppose the parametrized curve C: $(f(u), g(u))$ is revolved about the x-axis, where $g(u) > 0$ for $a \le u \le b$.

a) Show that

$$\mathbf{r}(u, v) = f(u)\,\mathbf{i} + (g(u)\cos v)\,\mathbf{j} + (g(u)\sin v)\,\mathbf{k}$$

is a parametrization of the resulting surface of revolution, where $0 \le v \le 2\pi$ is the angle from the xy-plane to the point $\mathbf{r}(u, v)$ on the surface. (See the accompanying figure.) Notice that $f(u)$ measures distance *along* the axis of revolution and $g(u)$ measures distance *from* the axis of revolution.



b) Find a parametrization for the surface obtained by revolving the curve $x = y^2$, $y \ge 0$, about the x-axis.

55. a) Recall the parametrization $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$ for the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ (Section 9.4, Example 5). Using the angles θ and ϕ as defined in spherical coordinates, show that

$$\mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi)\,\mathbf{i} + (b\sin\theta\cos\phi)\,\mathbf{j} + (c\sin\phi)\,\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b) Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

56. a) Find a parametrization for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ in terms of the angle θ associated with the circle $x^2 + y^2 = r^2$ and the hyperbolic parameter u associated with the hyperbolic function $r^2 - z^2 = 1$. (See Section 6.10. Exercise 86.)

b) Generalize the result in (a) to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

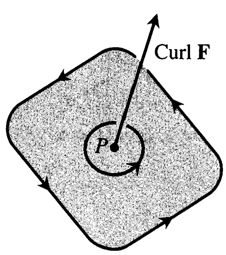
57. (*Continuation of Exercise 56.*) Find a Cartesian equation for the plane tangent to the hyperboloid $x^2 + y^2 - z^2 = 25$ at the point $(x_0, y_0, 0)$, where ${x_0}^2 + {y_0}^2 = 25$.

58. Find a parametrization of the hyperboloid of two sheets $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$.

14.7 Stokes's Theorem

As we saw in Section 14.4, the circulation density or curl of a two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at a point (x, y) is described by the scalar quantity $(\partial N/\partial x - \partial M/\partial y)$. In three dimensions, the circulation around a point P in a plane is described with a vector. This vector is normal to the plane of the circulation (Fig. 14.63) and points in the direction that gives it a right-hand relation to the circulation line. The length of the vector gives the rate of the fluid's rotation, which usually varies as the circulation plane is tilted about P. It turns out that the vector of greatest circulation in a flow with velocity field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{curl }\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}. \tag{1}$$



14.63 The circulation vector at a point P in a plane in a three-dimensional fluid flow. Notice its right-hand relation to the circulation line.

We get this information from Stokes's theorem, the generalization of the circulation-curl form of Green's theorem to space.

Del Notation

The formula for curl $\mathbf{F}$ in Eq. (1) is usually written using the symbolic operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{2}$$

George Gabriel Stokes

Sir George Gabriel Stokes (1819–1903), one of the most influential scientific figures of his century, was Lucasian Professor of Mathematics at Cambridge University from 1849 until his death in 1903. His theoretical and experimental investigations covered hydrodynamics, elasticity, light, gravity, sound, heat, meteorology, and solar physics. He left electricity and magnetism to his friend William Thomson, Baron Kelvin of Largs. It is another one of those delightful quirks of history that the theorem we call Stokes's theorem isn't his theorem at all. He learned of it from Thomson in 1850 and a few years later included it among the questions on an examination he wrote for the Smith Prize. It has been known as Stokes's theorem ever since. As usual, things have balanced out. Stokes was the original discoverer of the principles of spectrum analysis that we now credit to Bunsen and Kirchhoff.

(The symbol ∇ is pronounced "del.") The curl of $\mathbf{F}$ is $\nabla \times \mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \qquad (3) \\ &= \operatorname{curl} \mathbf{F}. \end{aligned}$$

$$\boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} \qquad (4)}$$

EXAMPLE 1 Find the curl of $\mathbf{F} = (x^2 - y)\,\mathbf{i} + 4z\,\mathbf{j} + x^2\,\mathbf{k}$.

Solution

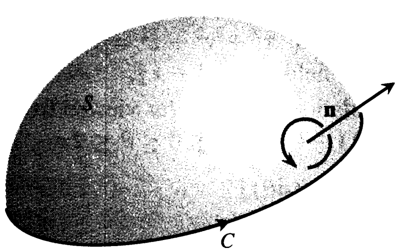
$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \qquad \text{Eq. (4)} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2 - y)\right)\mathbf{j} \\ &\qquad + \left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y)\right)\mathbf{k} \\ &= (0 - 4)\,\mathbf{i} - (2x - 0)\,\mathbf{j} + (0 + 1)\,\mathbf{k} \\ &= -4\,\mathbf{i} - 2x\,\mathbf{j} + \mathbf{k} \end{aligned}$$

❑

As we will see, the operator ∇ has a number of other applications. For instance, when applied to a scalar function $f(x, y, z)$, it gives the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

This may now be read as "del f" as well as "grad f."



14.64 The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$.

Stokes's Theorem

Stokes's theorem says that, under conditions normally met in practice, the circulation of a vector field around the boundary of an oriented surface in space in the direction counterclockwise with respect to the surface's unit normal vector field $\mathbf{n}$ (Fig. 14.64) equals the integral of the normal component of the curl of the field over the surface.

Theorem 5
Stokes's Theorem

The circulation of $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

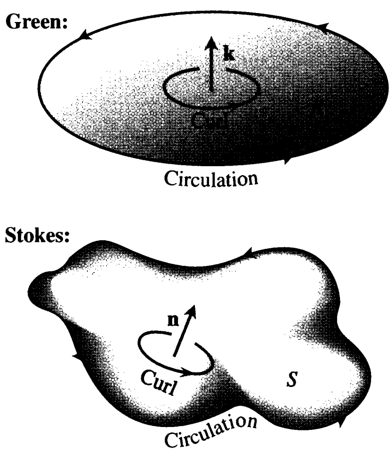
$$\underbrace{\oint_C \mathbf{F} \cdot d\,\mathbf{r}}_{\text{counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{curl integral}} \tag{5}$$

Notice from Eq. (5) that if two different oriented surfaces S_1 and S_2 have the same boundary C, then their curl integrals are equal:

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1\, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2\, d\sigma.$$

Both curl integrals equal the counterclockwise circulation integral on the left side of Eq. (5) as long as the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ correctly orient the surfaces.

Naturally, we need some mathematical restrictions on $\mathbf{F}$, C, and S to ensure the existence of the integrals in Stokes's equation. The usual restrictions are that all the functions and derivatives involved be continuous.



14.65 Green's theorem vs. Stokes's theorem.

If C is a curve in the xy-plane, oriented counterclockwise, and R is the region in the xy-plane bounded by C, then $d\sigma = dx\,dy$ and

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right). \tag{6}$$

Under these conditions, Stokes's equation becomes

$$\oint_C \mathbf{F} \cdot d\,\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

which is the circulation-curl form of the equation in Green's theorem. Conversely, by reversing these steps we can rewrite the circulation-curl form of Green's theorem for two-dimensional fields in del notation as

$$\oint_C \mathbf{F} \cdot d\,\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA. \tag{7}$$

See Fig. 14.65.

EXAMPLE 2 Evaluate Eq. (5) for the hemisphere S: $x^2 + y^2 + z^2 = 9$, $z \geq 0$, its bounding circle C: $x^2 + y^2 = 9$, $z = 0$, and the field $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$.

Solution We calculate the counterclockwise circulation around C (as viewed from above) using the parametrization $\mathbf{r}(\theta) = (3 \cos \theta)\,\mathbf{i} + (3 \sin \theta)\,\mathbf{j}$, $0 \leq \theta \leq 2\pi$:

$$d\,\mathbf{r} = (-3 \sin \theta\, d\theta)\,\mathbf{i} + (3 \cos \theta\, d\theta)\,\mathbf{j}$$

$$\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j} = (3 \sin \theta)\,\mathbf{i} - (3 \cos \theta)\,\mathbf{j}$$

$$\mathbf{F} \cdot d\,\mathbf{r} = -9 \sin^2 \theta\, d\theta - 9 \cos^2 \theta\, d\theta = -9\, d\theta$$

$$\oint_C \mathbf{F} \cdot d\,\mathbf{r} = \int_0^{2\pi} -9\, d\theta = -18\pi.$$

For the curl integral of **F**, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (-1-1)\,\mathbf{k} = -2\,\mathbf{k}$$

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{3} \qquad \text{Outer unit normal}$$

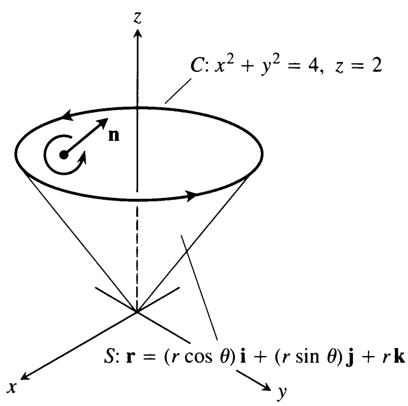
$$d\sigma = \frac{3}{z}\,dA \qquad \text{Section 14.5, Example 5, with } a = 3$$

$$\nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = -\frac{2z}{3}\frac{3}{z}\,dA = -2\,dA$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_{x^2+y^2 \le 9} -2\,dA = -18\pi.$$

The circulation around the circle equals the integral of the curl over the hemisphere, as it should. ❑



14.66 The curve C and cone S in Example 3.

EXAMPLE 3 Find the circulation of the field $\mathbf{F} = (x^2 - y)\,\mathbf{i} + 4z\,\mathbf{j} + x^2\,\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2+y^2}$, counterclockwise as viewed from above (Fig. 14.66).

Solution Stokes's theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the counterclockwise direction viewed from above corresponds to taking the *inner* normal **n** to the cone (which has a positive z-component).

We parametrize the cone as

$$\mathbf{r}(r,\theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + r\,\mathbf{k}, \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

We then have

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} = \frac{-(r\cos\theta)\,\mathbf{i} - (r\sin\theta)\,\mathbf{j} + r\,\mathbf{k}}{r\sqrt{2}} \qquad \text{Section 14.6, Example 4}$$

$$= \frac{1}{\sqrt{2}}(-(\cos\theta)\,\mathbf{i} - (\sin\theta)\,\mathbf{j} + \mathbf{k})$$

$$d\sigma = r\sqrt{2}\,dr\,d\theta \qquad \text{Section 14.6, Example 4}$$

$$\nabla \times \mathbf{F} = -4\,\mathbf{i} - 2x\,\mathbf{j} + \mathbf{k} \qquad \text{Example 1}$$

$$= -4\,\mathbf{i} - 2r\cos\theta\,\mathbf{j} + \mathbf{k}. \qquad x = r\cos\theta$$

Accordingly,

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{2}}(4\cos\theta + 2r\cos\theta\sin\theta + 1)$$

$$= \frac{1}{\sqrt{2}}(4\cos\theta + r\sin 2\theta + 1)$$

and the circulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{Stokes's theorem}$$

$$= \int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{2}}\big(4 \cos \theta + r \sin 2\theta + 1\big)\big(r\sqrt{2}\, dr\, d\theta\big) = 4\pi.$$

❑

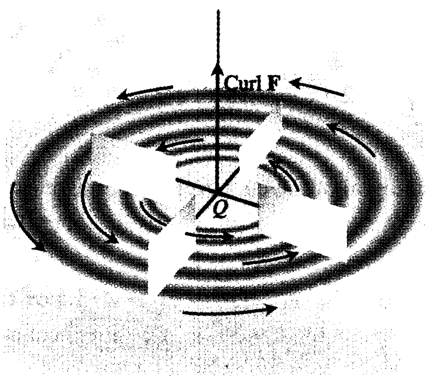
An Interpretation of $\nabla \times \mathbf{F}$

Suppose that $\mathbf{v}(x, y, z)$ is the velocity of a moving fluid whose density at (x, y, z) is $\delta(x, y, z)$, and let $\mathbf{F} = \delta\mathbf{v}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is the circulation of the fluid around the closed curve C. By Stokes's theorem, the circulation is equal to the flux of $\nabla \times \mathbf{F}$ through a surface S spanning C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$



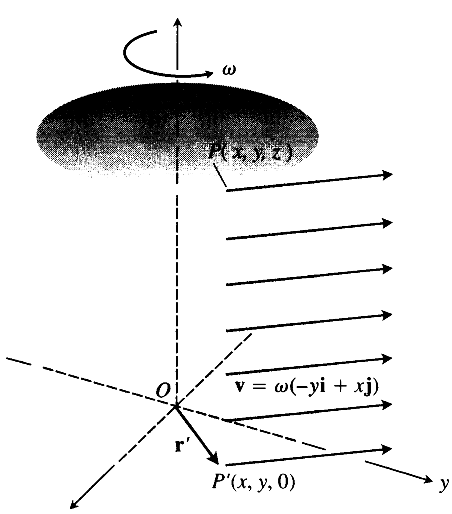
14.67 The paddle wheel interpretation of curl **F**.

Suppose we fix a point Q in the domain of $\mathbf{F}$ and a direction $\mathbf{u}$ at Q. Let C be a circle of radius ρ, with center at Q, whose plane is normal to $\mathbf{u}$. If $\nabla \times \mathbf{F}$ is continuous at Q, then the average value of the $\mathbf{u}$-component of $\nabla \times \mathbf{F}$ over the circular disk S bounded by C approaches the $\mathbf{u}$-component of $\nabla \times \mathbf{F}$ at Q as $\rho \to 0$:

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \to 0} \frac{1}{\pi\rho^2} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{u}\, d\sigma. \tag{8}$$

If we replace the double integral in Eq. (8) by the circulation, we get

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \to 0} \frac{1}{\pi\rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r}. \tag{9}$$



14.68 A steady rotational flow parallel to the *xy*-plane, with constant angular velocity ω in the positive (counterclockwise) direction.

The left-hand side of Eq. (9) has its maximum value when $\mathbf{u}$ is the direction of $\nabla \times \mathbf{F}$. When ρ is small, the limit on the right-hand side of Eq. (9) is approximately

$$\frac{1}{\pi\rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

which is the circulation around C divided by the area of the disk (circulation density). Suppose that a small paddle wheel of radius ρ is introduced into the fluid at Q, with its axle directed along $\mathbf{u}$. The circulation of the fluid around C will affect the rate of spin of the paddle wheel. The wheel will spin fastest when the circulation integral is maximized; therefore it will spin fastest when the axle of the paddle wheel points in the direction of $\nabla \times \mathbf{F}$ (Fig. 14.67).

EXAMPLE 4 A fluid of constant density δ rotates around the z-axis with velocity $\mathbf{v} = \omega(-y\,\mathbf{i} + x\,\mathbf{j})$, where ω is a positive constant called the *angular velocity* of the rotation (Fig. 14.68). If $\mathbf{F} = \delta\mathbf{v}$, find $\nabla \times \mathbf{F}$ and relate it to the circulation density.

Solution With $\mathbf{F} = \delta\mathbf{v} = -\delta\omega y\,\mathbf{i} + \delta\omega x\,\mathbf{j}$,

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

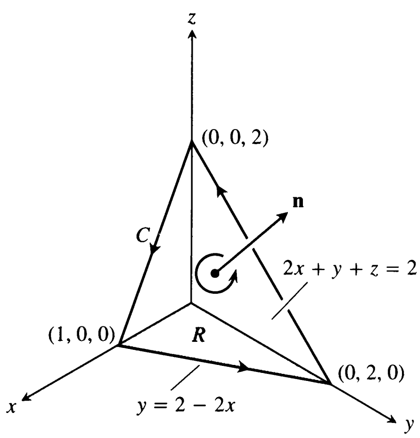
$$= (0 - 0)\,\mathbf{i} + (0 - 0)\,\mathbf{j} + (\delta\omega - (-\delta\omega))\,\mathbf{k} = 2\,\delta\omega\,\mathbf{k}.$$

By Stokes's theorem, the circulation of $\mathbf{F}$ around a circle C of radius ρ bounding a disk S in a plane normal to $\nabla \times \mathbf{F}$, say the xy-plane, is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 2\,\delta\omega\,\mathbf{k} \cdot \mathbf{k}\, dx\, dy = (2\,\delta\omega)(\pi\rho^2).$$

Thus,
$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = 2\,\delta\omega = \frac{1}{\pi\rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

in agreement with Eq. (9) with $\mathbf{u} = \mathbf{k}$.



14.69 The planar surface in Example 5.

EXAMPLE 5 Use Stokes's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F} = xz\,\mathbf{i} + xy\,\mathbf{j} + 3xz\,\mathbf{k}$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise as viewed from above (Fig. 14.69).

Solution The plane is the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = 2x + y + z$. The unit normal vector

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2\,\mathbf{i} + \mathbf{j} + \mathbf{k})}{|2\,\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{6}}(2\,\mathbf{i} + \mathbf{j} + \mathbf{k})$$

is consistent with the counterclockwise motion around C. To apply Stokes's theorem, we find

$$\text{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\,\mathbf{j} + y\,\mathbf{k}.$$

On the plane, z equals $2 - 2x - y$, so

$$\nabla \times \mathbf{F} = (x - 3(2 - 2x - y))\,\mathbf{j} + y\,\mathbf{k} = (7x + 3y - 6)\,\mathbf{j} + y\,\mathbf{k}$$

and

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}(7x + 3y - 6 + y) = \frac{1}{\sqrt{6}}(7x + 4y - 6).$$

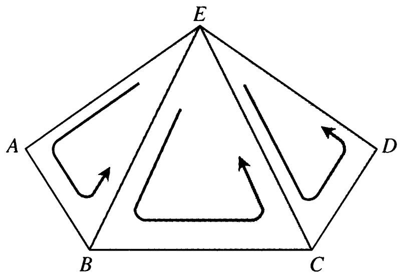
The surface area element is

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA = \frac{\sqrt{6}}{1}\, dx\, dy.$$

The circulation is
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{Stokes's theorem}$$
$$= \int_0^1 \int_0^{2-2x} \frac{1}{\sqrt{6}}(7x + 4y - 6)\sqrt{6}\, dy\, dx$$
$$= \int_0^1 \int_0^{2-2x} (7x + 4y - 6)\, dy\, dx = -1.$$

Proof of Stokes's Theorem for Polyhedral Surfaces

Let S be a polyhedral surface consisting of a finite number of plane regions. (Think of one of Buckminster Fuller's geodesic domes.) We apply Green's theorem to each



14.70 Part of a polyhedral surface.

separate panel of S. There are two types of panels:

1. those that are surrounded on all sides by other panels and
2. those that have one or more edges that are not adjacent to other panels.

The boundary Δ of S consists of those edges of the type 2 panels that are not adjacent to other panels. In Fig. 14.70, the triangles EAB, BCE, and CDE represent a part of S, with $ABCD$ part of the boundary Δ. Applying Green's theorem to the three triangles in turn and adding the results, we get

$$\left(\oint_{EAB} + \oint_{BCE} + \oint_{CDE}\right) \mathbf{F} \cdot d\mathbf{r} = \left(\iint\limits_{EAB} + \iint\limits_{BCE} + \iint\limits_{CDE}\right) \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma. \quad (10)$$

The three line integrals on the left-hand side of Eq. (10) combine into a single line integral taken around the periphery $ABCDE$ because the integrals along interior segments cancel in pairs. For example, the integral along segment BE in triangle ABE is opposite in sign to the integral along the same segment in triangle EBC. Similarly for segment CE. Hence (10) reduces to

$$\oint_{ABCDE} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{ABCDE} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

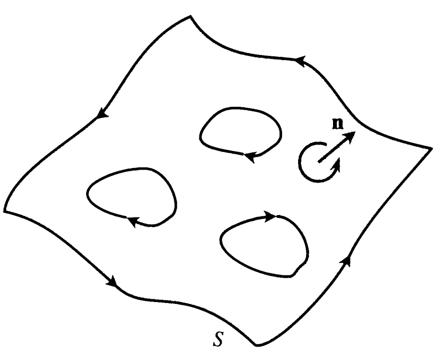
When we apply Green's theorem to all the panels and add the results, we get

$$\oint\limits_{\Delta} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{S} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma. \quad (11)$$

This is Stokes's theorem for a polyhedral surface S. You can find proofs for more general surfaces in advanced calculus texts. ❑

Stokes's Theorem for Surfaces with Holes

Stokes's theorem can be extended to an oriented surface S that has one or more holes (Fig. 14.71), in a way analogous to the extension of Green's theorem: The surface integral over S of the normal component of $\nabla \times \mathbf{F}$ equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of S.



14.71 Stokes's theorem also holds for oriented surfaces with holes.

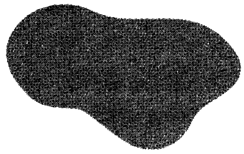
An Important Identity

The following identity arises frequently in mathematics and the physical sciences.

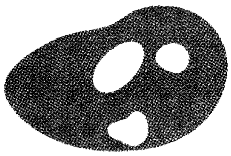
$$\boxed{\text{curl grad } f = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0} \qquad (12)}$$

This identity holds for any function $f(x, y, z)$ whose second partial derivatives are continuous. The proof goes like this:

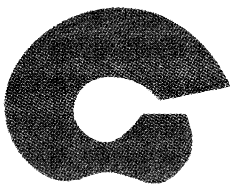
$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\,\mathbf{i} - (f_{zx} - f_{xz})\,\mathbf{j} + (f_{yx} - f_{xy})\,\mathbf{k}.$$



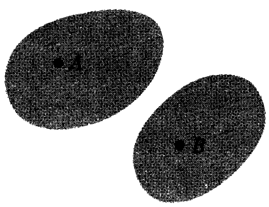
Connected and simply connected.



Connected but not simply connected.



Connected and simply connected.



Simply connected but not connected.
No path from A to B lies entirely in the region.

14.72 Connectivity and simple connectivity are not the same. Neither implies the other, as these pictures of plane regions illustrate. To make three-dimensional regions with these properties, thicken the plane regions into cylinders.

If the second partial derivatives are continuous, the mixed second derivatives in parentheses are equal (Euler's theorem, Section 12.3) and the vector is zero.

Conservative Fields and Stokes's Theorem

In Section 14.3, we found that saying that a field $\mathbf{F}$ is conservative in an open region D in space is equivalent to saying that the integral of $\mathbf{F}$ around every closed loop in D is zero. This, in turn, is equivalent in *simply connected* open regions to saying that $\nabla \times \mathbf{F} = \mathbf{0}$. A region D is **simply connected** if every closed path in D can be contracted to a point in D without ever leaving D. If D consisted of space with a line removed, for example, D would not be simply connected. There would be no way to contract a loop around the line to a point without leaving D. On the other hand, space itself *is* simply connected (Fig. 14.72).

Theorem 6

If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise smooth closed path C in D,

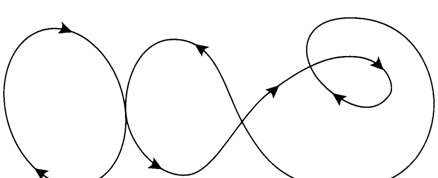
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Sketch of a Proof Theorem 6 is usually proved in two steps. The first step is for simple closed curves. A theorem from topology, a branch of advanced mathematics, states that every differentiable simple closed curve C in a simply connected open region D is the boundary of a smooth two-sided surface S that also lies in D. Hence, by Stokes's theorem,

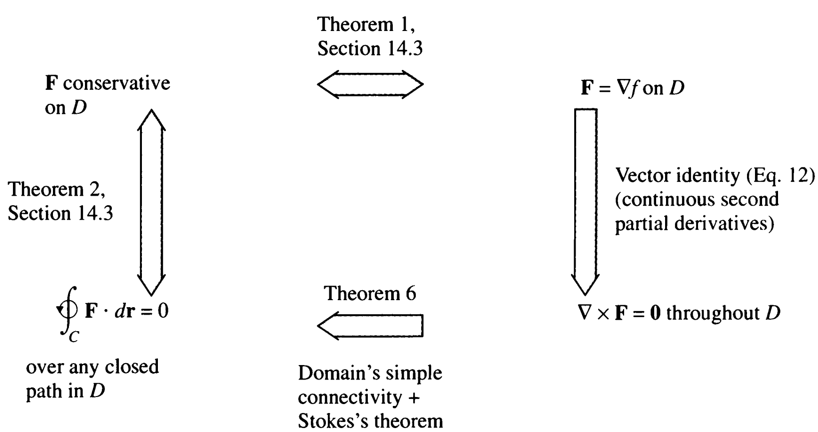
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

The second step is for curves that cross themselves, like the one in Fig. 14.73. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes's theorem one loop at a time, and add the results. ❑

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions.



14.73 In a simply connected open region in space, differentiable curves that cross themselves can be divided into loops to which Stokes's theorem applies.



Exercises 14.7

Using Stokes's Theorem to Calculate Circulation

In Exercises 1–6, use the surface integral in Stokes's theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$
   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$
   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above

3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above

6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$
   C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$

Flux of the Curl

7. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S:\quad 4x^2 + 9y^2 + 36z^2 = 36, \qquad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2}\sin e^{\sqrt{xyz}}\,\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$.)

8. Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell

$$S:\quad 4x^2 + y + z^2 = 4, \quad y \geq 0$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes's theorem to calculate the flux of $\nabla \times \mathbf{F}$ outward through S.

10. Evaluate

$$\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n}\, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

11. Show that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

has the same value for all oriented surfaces S that span C and that induce the same positive direction on C.

12. Let $\mathbf{F}$ be a differentiable vector field defined on a region containing a smooth closed oriented surface S and its interior. Let $\mathbf{n}$ be the unit normal vector field on S. Suppose that S is the union of two surfaces S_1 and S_2 joined along a smooth simple closed curve C. Can anything be said about

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma?$$

Give reasons for your answer.

Stokes's Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes's theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$
    S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4 - r^2)\mathbf{k}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$
    S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$

15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$
    S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$
    S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$, $0 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$

17. $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$
    S: $\mathbf{r}(\phi, \theta) = (\sqrt{3}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{3}\sin\phi\sin\theta)\mathbf{j} + (\sqrt{3}\cos\phi)\mathbf{k}$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$

18. $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$
    S: $\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$

Theory and Examples

19. Use the identity $\nabla \times \nabla f = \mathbf{0}$ (Eq. 12 in the text) and Stokes's theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

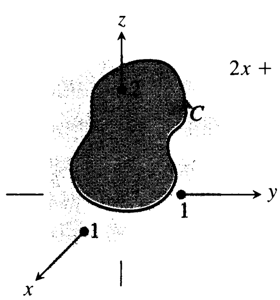
a) $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$
b) $\mathbf{F} = \nabla(xy^2z^3)$
c) $\mathbf{F} = \nabla \times (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$
d) $\mathbf{F} = \nabla f$

20. Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy-plane is zero

a) by taking $\mathbf{r} = (a \cos t)\,\mathbf{i} + (a \sin t)\,\mathbf{j}$, $0 \le t \le 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle, and
b) by applying Stokes's theorem.

21. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\oint_C 2y\,dx + 3z\,dy - x\,dz$$



depends only on the area of the region enclosed by C and not on the position or shape of C.

22. Show that if $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\nabla \times \mathbf{F} = \mathbf{0}$.

23. Find a vector field with twice-differentiable components whose curl is $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ or prove that no such field exists.

24. Does Stokes's theorem say anything special about circulation in a field whose curl is zero? Give reasons for your answer.

25. Let R be a region in the xy-plane that is bounded by a piecewise smooth simple closed curve C, and suppose that the moments of inertia of R about the x- and y-axes are known to be I_x and I_y. Evaluate the integral

$$\oint_C \nabla(r^4) \cdot \mathbf{n}\,ds,$$

where $r = \sqrt{x^2 + y^2}$, in terms of I_x and I_y.

26. Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j} + z\,\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy-plane. (Theorem 6 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

14.8 The Divergence Theorem and a Unified Theory

The divergence form of Green's theorem in the plane states that the net outward flux of a vector field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the Divergence Theorem, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface. In this section, we prove the Divergence Theorem and show how it simplifies the calculation of flux. We also derive Gauss's law for flux in an electric field and the continuity equation of hydrodynamics. Finally, we unify the chapter's vector integral theorems into a single fundamental theorem.

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ is the scaler function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \tag{1}$$

The Divergence Theorem

Mikhail Vassilievich Ostrogradsky (1801–1862) was the first mathematician to publish a proof of the Divergence Theorem. Upon being denied his degree at Kharkhov University by the minister for religious affairs and national education (for atheism), Ostrogradsky left Russia for Paris in 1822, attracted by the presence of Laplace, Legendre, Fourier, Poisson, and Cauchy. While working on the theory of heat in the mid-1820s, he formulated the Divergence Theorem as a tool for converting volume integrals to surface integrals.

Carl Friedrich Gauss (1777–1855) had already proved the theorem while working on the theory of gravitation, but his notebooks were not to be published until many years later. (The theorem is sometimes called Gauss's theorem.) The list of Gauss's accomplishments in science and mathematics is truly astonishing, ranging from the invention of the electric telegraph (with Wilhelm Weber in 1833) to the development of a wonderfully accurate theory of planetary orbits and to work in non-Euclidean geometry that later became fundamental to Einstein's general theory of relativity.

The symbol "div **F**" is read as "divergence of **F**" or "div **F**." The notation $\nabla \cdot \mathbf{F}$ is read "del dot **F**."

Div **F** has the same physical interpretation in three dimensions that it does in two. If **F** is the velocity field of a fluid flow, the value of div **F** at a point (x, y, z) is the rate at which fluid is being piped in or drained away at (x, y, z). The divergence is the flux per unit volume or flux density at the point.

EXAMPLE 1 Find the divergence of $\mathbf{F} = 2xz\,\mathbf{i} - xy\,\mathbf{j} - z\,\mathbf{k}$.

Solution The divergence of **F** is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z) = 2z - x - 1.$$

❑

The Divergence Theorem

The Divergence Theorem says that under suitable conditions the outward flux of a vector field across a closed surface (oriented outward) equals the triple integral of the divergence of the field over the region enclosed by the surface.

Theorem 7

The Divergence Theorem

The flux of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ across a closed oriented surface S in the direction of the surface's outward unit normal field **n** equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{divergence integral}}. \qquad (2)$$

EXAMPLE 2 Evaluate both sides of Eq. (2) for the field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

Solution The outer unit normal to S, calculated from the gradient of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$, is

$$\mathbf{n} = \frac{2(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}.$$

Hence

$$\mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{x^2 + y^2 + z^2}{a}\, d\sigma = \frac{a^2}{a}\, d\sigma = a\, d\sigma$$

because $x^2 + y^2 + z^2 = a^2$ on the surface. Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S a\, d\sigma = a \iint_S d\sigma = a\,(4\pi a^2) = 4\pi a^3.$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

so
$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3.$$

❑

EXAMPLE 3 Find the flux of $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution Instead of calculating the flux as a sum of six separate integrals, one for each face of the cube, we can calculate the flux by integrating the divergence

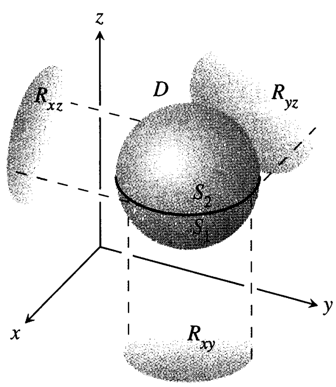
$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

over the cube's interior:

$$\text{Flux} = \iint_{\substack{\text{cube}\\\text{surface}}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_{\substack{\text{cube}\\\text{interior}}} \nabla \cdot \mathbf{F}\, dV \qquad \text{The Divergence Theorem}$$

$$= \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dx\, dy\, dz = \frac{3}{2}. \qquad \text{Routine integration}$$

❑



14.74 We first prove the Divergence Theorem for the kind of three-dimensional region shown here. We then extend the theorem to other regions.

Proof of the Divergence Theorem (Special Regions)

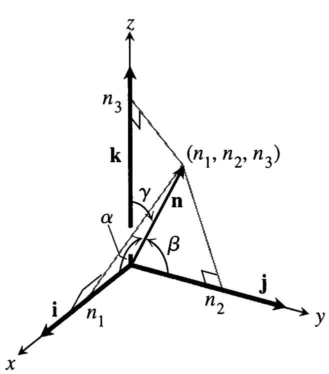
To prove the Divergence Theorem, we assume that the components of $\mathbf{F}$ have continuous first partial derivatives. We also assume that D is a convex region with no holes or bubbles, such as a solid sphere, cube, or ellipsoid, and that S is a piecewise smooth surface. In addition, we assume that any line perpendicular to the xy-plane at an interior point of the region R_{xy} that is the projection of D on the xy-plane intersects the surface S in exactly two points, producing surfaces

$$S_1: \quad z = f_1(x, y), \quad (x, y) \text{ in } R_{xy}$$
$$S_2: \quad z = f_2(x, y), \quad (x, y) \text{ in } R_{xy},$$

with $f_1 \le f_2$. We make similar assumptions about the projection of D onto the other coordinate planes. See Fig. 14.74.

The components of the unit normal vector $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$ are the cosines of the angles α, β, and γ that $\mathbf{n}$ makes with **i, j,** and **k** (Fig. 14.75). This is true because all the vectors involved are unit vectors. We have

$$n_1 = \mathbf{n} \cdot \mathbf{i} = |\mathbf{n}|\,|\mathbf{i}| \cos\alpha = \cos\alpha$$
$$n_2 = \mathbf{n} \cdot \mathbf{j} = |\mathbf{n}|\,|\mathbf{j}| \cos\beta = \cos\beta$$
$$n_3 = \mathbf{n} \cdot \mathbf{k} = |\mathbf{n}|\,|\mathbf{k}| \cos\gamma = \cos\gamma.$$



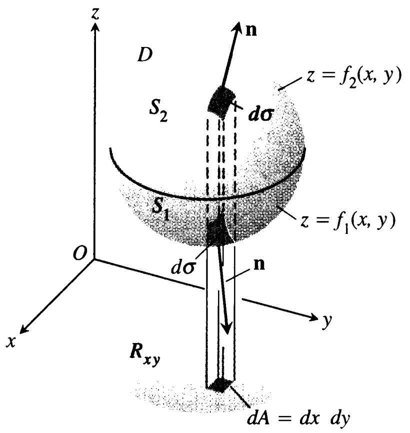
14.75 The scalar components of a unit normal vector **n** are the cosines of the angles α, β, and γ that it makes with **i, j,** and **k.**

Thus,

$$\mathbf{n} = (\cos\alpha)\,\mathbf{i} + (\cos\beta)\,\mathbf{j} + (\cos\gamma)\,\mathbf{k}$$

and

$$\mathbf{F} \cdot \mathbf{n} = M \cos\alpha + N \cos\beta + P \cos\gamma.$$



14.76 The three-dimensional region D enclosed by the surfaces S_1 and S_2 shown here projects vertically onto a two-dimensional region R_{xy} in the xy-plane.

In component form, the Divergence Theorem states that

$$\iint_S (M\cos\alpha + N\cos\beta + P\cos\gamma)\,d\sigma = \iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx\,dy\,dz.$$

We prove the theorem by proving the three following equalities:

$$\iint_S M\cos\alpha\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dx\,dy\,dz \tag{3}$$

$$\iint_S N\cos\beta\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dx\,dy\,dz \tag{4}$$

$$\iint_S P\cos\gamma\,d\sigma = \iiint_D \frac{\partial P}{\partial z}\,dx\,dy\,dz \tag{5}$$

We prove Eq. (5) by converting the surface integral on the left to a double integral over the projection R_{xy} of D on the xy-plane (Fig. 14.76). The surface S consists of an upper part S_2 whose equation is $z = f_2(x, y)$ and a lower part S_1 whose equation is $z = f_1(x, y)$. On S_2, the outer normal $\mathbf{n}$ has a positive $\mathbf{k}$-component and

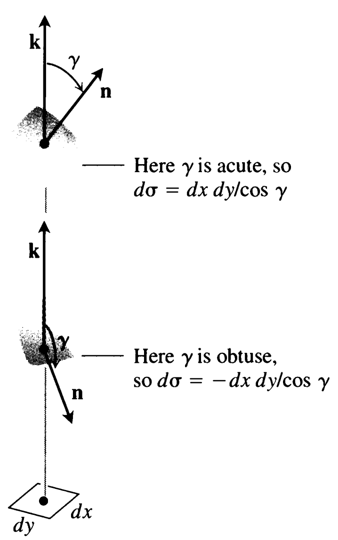
$$\cos\gamma\,d\sigma = dx\,dy \quad \text{because} \quad d\sigma = \frac{dA}{|\cos\gamma|} = \frac{dx\,dy}{\cos\gamma}.$$

See Fig. 14.77. On S_1, the outer normal $\mathbf{n}$ has a negative $\mathbf{k}$-component and

$$\cos\gamma\,d\sigma = -dx\,dy.$$

Therefore,

$$\begin{aligned}
\iint_S P\cos\gamma\,d\sigma &= \iint_{S_2} P\cos\gamma\,d\sigma + \iint_{S_1} P\cos\gamma\,d\sigma \\
&= \iint_{R_{xy}} P(x, y, f_2(x, y))\,dx\,dy - \iint_{R_{xy}} P(x, y, f_1(x, y))\,dx\,dy \\
&= \iint_{R_{xy}} [P(x, y, f_2(x, y)) - P(x, y, f_1(x, y))]\,dx\,dy \\
&= \iint_{R_{xy}} \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z}\,dz\right] dx\,dy = \iiint_D \frac{\partial P}{\partial z}\,dz\,dx\,dy.
\end{aligned}$$



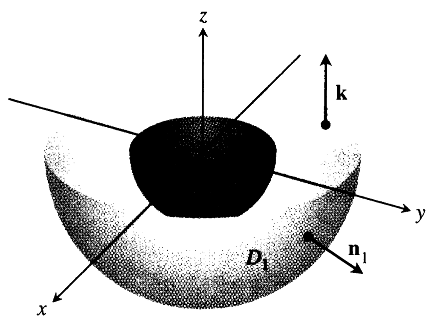
14.77 An enlarged view of the area patches in Fig. 14.76. The relations $d\sigma = \pm dx\,dy/\cos\gamma$ are derived in Section 14.5.

This proves Eq. (5).

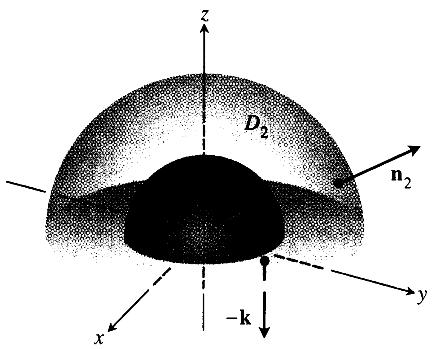
The proofs for Eqs. (3) and (4) follow the same pattern; or just permute x, y, z; M, N, P; α, β, γ, in order, and get those results from Eq. (5).

The Divergence Theorem for Other Regions

The Divergence Theorem can be extended to regions that can be partitioned into a finite number of simple regions of the type just discussed and to regions that can be defined as limits of simpler regions in certain ways. For example, suppose that D is the region between two concentric spheres and that $\mathbf{F}$ has continuously differentiable components throughout D and on the bounding surfaces. Split D by an equatorial plane and apply the Divergence Theorem to each half separately. The



14.78 The lower half of the solid region between two concentric spheres.



14.79 The upper half of the solid region between two concentric spheres.

bottom half, D_1, is shown in Fig. 14.78. The surface that bounds D_1 consists of an outer hemisphere, a plane washer-shaped base, and an inner hemisphere. The Divergence Theorem says that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma_1 = \iiint_{D_1} \nabla \cdot \mathbf{F} \, dV_1. \tag{6}$$

The unit normal $\mathbf{n}_1$ that points outward from D_1 points away from the origin along the outer surface, equals $\mathbf{k}$ along the flat base, and points toward the origin along the inner surface. Next apply the Divergence Theorem to D_2, as shown in Fig. 14.79:

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma_2 = \iiint_{D_2} \nabla \cdot \mathbf{F} \, dV_2. \tag{7}$$

As we follow $\mathbf{n}_2$ over S_2, pointing outward from D_2, we see that $\mathbf{n}_2$ equals $-\mathbf{k}$ along the washer-shaped base in the xy-plane, points away from the origin on the outer sphere, and points toward the origin on the inner sphere. When we add Eqs. (6) and (7), the integrals over the flat base cancel because of the opposite signs of $\mathbf{n}_1$ and $\mathbf{n}_2$. We thus arrive at the result

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_{D} \nabla \cdot \mathbf{F} \, dV,$$

with D the region between the spheres, S the boundary of D consisting of two spheres, and $\mathbf{n}$ the unit normal to S directed outward from D.

EXAMPLE 4 Find the net outward flux of the field

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region D: $0 < a^2 \le x^2 + y^2 + z^2 \le b^2$.

Solution The flux can be calculated by integrating $\nabla \cdot \mathbf{F}$ over D. We have

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\rho}$$

and

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x\rho^{-3}) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly,

$$\frac{\partial N}{\partial y} = \frac{1}{\rho^3} - \frac{3y^2}{\rho^5} \quad \text{and} \quad \frac{\partial P}{\partial z} = \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}.$$

Hence,

$$\text{div}\, \mathbf{F} = \frac{3}{\rho^3} - \frac{3}{\rho^5}(x^2 + y^2 + z^2) = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0$$

and

$$\iiint_{D} \nabla \cdot \mathbf{F} \, dV = 0.$$

So the integral of $\nabla \cdot \mathbf{F}$ over D is zero and the net outward flux across the boundary of D is zero. But there is more to learn from this example. The flux leaving D across the inner sphere S_a is the negative of the flux leaving D across the outer sphere S_b (because the sum of these fluxes is zero). This means that the flux of $\mathbf{F}$ across S_a in the direction away from the origin equals the flux of $\mathbf{F}$

across S_b in the direction away from the origin. Thus, the flux of $\mathbf{F}$ across a sphere centered at the origin is independent of the radius of the sphere. What is this flux?

To find it, we evaluate the flux integral directly. The outward unit normal on the sphere of radius a is

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}.$$

Hence, on the sphere,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a^3} \cdot \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and
$$\iint\limits_{S_a} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{a^2} \iint\limits_{S_a} d\sigma = \frac{1}{a^2}(4\pi a^2) = 4\pi.$$

The outward flux of $\mathbf{F}$ across any sphere centered at the origin is 4π. ❑

Gauss's Law—One of the Four Great Laws of Electromagnetic Theory

There is more to be learned from Example 4. In electromagnetic theory, the electric field created by a point charge q located at the origin is the inverse square field

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r}|^2} \left(\frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{q}{4\pi\epsilon_0} \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}$$

where ϵ_0 is a physical constant, $\mathbf{r}$ is the position vector of the point (x, y, z), and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. In the notation of Example 4,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\mathbf{F}.$$

The calculations in Example 4 show that the outward flux of $\mathbf{E}$ across any sphere centered at the origin is q/ϵ_0. But this result is not confined to spheres. The outward flux of $\mathbf{E}$ across any closed surface S that encloses the origin (and to which the Divergence Theorem applies) is also q/ϵ_0. To see why, we have only to imagine a large sphere S_a centered at the origin and enclosing the surface S. Since

$$\nabla \cdot \mathbf{E} = \nabla \cdot \frac{q}{4\pi\epsilon_0}\mathbf{F} = \frac{q}{4\pi\epsilon_0} \nabla \cdot \mathbf{F} = 0$$

when $\rho > 0$, the integral of $\nabla \cdot \mathbf{E}$ over the region D between S and S_a is zero. Hence, by the Divergence Theorem,

$$\iint\limits_{\substack{\text{boundary}\\ \text{of } D}} \mathbf{E} \cdot \mathbf{n}\, d\sigma = 0,$$

and the flux of $\mathbf{E}$ across S in the direction away from the origin must be the same as the flux of $\mathbf{E}$ across S_a in the direction away from the origin, which is q/ϵ_0. This statement, called *Gauss's law,* also applies to charge distributions that are more general than the one assumed here, as you will see in nearly any physics text.

$$\text{Gauss's Law:} \qquad \iint\limits_{S} \mathbf{E} \cdot \mathbf{n}\, d\sigma = \frac{q}{\epsilon_0}$$

The Continuity Equation of Hydrodynamics

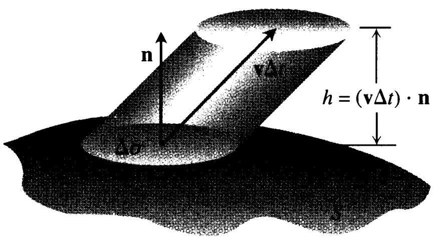
Let D be a region in space bounded by a closed oriented surface S. If $\mathbf{v}(x, y, z)$ is the velocity field of a fluid flowing smoothly through D, $\delta = \delta(t, x, y, z)$ is the fluid's density at (x, y, z) at time t, and $\mathbf{F} = \delta\mathbf{v}$, then the **continuity equation** of hydrodynamics states that

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0.$$

If the functions involved have continuous first partial derivatives, the equation evolves naturally from the Divergence Theorem, as we will now see.

First, the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$



14.80 The fluid that flows upward through the patch $\Delta\sigma$ in a short time Δt fills a "cylinder" whose volume is approximately base × height $= \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma \Delta t$.

is the rate at which mass leaves D across S (leaves because $\mathbf{n}$ is the outer normal). To see why, consider a patch of area $\Delta\sigma$ on the surface (Fig. 14.80). In a short time interval Δt, the volume ΔV of fluid that flows across the patch is approximately equal to the volume of a cylinder with base area $\Delta\sigma$ and height $(\mathbf{v}\,\Delta t) \cdot \mathbf{n}$, where $\mathbf{v}$ is a velocity vector rooted at a point of the patch:

$$\Delta V \approx \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t.$$

The mass of this volume of fluid is about

$$\Delta m \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t,$$

so the rate at which mass is flowing out of D across the patch is about

$$\frac{\Delta m}{\Delta t} \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma.$$

This leads to the approximation

$$\frac{\Sigma\, \Delta m}{\Delta t} \approx \sum \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma$$

as an estimate of the average rate at which mass flows across S. Finally, letting $\Delta\sigma \to 0$ and $\Delta t \to 0$ gives the instantaneous rate at which mass leaves D across S as

$$\frac{dm}{dt} = \iint_S \delta\mathbf{v} \cdot \mathbf{n}\, d\sigma,$$

which for our particular flow is

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

Now let B be a solid sphere centered at a point Q in the flow. The average value of $\nabla \cdot \mathbf{F}$ over B is

$$\frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV.$$

It is a consequence of the continuity of the divergence that $\nabla \cdot \mathbf{F}$ actually takes on this value at some point P in B. Thus,

$$(\nabla \cdot \mathbf{F})_P = \frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV = \frac{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}{\text{volume of } B}$$

$$= \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of } B}. \tag{8}$$

The fraction on the right describes decrease in mass per unit volume.

Now let the radius of B approach zero while the center Q stays fixed. The left-hand side of Eq. (8) converges to $(\nabla \cdot \mathbf{F})_Q$, the right side to $(-\partial\delta/\partial t)_Q$. The equality of these two limits is the continuity equation

$$\nabla \cdot \mathbf{F} = -\frac{\partial \delta}{\partial t}.$$

The continuity equation "explains" $\nabla \cdot \mathbf{F}$: The divergence of $\mathbf{F}$ at a point is the rate at which the density of the fluid is decreasing there.

The Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

now says that the net decrease in density of the fluid in region D is accounted for by the mass transported across the surface S. In a way, the theorem is a statement about conservation of mass.

Unifying the Integral Theorems

If we think of a two-dimensional field $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero, then $\nabla \cdot \mathbf{F} = (\partial M/\partial x) + (\partial N/\partial y)$ and the normal form of Green's theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

Similarly, $\nabla \times \mathbf{F} \cdot \mathbf{k} = (\partial N/\partial x) - (\partial M/\partial y)$, so the tangential form of Green's theorem can be written as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA.$$

With the equations of Green's theorem now in del notation, we can see their relationships to the equations in Stokes's theorem and the Divergence Theorem.

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's theorem: $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Divergence Theorem: $$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Tangential form of Green's theorem:	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$
Stokes's theorem:	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

Notice how Stokes's theorem generalizes the tangential (curl) form of Green's theorem from a flat surface in the plane to a surface in three-dimensional space. In each case, the integral of the normal component of curl **F** over the interior of the surface equals the circulation of **F** around the boundary.

Likewise, the Divergence Theorem generalizes the normal (flux) form of Green's theorem from a two-dimensional region in the plane to a three-dimensional region in space. In each case, the integral of $\nabla \cdot \mathbf{F}$ over the interior of the region equals the total flux of the field across the boundary.

There is still more to be learned here. All of these results can be thought of as forms of a *single fundamental theorem.* Think back to the Fundamental Theorem of Calculus in Section 4.7. It says that if $f(x)$ is differentiable on $[a, b]$ then

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a).$$

14.81 The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

If we let $\mathbf{F} = f(x)\,\mathbf{i}$ throughout $[a, b]$, then $(df/dx) = \nabla \cdot \mathbf{F}$. If we define the unit vector **n** normal to the boundary of $[a, b]$ to be **i** at b and $-\mathbf{i}$ at a (Fig. 14.81) then

$$\begin{aligned} f(b) - f(a) &= f(b)\,\mathbf{i} \cdot (\mathbf{i}) + f(a)\,\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{total outward flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

The Fundamental Theorem now says that

$$\int_{[a,b]} \nabla \cdot \mathbf{F}\, dx = \text{total outward flux of } \mathbf{F} \text{ across the boundary.}$$

The Fundamental Theorem of Calculus, the flux form of Green's theorem, and the Divergence Theorem all say that the integral of the differential operator $\nabla \cdot$ operating on a field **F** over a region equals the sum of the normal field components over the boundary of the region.

Stokes's theorem and the circulation form of Green's theorem say that, when things are properly oriented, the integral of the normal component of the curl operating on a field equals the sum of the tangential field components on the boundary of the surface.

The beauty of these interpretations is the observance of a marvelous underlying principle, which we might state as follows.

> The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

Exercises 14.8

Calculating Divergence

In Exercises 1–4, find the divergence of the field.

1. The spin field in Fig. 14.15.
2. The radial field in Fig. 14.14.
3. The gravitational field in Fig. 14.13.
4. The velocity field in Fig. 14.10.

Using the Divergence Theorem to Calculate Outward Flux

In Exercises 5–16, use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

5. $\mathbf{F} = (y - x)\,\mathbf{i} + (z - y)\,\mathbf{j} + (y - x)\,\mathbf{k}$
   D: The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$

6. $\mathbf{F} = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$
   a) D: The cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$
   b) D: The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$
   c) D: The region cut from the solid cylinder $x^2 + y^2 \le 4$ by the planes $z = 0$ and $z = 1$

7. $\mathbf{F} = y\,\mathbf{i} + xy\,\mathbf{j} - z\,\mathbf{k}$
   D: The region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

8. $\mathbf{F} = x^2\,\mathbf{i} + xz\,\mathbf{j} + 3z\,\mathbf{k}$
   D: the solid sphere $x^2 + y^2 + z^2 \le 4$

9. $\mathbf{F} = x^2\,\mathbf{i} - 2xy\,\mathbf{j} + 3xz\,\mathbf{k}$
   D: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$

10. $\mathbf{F} = (6x^2 + 2xy)\,\mathbf{i} + (2y + x^2z)\,\mathbf{j} + 4x^2y^3\,\mathbf{k}$
    D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

11. $\mathbf{F} = 2xz\,\mathbf{i} - xy\,\mathbf{j} - z^2\,\mathbf{k}$
    D: The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$

12. $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$
    D: The solid sphere $x^2 + y^2 + z^2 \le a^2$

13. $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$
    D: The region $1 \le x^2 + y^2 + z^2 \le 2$

14. $\mathbf{F} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$
    D: The region $1 \le x^2 + y^2 + z^2 \le 4$

15. $\mathbf{F} = (5x^3 + 12xy^2)\,\mathbf{i} + (y^3 + e^y \sin z)\,\mathbf{j} + (5z^3 + e^y \cos z)\,\mathbf{k}$
    D: The solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$

16. $\mathbf{F} = \ln(x^2 + y^2)\,\mathbf{i} - \left(\dfrac{2z}{x}\tan^{-1}\dfrac{y}{x}\right)\mathbf{j} + z\sqrt{x^2 + y^2}\,\mathbf{k}$
    D: The thick-walled cylinder $1 \le x^2 + y^2 \le 2, \quad -1 \le z \le 2$

Properties of Curl and Divergence

17. div (curl G) $= 0$
    a) Show that if the necessary partial derivatives of the components of the field $\mathbf{G} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ are continuous, then $\nabla \cdot \nabla \times \mathbf{G} = 0$.
    b) What, if anything, can you conclude about the flux of the field $\nabla \times \mathbf{G}$ across a closed surface? Give reasons for your answer.

18. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields, and let a and b be arbitrary real constants. Verify the following identities.
    a) $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\,\nabla \cdot \mathbf{F}_1 + b\,\nabla \cdot \mathbf{F}_2$
    b) $\nabla \times (a\,\mathbf{F}_1 + b\,\mathbf{F}_2) = a\,\nabla \times \mathbf{F}_1 + b\,\nabla \times \mathbf{F}_2$
    c) $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

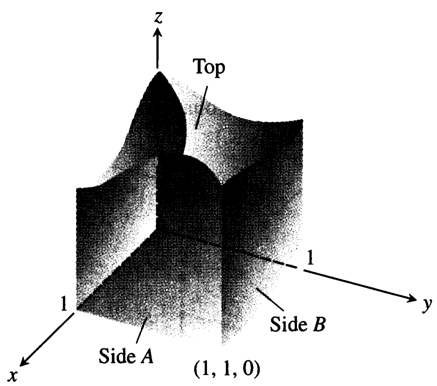
19. Let $\mathbf{F}$ be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar function. Verify the following identities.
    a) $\nabla \cdot (g\,\mathbf{F}) = g\,\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$
    b) $\nabla \times (g\,\mathbf{F}) = g\,\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

20. If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean
    $$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$
    For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$ verify the following identities.
    a) $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\,\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\,\mathbf{F}_2$
    $+ (\nabla \cdot \mathbf{F}_2)\,\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\,\mathbf{F}_2$
    b) $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\,\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\,\mathbf{F}_1$
    $+ \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$

Theory and Examples

21. Let $\mathbf{F}$ be a field whose components have continuous first partial derivatives throughout a portion of space containing a region D bounded by a smooth closed surface S. If $|\mathbf{F}| \le 1$, can any bound be placed on the size of
    $$\iiint\limits_D \nabla \cdot \mathbf{F}\,dV?$$
    Give reasons for your answer.

22. The base of the closed cubelike surface shown here is the unit square in the xy-plane. The four sides lie in the planes $x = 0, x = 1, y = 0$, and $y = 1$. The top is an arbitrary smooth surface whose identity is unknown. Let $\mathbf{F} = x\,\mathbf{i} - 2y\,\mathbf{j} + (z + 3)\,\mathbf{k}$, and suppose the outward flux of $\mathbf{F}$ through side A is 1 and through side B is -3. Can you conclude anything about the outward flux through the top? Give reasons for your answer.



23. a) Show that the flux of the position vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a smooth closed surface S is three times the volume of the region enclosed by the surface.

b) Let $\mathbf{n}$ be the outward unit normal vector field on S. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of S.

24. Among all rectangular solids defined by the inequalities $0 \le x \le a, 0 \le y \le b, 0 \le z \le 1$, find the one for which the total flux of $\mathbf{F} = (-x^2 - 4xy)\,\mathbf{i} - 6yz\,\mathbf{j} + 12z\,\mathbf{k}$ outward through the six sides is greatest. What *is* the greatest flux?

25. Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and suppose that the surface S and region D satisfy the hypotheses of the Divergence Theorem. Show that the volume of D is given by the formula

$$\text{Volume of } D = \frac{1}{3}\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

26. Show that the outward flux of a constant vector field $\mathbf{F} = \mathbf{C}$ across any closed surface to which the Divergence Theorem applies is zero.

27. *Harmonic functions.* A function $f(x, y, z)$ is said to be **harmonic** in a region D in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

a) Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

b) Show that if f is harmonic on D, then

$$\iint_S f\,\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$

28. Let S be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \le a^2$ that lies in the first octant and let $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$. Calculate

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma.$$

($\nabla f \cdot \mathbf{n}$ is the derivative of f in the direction of $\mathbf{n}$.)

29. *Green's first formula.* Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a closed region D that is bounded by a piecewise smooth surface S. Show that

$$\iint_S f\,\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D (f\,\nabla^2 g + \nabla f \cdot \nabla g)\, dV. \qquad \textbf{(9)}$$

Equation (9) is **Green's first formula.** (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f\,\nabla g$.)

30. *Green's second formula.* (*Continuation of Exercise 29*). Interchange f and g in Eq. (9) to obtain a similar formula. Then subtract this formula from Eq. (9) to show that

$$\iint_S (f\,\nabla g - g\,\nabla f) \cdot \mathbf{n}\, d\sigma$$

$$= \iiint_D (f\,\nabla^2 g - g\,\nabla^2 f)\, dV. \qquad \textbf{(10)}$$

This equation is **Green's second formula.**

31. *Conservation of mass.* Let $\mathbf{v}(t, x, y, z)$ be a continuously differentiable vector field over the region D in space and let $p(t, x, y, z)$ be a continuously differentiable scalar function. The variable t represents the time domain. The Law of Conservation of Mass asserts that

$$\frac{d}{dt}\iiint_D p(t, x, y, z)\, dV = -\iint_S p\,\mathbf{v} \cdot \mathbf{n}\, d\sigma,$$

where S is the surface enclosing D.

a) Give a physical interpretation of the conservation of mass law if $\mathbf{v}$ is a velocity flow field and p represents the density of the fluid at point (x, y, z) at time t.

b) Use the Divergence Theorem and Leibniz's rule,

$$\frac{d}{dt}\iiint_D p(t, x, y, z)\, dV = \iiint_D \frac{\partial p}{\partial t}\, dV,$$

to show that the Law of Conservation of Mass is equivalent to the continuity equation,

$$\nabla \cdot p\,\mathbf{v} + \frac{\partial p}{\partial t} = 0.$$

(In the first term $\nabla \cdot p\,\mathbf{v}$ the variable t is held fixed and in the second term $\partial p/\partial t$ it is assumed that the point (x, y, z) in D is held fixed.)

32. *General diffusion equation.* Let $T(t, x, y, z)$ be a function with continuous second derivatives giving the temperature at time t at the point (x, y, z) of a solid occupying a region D in space. If the solid's specific heat and mass density are denoted by the constants c and ρ respectively, the quantity $c\rho T$ is called the solid's **heat energy per unit volume.**

a) Explain why $-\nabla T$ points in the direction of heat flow.

b) Let $-k\nabla T$ denote the **energy flux vector.** (Here the constant k is called the **conductivity.**) Assuming the Law of

Conservation of Mass with $-k\nabla T = \mathbf{v}$ and $c\,\rho\,T = p$ in Exercise 31, derive the diffusion (heat) equation

$$\frac{\partial T}{\partial t} = K\nabla^2 T,$$

where $K = k/(c\rho) > 0$ is the *diffusivity* constant. (Notice that if $T\,(t,\ x)$ represents the temperature at time t at position x in a uniform conducting rod with perfectly insulated sides, then $\nabla^2 T = \partial^2 T/\partial x^2$ and the diffusion equation reduces to the one-dimensional heat equation in the Chapter 12 Additional Exercises.)

CHAPTER 14 QUESTIONS TO GUIDE YOUR REVIEW

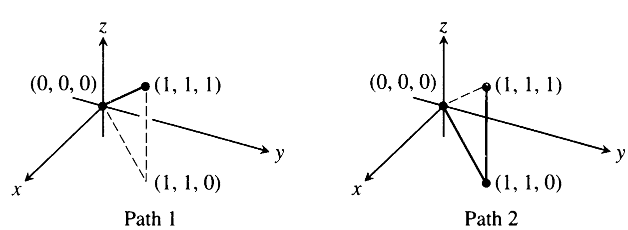
1. What are line integrals? How are they evaluated? Give examples.
2. How can you use line integrals to find the centers of mass of springs? Explain.
3. What is a vector field? a gradient field? Give examples.
4. How do you calculate the work done by a force in moving a particle along a curve? Give an example.
5. What are flow, circulation, and flux?
6. What is special about path independent fields?
7. How can you tell when a field is conservative?
8. What is a potential function? Show by example how to find a potential function for a conservative field.
9. What is a differential form? What does it mean for such a form to be exact? How do you test for exactness? Give examples.
10. What is the divergence of a vector field? How can you interpret it?
11. What is the curl of a vector field? How can you interpret it?
12. What is Green's theorem? How can you interpret it?
13. How do you calculate the area of a curved surface in space? Give an example.
14. What is an oriented surface? How do you calculate the flux of a three-dimensional vector field across an oriented surface? Give an example.
15. What are surface integrals? What can you calculate with them? Give an example.
16. What is a parametrized surface? How do you find the area of such a surface? Give examples.
17. How do you integrate a function over a parametrized surface? Give an example.
18. What is Stokes's theorem? How can you interpret it?
19. Summarize the chapter's results on conservative fields.
20. What is the Divergence Theorem? How can you interpret it?
21. How does the Divergence Theorem generalize Green's theorem?
22. How does Stokes's theorem generalize Green's theorem?
23. How can Green's theorem, Stokes's theorem, and the Divergence Theorem be thought of as forms of a single fundamental theorem?

CHAPTER 14 PRACTICE EXERCISES

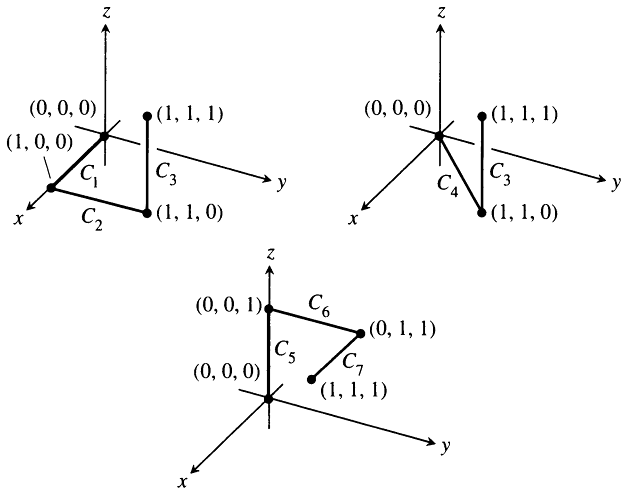
Evaluating Line Integrals

1. Figure 14.82 shows two polygonal paths in space joining the origin to the point $(1, 1, 1)$. Integrate $f\,(x, y, z) = 2x - 3y^2 - 2z + 3$ over each path.
2. Figure 14.83 shows three polygonal paths joining the origin to the point $(1, 1, 1)$. Integrate $f\,(x, y, z) = x^2 + y - z$ over each path.
3. Integrate $f\,(x, y, z) = \sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{j} + (a\sin t)\,\mathbf{k},\ 0 \le t \le 2\pi.$$



14.82 The paths in Exercise 1.



14.83 The paths in Exercise 2.

4. Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the involute curve $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j},\ 0 \le t \le \sqrt{3}$. (See Fig. 11.20.)

Evaluate the integrals in Exercises 5 and 6.

5. $\displaystyle\int_{(-1,1,1)}^{(4,-3,0)} \frac{dx + dy + dz}{\sqrt{x + y + z}}$

6. $\displaystyle\int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\, dy - \sqrt{\frac{y}{z}}\, dz$

7. Integrate $\mathbf{F} = -(y \sin z)\,\mathbf{i} + (x \sin z)\,\mathbf{j} + (xy \cos z)\,\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 5$ by the plane $z = -1$, clockwise as viewed from above.

8. Integrate $\mathbf{F} = 3x^2 y\,\mathbf{i} + (x^3 + 1)\,\mathbf{j} + 9z^2\,\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 9$ by the plane $x = 2$.

Evaluate the line integrals in Exercises 9 and 10.

9. $\displaystyle\int_C 8x \sin y\, dx - 8y \cos x\, dy$

C is the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

10. $\displaystyle\int_C y^2\, dx + x^2\, dy$

C is the circle $x^2 + y^2 = 4$.

Evaluating Surface Integrals

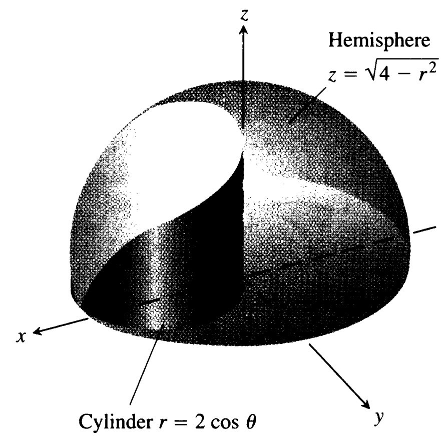
11. Find the area of the elliptical region cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$.

12. Find the area of the cap cut from the paraboloid $y^2 + z^2 = 3x$ by the plane $x = 1$.

13. Find the area of the cap cut from the top of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \sqrt{2}/2$.

14. a) Find the area of the surface cut from the hemisphere $x^2 + y^2 + z^2 = 4,\ z \ge 0$, by the cylinder $x^2 + y^2 = 2x$.

b) Find the area of the portion of the cylinder that lies inside the hemisphere. (*Hint:* Project onto the xz-plane. Or evaluate the integral $\int h\, ds$, where h is the altitude of the cylinder and ds is the element of arc length on the circle $x^2 + y^2 = 2x$ in the xy-plane.)



15. Find the area of the triangle in which the plane $(x/a) + (y/b) + (z/c) = 1\ (a, b, c > 0)$ intersects the first octant. Check your answer with an appropriate vector calculation.

16. Integrate

a) $g(x, y, z) = \dfrac{yz}{\sqrt{4y^2 + 1}}$ **b)** $g(x, y, z) = \dfrac{z}{\sqrt{4y^2 + 1}}$

over the surface cut from the parabolic cylinder $y^2 - z = 1$ by the planes $x = 0, x = 3$, and $z = 0$.

17. Integrate $g(x, y, z) = x^4 y(y^2 + z^2)$ over the portion of the cylinder $y^2 + z^2 = 25$ that lies in the first octant between the planes $x = 0$ and $x = 1$ and above the plane $z = 3$.

18. CALCULATOR The state of Wyoming is bounded by the meridians 111° 3′ and 104° 3′ west longitude and by the circles 41° and 45° north latitude. Assuming that the earth is a sphere of radius $R = 3959$ mi, find the area of Wyoming.

Parametrized Surfaces

Find the parametrizations for the surfaces in Exercises 19–24. (There are many ways to do these, so your answers may not be the same as those in the back of the book.)

19. The portion of the sphere $x^2 + y^2 + z^2 = 36$ between the planes $z = -3$ and $z = 3\sqrt{3}$

20. The portion of the paraboloid $z = -(x^2 + y^2)/2$ above the plane $z = -2$

21. The cone $z = 1 + \sqrt{x^2 + y^2},\ z \le 3$

22. The portion of the plane $4x + 2y + 4z = 12$ that lies above the square $0 \le x \le 2, 0 \le y \le 2$ in the first quadrant

23. The portion of the paraboloid $y = 2(x^2 + z^2),\ y \le 2$, that lies above the xy-plane

24. The portion of the hemisphere $x^2 + y^2 + z^2 = 10$, $y \geq 0$, in the first octant

25. Find the area of the surface

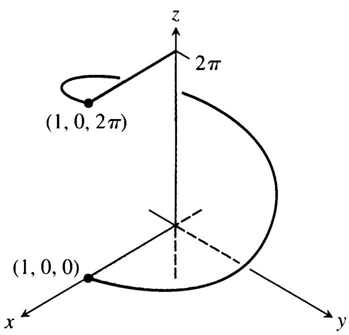
$$\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (u - v)\,\mathbf{j} + v\,\mathbf{k}, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

26. Integrate $f(x, y, z) = xy - z^2$ over the surface in Exercise 25.

27. Find the surface area of the helicoid

$$\mathbf{r}(r, \theta) = r \cos\theta\,\mathbf{i} + r \sin\theta\,\mathbf{j} + \theta\mathbf{k}, \quad 0 \leq \theta \leq 2\pi \text{ and } 0 \leq r \leq 1,$$

in the accompanying figure.



28. Evaluate the integral $\iint_S \sqrt{x^2 + y^2 + 1}\, d\sigma$, where S is the helicoid in Exercise 27.

Conservative Fields

Which of the fields in Exercises 29–32 are conservative, and which are not?

29. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

30. $\mathbf{F} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$

31. $\mathbf{F} = x\,e^y\,\mathbf{i} + y\,e^z\,\mathbf{j} + z\,e^x\,\mathbf{k}$

32. $\mathbf{F} = (\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k})/(x + yz)$

Find potential functions for the fields in Exercises 33 and 34.

33. $\mathbf{F} = 2\,\mathbf{i} + (2y + z)\,\mathbf{j} + (y + 1)\,\mathbf{k}$

34. $\mathbf{F} = (z \cos xz)\,\mathbf{i} + e^y\,\mathbf{j} + (x \cos xz)\,\mathbf{k}$

Work and Circulation

In Exercises 35 and 36, find the work done by each field along the paths from $(0, 0, 0)$ to $(1, 1, 1)$ in Fig. 14.82.

35. $\mathbf{F} = 2xy\,\mathbf{i} + \mathbf{j} + x^2\,\mathbf{k}$

36. $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j} + \mathbf{k}$

37. Find the work done by

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{3/2}}$$

over the plane curve $\mathbf{r}(t) = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j}$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$ in two ways:

a) by using the parametrization of the curve to evaluate the work integral.
b) by evaluating a potential function for **F**.

38. Find the flow of the field $\mathbf{F} = \nabla(x^2 z e^y)$

a) once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y-axis.
b) along the curved boundary of the helicoid in Exercise 27 from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

39. Suppose $\mathbf{F}(x, y) = (x + y)\,\mathbf{i} - (x^2 + y^2)\,\mathbf{j}$ is the velocity field of a fluid flowing across the xy-plane. Find the flow along each of the following paths from $(1, 0)$ to $(-1, 0)$.

a) The upper half of the circle $x^2 + y^2 = 1$
b) The line segment from $(1, 0)$ to $(-1, 0)$
c) The line segment from $(1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$

40. Find the circulation of $\mathbf{F} = 2x\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ along the closed path consisting of the helix $\mathbf{r}_1(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \leq t \leq \pi/2$, followed by the line segments $\mathbf{r}_2(t) = \mathbf{j} + (\pi/2)(1 - t)\,\mathbf{k}$, $0 \leq t \leq 1$, and $\mathbf{r}_3(t) = t\,\mathbf{i} + (1 - t)\,\mathbf{j}$, $0 \leq t \leq 1$.

In Exercises 41 and 42, use the surface integral in Stokes's theorem to find the circulation of the field **F** around the curve C in the indicated direction.

41. $\mathbf{F} = y^2\,\mathbf{i} - y\,\mathbf{j} + 3z^2\,\mathbf{k}$
C: The ellipse in which the plane $2x + 6y - 3z = 6$ meets the cylinder $x^2 + y^2 = 1$, counterclockwise as viewed from above

42. $\mathbf{F} = (x^2 + y)\,\mathbf{i} + (x + y)\,\mathbf{j} + (4y^2 - z)\,\mathbf{k}$
C: The circle in which the plane $z = -y$ meets the sphere $x^2 + y^2 + z^2 = 4$, counterclockwise as viewed from above

Mass and Moments

43. Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + (4 - t^2)\,\mathbf{k}$, $0 \leq t \leq 1$, if the density at t is (a) $\delta = 3t$, (b) $\delta = 1$.

44. Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \leq t \leq 2$, if the density at t is $\delta = 3\sqrt{5 + t}$.

45. Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{2\sqrt{2}}{3}\,t^{3/2}\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}, \quad 0 \leq t \leq 2,$$

if the density at t is $\delta = 1/(t + 1)$.

46. A slender metal arch lies along the semicircle $y = \sqrt{a^2 - x^2}$ in the xy-plane. The density at the point (x, y) on the arch is $\delta(x, y) = 2a - y$. Find the center of mass.

47. A wire of constant density $\delta = 1$ lies along the curve $\mathbf{r}(t) = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} + e^t\,\mathbf{k}$, $0 \leq t \leq \ln 2$. Find $\overline{z}$, I_z, and R_z.

48. Find the mass and center of mass of a wire of constant density δ that lies along the helix $\mathbf{r}(t) = (2 \sin t)\,\mathbf{i} + (2 \cos t)\,\mathbf{j} + 3t\,\mathbf{k}$, $0 \leq t \leq 2\pi$.

49. Find I_z, R_z, and the center of mass of a thin shell of density $\delta(x, y, z) = z$ cut from the upper portion of the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$.

50. Find the moment of inertia about the z-axis of the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ if the density is $\delta = 1$.

Flux Across a Plane Curve or Surface

Use Green's theorem to find the counterclockwise circulation and outward flux for the fields and curves in Exercises 51 and 52.

51. $\mathbf{F} = (2xy + x)\,\mathbf{i} + (xy - y)\,\mathbf{j}$
C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

52. $\mathbf{F} = (y - 6x^2)\,\mathbf{i} + (x + y^2)\,\mathbf{j}$
C: The triangle made by the lines $y = 0$, $y = x$, and $x = 1$

53. Show that

$$\oint_C \ln x \sin y\, dy - \frac{\cos y}{x}\, dx = 0$$

for any closed curve C to which Green's theorem applies.

54. a) Show that the outward flux of the position vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ across any closed curve to which Green's theorem applies is twice the area of the region enclosed by the curve.

b) Let $\mathbf{n}$ be the outward unit normal vector to a closed curve to which Green's theorem applies. Show that it is not possible for $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ to be orthogonal to $\mathbf{n}$ at every point of C.

In Exercises 55–58, find the outward flux of $\mathbf{F}$ across the boundary of D.

55. $\mathbf{F} = 2xy\,\mathbf{i} + 2yz\,\mathbf{j} + 2xz\,\mathbf{k}$
D: The cube cut from the first octant by the planes $x = 1$, $y = 1, z = 1$

56. $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + \mathbf{k}$
D: The entire surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \le 25$ by the plane $z = 3$

57. $\mathbf{F} = -2x\,\mathbf{i} - 3y\,\mathbf{j} + z\,\mathbf{k}$
D: The upper region cut from the solid sphere $x^2 + y^2 + z^2 \le 2$ by the paraboloid $z = x^2 + y^2$

58. $\mathbf{F} = (6x + y)\,\mathbf{i} - (x + z)\,\mathbf{j} + 4yz\,\mathbf{k}$
D: The region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes

59. Let S be the surface that is bounded on the left by the hemisphere $x^2 + y^2 + z^2 = a^2$, $y \le 0$, in the middle by the cylinder $x^2 + z^2 = a^2$, $0 \le y \le a$, and on the right by the plane $y = a$. Find the flux of the field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ outward across S.

60. Find the outward flux of the field $\mathbf{F} = 3xz^2\,\mathbf{i} + y\,\mathbf{j} - z^3\,\mathbf{k}$ across the surface of the solid in the first octant that is bounded by the cylinder $x^2 + 4y^2 = 16$ and the planes $y = 2z$, $x = 0$, and $z = 0$.

61. Use the Divergence Theorem to find the flux of $\mathbf{F} = xy^2\,\mathbf{i} + x^2y\,\mathbf{j} + y\,\mathbf{k}$ outward through the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = -1$.

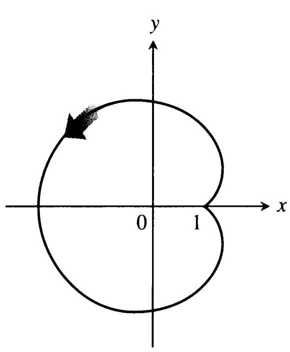
62. Find the flux of $\mathbf{F} = (3z + 1)\,\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$ (a) with the Divergence Theorem, (b) by evaluating the flux integral directly.

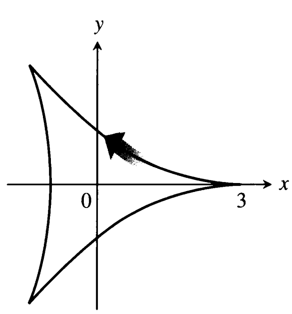
CHAPTER 14 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Finding Areas with Green's Theorem

Use the Green's theorem area formula, Eq. (22) in Exercises 14.4, to find the areas of the regions enclosed by the curves in Exercises 1–4.

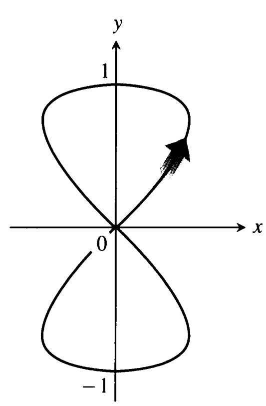
1. The limaçon $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, $0 \le t \le 2\pi$

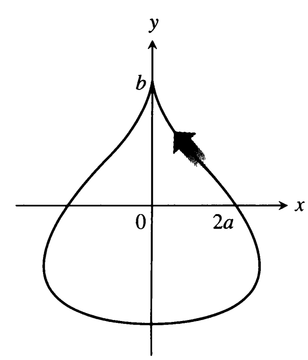




2. The deltoid $x = 2\cos t + \cos 2t$, $y = 2\sin t - \sin 2t$, $0 \le t \le 2\pi$

3. The eight curve $x = (1/2)\sin 2t$, $y = \sin t$, $0 \le t \le \pi$ (one loop)





4. The teardrop $x = 2a\cos t - a\sin 2t$, $y = b\sin t$, $0 \le t \le 2\pi$

Theory and Applications

5. a) Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ at only one point and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the point and compute the curl.

b) Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on precisely one line and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the line and compute the curl.

c) Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on a surface and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the surface and compute the curl.

6. Find all points (a, b, c) on the sphere $x^2 + y^2 + z^2 = R^2$ where the vector field $\mathbf{F} = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$ is normal to the surface and $\mathbf{F}(a, b, c) \neq \mathbf{0}$.

7. Find the mass of a spherical shell of radius R such that at each point (x, y, z) on the surface the mass density $\delta(x, y, z)$ is its distance to some fixed point (a, b, c) on the surface.

8. Find the mass of a helicoid

$$\mathbf{r}(r, \theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + \theta\,\mathbf{k},$$

$0 \le r \le 1$, $0 \le \theta \le 2\pi$, if the density function is $\delta(x, y, z) = 2\sqrt{x^2 + y^2}$. See Practice Exercise 27 for a figure.

9. Among all rectangular regions $0 \le x \le a$, $0 \le y \le b$, find the one for which the total outward flux of $\mathbf{F} = (x^2 + 4xy)\,\mathbf{i} - 6y\,\mathbf{j}$ across the four sides is least. What *is* the least flux?

10. Find an equation for the plane through the origin such that the circulation of the flow field $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ around the circle of intersection of the plane with the sphere $x^2 + y^2 + z^2 = 4$ is a maximum.

11. A string lies along the circle $x^2 + y^2 = 4$ from (2, 0) to (0, 2) in the first quadrant. The density of the string is $\rho(x, y) = xy$.

a) Partition the string into a finite number of subarcs to show that the work done by gravity to move the string straight down to the x-axis is given by

$$\text{Work} = \lim_{n\to\infty} \sum_{k=1}^{n} g\, x_k\, {y_k}^2 \Delta s_k = \int_C g\,xy^2\,ds,$$

where g is the gravitational constant.

b) Find the total work done by evaluating the line integral in part (a).

c) Show that the total work done equals the work required to move the string's center of mass $(\bar{x}, \bar{y})$ straight down to the x-axis.

12. A thin sheet lies along the portion of the plane $x + y + z = 1$ in the first octant. The density of the sheet is $\delta(x, y, z) = xy$.

a) Partition the sheet into a finite number of subpieces to show that the work done by gravity to move the sheet straight down to the xy-plane is given by

$$\text{Work} = \lim_{n\to\infty} \sum_{k=1}^{n} g\, x_k\, y_k\, z_k \Delta\sigma_k = \iint_S g\,xyz\,d\sigma,$$

where g is the gravitational constant.

b) Find the total work done by evaluating the surface integral in part (a).

c) Show that the total work done equals the work required to move the sheet's center of mass $(\bar{x}, \bar{y}, \bar{z})$ straight down to the xy-plane.

13. *Archimedes' principle.* If an object such as a ball is placed in a liquid, it will either sink to the bottom, float, or sink a certain distance and remain suspended in the liquid. Suppose a fluid has constant weight density w and that the fluid's surface coincides with the plane $z = 4$. A spherical ball remains suspended in the fluid and occupies the region $x^2 + y^2 + (z - 2)^2 \le 1$.

a) Show that the surface integral giving the magnitude of the total force on the ball due to the fluid's pressure is

$$\text{Force} = \lim_{n\to\infty} \sum_{k=1}^{n} w\,(4 - z_k)\,\Delta\sigma_k = \iint_S w\,(4 - z)\,d\sigma.$$

b) Since the ball is not moving, it is being held up by the buoyant force of the liquid. Show that the magnitude of the buoyant force on the sphere is

$$\text{Buoyant force} = \iint_S w\,(z - 4)\,\mathbf{k}\cdot\mathbf{n}\,d\sigma,$$

where $\mathbf{n}$ is the outer unit normal at (x, y, z). This illustrates Archimedes' principle that the magnitude of the buoyant force on a submerged solid equals the weight of the displaced fluid.

c) Use the Divergence Theorem to find the magnitude of the buoyant force in part (b).

14. *Fluid force on a curved surface.* A cone in the shape of the surface $z = \sqrt{x^2 + y^2}$, $0 \le z \le 2$, is filled with a liquid of constant weight density w. Assuming the xy-plane is "ground level," show that the total force on the portion of the cone from $z = 1$ to $z = 2$ due to liquid pressure is the surface integral

$$F = \iint_S w\,(2 - z)\,d\sigma.$$

Evaluate the integral.

15. *Faraday's law.* If $\mathbf{E}(t, x, y, z)$ and $\mathbf{B}(t, x, y, z)$ represent the electric and magnetic fields at point (x, y, z) at time t, a basic principle of electromagnetic theory says that $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. In this expression $\nabla \times \mathbf{E}$ is computed with t held fixed and $\partial\mathbf{B}/\partial t$ is calculated with (x, y, z) fixed. Use Stokes's theorem to derive Faraday's law

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\,d\sigma,$$

where C represents a wire loop through which current flows counterclockwise with respect to the surface's unit normal $\mathbf{n}$, giving rise to the voltage

$$\oint_C \mathbf{E}\cdot d\mathbf{r}$$

around C. The surface integral on the right side of the equation is called the **magnetic flux,** and S is any oriented surface with boundary C.

16. Let

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$$

be the gravitational force field defined for $\mathbf{r} \neq \mathbf{0}$. Use Gauss's law in Section 14.8 to show that there is no continuously differentiable vector field $\mathbf{H}$ satisfying $\mathbf{F} = \nabla \times \mathbf{H}$.

17. If $f(x, y, z)$ and $g(x, y, z)$ are continuously differentiable scalar functions defined over the oriented surface S with boundary curve C, prove that

$$\iint\limits_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma = \oint_C f \nabla g \cdot d\mathbf{r}.$$

18. Suppose that $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2$ and $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2$ over a region D enclosed by the oriented surface S with outward unit normal $\mathbf{n}$ and that $\mathbf{F}_1 \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n}$ on S. Prove that $\mathbf{F}_1 = \mathbf{F}_2$ throughout D.

19. Prove or disprove that if $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \mathbf{0}$.

20. Let S be an oriented surface parametrized by $\mathbf{r}(u, v)$. Define the notation $d\boldsymbol{\sigma} = \mathbf{r}_u\, du \times \mathbf{r}_v\, dv$ so that $d\boldsymbol{\sigma}$ is a vector normal to the surface. Also, the magnitude $d\sigma = |d\boldsymbol{\sigma}|$ is the element of surface area (by Eq. 5 in Section 14.6). Derive the identity

$$d\sigma = (EG - F^2)^{1/2}\, du\, dv$$

where

$$E = |\mathbf{r}_u|^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad \text{and} \quad G = |\mathbf{r}_v|^2.$$

21. Show that the volume V of a region D in space enclosed by the oriented surface S with outward normal $\mathbf{n}$ satisfies the identity

$$V = \frac{1}{3}\iint\limits_S \mathbf{r} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{r}$ is the position vector of the point (x, y, z) in D.

Appendices

A.1 Mathematical Induction

Many formulas, like

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

can be shown to hold for every positive integer n by applying an axiom called the *mathematical induction principle.* A proof that uses this axiom is called a *proof by mathematical induction* or a *proof by induction.*

The steps in proving a formula by induction are the following.

Step 1: Check that the formula holds for $n = 1$.

Step 2: Prove that if the formula holds for any positive integer $n = k$, then it also holds for the next integer, $n = k + 1$.

Once these steps are completed (the axiom says), we know that the formula holds for all positive integers n. By step 1 it holds for $n = 1$. By step 2 it holds for $n = 2$, and therefore by step 2 also for $n = 3$, and by step 2 again for $n = 4$, and so on. If the first domino falls, and the kth domino always knocks over the $(k + 1)$st when it falls, all the dominoes fall.

From another point of view, suppose we have a sequence of statements S_1, $S_2, \ldots, S_n, \ldots$, one for each positive integer. Suppose we can show that assuming any one of the statements to be true implies that the next statement in line is true. Suppose that we can also show that S_1 is true. Then we may conclude that the statements are true from S_1 on.

EXAMPLE 1 Show that for every positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution We accomplish the proof by carrying out the two steps above.

Step 1: The formula holds for $n = 1$ because

$$1 = \frac{1(1+1)}{2}.$$

Step 2: If the formula holds for $n = k$, does it also hold for $n = k + 1$? The answer is yes, and here's why: If

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2},$$

then

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2 + k + 2k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}.$$

The last expression in this string of equalities is the expression $n(n+1)/2$ for $n = (k+1)$.

The mathematical induction principle now guarantees the original formula for all positive integers n. Notice that all *we* have to do is carry out steps 1 and 2. The mathematical induction principle does the rest. ❑

EXAMPLE 2 Show that for all positive integers n,

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Solution We accomplish the proof by carrying out the two steps of mathematical induction.

Step 1: The formula holds for $n = 1$ because

$$\frac{1}{2^1} = 1 - \frac{1}{2^1}.$$

Step 2: If

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k},$$

then

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1 \cdot 2}{2^k \cdot 2} + \frac{1}{2^{k+1}}$$
$$= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.$$

Thus, the original formula holds for $n = (k+1)$ whenever it holds for $n = k$.

With these steps verified, the mathematical induction principle now guarantees the formula for every positive integer n. ❑

Other Starting Integers

Instead of starting at $n = 1$, some induction arguments start at another integer. The steps for such an argument are as follows.

Step 1: Check that the formula holds for $n = n_1$ (the first appropriate integer).

Step 2: Prove that if the formula holds for any integer $n = k \geq n_1$, then it also holds for $n = (k+1)$.

Once these steps are completed, the mathematical induction principle guarantees the formula for all $n \geq n_1$.

EXAMPLE 3 Show that $n! > 3^n$ if n is large enough.

Solution How large is large enough? We experiment:

n	1	2	3	4	5	6	7
$n!$	1	2	6	24	120	720	5040
3^n	3	9	27	81	243	729	2187

It looks as if $n! > 3^n$ for $n \geq 7$. To be sure, we apply mathematical induction. We take $n_1 = 7$ in step 1 and try for step 2.

Suppose $k! > 3^k$ for some $k \geq 7$. Then

$$(k+1)! = (k+1)(k!) > (k+1)3^k > 7 \cdot 3^k > 3^{k+1}.$$

Thus, for $k \geq 7$,

$$k! > 3^k \quad \Rightarrow \quad (k+1)! > 3^{k+1}.$$

The mathematical induction principle now guarantees $n! \geq 3^n$ for all $n \geq 7$. ❑

Exercises A.1

1. Assuming that the triangle inequality $|a+b| \leq |a| + |b|$ holds for any two numbers a and b, show that
$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$
for any n numbers.

2. Show that if $r \neq 1$, then
$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$
for every positive integer n.

3. Use the Product Rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$,
and the fact that $\frac{d}{dx}(x) = 1$
to show that $\frac{d}{dx}(x^n) = nx^{n-1}$
for every positive integer n.

4. Suppose that a function $f(x)$ has the property that $f(x_1x_2) = f(x_1) + f(x_2)$ for any two positive numbers x_1 and x_2. Show that
$$f(x_1x_2\cdots x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$$
for the product of any n positive numbers $x_1, x_2 \ldots, x_n$.

5. Show that
$$\frac{2}{3^1} + \frac{2}{3^2} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}$$
for all positive integers n.

6. Show that $n! > n^3$ if n is large enough.

7. Show that $2^n > n^2$ if n is large enough.

8. Show that $2^n \geq 1/8$ for $n \geq -3$.

9. *Sums of squares.* Show that the sum of the squares of the first n positive integers is
$$\frac{n\left(n + \frac{1}{2}\right)(n+1)}{3}.$$

10. *Sums of cubes.* Show that the sum of the cubes of the first n positive integers is $(n(n+1)/2)^2$.

11. *Rules for finite sums.* Show that the following finite sum rules hold for every positive integer n.

    a) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

    b) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

    c) $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)

    d) $\sum_{k=1}^{n} a_k = n \cdot c$ (if a_k has the constant value c)

12. Show that $|x^n| = |x|^n$ for every positive integer n and every real number x.

A.2 Proofs of Limit Theorems in Section 1.2

This appendix proves Theorem 1, Parts 2–5, and Theorem 4 from Section 1.2.

Theorem 1
Properties of Limits

The following rules hold if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim_{x\to c} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim_{x\to c} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c} kf(x) = kL$ (any number k)
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. *Power Rule:* If m and n are integers, then

$$\lim_{x\to c} [f(x)]^{m/n} = L^{m/n}$$

provided $L^{m/n}$ is a real number.

We proved the Sum Rule in Section 1.3 and the Power Rule is proved in more advanced texts. We obtain the Difference Rule by replacing $g(x)$ by $-g(x)$ and M by $-M$ in the Sum Rule. The Constant Multiple Rule is the special case $g(x) = k$ of the Product Rule. This leaves only the Product and Quotient Rules.

Proof of the Limit Product Rule We show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x in the intersection D of the domains of f and g,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x)\,g(x) - LM| < \epsilon.$$

Suppose then that ϵ is a positive number, and write $f(x)$ and $g(x)$ as

$$f(x) = L + (f(x) - L), \qquad g(x) = M + (g(x) - M).$$

Multiply these expressions together and subtract LM:

$$\begin{aligned} f(x) \cdot g(x) - LM &= (L + (f(x) - L))(M + (g(x) - M)) - LM \\ &= LM + L(g(x) - M) + M(f(x) - L) \\ &\quad + (f(x) - L)(g(x) - M) - LM \\ &= L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M). \end{aligned} \tag{1}$$

Since f and g have limits L and M as $x \to c$, there exist positive numbers δ_1, δ_2, δ_3, and δ_4 such that for all x in D

$$\begin{aligned}
0 < |x-c| < \delta_1 &\Rightarrow |f(x)-L| < \sqrt{\epsilon/3} \\
0 < |x-c| < \delta_2 &\Rightarrow |g(x)-M| < \sqrt{\epsilon/3} \\
0 < |x-c| < \delta_3 &\Rightarrow |f(x)-L| < \epsilon/(3(1+|M|)) \\
0 < |x-c| < \delta_4 &\Rightarrow |g(x)-M| < \epsilon/(3(1+|L|))
\end{aligned} \tag{2}$$

If we take δ to be the smallest numbers δ_1 through δ_4, the inequalities on the right-hand side of (2) will hold simultaneously for $0 < |x-c| < \delta$. Therefore, for all x in D, $0 < |x-c| < \delta$ implies

$$\begin{aligned}
&|f(x) \cdot g(x) - LM| \\
&\quad \le |L||g(x)-M| + |M||f(x)-L| + |f(x)-L||g(x)-M| && \text{Triangle inequality applied to Eq. (1)} \\
&\quad \le (1+|L|)|g(x)-M| + (1+|M|)|f(x)-L| + |f(x)-L||g(x)-M| \\
&\quad \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sqrt{\frac{\epsilon}{3}}\sqrt{\frac{\epsilon}{3}} = \epsilon. && \text{Values from (2)}
\end{aligned}$$

This completes the proof of the Limit Product Rule. ❑

Proof of the Limit Quotient Rule We show that $\lim_{x\to c}(1/g(x)) = 1/M$. We can then conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c}\left(f(x)\cdot\frac{1}{g(x)}\right) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = L\cdot\frac{1}{M} = \frac{L}{M}$$

by the Limit Product Rule.

Let $\epsilon > 0$ be given. To show that $\lim_{x\to c}(1/g(x)) = 1/M$, we need to show that there exists a $\delta > 0$ such that for all x

$$0 < |x-c| < \delta \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon.$$

Since $|M| > 0$, there exists a positive number δ_1 such that for all x

$$0 < |x-c| < \delta_1 \quad \Rightarrow \quad |g(x)-M| < \frac{M}{2}. \tag{3}$$

For any numbers A and B it can be shown that $|A| - |B| \le |A-B|$ and $|B| - |A| \le |A-B|$, from which it follows that $||A| - |B|| \le |A-B|$. With $A = g(x)$ and $B = M$, this becomes

$$||g(x)| - |M|| \le |g(x) - M|,$$

which can be combined with the inequality on the right in (3) to get, in turn,

$$\begin{aligned}
||g(x)| - |M|| &< \frac{|M|}{2} \\
-\frac{|M|}{2} < |g(x)| - |M| &< \frac{|M|}{2} \\
\frac{|M|}{2} < |g(x)| &< \frac{3|M|}{2} \\
|M| < 2|g(x)| &< 3|M| \\
\frac{1}{|g(x)|} < \frac{2}{|M|} &< \frac{3}{|g(x)|}
\end{aligned} \tag{4}$$

Therefore, $0 < |x - c| < \delta_1$ implies that

$$\begin{aligned}\left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \left|\frac{M - g(x)}{Mg(x)}\right| \le \frac{1}{|M|} \cdot \frac{1}{|g(x)|} \cdot |M - g(x)| \\ &< \frac{1}{|M|} \cdot \frac{2}{|M|} \cdot |M - g(x)|. \qquad \text{Inequality (4)}\end{aligned} \tag{5}$$

Since $(1/2)|M|^2 \epsilon > 0$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |M - g(x)| < \frac{\epsilon}{2}|M|^2. \tag{6}$$

If we take δ to be the smaller of δ_1 and δ_2, the conclusions in (5) and (6) both hold for all x such that $0 < |x - c| < \delta$. Combining these conclusions gives

$$0 < |x - c| < \delta \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon.$$

This concludes the proof of the Limit Quotient Rule. ❑

Theorem 4
The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$.

Proof for Right-hand Limits Suppose $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} h(x) = L$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x the inequality $c < x < c + \delta$ implies

$$L - \epsilon < g(x) < L + \epsilon \qquad \text{and} \qquad L - \epsilon < h(x) < L + \epsilon. \tag{7}$$

These inequalities combine with the inequality $g(x) \le f(x) \le h(x)$ to give

$$\begin{aligned} L - \epsilon < g(x) \le f(x) \le h(x) < L + \epsilon, \\ L - \epsilon < f(x) < L + \epsilon, \\ -\epsilon < f(x) - L < \epsilon. \end{aligned} \tag{8}$$

Therefore, for all x, the inequality $c < x < c + \delta$ implies $|f(x) - L| < \epsilon$. ❑

Proof for Left-hand Limits Suppose $\lim_{x\to c^-} g(x) = \lim_{x\to c^-} h(x) = L$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x the inequality $c - \delta < x < c$ implies

$$L - \epsilon < g(x) < L + \epsilon \qquad \text{and} \qquad L - \epsilon < h(x) < L + \epsilon. \tag{9}$$

We conclude as before that for all x, $c - \delta < x < c$ implies $|f(x) - L| < \epsilon$. ❑

Proof for Two-sided Limits If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $g(x)$ and $h(x)$ both approach L as $x \to c^+$ and as $x \to c^-$; so $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$. Hence $\lim_{x\to c} f(x)$ exists and equals L. ❑

Exercises A.2

1. Suppose that functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ have limits L_1, L_2, and L_3, respectively, as $x \to c$. Show that their sum has limit $L_1 + L_2 + L_3$. Use mathematical induction (Appendix 1) to generalize this result to the sum of any finite number of functions.

2. Use mathematical induction and the Limit Product Rule in Theorem 1 to show that if functions $f_1(x), f_2(x), \ldots, f_n(x)$ have limits $L_1, L_2, \ldots, L_n$ as $x \to c$, then
$$\lim_{x\to c} f_1(x) f_2(x) \cdot \cdots \cdot f_n(x) = L_1 \cdot L_2 \cdot \cdots \cdot L_n.$$

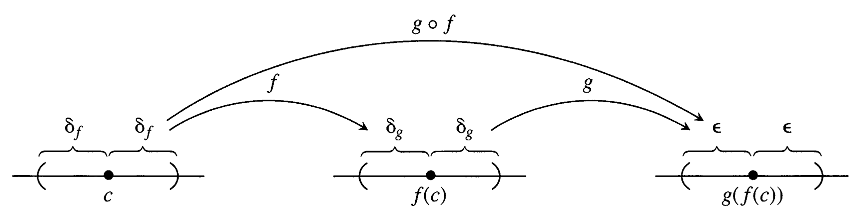
3. Use the fact that $\lim_{x\to c} x = c$ and the result of Exercise 2 to show that $\lim_{x\to c} x^n = c^n$ for any integer $n > 1$.

4. *Limits of polynomials.* Use the fact that $\lim_{x\to c}(k) = k$ for any number k together with the results of Exercises 1 and 3 to show that $\lim_{x\to c} f(x) = f(c)$ for any polynomial function
$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

5. *Limits of rational functions.* Use Theorem 1 and the result of Exercise 4 to show that if $f(x)$ and $g(x)$ are polynomial functions and $g(c) \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

6. *Composites of continuous functions.* Figure A.1 gives the diagram for a proof that the composite of two continuous functions is continuous. Reconstruct the proof from the diagram. The statement to be proved is this: If f is continuous at $x = c$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

   Assume that c is an interior point of the domain of f and that $f(c)$ is an interior point of the domain of g. This will make the limits involved two-sided. (The arguments for the cases that involve one-sided limits are similar.)



A.1 The diagram for a proof that the composite of two continuous functions is continuous. The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied. In the figure, f is to be continuous at c and g at $f(c)$.

A.3

Complex Numbers

Complex numbers are expressions of the form $a + ib$, where a and b are real numbers and i is a symbol for $\sqrt{-1}$. Unfortunately, the words "real" and "imaginary" have connotations that somehow place $\sqrt{-1}$ in a less favorable position in our minds than $\sqrt{2}$. As a matter of fact, a good deal of imagination, in the sense of *inventiveness,* has been required to construct the *real* number system, which forms the basis of the calculus. In this appendix we review the various stages of this invention. The further invention of a complex number system will then not seem so strange.

The Development of the Real Numbers

The earliest stage of number development was the recognition of the **counting numbers** 1, 2, 3, . . . , which we now call the **natural numbers** or the **positive integers.** Certain simple arithmetical operations can be performed with these numbers without getting outside the system. That is, the system of positive integers is **closed** under the operations of addition and multiplication. By this we mean that if m and n are any positive integers, then

$$m + n = p \qquad \text{and} \qquad mn = q \tag{1}$$

are also positive integers. Given the two positive integers on the left-hand side of either equation in (1), we can find the corresponding positive integer on the right. More than this, we can sometimes specify the positive integers m and p and find a positive integer n such that $m + n = p$. For instance, $3 + n = 7$ can be solved when the only numbers we know are the positive integers. But the equation $7 + n = 3$ cannot be solved unless the number system is enlarged.

The number zero and the negative integers were invented to solve equations like $7 + n = 3$. In a civilization that recognizes all the **integers**

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \tag{2}$$

an educated person can always find the missing integer that solves the equation $m + n = p$ when given the other two integers in the equation.

Suppose our educated people also know how to multiply any two of the integers in (2). If, in Eqs. (1), they are given m and q, they discover that sometimes they can find n and sometimes they cannot. If their imagination is still in good working order, they may be inspired to invent still more numbers and introduce fractions, which are just ordered pairs m/n of integers m and n. The number zero has special properties that may bother them for a while, but they ultimately discover that it is handy to have all ratios of integers m/n, excluding only those having zero in the denominator. This system, called the set of **rational numbers,** is now rich enough for them to perform the so-called **rational operations** of arithmetic:

1. a) addition
b) subtraction

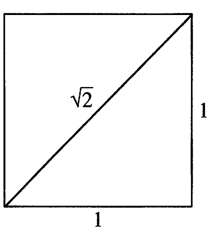
2. a) multiplication
b) division

on any two numbers in the system, *except that they cannot divide by zero.*

The geometry of the unit square (Fig. A.2) and the Pythagorean theorem showed that they could construct a geometric line segment that, in terms of some basic unit of length, has length equal to $\sqrt{2}$. Thus they could solve the equation

$$x^2 = 2$$

by a geometric construction. But then they discovered that the line segment representing $\sqrt{2}$ and the line segment representing the unit of length 1 were incommensurable quantities. This means that the ratio $\sqrt{2}/1$ cannot be expressed as the ratio of two *integer* multiples of some other, presumably more fundamental, unit of length. That is, our educated people could not find a rational number solution of the equation $x^2 = 2$.



A.2 With a straightedge and compass, it is possible to construct a segment of irrational length.

There *is* no rational number whose square is 2. To see why, suppose that there were such a rational number. Then we could find integers p and q with no common factor other than 1, and such that

$$p^2 = 2q^2. \tag{3}$$

Since p and q are integers, p must be even; otherwise its product with itself would be odd. In symbols, $p = 2p_1$, where p_1 is an integer. This leads to $2p_1{}^2 = q^2$, which says q must be even, say $q = 2q_1$, where q_1 is an integer. This makes 2 a factor of both p and q, contrary to our choice of p and q as integers with no common factor other than 1. Hence there is no rational number whose square is 2.

Although our educated people could not find a rational solution of the equation $x^2 = 2$, they could get a sequence of rational numbers

$$\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \ldots, \tag{4}$$

whose squares form a sequence

$$\frac{1}{1}, \quad \frac{49}{25}, \quad \frac{1681}{841}, \quad \frac{57{,}121}{28{,}561}, \quad \ldots, \tag{5}$$

that converges to 2 as its limit. This time their imagination suggested that they needed the concept of a limit of a sequence of rational numbers. If we accept the fact that an increasing sequence that is bounded from above always approaches a limit and observe that the sequence in (4) has these properties, then we want it to have a limit L. This would also mean, from (5), that $L^2 = 2$, and hence L is *not* one of our rational numbers. If to the rational numbers we further add the limits of all bounded increasing sequences of rational numbers, we arrive at the system of all "real" numbers. The word *real* is placed in quotes because there is nothing that is either "more real" or "less real" about this system than there is about any other mathematical system.

The Complex Numbers

Imagination was called upon at many stages during the development of the real number system. In fact, the art of invention was needed at least three times in constructing the systems we have discussed so far:

1. The *first invented* system: the set of *all integers* as constructed from the counting numbers.
2. The *second invented* system: the set of *rational numbers* m/n as constructed from the integers.
3. The *third invented* system: the set of all *real numbers* x as constructed from the rational numbers.

These invented systems form a hierarchy in which each system contains the previous system. Each system is also richer than its predecessor in that it permits additional operations to be performed without going outside the system:

1. In the system of all integers, we can solve all equations of the form

$$x + a = 0, \tag{6}$$

where a can be any integer.

2. In the system of all rational numbers, we can solve all equations of the form

$$ax + b = 0, \tag{7}$$

provided a and b are rational numbers and $a \neq 0$.

3. In the system of all real numbers, we can solve all of the equations in (6) and (7) and, in addition, all quadratic equations

$$ax^2 + bx + c = 0 \quad \text{having} \quad a \neq 0 \quad \text{and} \quad b^2 - 4ac \geq 0. \tag{8}$$

You are probably familiar with the formula that gives the solutions of (8), namely,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{9}$$

and are familiar with the further fact that when the discriminant, $d = b^2 - 4ac$, is negative, the solutions in (9) do *not* belong to any of the systems discussed above. In fact, the very simple quadratic equation

$$x^2 + 1 = 0$$

is impossible to solve if the only number systems that can be used are the three invented systems mentioned so far.

Thus we come to the *fourth invented* system, the set of all complex numbers $a + ib$. We could dispense entirely with the symbol i and use a notation like (a, b). We would then speak simply of a pair of real numbers a and b. Since, under algebraic operations, the numbers a and b are treated somewhat differently, it is essential to keep the *order* straight. We therefore might say that the **complex number system** consists of the set of all ordered pairs of real numbers (a, b), together with the rules by which they are to be equated, added, multiplied, and so on, listed below. We will use both the (a, b) notation and the notation $a + ib$ in the discussion that follows. We call a the **real part** and b the **imaginary part** of the complex number (a, b).

We make the following definitions.

Equality	
$a + ib = c + id$ if and only if $a = c$ and $b = d$	Two complex numbers (a, b) and (c, d) are *equal* if and only if $a = c$ and $b = d$.
Addition	
$(a + ib) + (c + id)$ $= (a + c) + i(b + d)$	The sum of the two complex numbers (a, b) and (c, d) is the complex number $(a + c, b + d)$.
Multiplication	
$(a + ib)(c + id)$ $= (ac - bd) + i(ad + bc)$	The product of two complex numbers (a, b) and (c, d) is the complex number $(ac - bd, ad + bc)$.
$c(a + ib) = ac + i(bc)$	The product of a real number c and the complex number (a, b) is the complex number (ac, bc).

The set of all complex numbers (a, b) in which the second number b is zero has all the properties of the set of real numbers a. For example, addition and multiplication of $(a, 0)$ and $(c, 0)$ give

$$(a, 0) + (c, 0) = (a + c, 0),$$

$$(a, 0) \cdot (c, 0) = (ac, 0),$$

which are numbers of the same type with imaginary part equal to zero. Also, if we multiply a "real number" $(a, 0)$ and the complex number (c, d), we get

$$(a, 0) \cdot (c, d) = (ac, ad) = a(c, d).$$

In particular, the complex number (0, 0) plays the role of zero in the complex number system, and the complex number (1, 0) plays the role of unity.

The number pair (0, 1), which has real part equal to zero and imaginary part equal to one, has the property that its square,

$$(0, 1)(0, 1) = (-1, 0),$$

has real part equal to minus one and imaginary part equal to zero. Therefore, in the system of complex numbers (a, b), there is a number $x = (0, 1)$ whose square can be added to unity $= (1, 0)$ to produce zero $= (0, 0)$; that is,

$$(0, 1)^2 + (1, 0) = (0, 0).$$

The equation

$$x^2 + 1 = 0$$

therefore has a solution $x = (0, 1)$ in this new number system.

You are probably more familiar with the $a + ib$ notation than you are with the notation (a, b). And since the laws of algebra for the ordered pairs enable us to write

$$(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1),$$

while (1, 0) behaves like unity and (0, 1) behaves like a square root of minus one, we need not hesitate to write $a + ib$ in place of (a, b). The i associated with b is like a tracer element that tags the imaginary part of $a + ib$. We can pass at will from the realm of ordered pairs (a, b) to the realm of expressions $a + ib$, and conversely. But there is nothing less "real" about the symbol $(0, 1) = i$ than there is about the symbol $(1, 0) = 1$, once we have learned the laws of algebra in the complex number system (a, b).

To reduce any rational combination of complex numbers to a single complex number, we apply the laws of elementary algebra, replacing i^2 wherever it appears by -1. Of course, we cannot divide by the complex number $(0, 0) = 0 + i\,0$. But if $a + ib \neq 0$, then we may carry out a division as follows:

$$\frac{c + id}{a + ib} = \frac{(c + id)(a - ib)}{(a + ib)(a - ib)} = \frac{(ac + bd) + i(ad - bc)}{a^2 + b^2}.$$

The result is a complex number $x + iy$ with

$$x = \frac{ac + bd}{a^2 + b^2}, \qquad y = \frac{ad - bc}{a^2 + b^2},$$

and $a^2 + b^2 \neq 0$, since $a + ib = (a, b) \neq (0, 0)$.

The number $a - ib$ that is used as multiplier to clear the i from the denominator is called the **complex conjugate** of $a + ib$. It is customary to use $\bar{z}$ (read "z bar") to denote the complex conjugate of z; thus

$$z = a + ib, \qquad \bar{z} = a - ib.$$

Multiplying the numerator and denominator of the fraction $(c + id)/(a + ib)$ by the complex conjugate of the denominator will always replace the denominator by a real number.

EXAMPLE 1

a) $(2 + 3i) + (6 - 2i) = (2 + 6) + (3 - 2)i = 8 + i$

b) $(2 + 3i) - (6 - 2i) = (2 - 6) + (3 - (-2))i = -4 + 5i$

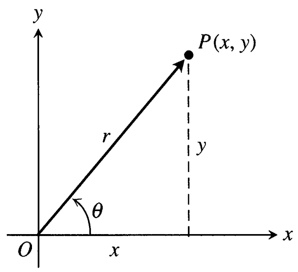
c) $(2 + 3i)(6 - 2i) = (2)(6) + (2)(-2i) + (3i)(6) + (3i)(-2i)$
$= 12 - 4i + 18i - 6i^2 = 12 + 14i + 6 = 18 + 14i$

d) $\dfrac{2 + 3i}{6 - 2i} = \dfrac{2 + 3i}{6 - 2i}\,\dfrac{6 + 2i}{6 + 2i}$

$$= \frac{12 + 4i + 18i + 6i^2}{36 + 12i - 12i - 4i^2}$$

$$= \frac{6 + 22i}{40} = \frac{3}{20} + \frac{11}{20}i$$

❑



A.3 This Argand diagram represents $z = x + iy$ both as a point $P(x, y)$ and as a vector $\overrightarrow{OP}$.

Argand Diagrams

There are two geometric representations of the complex number $z = x + iy$:

a) as the point $P(x, y)$ in the xy-plane and

b) as the vector $\overrightarrow{OP}$ from the origin to P.

In each representation, the x-axis is called the **real axis** and the y-axis is the **imaginary axis.** Both representations are **Argand diagrams** for $x + iy$ (Fig. A.3).

In terms of the polar coordinates of x and y, we have

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and

$$z = x + iy = r(\cos\theta + i\sin\theta). \tag{10}$$

We define the **absolute value** of a complex number $x + iy$ to be the length r of a vector $\overrightarrow{OP}$ from the origin to $P(x, y)$. We denote the absolute value by vertical bars, thus:

$$|x + iy| = \sqrt{x^2 + y^2}.$$

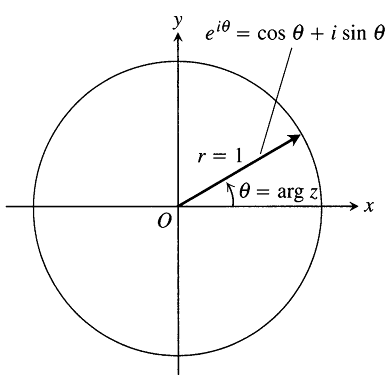
If we always choose the polar coordinates r and θ so that r is nonnegative, then

$$r = |x + iy|.$$

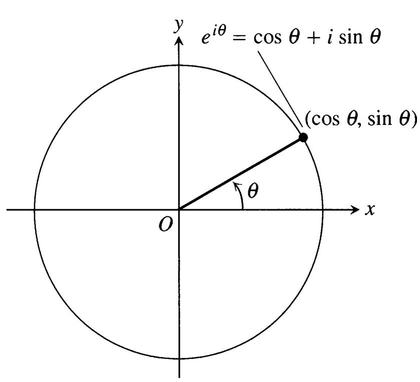
The polar angle θ is called the **argument** of z and is written $\theta = \arg z$. Of course, any integer multiple of 2π may be added to θ to produce another appropriate angle.

The following equation gives a useful formula connecting a complex number z, its conjugate $\overline{z}$, and its absolute value $|z|$, namely,

$$z \cdot \overline{z} = |z|^2.$$



(a)



(b)

A.4 Argand diagrams for $e^{i\theta} = \cos\theta + i\sin\theta$ (a) as a vector, (b) as a point.

Euler's Formula, Products, and Quotients

The identity

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

called **Euler's formula,** enables us to rewrite Eq. (10) as

$$z = re^{i\theta}.$$

This, in turn, leads to the following rules for calculating products, quotients, powers, and roots of complex numbers. It also leads to Argand diagrams for $e^{i\theta}$. Since $\cos\theta + i\sin\theta$ is what we get from Eq. (10) by taking $r = 1$, we can say that $e^{i\theta}$ is represented by a unit vector that makes an angle θ with the positive x-axis, as shown in Fig. A.4.

Products To multiply two complex numbers, we multiply their absolute values and add their angles. Let

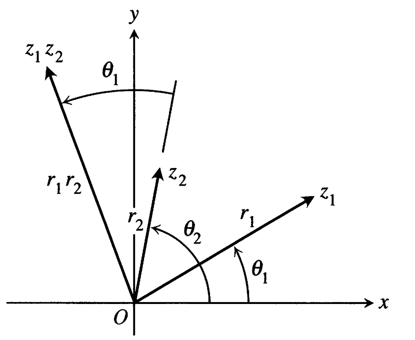
$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}, \tag{11}$$

so that

$$|z_1| = r_1, \quad \arg z_1 = \theta_1; \qquad |z_2| = r_2, \quad \arg z_2 = \theta_2.$$

Then

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$



A.5 When z_1 and z_2 are multiplied, $|z_1 z_2| = r_1 \cdot r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

and hence

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|,$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2. \tag{12}$$

Thus the product of two complex numbers is represented by a vector whose length is the product of the lengths of the two factors and whose argument is the sum of their arguments (Fig. A.5). In particular, a vector may be rotated counterclockwise through an angle θ by multiplying it by $e^{i\theta}$. Multiplication by i rotates 90°, by -1 rotates 180°, by $-i$ rotates 270°, etc.

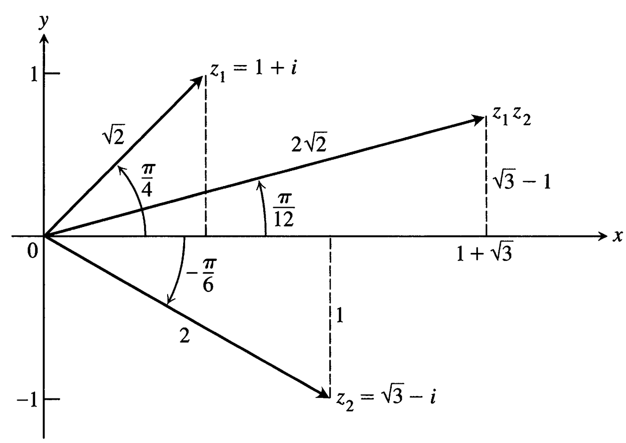
EXAMPLE 2 Let $z_1 = 1 + i$, $z_2 = \sqrt{3} - i$. We plot these complex numbers in an Argand diagram (Fig. A.6) from which we read off the polar representations

$$z_1 = \sqrt{2}e^{i\pi/4}, \qquad z_2 = 2e^{-i\pi/6}.$$

Then

$$z_1 z_2 = 2\sqrt{2}\exp\left(\frac{i\pi}{4} - \frac{i\pi}{6}\right) = 2\sqrt{2}\exp\left(\frac{i\pi}{12}\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \approx 2.73 + 0.73i.$$

exp (A) stands for e^A.



A.6 To multiply two complex numbers, multiply their absolute values and add their arguments. ❑

Quotients

Suppose $r_2 \neq 0$ in Eq. (11). Then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}.$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

That is, we divide lengths and subtract angles.

EXAMPLE 3 Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, as in Example 2. Then

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{5\pi i/12} \approx 0.707\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$
$$\approx 0.183 + 0.\overline{683}i.$$ ❑

Powers

If n is a positive integer, we may apply the product formulas in (12) to find

$$z^n = z \cdot z \cdot \cdots \cdot z. \qquad n \text{ factors}$$

With $z = re^{i\theta}$, we obtain

$$\begin{aligned} z^n = (re^{i\theta})^n &= r^n e^{i(\theta+\theta+\cdots+\theta)} \qquad n \text{ summands} \\ &= r^n e^{in\theta}. \end{aligned} \tag{13}$$

The length $r = |z|$ is raised to the nth power and the angle $\theta = \arg z$ is multiplied by n.

If we take $r = 1$ in Eq. (13), we obtain De Moivre's theorem.

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{14}$$

If we expand the left-hand side of De Moivre's equation (Eq. 14) by the binomial theorem and reduce it to the form $a + ib$, we obtain formulas for $\cos n\theta$ and $\sin n\theta$ as polynomials of degree n in $\cos\theta$ and $\sin\theta$.

EXAMPLE 4 If $n = 3$ in Eq. (14), we have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left-hand side of this equation is

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

The real part of this must equal $\cos 3\theta$ and the imaginary part must equal $\sin 3\theta$. Therefore,

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta, \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta. \end{aligned}$$ ❑

Roots If $z = re^{i\theta}$ is a complex number different from zero and n is a positive integer, then there are precisely n different complex numbers $w_0, w_1, \ldots, w_{n-1}$, that are nth roots of z. To see why, let $w = \rho e^{i\alpha}$ be an nth root of $z = re^{i\theta}$, so that

$$w^n = z$$

or

$$\rho^n e^{in\alpha} = re^{i\theta}.$$

Then

$$\rho = \sqrt[n]{r}$$

is the real, positive nth root of r. As regards the angle, although we cannot say that $n\alpha$ and θ must be equal, we can say that they may differ only by an integer multiple of 2π. That is,

$$n\alpha = \theta + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Therefore,

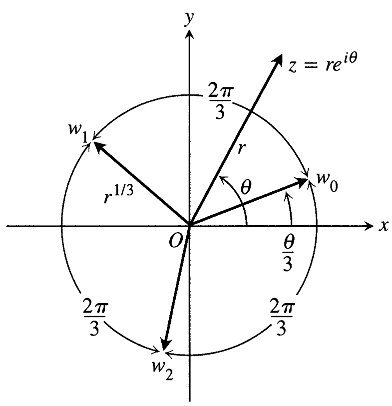
$$\alpha = \frac{\theta}{n} + k\frac{2\pi}{n}.$$

Hence all nth roots of $z = re^{i\theta}$ are given by

$$\boxed{\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}\,\exp i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right), \qquad k = 0, \pm 1, \pm 2, \ldots . \qquad (15)}$$

There might appear to be infinitely many different answers corresponding to the infinitely many possible values of k. But $k = n + m$ gives the same answer as $k = m$ in Eq. (15). Thus we need only take n consecutive values for k to obtain all the different nth roots of z. For convenience, we take

$$k = 0, 1, 2, \ldots, n - 1.$$



A.7 The three cube roots of $z = re^{i\theta}$.

All the nth roots of $re^{i\theta}$ lie on a circle centered at the origin O and having radius equal to the real, positive nth root of r. One of them has argument $\alpha = \theta/n$. The others are uniformly spaced around the circle, each being separated from its neighbors by an angle equal to $2\pi/n$. Figure A.7 illustrates the placement of the three cube roots, w_0, w_1, w_2, of the complex number $z = re^{i\theta}$.

EXAMPLE 5 Find the four fourth roots of -16.

Solution As our first step, we plot the number -16 in an Argand diagram (Fig. A.8) and determine its polar representation $re^{i\theta}$. Here, $z = -16, r = +16$, and $\theta = \pi$. One of the fourth roots of $16e^{i\pi}$ is $2e^{i\pi/4}$. We obtain others by successive additions of $2\pi/4 = \pi/2$ to the argument of this first one. Hence

$$\sqrt[4]{16 \exp i\pi} = 2 \exp i\left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right),$$

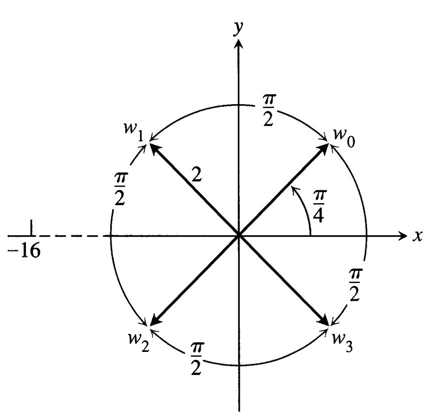
and the four roots are

$$w_0 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}(1 + i),$$

$$w_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = \sqrt{2}(-1 + i),$$

$$w_2 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = \sqrt{2}(-1 - i),$$

$$w_3 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = \sqrt{2}(1 - i).$$ ❑



A.8 The four fourth roots of -16.

The Fundamental Theorem of Algebra One may well say that the invention of $\sqrt{-1}$ is all well and good and leads to a number system that is richer than the real number system alone; but where will this process end? Are we also going to invent still more systems so as to obtain $\sqrt[4]{-1}$, $\sqrt[6]{-1}$, and so on? By now it should be clear that this is not necessary. These numbers are already expressible in terms of the complex number system $a + ib$. In fact, the Fundamental Theorem of Algebra says that with the introduction of the complex numbers we now have enough numbers to factor every polynomial into a product of linear factors and hence enough numbers to solve every possible polynomial equation.

The Fundamental Theorem of Algebra

Every polynomial equation of the form

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0,$$

in which the coefficients $a_0, a_1, \ldots, a_n$ are any complex numbers, whose degree n is greater than or equal to one, and whose leading coefficient a_0 is not zero, has exactly n roots in the complex number system, provided each multiple root of multiplicity m is counted as m roots.

A proof of this theorem can be found in almost any text on the theory of functions of a complex variable.

Exercises A.3

Operations with Complex Numbers

1. *How computers multiply complex numbers*
Find $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

   a) $(2, 3) \cdot (4, -2)$ **b)** $(2, -1) \cdot (-2, 3)$
   c) $(-1, -2) \cdot (2, 1)$

   (This is how complex numbers are multiplied by computers.)

2. Solve the following equations for the real numbers, x and y.

   a) $(3 + 4i)^2 - 2(x - iy) = x + iy$

   b) $\left(\dfrac{1+i}{1-i}\right)^2 + \dfrac{1}{x+iy} = 1 + i$

   c) $(3 - 2i)(x + iy) = 2(x - 2iy) + 2i - 1$

Graphing and Geometry

3. How may the following complex numbers be obtained from $z = x + iy$ geometrically? Sketch.

   a) $\overline{z}$ **b)** $\overline{(-z)}$
   c) $-z$ **d)** $1/z$

4. Show that the distance between the two points z_1 and z_2 in an Argand diagram is equal to $|z_1 - z_2|$.

In Exercises 5–10, graph the points $z = x + iy$ that satisfy the given conditions.

5. **a)** $|z| = 2$ **b)** $|z| < 2$ **c)** $|z| > 2$
6. $|z - 1| = 2$
7. $|z + 1| = 1$
8. $|z + 1| = |z - 1|$
9. $|z + i| = |z - 1|$
10. $|z + 1| \geq |z|$

Express the complex numbers in Exercises 11–14 in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. Draw an Argand diagram for each calculation.

11. $(1 + \sqrt{-3})^2$
12. $\dfrac{1+i}{1-i}$
13. $\dfrac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$
14. $(2 + 3i)(1 - 2i)$

Theory and Examples

15. Show with an Argand diagram that the law for adding complex numbers is the same as the parallelogram law for adding vectors.
16. Show that the conjugate of the sum (product, or quotient) of two complex numbers z_1 and z_2 is the same as the sum (product, or quotient) of their conjugates.
17. *Complex roots of polynomials with real coefficients come in complex-conjugate pairs.*

    a) Extend the results of Exercise 16 to show that $f(\overline{z}) = \overline{f(z)}$

if

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

is a polynomial with real coefficients $a_0, \ldots, a_n$.

b) If z is a root of the equation $f(z) = 0$, where $f(z)$ is a polynomial with real coefficients as in part (a), show that the conjugate $\overline{z}$ is also a root of the equation. (*Hint:* Let $f(z) = u + iv = 0$; then both u and v are zero. Now use the fact that $f(\overline{z}) = \overline{f(z)} = u - iv$.)

18. Show that $|\overline{z}| = |z|$.

19. If z and $\overline{z}$ are equal, what can you say about the location of the point z in the complex plane?

20. Let $Re(z)$ denote the real part of z and $Im(z)$ the imaginary part. Show that the following relations hold for any complex numbers z, z_1, and z_2.

a) $z + \overline{z} = 2Re(z)$ **b)** $z - \overline{z} = 2i\,Im(z)$

c) $|Re(z)| \leq |z|$

d) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z}_2)$

e) $|z_1 + z_2| \leq |z_1| + |z_2|$

Use De Moivre's theorem to express the trigonometric functions in Exercises 21 and 22 in terms of $\cos\theta$ and $\sin\theta$.

21. $\cos 4\theta$ **22.** $\sin 4\theta$

Roots

23. Find the three cube roots of 1.

24. Find the two square roots of i.

25. Find the three cube roots of $-8i$.

26. Find the six sixth roots of 64.

27. Find the four solutions of the equation $z^4 - 2z^2 + 4 = 0$.

28. Find the six solutions of the equation $z^6 + 2z^3 + 2 = 0$.

29. Find all solutions of the equation $x^4 + 4x^2 + 16 = 0$.

30. Solve the equation $x^4 + 1 = 0$.

A.4 Simpson's One-Third Rule

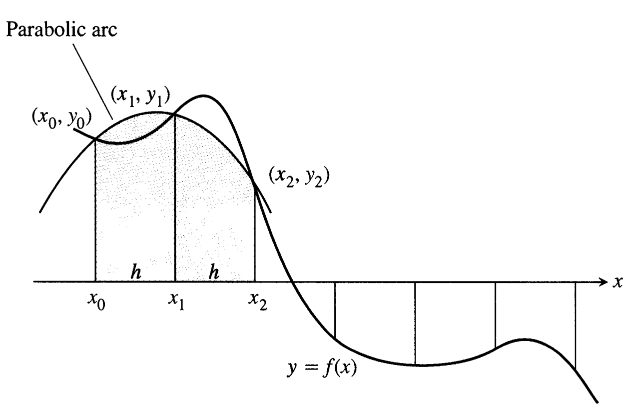
Simpson's rule for approximating $\int_a^b f(x)\,dx$ is based on approximating the graph of f with parabolic arcs.

The area of the shaded region under the parabola in Fig. A.9 is

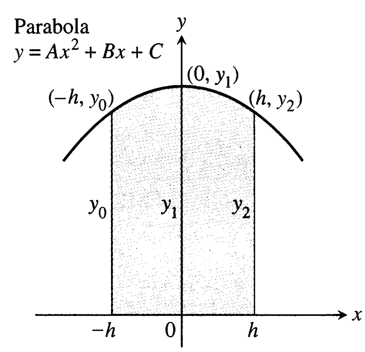
$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

This formula is known as Simpson's one-third rule.

We can derive the formula as follows. To simplify the algebra, we use the coordinate system in Fig. A.10. The area under the parabola is the same no matter where the y-axis is, as long as we preserve the vertical scale. The parabola has an equation of the form $y = Ax^2 + Bx + C$, so the area under it from $x = -h$ to



A.9 Simpson's rule approximates short stretches of curve with parabolic arcs.



A.10 By integrating from $-h$ to h, the shaded area is found to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

$x = h$ is

$$\text{Area} = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$
$$= \frac{2Ah^3}{3} + 2Ch$$
$$= \frac{h}{3}(2Ah^2 + 6C).$$

Since the curve passes through $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C.$$

From these equations we obtain

$$\begin{aligned} C &= y_1, \\ Ah^2 - Bh &= y_0 - y_1, \\ Ah^2 + Bh &= y_2 - y_1, \\ 2Ah^2 &= y_0 + y_2 - 2y_1. \end{aligned}$$

These substitutions for C and $2Ah^2$ give

$$\text{Area} = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

A.5

Cauchy's Mean Value Theorem and the Stronger Form of l'Hôpital's Rule

This appendix proves the finite-limit case of the stronger form of l'Hôpital's Rule (Section 6.6, Theorem 3).

L'Hôpital's Rule (Stronger Form)

Suppose that

$$f(x_0) = g(x_0) = 0$$

and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_0. Suppose also that $g' \neq 0$ at every point in (a, b) except possibly x_0. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}, \tag{1}$$

provided the limit on the right exists.

The proof of the stronger form of l'Hôpital's rule is based on Cauchy's Mean Value Theorem, a mean value theorem that involves two functions instead of one. We prove Cauchy's theorem first and then show how it leads to l'Hôpital's rule.

Cauchy's Mean Value Theorem

Suppose that functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and suppose also that $g' \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{2}$$

The ordinary Mean Value Theorem (Section 3.2, Theorem 4) is the case $g(x) = x$.

Proof of Cauchy's Mean Value Theorem We apply the Mean Value Theorem of Section 3.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b. This cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore there is a number c between a and b for which $F'(c) = 0$. In terms of f and g this says

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0,$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$

which is Eq. (2). ❑

Proof of the Stronger Form of l'Hôpital's Rule We first establish Eq. (1) for the case $x \to x_0^+$. The method needs almost no change to apply to $x \to x_0^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of x_0. Then $g'(x) \neq 0$ and we can apply Cauchy's Mean Value Theorem to the closed interval from x_0 to x. This produces a number c between x_0 and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}.$$

But $f(x_0) = g(x_0) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches x_0, c approaches x_0 because it lies between x and x_0. Therefore,

$$\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{c \to x_0^+} \frac{f'(c)}{g'(c)} = \lim_{x \to x_0^+} \frac{f'(x)}{g'(x)}.$$

This establishes l'Hôpital's rule for the case where x approaches x_0 from above. The case where x approaches x_0 from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, x_0]$, $x < x_0$. ❑

A.6 Limits That Arise Frequently

This appendix verifies limits (4)–(6) in Section 8.2, Table 1.

Limit 4: If $|x| < 1$, $\lim_{n\to\infty} x^n = 0$ We need to show that to each $\epsilon > 0$ there corresponds an integer N so large that $|x^n| < \epsilon$ for all n greater than N. Since $\epsilon^{1/n} \to 1$, while $|x| < 1$, there exists an integer N for which $\epsilon^{1/N} > |x|$. In other words,

$$|x^N| = |x|^N < \epsilon. \tag{1}$$

This is the integer we seek because, if $|x| < 1$, then

$$|x^n| < |x^N| \text{ for all } n > N. \tag{2}$$

Combining (1) and (2) produces $|x^n| < \epsilon$ for all $n > N$, concluding the proof.

Limit 5: For any number x, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ Let

$$a_n = \left(1+\frac{x}{n}\right)^n.$$

Then

$$\ln a_n = \ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right) \to x,$$

as we can see by the following application of l'Hôpital's rule, in which we differentiate with respect to n:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) &= \lim_{n\to\infty}\frac{\ln(1+x/n)}{1/n}\\ &= \lim_{n\to\infty}\frac{\left(\dfrac{1}{1+x/n}\right)\cdot\left(-\dfrac{x}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{x}{1+x/n} = x.\end{aligned}$$

Apply Theorem 4, Section 8.2, with $f(x) = e^x$ to conclude that

$$\left(1+\frac{x}{n}\right)^n = a_n = e^{\ln a_n} \to e^x.$$

Limit 6: For any number x, $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ Since

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!},$$

all we need to show is that $|x|^n/n! \to 0$. We can then apply the Sandwich Theorem for Sequences (Section 8.2, Theorem 3) to conclude that $x^n/n! \to 0$.

The first step in showing that $|x|^n/n! \to 0$ is to choose an integer $M > |x|$, so that $(|x|/M) < 1$. By Limit 4, just proved, we then have $(|x|/M)^n \to 0$. We then restrict our attention to values of $n > M$. For these values of n, we can write

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdot \cdots \cdot M \cdot \underbrace{(M+1)(M+2) \cdot \cdots \cdot n}_{(n-M)\text{ factors}}}$$

$$\leq \frac{|x|^n}{M!M^{n-M}} = \frac{|x|^n M^M}{M!M^n} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n.$$

Thus,

$$0 \leq \frac{|x|^n}{n!} \leq \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n.$$

Now, the constant $M^M/M!$ does not change as n increases. Thus the Sandwich Theorem tell us that $|x|^n/n! \to 0$ because $(|x|/M)^n \to 0$.

A.7 The Distributive Law for Vector Cross Products

In this appendix we prove the distributive law

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1}$$

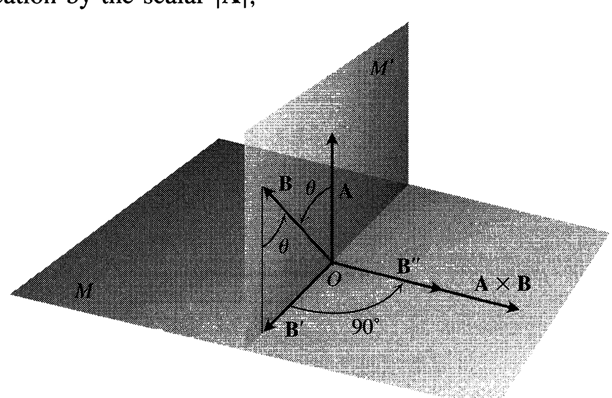
from Eq. (6) in Section 10.4.

Proof To derive Eq. (1), we construct $\mathbf{A} \times \mathbf{B}$ a new way. We draw $\mathbf{A}$ and $\mathbf{B}$ from the common point O and construct a plane M perpendicular to $\mathbf{A}$ at O (Fig. A.11). We then project $\mathbf{B}$ orthogonally onto M, yielding a vector $\mathbf{B}'$ with length $|\mathbf{B}| \sin\theta$. We rotate $\mathbf{B}'$ 90°about $\mathbf{A}$ in the positive sense to produce a vector $\mathbf{B}''$. Finally, we multiply $\mathbf{B}''$ by the length of $\mathbf{A}$. The resulting vector $|\mathbf{A}|\mathbf{B}''$ is equal to $\mathbf{A} \times \mathbf{B}$ since $\mathbf{B}''$ has the same direction as $\mathbf{A} \times \mathbf{B}$ by its construction (Fig. A.11) and

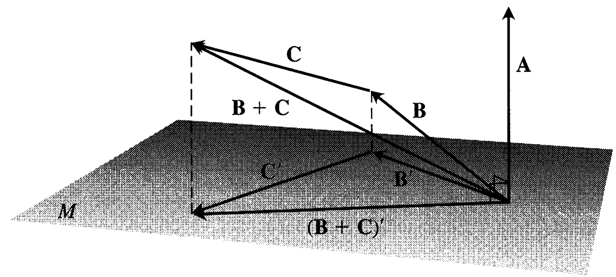
$$|\mathbf{A}||\mathbf{B}''| = |\mathbf{A}||\mathbf{B}'| = |\mathbf{A}||\mathbf{B}| \sin\theta = |\mathbf{A} \times \mathbf{B}|.$$

Now each of these three operations, namely,

1. projection onto M,
2. rotation about $\mathbf{A}$ through 90°,
3. multiplication by the scalar $|\mathbf{A}|$,



A.11 As explained in the text, $\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\mathbf{B}''$.



A.12 The vectors, **B**, **C**, **B** + **C**, and their projections onto a plane perpendicular to **A**.

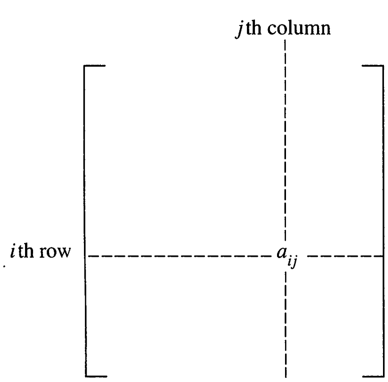
when applied to a triangle whose plane is not parallel to **A**, will produce another triangle. If we start with the triangle whose sides are **B**, **C**, and $\mathbf{B}+\mathbf{C}$ (Fig. A.12) and apply these three steps, we successively obtain

1. a triangle whose sides are $\mathbf{B}'$, $\mathbf{C}'$, and $(\mathbf{B}+\mathbf{C})'$ satisfying the vector equation
$$\mathbf{B}'+\mathbf{C}'=(\mathbf{B}+\mathbf{C})';$$
2. a triangle whose sides are $\mathbf{B}''$, $\mathbf{C}''$, and $(\mathbf{B}+\mathbf{C})''$ satisfying the vector equation
$$\mathbf{B}''+\mathbf{C}''=(\mathbf{B}+\mathbf{C})''$$
(the double prime on each vector has the same meaning as in Fig. A.11); and, finally,
3. a triangle whose sides are $|\mathbf{A}|\mathbf{B}''$, $|\mathbf{A}|\mathbf{C}''$, and $|\mathbf{A}|(\mathbf{B}+\mathbf{C})''$ satisfying the vector equation
$$|\mathbf{A}|\mathbf{B}''+|\mathbf{A}|\mathbf{C}''=|\mathbf{A}|(\mathbf{B}+\mathbf{C})''. \tag{2}$$

Substituting $|\mathbf{A}|\mathbf{B}''=\mathbf{A}\times\mathbf{B}$, $|\mathbf{A}|\mathbf{C}''=\mathbf{A}\times\mathbf{C}$, and $|\mathbf{A}|(\mathbf{B}+\mathbf{C})''=\mathbf{A}\times(\mathbf{B}+\mathbf{C})$ from our discussion above into Eq. (2) gives
$$\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}=\mathbf{A}\times(\mathbf{B}+\mathbf{C}),$$
which is the law we wanted to establish. ❑

A.8 Determinants and Cramer's Rule

A rectangular array of numbers like
$$A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}$$
is called a **matrix.** We call A a 2 by 3 matrix because it has two rows and three columns. An m by n matrix has m rows and n columns, and the **entry** or **element** (number) in the ith row and jth column is denoted by a_{ij}. The matrix



$$A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}$$
has
$$a_{11}=2,\quad a_{12}=1,\quad a_{13}=3,$$
$$a_{21}=1,\quad a_{22}=0,\quad a_{23}=-2.$$

The vertical bars in the notation $|a_{ij}|$ do not mean absolute value.

A matrix with the same number of rows as columns is a **square matrix.** It is a **matrix of order n** if the number of rows and columns is n.

With each square matrix A we associate a number det A or $|a_{ij}|$, called the **determinant** of A, calculated from the entries of A in the following way. For $n = 1$ and $n = 2$, we define

$$\det [a] = a, \tag{1}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{2}$$

For a matrix of order 3, we define

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{l}\text{Sum of all signed products} \\ \text{of the form } \pm a_{1i}a_{2j}a_{3k},\end{array} \tag{3}$$

where i, j, k is a permutation of 1, 2, 3 in some order. There are $3! = 6$ such permutations, so there are six terms in the sum. The sign is positive when the index of the permutation is even and negative when the index is odd.

Definition
Index of a Permutation

Given any permutation of the numbers $1, 2, 3, \ldots, n$, denote the permutation by $i_1, i_2, i_3, \ldots, i_n$. In this arrangement, some of the numbers following i_1 may be less than i_1, and the number of these is called the **number of inversions** in the arrangement pertaining to i_1. Likewise, there are a number of inversions pertaining to each of the other i's; it is the number of indices that come after that particular i in the arrangement and are less than it. The **index** of the permutation is the sum of all of the numbers of inversions pertaining to the separate indices.

EXAMPLE 1 For $n = 5$, the permutation

$$5 \quad 3 \quad 1 \quad 2 \quad 4$$

has 4 inversions pertaining to the first element, 5, 2 inversions pertaining to the second element, 3, and no further inversions, so the index is $4 + 2 = 6$. ❑

The following table shows the permutations of 1, 2, 3, the index of each permutation, and the signed product in the determinant of Eq. (3).

Permutation	Index	Signed product
1 2 3	0	$+a_{11}a_{22}a_{33}$
1 3 2	1	$-a_{11}a_{23}a_{32}$
2 1 3	1	$-a_{12}a_{21}a_{33}$
2 3 1	2	$+a_{12}a_{23}a_{31}$
3 1 2	2	$+a_{13}a_{21}a_{32}$
3 2 1	3	$-a_{13}a_{22}a_{31}$

(4)

The sum of the six signed products is

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{5}$$

The formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{6}$$

reduces the calculation of a 3 by 3 determinant to the calculation of three 2 by 2 determinants.

Many people prefer to remember the following scheme for calculating the six signed products in the determinant of a 3 by 3 matrix:

$$\begin{array}{ccc:cc} & & - & - & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ & & + & + & + \end{array} \tag{7}$$

◁ Change these signs

◁ Keep these signs

Minors and Cofactors

The second order determinants on the right-hand side of Eq. (6) are called the **minors** (short for "minor determinants") of the entries they multiply. Thus,

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ is the minor of } a_{11}, \qquad \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ is the minor of } a_{12},$$

and so on. The minor of the element a_{ij} in a matrix A is the determinant of the matrix that remains after we delete the row and column containing a_{ij}:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \qquad \text{The minor of } a_{22} \text{ is } \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \qquad \text{The minor of } a_{23} \text{ is } \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

The **cofactor** A_{ij} of a_{ij} is $(-1)^{i+j}$ times the minor of a_{ij}. Thus,

$$A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

The factor $(-1)^{i+j}$ changes the sign of the minor when $i + j$ is odd. There is a checkerboard pattern for remembering these changes:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

In the upper left corner, $i = 1$, $j = 1$ and $(-1)^{1+1} = +1$. In going from any cell to an adjacent cell in the same row or column, we change i by 1 or j by 1, but not both, so we change the exponent from even to odd or from odd to even, which changes the sign from $+$ to $-$ or from $-$ to $+$.

When we rewrite Eq. (6) in terms of cofactors we get

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}. \tag{8}$$

EXAMPLE 2 Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}.$$

Solution 1 The cofactors are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}, \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix},$$

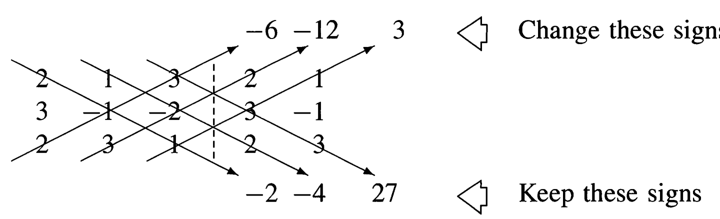
$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}.$$

To find det A, we multiply each element of the first row of A by its cofactor and add:

$$\det A = 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-1+6) - 1(3+4) + 3(9+2) = 10 - 7 + 33 = 36.$$

Solution 2 From (7) we find



$$\det A = -(-6) - (-12) - 3 + (-2) + (-4) + 27 = 36$$ ❑

Expanding by Columns or by Other Rows

The determinant of a square matrix can be calculated from the cofactors of any row or any column.

If we were to expand the determinant in Example 2 by cofactors according to elements of its third column, say, we would get

$$+3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 3(9+2) + 2(6-2) + 1(-2-3) = 33 + 8 - 5 = 36.$$

Useful Facts About Determinants

Fact 1: If two rows (or columns) are identical, the determinant is zero.

Fact 2: Interchanging two rows (or columns) changes the sign of the determinant.

Fact 3: The determinant is the sum of the products of the elements of the ith row (or column) by their cofactors, for any i.

Fact 4: The determinant of the transpose of a matrix is the same as the determinant of the original matrix. (The **transpose** of a matrix is obtained by writing the rows as columns.)

Fact 5: Multiplying each element of some row (or column) by a constant c multiplies the determinant by c.

Fact 6: If all elements above the main diagonal (or all below it) are zero, the determinant is the product of the elements on the main diagonal. (The **main diagonal** is the diagonal from upper left to lower right.)

EXAMPLE 3

$$\begin{vmatrix} 3 & 4 & 7 \\ 0 & -2 & 5 \\ 0 & 0 & 5 \end{vmatrix} = (3)(-2)(5) = -30$$

❑

Fact 7: If the elements of any row are multiplied by the cofactors of the corresponding elements of a different row and these products are summed, the sum is zero.

EXAMPLE 4 If A_{11}, A_{12}, A_{13} are the cofactors of the elements of the first row of $A = (a_{ij})$, then the sums

$$a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$$

(elements of second row times cofactors of elements of first row) and

$$a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13}$$

are both zero. ❑

Fact 8: If the elements of any column are multiplied by the cofactors of the corresponding elements of a different column and these products are summed, the sum is zero.

Fact 9: If each element of a row is multiplied by a constant c and the results added to a different row, the determinant is not changed. A similar result holds for columns.

EXAMPLE 5 If we start with

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

and add -2 times row 1 to row 2 (subtract 2 times row 1 from row 2), we get

$$B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -3 & -8 \\ 2 & 3 & 1 \end{bmatrix}.$$

Since $\det A = 36$ (Example 2), we should find that $\det B = 36$ as well. Indeed we do, as the following calculation shows:

$$\begin{array}{ccc:cccc}
 & & & -18 & -48 & -1 \\
2 & 1 & 3 & 2 & 1 & \\
-1 & -3 & -8 & -1 & -3 & \\
2 & 3 & 1 & 2 & 3 & \\
 & & & -6 & -16 & -9
\end{array}$$

$$\begin{aligned}
\det B &= -(-18) - (-48) - (-1) + (-6) + (-16) + (-9) \\
&= 18 + 48 + 1 - 6 - 16 - 9 = 67 - 31 = 36.
\end{aligned}$$

❑

EXAMPLE 6 Evaluate the fourth order determinant

$$D = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}.$$

Solution We subtract 2 times row 1 from row 2 and add row 1 to row 3 to get

$$D = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 1 & 2 & 1 \end{vmatrix}.$$

We then multiply the elements of the first column by their cofactors to get

$$D = \begin{vmatrix} 5 & -6 & 0 \\ 0 & 4 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 5(4+2) - (-6)(0+1) + 0 = 36.$$

❑

Cramer's Rule

If the determinant $D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$, the system

$$\begin{aligned} a_{11}x + a_{12}y &= b_1, \\ a_{21}x + a_{22}y &= b_2 \end{aligned} \tag{9}$$

has either infinitely many solutions or no solution at all. The system

$$\begin{aligned} x + y &= 0, \\ 2x + 2y &= 0 \end{aligned}$$

whose determinant is

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$

has infinitely many solutions. We can find an x to match any given y. The system

$$\begin{aligned} x + y &= 0, \\ 2x + 2y &= 2 \end{aligned}$$

has no solution. If $x + y = 0$, then $2x + 2y = 2(x + y)$ cannot be 2.

If $D \neq 0$, the system (9) has a unique solution, and Cramer's rule states that it may be found from the formulas

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}. \tag{10}$$

The numerator in the formula for x comes from replacing the first column in A (the x-column) by the column of constants b_1 and b_2 (the b-column). Replacing the y-column by the b-column gives the numerator of the y-solution.

EXAMPLE 7 Solve the system

$$\begin{aligned} 3x - y &= 9, \\ x + 2y &= -4. \end{aligned}$$

Solution We use Eqs. (10). The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6 + 1 = 7.$$

Hence,

$$x = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{D} = \frac{18 - 4}{7} = \frac{14}{7} = 2,$$

$$y = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{D} = \frac{-12 - 9}{7} = \frac{-21}{7} = -3. \qquad \square$$

Systems of three equations in three unknowns work the same way. If

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0,$$

the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1, \\ a_{21}x + a_{22}y + a_{23}z &= b_2, \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \tag{11}$$

has either infinitely many solutions or no solution at all. If $D \neq 0$, the system has a unique solution, given by Cramer's rule:

$$x = \frac{1}{D}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \qquad y = \frac{1}{D}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}.$$

$$z = \frac{1}{D}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

The pattern continues in higher dimensions.

Exercises A.8

Evaluating Determinants

Evaluate the following determinants.

1. $\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$

4. $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix}$

Evaluate the following determinants by expanding according to the cofactors of (a) the third row and (b) the second column.

5. $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix}$

8. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$

Systems of Equations

Solve the following systems of equations by Cramer's rule.

9. $\begin{aligned} x + 8y &= 4 \\ 3x - y &= -13 \end{aligned}$

10. $\begin{aligned} 2x + 3y &= 5 \\ 3x - y &= 2 \end{aligned}$

11. $\begin{aligned} 4x - 3y &= 6 \\ 3x - 2y &= 5 \end{aligned}$

12. $\begin{aligned} x + y + z &= 2 \\ 2x - y + z &= 0 \\ x + 2y - z &= 4 \end{aligned}$

13. $\begin{aligned} 2x + y - z &= 2 \\ x - y + z &= 7 \\ 2x + 2y + z &= 4 \end{aligned}$

14. $\begin{aligned} 2x - 4y &= 6 \\ x + y + z &= 1 \\ 5y + 7z &= 10 \end{aligned}$

15. $\begin{aligned} x - z &= 3 \\ 2y - 2z &= 2 \\ 2x + z &= 3 \end{aligned}$

16. $\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 - x_2 + x_3 + x_4 &= -1 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_3 + x_4 &= -1 \end{aligned}$

Theory and Examples

17. Find values of h and k for which the system

$$2x + hy = 8,$$
$$x + 3y = k$$

has (a) infinitely many solutions, (b) no solution at all.

18. For what value of x will

$$\begin{vmatrix} x & x & 1 \\ 2 & 0 & 5 \\ 6 & 7 & 1 \end{vmatrix} = 0?$$

19. Suppose u, v, and w are twice-differentiable functions of x that satisfy the relation $au + bv + cw = 0$, where a, b, and c are constants, not all zero. Show that

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = 0.$$

20. *Partial fractions.* Expanding the quotient

$$\frac{ax + b}{(x - r_1)(x - r_2)}$$

by partial fractions calls for finding the values of C and D that make the equation

$$\frac{ax + b}{(x - r_1)(x - r_2)} = \frac{C}{x - r_1} + \frac{D}{x - r_2}$$

hold for all x.

a) Find a system of linear equations that determines C and D.

b) Under what circumstances does the system of equations in part (a) have a unique solution? That is, when is the determinant of the coefficient matrix of the system different from zero?

A.9 Euler's Theorem and the Increment Theorem

This appendix derives Euler's Theorem (Theorem 2, Section 12.3) and the Increment Theorem for Functions of Two Variables (Theorem 3, Section 12.4). Euler first published his theorem in 1734, in a series of papers he wrote on hydrodynamics.

Euler's Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof The equality of $f_{xy}(a,b)$ and $f_{yx}(a,b)$ can be established by four applications of the Mean Value Theorem (Theorem 4, Section 3.2). By hypothesis, the point (a,b) lies in the interior of a rectangle R in the xy-plane on which f, f_x, f_y, f_{xy}, and f_{yx} are all defined. We let h and k be numbers such that the point $(a+h, b+k)$ also lies in the rectangle R, and we consider the difference

$$\Delta = F(a+h) - F(a), \tag{1}$$

where

$$F(x) = f(x, b+k) - f(x, b). \tag{2}$$

We apply the Mean Value Theorem to F (which is continuous because it is differentiable), and Eq. (1) becomes

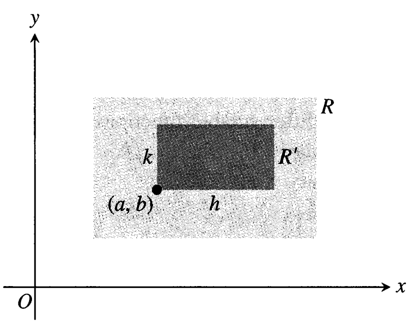
$$\Delta = hF'(c_1), \tag{3}$$

where c_1 lies between a and $a+h$. From Eq. (2),

$$F'(x) = f_x(x, b+k) - f_x(x, b),$$

so Eq. (3) becomes

$$\Delta = h[f_x(c_1, b+k) - f_x(c_1, b)]. \tag{4}$$



A.13 The key to proving $f_{xy}(a,b) = f_{yx}(a,b)$ is the fact that no matter how small R' is, f_{xy} and f_{yx} take on equal values somewhere inside R' (although not necessarily at the same point).

Now we apply the Mean Value Theorem to the function $g(y) = f_x(c_1, y)$ and have

$$g(b+k) - g(b) = kg'(d_1),$$

or

$$f_x(c_1, b+k) - f_x(c_1, b) = kf_{xy}(c_1, d_1),$$

for some d_1 between b and $b+k$. By substituting this into Eq. (4), we get

$$\Delta = hkf_{xy}(c_1, d_1), \tag{5}$$

for some point (c_1, d_1) in the rectangle R' whose vertices are the four points (a,b), $(a+h, b)$, $(a+h, b+k)$, and $(a, b+k)$. (See Fig. A.13.)

By substituting from Eq. (2) into Eq. (1), we may also write

$$\begin{aligned}\Delta &= f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)\\ &= [f(a+h, b+k) - f(a, b+k)] - [f(a+h, b) - f(a, b)] \qquad (6)\\ &= \phi(b+k) - \phi(b),\end{aligned}$$

where

$$\phi(y) = f(a+h, y) - f(a, y). \tag{7}$$

The Mean Value Theorem applied to Eq. (6) now gives

$$\Delta = k\phi'(d_2), \tag{8}$$

for some d_2 between b and $b+k$. By Eq. (7),

$$\phi'(y) = f_y(a+h, y) - f_y(a, y). \tag{9}$$

Substituting from Eq. (9) into Eq. (8) gives

$$\Delta = k[f_y(a+h, d_2) - f_y(a, d_2)].$$

Finally, we apply the Mean Value Theorem to the expression in brackets and get

$$\Delta = khf_{yx}(c_2, d_2), \tag{10}$$

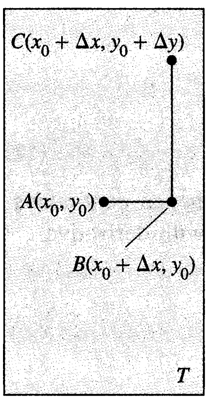
for some c_2 between a and $a+h$.

Together, Eqs. (5) and (10) show that

$$f_{xy}(c_1, d_1) = f_{yx}(c_2, d_2), \tag{11}$$

where (c_1, d_1) and (c_2, d_2) both lie in the rectangle R' (Fig. A.13). Equation (11) is not quite the result we want, since it says only that f_{xy} has the same value at (c_1, d_1) that f_{yx} has at (c_2, d_2). But the numbers h and k in our discussion may be made as small as we wish. The hypothesis that f_{xy} and f_{yx} are both continuous at (a, b) means that $f_{xy}(c_1, d_1) = f_{xy}(a, b) + \epsilon_1$ and $f_{yx}(c_2, d_2) = f_{yx}(a, b) + \epsilon_2$, where $\epsilon_1, \epsilon_2 \to 0$ as $h, k \to 0$. Hence, if we let h and $k \to 0$, we have $f_{xy}(a, b) = f_{yx}(a, b)$. ❑

The equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$ can be proved with hypotheses weaker than the ones we assumed. For example, it is enough for f, f_x, and f_y to exist in R and for f_{xy} to be continuous at (a, b). Then f_{yx} will exist at (a, b) and will equal f_{xy} at that point.



A.14 The rectangular region T in the proof of the Increment Theorem. The figure is drawn for Δx and Δy positive, but either increment might be zero or negative.

The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $z = f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

in which $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

Proof We work within a rectangle T centered at $A(x_0, y_0)$ and lying within R, and we assume that Δx and Δy are already so small that the line segment joining A to $B(x_0 + \Delta x, y_0)$ and the line segment joining B to $C(x_0 + \Delta x, y_0 + \Delta y)$ lie in the interior of T (Fig. A.14).

We may think of Δz as the sum $\Delta z = \Delta z_1 + \Delta z_2$ of two increments, where

$$\Delta z_1 = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

is the change in the value of f from A to B and

$$\Delta z_2 = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$$

is the change in the value of f from B to C (Fig. A.15, on the following page).

On the closed interval of x-values joining x_0 to $x_0 + \Delta x$, the function $F(x) = f(x, x_0)$ is a differentiable (and hence continuous) function of x, with derivative

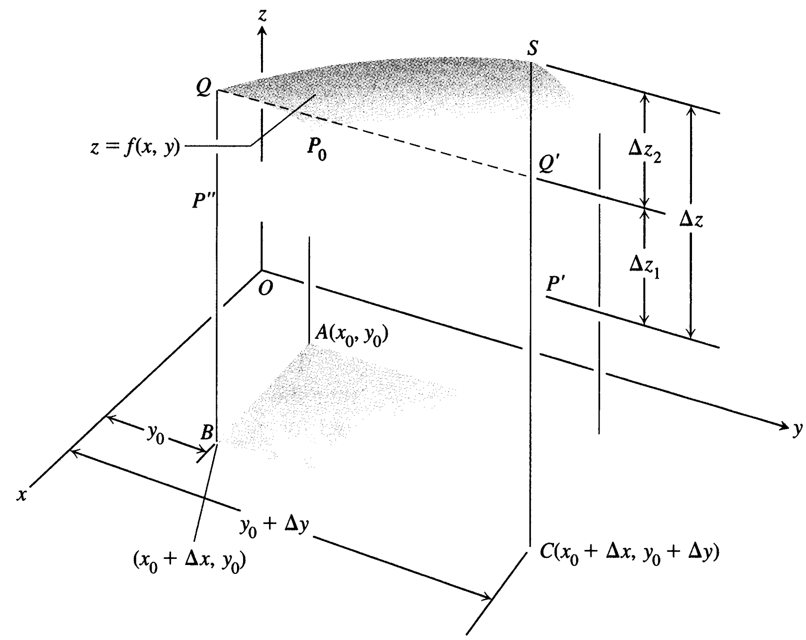
$$F'(x) = f_x(x, y_0).$$

By the Mean Value Theorem (Theorem 4, Section 3.2), there is an x-value c between x_0 and $x_0 + \Delta x$ at which

$$F(x_0 + \Delta x) - F(x_0) = F'(c)\Delta x$$

or

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0) = f_x(c, y_0)\Delta x$$



A.15 Part of the surface $z = f(x, y)$ near $P_0(x_0, y_0, f(x_0, y_0))$. The points P_0, P', and P'' have the same height $z_0 = f(x_0, y_0)$ above the xy-plane. The change in z is $\Delta z = P'S$. The change

$$\Delta z_1 = f(x_0 + \Delta x, y_0) - f(x_0, y_0),$$

shown as $P''Q = P'Q'$, is caused by changing x from x_0 to $x_0 + \Delta x$ while holding y equal to y_0. Then, with x held equal to $x_0 + \Delta x$,

$$\Delta z_2 = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$$

is the change in z caused by changing y from y_0 to $y_0 + \Delta y$. This is represented by $Q'S$. The total change in z is the sum of Δz_1 and Δz_2.

or

$$\Delta z_1 = f_x(c, y_0)\Delta x. \tag{12}$$

Similarly, $G(y) = f(x_0 + \Delta x, y)$ is a differentiable (and hence continuous) function of y on the closed y-interval joining y_0 and $y_0 + \Delta y$, with derivative

$$G'(y) = f_y(x_0 + \Delta x, y).$$

Hence there is a y-value d between y_0 and $y_0 + \Delta y$ at which

$$G(y_0 + \Delta y) - G(y_0) = G'(d)\Delta y$$

or

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y) = f_y(x_0 + \Delta x, d)\Delta y$$

or

$$\Delta z_2 = f_y(x_0 + \Delta x, d)\Delta y. \tag{13}$$

Now, as Δx and $\Delta y \to 0$, we know $c \to x_0$ and $d \to y_0$. Therefore, since f_x and f_y are continuous at (x_0, y_0), the quantities

$$\begin{aligned} \epsilon_1 &= f_x(c, y_0) - f_x(x_0, y_0), \\ \epsilon_2 &= f_y(x_0 + \Delta x, d) - f_y(x_0, y_0) \end{aligned} \tag{14}$$

both approach zero as Δx and $\Delta y \to 0$.

Finally,

$$\begin{aligned} \Delta z &= \Delta z_1 + \Delta z_2 \\ &= f_x(c, y_0)\Delta x + f_y(x_0 + \Delta x, d)\,\Delta y && \text{From (12) and (13)} \\ &= [f_x(x_0, y_0) + \epsilon_1]\,\Delta x + [f_y(x_0, y_0) + \epsilon_2]\,\Delta y && \text{From (14)} \\ &= f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y, \end{aligned}$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. This is what we set out to prove. ❑

Analogous results hold for functions of any finite number of independent variables. Suppose that the first partial derivatives of

$$w = f(x, y, z)$$

are defined throughout an open region containing the point (x_0, y_0, z_0) and that f_x, f_y, and f_z are continuous at (x_0, y_0). Then

$$\begin{aligned} \Delta w &= f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \\ &= f_x \Delta x + f_y \Delta y + f_z \Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z, \end{aligned} \tag{15}$$

where

$$\epsilon_1, \epsilon_2, \epsilon_3 \to 0 \quad \text{when} \quad \Delta x, \Delta y, \text{ and } \Delta z \to 0.$$

The partial derivatives f_x, f_y, f_z in this formula are to be evaluated at the point (x_0, y_0, z_0).

The result (15) can be proved by treating Δw as the sum of three increments,

$$\Delta w_1 = f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0), \tag{16}$$

$$\Delta w_2 = f(x_0 + \Delta x, y_0 + \Delta y, z_0) - f(x_0 + \Delta x, y_0, z_0), \tag{17}$$

$$\Delta w_3 = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0 + \Delta x, y_0 + \Delta y, z_0), \tag{18}$$

and applying the Mean Value Theorem to each of these separately. Two coordinates remain constant and only one varies in each of these partial increments $\Delta w_1, \Delta w_2, \Delta w_3$. In (17), for example, only y varies, since x is held equal to $x_0 + \Delta x$ and z is held equal to z_0. Since $f(x_0 + \Delta x, y, z_0)$ is a continuous function of y with a derivative f_y, it is subject to the Mean Value Theorem, and we have

$$\Delta w_2 = f_y(x_0 + \Delta x, y_1, z_0)\Delta y$$

for some y_1 between y_0 and $y_0 + \Delta y$.

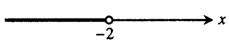
Answers

PRELIMINARY CHAPTER

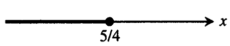
Section 1, pp. 7–8

1. $0.\overline{1}, 0.\overline{2}, 0.\overline{3}, 0.\overline{8}$ **3.** a) Not necessarily true b) True c) True d) True e) True f) True g) True h) True

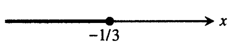
5. $x < -2$



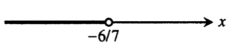
7. $x \le \frac{5}{4}$



9. $x \le -\frac{1}{3}$



11. $x < -\frac{6}{7}$



13. ± 3 **15.** $-\frac{1}{2}, -\frac{9}{2}$ **17.** $\frac{7}{6}, \frac{25}{6}$

19. $-2 < x < 2$

21. $-2 \le t \le 4$

23. $1 < y < \frac{11}{3}$

25. $0 \le z \le 10$

27. $\frac{2}{7} < x < \frac{2}{5}$ or $\frac{10}{35} < x < \frac{14}{35}$

29. $(-\infty, -2] \cup [2, \infty)$

31. $(-\infty, 0) \cup (2, \infty)$

33. $(-\infty, -3] \cup [1, \infty)$

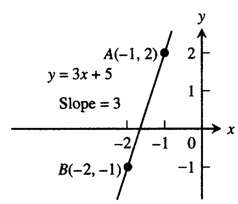
35. $(-\sqrt{2}, \sqrt{2})$ **37.** $(-3, -2) \cup (2, 3)$ **39.** $(-1, 3)$
41. (0, 1) **43.** $a \ge 0$; any negative real number

47. $-\frac{1}{2} < x \le 3$ **49.** a) $(-2, 0) \cup (4, \infty)$

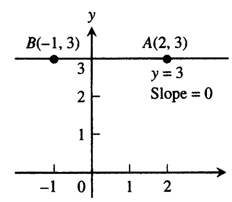
Section 2, pp. 15–17

1. $2, -4; 2\sqrt{5}$ **3.** $-4.9, 0; 4.9$ **5.** Unit circle
7. The circle centered at the origin with points less than a radius of $\sqrt{3}$ and its interior

9. $m_\perp = -\frac{1}{3}$



11. $m_\perp$ is undefined.



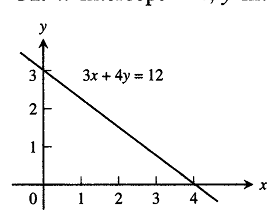
13. a) $x = -1$ b) $y = 4/3$ **15.** a) $x = 0$ b) $y = -\sqrt{2}$

17. $y = -x$ **19.** $y = -\frac{x}{5} + \frac{23}{5}$ **21.** $y = -\frac{5}{4}x + 6$

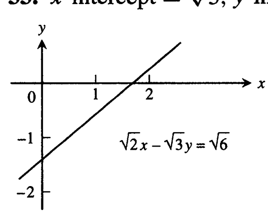
23. $y = -9$ **25.** $y = 4x + 4$ **27.** $y = -\frac{2}{5}x + 1$

29. $y = -\frac{x}{2} + 12$

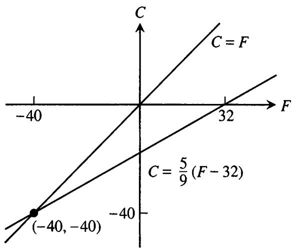
31. x-intercept = 4, y-intercept = 3



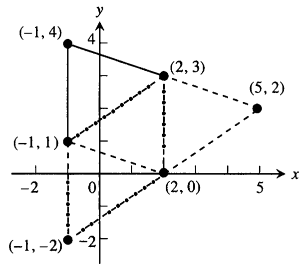
33. x-intercept = $\sqrt{3}$, y-intercept = $-\sqrt{2}$



35. Yes. The lines are perpendicular because their slopes, $-A/B$ and B/A, are negative reciprocals of one another.
37. $(3, -3)$ **39.** $(-2, -9)$ **41.** a) ≈ -2.5 degrees/inch b) ≈ -16.1 degrees/inch c) ≈ -8.3 degrees/inch
43. 5.97 atm
45. Yes: $C = F = -40°$



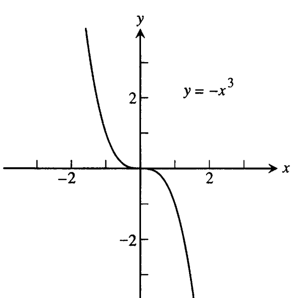
51.



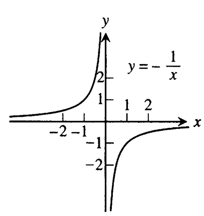
53. $k = -8, \quad k = 1/2$

Section 3, pp. 25–27

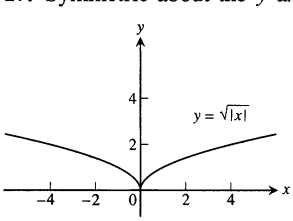
1. $D: (-\infty, \infty), \quad R: [1, \infty)$ **3.** $D: (0, \infty), \quad R: (0, \infty)$
5. $D: [-2, 2], \quad R: [0, 2]$
7. a) Not a function of x because some values of x have two values of y
b) A function of x because for every x there is only one possible y
9. $A = \dfrac{\sqrt{3}}{4}x^2, \quad p = 3x$ **11.** $x = \dfrac{d}{\sqrt{3}}, \quad A = 2d^2, \quad V = \dfrac{d^3}{3\sqrt{3}}$
13. Symmetric about the origin



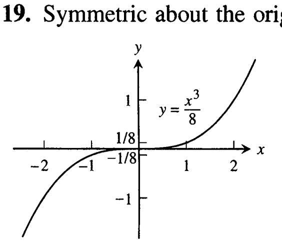
15. Symmetric about the origin



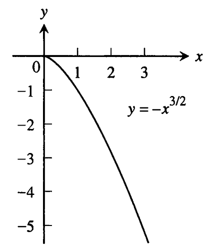
17. Symmetric about the y-axis



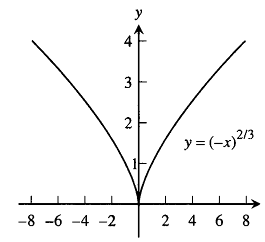
19. Symmetric about the origin



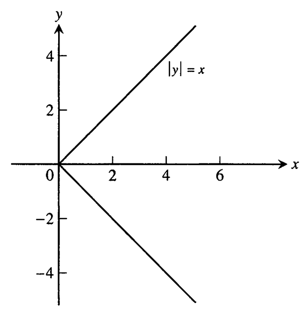
21. No symmetry



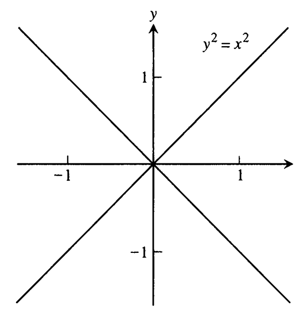
23. Symmetric about the y-axis



25. a) For each positive value of x, there are two values of y.



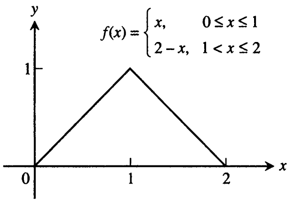
b) For each value of $x \neq 0$, there are two values of y.



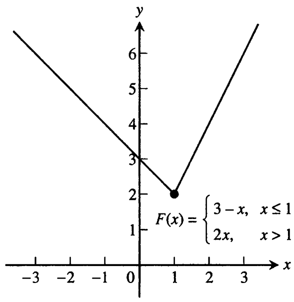
27. Even **29.** Even **31.** Odd **33.** Even **35.** Neither
37. Neither
39. $D_f: -\infty < x < \infty, \quad D_g: x \geq 1, \quad R_f: -\infty < y < \infty,$ $R_g: y \geq 0, \quad D_{f+g} = D_{f \cdot g} = D_g, \quad R_{f+g}: y \geq 1, \quad R_{f \cdot g}: y \geq 0$
41. $D_f: -\infty < x < \infty, \quad D_g: -\infty < x < \infty, \quad R_f: y = 2,$ $R_g: y \geq 1, \quad D_{f/g}: -\infty < x < \infty, \quad R_{f/g}: 0 < y \leq 2,$ $D_{g/f}: -\infty < x < \infty, \quad R_{g/f}: y \geq 1/2$
43. a) 2 b) 22 c) $x^2 + 2$ d) $x^2 + 10x + 22$ e) 5 f) -2 g) $x + 10$ h) $x^4 - 6x^2 + 6$
45. a) $\dfrac{4}{x^2} - 5$ b) $\dfrac{4}{x^2} - 5$ c) $\left(\dfrac{4}{x} - 5\right)^2$ d) $\left(\dfrac{1}{4x - 5}\right)^2$
e) $\dfrac{1}{4x^2 - 5}$ f) $\dfrac{1}{(4x - 5)^2}$
47. a) $f(g(x))$ b) $j(g(x))$ c) $g(g(x))$ d) $j(j(x))$ e) $g(h(f(x)))$ f) $h(j(f(x)))$
49.

	$g(x)$	$f(x)$	$f \circ g(x)$
a)	$x - 7$	$\sqrt{x}$	$\sqrt{x - 7}$
b)	$x + 2$	$3x$	$3x + 6$
c)	x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d)	$\dfrac{x}{x - 1}$	$\dfrac{x}{x - 1}$	x
e)	$\dfrac{1}{x - 1}$	$1 + \dfrac{1}{x}$	x
f)	$\dfrac{1}{x}$	$\dfrac{1}{x}$	x

51.



53.



55. a) $y = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

b) $y = \begin{cases} 2, & 0 \le x < 1 \text{ or } 2 \le x < 3 \\ 0, & 1 \le x < 2 \text{ or } 3 \le x \le 4 \end{cases}$

57. a) $0 \le x < 1$ b) $-1 < x \le 0$ **59.** Yes

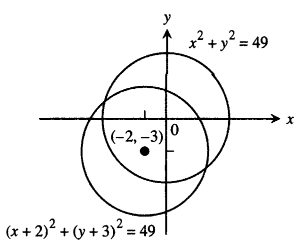
61. a) Odd b) Odd c) Odd d) Even e) Even f) Even g) Even h) Even i) Odd

Section 4, pp. 32–35

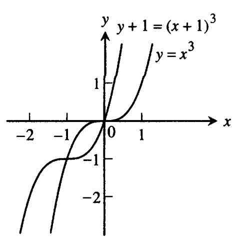
1. a) $y = -(x + 7)^2$ b) $y = -(x - 4)^2$

3. a) Position 4 b) Position 1 c) Position 2 d) Position 3

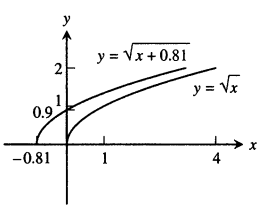
5. $(x + 2)^2 + (y + 3)^2 = 49$



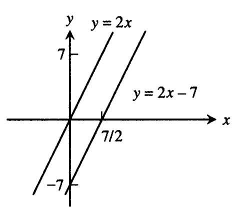
7. $y + 1 = (x + 1)^3$



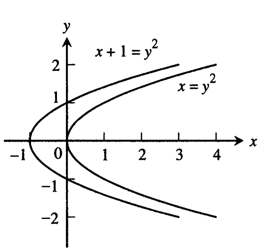
9. $y = \sqrt{x + 0.81}$



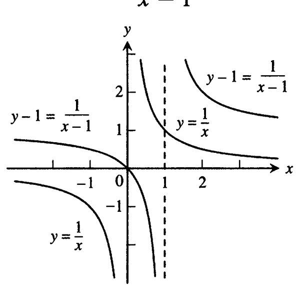
11. $y = 2x$



13. $x + 1 = y^2$



15. $y - 1 = \dfrac{1}{x - 1}$



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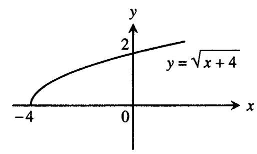

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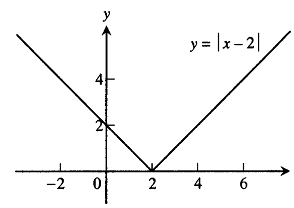

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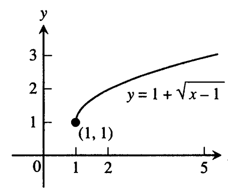

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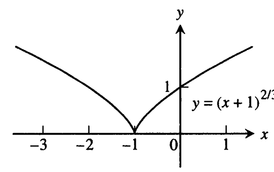

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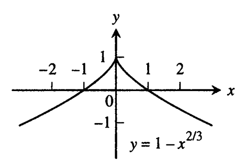

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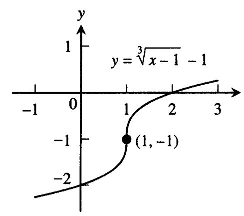

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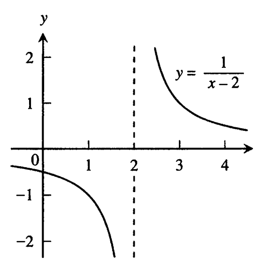

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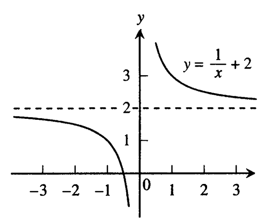

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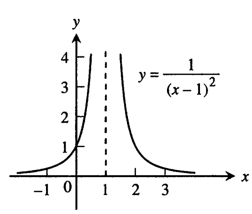
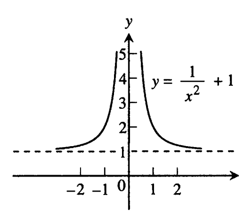
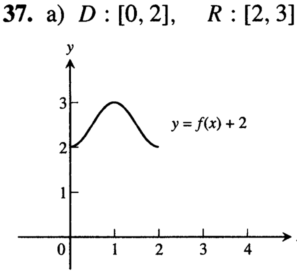


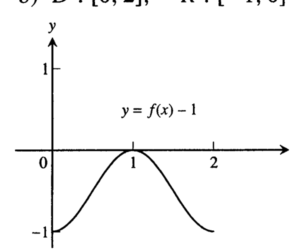
35.



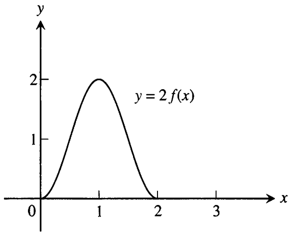
37. a) $D: [0, 2]$, $R: [2, 3]$



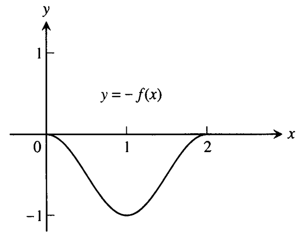
b) $D: [0, 2]$, $R: [-1, 0]$



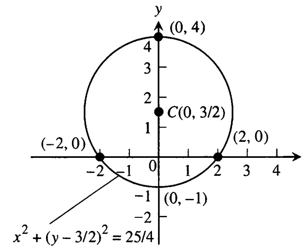
c) $D: [0, 2], \quad R: [0, 2]$



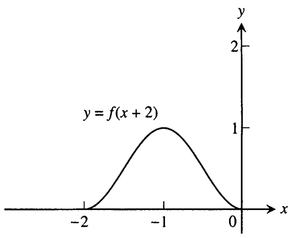
d) $D: [0, 2], \quad R: [-1, 0]$



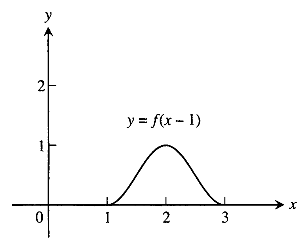
47. $x^2 + (y - 3/2)^2 = 25/4$



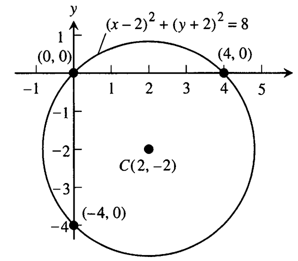
e) $D: [-2, 0], \quad R: [0, 1]$



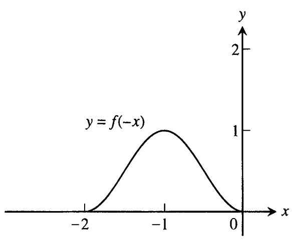
f) $D: [1, 3], \quad R: [0, 1]$



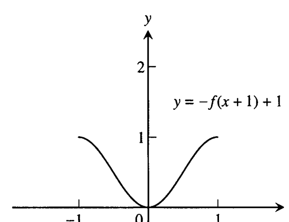
49. $(x - 2)^2 + (y + 2)^2 = 8$



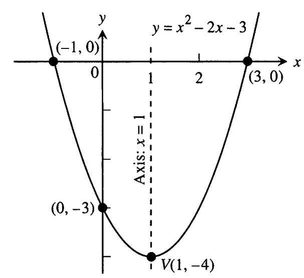
g) $D: [-2, 0], \quad R: [0, 1]$



h) $D: [-1, 1], \quad R: [0, 1]$



51.



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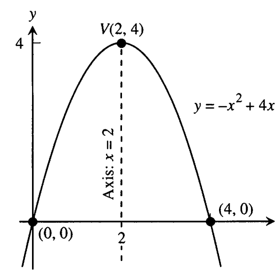
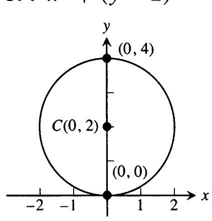
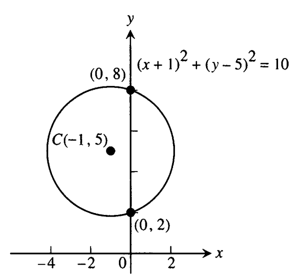


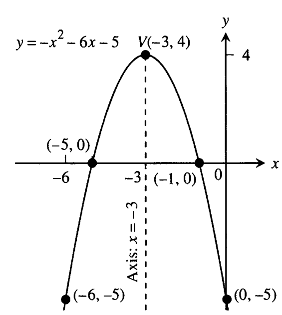
39. $x^2 + (y - 2)^2 = 4$



41. $(x + 1)^2 + (y - 5)^2 = 10$



55.



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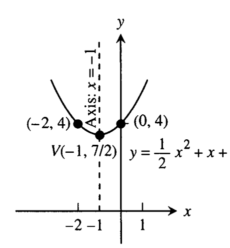
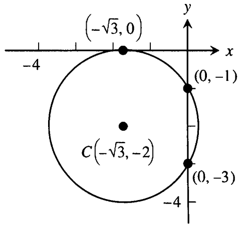
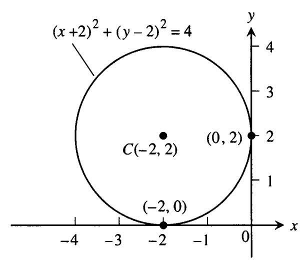


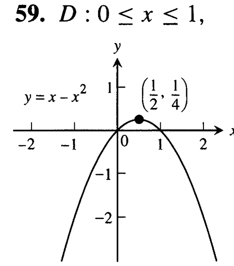
43. $(x + \sqrt{3})^2 + (y + 2)^2 = 4$



45. $(x + 2)^2 + (y - 2)^2 = 4$



59. $D: 0 \le x \le 1, \quad R: 0 \le y \le 1/2$



61. Exterior points of a circle of radius $\sqrt{7}$, centered at the origin
63. A circle of radius 2, centered at (1, 0), together with its interior
65. The washer between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (points with distance from the origin between 1 and 2)
67. The interior points of a circle centered at $(0, -3)$ with a radius of 3 that lie above the line $y = -3$
69. $(x + 2)^2 + (y - 1)^2 < 6$ **71.** $x^2 + y^2 \le 2, \quad x \ge 1$

73. $y = y_0 + m(x - x_0)$ **75.** $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \quad \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

77. $\left(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right), \quad \left(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$

79. $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right), \quad \left(\frac{1}{\sqrt{3}}, -\frac{1}{3}\right)$

81. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Section 5, pp. 43–47

1. a) 8π m b) $\frac{55\pi}{9}$ m **3.** 8.4 in.

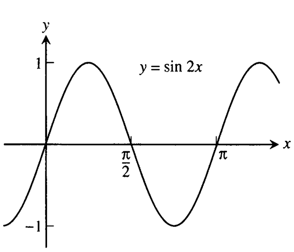
5.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$	0	$-\frac{\sqrt{3}}{2}$	0	1	$\frac{1}{\sqrt{2}}$
$\cos\theta$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	0	$\sqrt{3}$	0	UND	-1
$\cot\theta$	UND	$\frac{1}{\sqrt{3}}$	UND	0	-1
$\sec\theta$	-1	-2	1	UND	$-\sqrt{2}$
$\csc\theta$	UND	$-\frac{2}{\sqrt{3}}$	UND	1	$\sqrt{2}$

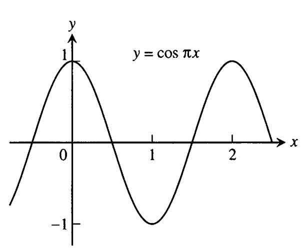
7. $\cos x = -4/5, \tan x = -3/4$ **9.** $\sin x = -\frac{\sqrt{8}}{3}, \tan x = -\sqrt{8}$

11. $\sin x = -\frac{1}{\sqrt{5}}, \cos x = -\frac{2}{\sqrt{5}}$

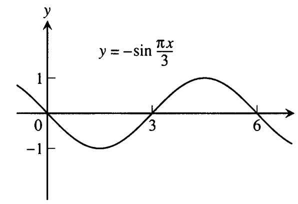
13. Period π



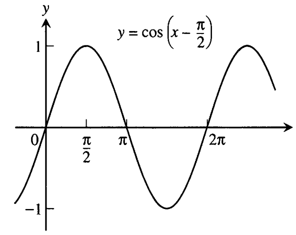
15. Period 2



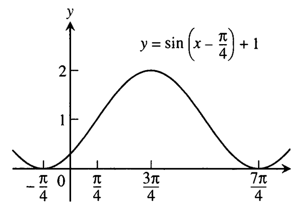
17. Period 6



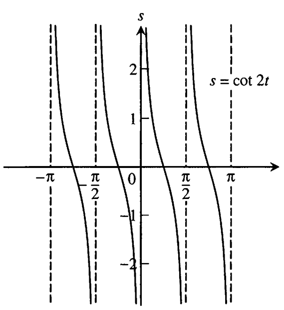
19. Period 2π



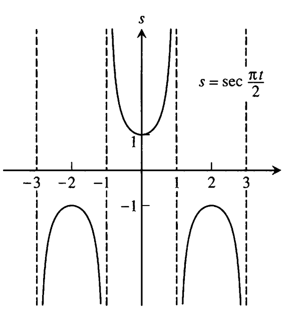
21. Period 2π



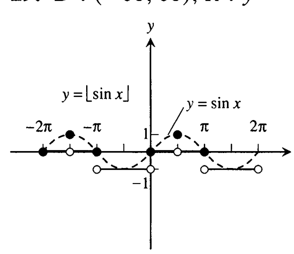
23. Period $\pi/2$, symmetric about the origin



25. Period 4, symmetric about the y-axis



29. $D: (-\infty, \infty), R: y = -1, 0, 1$

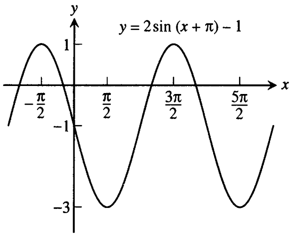


39. $-\cos x$ **41.** $-\cos x$ **43.** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **45.** $\frac{1+\sqrt{3}}{2\sqrt{2}}$

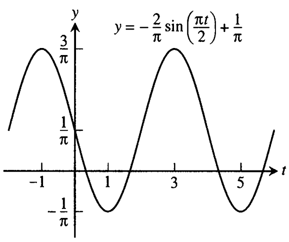
47. $\frac{2+\sqrt{2}}{4}$ **49.** $\frac{2-\sqrt{3}}{4}$ **55.** $c = \sqrt{7} \approx 2.646$

59. $a = 1.464$

61. $A = 2, B = 2\pi, C = -\pi, D = -1$



63. $A = -\dfrac{2}{\pi}, B = 4, C = 0, D = \dfrac{1}{\pi}$



65. a) 37 b) 365 c) Right 101 d) Up 25

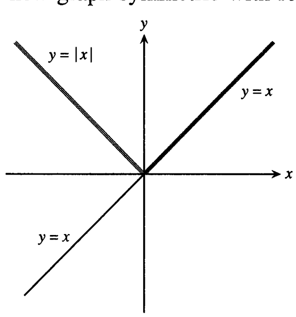
Practice Exercises, pp. 48–49

1. (0, 11)

3. No, no two sides have the same length; no, no two sides are perpendicular

5. $A = \pi r^2, C = 2\pi r, A = \dfrac{C^2}{4\pi}$ **7.** $x = \tan\theta, y = \tan^2\theta$

9. Replaces the portion for $x < 0$ with mirror image of portion for $x > 0$, to make the new graph symmetric with respect to the y-axis.



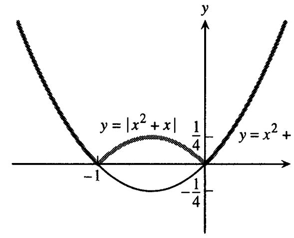
11. It does not change it.

13. Adds the mirror image of the portion for $x > 0$ to make the new graph symmetric with respect to the y-axis.

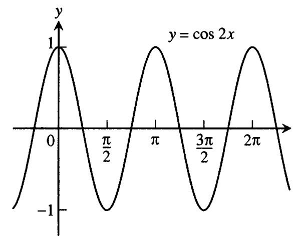
15. Reflects the portion for $y < 0$ across the x-axis.

17. Reflects the portion for $y < 0$ across the x-axis.

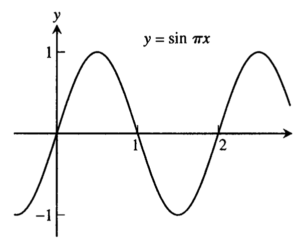
19. Reflects the portion for $y < 0$ across the x-axis.



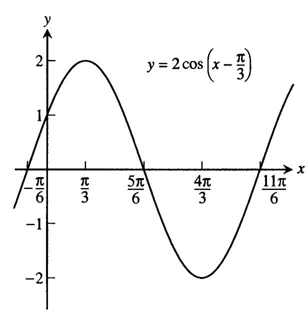
21. Period π



23. Period 2



25.



27. a) $a = 1, \quad b = \sqrt{3}$ b) $a = 2\sqrt{3}/3, \quad c = 4\sqrt{3}/3$

29. a) $a = \dfrac{b}{\tan B}$ b) $c = \dfrac{a}{\sin A}$ **31.** 16.98 m

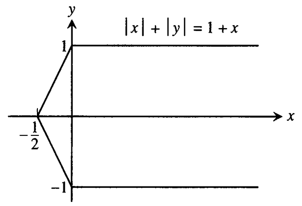
33. $3\sin x\cos^2 x - \sin^3 x$ **35.** b) 4π

Additional Exercises, pp. 49–50

3. Yes. For instance: $f(x) = 1/x$ and $g(x) = 1/x$, or $f(x) = 2x$ and $g(x) = x/2$, or $f(x) = e^x$ and $g(x) = \ln x$.

5. If $f(x)$ is odd, then $g(x) = f(x) - 2$ is not odd. Nor is $g(x)$ even, unless $f(x) = 0$ for all x. If f is even, then $g(x) = f(x) - 2$ is also even.

7.



9. $\sqrt{2}$ **11.** 3/4 **13.** $3\sqrt{15}/16$ **27.** $-4 < m < 0$

CHAPTER 1

Section 1.1, pp. 57–60

1. a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$. b) 1 c) 0

3. a) True b) True c) False d) False e) False f) True

5. As x approaches 0 from the left, $x/|x|$ approaches -1. As x approaches 0 from the right, $x/|x|$ approaches 1. There is no single number L that the function values all get arbitrarily close to as $x \to 0$.

11. a) $f(x) = (x^2 - 9)/(x + 3)$

x	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

x	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

c) $\lim_{x \to -3} f(x) = -6$

13. a) $G(x) = (x + 6)/(x^2 + 4x - 12)$

x	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999
$G(x)$	$-.126582$	$-.1251564$	$-.1250156$	$-.1250015$	$-.1250001$	$-.1250000$

x	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001
$G(x)$	$-.123456$	$-.124843$	$-.124984$	$-.124998$	$-.124999$	$-.124999$

c) $\lim_{x \to -6} G(x) = -1/8 = -0.125$

15. a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

c) $\lim_{x \to -1} f(x) = 2$

17. a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

19. a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$\lim_{x \to 1} f(x) \approx 0.36788$

21. 4 **23.** 0 **25.** 9 **27.** $\pi/2$ **29.** a) 19 b) 1

31. a) $-\dfrac{4}{\pi}$ b) $-\dfrac{3\sqrt{3}}{\pi}$ **33.** 1

35. Graphs can shift during a press run, so your estimates may not completely agree with these.

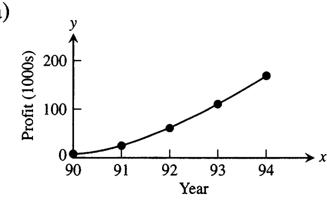
a)

PQ_1	PQ_2	PQ_3	PQ_4
43	46	49	50

The appropriate units are m/sec.

b) $\approx$ 50 m/sec or 180 km/h

37. a)



b) $\approx$ \$56,000/year

c) $\approx$ \$42,000/year

39. a) 0.414213, 0.449489, $(\sqrt{1+h}-1)/h$ b) $g(x)=\sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$(\sqrt{1+h}-1)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

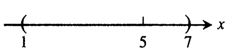
c) 0.5 d) 0.5

Section 1.2, pp. 65–66

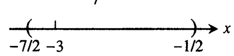
1. -9 **3.** 4 **5.** -8 **7.** 5/8 **9.** 5/2 **11.** 27 **13.** 16 **15.** 3/2 **17.** 1/10 **19.** -7 **21.** 3/2 **23.** $-1/2$ **25.** 4/3 **27.** 1/6 **29.** 4
31. a) Quotient rule b) Difference and Power rules c) Sum and Constant Multiple rules
33. a) -10 b) -20 c) -1 d) $5/7$
35. a) 4 b) -21 c) -12 d) $-7/3$
37. 2 **39.** 3 **41.** $1/(2\sqrt{7})$ **43.** $\sqrt{5}$ **45.** a) The limit is 1.
49. 7 **51.** a) 5 b) 5

Section 1.3, pp. 74–77

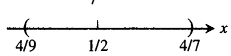
1. $\delta = 2$



3. $\delta = 1/2$



5. $\delta = 1/18$



7. $\delta = 0.1$ **9.** $\delta = 7/16$ **11.** $\delta = \sqrt{5}-2$ **13.** $\delta = 0.36$
15. $(3.99, 4.01)$, $\delta = 0.01$ **17.** $(-0.19, 0.21)$, $\delta = 0.19$
19. $(3, 15)$, $\delta = 5$ **21.** $(10/3, 5)$, $\delta = 2/3$
23. $(-\sqrt{4.5}, -\sqrt{3.5})$, $\delta = \sqrt{4.5}-2 \approx 0.12$
25. $(\sqrt{15}, \sqrt{17})$, $\delta = \sqrt{17}-4 \approx 0.12$

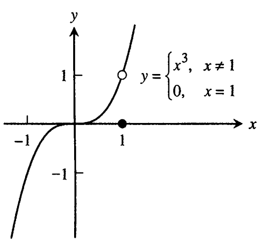
27. $\left(2-\dfrac{0.03}{m}, 2+\dfrac{0.03}{m}\right), \delta = \dfrac{0.03}{m}$

29. $\left(\dfrac{1}{2}-\dfrac{c}{m}, \dfrac{c}{m}+\dfrac{1}{2}\right), \delta = \dfrac{c}{m}$

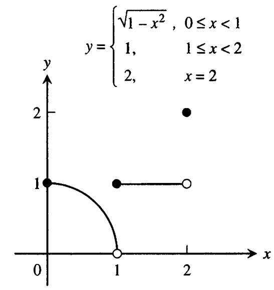
31. $L=-3, \delta = 0.01$ **33.** $L=4, \delta=0.05$
35. $L=4, \delta = 0.75$
55. [3.384, 3.387]. To be safe, the left endpoint was rounded up and the right endpoint rounded down.

Section 1.4, pp. 83–86

1. a) True b) True c) False d) True e) True f) True g) False h) False i) False j) False k) True l) False
3. a) 2, 1 b) No, $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$ c) 3, 3 d) Yes, 3
5. a) No b) Yes, 0 c) No
7. a)



b) 1, 1 c) Yes, 1
9. a) $D: 0 \le x \le 2$, $R: 0 < y \le 1$ and $y=2$ b) $(0,1)\cup(1,2)$ c) $x=2$ d) $x=0$



11. $\sqrt{3}$ **13.** 1 **15.** $2/\sqrt{5}$ **17.** a) 1 b) -1 **19.** a) 1 b) 2/3 **21.** ∞ **23.** $-\infty$ **25.** $-\infty$ **27.** ∞ **29.** a) ∞ b) $-\infty$ **31.** ∞ **33.** ∞ **35.** $-\infty$ **37.** a) ∞ b) $-\infty$ c) $-\infty$ d) ∞ **39.** a) $-\infty$ b) ∞ c) 0 d) 3/2
41. a) $-\infty$ b) 1/4 c) 1/4 d) 1/4 e) It will be $-\infty$.
43. a) $-\infty$ b) ∞ **45.** a) ∞ b) ∞ c) ∞ d) ∞
51. $\delta = \epsilon^2$, $\lim_{x\to 5^+} \sqrt{x-5} = 0$ **55.** a) 400 b) 399 c) The limit does not exist.
61. a) For every positive real number B there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Rightarrow f(x) > B.$$

b) For every negative real number $-B$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \Rightarrow f(x) < -B.$$

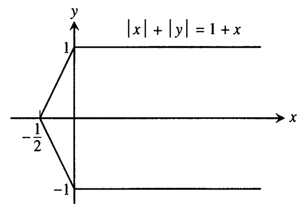
c) For every negative real number $-B$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Rightarrow f(x) < -B.$$

Section 1.5, pp. 95–97

1. No; discontinuous at $x=2$; not defined at $x=2$
3. Continuous **5.** a) Yes b) Yes c) Yes d) Yes
7. a) No b) No **9.** 0 **11.** 1, nonremovable; 0, removable
13. All x except $x=2$ **15.** All x except $x=3, x=1$
17. All x **19.** All x except $x=0$
21. All x except $x = n\pi/2$, n any integer
23. All x except $n\pi/2$, n an odd integer
25. All $x > -3/2$ **27.** All x **29.** 0 **31.** 1 **33.** $\sqrt{2}/2$
35. $g(3)=6$ **37.** $f(1)=3/2$ **39.** $a=4/3$
63. $x \approx 1.8794, -1.5321, -0.3473$ **65.** $x \approx 1.7549$
67. $x \approx 3.5156$ **69.** $x \approx 0.7391$

7.



9. $\sqrt{2}$ **11.** 3/4 **13.** $3\sqrt{15}/16$ **27.** $-4 < m < 0$

CHAPTER 1

Section 1.1, pp. 57–60

1. a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$. b) 1 c) 0

3. a) True b) True c) False d) False e) False f) True

5. As x approaches 0 from the left, $x/|x|$ approaches -1. As x approaches 0 from the right, $x/|x|$ approaches 1. There is no single number L that the function values all get arbitrarily close to as $x \to 0$.

11. a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
$f(x)$	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
$f(x)$	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

c) $\lim_{x \to -3} f(x) = -6$

13. a) $G(x) = (x + 6)/(x^2 + 4x - 12)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
$G(x)$	−.126582	−.1251564	−.1250156	−.1250015	−.1250001	−.1250000

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
$G(x)$	−.123456	−.124843	−.124984	−.124998	−.124999	−.124999

c) $\lim_{x \to -6} G(x) = -1/8 = -0.125$

15. a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	−1.1	−1.01	−1.001	−1.0001	−1.00001	−1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	−.9	−.99	−.999	−.9999	−.99999	−.999999
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

c) $\lim_{x \to -1} f(x) = 2$

17. a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	−.1	−.01	−.001	−.0001	−.00001	−.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

19. a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$\lim_{x \to 1} f(x) \approx 0.36788$

21. 4 **23.** 0 **25.** 9 **27.** $\pi/2$ **29.** a) 19 b) 1

31. a) $-\dfrac{4}{\pi}$ b) $-\dfrac{3\sqrt{3}}{\pi}$ **33.** 1

35. Graphs can shift during a press run, so your estimates may not completely agree with these.

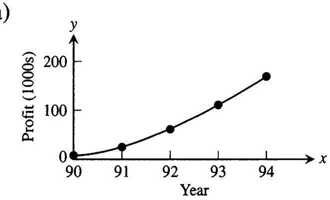
a)

PQ_1	PQ_2	PQ_3	PQ_4
43	46	49	50

The appropriate units are m/sec.

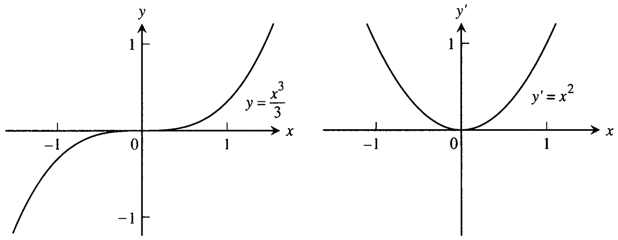
b) $\approx$ 50 m/sec or 180 km/h

37. a)



b) $\approx$ \$56,000/year
c) $\approx$ \$42,000/year

47. a) $y' = x^2$
b)



c) $x \neq 0, x = 0$, none d) $-\infty < x < \infty$, none
49. $y' = 3x^2$ is never negative
51. Yes, $y + 16 = -(x - 3)$ is tangent at $(3, -16)$
53. No, the function $y = \lfloor x \rfloor$ does not satisfy the intermediate value property of derivatives.
55. Yes, $(-f)'(x) = -(f'(x))$
57. For $g(t) = mt$ and $h(t) = t$, $\lim_{t \to 0} \dfrac{g(t)}{h(t)} = m$, which need not be zero.

Section 2.2, pp. 129–131

1. $\dfrac{dy}{dx} = -2x$, $\dfrac{d^2y}{dx^2} = -2$

3. $\dfrac{ds}{dt} = 15t^2 - 15t^4$, $\dfrac{d^2s}{dt^2} = 30t - 60t^3$

5. $\dfrac{dy}{dx} = 4x^2 - 1$, $\dfrac{d^2y}{dx^2} = 8x$

7. $\dfrac{dw}{dz} = -6z^{-3} + \dfrac{1}{z^2}$, $\dfrac{d^2w}{dz^2} = 18z^{-4} - \dfrac{2}{z^3}$

9. $\dfrac{dy}{dx} = 12x - 10 + 10x^{-3}$, $\dfrac{d^2y}{dx^2} = 12 - 30x^{-4}$

11. $\dfrac{dr}{ds} = \dfrac{-2}{3s^3} + \dfrac{5}{2s^2}$, $\dfrac{d^2r}{ds^2} = \dfrac{2}{s^4} - \dfrac{5}{s^3}$

13. $y' = -5x^4 + 12x^2 - 2x - 3$ **15.** $y' = 3x^2 + 10x + 2 - \dfrac{1}{x^2}$

17. $y' = \dfrac{-19}{(3x-2)^2}$ **19.** $g'(x) = \dfrac{x^2 + x + 4}{(x + 0.5)^2}$

21. $\dfrac{dv}{dt} = \dfrac{t^2 - 2t - 1}{(1 + t^2)^2}$ **23.** $f'(s) = \dfrac{1}{\sqrt{s}(\sqrt{s} + 1)^2}$

25. $v' = -\dfrac{1}{x^2} + 2x^{-3/2}$ **27.** $y' = \dfrac{-4x^3 - 3x^2 + 1}{(x^2 - 1)^2(x^2 + x + 1)^2}$

29. $y' = 2x^3 - 3x - 1$, $y'' = 6x^2 - 3$, $y''' = 12x$, $y^{(4)} = 12$, $y^{(n)} = 0$ for $n \geq 5$

31. $y' = 2x - 7x^{-2}$, $y'' = 2 + 14x^{-3}$

33. $\dfrac{dr}{d\theta} = 3\theta^{-4}$, $\dfrac{d^2r}{d\theta^2} = -12\theta^{-5}$

35. $\dfrac{dw}{dz} = -z^{-2} - 1$, $\dfrac{d^2w}{dz^2} = 2z^{-3}$

37. $\dfrac{dp}{dq} = \dfrac{1}{6}q + \dfrac{1}{6}q^{-3} + q^{-5}$, $\dfrac{d^2p}{dq^2} = \dfrac{1}{6} - \dfrac{1}{2}q^{-4} - 5q^{-6}$

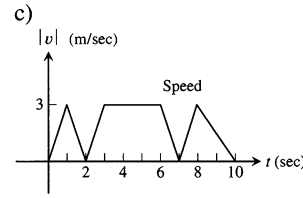
39. a) 13 b) −7 c) 7/25 d) 20 **41.** a) $y = -\dfrac{x}{8} + \dfrac{5}{4}$
b) $m = -4$ at $(0, 1)$ c) $y = 8x - 15$, $y = 8x + 17$
43. $y = 4x$, $y = 2$ **45.** $a = 1, b = 1, c = 0$ **47.** a) $y = 2x + 2$,
c) $(2, 6)$ **49.** $\dfrac{dP}{dV} = -\dfrac{nRT}{(V - nb)^2} + \dfrac{2an^2}{V^3}$

51. The Product Rule is then the Constant Multiple Rule, so the latter is a special case of the Product Rule.

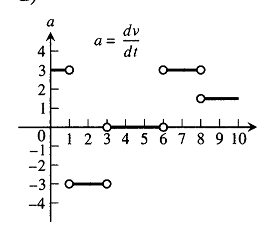
55. a) $\dfrac{3}{2}x^{1/2}$, b) $\dfrac{5}{2}x^{3/2}$, c) $\dfrac{7}{2}x^{5/2}$, d) $\dfrac{d}{dx}\left(x^{n/2}\right) = \dfrac{n}{2}x^{(n/2)-1}$

Section 2.3, pp. 139–143

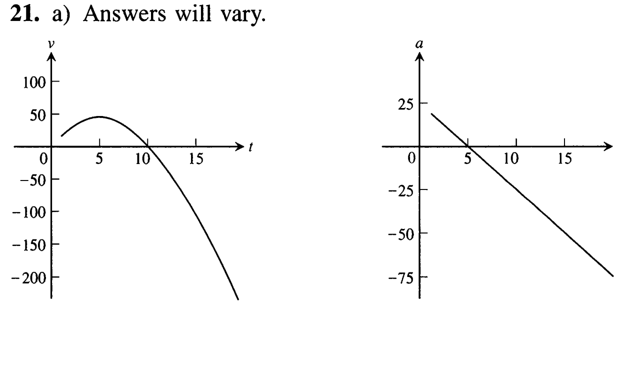
1. a) 80 m, 8 m/sec b) 0 m/sec, 16 m/sec; 1.6 m/sec², 1.6 m/sec²
c) no change in direction
3. a) −9 m, −3 m/sec b) 3 m/sec, 12 m/sec; 6 m/sec², −12 m/sec²
c) no change in direction
5. a) −20 m, − 5 m/sec b) 45 m/sec, (1/5) m/sec; 140 m/sec², (4/25) m/sec² c) no change in direction
7. a) $a(1) = -6$ m/sec², $a(3) = 6$ m/sec² b) $v(2) = 3$m/sec
c) 6 m
9. Mars: ≈ 7.5 sec, Jupiter: ≈ 1.2 sec
11. a) $24 - 9.8t$ m/sec, −9.8 m/sec² b) 2.4 sec c) 29.4 m
d) 0.7 sec going up, 4.2 sec going down e) 4.9 sec
13. 320 sec on the moon, 52 sec on Earth; ≈ 66,560 ft on the moon, ≈ 10, 816 ft on Earth
15. a) $9.8t$ m/sec, b) 9.8 m/sec²
17. a) $t = 2, t = 7$ b) $3 \leq t \leq 6$
c)



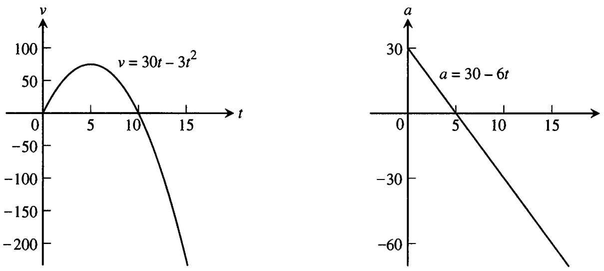
d)



19. a) 190 ft/sec b) 2 sec c) 8 sec, 0 ft/sec, d) 10.8 sec, 90 ft/sec e) 2.8 sec f) greatest acceleration happens 2 sec after launch g) constant acceleration between 2 and 10.8 sec, −32 ft/sec²
21. a) Answers will vary.



b)



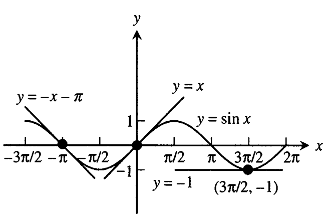
23. $C =$ position, $A =$ velocity, $B =$ acceleration
25. a) \$110/machine b) \$80 c) \$79.90
27. a) 10^4 bacteria/h b) 0 bacteria/h c) -10^4 bacteria/h
29. a) $\dfrac{t}{12} - 1$ b) Fastest ($dy/dt = -1$ m/h) when $t = 0$, Slowest ($dy/dt = 0$ m/h) when $t = 12$
31. $t = 25$ sec, $D = 6250/9$ m
33. a) $t = 6.25$ sec b) Up on $[0, 6.25)$, down on $(6.25, 12.5]$ c) $t = 6.25$ sec d) Speeds up on $(6.25, 12.5]$, slows down on $[0, 6.25)$ e) Fastest at $t = 0, 12.5$, slowest at $t = 6.25$ f) $t = 6.25$ sec
35. a) $t = (6 \pm \sqrt{15})/3$ b) left on $((6-\sqrt{15})/3, (6+\sqrt{15})/3)$); right on $[0, (6-\sqrt{15})/3) \cup ((6+\sqrt{15})/3, 4]$ c) $t = (6 \pm \sqrt{15})/3$ d) Speeds up on $((6-\sqrt{15})/3, 2) \cup ((6+\sqrt{15})/3, 4]$, slows down on $[0, (6-\sqrt{15})/3) \cup (2, (6+\sqrt{15})/3)$ e) Fastest at $t = 0, 4$; slowest at $t = (6 \pm \sqrt{15})/3$ f) $t = (6+\sqrt{15})/3$

Section 2.4, pp. 152–154

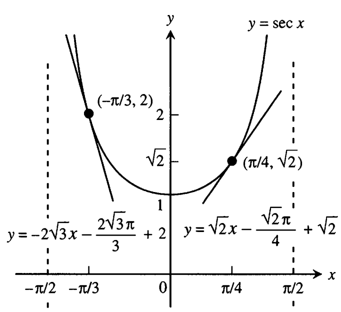
1. $-10 - 3\sin x$ **3.** $-\csc x \cot x - \dfrac{2}{\sqrt{x}}$ **5.** 0

7. $\dfrac{-\csc^2 x}{(1+\cot x)^2}$ **9.** $4\tan x \sec x - \csc^2 x$ **11.** $x^2 \cos x$

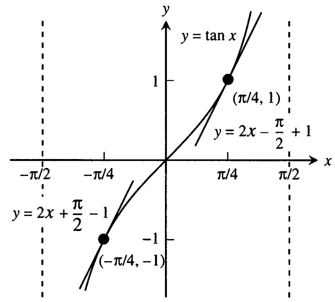
13. $\sec^2 t - 1$ **15.** $\dfrac{-2\csc t \cot t}{(1-\csc t)^2}$ **17.** $-\theta(\theta\cos\theta + 2\sin\theta)$
19. $\sec\theta \csc\theta (\tan\theta - \cot\theta) = \sec^2\theta - \csc^2\theta$ **21.** $\sec^2 q$
23. $\sec^2 q$ **25.** a) $2\csc^3 x - \csc x$ b) $2\sec^3 x - \sec x$
27. 0 **29.** -1 **31.** 0 **33.** 1 **35.** 3/4 **37.** 2 **39.** 1/2
41. 2 **43.** 1 **45.** 1/2 **47.** 3/8
49.



51.



53. Yes, at $x = \pi$ **55.** No
57. $\left(-\dfrac{\pi}{4}, -1\right)$; $\left(\dfrac{\pi}{4}, 1\right)$



59. a) $y = -x + \pi/2 + 2$ b) $y = 4 - \sqrt{3}$
61. $-\sqrt{2}$ m/sec, $\sqrt{2}$ m/sec, $\sqrt{2}$ m/sec^2, $\sqrt{2}$ m/sec^3 **63.** $c = 9$
65. $\sin x$

Section 2.5, pp. 160–163

1. $12x^3$ **3.** $3\cos(3x+1)$ **5.** $-\sin(\sin x)\cos x$
7. $10\sec^2(10x - 5)$

9. With $u = (2x+1)$, $y = u^5$: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 5u^4 \cdot 2 = 10(2x+1)^4$

11. With $u = (1 - (x/7))$, $y = u^{-7}$: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -7u^{-8} \cdot \left(-\dfrac{1}{7}\right) = \left(1 - \dfrac{x}{7}\right)^{-8}$

13. With $u = ((x^2/8) + x - (1/x))$, $y = u^4$: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 4u^3 \cdot \left(\dfrac{x}{4} + 1 + \dfrac{1}{x^2}\right) = 4\left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^3 \left(\dfrac{x}{4} + 1 + \dfrac{1}{x^2}\right)$

15. With $u = \tan x$, $y = \sec u$: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec u \tan u)(\sec^2 x) = \sec(\tan x)\tan(\tan x)\sec^2 x$

17. With $u = \sin x$, $y = u^3$: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 3u^2 \cos x = 3\sin^2 x(\cos x)$

19. $-\dfrac{1}{2\sqrt{3-t}}$ **21.** $\dfrac{4}{\pi}(\cos 3t - \sin 5t)$ **23.** $\dfrac{\csc\theta}{\cot\theta + \csc\theta}$
25. $2x\sin^4 x + 4x^2\sin^3 x\cos x + \cos^{-2} x + 2x\cos^{-3} x\sin x$

27. $(3x-2)^6 - \dfrac{1}{x^3\left(4 - \dfrac{1}{2x^2}\right)^2}$ **29.** $\dfrac{(4x+3)^3(4x+7)}{(x+1)^4}$

31. $\sqrt{x}\sec^2(2\sqrt{x}) + \tan(2\sqrt{x})$ **33.** $\dfrac{2\sin\theta}{(1+\cos\theta)^2}$

35. $\dfrac{dr}{d\theta} = -2\sin(\theta^2)\sin 2\theta + 2\theta\cos(2\theta)\cos(\theta^2)$

37. $\dfrac{dq}{dt} = \left(\dfrac{t+2}{2(t+1)^{3/2}}\right)\cos\left(\dfrac{t}{\sqrt{t+1}}\right)$

39. $2\pi \sin(\pi t - 2)\cos(\pi t - 2)$ **41.** $\dfrac{8\sin(2t)}{(1+\cos 2t)^5}$

43. $-2\cos(\cos(2t-5))(\sin(2t-5))$

45. $\left(1+\tan^4\left(\frac{t}{12}\right)\right)^2\left(\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\right)$

47. $-\dfrac{t\sin(t^2)}{\sqrt{1+\cos(t^2)}}$ **49.** $\dfrac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

51. $2\csc^2(3x-1)\cot(3x-1)$ **53.** $5/2$ **55.** $-\pi/4$ **57.** 0

59. a) $2/3$ b) $2\pi+5$ c) $15-8\pi$ d) $37/6$ e) -1 f) $\sqrt{2}/24$ g) $5/32$ h) $-5/(3\sqrt{17})$

61. 5 **63.** a) 1 b) 1 **65.** a) $y=\pi x+2-\pi$ b) $\pi/2$

67. It multiplies the velocity, acceleration, and jerk by 2, 4, and 8 respectively.

69. $v=\dfrac{2}{5}$ m/sec, $a=-\dfrac{4}{125}$ m/sec^2

Section 2.6, pp. 170–172

1. $\dfrac{9}{4}x^{5/4}$ **3.** $\dfrac{2^{1/3}}{3x^{2/3}}$ **5.** $\dfrac{7}{2(x+6)^{1/2}}$ **7.** $-(2x+5)^{-3/2}$

9. $\dfrac{2x^2+1}{(x^2+1)^{1/2}}$ **11.** $\dfrac{ds}{dt}=\dfrac{2}{7}t^{-5/7}$

13. $\dfrac{dy}{dt}=-\dfrac{4}{3}(2t+5)^{-5/3}\cos\left[(2t+5)^{-2/3}\right]$

15. $f'(x)=\dfrac{-1}{4\sqrt{x(1-\sqrt{x})}}$

17. $h'(\theta)=-\dfrac{2}{3}(\sin 2\theta)(1+\cos 2\theta)^{-2/3}$ **19.** $\dfrac{-2xy-y^2}{x^2+2xy}$

21. $\dfrac{1-2y}{2x+2y-1}$ **23.** $\dfrac{-2x^3+3x^2y-xy^2+x}{x^2y-x^3+y}$ **25.** $\dfrac{1}{y(x+1)^2}$

27. $\cos^2 y$ **29.** $\dfrac{-\cos^2(xy)-y}{x}$

31. $\dfrac{-y^2}{y\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)+xy}$ **33.** $-\dfrac{\sqrt{r}}{\sqrt{\theta}}$ **35.** $\dfrac{-r}{\theta}$

37. $y'=-\dfrac{x}{y}$, $y''=\dfrac{-y^2-x^2}{y^3}$

39. $y'=\dfrac{x+1}{y}$, $y''=\dfrac{y^2-(x+1)^2}{y^3}$

41. $y'=\dfrac{\sqrt{y}}{\sqrt{y}+1}$, $y''=\dfrac{1}{2(\sqrt{y}+1)^3}$ **43.** -2

45. $(-2,1): m=-1$, $(-2,-1): m=1$ **47.** a) $y=\dfrac{7}{4}x-\dfrac{1}{2}$, b) $y=-\dfrac{4}{7}x+\dfrac{29}{7}$ **49.** a) $y=3x+6$, b) $y=-\dfrac{1}{3}x+\dfrac{8}{3}$

51. a) $y=\dfrac{6}{7}x+\dfrac{6}{7}$, b) $y=-\dfrac{7}{6}x-\dfrac{7}{6}$

53. a) $y=-\dfrac{\pi}{2}x+\pi$, b) $y=\dfrac{2}{\pi}x-\dfrac{2}{\pi}+\dfrac{\pi}{2}$

55. a) $y=2\pi x-2\pi$, b) $y=-\dfrac{x}{2\pi}+\dfrac{1}{2\pi}$

57. Points: $(-\sqrt{7},0)$ and $(\sqrt{7},0)$, Slope: -2

59. $m=-1$ at $\left(\dfrac{\sqrt{3}}{4},\dfrac{\sqrt{3}}{2}\right)$, $m=\sqrt{3}$ at $\left(\dfrac{\sqrt{3}}{4},\dfrac{1}{2}\right)$

61. $(-3,2): m=-\dfrac{27}{8}$; $(-3,-2): m=\dfrac{27}{8}$; $(3,2): m=\dfrac{27}{8}$; $(3,-2): m=-\dfrac{27}{8}$

63. a) False b) True c) True d) True **65.** $(3,-1)$

69. $\dfrac{dy}{dx}=-\dfrac{y^3+2xy}{x^2+3xy^2}$, $\dfrac{dx}{dy}=-\dfrac{x^2+3xy^2}{y^3+2xy}$, $\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$

Section 2.7, pp. 176–180

1. $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}$ **3.** a) $\dfrac{dV}{dt}=\pi r^2\dfrac{dh}{dt}$ b) $\dfrac{dV}{dt}=2\pi hr\dfrac{dr}{dt}$ c) $\dfrac{dV}{dt}=\pi r^2\dfrac{dh}{dt}+2\pi hr\dfrac{dr}{dt}$

5. a) 1 volt/sec b) $-\dfrac{1}{3}$ amp/sec c) $\dfrac{dR}{dt}=\dfrac{1}{I}\left(\dfrac{dV}{dt}-\dfrac{V}{I}\dfrac{dI}{dt}\right)$ d) 3/2 ohms/sec, R is increasing.

7. a) $\dfrac{dS}{dt}=\dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt}$

b) $\dfrac{dS}{dt}=\dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt}+\dfrac{y}{\sqrt{x^2+y^2}}\dfrac{dy}{dt}$ c) $\dfrac{dx}{dt}=-\dfrac{y}{x}\dfrac{dy}{dt}$

9. a) $\dfrac{dA}{dt}=\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}$

b) $\dfrac{dA}{dt}=\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}+\dfrac{1}{2}b\sin\theta\dfrac{da}{dt}$

c) $\dfrac{dA}{dt}=\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}+\dfrac{1}{2}b\sin\theta\dfrac{da}{dt}+\dfrac{1}{2}a\sin\theta\dfrac{db}{dt}$

11. a) 14 cm^2/sec, increasing b) 0 cm/sec, constant c) $-14/13$ cm/sec, decreasing

13. a) -12 ft/sec b) -59.5 ft^2/sec c) -1 rad/sec

15. 20 ft/sec **17.** a) $\dfrac{dh}{dt}=11.19$ cm/min b) $\dfrac{dr}{dt}=14.92$ cm/min **19.** a) $\dfrac{-1}{24\pi}$ m/min

b) $r=\sqrt{26y-y^2}$ m, c) $\dfrac{dr}{dt}=-\dfrac{5}{288\pi}$ m/min

21. 1 ft/min, 40π ft^2/min **23.** 11 ft/sec

25. Increasing at 466/1681 L/min **27.** 1 rad/sec **29.** -5 m/sec

31. -1500 ft/sec **33.** $\dfrac{5}{72\pi}$ in/min, $\dfrac{10}{3}$ in^2/min **35.** 7.1 in/min

37. a) $-32\sqrt{13}\approx -8.875$ ft/sec, b) $d\theta_1/dt=-8/65$ rad/sec, $d\theta_2/dt=8/65$ rad/sec

c) $d\theta_1/dt = -1/6$ rad/sec, $d\theta_2/dt = 1/6$ rad/sec
39. 29.5 knots

Chapter 2 Practice Exercises, pp. 181–185

1. $5x^4 - .25x + .25$ **3.** $3x(x-2)$ **5.** $2(x+1)(2x^2+4x+1)$

7. $3(\theta^2 + \sec\theta + 1)^2(2\theta + \sec\theta\tan\theta)$ **9.** $\dfrac{1}{2\sqrt{t}(1+\sqrt{t})^2}$

11. $2\sec^2 x \tan x$ **13.** $8\cos^3(1-2t)\sin(1-2t)$

15. $5(\sec t)(\sec t + \tan t)^5$ **17.** $\dfrac{\theta\cos\theta + \sin\theta}{\sqrt{2\theta\sin\theta}}$ **19.** $\dfrac{\cos\sqrt{2\theta}}{\sqrt{2\theta}}$

21. $x\csc\left(\dfrac{2}{x}\right) + \csc\left(\dfrac{2}{x}\right)\cot\left(\dfrac{2}{x}\right)$

23. $\dfrac{1}{2}x^{1/2}\sec(2x)^2\left[16\tan(2x)^2 - x^{-2}\right]$ **25.** $-10x\csc^2(x^2)$

27. $8x^3\sin(2x^2)\cos(2x^2) + 2x\sin^2(2x^2)$ **29.** $\dfrac{-(t+1)}{8t^3}$

31. $\dfrac{1-x}{(x+1)^3}$ **33.** $-\dfrac{1}{2x^2\left(1+\dfrac{1}{x}\right)^{1/2}}$ **35.** $\dfrac{-2\sin\theta}{(\cos\theta - 1)^2}$

37. $3\sqrt{2x+1}$ **39.** $-9\left(\dfrac{5x+\cos 2x}{(5x^2+\sin 2x)^{5/2}}\right)$ **41.** $-\dfrac{y+2}{x+3}$

43. $\dfrac{-3x^2-4y+2}{4x-4y^{1/3}}$ **45.** $-y/x$ **47.** $\dfrac{1}{2y(x+1)^2}$

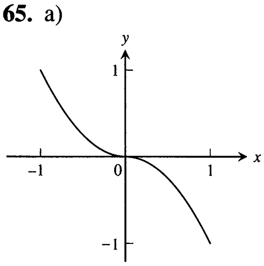
49. $\dfrac{dp}{dq} = \dfrac{6q-4p}{3p^2+4q}$ **51.** $\dfrac{dr}{ds} = (2r-1)(\tan 2s)$

53. a) $\dfrac{d^2y}{dx^2} = \dfrac{-2xy^3-2x^4}{y^5}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{-2xy^2-1}{x^4y^3}$

55. a) 1 b) 6 c) 1 d) $-1/9$ e) $-40/3$ f) 2 g) $-4/9$

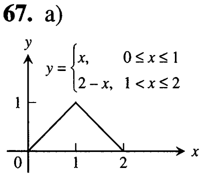
57. 0 **59.** $\sqrt{3}$ **61.** $-1/2$ **63.** $\dfrac{-2}{(2t+1)^2}$

65. a) b) Yes c) Yes



$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1 \end{cases}$

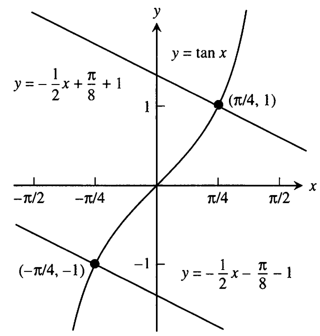
67. a) b) Yes c) No



69. $\left(\dfrac{5}{2}, \dfrac{9}{4}\right)$ and $\left(\dfrac{3}{2}, -\dfrac{1}{4}\right)$ **71.** $(-1, 27)$ and $(2, 0)$

73. a) $(-2, 16), (3, 11)$ b) $(0, 20), (1, 7)$

75.



77. 1/4 **79.** 4

81. Tangent: $y = -\dfrac{1}{4}x + \dfrac{9}{4}$; Normal: $y = 4x - 2$

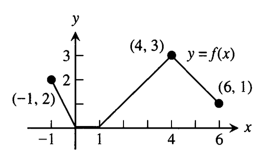
83. Tangent: $y = 2x - 4$; Normal: $y = -\dfrac{1}{2}x + \dfrac{7}{2}$

85. Tangent: $y = -\dfrac{5}{4}x + 6$; Normal: $y = \dfrac{4}{5}x - \dfrac{11}{5}$

87. $(1, 1): m = -1/2$, $(1, -1)$; m not defined

89. B = graph of f, A = graph of f'

91.



93. a) 0, 0 b) 1700 rabbits, ≈1400 rabbits **95.** 3/2 **97.** −1

99. 1/2 **101.** 4 **103.** 1 **107.** Yes, $k = 1/2$

109. a) $\dfrac{dS}{dt} = (4\pi r + 2\pi h)\dfrac{dr}{dt}$ b) $\dfrac{dS}{dt} = 2\pi r\dfrac{dh}{dt}$

c) $\dfrac{dS}{dt} = (4\pi r + 2\pi h)\dfrac{dr}{dt} + 2\pi r\dfrac{dh}{dt}$ d) $\dfrac{dr}{dt} = -\dfrac{r}{2r+h}\dfrac{dh}{dt}$

111. $-40\ \text{m}^2/\text{sec}$ **113.** 0.02 ohm/sec **115.** $5\ \text{m/sec}^2$

117. a) $r = \dfrac{2}{5}$ h b) $-\dfrac{125}{144\pi}$ ft/min

119. a) $\dfrac{3}{5}$ km/sec or 600 m/sec b) $\dfrac{18}{\pi}$ RPM

Chapter 2 Additional Exercises, pp. 185–187

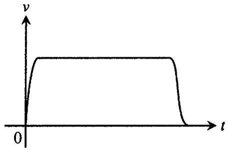
1. a) $\sin 2\theta = 2\sin\theta\cos\theta$; $2\cos 2\theta$
$= 2\sin\theta(-\sin\theta) + \cos\theta(2\cos\theta)$; $2\cos 2\theta$
$= -2\sin^2\theta + 2\cos^2\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta$
b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$; $-2\sin 2\theta$
$= 2\cos\theta(-\sin\theta) - 2\sin\theta(\cos\theta)$; $\sin 2\theta$
$= \cos\theta\sin\theta + \sin\theta\cos\theta$; $\sin 2\theta = 2\sin\theta\cos\theta$

3. a) $a = 1, b = 0, c = -1/2$ b) $b = \cos a, c = \sin a$

5. $h = -4, k = 9/2, a = \dfrac{5\sqrt{5}}{2}$

7. a) $0.09y$ b) increasing at 1% per year

9. Answers will vary. Here is one possibility.



11. a) 2 sec, 64 ft/sec b) 12.31 sec, 393.85 ft.

15. a) $m = -\dfrac{b}{\pi}$, b) $m = -1$, $b = \pi$ **17.** a) $a = 3/4$, $b = 9/4$

23. h' is defined but not continuous at $x = 0$; k' is defined *and* continuous at $x = 0$.

CHAPTER 3

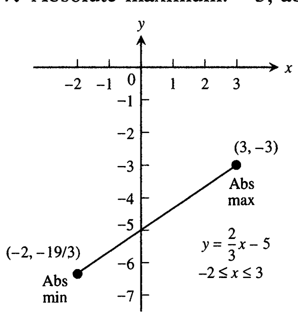
Section 3.1, pp. 195–196

1. Absolute minimum at $x = c_2$, absolute maximum at $x = b$

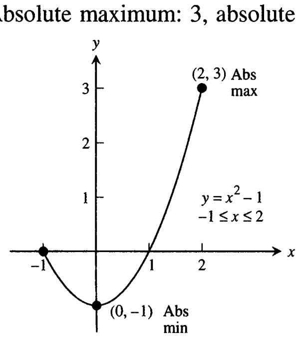
3. Absolute maximum at $x = c$, no absolute minimum

5. Absolute minimum at $x = a$, absolute maximum at $x = c$

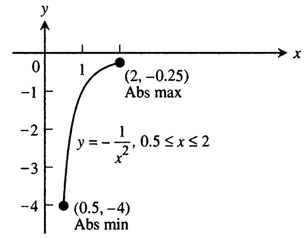
7. Absolute maximum: -3, absolute minimum: $-19/3$



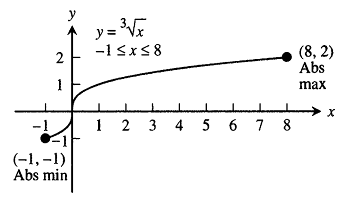
9. Absolute maximum: 3, absolute minimum: -1



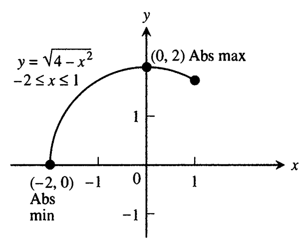
11. Absolute maximum: -0.25, absolute minimum: -4



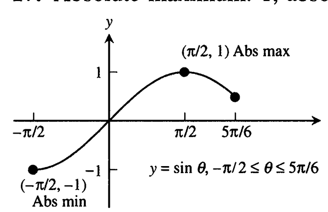
13. Absolute maximum: 2, absolute minimum: -1



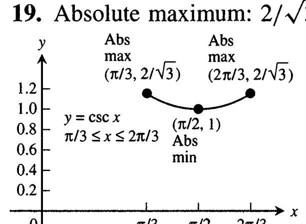
15. Absolute maximum: 2, absolute minimum: 0



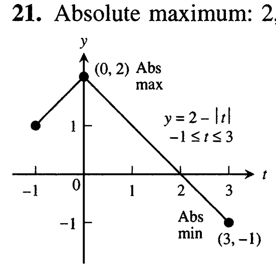
17. Absolute maximum: 1, absolute minimum: -1



19. Absolute maximum: $2/\sqrt{3}$, absolute minimum: 1



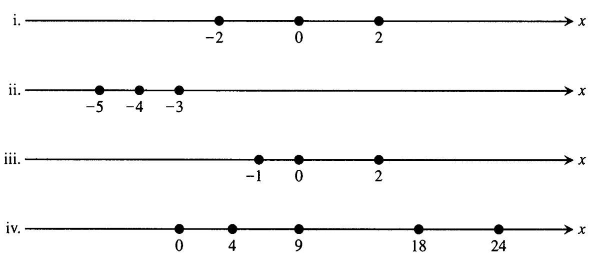
21. Absolute maximum: 2, absolute minimum: -1



23. Increasing on (0, 8), decreasing on $(-1, 0)$, absolute maximum: 16 at $x = 8$, absolute minimum: 0 at $x = 0$
25. Increasing on $(-32, 1)$, absolute maximum: 1 at $\theta = 1$, absolute minimum: -8 at $\theta = -32$
27. a) Local maximum: 0 at $x = \pm 2$, local minimum: -4 at $x = 0$, absolute maximum: 0, absolute minimum: -4
b) Local maximum: 0 at $x = -2$, local minimum: -4 at $x = 0$, absolute maximum: 0, absolute minimum: -4
c) No local maximum, local minimum: -4 at $x = 0$, absolute minimum: -4
d) Local maximum: 0 at $x = -2$, local minimum: -4 at $x = 0$, absolute minimum: -4
e) No local extrema, no absolute extrema **29.** Yes

Section 3.2, pp. 203–205

1. 1/2 **3.** 1
5. Does not; f is not differentiable at the interior domain point $x = 0$.
7. Does
11. a)



27. $1.09999 \le f(0.1) \le 1.1$ **31.** Yes **33.** a) 4 b) 3 c) 3

35. a) $\dfrac{x^2}{2} + C$ b) $\dfrac{x^3}{3} + C$ c) $\dfrac{x^4}{4} + C$

37. a) $\dfrac{1}{x} + C$ b) $x + \dfrac{1}{x} + C$ c) $5x - \dfrac{1}{x} + C$

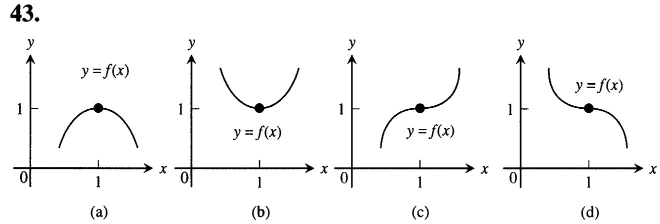
39. a) $-\dfrac{1}{2}\cos 2t + C$ b) $2\sin\dfrac{t}{2} + C$

c) $-\dfrac{1}{2}\cos 2t + 2\sin\dfrac{t}{2} + C$

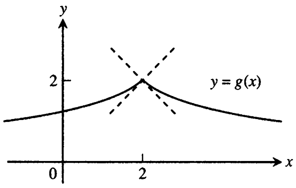
41. $f(x) = x^2 - x$ **43.** $r(\theta) = 8\theta + \cot\theta - 2\pi - 1$

Section 3.3, pp. 208–209

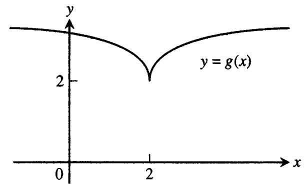
1. a) 0, 1 b) increasing on $(-\infty, 0)$ and $(1, \infty)$, decreasing on (0, 1) c) local maximum at $x = 0$, local minimum at $x = 1$
3. a) -2, 1 b) increasing on $(-2, 1)$ and $(1, \infty)$, decreasing on $(-\infty, -2)$ c) No local maximum, local minimum at $x = -2$
5. a) -2, 1, 3 b) increasing on $(-2, 1)$ and $(3, \infty)$, decreasing on $(-\infty, -2)$ and (1, 3) c) local maximum at $x = 1$, local minima at $x = -2, 3$
7. a) -2, 0 b) increasing on $(-\infty, -2)$ and $(0, \infty)$, decreasing on $(-2, 0)$ c) Local maximum at $x = -2$, local minimum at $x = 0$
9. a) increasing on $(-\infty, -1.5)$, decreasing on $(-1.5, \infty)$
b) local maximum: 5.25 at $t = -1.5$
c) Absolute maximum: 5.25 at $t = -1.5$
11. a) Decreasing on $(-\infty, 0)$, increasing on (0, 4/3), decreasing on $(4/3, \infty)$ b) local minimum at $x = 0$ (0, 0), local maximum at $x = 4/3$ (4/3, 32/27) c) no absolute extrema
13. a) Decreasing on $(-\infty, 0)$, increasing on (0, 1/2), decreasing on $(1/2, \infty)$ b) local minimum at $\theta = 0$ (0, 0), local maximum at $\theta = 1/2$ (1/2, 1/4) c) no absolute extrema
15. a) Increasing on $(-\infty, \infty)$, never decreasing b) no local extrema c) no absolute extrema
17. a) Increasing on $(-2, 0)$ and $(2, \infty)$, decreasing on $(-\infty, -2)$ and (0, 2) b) local maximum: 16 at $x = 0$, local minimum: 0 at $x = \pm 2$ c) no absolute maximum, absolute minimum: 0 at $x = \pm 2$
19. a) Increasing on $(-\infty, -1)$, decreasing on $(-1, 0)$, increasing on (0, 1), decreasing on $(1, \infty)$
b) local maximum at $x = \pm 1$ $(1, 0.5), (-1, 0.5)$, local minimum at $x = 0$ $(0, 0)$ c) absolute maximum: 1/2 at $x = \pm 1$, no absolute minimum
21. a) Decreasing on $(-2\sqrt{2}, -2)$, increasing on $(-2, 2)$, decreasing on $(2, 2\sqrt{2})$ b) local minima: $g(-2) = -4$, $g(2\sqrt{2}) = 0$; local maxima: $g(-2\sqrt{2}) = 0$, $g(2) = 4$ c) absolute maximum: 4 at $x = 2$, absolute minimum: -4 at $x = -2$
23. a) Increasing on $(-\infty, 1)$, decreasing when $1 < x < 2$, decreasing when $2 < x < 3$, discontinuous at $x = 2$, increasing on $(3, \infty)$, b) local minimum at $x = 3$ (3, 6), local maximum at $x = 1$ (1, 2) c) no absolute extrema
25. a) Increasing on $(-2, 0)$ and $(0, \infty)$, decreasing on $(-\infty, -2)$ b) local minimum: $-6\sqrt[3]{2}$ at $x = -2$ c) no absolute maximum, absolute minimum: $-6\sqrt[3]{2}$ at $x = -2$
27. a) Increasing on $(-\infty, -2/\sqrt{7})$ and $(2/\sqrt{7}, \infty)$, decreasing on $(-2/\sqrt{7}, 2/\sqrt{7})$ b) local maximum: $24\sqrt[3]{2}/7^{7/6} \approx 3.12$ at $x = -2/\sqrt{7}$, local minimum: $-24\sqrt[3]{2}/7^{7/6} \approx -3.12$ at $x = 2/\sqrt{7}$ c) no absolute extrema
29. a) Local maximum: 1 at $x = 1$, local minimum: 0 at $x = 2$
b) absolute maximum: 1 at $x = 1$; no absolute minimum
31. a) Local maximum: 1 at $x = 1$, local minimum: 0 at $x = 2$
b) no absolute maximum, absolute minimum: 0 at $x = 2$
33. a) Local maxima: -9 at $t = -3$ and 16 at $t = 2$, local minimum: -16 at $t = -2$ b) absolute maximum: 16 at $t = 2$, no absolute minimum
35. a) Local minimum: 0 at $x = 0$ b) no absolute maximum, absolute minimum: 0 at $x = 0$
37. a) Local minimum: $(\pi/3) - \sqrt{3}$ at $x = 2\pi/3$, local maximum: 0 at $x = 0$, local maximum: π at $x = 2\pi$
39. a) Local minimum: 0 at $x = \pi/4$
41. Local maximum: 3 at $\theta = 0$, Local minimum: -3 at $\theta = 2\pi$
43.



45. a)



b)



47. Rising

Section 3.4, pp. 217–220

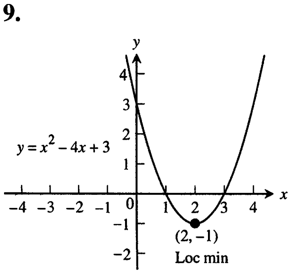
1. Local maximum: 3/2 at $x=-1$, local minimum: -3 at $x=2$, point of inflection at $(1/2, -3/4)$, rising on $(-\infty, -1)$ and $(2, \infty)$, falling on $(-1, 2)$, concave up on $(1/2, \infty)$, concave down on $(-\infty, 1/2)$

3. Local maximum: 3/4 at $x=0$, local minimum: 0 at $x=\pm 1$, points of inflection at $\left(-\sqrt{3}, \dfrac{3\sqrt[3]{4}}{4}\right)$ and $\left(\sqrt{3}, \dfrac{3\sqrt[3]{4}}{4}\right)$, rising on $(-1, 0)$ and $(1, \infty)$, falling on $(-\infty, -1)$ and $(0, 1)$, concave up on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$, concave down on $(-\sqrt{3}, \sqrt{3})$

5. Local maxima: $-2\pi/3+\sqrt{3}/2$ at $x=-2\pi/3$; $\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}$ at $x=\dfrac{\pi}{3}$, local minima: $-\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{2}$ at $x=-\dfrac{\pi}{3}$; $2\pi/3-\sqrt{3}/2$ at $x=\dfrac{2\pi}{3}$, points of inflection at $(-\pi/2, -\pi/2)$, $(0, 0)$, and $(\pi/2, \pi/2)$, rising on $(-\pi/3, \pi/3)$, falling on $(-2\pi/3, -\pi/3)$ and $(\pi/3, 2\pi/3)$, concave up on $(-\pi/2, 0)$ and $(\pi/2, 2\pi/3)$, concave down on $(-2\pi/3, -\pi/2)$ and $(0, \pi/2)$

7. Local maxima: 1 at $x=-\dfrac{\pi}{2}$ and $x=\dfrac{\pi}{2}$; 0 at $x=-2\pi$ and $x=2\pi$; local minima: -1 at $x=-\dfrac{3\pi}{2}$ and $x=\dfrac{3\pi}{2}$, 0 at $x=0$, points of inflection at $(-\pi, 0)$ and $(\pi, 0)$, rising on $(-3\pi/2, -\pi/2)$, $(0, \pi/2)$ and $(3\pi/2, 2\pi)$, falling on $(-2\pi, -3\pi/2)$, $(-\pi/2, 0)$ and $(\pi/2, 3\pi/2)$, concave up on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$, concave down on $(-\pi, \pi)$

9.



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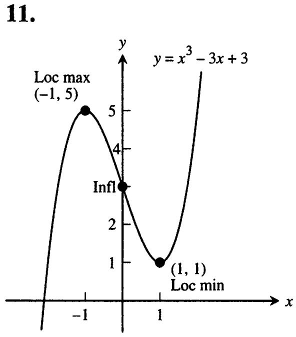

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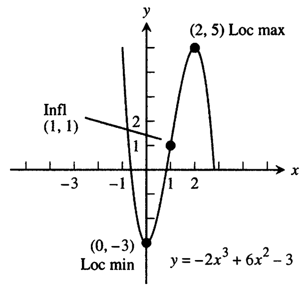

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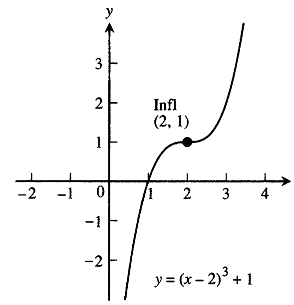

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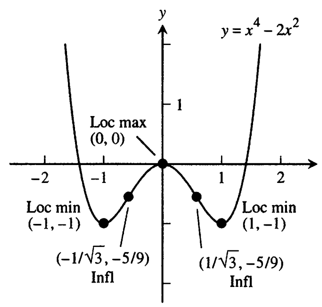

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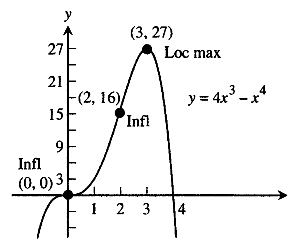

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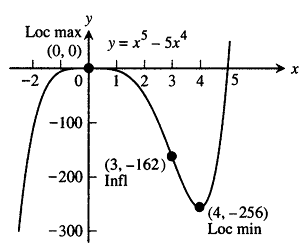

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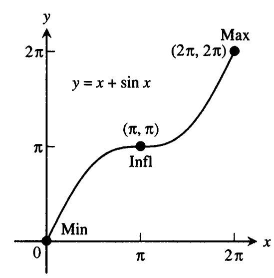

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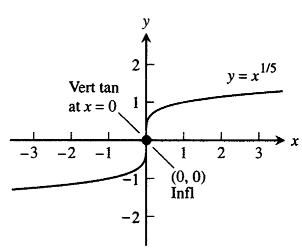

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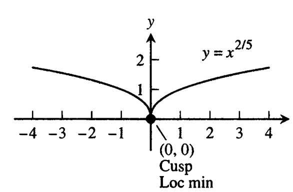

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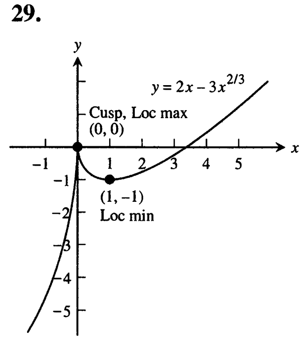

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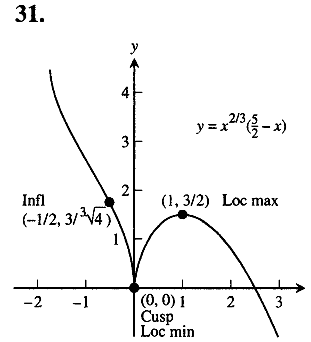

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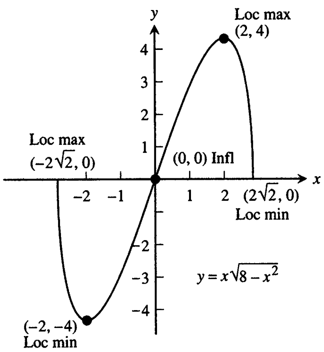

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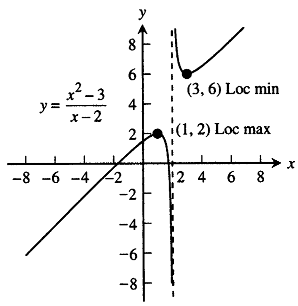

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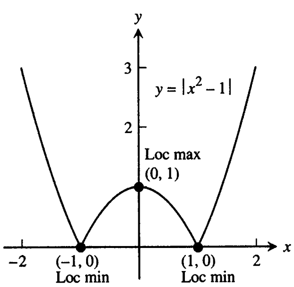
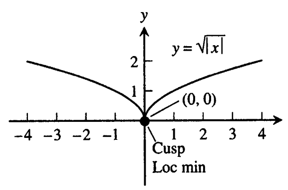
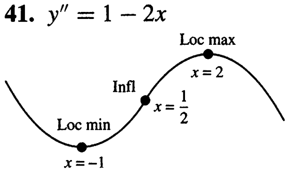


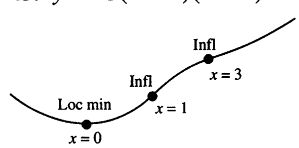
39.



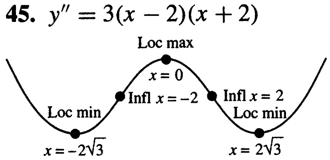
41. $y'' = 1 - 2x$



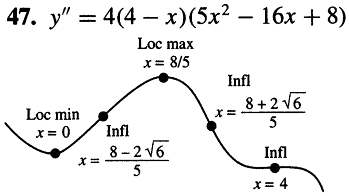
43. $y'' = 3(x-3)(x-1)$



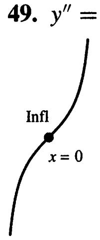
45. $y'' = 3(x-2)(x+2)$



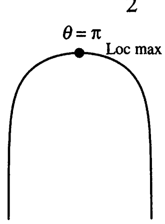
47. $y'' = 4(4-x)(5x^2 - 16x + 8)$



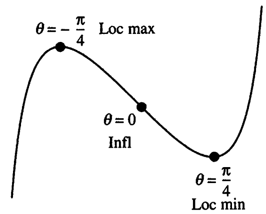
49. $y'' = 2\sec^2 x \tan x$



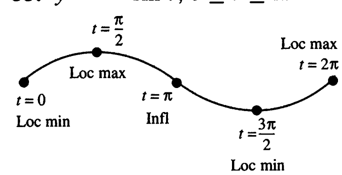
51. $y'' = -\dfrac{1}{2}\csc^2 \dfrac{\theta}{2}, 0 < \theta < 2\pi$



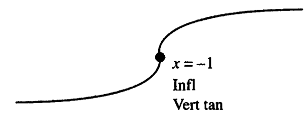
53. $y'' = 2\tan\theta\sec^2\theta, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



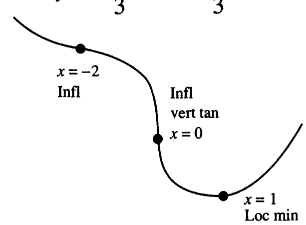
55. $y'' = -\sin t, 0 \le t \le 2\pi$



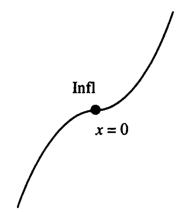
57. $y'' = -\dfrac{2}{3}(x+1)^{-5/3}$



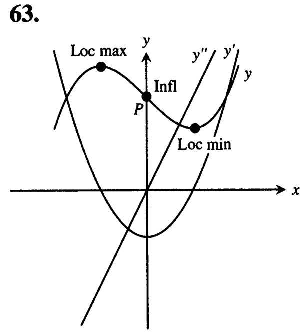
59. $y'' = \dfrac{1}{3}x^{-2/3} + \dfrac{2}{3}x^{-5/3}$



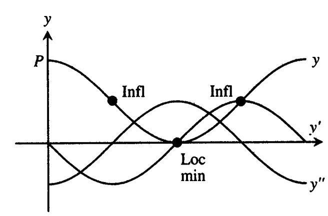
61. $y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$



63.



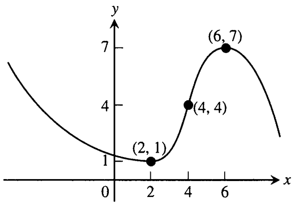
65.



67.

Point	y'	y''
P	−	+
Q	+	0
R	+	−
S	0	−
T	−	−

69.



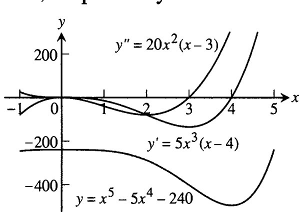
73. ≈ 60 thousand units

75. Local minimum at $x = 2$, inflection points at $x = 1$ and $x = 5/3$

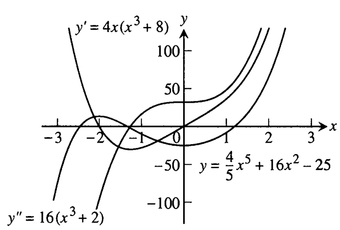
79. $b = -3$

81. a) $\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$ b) concave up if $a > 0$, concave down if $a < 0$

85. The zeros of $y' = 0$ and $y'' = 0$ are extrema and points of inflection, respectively.



87. The zeros of $y' = 0$ and $y'' = 0$ are extrema and points of inflection, respectively. Inflection at $x = -\sqrt[3]{2}$, local maximum at $x = -2$, local minimum at $x = 0$.



91. b) $f'(x) = 3x^2 + k$; $-12k$; positive if $k < 0$, negative if $k > 0$, 0 if $k = 0$; f' has two zeros if $k < 0$, one zero if $k = 0$, no zeros if $k > 0$

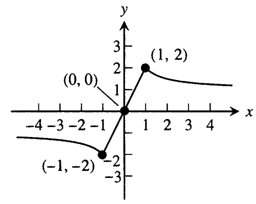
93. b) A cusp since $\lim\limits_{x\to 0^-} y' = \infty$ and $\lim\limits_{x\to 0^+} y' = -\infty$

95. Yes, the graph of y' crosses through zero near -3, so y has a horizontal tangent near -3.

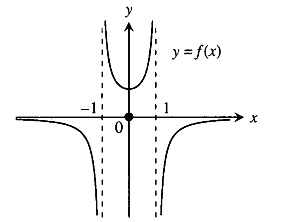
Section 3.5, pp. 230–233

1. a) -3 b) -3 **3.** a) $1/2$ b) $1/2$ **5.** a) $-5/3$ b) $-5/3$
7. 0 **9.** -1 **11.** a) $2/5$ b) $2/5$ **13.** a) 0 b) 0
15. a) $-\infty$ b) ∞ **17.** a) 7 b) 7 **19.** a) $-\infty$ b) ∞
21. a) ∞ b) $-\infty$ **23.** a) $-2/3$ b) $-2/3$ **25.** 0 **27.** 1
29. ∞

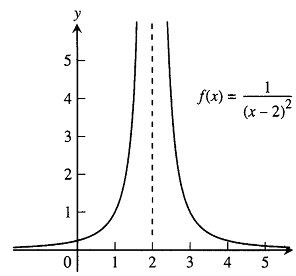
31. Here is one possibility.



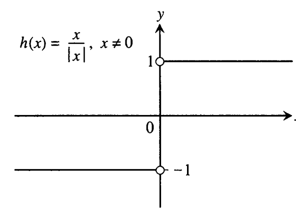
33. Here is one possibility.



35. Here is one possibility.



37. Here is one possibility.



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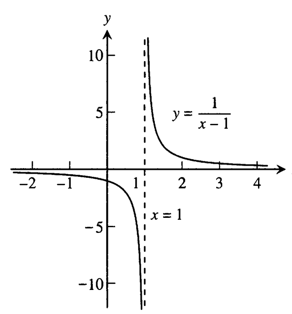

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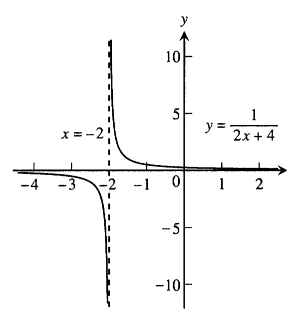

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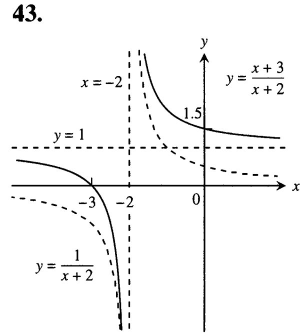

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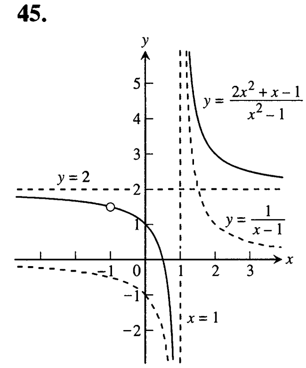

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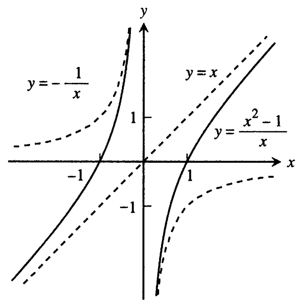

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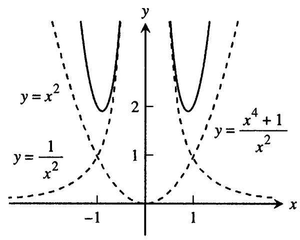

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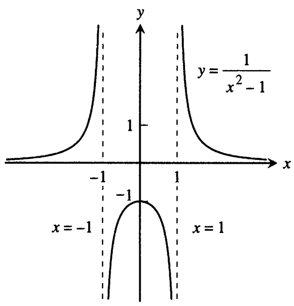

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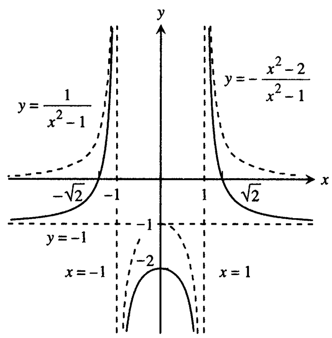

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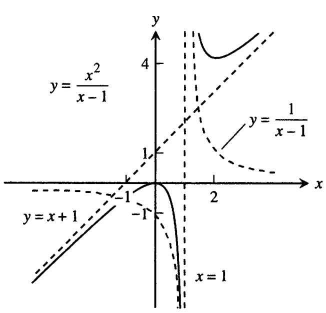

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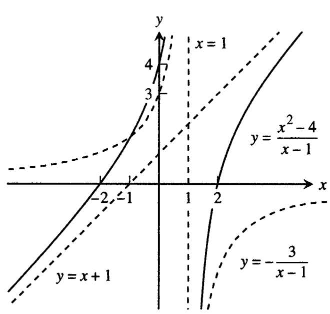

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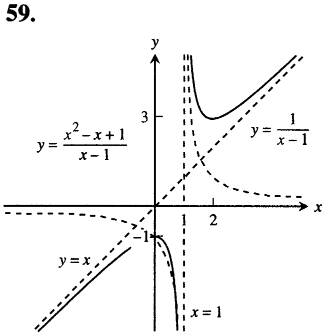

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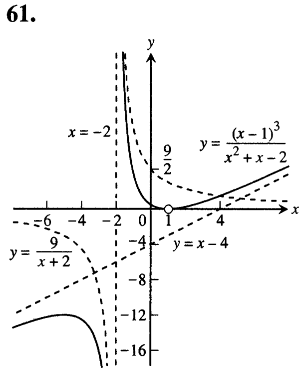

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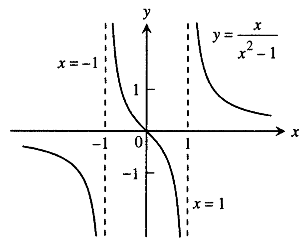

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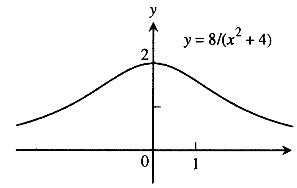

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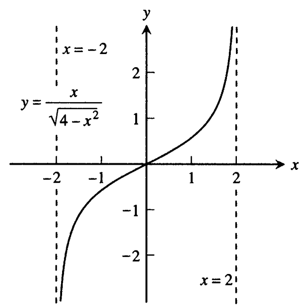

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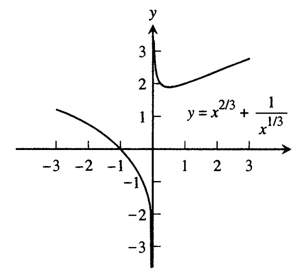

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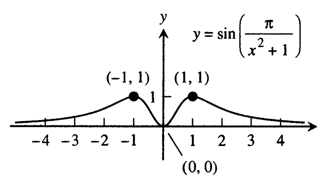

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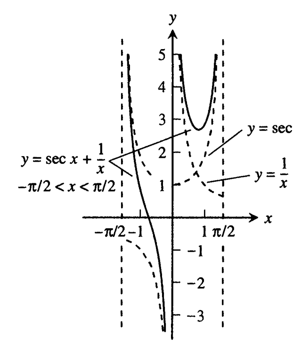

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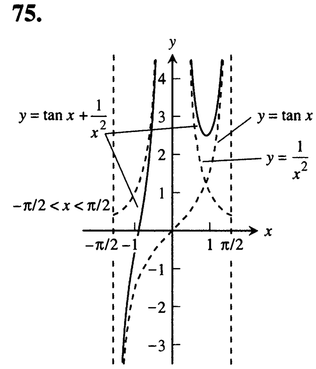


79. Increasing **83.** 2

85. b) One possibility is $f(x) = 2 + (1/x)\sin(x^2)$.

89. $x = -1,\ y = 1 - x$ **91.** $x = 1,\ x = -1, y = x - 1$

99. a) $y \to \infty$, b) $y \to \infty$, c) cusp at $x = \pm 1$

101. The distance in part (c) is so great that small movements are not visible.

103. 1 **105.** 3/2 **107.** 3

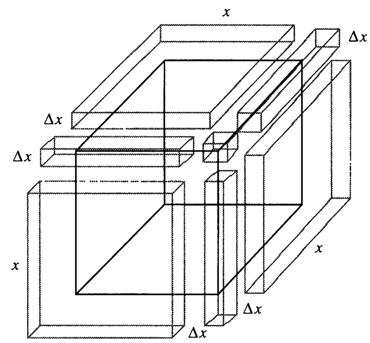
Section 3.6, pp. 242–247

1. $r = 25$ m, $s = 50$ m **3.** 16 in. **5.** a) $(x, 1-x)$ b) $A(x) = 2x(1-x)$ c) 1/2 square units **7.** $\frac{14}{3} \times \frac{35}{3} \times \frac{5}{3}$ in. **9.** 80,000 m^2 **11.** base: 10 ft, height: 5 ft **13.** 9×18 in. **15.** $\pi/2$ **17.** $r = h = \frac{10}{\sqrt[3]{\pi}}$ cm **19.** a) $18 \times 18 \times 36$ in.
21. a) 12 cm, 6 cm b) 12 cm, 6 cm
23. a) The circumference of the circle is 4 m.
25. If r is the radius of the semicircle, $2r$ is the base of the rectangle, and h is the height of the rectangle, then $(2r)/h = 8/(4+\pi)$.
27. $\pi/6$ **29.** $\frac{v_0^2}{2g} + s_0$ **31.** a) $4\sqrt{3} \times 4\sqrt{6}$ in. **33.** $2\sqrt{2}$ amps
35. a) When t is an integer multiple of π b) $t = \frac{2\pi}{3}, t = \frac{4\pi}{3}, 3\frac{\sqrt{3}}{2}$ **37.** a) $t = \frac{8}{5}, t = 4$ b) $\frac{8}{5} < t < 4$ c) $\frac{2187}{125}$ units/time
41. No. The function has an absolute minimum of 3/4.
43. a) $\left(c - \frac{1}{2}, \sqrt{c - \frac{1}{2}}\right)$ b) (0, 0) **45.** a) $a = -3, b = -9$ b) $a = -3, b = -24$ **47.** a) $y = -1$ **49.** $(7/2)\sqrt{17}$
53. $M = c/2$ **55.** $\frac{c}{2} + 50$ **57.** $\sqrt{\frac{2km}{h}}$

Section 3.7, pp. 257–260

1. $4x - 3$ **3.** $2x - 2$ **5.** $\frac{1}{4}x + 1$ **7.** $2x$ **9.** -5
11. $\frac{1}{12}x + \frac{4}{3}$ **13.** a) $L(x) = x$ b) $L(x) = \pi - x$
15. a) $L(x) = 1$ b) $L(x) = 2 - 2\sqrt{3}\left(x + \frac{\pi}{3}\right)$ **17.** a) $1 + 2x$ b) $1 - 5x$ c) $2 + 2x$ d) $1 - 6x$ e) $3 + x$ f) $1 - \frac{x}{2}$
19. $\frac{3}{2}x + 1$. It is equal to their sum. **21.** $\left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx$
23. $\frac{2 - 2x^2}{(1+x^2)^2} dx$ **25.** $\frac{1-y}{3\sqrt{y} + x} dx$ **27.** $\frac{5}{2\sqrt{x}} \cos(5\sqrt{x})\, dx$
29. $(4x^2) \sec^2\left(\frac{x^3}{3}\right) dx$
31. $\frac{3}{\sqrt{x}}(\csc(1 - 2\sqrt{x}) \cot(1 - 2\sqrt{x}))\, dx$ **33.** a) .21 b) .2 c) .01 **35.** a) .231 b) .2 c) .031 **37.** a) $-1/3$ b) $-2/5$ c) 1/15 **39.** $dV = 4\pi r_0^2\, dr$ **41.** $dS = 12x_0\, dx$
43. $dV = 2\pi r_0 h\, dr$ **45.** a) $.08\pi$ m^2 b) 2% **47.** 3%
49. 3% **51.** 1/3% **53.** .05%

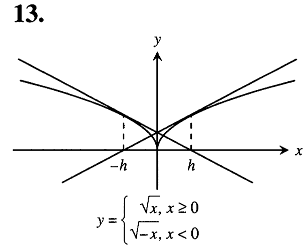
57. Volume $= (x + \Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$



59. $\lim_{x \to 0} \frac{\sqrt{1+x}}{1 + \left(\frac{x}{2}\right)} = \frac{\sqrt{1+0}}{1 + \left(\frac{0}{2}\right)} = \frac{1}{1} = 1$

Section 3.8, pp. 266–268

1. $x_2 = 13/21, -5/3$ **3.** $x_2 = 5763/4945, -51/31$
5. $x_2 = 2387/2000$ **7.** $x \approx 0.45$ **9.** The root is 1.17951
13.



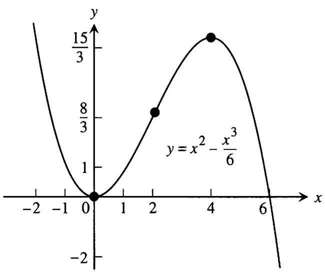
15. a) The points of intersection of $y = x^3$ and $y = 3x + 1$ or $y = x^3 - 3x$ and $y = 1$ have the same x-values as the roots of part (i) or the solutions of part (iv). b) $-1.53209, -0.34730$
17. 2.45, 0.000245 **19.** 1.1655 61185 **21.** a) Two b) 0.3500 35015 05249 and -1.0261 73161 5301
23. ± 1.3065 62964 8764, ± 0.5411 96100 14619
25. Answers will vary with machine used.
27.

x_0	Approximation of corresponding root
-1.0	-0.976823589
0.1	0.100363332
0.6	0.642746671
2.0	1.98371387

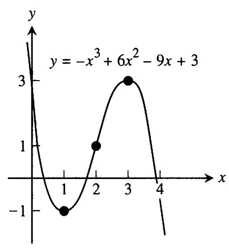
Chapter 3 Practice Exercises, pp. 269–272

1. No
3. No minimum, absolute maximum: $f(1) = 16$, critical points: $x = 1$ and 11/3
7. No **11.** b) one **13.** b) 0.8555 99677 2

19.



21.



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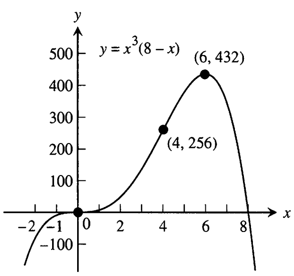

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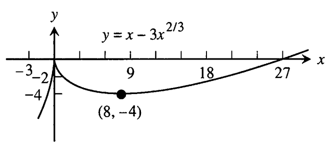

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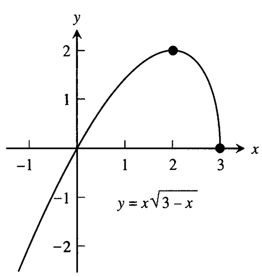
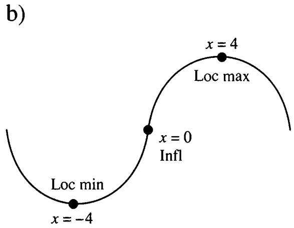
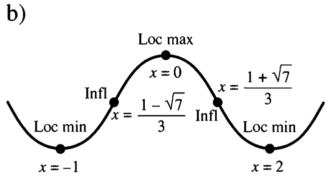


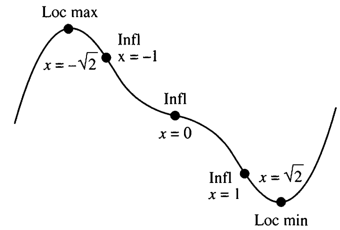
29. a) Local maximum at $x = 4$, local minimum at $x = -4$, inflection point at $x = 0$
b)



31. a) Local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$, inflection points at $x = (1 \pm \sqrt{7})/3$
b)



33. a) Local maximum at $x = -\sqrt{2}$, local minimum at $x = \sqrt{2}$, inflection points at $x = \pm 1$ and 0
b)



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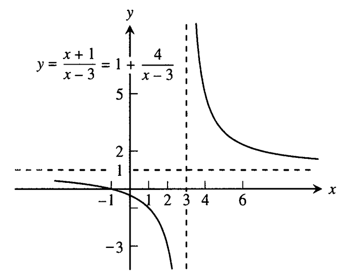

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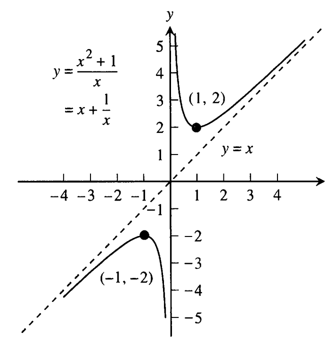

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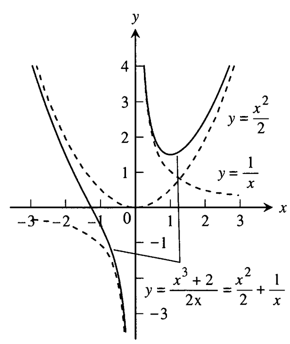
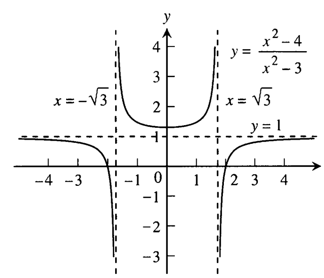
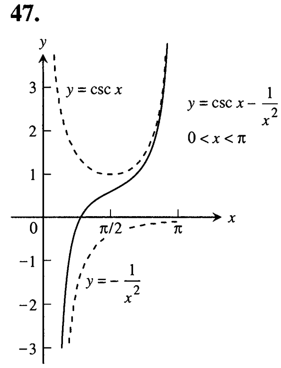


45.



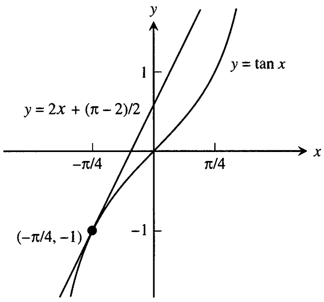
47.



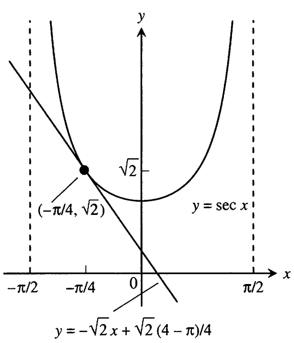
49. a) $t = 0,\ 6,\ 12$ b) $t = 3,\ 9$ c) $6 < t < 12$ d) $0 < t < 6$, $12 < t < 14$
51. 2/5 **53.** 0 **55.** $-\infty$ **57.** 0 **59.** 1 **61.** a) 0, 36 b) 18, 18 **63.** 54 square units **65.** height = 2, radius = $\sqrt{2}$

67. $x = 15$ mi, $y = 9$ mi **69.** $x = 5 - \sqrt{5}$ hundred ≈ 276 tires, $y = 2(5 - \sqrt{5})$ hundred ≈ 553 tires

71. a) $L(x) = 2x + (\pi - 2)/2$



b) $L(x) = -\sqrt{2}x + \sqrt{2}(4 - \pi)/4$



73. $L(x) = 1.5x + 0.5$ **75.** $dV = \frac{2}{3}\pi r_0 h\, dr$

77. a) error $< 1\%$ b) 3% **79.** $dh \approx \pm 2.3271$ ft

81. $x_5 = 2.1958\ 23345$

Chapter 3 Additional Exercises, pp. 272–274

3. The extreme points will not be at the end of an open interval.

5. a) A local minimum at $x = -1$, points of inflection at $x = 0$ and $x = 2$ b) A local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$, points of inflection at $x = \dfrac{1 \pm \sqrt{7}}{3}$

11. $a = 1, b = 0, c = 1$ **13.** Yes **15.** Drill the hole at $y = h/2$.

17. $r = \dfrac{RH}{2(H - R)}$ for $H < 2R, r = R$ if $H \le 2R$

21. a) 0.8156 ft b) 0.00613 sec c) It will lose about 8.83 min/day.

CHAPTER 4

Section 4.1, pp. 280–282

1. a) x^2 b) $\dfrac{x^3}{3}$ c) $\dfrac{x^3}{3} - x^2 + x$ **3.** a) x^{-3} b) $-\frac{1}{3}x^{-3}$ c) $-\frac{1}{3}x^{-3} + x^2 + 3x$ **5.** a) $-\dfrac{1}{x}$ b) $-\dfrac{5}{x}$ c) $2x + \dfrac{5}{x}$

7. a) $\sqrt{x^3}$ b) $\sqrt{x}$ c) $\dfrac{2\sqrt{x^3}}{3} + 2\sqrt{x}$ **9.** a) $x^{2/3}$ b) $x^{1/3}$ c) $x^{-1/3}$ **11.** a) $\cos(\pi x)$ b) $-3\cos x$ c) $-\dfrac{1}{\pi}\cos(\pi x) + \cos(3x)$ **13.** a) $\tan x$ b) $2\tan\left(\dfrac{x}{3}\right)$ c) $-\dfrac{2}{3}\tan\left(\dfrac{3x}{2}\right)$ **15.** a) $-\csc x$ b) $\dfrac{1}{5}\csc(5x)$ c) $2\csc\left(\dfrac{\pi x}{2}\right)$

17. $x + \dfrac{\cos(2x)}{2}$ **19.** $\dfrac{x^2}{2} + x + C$ **21.** $t^3 + \dfrac{t^2}{4} + C$

23. $\dfrac{x^4}{2} - \dfrac{5x^2}{2} + 7x + C$ **25.** $-\dfrac{1}{x} - \dfrac{x^3}{3} - \dfrac{x}{3} + C$

27. $\frac{3}{2}x^{2/3} + C$ **29.** $\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$ **31.** $4y^2 - \frac{8}{3}y^{3/4} + C$

33. $x^2 + \dfrac{2}{x} + C$ **35.** $2\sqrt{t} - \dfrac{2}{\sqrt{t}} + C$ **37.** $-2\sin t + C$

39. $-21\cos\dfrac{\theta}{3} + C$ **41.** $3\cot x + C$ **43.** $-\dfrac{1}{2}\csc\theta + C$

45. $4\sec x - 2\tan x + C$ **47.** $-\dfrac{1}{2}\cos 2x + \cot x + C$

49. $2y - \sin 2y + C$ **51.** $\dfrac{t}{2} + \dfrac{\sin 4t}{8} + C$ **53.** $\tan\theta + C$

55. $-\cot x - x + C$ **57.** $-\cos\theta + \theta + C$

65. a) Wrong: $\dfrac{d}{dx}\left(\dfrac{x^2}{2}\sin x + C\right) = \dfrac{2x}{2}\sin x + \dfrac{x^2}{2}\cos x = x\sin x + \dfrac{x^2}{2}\cos x$

b) Wrong: $\dfrac{d}{dx}(-x\cos x + C) = -\cos x + x\sin x$

c) Right: $\dfrac{d}{dx}(-x\cos x + \sin x + C) = -\cos x + x\sin x + \cos x = x\sin x$

67. a) Wrong: $\dfrac{d}{dx}\left(\dfrac{(2x+1)^3}{3} + C\right) = \dfrac{3(2x+1)^2(2)}{3} = 2(2x+1)^2$

b) Wrong: $\dfrac{d}{dx}((2x+1)^3 + C) = 3(2x+1)^2(2) = 6(2x+1)^2$

c) Right: $\dfrac{d}{dx}((2x+1)^3 + C) = 6(2x+1)^2$

69. a) $-\sqrt{x} + C$ b) $x + C$ c) $\sqrt{x} + C$ d) $-x + C$ e) $x - \sqrt{x} + C$ f) $-x - \sqrt{x} + C$ g) $\dfrac{x^2}{2} - \sqrt{x} + C$ h) $-3x + C$

Section 4.2, pp. 288–290

1. b **3.** $y = x^2 - 7x + 10$ **5.** $y = -\dfrac{1}{x} + \dfrac{x^2}{2} - \dfrac{1}{2}$

7. $y = 9x^{1/3} + 4$ **9.** $s = t + \sin t + 4$ **11.** $r = \cos(\pi\theta) - 1$

13. $v = \frac{1}{2}\sec t + \frac{1}{2}$ **15.** $y = x^2 - x^3 + 4x + 1$

17. $r = \frac{1}{t} + 2t - 2$ **19.** $y = x^3 - 4x^2 + 5$

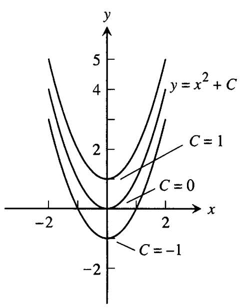
21. $y = -\sin t + \cos t + t^3 - 1$ **23.** $s = 4.9t^2 + 5t + 10$

25. $s = \frac{1 - \cos(\pi t)}{\pi}$ **27.** $s = 16t^2 + 20t + 5$

29. $s = \sin(2t) - 3$ **31.** $y = 2x^{3/2} - 50$ **33.** $y = x - x^{4/3} + \frac{1}{2}$

35. $y = -\sin x - \cos x - 2$

37.



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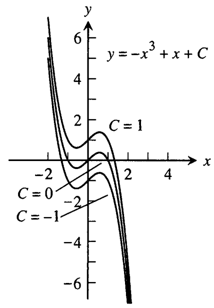

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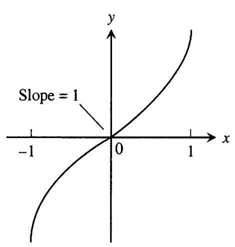
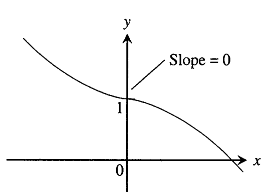


43.



45. 48 m/sec **47.** 14 m/sec **49.** $t = 88/k, k = 16$

51. a) $v = 10t^{3/2} - 6t^{1/2}$ b) $s = 4t^{5/2} - 4t^{3/2}$

55. a) 1: 33.2 units, 2: 33.2 units, 3: 33.2 units b) True

Section 4.3, p. 296

1. $-\frac{1}{3}\cos 3x + C$ **3.** $\frac{1}{2}\sec 2t + C$ **5.** $-(7x-2)^{-4} + C$

7. $-6(1 - r^3)^{1/2} + C$ **9.** $\frac{1}{3}(x^{3/2} - 1) - \frac{1}{6}\sin(2x^{3/2} - 2) + C$

11. a) $-\frac{1}{4}(\cot^2 2\theta) + C$ b) $-\frac{1}{4}(\csc^2 2\theta) + C$

13. $-\frac{1}{3}(3 - 2s)^{3/2} + C$ **15.** $\frac{2}{5}(5s + 4)^{1/2} + C$

17. $-\frac{2}{5}(1 - \theta^2)^{5/4} + C$ **19.** $-\frac{1}{3}(7 - 3y^2)^{3/2} + C$

21. $(-2/(1 + \sqrt{x})) + C$ **23.** $\frac{1}{3}\sin(3z + 4) + C$

25. $\frac{1}{3}\tan(3x + 2) + C$ **27.** $\frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$

29. $\left(\frac{r^3}{18} - 1\right)^6 + C$ **31.** $-\frac{2}{3}\cos(x^{3/2} + 1) + C$

33. $\sec\left(v + \frac{\pi}{2}\right) + C$ **35.** $\frac{1}{2\cos(2t + 1)} + C$

37. $-\frac{2}{3}(\cot^3 y)^{1/2} + C$ **39.** $-\sin\left(\frac{1}{t} - 1\right) + C$

41. $-\frac{\sin^2(1/\theta)}{2} + C$ **43.** $\frac{(s^3 + 2s^2 - 5s + 5)^2}{2} + C$

45. $\frac{1}{16}(1 + t^4)^4 + C$ **47.** a) $-\frac{6}{2 + \tan^3 x} + C$

b) $-\frac{6}{2 + \tan^3 x} + C$ c) $-\frac{6}{2 + \tan^3 x} + C$

49. $\frac{1}{6}\sin\sqrt{3(2r-1)^2 + 6} + C$ **51.** $s = \frac{1}{2}(3t^2 - 1)^4 - 5$

53. $s = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

55. $s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$ **57.** 6 m

Section 4.4, pp. 305–309

1. ≈ 44.8, 6.7 L/min **3.** a) 87 in. b) 87 in. **5.** a) 3,490 ft b) 3,840 ft **7.** a) 112 b) 9% **9.** a) 80π b) 6% **11.** a) $93\pi/2$, overestimate b) 9% **13.** a) 40 b) 25% c) 36, 12.5% **15.** a) 118.5π or ≈ 372.28 m^3 b) error $\approx$ 11% **17.** a) 10π, underestimate b) 20% **19.** 31/16 **21.** 1 **23.** a) 74.65 ft/sec b) 45.28 ft/sec c) 146.59 ft **25.** a) upper = 758 gal, lower = 543 gal b) upper = 2363 gal, lower = 1693 gal c) $\approx$ 31.4 hours, $\approx$ 32.4 hours

Section 4.5, pp. 320–323

1. $\frac{6(1)}{1+1} + \frac{6(2)}{2+1} = 7$

3. $\cos(1)\pi + \cos(2)\pi + \cos(3)\pi + \cos(4)\pi = 0$

5. $\sin\pi - \sin\frac{\pi}{2} + \sin\frac{\pi}{3} = \frac{\sqrt{3} - 2}{2}$

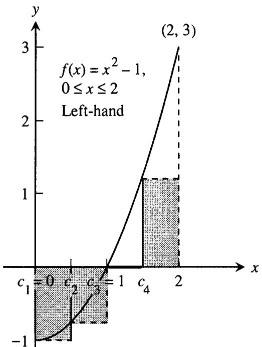
7. All of them **9.** b **11.** $\sum_{k=1}^{6} k$ **13.** $\sum_{k=1}^{4} \frac{1}{2^k}$

15. $\sum_{k=1}^{5}(-1)^{k+1}\frac{1}{k}$ **17.** a) −15 b) 1 c) 1 d) −11 e) 16

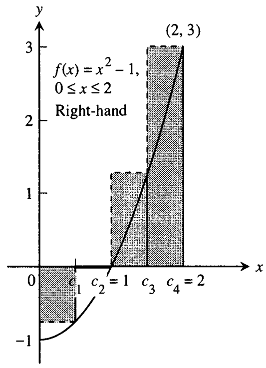
19. a) 55 b) 385 c) 3025 **21.** −56 **23.** −73 **25.** 240

27. 3376

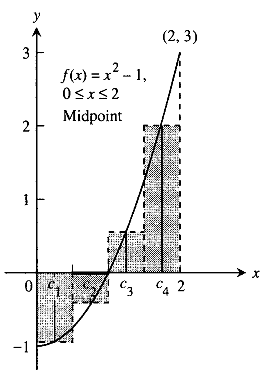
29. a)



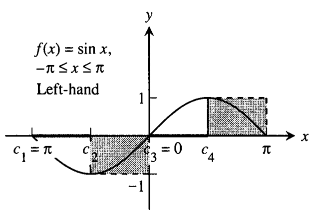
b)



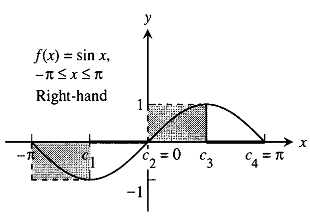
c)



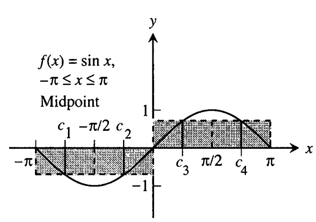
31. a)



b)



c)



33. 1.2 **35.** $\int_0^2 x^2\,dx$ **37.** $\int_{-7}^{5} (x^2 - 3x)\,dx$

39. $\int_2^3 \frac{1}{1-x}\,dx$ **41.** $\int_{-\pi/4}^{0} \sec x\,dx$ **43.** 15 **45.** -480
47. 2.75 **49.** Area = 21 square units
51. Area = $9\pi/2$ square units **53.** Area = 2.5 square units
55. Area = 3 square units **57.** $b^2/2$ **59.** $b^2 - a^2$ **61.** 1/2
63. $3\pi^2/2$ **65.** 7/3 **67.** 1/24 **69.** $3a^2/2$ **71.** $b/3$
73. Using n subintervals of length $\Delta x = b/n$ and right-endpoint values:

$$\text{Area} = \int_0^b 3x^2\,dx = b^3$$

75. Using n subintervals of length $\Delta x = b/n$ and right-endpoint values:

$$\text{Area} = \int_0^b 2x\,dx = b^2$$

77. $a = 0$ and $b = 1$ maximize the integral. **81.** $b^3/3$

Section 4.6, pp. 330–332

1. a) 0 b) -8 c) -12 d) 10 e) -2 f) 16 **3.** a) 5 b) $5\sqrt{3}$ c) -5 d) -5 **5.** a) 4 b) -4 **7.** -14 **9.** 10
11. -2 **13.** $-7/4$ **15.** 7 **17.** 0 **19.** 5 1/3 **21.** 19/3
23. a) 6 b) $7\frac{1}{3}$ **25.** a) 0 b) 8/3
27. $\text{av}(f) = 0$, assumed at $x = 1$
29. $\text{av}(f) = -2$, assumed at $x = \sqrt{3}/3$
31. $\text{av}(f) = 1$, assumed at $t = 0$ and $t = 2$
33. a) $\text{av}(g) = -1/2$, assumed at $x = \pm 1/2$ b) $\text{av}(g) = 1$, assumed at $x = 2$ c) $\text{av}(g) = 1/4$, assumed at $x = 5/4$
35. 3/2 **37.** 0 **39.** Upper bound = 1, lower bound = 1/2
47. Upper bound = 1/2 **51.** 37.5 mi/hr

Section 4.7, pp. 338–342

1. 6 **3.** 8 **5.** 1 **7.** 5/2 **9.** 2 **11.** $2\sqrt{3}$ **13.** 0
15. $-\pi/4$ **17.** $\frac{2\pi^3}{3}$ **19.** $-8/3$ **21.** $-3/4$
23. $\sqrt{2} - \sqrt[4]{8} + 1$ **25.** 16 **27.** 0 **29.** $\frac{1}{3}(2\sqrt{2} - 1)$
31. $\frac{\pi}{2} + \sin 2$ **33.** $\sqrt{2}/3$ **35.** 28/3 **37.** 1/2 **39.** 51/4
41. π **43.** $\frac{\sqrt{2}\,\pi}{2}$ **45.** $(\cos\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$ **47.** $4t^5$
49. $\sqrt{1+x^2}$ **51.** $\frac{1}{2}x^{-1/2}\sin x$ **53.** 1
55. d, since $y' = \frac{1}{x}$ and $y(\pi) = \int_\pi^\pi \frac{1}{t}\,dt - 3 = -3$
57. b, since $y' = \sec x$ and $y(0) = \int_0^0 \sec t\,dt + 4 = 4$
59. $y = \int_2^x \sec t\,dt + 3$ **61.** $s = \int_{t_0}^{t} f(x)\,dx + s_0$
63. a) 125/6 b) $h = 25/4$ d) $\frac{2}{3}bh$ **65.** a) \$9.00 b) \$10.00
67. a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(5) = f(5) = 2$ m/sec
b) $a = df/dt$ is negative since the slope of the tangent line at $t = 5$ is negative.

c) $s = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = \frac{9}{2}$ m since the integral is the area of the triangle formed by $y = f(x)$, the x-axis, and $x = 3$.
d) $t = 6$ since after $t = 6$ to $t = 9$, the region lies below the x-axis.
e) At $t = 4$ and $t = 7$, since there are horizontal tangents there.
f) Toward the origin between $t = 6$ and $t = 9$ since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$ since the velocity is positive there.
g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x-axis than below.

69. $\int_{-2}^{2} 4(9 - x^2)\,dx = 368/3$ **71.** $\int_4^8 \pi(64 - x^2)\,dx = 320\pi/3$
75. $2x - 2$ **77.** $-3x + 5$
79. a) True. Since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
b) True: g is continuous because it is differentiable.
c) True, since $g'(1) = f(1) = 0$.
d) False, since $g''(1) = f'(1) > 0$.
e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

Section 4.8, pp. 344–345

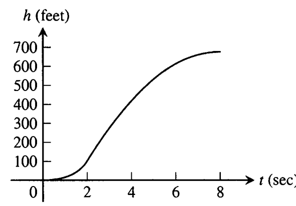
1. a) 14/3 b) 2/3 **3.** a) 1/2 b) −1/2 **5.** a) 15/16 b) 0 **7.** a) 0 b) 1/8 **9.** a) 4 b) 0 **11.** a) 1/6 b) 1/2 **13.** a) 0 b) 0 **15.** $2\sqrt{3}$ **17.** 3/4 **19.** $9^{5/4} - 1$ **21.** 3 **23.** $\pi/3$ **25.** 16/3 **27.** $2^{5/2}$ **29.** $F(6) - F(2)$ **31.** a) −3 b) 3 **33.** $I = a/2$

Section 4.9, pp. 353–356

1. I: a) 1.5, 0 b) 1.5, 0 c) 0%
II: a) 1.5, 0 b) 1.5, 0 c) 0%
3. I: a) 2.75, 0.08 b) 2.67, 0.08 c) $0.0312 \approx 3\%$
II: a) 2.67, 0 b) 2.67, 0 c) 0%
5. I: a) 6.25, 0.5 b) 6, 0.25 c) $0.0417 \approx 4\%$
II: a) 6, 0 b) 6, 0 c) 0%
7. I: a) 0.509, 0.03125 b) 0.5, 0.009 c) $0.018 \approx 2\%$
II: a) 0.5, 0.002604 b) 0.5, 0.0004 c) 0%
9. I: a) 1.8961, 0.161 b) 2, 0.1039 c) $0.052 \approx 5\%$
II: a) 2.0045, 0.0066 b) 2, 0.00454 c) 0%
11. a) 0.31929 b) 0.32812 c) 1/3, 0.01404, 0.00521
13. a) 1.95643 b) 2.00421 c) 2, 0.04357, −0.00421 **15.** a) 1 b) 2 **17.** a) 116 b) 2 **19.** a) 283 b) 2 **21.** a) 71 b) 10 **23.** a) 76 b) 12 **25.** a) 82 b) 8 **27.** 1013 **29.** ≈ 466.7 in^2 **31.** 4, 4 **33.** a) 3.11571 b) 0.02588 c) With $M = 3.11$, we get $|E_T| \le (\pi^3/1200)(3.11) < 0.081$
37. 1.08943 **39.** 0.82812

Chapter 4 Practice Exercises, pp. 357–360

1. a) about 680 ft
b)

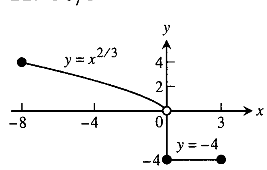


3. a) −1/2 b) 31 c) 13 d) 0
5. $\int_1^5 (2x - 1)^{-1/2}\,dx = 2$
7. $\int_{-\pi}^{0} \cos\frac{x}{2}\,dx = 2$

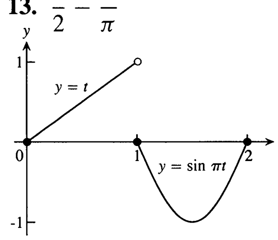
9. a) 4 b) 2 c) −2 d) -2π e) 8/5
11. 8/3 **13.** 62 **15.** $y = x - \frac{1}{x} - 1$
17. $r = 4t^{5/2} + 4t^{3/2} - 8t$ **21.** $y = \int_5^x \left(\frac{\sin t}{t}\right) dt - 3$
23. $\frac{x^4}{4} + \frac{5}{2}x^2 - 7x + C$ **25.** $2t^{3/2} - \frac{4}{t} + C$
27. $-\frac{1}{2(r^2 + 5)} + C$ **29.** $-(2 - \theta^2)^{3/2} + C$
31. $\frac{1}{3}(1 + x^4)^{3/4} + C$ **33.** $10\tan\frac{s}{10} + C$
35. $-\frac{1}{\sqrt{2}}\csc\sqrt{2}\theta + C$ **37.** $\frac{1}{2}x - \sin\frac{x}{2} + C$
39. $-4(\cos x)^{1/2} + C$ **41.** $\theta^2 + \theta + \sin(2\theta + 1) + C$
43. $\frac{t^3}{3} + \frac{4}{t} + C$ **45.** 16 **47.** 2 **49.** 1 **51.** 8
53. $27\sqrt{3}/160$ **55.** $\pi/2$ **57.** $\sqrt{3}$ **59.** $6\sqrt{3} - 2\pi$ **61.** −1 **63.** 2 **65.** −2 **67.** 1 **69.** $\sqrt{2} - 1$ **71.** a) b b) b **75.** At least 16 **77.** $T = \pi, S = \pi$ **79.** 25°F **81.** Yes **83.** $-\sqrt{1 + x^2}$ **85.** cost $\approx$ \$12, 518.10 (trapezoidal rule), no **87.** 600, \$18.00 **89.** 300, \$6.00

Chapter 4 Additional Exercises, pp. 360–364

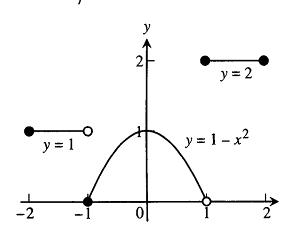
1. a) Yes b) No **5.** a) 1/4 b) $\sqrt[3]{12}$ **7.** $f(x) = \frac{x}{\sqrt{x^2 + 1}}$
9. $y = x^3 + 2x - 4$
11. 36/5



13. $\frac{1}{2} - \frac{2}{\pi}$



15. 13/3



17. $1/2$ **19.** $2/x$ **21.** $\dfrac{\sin 4y}{\sqrt{y}} - \dfrac{\sin y}{2\sqrt{y}}$ **23.** $1/6$

25. $\displaystyle\int_0^1 f(x)\,dx$

CHAPTER 5

Section 5.1, pp. 371–373

1. $\pi/2$ **3.** $1/12$ **5.** $128/15$ **7.** $5/6$ **9.** $38/3$ **11.** $49/6$
13. $32/3$ **15.** $48/5$ **17.** $8/3$ **19.** 8
21. $5/3$ (There are three intersection points.) **23.** 18 **25.** $243/8$
27. $8/3$ **29.** 2 **31.** $104/15$ **33.** $56/15$ **35.** 4
37. $\dfrac{4}{3} - \dfrac{4}{\pi}$ **39.** $\pi/2$ **41.** 2 **43.** $1/2$ **45.** 1
47. a) $(\pm\sqrt{c}, c)$ b) $c = 4^{2/3}$ c) $c = 4^{2/3}$ **49.** $11/3$ **51.** $3/4$
53. Neither

Section 5.2, pp. 377–378

1. a) $A(x) = \pi(1 - x^2)$ b) $A(x) = 4(1 - x^2)$
c) $A(x) = 2(1 - x^2)$ d) $A(x) = \sqrt{3}(1 - x^2)$ **3.** 16 **5.** $16/3$
7. a) $2\sqrt{3}$ b) 8 **9.** 8π **11.** a) $s^2 h$ b) $s^2 h$

Section 5.3, pp. 385–387

1. $2\pi/3$ **3.** $4 - \pi$ **5.** $32\pi/5$ **7.** 36π **9.** π
11. $\pi\left(\dfrac{\pi}{2} + 2\sqrt{2} - \dfrac{11}{3}\right)$ **13.** 2π **15.** 2π **17.** 3π
19. $\pi^2 - 2\pi$ **21.** $\dfrac{2\pi}{3}$ **23.** 2π **25.** $117\pi/5$ **27.** $\pi(\pi - 2)$
29. $4\pi/3$ **31.** 8π **33.** $\pi(\sqrt{3})$ **35.** $7\pi/6$ **37.** a) 8π
b) $32\pi/5$ c) $8\pi/3$ d) $224\pi/15$ **39.** a) $16\pi/15$ b) $56\pi/15$
c) $64\pi/15$ **41.** $V = 1053\pi$ cm^3 **43.** a) $c = 2/\pi$ b) $c = 0$
45. $V = 2a^2 b\pi^2$ **47.** b) $V = \dfrac{\pi r^2 h}{3}$

Section 5.4, pp. 392–393

1. 6π **3.** 2π **5.** $14\pi/3$ **7.** 8π **9.** $5\pi/6$ **11.** $128\pi/5$
13. 3π **15.** $\dfrac{16\pi}{15}(3\sqrt{2} + 5)$ **17.** $8\pi/3$ **19.** $4\pi/3$
21. $16\pi/3$ **23.** a) $6\pi/5$ b) $4\pi/5$ c) 2π d) 2π
25. a) $5\pi/3$ b) $4\pi/3$ c) 2π d) $2\pi/3$ **27.** a) $11\pi/15$
b) $97\pi/105$ c) $121\pi/210$ d) $23\pi/30$ **29.** a) $512\pi/21$
b) $832\pi/21$ **31.** a) $\pi/6$ b) $\pi/6$ **33.** $9\pi/16$ **35.** b) 4π
37. Disk: 2 integrals; washer: 2 integrals; shell: 1 integral **39.** $3x$

Section 5.5, pp. 398–400

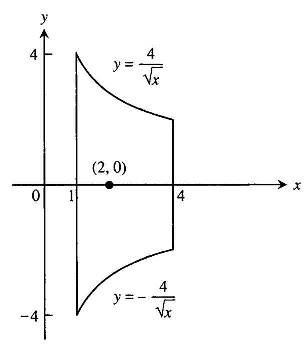
1. a) $\displaystyle\int_{-1}^{2} \sqrt{1 + 4x^2}\,dx$ c) ≈ 6.13 **3.** a) $\displaystyle\int_0^{\pi} \sqrt{1 + \cos^2 y}\,dy$
c) ≈ 3.82 **5.** a) $\displaystyle\int_{-1}^{3} \sqrt{1 + (y + 1)^2}\,dy$ c) ≈ 9.29
7. a) $\displaystyle\int_0^{\pi/6} \sec x\,dx$ c) ≈ 0.55 **9.** 12 **11.** $53/6$
13. $123/32$ **15.** $99/8$ **17.** 2
19. a) $y = \sqrt{x}$ or $y = -\sqrt{x} + 2$ b) Two **21.** 1 **23.** 21.07 in.

Section 5.6, pp. 405–407

1. a) $\displaystyle 2\pi\int_0^{\pi/4} \tan x\sqrt{1 + \sec^4 x}\,dx$ c) ≈ 3.84
3. a) $\displaystyle 2\pi\int_1^{2} \frac{1}{y}\sqrt{1 + y^{-4}}\,dy$ c) ≈ 5.02
5. a) $\displaystyle 2\pi\int_1^{4} (3 - \sqrt{x})^2\sqrt{1 + (1 - 3x^{-1/2})^2}\,dx$ c) ≈ 63.37
7. a) $\displaystyle 2\pi\int_0^{\pi/3}\left(\int_0^{y} \tan t\,dt\right)\sec y\,dy$ c) ≈ 2.08 **9.** $4\pi\sqrt{5}$
11. $3\pi\sqrt{5}$ **13.** $98\pi/81$ **15.** 2π **17.** $\pi(\sqrt{8} - 1)/9$
19. $35\pi\sqrt{5}/3$ **21.** $253\pi/20$
25. a) $\displaystyle 2\pi\int_{-\pi/2}^{\pi/2} (\cos x)\sqrt{1 + \sin^2 x}\,dx$ b) ≈ 14.4236
27. Order 226.2 liters of each color. **31.** $5\sqrt{2}\pi$ **33.** 14.4
35. 54.9

Section 5.7, pp. 416–418

1. 4 ft **3.** $(L/4,\ L/4)$ **5.** $M_0 = 8, M = 8,\ \bar{x} = 1$
7. $M_0 = 15/2, M = 9/2,\ \bar{x} = 5/3$
9. $M_0 = 73/6, M = 5,\ \bar{x} = 73/30$ **11.** $M_0 = 3, M = 3,\ \bar{x} = 1$
13. $\bar{x} = 0,\ \bar{y} = 12/5$ **15.** $\bar{x} = 1,\ \bar{y} = -3/5$
17. $\bar{x} = 16/105,\ \bar{y} = 8/15$ **19.** $\bar{x} = 0,\ \bar{y} = \pi/8$
21. $\bar{x} = 1,\ \bar{y} = -2/5$ **23.** $\bar{x} = \bar{y} = \dfrac{2}{4 - \pi}$
25. $\bar{x} = 3/2,\ \bar{y} = 1/2$ **27.** a) $\dfrac{224\pi}{3}$ b) $\bar{x} = 2,\ \bar{y} = 0$
c)



31. $\bar{x} = \bar{y} = 1/3$ **33.** $\bar{x} = a/3,\ \bar{y} = b/3$ **35.** $13\delta/6$
37. $\bar{x} = 0,\ \bar{y} = \dfrac{a\pi}{4}$

Section 5.8, pp. 424–427

1. 400 ft • lb **3.** 780 J **5.** 72,900 ft • lb **9.** 400 N/m
11. 4 cm, 0.08 J **13.** a) 7238 lb/in.
b) 905 in • lb, 2714 in • lb
15. a) 1,497,600 ft • lb b) 1 hr, 40 min
d) At 62.26 lb/ft^3: a) 1,494,240 ft • lb b) 1 hr, 40 min
At 62.59 lb/ft^3: a) 1,502,160 ft • lb b) 1 hr, 40 min
17. 38,484,510 J **19.** 7,238,229.48 ft • lb **21.** 91.32 in • oz
23. 21,446,605.9 J **25.** 967,611 ft • lb, at a cost of \$4838.05
27. 5.144×10^{10} J **31.** ≈ 85.1 ft • lb **33.** ≈ 64.6 ft • lb
35. ≈ 110.6 ft • lb

Section 5.9, pp. 432–434

1. 114,511,052 lb, 28,627,763 lb **5.** 2808 lb **7.** a) 1164.8 lb
b) 1194.7 lb **9.** a) 374.4 lb b) 7.5 in. c) No **11.** 1309 lb
13. 4.2 lb **15.** 41.6 lb **17.** a) 93.33 lb b) 3 ft **19.** 1035 ft^3
21. $wb/2$
23. No. The tank will overflow because the movable end will have moved only $3\frac{1}{3}$ ft by the time the tank is full.

Section 5.10, pp. 441–443

1. b) 20 m c) 0 m **3.** b) 6 m c) 2 m **5.** b) 245 m
c) 0 m **7.** b) 6 m c) 4 m **9.** b) $2 < t < 4$ c) 6 m
d) $\dfrac{22}{3}$ m **11.** a) Total distance = 7, displacement = 3
b) Total distance = 19.5, displacement = −4.5 **13.** About 65%
15. $\sqrt{3}\pi$ **17.** a) 210 ft^3 b) 13,440 lb
19. $V = 32\pi,\ S = 32\sqrt{2}\pi$ **21.** $4\pi^2$ **23.** $\bar{x} = 0, \bar{y} = \dfrac{2a}{\pi}$
25. $\bar{x} = 0, \bar{y} = \dfrac{4b}{3\pi}$ **27.** $\sqrt{2}\pi a^3(4 + 3\pi)/6$ **29.** $\dfrac{2a^3}{3}$

Chapter 5 Practice Exercises, pp. 444–447

1. 1 **3.** 1/6 **5.** 18 **7.** 9/8 **9.** $\dfrac{\pi^2}{32} + \dfrac{\sqrt{2}}{2} - 1$ **11.** 4
13. $\dfrac{8\sqrt{2} - 7}{6}$ **15.** Min: −4, max: 0, area: 27/4 **17.** 6/5
19. $9\pi/280$ **21.** π^2 **23.** $72\pi/35$ **25.** a) 2π b) π
c) $12\pi/5$ d) $26\pi/5$ **27.** a) 8π b) $1088\pi/15$ c) $512\pi/15$
29. $\pi(3\sqrt{3} - \pi)/3$ **31.** a) $16\pi/15$ b) $8\pi/5$ c) $8\pi/3$
d) $32\pi/5$ **33.** $28\pi/3$ **35.** 10/3 **37.** 285/8
39. $28\pi\sqrt{2}/3$ **41.** 4π **43.** $\bar{x} = 0, \bar{y} = 8/5$
45. $\bar{x} = 3/2, \bar{y} = 12/5$ **47.** $\bar{x} = 9/5, \bar{y} = 11/10$ **49.** 4640 J
51. 10 ft • lb, 30 ft • lb **53.** 418,208.81 ft • lb
55. $22,500\pi$ ft • lb, 257 sec **57.** 332.8 lb **59.** 2196.48 lb
61. $216w_1 + 360w_2$ **63.** a) 64/3 m b) 0 m
65. a) 15 m b) −5 m

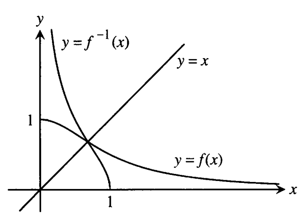
Chapter 5 Additional Exercises, pp. 447–448

1. $f(x) = \sqrt{\dfrac{2x - a}{\pi}}$ **3.** $f(x) = \sqrt{C^2 - 1}\, x + a$, where $C \geq 1$
5. $\bar{x} = 0, \bar{y} = \dfrac{n}{2n + 1}$, (0, 1/2)
9. a) $\bar{x} = \bar{y} = 4(a^2 + ab + b^2)/(3\pi(a + b))$ b) $(2a/\pi, 2a/\pi)$
11. 28/3 **13.** $\dfrac{4h\sqrt{3mh}}{3}$ **15.** $\approx 2{,}329.6$ lb
17. a) $2h/3$ b) $(6a^2 + 8ah + 3h^2)/(6a + 4h)$

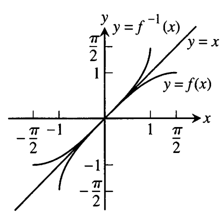
CHAPTER 6

Section 6.1, pp. 454–457

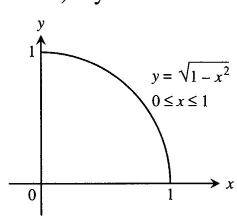
1. One-to-one **3.** Not one-to-one **5.** One-to-one
7. D: (0, 1] R: [0, ∞)



9. D: [−1, 1] R: $[-\pi/2, \pi/2]$

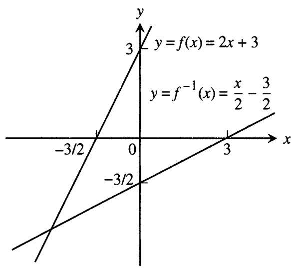


11. a) Symmetric about the line $y = x$



13. $f^{-1}(x) = \sqrt{x - 1}$ **15.** $f^{-1}(x) = \sqrt[3]{x + 1}$
17. $f^{-1}(x) = \sqrt{x} - 1$
19. $f^{-1}(x) = \sqrt[5]{x}$; domain: $-\infty < x < \infty$, range: $-\infty < y < \infty$
21. $f^{-1}(x) = \sqrt[3]{x - 1}$; domain: $-\infty < x < \infty$, range: $-\infty < y < \infty$
23. $f^{-1}(x) = \dfrac{1}{\sqrt{x}}$; domain: $x > 0$, range: $y > 0$
25. a) $f^{-1}(x) = \dfrac{x}{2} - \dfrac{3}{2}$

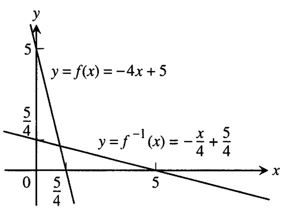
b)



c) 2, 1/2

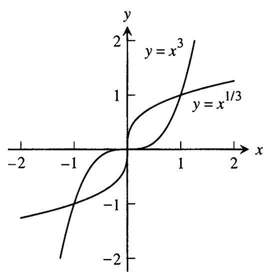
27. a) $f^{-1}(x) = -\frac{x}{4} + \frac{5}{4}$

b)



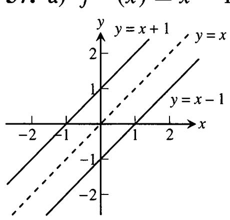
c) −4, −1/4

29. b)



c) Slope of f at (1, 1) : 3, slope of g at (1, 1): 1/3, slope of f at (−1, −1) : 3, slope of g at (−1, −1) : 1/3
d) $y = 0$ is tangent to $y = x^3$ at $x = 0$; $x = 0$ is tangent to $y = \sqrt[3]{x}$ at $x = 0$

31. 1/9 **33.** 3 **35.** a) $f^{-1}(x) = \frac{1}{m}x$
b) The graph of f^{-1} is the line through the origin with slope $1/m$.
37. a) $f^{-1}(x) = x - 1$



b) $f^{-1}(x) = x - b$. The graph of f^{-1} is a line parallel to the graph of f. The graphs of f and f^{-1} lie on opposite sides of the line $y = x$ and are equidistant from that line.
c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.

41. Increasing, therefore one-to-one; $df^{-1}/dx = \frac{1}{9}x^{-2/3}$

43. Decreasing, therefore one-to-one; $df^{-1}/dx = -\frac{1}{3}x^{-2/3}$

Section 6.2, pp. 465–467

1. a) $\ln 3 - 2\ln 2$ b) $2(\ln 2 - \ln 3)$ c) $-\ln 2$, d) $\frac{2}{3}\ln 3$

e) $\ln 3 + \frac{1}{2}\ln 2$ f) $\frac{1}{2}(3\ln 3 - \ln 2)$ **3.** a) $\ln 5$ b) $\ln(x-3)$
c) $\ln(t^2)$ **5.** $1/x$ **7.** $2/t$ **9.** $-1/x$ **11.** $\frac{1}{\theta + 1}$

13. $3/x$ **15.** $2(\ln t) + (\ln t)^2$ **17.** $x^3 \ln x$ **19.** $\frac{1 - \ln t}{t^2}$

21. $\frac{1}{x(1 + \ln x)^2}$ **23.** $\frac{1}{x \ln x}$ **25.** $2\cos(\ln\theta)$

27. $-\frac{3x+2}{2x(x+1)}$ **29.** $\frac{2}{t(1-\ln t)^2}$ **31.** $\frac{\tan(\ln\theta)}{\theta}$

33. $\frac{10x}{x^2+1} + \frac{1}{2(1-x)}$ **35.** $2x\ln|x| - x\ln\frac{|x|}{\sqrt{2}}$

37. $\left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{2x+1}{2\sqrt{x(x+1)}}$

39. $\left(\frac{1}{2}\right)\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2\sqrt{t}(t+1)^{3/2}}$

41. $\sqrt{\theta+3}\,(\sin\theta)\left(\frac{1}{2(\theta+3)} + \cot\theta\right)$

43. $t(t+1)(t+2)\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right] = 3t^2 + 6t + 2$

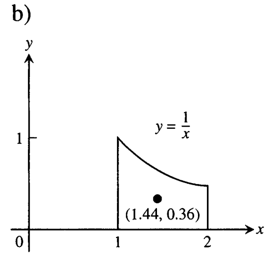
45. $\frac{\theta+5}{\theta\cos\theta}\left[\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right]$

47. $\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

49. $\frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$ **51.** $\ln\left(\frac{2}{3}\right)$

53. $\ln|y^2 - 25| + C$ **55.** $\ln 3$ **57.** $(\ln 2)^2$ **59.** $\frac{1}{\ln 4}$

61. $\ln|6 + 3\tan t| + C$ **63.** $\ln 2$ **65.** $\ln 27$
67. $\ln(1 + \sqrt{x}) + C$
69. a) Max = 0 at $x = 0$, min = $-\ln 2$ at $x = \pi/3$
b) Max = 1 at $x = 1$, min = $\cos(\ln 2)$ at $x = 1/2$ and $x = 2$
71. $\ln 16$ **73.** $4\pi\ln 4$ **75.** $\pi\ln 16$
77. a) $6 + \ln 2$ b) $8 + \ln 9$
79. a) $\bar{x} \approx 1.44, \bar{y} \approx 0.36$
b)



81. $y = x + \ln|x| + 2$ **83.** b) 0.00469 **85.** 2

Section 6.3, pp. 472–474

1. a) 7.2 b) $\frac{1}{x^2}$ c) $\frac{x}{y}$ **3.** a) 1 b) 1 c) $-x^2 - y^2$

5. e^{2t+4} **7.** $e^{5t}+40$ **9.** $y=2xe^x+1$ **11.** a) $k=\ln 2$ b) $k=(1/10)\ln 2$ c) $k=1000\ln a$ **13.** a) $t=-10\ln 3$ b) $t=-\dfrac{\ln 2}{k}$ c) $t=\dfrac{\ln .4}{\ln .2}$ **15.** $4(\ln x)^2$ **17.** $-5e^{-5x}$
19. $-7e^{(5-7x)}$ **21.** xe^x **23.** x^2e^x **25.** $2e^{\theta}\cos\theta$
27. $2\theta e^{-\theta^2}\sin(e^{-\theta^2})$ **29.** $\dfrac{1-t}{t}$ **31.** $1/(1+e^{\theta})$
33. $e^{\cos t}(1-t\sin t)$ **35.** $(\sin x)/x$ **37.** $\dfrac{ye^y\cos x}{1-ye^y\sin x}$
39. $\dfrac{2e^{2x}-\cos(x+3y)}{3\cos(x+3y)}$ **41.** $\frac{1}{3}e^{3x}-5e^{-x}+C$ **43.** 1
45. $8e^{(x+1)}+C$ **47.** 2 **49.** $2e^{\sqrt{r}}+C$ **51.** $-e^{-t^2}+C$
53. $-e^{1/x}+C$ **55.** e **57.** $\dfrac{1}{\pi}e^{\sec\pi t}+C$ **59.** 1
61. $\ln(1+e^r)+C$ **63.** $y=1-\cos(e^t-2)$
65. $y=2(e^{-x}+x)-1$
67. Maximum: 1 at $x=0$, minimum: $2-2\ln 2$ at $x=\ln 2$
69. Abs max of $1/(2e)$ assumed at $x=1/\sqrt{e}$ **71.** 2
73. $y=e^{x/2}-1$
75. a) $\dfrac{d}{dx}(x\ln x-x+C)=x\cdot\dfrac{1}{x}+\ln x-1+0=\ln x$
b) $\dfrac{1}{e-1}$ **77.** b) $|\text{error}|\approx 0.02140$ **79.** 2.71828183

Section 6.4, pp. 480–482

1. a) 7 b) $\sqrt{2}$ c) 75 d) 2 e) 0.5 f) -1 **3.** a) $\sqrt{x}$ b) x^2 c) $\sin x$ **5.** a) $\dfrac{\ln 3}{\ln 2}$ b) 3 c) 2 **7.** $x=12$
9. $x=3$ or $x=2$ **11.** $2^x\ln x$ **13.** $\left(\dfrac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$ **15.** $\pi x^{(\pi-1)}$
17. $-\sqrt{2}\cos\theta^{(\sqrt{2}-1)}\sin\theta$ **19.** $7^{\sec\theta}(\ln 7)^2(\sec\theta\tan\theta)$
21. $(3\cos 3t)(2^{\sin 3t})\ln 2$ **23.** $\dfrac{1}{\theta\ln 2}$ **25.** $\dfrac{3}{x\ln 4}$
27. $\dfrac{2(\ln r)}{r(\ln 2)(\ln 4)}$ **29.** $\dfrac{-2}{(x+1)(x-1)}$
31. $\sin(\log_7\theta)+\dfrac{1}{\ln 7}\cos(\log_7\theta)$ **33.** $\dfrac{1}{\ln 5}$ **35.** $\dfrac{1}{t}(\log_2 3)3^{\log_2 t}$
37. $\dfrac{1}{t}$ **39.** $(x+1)^x\left(\dfrac{x}{x+1}+\ln(x+1)\right)$
41. $(\sqrt{t})^t\left(\dfrac{\ln t}{2}+\dfrac{1}{2}\right)$ **43.** $(\sin x)^x(\ln\sin x+x\cot x)$
45. $\left(x^{\ln x}\right)\left(\dfrac{\ln x^2}{x}\right)$ **47.** $\dfrac{5^x}{\ln 5}+C$ **49.** $\dfrac{1}{2\ln 2}$ **51.** $\dfrac{1}{\ln 2}$
53. $\dfrac{6}{\ln 7}$ **55.** 32760 **57.** $\dfrac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1}+C$ **59.** $3^{\sqrt{2}+1}$
61. $\dfrac{1}{\ln 10}\left(\dfrac{(\ln x)^2}{2}\right)+C$ **63.** $2(\ln 2)^2$ **65.** $\dfrac{3\ln 2}{2}$
67. $\ln 10$ **69.** $(\ln 10)\ln|\ln x|+C$ **71.** $\ln(\ln x)$, $x>1$
73. $-\ln x$ **75.** $2\ln 5$ **77.** $\left[10^{-7.44}, 10^{-7.37}\right]$ **79.** $k=10$
81. a) 10^{-7} b) 7 c) 1 : 1 **83.** $x\approx -0.76666$
85. a) $L(x)=1+(\ln 2)x\approx 0.69x+1$
87. a) 1.89279 b) -0.35621 c) 0.94575 d) -2.80735 e) 5.29595 f) 0.97041 g) -1.03972 h) -1.61181

Section 6.5, pp. 488–491

1. a) -0.00001 b) 10,536 years c) 82% **3.** 54.88 g
5. 59.8 ft **7.** 2.8147497×10^{14} **9.** a) 8 years b) 32.02 years
11. 15.28 years **13.** a) $A_0e^{0.2}$ b) 17.33 years c) 27.47 years
15. 4.50% **17.** 0.585 days **21.** a) 17.5 min. b) 13.26 min.
23. -3°C **25.** About 6658 years **27.** 41 years old

Section 6.6, pp. 496–498

1. 1/4 **3.** $-23/7$ **5.** 5/7 **7.** 0 **9.** -16 **11.** -2
13. 1/4 **15.** 2 **17.** 3 **19.** -1 **21.** $\ln 3$ **23.** $\dfrac{1}{\ln 2}$
25. $\ln 2$ **27.** 1 **29.** 1/2 **31.** $\ln 2$ **33.** 0 **35.** $-1/2$
37. $\ln 2$ **39.** -1 **41.** 1 **43.** $1/e$ **45.** 1 **47.** $1/e$
49. $e^{1/2}$ **51.** 1 **53.** 3 **55.** 1 **57.** (b) is correct.
59. (d) is correct. **61.** $c=\dfrac{27}{10}$

Section 6.7, pp. 503–504

1. a) Slower b) slower c) slower d) faster e) slower f) slower g) same h) slower
3. a) Same b) faster c) same d) same e) slower f) faster g) slower h) same
5. a) Same b) same c) same d) faster e) faster f) same g) slower h) faster **7.** d, a, c, b
9. a) False b) false c) true d) true e) true f) true g) false h) true
13. When the degree of f is less than or equal to the degree of g.
15. Polynomials of a greater degree grow at a greater rate than polynomials of a lesser degree. Polynomials of the same degree grow at the same rate.
21. b) $\ln\left(e^{17000000}\right)=17,000,000<\left(e^{17\times 10^6}\right)^{1/10^6}=e^{17}\approx 24,154,952.75$
c) $x\approx 3.4306311\times 10^{15}$ d) They cross at $x\approx 3.4306311\times 10^{15}$
23. a) The algorithm that takes O $(n\log_2 n)$ steps
25. It could take one million for a sequential search; at most 20 steps for a binary search.

Section 6.8, pp. 510–513

1. a) $\pi/4$ b) $-\pi/3$ c) $\pi/6$ **3.** a) $-\pi/6$ b) $\pi/4$ c) $-\pi/3$
5. a) $\pi/3$ b) $3\pi/4$ c) $\pi/6$ **7.** a) $3\pi/4$ b) $\pi/6$ c) $2\pi/3$
9. a) $\pi/4$ b) $-\pi/3$ c) $\pi/6$ **11.** a) $3\pi/4$ b) $\pi/6$ c) $2\pi/3$
13. $\cos\alpha=\dfrac{12}{13}$, $\tan\alpha=\dfrac{5}{12}$, $\sec\alpha=\dfrac{13}{12}$, $\csc\alpha=\dfrac{13}{5}$, $\cot\alpha=\dfrac{12}{5}$

15. $\sin\alpha = \frac{2}{\sqrt{5}}$, $\cos\alpha = -\frac{1}{\sqrt{5}}$, $\tan\alpha = -2$, $\csc\alpha = \frac{\sqrt{5}}{2}$, $\cot\alpha = -\frac{1}{2}$

17. $1/\sqrt{2}$ **19.** $-1/\sqrt{3}$ **21.** $\frac{4+\sqrt{3}}{2\sqrt{3}}$ **23.** 1 **25.** $-\sqrt{2}$

27. $\pi/6$ **29.** $\frac{\sqrt{x^2+4}}{2}$ **31.** $\sqrt{9y^2-1}$ **33.** $\sqrt{1-x^2}$

35. $\frac{\sqrt{x^2-2x}}{x-1}$ **37.** $\frac{\sqrt{9-4y^2}}{3}$ **39.** $\frac{\sqrt{x^2-16}}{x}$ **41.** $\pi/2$

43. $\pi/2$ **45.** $\pi/2$ **47.** 0 **51.** $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$

57. a) Defined; there is an angle whose tangent is 2.
b) Not defined; there is no angle whose cosine is 2.
59. a) Not defined; no angle has secant 0.
b) Not defined; no angle has sine $\sqrt{2}$.
61. a) 0.84107 b) -0.72973 c) 0.46365
63. a) Domain; all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer; range: $-\pi/2 < y < \pi/2$.
b) Domain: $-\infty < x < \infty$; range: $-\infty < y < \infty$
65. a) Domain; $-\infty < x < \infty$; range: $0 \le y \le \pi$
b) Domain: $-1 \le x \le 1$; range: $-1 \le y \le 1$
67. The graphs are identical.

Section 6.9, pp. 518–520

1. $\frac{-2x}{\sqrt{1-x^4}}$ **3.** $\frac{\sqrt{2}}{\sqrt{1-2t^2}}$ **5.** $\frac{1}{|2s+1|\sqrt{s^2+s}}$

7. $\frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$ **9.** $\frac{-1}{\sqrt{1-t^2}}$ **11.** $\frac{-1}{2\sqrt{t}(1+t)}$

13. $\frac{1}{\tan^{-1}(x(1+x^2))}$ **15.** $\frac{-e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

17. $\frac{-2s^2}{\sqrt{1-s^2}}$ **19.** 0 **21.** $\sin^{-1} x$ **23.** $\sin^{-1}\frac{x}{7} + C$

25. $\frac{1}{\sqrt{17}}\tan^{-1}\frac{x}{\sqrt{17}} + C$ **27.** $\frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$ **29.** $2\pi/3$

31. $\pi/16$ **33.** $-\pi/12$ **35.** $\frac{3}{2}\sin^{-1} 2(r-1) + C$

37. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$ **39.** $\frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + C$

41. π **43.** $\pi/12$ **45.** $\frac{1}{2}\sin^{-1} y^2 + C$ **47.** $\sin^{-1}(x-2) + C$

49. π **51.** $\frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) + C$ **53.** 2π

55. $\sec^{-1}|x+1| + C$ **57.** $e^{\sin^{-1} x} + C$ **59.** $\frac{1}{3}(\sin^{-1} x)^3 + C$

61. $\ln|\tan^{-1} y| + C$ **63.** $\sqrt{3}-1$ **65.** 5 **67.** 2

73. $y = \sin^{-1}(x)$ **75.** $y = \sec^{-1}(x) + \frac{2\pi}{3}, x > 1$ **77.** $3\sqrt{5}$ ft.

79. Yes, $\sin^{-1}(x)$ and $\cos^{-1}(x)$ differ by the constant $\pi/2$.
89. $\pi^2/2$ **91.** a) $\pi^2/2$ b) 2π

Section 6.10, pp. 525–529

1. $\cosh x = 5/4$, $\tanh x = -3/5$, $\coth x = -5/3$, $\operatorname{sech} x = 4/5$, $\operatorname{csch} x = -4/3$
3. $\sinh x = 8/15$, $\tanh x = 8/17$, $\coth x = 17/8$, $\operatorname{sech} x = 15/17$, $\operatorname{csch} x = 15/8$

5. $x + \frac{1}{x}$ **7.** e^{5x} **9.** e^{4x} **13.** $2\cosh\frac{x}{3}$

15. $\operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$ **17.** $\coth z$

19. $(\ln \operatorname{sech}\theta)(\operatorname{sech}\theta \tanh\theta)$ **21.** $\tanh^3 v$ **23.** 2

25. $\frac{1}{2\sqrt{x(1+x)}}$ **27.** $\frac{1}{1+\theta} - \tanh^{-1}\theta$ **29.** $\frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$

31. $-\operatorname{sech}^{-1} x$ **33.** $\frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$ **35.** $|\sec x|$

41. $\frac{\cosh 2x}{2} + C$ **43.** $12\sinh\left(\frac{x}{2} - \ln 3\right) + C$

45. $7\ln\left|e^{x/7} + e^{-x/7}\right| + C$ **47.** $\tanh\left(x - \frac{1}{2}\right) + C$

49. $-2\operatorname{sech}\sqrt{t} + C$ **51.** $\ln\frac{5}{2}$ **53.** $\frac{3}{32} + \ln 2$ **55.** $e - e^{-1}$

57. 3/4 **59.** $\frac{3}{8} + \ln\sqrt{2}$ **61.** $\ln(2/3)$ **63.** $\frac{-\ln 3}{2}$ **65.** $\ln 3$

67. a) $\sinh^{-1}(\sqrt{3})$ b) $\ln(\sqrt{3}+2)$

69. a) $\coth^{-1}(2) - \coth^{-1}(5/4)$ b) $\left(\frac{1}{2}\right)\ln\left(\frac{1}{3}\right)$

71. a) $-\operatorname{sech}^{-1}\left(\frac{12}{13}\right) + \operatorname{sech}^{-1}\left(\frac{4}{5}\right)$

b) $-\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right)$

$= -\ln\left(\frac{3}{2}\right) + \ln(2) = \ln(4/3)$

73. a) 0 b) 0

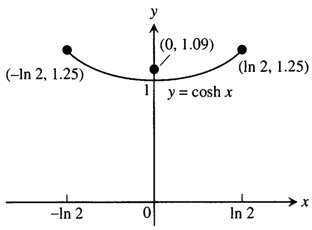
75. b) i) $f(x) = \frac{2f(x)}{2} + 0 = f(x)$, ii) $f(x) = 0 + \frac{2f(x)}{2} = f(x)$

77. b) $\sqrt{\frac{mg}{k}}$ c) $80\sqrt{5} \approx 178.89$ ft/sec

79. $y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2}$ **81.** 2π **83.** a) $\frac{6}{5}$ b) $\frac{\sinh ab}{a}$

85. a) $\bar{x} = 0$, $\bar{y} = \frac{5}{8} + \frac{\ln 4}{3} \approx 1.09$

b)



89. c) $a \approx 0.0417525$ d) ≈ 47.90 lb

Section 6.11, pp. 537–540

9. $y = \tan(x^2 + C)$ **11.** $\frac{2}{3}y^{3/2} - x^{1/2} = C$ **13.** $e^y - e^x = C$

15. $y = \frac{e^x + C}{x}$ **17.** $y = \frac{C - \cos x}{x^3}, x > 0$

19. $y = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}, x > 0$ **21.** $y = \frac{1}{2}xe^{x/2} + Ce^{x/2}$

23. $y = x^2e^{-2x} + Ce^{-2x}$ **25.** $-e^{-y} - e^{\sin x} = C$

27. $s = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$ **29.** $2\tan\sqrt{x} = t + C$

31. $r = \csc\theta\,(\ln|\sec\theta| + C)$ **33.** $y = -e^{-x}\operatorname{sech} x + C\operatorname{sech} x$

35. $y = \frac{3}{2} - \frac{1}{2}e^{-2t}$ **37.** $y = -\frac{1}{\theta}\cos\theta + \frac{\pi}{2\theta}$

39. $y = 6e^{x^2} - \frac{e^{x^2}}{x+1}$ **41.** $y = y_0e^{kt}$

43. (b) is correct, but (a) is not.

45. a) $c = \frac{G}{100Vk} + \left(c_0 - \frac{G}{100Vk}\right)e^{-kt}$ b) $\frac{G}{100Vk}$

47. 1 hour **49.** a) 550 ft b) $25\ln 22 \approx 77$ sec

51. $t = \frac{L}{R}\ln 2$ seconds

53. a) $i = \frac{V}{R} - \frac{V}{R}e^{-3} = \frac{V}{R}(1 - e^{-3}) \approx 0.95\frac{V}{R}$ amp b) 86%

55. a) 10 lb/min b) $100 + t$ gal c) $4\left(\frac{y}{100+t}\right)$ lb/min

d) $\frac{dy}{dt} = 10 - \frac{4y}{100+t}$, $y(0) = 50$, $y = 2(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^4}$

e) concentration $= \frac{y(25)}{\text{amt. brine in tank}} = \frac{188.6}{125} \approx 1.5$ lb/gal

57. $y(27.8) \approx 14.8$ lb, $t \approx 27.8$ min

Section 6.12, pp. 545–546

1. y (exact) $= \frac{x}{2} - \frac{4}{x}$, $y_1 = -0.25$, $y_2 = 0.3$, $y_3 = 0.75$

3. y (exact) $= 3e^{x(x+2)}$, $y_1 = 4.2$, $y_2 = 6.216$, $y_3 = 9.697$

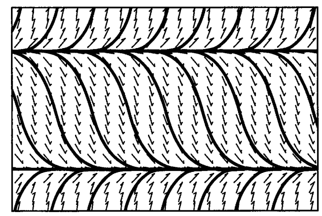
5. y (exact) $= e^{x^2} + 1$, $y_1 = 2.0$, $y_2 = 2.0202$, $y_3 = 2.0618$

7. $y \approx 2.48832$, exact value is e

9. $y \approx -0.2272$, exact value is $1/(1 - 2\sqrt{5}) \approx -0.2880$

11. b **13.** a

15.



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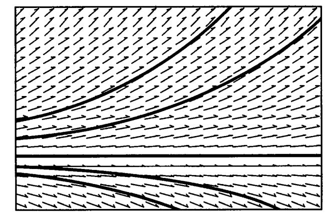

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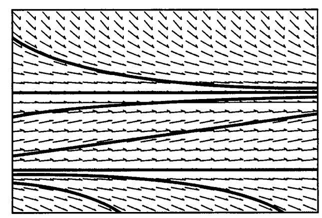


Chapter 6 Practice Exercises, pp. 548–551

1. $-2e^{-x/5}$ **3.** xe^{4x} **5.** $\frac{2\sin\theta\cos\theta}{\sin^2\theta} = 2\cot\theta$ **7.** $\frac{2}{(\ln 2)x}$

9. $-8^{-t}(\ln 8)$ **11.** $18x^{2.6}$ **13.** $(x+2)^{x+2}(\ln(x+2) + 1)$

15. $-\frac{1}{\sqrt{1-u^2}}$ **17.** $\frac{-1}{\sqrt{1-x^2}\cos^{-1}x}$

19. $\tan^{-1}(t) + \frac{t}{1+t^2} - \frac{1}{2t}$ **21.** $\frac{1-z}{\sqrt{z^2-1}} + \sec^{-1}z$ **23.** -1

25. $\frac{2(x^2+1)}{\sqrt{\cos 2x}}\left[\frac{2x}{x^2+1} + \tan 2x\right]$

27. $5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5\left[\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t-3}\right]$

29. $\frac{1}{\sqrt{\theta}}(\sin\theta)^{\sqrt{\theta}}(\ln\sqrt{\sin\theta} + \theta\cot\theta)$ **31.** $-\cos e^x + C$

33. $\tan(e^x - 7) + C$ **35.** $e^{\tan x} + C$ **37.** $\frac{-\ln 7}{3}$ **39.** $\ln 8$

41. $\ln(9/25)$ **43.** $-[\ln|\cos(\ln v)|] + C$ **45.** $-\frac{1}{2}(\ln x)^{-2} + C$

47. $-\cot(1 + \ln r) + C$ **49.** $\frac{1}{2\ln 3}\left(3^{x^2}\right) + C$ **51.** $3\ln 7$

53. $15/16 + \ln 2$ **55.** $e - 1$ **57.** $1/6$ **59.** $9/14$

61. $\frac{1}{3}[(\ln 4)^3 - (\ln 2)^3]$ or $\frac{7}{3}(\ln 2)^3$ **63.** $\frac{9\ln 2}{4}$ **65.** π

67. $\pi/\sqrt{3}$ **69.** $\sec^{-1}2y + C$ **71.** $\pi/12$

73. $\sin^{-1}(x+1) + C$ **75.** $\pi/2$ **77.** $\frac{1}{3}\sec^{-1}\left(\frac{t+1}{3}\right) + C$

79. $y = \frac{\ln 2}{\ln(3/2)}$ **81.** $y = \ln x - \ln 3$ **83.** $y = \frac{1}{1 - e^x}$

85. $\ln 10$ **87.** $\ln 2$ **89.** 5 **91.** $-\infty$ **93.** 1 **95.** e^3

97. a) Same rate b) same rate c) faster d) faster e) same rate f) same rate **99.** a) True b) false c) false d) true e) true f) true **101.** 1/3
103. Absolute maximum $= 0$ at $x = e/2$, absolute minimum $= -0.5$ at $x = 0.5$
105. 1 **107.** $1/e$ m/sec **109.** $1/\sqrt{2}$ units long by $1/\sqrt{e}$ units high, $A = 1/\sqrt{2e} \approx 0.43$ units^2
111. $\ln 5x - \ln 3x = \ln(5/3)$ **113.** 1/2
115. a) Absolute maximum of $2/e$ at $x = e^2$, inflection point $(e^{8/3}, (8/3)e^{-4/3})$, concave up on $(e^{8/3}, \infty)$, concave down on $(0, e^{8/3})$
b) Absolute maximum of 1 at $x = 0$, inflection points $(\pm 1/\sqrt{2}, 1/\sqrt{e})$, concave up on $(-\infty, -1/\sqrt{2}) \cup (1/\sqrt{2}, \infty)$, concave down on $(-1/\sqrt{2}, 1/\sqrt{2})$
c) Absolute maximum of 1 at $x = 0$, inflection point $(1, 2/e)$, concave up on $(1, \infty)$, concave down on $(-\infty, 1)$
117. 18,935 years **119.** $20(5 - \sqrt{17})$ m
121. $y = \ln(-e^{-x-2} + 2e^{-2})$
123. $y = \dfrac{1}{(x+1)^2} \cdot \left(\dfrac{x^3}{3} + \dfrac{x^2}{2} + 1\right)$
125. y (exact) $= \dfrac{1}{2}x^2 - \dfrac{3}{2}$; $y \approx 0.4$; exact value is $1/2$
127. y (exact) $= -e^{(x^2-1)/2}$; $y \approx -3.4192$; exact value is $-e^{3/2} \approx -4.4817$

Chapter 6 Additional Exercises, pp. 551–553

1. $\pi/2$ **3.** $1/\sqrt{e}$ **5.** $\ln 2$ **7.** a) 1 b) $\pi/2$ c) π
9. $a = 2,\ b = -2$ **11.** $\dfrac{1}{\ln 2}, \dfrac{1}{2\ln 2}$, 2 : 1 **13.** $x = 2$
15. 2/17 **23.** $\bar{x} = \dfrac{\ln 4}{\pi},\ \bar{y} = 0$ **27.** b) 61°
29. a) $c - (c - y_0)e^{-(kA/V)t}$ b) c

CHAPTER 7

Section 7.1, pp. 560–561

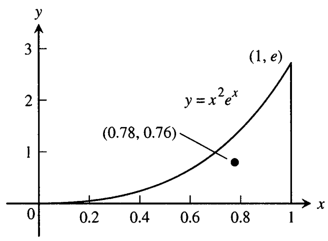
1. $2\sqrt{8x^2+1} + C$ **3.** $2(\sin v)^{3/2} + C$ **5.** $\ln 5$
7. $2\ln(\sqrt{x}+1) + C$ **9.** $-\dfrac{1}{7}\ln|\sin(3-7x)| + C$
11. $-\ln|\csc(e^\theta+1) + \cot(e^\theta+1)| + C$
13. $3\ln\left|\sec\dfrac{t}{3} + \tan\dfrac{t}{3}\right| + C$
15. $-\ln|\csc(s-\pi) + \cot(s-\pi)| + C$ **17.** 1 **19.** $e^{\tan v} + C$
21. $\dfrac{3^{(x+1)}}{\ln 3} + C$ **23.** $\dfrac{2^{\sqrt{w}}}{\ln 2} + C$ **25.** $3\tan^{-1}3u + C$
27. $\pi/18$ **29.** $\sin^{-1}s^2 + C$ **31.** $6\sec^{-1}|5x| + C$
33. $\tan^{-1}e^x + C$ **35.** $\ln(2+\sqrt{3})$ **37.** 2π
39. $\sin^{-1}(t-2) + C$
41. $\sec^{-1}|x+1| + C$, when $|x+1| > 1$
43. $\tan x - 2\ln|\csc x + \cot x| - \cot x - x + C$
45. $x + \sin 2x + C$ **47.** $x - \ln|x+1| + C$ **49.** $7 + \ln 8$
51. $2t^2 - t + 2\tan^{-1}\left(\dfrac{t}{2}\right) + C$ **53.** $\sin^{-1}x + \sqrt{1-x^2} + C$
55. $\sqrt{2}$ **57.** $\tan x - \sec x + C$ **59.** $\ln|1+\sin\theta| + C$
61. $\cot x + x + \csc x + C$ **63.** 4 **65.** $\sqrt{2}$ **67.** 2
69. $\ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|$ **71.** $4 - \dfrac{\pi}{2}$
73. $-\ln|\csc(\sin\theta) + \cot(\sin\theta)| + C$
75. $\ln|\sin x| + \ln|\cos x| + C$ **77.** $12\tan^{-1}(\sqrt{y}) + C$
79. $\sec^{-1}\left|\dfrac{x-1}{7}\right| + C$ **81.** $\ln|\sec(\tan t)| + C$
83. a) $\sin\theta - \dfrac{1}{3}\sin^3\theta + C$ b) $\sin\theta - \dfrac{2}{3}\sin^3\theta + \dfrac{1}{5}\sin^5\theta + C$
c) $\displaystyle\int \cos^9\theta\,d\theta = \int \cos^8\theta(\cos\theta)\,d\theta = \int (1-\sin^2\theta)^4(\cos\theta)\,d\theta$
85. a) $\displaystyle\int \tan^3\theta\,d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta\,d\theta = \frac{1}{2}\tan^2\theta +$ $\ln|\cos\theta| + C$
b) $\displaystyle\int \tan^5\theta\,d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\,d\theta$
c) $\displaystyle\int \tan^7\theta\,d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\,d\theta$
d) $\displaystyle\int \tan^{2k+1}\theta\,d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta\,d\theta$
87. $2\sqrt{2} - \ln(3+2\sqrt{2})$ **89.** π^2 **91.** $\ln(2+\sqrt{3})$
93. $\bar{x} = 0, \bar{y} = \dfrac{1}{\ln(2\sqrt{2}+3)}$

Section 7.2, pp. 567–569

1. $-2x\cos(x/2) + 4\sin(x/2) + C$
3. $t^2\sin t + 2t\cos t - 2\sin t + C$ **5.** $\ln 4 - \dfrac{3}{4}$
7. $y\tan^{-1}(y) - \ln\sqrt{1+y^2} + C$ **9.** $x\tan x + \ln|\cos x| + C$
11. $(x^3 - 3x^2 + 6x - 6)e^x + C$ **13.** $(x^2 - 7x + 7)e^x + C$
15. $(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$ **17.** $\dfrac{\pi^2 - 4}{8}$
19. $\dfrac{5\pi - 3\sqrt{3}}{9}$ **21.** $\dfrac{1}{2}(-e^\theta\cos\theta + e^\theta\sin\theta) + C$
23. $\dfrac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$
25. $\dfrac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$ **27.** $\dfrac{\pi\sqrt{3}}{3} - \ln(2) - \dfrac{\pi^2}{18}$
29. $\dfrac{1}{2}[-x\cos(\ln x) + x\sin(\ln x)] + C$ **31.** a) π b) 3π c) 5π d) $(2n+1)\pi$ **33.** $2\pi(1 - \ln 2)$ **35.** a) $\pi(\pi - 2)$ b) 2π

37. a) $\bar{x} = \dfrac{6-2e}{e-2} \approx 0.78$, $\bar{y} = \dfrac{e^2-3}{8(e-2)} \approx 0.76$

b)



39. $\pi^2 + \pi - 4$ **41.** a) $\dfrac{1}{2\pi}\left(1 - e^{-2\pi}\right)$

43. $x \sin^{-1} x + \cos(\sin^{-1} x) + C$

45. $x \sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C$ **47.** Yes

49. a) $x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$
b) $x \sinh^{-1} x + (1+x^2)^{1/2} + C$

Section 7.3, pp. 576–578

1. $\dfrac{2}{x-3} + \dfrac{3}{x-2}$ **3.** $\dfrac{1}{x+1} + \dfrac{3}{(x+1)^2}$

5. $\dfrac{-2}{z} + \dfrac{-1}{z^2} + \dfrac{2}{z-1}$ **7.** $1 + \dfrac{17}{t-3} + \dfrac{-12}{t-2}$

9. $\dfrac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

11. $\dfrac{1}{7}\ln|(x+6)^2(x-1)^5| + C$ **13.** $(\ln 15)/2$

15. $-\dfrac{1}{2}\ln|t| + \dfrac{1}{6}\ln|t+2| + \dfrac{1}{3}\ln|t-1| + C$ **17.** $3\ln 2 - 2$

19. $\dfrac{1}{4}\ln\left|\dfrac{x+1}{x-1}\right| - \dfrac{x}{2(x^2-1)} + C$ **21.** $(\pi + 2\ln 2)/8$

23. $\tan^{-1} y - \dfrac{1}{y^2+1} + C$

25. $-(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

27. $\dfrac{-1}{\theta^2+2\theta+2} + \ln|\theta^2+2\theta+2| - \tan^{-1}(\theta+1) + C$

29. $x^2 + \ln\left|\dfrac{x-1}{x}\right| + C$

31. $9x + 2\ln|x| + \dfrac{1}{x} + 7\ln|x-1| + C$

33. $\dfrac{y^2}{2} - \ln|y| + \dfrac{1}{2}\ln(1+y^2) + C$ **35.** $\ln\left|\dfrac{e^t+1}{e^t+2}\right| + C$

37. $\dfrac{1}{5}\ln\left|\dfrac{\sin y - 2}{\sin y + 3}\right| + C$

39. $\dfrac{(\tan^{-1} 2x)^2}{4} - 3\ln|x-2| + \dfrac{6}{x-2} + C$

41. $x = \ln|t-2| - \ln|t-1| + \ln 2$ **43.** $x = \dfrac{6t}{t+2} - 1$

45. $3\pi \ln 25$ **47.** 1.10 **49.** a) $x = \dfrac{1000e^{4t}}{499 + e^{4t}}$ b) 1.55 days

51. a) $\dfrac{22}{7} - \pi$ b) 0.04% c) The area is less than 0.003.

Section 7.4, pp. 582–583

1. $\ln\left|\sqrt{9+y^2} + y\right| + C$ **3.** $\pi/4$ **5.** $\pi/6$

7. $\dfrac{25}{2}\sin^{-1}\left(\dfrac{t}{5}\right) + \dfrac{t\sqrt{25-t^2}}{2} + C$

9. $\dfrac{1}{2}\ln\left|\dfrac{2x}{7} + \dfrac{\sqrt{4x^2-49}}{7}\right| + C$

11. $7\left[\dfrac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\dfrac{y}{7}\right)\right] + C$ **13.** $\dfrac{\sqrt{x^2-1}}{x} + C$

15. $\dfrac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$ **17.** $\dfrac{-2\sqrt{4-w^2}}{w} + C$

19. $4\sqrt{3} - 4\pi/3$ **21.** $-\dfrac{x}{\sqrt{x^2-1}} + C$

23. $-\dfrac{1}{5}\left(\dfrac{\sqrt{1-x^2}}{x}\right)^5 + C$ **25.** $2\tan^{-1} 2x + \dfrac{4x}{(4x^2+1)} + C$

27. $\dfrac{1}{3}\left(\dfrac{v}{\sqrt{1-v^2}}\right)^3 + C$ **29.** $\ln 9 - \ln(1+\sqrt{10})$ **31.** $\pi/6$

33. $\sec^{-1}|x| + C$ **35.** $\sqrt{x^2-1} + C$

37. $y = 2\left[\dfrac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\dfrac{x}{2}\right)\right]$ **39.** $y = \dfrac{3}{2}\tan^{-1}\left(\dfrac{x}{2}\right) - \dfrac{3\pi}{8}$

41. $3\pi/4$ **43.** $\dfrac{2}{1-\tan(x/2)} + C$ **45.** 1 **47.** $\dfrac{\sqrt{3}\pi}{9}$

49. $\dfrac{1}{\sqrt{2}}\ln\left|\dfrac{\tan(t/2) + 1 - \sqrt{2}}{\tan(t/2) + 1 + \sqrt{2}}\right| + C$ **51.** $\ln\left|\dfrac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right| + C$

Section 7.5, pp. 591–594

1. $\dfrac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{\dfrac{x-3}{3}}\right) + C$ **3.** $\sqrt{x-2}\left(\dfrac{2(x-2)}{3} + 4\right) + C$

5. $\dfrac{(2x-3)^{3/2}(x+1)}{5} + C$

7. $\dfrac{-\sqrt{9-4x}}{x} - \dfrac{2}{3}\ln\left|\dfrac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right| + C$

9. $\dfrac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\dfrac{x-2}{2}\right) + C$

11. $-\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right|+C$

13. $\sqrt{4-x^2}-2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right|+C$

15. $\frac{p}{2}\sqrt{25-p^2}+\frac{25}{2}\sin^{-1}\frac{p}{5}+C$

17. $2\sin^{-1}\frac{r}{2}-\frac{1}{2}r\sqrt{4-r^2}+C$

19. $-\frac{1}{3}\tan^{-1}\left[\frac{1}{3}\tan\left(\frac{\pi}{4}-\theta\right)\right]+C$

21. $\frac{e^{2t}}{13}(2\cos 3t+3\sin 3t)+C$

23. $\frac{x^2}{2}\cos^{-1}(x)+\frac{1}{4}\sin^{-1}(x)-\frac{1}{4}x\sqrt{1-x^2}+C$

25. $\frac{s}{18(9-s^2)}+\frac{1}{108}\ln\left|\frac{s+3}{s-3}\right|+C$

27. $-\frac{\sqrt{4x+9}}{x}+\frac{2}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right|+C$

29. $2\sqrt{3t-4}-4\tan^{-1}\sqrt{\frac{3t-4}{4}}+C$

31. $\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln(1+x^2)+C$

33. $-\frac{\cos 5x}{10}-\frac{\cos x}{2}+C$ **35.** $8\left[\frac{\sin(7t/2)}{7}-\frac{\sin(9t/2)}{9}\right]+C$

37. $6\sin(\theta/12)+\frac{6}{7}\sin(7\theta/12)+C$

39. $\frac{1}{2}\ln\left|x^2+1\right|+\frac{x}{2(1+x^2)}+\frac{1}{2}\tan^{-1}x+C$

41. $\left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+C$

43. $\sin^{-1}\sqrt{x}-\sqrt{x-x^2}+C$

45. $\sqrt{1-\sin^2 t}-\ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right|+C$

47. $\ln\left|\ln y+\sqrt{3+(\ln y)^2}\right|+C$ **49.** $\ln\left|3r+\sqrt{9r^2-1}\right|+C$

51. $x\cos^{-1}\sqrt{x}+\frac{1}{2}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x-x^2}+C$

53. $-\frac{\sin^4 2x\cos 2x}{10}-\frac{2\sin^2 2x\cos 2x}{15}-\frac{4\cos 2x}{15}+C$

55. $\frac{\cos^3 2\pi t\sin 2\pi t}{\pi}+\frac{3}{2}\frac{\cos 2\pi t\sin 2\pi t}{\pi}+3t+C$

57. $\frac{\sin^3 2\theta\cos^2 2\theta}{10}+\frac{\sin^3 2\theta}{15}+C$ **59.** $\frac{2}{3}\tan^3 t+C$

61. $\tan^2 2x-2\ln|\sec 2x|+C$

63. $8\left[-\frac{1}{3}\cot^3 t+\cot t+t\right]+C$

65. $\frac{(\sec\pi x)(\tan\pi x)}{\pi}+\frac{1}{\pi}\ln|\sec\pi x+\tan\pi x|+C$

67. $\frac{\sec^2 3x\tan 3x}{3}+\frac{2}{3}\tan 3x+C$

69. $\frac{-\csc^3 x\cot x}{4}-\frac{3\csc x\cot x}{8}-\frac{3}{8}\ln|\csc x+\cot x|+C$

71. $4x^4(\ln x)^2-2x^4(\ln x)+\frac{x^2}{2}+C$ **73.** $\frac{e^{3x}}{9}(3x-1)+C$

75. $2x^3e^{x/2}-12x^2e^{x/2}+96e^{x/2}\left(\frac{x}{2}-1\right)+C$

77. $\frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left[\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}\right]+C$ **79.** $\frac{x\pi^x}{\ln\pi}-\frac{\pi^x}{(\ln\pi)^2}+C$

81. $\frac{1}{2}\left[\sec(e^t-1)\tan(e^t-1)+\ln|\sec(e^t-1)+\tan(e^t-1)|\right]+C$

83. $\sqrt{2}+\ln(\sqrt{2}+1)$ **85.** $\pi/3$

87. $\frac{1}{120}\sinh^4 3x\cosh 3x-\frac{1}{90}\sinh^2 3x\cosh 3x+\frac{2}{90}\cosh 3x+C$

89. $\frac{x^2}{3}\sinh 3x-\frac{2x}{9}\cosh 3x+\frac{2}{27}\sinh 3x+C$

91. $-\frac{\operatorname{sech}^7 x}{7}+C$ **101.** $\pi(2\sqrt{3}+\sqrt{2})\ln(\sqrt{2}+\sqrt{3})$

103. $\bar{x}=4/3,\ \bar{y}=\ln\sqrt{2}$ **105.** 7.62 **107.** $\pi/8$ **111.** $\pi/4$

Section 7.6, pp. 603–605

1. $\pi/2$ **3.** 2 **5.** 6 **7.** $\pi/2$ **9.** ln 3 **11.** ln 4 **13.** 0
15. $\sqrt{3}$ **17.** π **19.** $\ln\left(1+\frac{\pi}{2}\right)$ **21.** -1 **23.** 1
25. $-1/4$ **27.** $\pi/2$ **29.** $\pi/3$ **31.** 6 **33.** ln 2
35. Diverges **37.** Converges **39.** Converges **41.** Converges
43. Diverges **45.** Converges **47.** Converges **49.** Diverges
51. Converges **53.** Converges **55.** Diverges **57.** Converges
59. Diverges **61.** Converges **63.** Converges
65. b) ≈ 0.88621 **69.** 1 **71.** 2π **73.** ln 2 **79.** Diverges
81. Converges **83.** Converges **85.** Diverges **91.** b) $\pi/2$

Chapter 7 Practice Exercises, pp. 606–609

1. $\frac{1}{12}(4x^2-9)^{3/2}+C$ **3.** $\frac{(2x+1)^{5/2}}{10}-\frac{(2x+1)^{3/2}}{6}+C$

5. $\frac{\sqrt{8x^2+1}}{8}+C$ **7.** $\frac{1}{2}\ln(25+y^2)+C$

9. $\frac{-\sqrt{9-4t^4}}{8}+C$ **11.** $\frac{9}{25}(z^{5/3}+1)^{5/3}+C$

13. $-\dfrac{1}{2(1-\cos 2\theta)}+C$ **15.** $-\dfrac{1}{4}\ln|3+4\cos t|+C$

17. $-\dfrac{1}{2}e^{\cos 2x}+C$ **19.** $-\dfrac{1}{3}\cos^3(e^\theta)+C$ **21.** $\dfrac{2^{x-1}}{\ln 2}+C$

23. $\ln|\ln v|+C$ **25.** $\ln|2+\tan^{-1}x|+C$ **27.** $\sin^{-1}(2x)+C$

29. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3t}{4}\right)+C$ **31.** $\dfrac{1}{3}\tan^{-1}\left(\dfrac{t}{3}\right)+C$

33. $\dfrac{1}{5}\sec^{-1}\left|\dfrac{5x}{4}\right|+C$ **35.** $\sin^{-1}\left(\dfrac{x-2}{2}\right)+C$

37. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{y-2}{2}\right)+C$ **39.** $\sec^{-1}|x-1|+C$

41. $\dfrac{x}{2}-\dfrac{\sin 2x}{4}+C$

43. $\dfrac{2}{3}\cos^3\left(\dfrac{\theta}{2}\right)-2\cos\left(\dfrac{\theta}{2}\right)+C$

45. $\dfrac{\tan^2(2t)}{4}-\dfrac{1}{2}\ln|\sec 2t|+C$

47. $-\dfrac{1}{2}\ln|\csc(2x)+\cot(2x)|+C$ **49.** $\ln\sqrt{2}$ **51.** 2

53. $2\sqrt{2}$ **55.** $x-2\tan^{-1}\left(\dfrac{x}{2}\right)+C$

57. $x+x^2+2\ln|2x-1|+C$

59. $\ln(y^2+4)-\dfrac{1}{2}\tan^{-1}\left(\dfrac{y}{2}\right)+C$

61. $-\sqrt{4-t^2}+2\sin^{-1}\left(\dfrac{t}{2}\right)+C$ **63.** $x-\tan x+\sec x+C$

65. $-\dfrac{1}{3}\ln|\sec(5-3x)+\tan(5-3x)|+C$

67. $4\ln\left|\sin\left(\dfrac{x}{4}\right)\right|+C$ **69.** $-2\left(\dfrac{(\sqrt{1-x})^3}{3}-\dfrac{(\sqrt{1-x})^5}{5}\right)+C$

71. $\dfrac{1}{2}\left(z\sqrt{z^2+1}+\ln\left|z+\sqrt{z^2+1}\right|\right)+C$

73. $\ln\left|y+\sqrt{25+y^2}\right|+C$ **75.** $\dfrac{-\sqrt{1-x^2}}{x}+C$

77. $\dfrac{\sin^{-1}x}{2}-\dfrac{x\sqrt{1-x^2}}{2}+C$ **79.** $\ln\left|\dfrac{x}{3}+\dfrac{\sqrt{x^2-9}}{3}\right|+C$

81. $\sqrt{w^2-1}-\sec^{-1}(w)+C$

83. $[(x+1)(\ln(x+1))-(x+1)]+C$

85. $x\tan^{-1}(3x)-\dfrac{1}{6}\ln(1+9x^2)+C$

87. $(x+1)^2e^x-2(x+1)e^x+2e^x+C$

89. $\dfrac{2e^x\sin 2x}{5}+\dfrac{e^x\cos 2x}{5}+C$

91. $2\ln|x-2|-\ln|x-1|+C$

93. $\ln|x|-\ln|x+1|+\dfrac{1}{x+1}+C$ **95.** $-\dfrac{1}{3}\ln\left|\dfrac{\cos\theta-1}{\cos\theta+2}\right|+C$

97. $4\ln|x|-\dfrac{1}{2}\ln(x^2+1)+4\tan^{-1}x+C$

99. $\dfrac{1}{16}\ln\left|\dfrac{(v-2)^5(v+2)}{v^6}\right|+C$

101. $\dfrac{1}{2}\tan^{-1}t-\dfrac{\sqrt{3}}{6}\tan^{-1}\dfrac{t}{\sqrt{3}}+C$

103. $\dfrac{x^2}{2}+\dfrac{4}{3}\ln|x+2|+\dfrac{2}{3}\ln|x-1|+C$

105. $\dfrac{x^2}{2}-\dfrac{9}{2}\ln|x+3|+\dfrac{3}{2}\ln|x+1|+C$

107. $\dfrac{1}{3}\ln\left|\dfrac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$ **109.** $\ln|1-e^{-x}|+C$ **111.** $\pi/2$

113. 6 **115.** $\ln 3$ **117.** 2 **119.** $\pi/6$ **121.** Diverges

123. Diverges **125.** Converges **127.** $-\sqrt{16-y^2}+C$

129. $-\dfrac{1}{2}\ln|4-x^2|+C$ **131.** $\ln\dfrac{1}{\sqrt{9-x^2}}+C$

133. $\dfrac{1}{6}\ln\left|\dfrac{x+3}{x-3}\right|+C$

135. $\dfrac{2x^{3/2}}{3}-x+2\sqrt{x}-2\ln(\sqrt{x}+1)+C$

137. $\ln\left|\dfrac{x}{\sqrt{x^2+1}}\right|-\dfrac{1}{2}\left(\dfrac{x}{\sqrt{x^2+1}}\right)^2+C$ **139.** $\sin^{-1}(x+1)+C$

141. $\ln|u+\sqrt{1+u^2}|+C$

143. $-2\cot x-\ln|\csc x+\cot x|+\csc x+C$

145. $\dfrac{1}{12}\ln\left|\dfrac{3+v}{3-v}\right|+\dfrac{1}{6}\tan^{-1}\dfrac{v}{3}+C$

147. $\dfrac{\theta\sin(2\theta+1)}{2}+\dfrac{\cos(2\theta+1)}{4}+C$

149. $\dfrac{x^2}{2}+2x+3\ln|x-1|-\dfrac{1}{x-1}+C$ **151.** $-\cos(2\sqrt{x})+C$

153. $-\ln|\csc(2y)+\cot(2y)|+C$ **155.** $\dfrac{1}{2}\tan^2 x+C$

157. $-\sqrt{4-(r+2)^2}+C$ **159.** $\dfrac{1}{4}\sec^2\theta+C$ **161.** $\dfrac{\sqrt{2}}{2}$

163. $2\left(\dfrac{(\sqrt{2-x})^3}{3}-2\sqrt{2-x}\right)+C$ **165.** $\tan^{-1}(y-1)+C$

167. $\dfrac{1}{3}\ln|\sec\theta^3|+C$

169. $\dfrac{1}{4}\ln|z|-\dfrac{1}{4z}-\dfrac{1}{4}\left[\dfrac{1}{2}\ln(z^2+4)+\dfrac{1}{2}\tan^{-1}\left(\dfrac{z}{2}\right)\right]+C$

171. $-\dfrac{1}{4}\sqrt{9-4t^2}+C$ **173.** $\ln|\sin\theta|-\dfrac{1}{2}\ln(1+\sin^2\theta)+C$

175. $\ln|\sec\sqrt{y}|+C$ **177.** $-\theta+\ln\left|\dfrac{\theta+2}{\theta-2}\right|+C$ **179.** $x+C$

181. $-\dfrac{\cos x}{2}+C$ **183.** $\ln(1+e^t)+C$ **185.** $1/4$

187. $\ln|\ln\sin v|+C$ **189.** $\dfrac{2}{3}x^{3/2}+C$

191. $-\dfrac{1}{5}\tan^{-1}\cos(5t)+C$ **193.** $\dfrac{1}{3}\left(\dfrac{27^{3\theta+1}}{\ln 27}\right)+C$

195. $2\sqrt{r}-2\ln(1+\sqrt{r})+C$ **197.** $\ln\left|\dfrac{y}{y+2}\right|+\dfrac{2}{y}-\dfrac{2}{y^2}+C$

199. $4\sec^{-1}\left(\dfrac{7m}{2}\right)+C$ **201.** $\dfrac{\sqrt{8}-1}{6}$ **203.** $\dfrac{\pi}{2}(3b-a)+2$

Chapter 7 Additional Exercises, pp. 609–612

1. $x(\sin^{-1}x)^2+2(\sin^{-1}x)\sqrt{1-x^2}-2x+C$

3. $\dfrac{x^2\sin^{-1}x}{2}+\dfrac{x\sqrt{1-x^2}-\sin^{-1}x}{4}+C$

5. $\dfrac{\ln|\sec 2\theta+\tan 2\theta|+2\theta}{4}+C$

7. $\dfrac{1}{2}\left(\ln\left(t-\sqrt{1-t^2}\right)-\sin^{-1}t\right)+C$

9. $\dfrac{1}{16}\ln\left|\dfrac{x^2+2x+2}{x^2-2x+2}\right|+\dfrac{1}{8}\left(\tan^{-1}(x+1)+\tan^{-1}(x-1)\right)+C$

11. 0 **13.** $\ln(4)-1$ **15.** 1 **17.** $32\pi/35$ **19.** 2π

21. a) π b) $\pi(2e-5)$ **23.** b) $\pi\left(\dfrac{8(\ln 2)^2}{3}-\dfrac{16(\ln 2)}{9}+\dfrac{16}{27}\right)$

25. $\left(\dfrac{e^2+1}{4},\dfrac{e-2}{2}\right)$

27. $\sqrt{1+e^2}-\ln\left(\dfrac{\sqrt{1+e^2}}{e}+\dfrac{1}{e}\right)-\sqrt{2}+\ln(1+\sqrt{2})$ **29.** 6

31. $y=\sqrt{x},\quad 0\le x\le 4$ **33.** b) 1 **37.** $a=\dfrac{1}{2},-\dfrac{\ln 2}{4}$

39. $\dfrac{1}{2}<p\le 1$

41. $\dfrac{e^{2x}}{13}(3\sin 3x+2\cos 3x)+C$

43. $\dfrac{\cos x\sin 3x-3\sin x\cos 3x}{8}+C$

45. $\dfrac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

47. $x\ln(ax)-x+C$

CHAPTER 8

Section 8.1, pp. 619–622

1. $a_1=0,\ a_2=-1/4,\ a_3=-2/9,\ a_4=-3/16$
3. $a_1=1,\ a_2=-1/3,\ a_3=1/5,\ a_4=-1/7$
5. $a_1=1/2,\ a_2=1/2,\ a_3=1/2,\ a_4=1/2$
7. $1,\ \dfrac{3}{2},\ \dfrac{7}{4},\ \dfrac{15}{8},\ \dfrac{31}{16},\ \dfrac{63}{32},\ \dfrac{127}{64},\ \dfrac{255}{128},\ \dfrac{511}{256},\ \dfrac{1023}{512}$
9. $2,\ 1,\ -\dfrac{1}{2},\ -\dfrac{1}{4},\ \dfrac{1}{8},\ \dfrac{1}{16},\ -\dfrac{1}{32},\ -\dfrac{1}{64},\ \dfrac{1}{128},\ \dfrac{1}{256}$
11. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 **13.** $a_n=(-1)^{n+1},\ n\ge 1$
15. $a_n=(-1)^{n+1}(n)^2,\ n\ge 1$ **17.** $a_n=n^2-1,\ n\ge 1$
19. $a_n=4n-3,\ n\ge 1$ **21.** $a_n=\dfrac{1+(-1)^{n+1}}{2},\ n\ge 1$
23. $N=692,\ a_n=\sqrt[n]{0.5},\ L=1$
25. $N=65,\ a_n=(0.9)^n,\ L=0$ **27.** b) $\sqrt{3}$
31. Nondecreasing, bounded **33.** Not nondecreasing, bounded
35. Converges, nondecreasing sequence theorem
37. Converges, nondecreasing sequence theorem
39. Diverges, definition of divergence **43.** Converges
45. Converges

Section 8.2, pp. 628–630

1. Converges, 2 **3.** Converges, -1 **5.** Converges, -5
7. Diverges **9.** Diverges **11.** Converges, $1/2$
13. Converges, 0 **15.** Converges, $\sqrt{2}$ **17.** Converges, 1
19. Converges, 0 **21.** Converges, 0 **23.** Converges, 0
25. Converges, 1 **27.** Converges, e^7 **29.** Converges, 1
31. Converges, 1 **33.** Diverges **35.** Converges, 4
37. Converges, 0 **39.** Diverges **41.** Converges, e^{-1}
43. Converges, $e^{2/3}$ **45.** Converges, $x\ (x>0)$ **47.** Converges, 0
49. Converges, 1 **51.** Converges, $1/2$ **53.** Converges, $\pi/2$
55. Converges, 0 **57.** Converges, 0 **59.** Converges, $1/2$
61. Converges, 0 **63.** $x_n=2^{n-2}$
65. a) $f(x)=x^2-2,\ 1.414213562\approx\sqrt{2}$
b) $f(x)=\tan(x)-1,\ 0.7853981635\approx\pi/4$
c) $f(x)=e^x$, diverges **67.** b) 1 **75.** 1 **77.** -0.73908456
79. 0.853748068 **83.** -3

Section 8.3, pp. 638–640

1. $s_n=\dfrac{2(1-(1/3)^n)}{1-(1/3)},\ 3$ **3.** $s_n=\dfrac{1-(-1/2)^n}{1-(-1/2)},\ 2/3$

5. $s_n=\dfrac{1}{2}-\dfrac{1}{n+2},\ \dfrac{1}{2}$ **7.** $1-\dfrac{1}{4}+\dfrac{1}{16}-\dfrac{1}{64}+\cdots,\ \dfrac{4}{5}$

9. $\dfrac{7}{4}+\dfrac{7}{16}+\dfrac{7}{64}+\cdots,\ \dfrac{7}{3}$

11. $(5+1)+\left(\dfrac{5}{2}+\dfrac{1}{3}\right)+\left(\dfrac{5}{4}+\dfrac{1}{9}\right)+\left(\dfrac{5}{8}+\dfrac{1}{27}\right)+\cdots,\ \dfrac{23}{2}$

13. $(1+1)+\left(\dfrac{1}{2}-\dfrac{1}{5}\right)+\left(\dfrac{1}{4}+\dfrac{1}{25}\right)+\left(\dfrac{1}{8}-\dfrac{1}{125}\right)+\cdots,\ \dfrac{17}{6}$

15. 1 **17.** 5 **19.** Converges, 1 **21.** Converges, $-\dfrac{1}{\ln 2}$

23. Converges, $2+\sqrt{2}$ **25.** Converges, 1 **27.** Diverges

29. Converges, $\dfrac{e^2}{e^2-1}$ **31.** Converges, $2/9$ **33.** Converges, $3/2$

35. Diverges **37.** Diverges **39.** Converges, $\dfrac{\pi}{\pi - e}$

41. $a = 1, r = -x$; converges to $1/(1 + x)$ for $|x| < 1$

43. $a = 3, r = (x - 1)/2$; converges to $6/(3 - x)$ for x in $(-1, 3)$

45. $|x| < \dfrac{1}{2}, \dfrac{1}{1 - 2x}$ **47.** $-2 < x < 0$, $\dfrac{1}{2 + x}$

49. $x \neq (2k + 1)\dfrac{\pi}{2}$, k an integer; $\dfrac{1}{1 - \sin x}$ **51.** $23/99$

53. $7/9$ **55.** $1/15$ **57.** $41251/33300$

59. a) $\displaystyle\sum_{n=-2}^{\infty} \frac{1}{(n + 4)(n + 5)}$ b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n + 2)(n + 3)}$

c) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{(n - 3)(n - 2)}$ **61.** a) Answers may vary.

b) Answers may vary. c) Answers may vary. **69.** a) $r = 3/5$

b) $r = -3/10$ **71.** $|r| < 1$, $\dfrac{1 + 2r}{1 - r^2}$ **73.** 28 m **75.** 8 m^2

77. a) $3\left(\dfrac{4}{3}\right)^{n-1}$

b) $A_n = A + \dfrac{1}{3}A + \dfrac{1}{3}\left(\dfrac{4}{9}\right)A + \cdots + \dfrac{1}{3}\left(\dfrac{4}{9}\right)^{n-2} A$,

$\lim_{n\to\infty} A_n = 2\sqrt{3}/5$

Section 8.4, pp. 643–644

1. Converges; geometric series, $r = \dfrac{1}{10} < 1$

3. Diverges; $\lim_{n\to\infty} \dfrac{n}{n + 1} = 1 \neq 0$

5. Diverges; p-series, $p < 1$

7. Converges; geometric series, $r = \dfrac{1}{8} < 1$

9. Diverges; Integral Test

11. Converges; geometric series, $r = 2/3 < 1$

13. Diverges; Integral Test

15. Diverges; $\lim_{n\to\infty} \dfrac{2^n}{n + 1} \neq 0$

17. Diverges; $\lim_{n\to\infty} (\sqrt{n}/\ln n) \neq 0$

19. Diverges; geometric series, $r = \dfrac{1}{\ln 2} > 1$

21. Converges; Integral Test

23. Diverges; nth-Term Test

25. Converges; Integral Test

27. Converges; Integral Test

29. Converges; Integral Test **31.** $a = 1$ **33.** b) About 41.55

35. True

Section 8.5, p. 649

1. Diverges; limit comparison with $\sum(1/\sqrt{n})$

3. Converges; compare with $\sum(1/2^n)$ **5.** Diverges; nth-Term Test

7. Converges; $\left(\dfrac{n}{3n + 1}\right)^n < \left(\dfrac{n}{3n}\right)^n = \left(\dfrac{1}{3}\right)^n$

9. Diverges; direct comparison with $\sum(1/n)$

11. Converges; limit comparison with $\sum(1/n^2)$

13. Diverges; limit comparison with $\sum(1/n)$

15. Diverges; limit comparison with $\sum(1/n)$

17. Diverges; Integral Test

19. Converges; compare with $\sum(1/n^{3/2})$

21. Converges; $\dfrac{1}{n2^n} \leq \dfrac{1}{2^n}$

23. Converges; $\dfrac{1}{3^{n-1} + 1} < \dfrac{1}{3^{n-1}}$

25. Diverges; limit comparison with $\sum(1/n)$

27. Converges; compare with $\sum(1/n^2)$

29. Converges; $\dfrac{\tan^{-1} n}{n^{1.1}} < \dfrac{\pi/2}{n^{1.1}}$

31. Converges; compare with $\sum(1/n^2)$

33. Diverges; $3n > n\sqrt[n]{n} \Rightarrow \dfrac{1}{3n} < \dfrac{1}{n\sqrt[n]{n}} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ diverges

35. Converges; limit comparison with $\sum(1/n^2)$

Section 8.6, pp. 654–655

1. Converges; Ratio Test **3.** Diverges; Ratio Test

5. Converges; Ratio Test **7.** Converges; compare with $\sum(3/(1.25)^n)$

9. Diverges; $\lim_{n\to\infty} \left(1 - \dfrac{3}{n}\right)^n = e^{-3} \neq 0$

11. Converges; compare with $\sum(1/n^2)$

13. Diverges; compare with $\sum(1/(2n))$

15. Diverges; compare with $\sum(1/n)$ **17.** Converges; Ratio Test

19. Converges; Ratio Test **21.** Converges; Ratio Test

23. Converges; Root Test **25.** Converges; compare with $\sum(1/n^2)$

27. Converges; Ratio Test **29.** Diverges; Ratio Test

31. Converges; Ratio Test **33.** Converges; Ratio Test

35. Diverges; $a_n = \left(\dfrac{1}{3}\right)^{(1/n!)} \to 1$ **37.** Converges; Ratio Test

39. Diverges; Root Test **41.** Converges; Root Test

43. Converges; Ratio Test **47.** Yes

Section 8.7, pp. 661–663

1. Converges by Theorem 8 **3.** Diverges; $a_n \not\to 0$

5. Converges by Theorem 8 **7.** Diverges; $a_n \to 1/2$

9. Converges by Theorem 8

11. Converges absolutely. Series of absolute values is a convergent geometric series.

13. Converges conditionally. $1/\sqrt{n} \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

15. Converges absolutely. Compare with $\sum_{n=1}^{\infty}(1/n^2)$.

17. Converges conditionally. $1/(n+3) \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges (compare with $\sum_{n=1}^{\infty}(1/n)$).

19. Diverges; $\frac{3+n}{5+n} \to 1$

21. Converges conditionally; $\left(\frac{1}{n^2}+\frac{1}{n}\right) \to 0$ but $(1+n)/n^2 > 1/n$

23. Converges absolutely; Root Test

25. Converges absolutely by Integral Test **27.** Diverges; $a_n \not\to 0$

29. Converges absolutely by the Ratio Test

31. Converges absolutely; $\frac{1}{n^2+2n+1} < \frac{1}{n^2}$

33. Converges absolutely since $\left|\frac{\cos n\pi}{n\sqrt{n}}\right| = \left|\frac{(-1)^{n+1}}{n^{3/2}}\right| = \frac{1}{n^{3/2}}$ (convergent p–series)

35. Converges absolutely by Root Test **37.** Diverges; $a_n \to \infty$

39. Converges conditionally; $\sqrt{n+1}-\sqrt{n} = 1/(\sqrt{n}+\sqrt{n+1}) \to 0$, but series of absolute values diverges (compare with $\sum(1/\sqrt{n})$)

41. Diverges, $a_n \to 1/2 \neq 0$

43. Converges absolutely; $\operatorname{sech} n = \frac{2}{e^n+e^{-n}} = \frac{2e^n}{e^{2n}+1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$, a term from a convergent geometric series.

45. $|\text{Error}| < 0.2$ **47.** $|\text{Error}| < 2 \times 10^{-11}$ **49.** 0.54030

51. a) $a_n \geq a_{n+1}$ b) $-1/2$

Section 8.8, pp. 671–672

1. a) $1,\ -1 < x < 1$ b) $-1 < x < 1$ c) none

3. a) $1/4,\ -1/2 < x < 0$ b) $-1/2 < x < 0$ c) none

5. a) $10,\ -8 < x < 12$ b) $-8 < x < 12$ c) none

7. a) $1,\ -1 < x < 1$ b) $-1 < x < 1$ c) none

9. a) $3,\ [-3, 3]$ b) $[-3, 3]$ c) none **11.** a) ∞, for all x b) for all x c) none **13.** a) ∞, for all x b) for all x c) none

15. a) $1,\ -1 \leq x < 1$ b) $-1 < x < 1$ c) $x = -1$

17. a) $5,\ -8 < x < 2$ b) $-8 < x < 2$ c) none

19. a) $3,\ -3 < x < 3$ b) $-3 < x < 3$ c) none

21. a) $1,\ -1 < x < 1$ b) $-1 < x < 1$ c) none

23. a) $0,\ x = 0$ b) $x = 0$ c) none **25.** a) $2,\ -4 < x \leq 0$ b) $-4 < x < 0$ c) $x = 0$ **27.** a) $1,\ -1 \leq x \leq 1$ b) $-1 \leq x \leq 1$ c) none **29.** a) $1/4,\ 1 \leq x \leq 3/2$ b) $1 \leq x \leq 3/2$ c) none **31.** a) $1,\ (-1-\pi) \leq x < (1-\pi)$ b) $(-1-\pi) < x < (1-\pi)$ c) $x = -1-\pi$

33. $-1 < x < 3,\ 4/(3+2x-x^2)$ **35.** $0 < x < 16,\ 2/(4-\sqrt{x})$

37. $-\sqrt{2} < x < \sqrt{2},\ 3/(2-x^2)$

39. $1 < x < 5,\ 2/(x-1),\ 1 < x < 5,\ -2/(x-1)^2$

41. a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$; converges for all x

b) and c) $2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \cdots$

43. a) $\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175},\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

b) $1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \cdots,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

Section 8.9, pp. 677–678

1. $P_0(x) = 0,\ P_1(x) = x - 1,\ P_2(x) = (x-1) - \frac{1}{2}(x-1)^2,$
$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

3. $P_0(x) = \frac{1}{2},\ P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2),$
$P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2,$
$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

5. $P_0(x) = \frac{\sqrt{2}}{2},\ P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right),$
$P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2,$
$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

7. $P_0(x) = 2,\ P_1(x) = 2 + \frac{1}{4}(x-4),$
$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2,$
$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

9. $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

11. $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$

13. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$ **15.** $7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ **17.** $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

19. $x^4 - 2x^3 - 5x + 4$ **21.** $8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

23. $21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

25. $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$ **27.** $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

33. $L(x) = 0,\ Q(x) = -x^2/2$ **35.** $L(x) = 1,\ Q(x) = 1 + x^2/2$

37. $L(x) = x,\ Q(x) = x$

Section 8.10, pp. 686–688

1. $\sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} = 1 - 5x + \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!} + \cdots$

3. $\sum_{n=0}^{\infty} \frac{5(-1)^n(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{5(-1)^{n+1}x^{2n+1}}{(2n+1)!}$
$= -5x + \frac{5x^3}{3!} - \frac{5x^5}{5!} + \frac{5x^7}{7!} + \cdots$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$ **7.** $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$

9. $\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$

11. $x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!}$

13. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!} = 1 - \frac{(2x)^2}{2 \cdot 2!} +$
$\frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)8}{2 \cdot 8!} - \cdots$

15. $\sum_{n=0}^{\infty} (2x)^{n+2} = 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots$

17. $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

19. $|x| < (0.06)^{1/5} < 0.56968$

21. $|\text{Error}| < (10^{-3})^3/6 < 1.67 \times 10^{-10}, \quad -10^{-3} < x < 0$

23. $|\text{Error}| < (3^{0.1})(0.1)^3/6 < 1.87 \times 10^{-5}$ **25.** 0.000293653

27. $|x| < 0.02$ **31.** $\sin x,\ x = 0.1; \sin(0.1)$

33. $\tan^{-1} x,\ x = \pi/3; \sqrt{3}$

35. $e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} \cdots$

43. a) $Q(x) = 1 + kx + \frac{k(k-1)}{2}x^2$ b) for $0 \le x < 100^{-1/3}$

49. a) -1 b) $(1/\sqrt{2})(1+i)$ c) $-i$

53. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 \cdots$; will converge for all x

Section 8.11, pp. 697–699

1. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ **3.** $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$

5. $1 - x + \frac{3x^2}{4} - \frac{x^3}{2}$ **7.** $1 - \frac{x^3}{2} + \frac{3x^6}{8} - \frac{5x^9}{16}$

9. $1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$

11. $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

13. $(1-2x)^3 = 1 - 6x + 12x^2 - 8x^3$

15. $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$ **17.** $y = \sum_{n=1}^{\infty} (x^n/n!) = e^x - 1$

19. $y = \sum_{n=2}^{\infty} (x^n/n!) = e^x - x - 1$ **21.** $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$

23. $y = \sum_{n=0}^{\infty} 2x^n = \frac{2}{1-x}$ **25.** $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$

27. $y = 2 + x - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$

29. $y = \sum_{n=0}^{\infty} \frac{-2(x-2)^{2n+1}}{(2n+1)!}$

31. $y = a + bx + \frac{1}{6}x^3 - \frac{ax^4}{3 \cdot 4} - \frac{bx^5}{4 \cdot 5} - \frac{x^7}{6 \cdot 6 \cdot 7} + \frac{ax^8}{3 \cdot 4 \cdot 7 \cdot 8}$
$+ \frac{bx^9}{4 \cdot 5 \cdot 8 \cdot 9} \cdots$. For $n \ge 6$, $a_n = (n-2)(n-3)\,a_{n-4}$.

33. 0.00267 **35.** 0.1 **37.** 0.0999 44461 1 **39.** 0.1000 01

41. $1/(13 \cdot 6!) \approx 0.00011$ **43.** $\frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!}$

45. a) $\frac{x^2}{2} - \frac{x^4}{12}$
b) $\frac{x^2}{2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \frac{x^8}{7 \cdot 8} + \cdots + (-1)^{15}\frac{x^{32}}{31 \cdot 32}$

47. 1/2 **49.** $-1/24$ **51.** 1/3 **53.** -1 **55.** 2

59. 500 terms

61. 3 terms

63. a) $x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$, radius of convergence = 1
b) $\frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$ **65.** $1 - 2x + 3x^2 - 4x^3 + \cdots$

71. c) $3\pi/4$

Chapter 8 Practice Exercises, pp. 700–702

1. Converges to 1 **3.** Converges to -1 **5.** Diverges
7. Converges to 0 **9.** Converges to 1 **11.** Converges to e^{-5}
13. Converges to 3 **15.** Converges to ln 2 **17.** Diverges
19. 1/6 **21.** 3/2 **23.** $e/(e-1)$ **25.** Diverges
27. Converges conditionally **29.** Converges conditionally
31. Converges absolutely **33.** Converges absolutely
35. Converges absolutely **37.** Converges absolutely
39. Converges absolutely **41.** a) 3, $-7 \le x < -1$
b) $-7 < x < -1$ c) $x = -7$ **43.** a) 1/3, $0 \le x \le 2/3$
b) $0 \le x \le 2/3$ c) none **45.** a) ∞, for all x b) for all x
c) none **47.** a) $\sqrt{3}$, $-\sqrt{3} < x < \sqrt{3}$ b) $-\sqrt{3} < x < \sqrt{3}$
c) none **49.** a) e, $(-e, e)$ b) $(-e, e)$ c) { }

51. $\frac{1}{1+x}, \frac{1}{4}, \frac{4}{5}$ **53.** $\sin x,\ \pi,\ 0$ **55.** $e^x,\ \ln 2,\ 2$

57. $\sum_{n=0}^{\infty} 2^n x^n$ **59.** $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$ **61.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$

63. $\sum_{n=0}^{\infty} \frac{((\pi x)/2)^n}{n!}$

65. $2 - \frac{(x+1)}{2 \cdot 1!} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!} + \cdots$

67. $\frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3$

69. $y = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n = -e^{-x}$

71. $y = 3 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n = 3e^{-2x}$

73. $y = -1 - x + 2 \sum_{n=2}^{\infty} (x^n/n!) = 2e^x - 3x - 3$

75. $y = 1 + x + 2 \sum_{n=0}^{\infty} (x^n/n!) = 2e^x - 1 - x$ **77.** 0.4849 17143 1

79. $\approx 0.4872\ 22358\ 3$ **81.** $7/2$ **83.** $1/12$ **85.** -2

87. $r = -3,\ s = 9/2$

89. b) $|\text{error}| < |\sin(1/42)| < 0.02381$; an underestimate because the remainder is positive

91. $2/3$ **93.** $\ln\left(\frac{n+1}{2n}\right)$; the series converges to $\ln\left(\frac{1}{2}\right)$.

95. a) ∞ b) $a = 1,\quad b = 0$ **97.** It converges.

Chapter 8 Additional Exercises, pp. 703–707

1. Converges; Comparison Test **3.** Diverges; nth Term Test
5. Converges; Comparison Test **7.** Diverges; nth Term Test

9. With $a = \pi/3$, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \pi/3) - \frac{1}{4}(x - \pi/3)^2 + \frac{\sqrt{3}}{12}(x - \pi/3)^3 + \cdots$

11. With $a = 0$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

13. With $a = 22\pi$, $\cos x = 1 - \frac{1}{2}(x - 22\pi)^2 + \frac{1}{4!}(x - 22\pi)^4 - \frac{1}{6!}(x - 22\pi)^6 + \cdots$

15. Converges, limit $= b$ **17.** $\pi/2$ **23.** $b = \pm\frac{1}{5}$

25. $a = 2,\ L = -7/6$ **29.** b) Yes

35. a) $\sum_{n=1}^{\infty} nx^{n-1}$ b) 6 c) $1/q$

37. a) $R_n = C_0 e^{-kt_0}\left(1 - e^{-nkt_0}\right) / \left(1 - e^{-kt_0}\right)$,
$R = C_0\left(e^{-kt_0}\right) / \left(1 - e^{-kt_0}\right) = C_0 / \left(e^{kt_0} - 1\right)$
b) $R_1 = 1/e \approx 0.368$,
$R_{10} = R(1 - e^{-10}) \approx R(0.9999546) \approx 0.58195$;
$R \approx 0.58198$; $0 < (R - R_{10})/R < 0.0001$ c) 7

CHAPTER 9

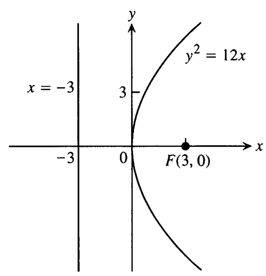
Section 9.1, pp. 719–722

1. $y^2 = 8x$, $F(2, 0)$, directrix: $x = -2$
3. $x^2 = -6y$, $F(0, -3/2)$, directrix: $y = 3/2$

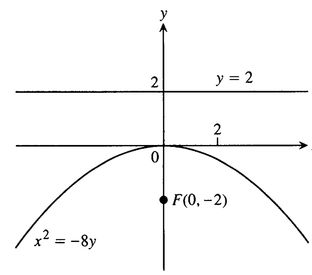
5. $\frac{x^2}{4} - \frac{y^2}{9} = 1$, $F(\pm\sqrt{13}, 0)$, $V(\pm 2, 0)$, asymptotes: $y = \pm\frac{3}{2}x$

7. $\frac{x^2}{2} + y^2 = 1$, $F(\pm 1, 0)$, $V(\pm\sqrt{2}, 0)$

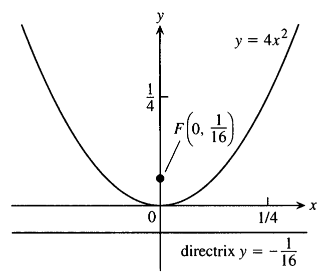
9.



11.



13.



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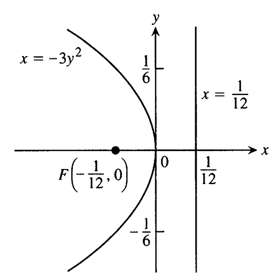

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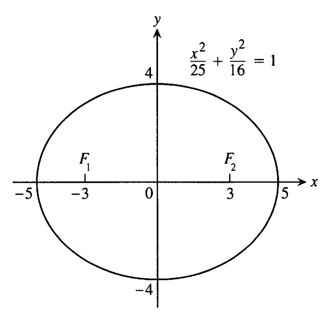

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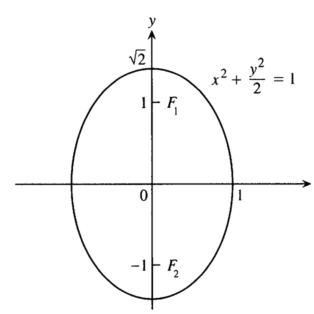

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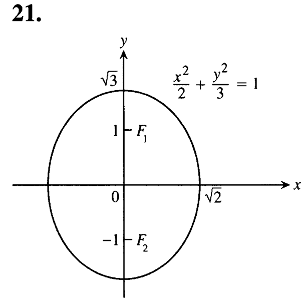

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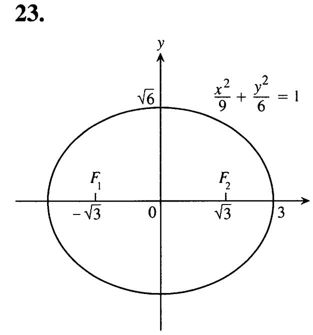
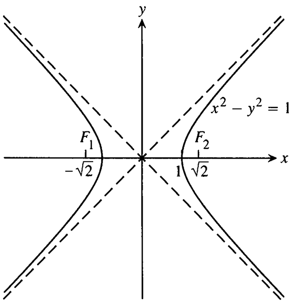
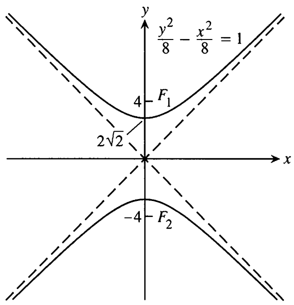


25. $\frac{x^2}{4} + \frac{y^2}{2} = 1$

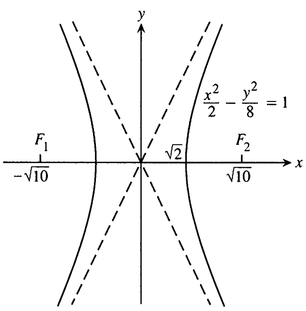
27. Asymptotes: $y = \pm x$



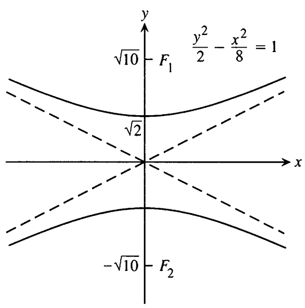
29. Asymptotes: $y = \pm x$



31. Asymptotes: $y = \pm 2x$

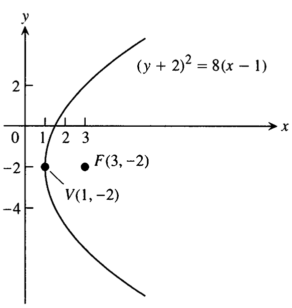


33. Asymptotes: $y = \pm\dfrac{x}{2}$

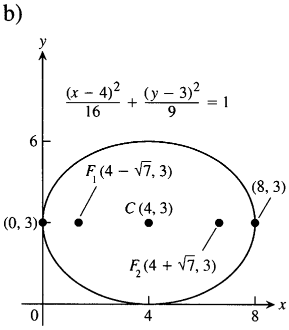


35. $y^2 - x^2 = 1$ **37.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

39. a) Vertex: $(1, -2)$; focus: $(3, -2)$; directrix: $x = -1$
b)

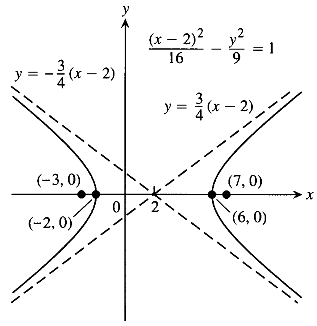


41. a) Foci: $(4 \pm \sqrt{7}, 3)$; vertices: $(8, 3)$ and $(0, 3)$; center: $(4, 3)$
b)



43. a) Center: $(2, 0)$; foci: $(7, 0)$ and $(-3, 0)$; vertices: $(6, 0)$ and $(-2, 0)$; asymptotes: $y = \pm\dfrac{3}{4}(x - 2)$

b)



45. $(y + 3)^2 = 4(x + 2)$, $V(-2, -3)$, $F(-1, -3)$, directrix: $x = -3$

47. $(x - 1)^2 = 8(y + 7)$, $V(1, -7)$, $F(1, -5)$, directrix: $y = -9$

49. $\dfrac{(x + 2)^2}{6} + \dfrac{(y + 1)^2}{9} = 1$, $F(-2, \pm\sqrt{3} - 1)$, $V(-2, \pm 3 - 1)$, $C(-2, -1)$

51. $\dfrac{(x - 2)^2}{3} + \dfrac{(y - 3)^2}{2} = 1$, $F(3, 3)$ and $F(1, 3)$, $V(\pm\sqrt{3} + 2, 3)$, $C(2, 3)$

53. $\dfrac{(x - 2)^2}{4} - \dfrac{(y - 2)^2}{5} = 1$, $C(2, 2)$, $F(5, 2)$ and $F(-1, 2)$, $V(4, 2)$ and $V(0, 2)$; asymptotes: $(y - 2) = \pm\dfrac{\sqrt{5}}{2}(x - 2)$

55. $(y + 1)^2 - (x + 1)^2 = 1$, $C(-1, -1)$, $F(-1, \sqrt{2} - 1)$ and $F(-1, -\sqrt{2} - 1)$, $V(-1, 0)$ and $V(-1, -2)$; asymptotes: $(y + 1) = \pm(x + 1)$

57. $C(-2, 0)$, $a = 4$ **59.** $V(-1, 1)$, $F(-1, 0)$

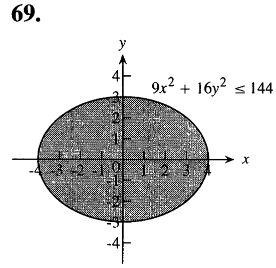
61. Ellipse: $\dfrac{(x + 2)^2}{5} + y^2 = 1, C(-2, 0)$, $F(0, 0)$ and $F(-4, 0)$, $V(\sqrt{5} - 2, 0)$ and $V(-\sqrt{5} - 2, 0)$

63. Ellipse: $\dfrac{(x - 1)^2}{2} + (y - 1)^2 = 1$, $C(1, 1)$, $F(2, 1)$ and $F(0, 1)$, $V(\sqrt{2} + 1, 1)$ and $V(-\sqrt{2} + 1, 1)$

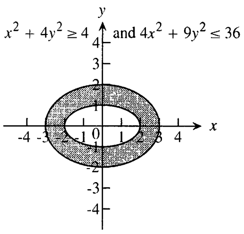
65. Hyperbola: $(x - 1)^2 - (y - 2)^2 = 1, C(1, 2), F(1 + \sqrt{2}, 2)$ and $F(1 - \sqrt{2}, 2), V(2, 2)$ and $V(0, 2)$; asymptotes: $(y - 2) = \pm(x - 1)$

67. Hyperbola: $\dfrac{(y - 3)^2}{6} - \dfrac{x^2}{3} = 1, C(0, 3), F(0, 6)$ and $F(0, 0)$, $V(0, \sqrt{6} + 3)$ and $V(0, -\sqrt{6} + 3)$; asymptotes: $y = \sqrt{2}x + 3$ or $y = -\sqrt{2}x + 3$

69.



71.



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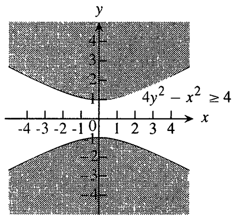


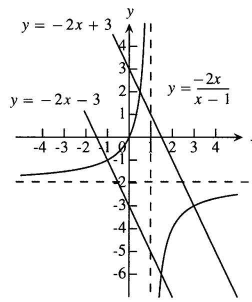
77. $3x^2 + 3y^2 - 7x - 7y + 4 = 0$
79. $(x + 2)^2 + (y - 1)^2 = 13$. The point is inside the circle.
81. b) 1 : 1 **83.** Length $= 2\sqrt{2}$, width $= \sqrt{2}$, area $= 4$
85. 24π **87.** $(0, 16/(3\pi))$

Section 9.2, pp. 726–727

1. $e = 3/5,\ F(\pm 3, 0),\ x = \pm 25/3$
3. $e = 1/\sqrt{2},\ F(0, \pm 1),\ y = \pm 2$
5. $e = 1/\sqrt{3},\ F(0, \pm 1),\ y = \pm 3$
7. $e = \sqrt{3}/3,\ F(\pm\sqrt{3}, 0),\ x = \pm 3\sqrt{3}$
9. $\dfrac{x^2}{27} + \dfrac{y^2}{36} = 1$ **11.** $\dfrac{x^2}{4851} + \dfrac{y^2}{4900} = 1$
13. $e = \dfrac{\sqrt{5}}{3},\ \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **15.** $e = 1/2,\ \dfrac{x^2}{64} + \dfrac{y^2}{48} = 1$
19. $\dfrac{(x-1)^2}{4} + \dfrac{(y-4)^2}{9} = 1,\ F(1, 4 \pm \sqrt{5}),\ e = \sqrt{5}/3,$
$y = 4 \pm (9\sqrt{5}/5)$
21. $a = 0,\ b = -4,\ c = 0,\ e = \sqrt{3}/2$
23. $e = \sqrt{2},\ F(\pm\sqrt{2}, 0),\ x = \pm 1/\sqrt{2}$
25. $e = \sqrt{2},\ F(0, \pm 4),\ y = \pm 2$
27. $e = \sqrt{5},\ F(\pm\sqrt{10}, 0),\ x = \pm 2/\sqrt{10}$
29. $e = \sqrt{5},\ F(0, \pm\sqrt{10}),\ y = \pm 2/\sqrt{10}$ **31.** $y^2 - \dfrac{x^2}{8} = 1$
33. $x^2 - \dfrac{y^2}{8} = 1$ **35.** $e = \sqrt{2},\ \dfrac{x^2}{8} - \dfrac{y^2}{8} = 1$
37. $e = 2,\ x^2 - \dfrac{y^2}{3} = 1$ **39.** $\dfrac{(y-6)^2}{36} - \dfrac{(x-1)^2}{45} = 1$

Section 9.3, pp. 733–734

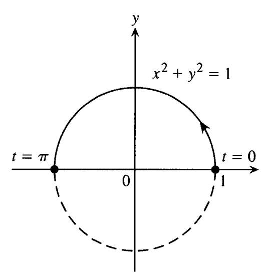
1. Hyperbola **3.** Ellipse **5.** Parabola **7.** Parabola
9. Hyperbola **11.** Hyperbola **13.** Ellipse **15.** Ellipse
17. $x'^2 - y'^2 = 4$, hyperbola **19.** $4x'^2 + 16y' = 0$, parabola
21. $y'^2 = 1$, parallel lines **23.** $2\sqrt{2}x'^2 + 8\sqrt{2}y' = 0$, parabola
25. $4x'^2 + 2y'^2 = 19$, ellipse
27. $\sin\alpha = 1/\sqrt{5},\ \cos\alpha = 2/\sqrt{5}$; or $\sin\alpha = -2/\sqrt{5}$,
$\cos\alpha = 1/\sqrt{5}$
29. $A' = 0.88,\ B' = 0.00,\ C' = 3.10,\ D' = 0.74,\ E' = -1.20,$
$F' = -3, 0.88x'^2 + 3.10y'^2 + 0.74x' - 1.20y' - 3 = 0$, ellipse
31. $A' = 0.00,\ B' = 0.00,\ C' = 5.00,\ D' = 0,\ E' = 0,\ F' = -5,$
$5.00y'^2 - 5 = 0$ or $y' = \pm 1.00$, parallel lines
33. $A' = 5.05,\ B' = 0.00,\ C' = -0.05,\ D' = -5.07,\ E' = -6.18,$
$F' = -1,\ 5.05x'^2 - 0.05y'^2 - 5.07x' - 6.18y' - 1 = 0$, hyperbola
35. a) $\dfrac{x'^2}{b^2} + \dfrac{y'^2}{a^2} = 1$ b) $\dfrac{y'^2}{a^2} - \dfrac{x'^2}{b^2} = 1$ c) $x'^2 + y'^2 = a^2$
d) $y' = -\dfrac{1}{m}x'$ e) $y' = -\dfrac{1}{m}x' + \dfrac{b}{m}$
37. a) $x'^2 - y'^2 = 2$ b) $x'^2 - y'^2 = 2a$ **43.** a) Parabola
45. a) Hyperbola
b)



c) $y = -2x - 3,\ y = -2x + 3$

Section 9.4, pp. 741–744

1.



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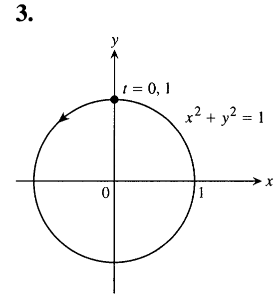

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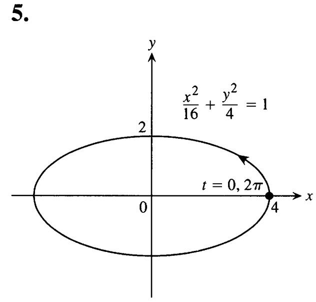

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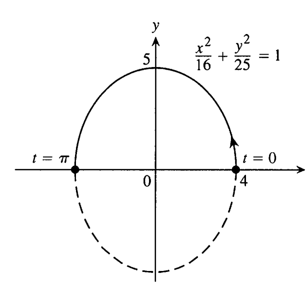

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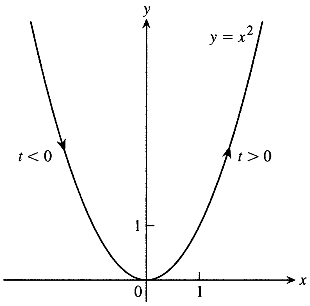

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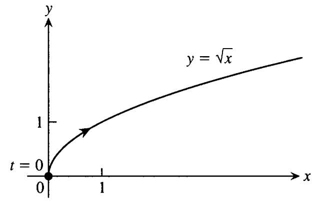

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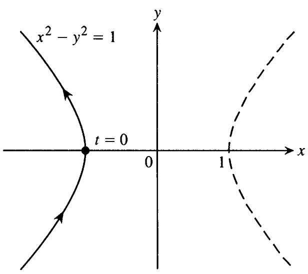

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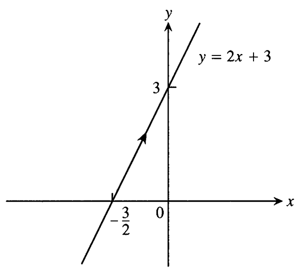

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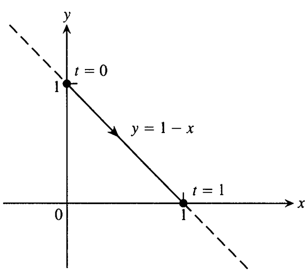

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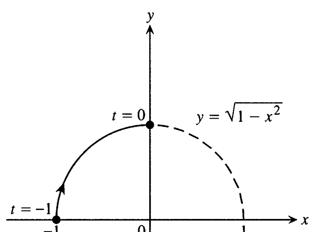

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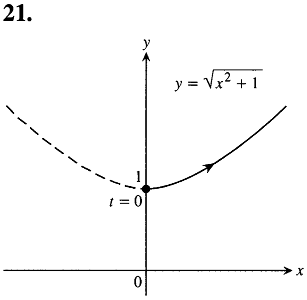

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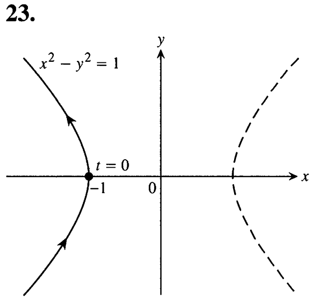


25. a) $x = a \cos t,\ y = -a \sin t,\ 0 \le t \le 2\pi$ b) $x = a \cos t,\ y = a \sin t,\ 0 \le t \le 2\pi$ c) $x = a \cos t,\ y = -a \sin t,\ 0 \le t \le 4\pi$ d) $x = a \cos t,\ y = a \sin t,\ 0 \le t \le 4\pi$

27. $x = \dfrac{-at}{\sqrt{1+t^2}},\ y = \dfrac{a}{\sqrt{1+t^2}},\ -\infty < t < \infty$

29. $x = 2 \cot t,\ y = 2 \sin^2 t,\ 0 < t < \pi$

31. b) $x = x_1 t,\ y = y_1 t$ (answer not unique) c) $x = -1 + t,\ y = t$ (answer not unique)

33. $x = (a-b)\cos\theta + b\cos\left(\dfrac{a-b}{b}\theta\right)$,
$y = (a-b)\sin\theta - b\sin\left(\dfrac{a-b}{b}\theta\right)$

35. $x = a \sin^2 t \tan t,\ y = a \sin^2 t$ **37.** $(1, 1)$

Section 9.5, pp. 749–751

1. $y = -x + 2\sqrt{2},\ \dfrac{d^2y}{dx^2} = -\sqrt{2}$

3. $y = -\dfrac{1}{2}x + 2\sqrt{2},\ \dfrac{d^2y}{dx^2} = -\dfrac{\sqrt{2}}{4}$ **5.** $y = x + \dfrac{1}{4},\ \dfrac{d^2y}{dx^2} = -2$

7. $y = 2x - \sqrt{3},\ \dfrac{d^2y}{dx^2} = -3\sqrt{3}$ **9.** $y = x - 4,\ \dfrac{d^2y}{dx^2} = \dfrac{1}{2}$

11. $y = \sqrt{3}x - \dfrac{\pi\sqrt{3}}{3} + 2,\ \dfrac{d^2y}{dx^2} = -4$ **13.** 0 **15.** -6 **17.** 4

19. 12 **21.** π^2 **23.** $8\pi^2$ **25.** $52\pi/3$ **27.** $3\pi\sqrt{5}$

29. a) $(\bar{x}, \bar{y}) = \left(\dfrac{12}{\pi} - \dfrac{24}{\pi^2}, \dfrac{24}{\pi^2} - 2\right)$ b) Centroid: $(1.4, 0.4)$

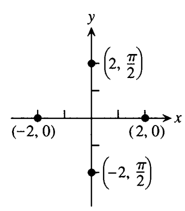
31. a) $(\bar{x}, \bar{y}) = \left(\dfrac{1}{3}, \pi - \dfrac{4}{3}\right)$ **33.** a) π b) π **37.** $3\pi a^2$

39. $64\pi/3$ **41.** $\left(\dfrac{\sqrt{2}}{2}, 1\right)$, $y = 2x$ at $t = 0$, $y = -2x$ at $t = \pi$

Section 9.6, pp. 755–756

1. a) e; b) g; c) h; d) f

3.



a) $\left(2, \dfrac{\pi}{2} + 2n\pi\right)$ and $\left(-2, \dfrac{\pi}{2} + (2n+1)\pi\right)$, n an integer

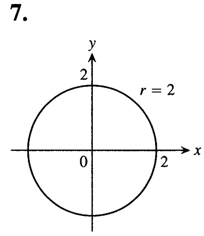
b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer

c) $\left(2, \dfrac{3\pi}{2} + 2n\pi\right)$ and $\left(-2, \dfrac{3\pi}{2} + (2n+1)\pi\right)$, n an integer

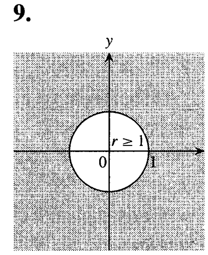
d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer

5. a) $(3, 0)$ b) $(-3, 0)$ c) $(-1, \sqrt{3})$ d) $(1, \sqrt{3})$, e) $(3, 0)$ f) $(1, \sqrt{3})$ g) $(-3, 0)$ h) $(-1, \sqrt{3})$

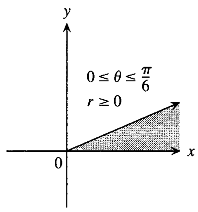
7.



9.



11.



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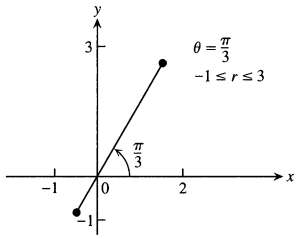

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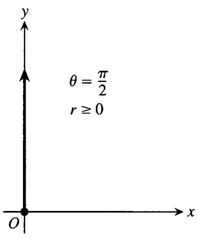

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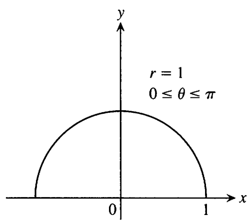

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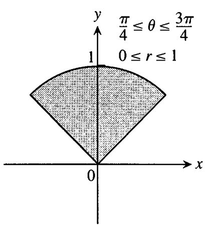
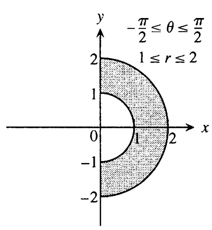


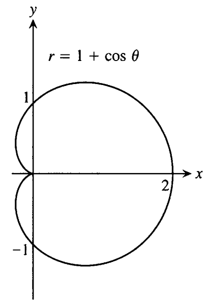
21.



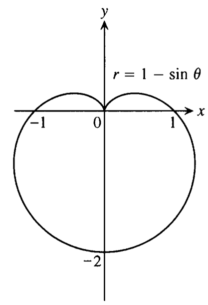
23. $x = 2$, vertical line through $(2, 0)$ **25.** $y = 0$, the x-axis
27. $y = 4$, horizontal line through $(0, 4)$
29. $x + y = 1$, line, $m = -1$, $b = 1$
31. $x^2 + y^2 = 1$, circle, $C(0, 0)$, radius 1
33. $y - 2x = 5$, line, $m = 2$, $b = 5$
35. $y^2 = x$, parabola, vertex $(0, 0)$, opens right
37. $y = e^x$, graph of natural exponential function
39. $x + y = \pm 1$, two straight lines of slope -1, y-intercepts $b = \pm 1$
41. $(x + 2)^2 + y^2 = 4$, circle, $C(-2, 0)$, radius 2
43. $x^2 + (y - 4)^2 = 16$, circle, $C(0, 4)$, radius 4
45. $(x - 1)^2 + (y - 1)^2 = 2$, circle, $C(1, 1)$, radius $\sqrt{2}$
47. $\sqrt{3}y + x = 4$ **49.** $r \cos \theta = 7$ **51.** $\theta = \pi/4$
53. $r = 2$ or $r = -2$ **55.** $4r^2 \cos^2 \theta + 9r^2 \sin^2 \theta = 36$
57. $r \sin^2 \theta = 4 \cos \theta$ **59.** $r = 4 \sin \theta$
61. $r^2 = 6r \cos \theta - 2r \sin \theta - 6$ **63.** $(0, \theta)$, where θ is any angle

Section 9.7, pp. 763–764

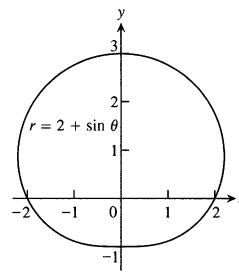
1. x-axis



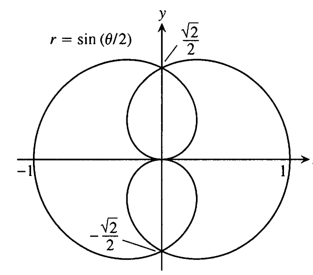
3. y-axis



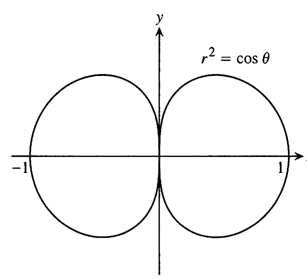
5. y-axis



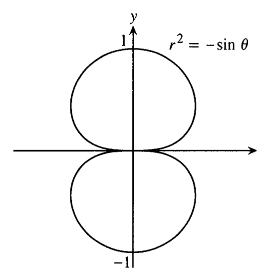
7. x-axis



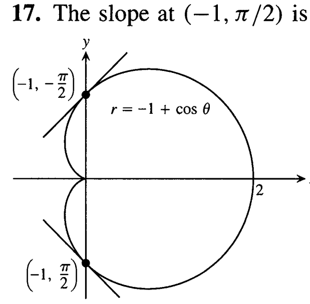
9. x-axis, y-axis, origin



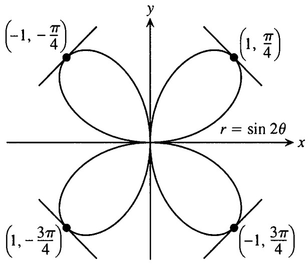
11. y-axis, x-axis, origin



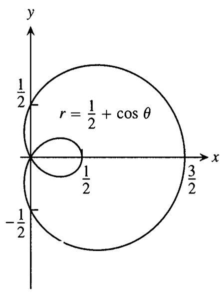
13. x-axis, y-axis, origin **15.** Origin
17. The slope at $(-1, \pi/2)$ is -1, at $(-1, -\pi/2)$ is 1.



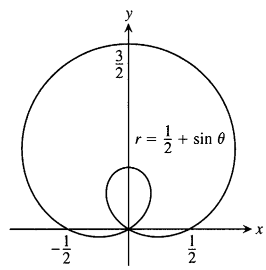
19. The slope at $(1, \pi/4)$ is -1, at $(-1, -\pi/4)$ is 1, at $(-1, 3\pi/4)$ is 1, at $(1, -3\pi/4)$ is -1.



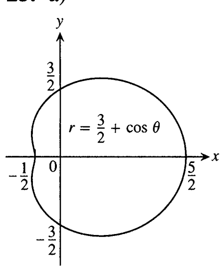
21. a)



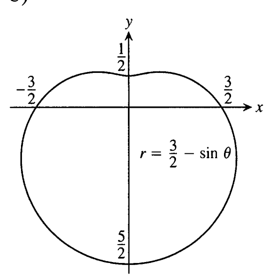
b)



23. a)



b)



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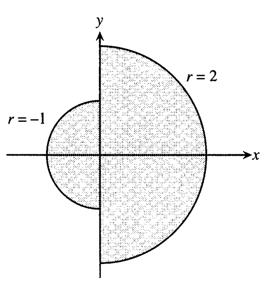
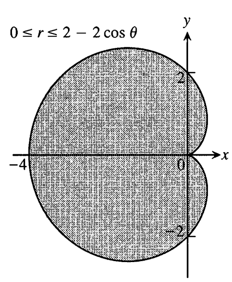


27.



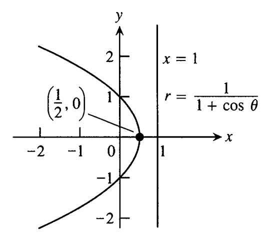
31. $(0, 0)$, $(1, \pi/2)$, $(1, 3\pi/2)$
33. $(0, 0)$, $(\sqrt{3}, \pi/3)$, $(-\sqrt{3}, -\pi/3)$
35. $(\sqrt{2}, \pm\pi/6)$, $(\sqrt{2}, \pm 5\pi/6)$
37. $(1, \pi/12)$, $(1, 5\pi/12)$, $(1, 13\pi/12)$, $(1, 17\pi/12)$ **43.** a

51. $2y = \dfrac{2\sqrt{6}}{9}$

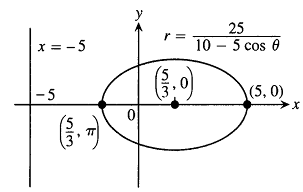
Section 9.8, pp. 768-770

1. $r\cos(\theta - \pi/6) = 5$, $y = -\sqrt{3}x + 10$
3. $r\cos(\theta - 4\pi/3) = 3$, $y = -(\sqrt{3}/3)x - 2\sqrt{3}$ **5.** $y = 2 - x$
7. $y = (\sqrt{3}/3)x + 2\sqrt{3}$ **9.** $r\cos\left(\theta - \dfrac{\pi}{4}\right) = 3$
11. $r\cos\left(\theta + \dfrac{\pi}{2}\right) = 5$ **13.** $r = 8\cos\theta$ **15.** $r = 2\sqrt{2}\sin\theta$
17. $C(2, 0)$, radius $= 2$ **19.** $C(1, \pi)$, radius $= 1$
21. $(x - 6)^2 + y^2 = 36$, $r = 12\cos\theta$
23. $x^2 + (y - 5)^2 = 25$, $r = 10\sin\theta$
25. $(x + 1)^2 + y^2 = 1$, $r = -2\cos\theta$
27. $x^2 + (y + 1/2)^2 = 1/4$, $r = -\sin\theta$ **29.** $r = 2/(1 + \cos\theta)$
31. $r = 30/(1 - 5\sin\theta)$ **33.** $r = 1/(2 + \cos\theta)$
35. $r = 10/(5 - \sin\theta)$

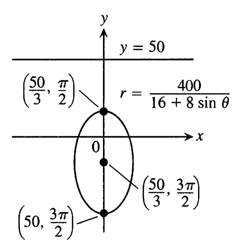
37.



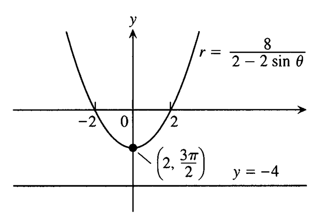
39.



41.



43.



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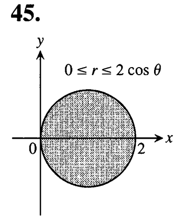
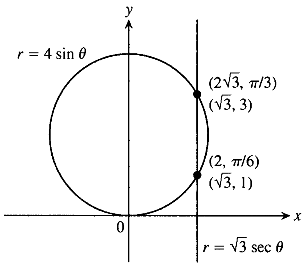


57. b)

Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU
Pluto	29.6549 AU	49.2251 AU

59. a) $x^2 + (y - 2)^2 = 4$, $x = \sqrt{3}$

b)



61. $r = 4/(1 + \cos \theta)$ **63.** b) The pins should be 2 in. apart. **65.** $r = 2a \sin \theta$ (a circle) **67.** $r \cos(\theta - \alpha) = p$ (a line)

Section 9.9, pp. 775–777

1. 18π **3.** $\pi/8$ **5.** 2 **7.** $\frac{\pi}{2} - 1$ **9.** $5\pi - 8$

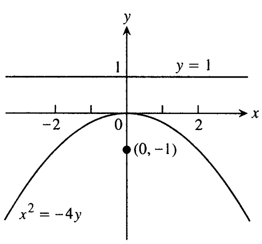
11. $3\sqrt{3} - \pi$ **13.** $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ **15.** $12\pi - 9\sqrt{3}$ **17.** a) $\frac{3}{2} - \frac{\pi}{4}$

19. $19/3$ **21.** 8 **23.** $3(\sqrt{2} + \ln(1 + \sqrt{2}))$ **25.** $\frac{\pi}{8} + \frac{3}{8}$

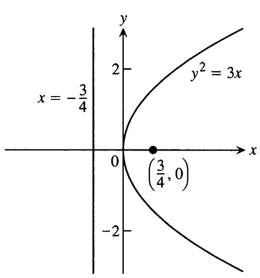
27. 2π **29.** $\pi\sqrt{2}$ **31.** $2\pi(2 - \sqrt{2})$ **37.** $\left(\frac{5}{6}a, 0\right)$

Chapter 9 Practice Exercises, pp. 778–782

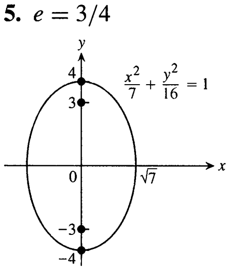
1.



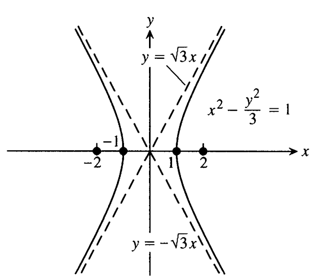
3.



5. $e = 3/4$



7. $e = 2$



9. $(x - 2)^2 = -12(y - 3)$, $V(2, 3)$, $F(2, 0)$; directrix: $y = 6$

11. $\frac{(x + 3)^2}{9} + \frac{(y + 5)^2}{25} = 1$, $C(-3, -5)$, $V(-3, 0)$ and $V(-3, -10)$, $F(-3, -1)$ and $F(-3, -9)$

13. $\frac{(y - 2\sqrt{2})^2}{8} - \frac{(x - 2)^2}{2} = 1$, $C(2, 2\sqrt{2})$, $V(2, 4\sqrt{2})$ and $V(2, 0)$, $F(2, \sqrt{10} + 2\sqrt{2})$ and $F(2, -\sqrt{10} + 2\sqrt{2})$; asymptotes: $y = 2x - 4 + 2\sqrt{2}$ and $y = -2x + 4 + 2\sqrt{2}$

15. Hyperbola: $\frac{(x - 2)^2}{4} - y^2 = 1$, $F(2 \pm \sqrt{5}, 0)$, $V(2 \pm 2, 0)$, $C(2, 0)$; asymptotes: $y = \pm\frac{1}{2}(x - 2)$

17. Parabola: $(y - 1)^2 = -16(x + 3)$, $V(-3, 1)$, $F(-7, 1)$; directrix: $x = 1$

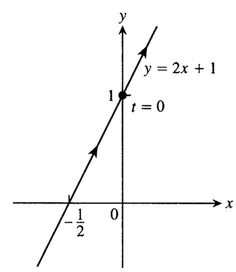
19. Ellipse: $\frac{(x + 3)^2}{16} + \frac{(y - 2)^2}{9} = 1$, $F(\pm\sqrt{7} - 3, 2)$, $V(\pm 4 - 3, 2)$, $C(-3, 2)$

21. Circle: $(x - 1)^2 + (y - 1)^2 = 2$, $C(1, 1)$, radius $= \sqrt{2}$

23. Ellipse **25.** Hyperbola **27.** Line

29. Ellipse, $5x'^2 + 3y'^2 = 30$ **31.** Hyperbola, $x'^2 - y'^2 = 2$

33.



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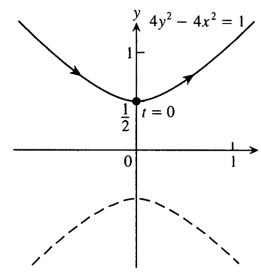

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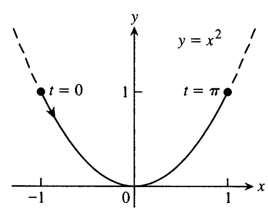
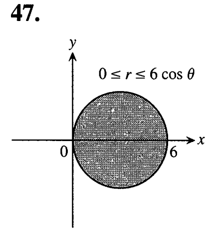


39. $x = 3 \cos t$, $y = 4 \sin t$, $0 \le t \le 2\pi$ **41.** $y = \frac{\sqrt{3}}{2}x + \frac{1}{4}, \frac{1}{4}$

43. $3 + \frac{\ln 2}{8}$ **45.** $76\pi/3$

47.

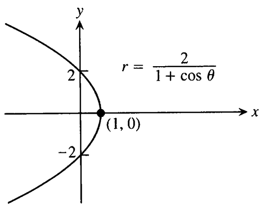


49. d **51.** l **53.** k **55.** i **57.** $(0, 0)$

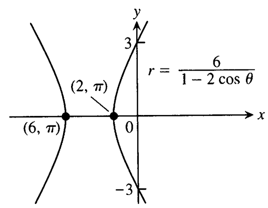
59. $(0, 0)$, $(1, \pm\pi/2)$ **61.** The graphs coincide. **63.** $(\sqrt{2}, \pi/4)$

65. $y = x$, $y = -x$

67. At $(1, \pi/4): r\cos(\theta - \pi/4) = 1$, At $(1, 3\pi/4)$: $r\cos(\theta - 3\pi/4) = 1$, At $(1, 5\pi/4): r\cos(\theta - 5\pi/4) = 1$, At $(1, 7\pi/4): r\cos(\theta - 7\pi/4) = 1$
69. $y = (\sqrt{3}/3)x - 4$ **71.** $x = 2$ **73.** $y = -3/2$
75. $x^2 + (y+2)^2 = 4$ **77.** $(x - \sqrt{2})^2 + y^2 = 2$
79. $r = -5\sin\theta$ **81.** $r = 3\cos\theta$
83.



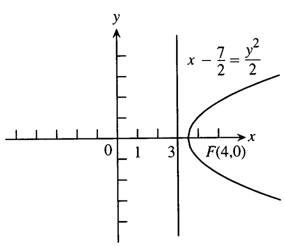
85.



87. $r = \dfrac{4}{1 + 2\cos\theta}$ **89.** $r = \dfrac{2}{2 + \sin\theta}$ **91.** $9\pi/2$
93. $2 + \pi/4$ **95.** 8 **97.** $\pi - 3$ **99.** $(2 - \sqrt{2})\pi$
101. a) 24π b) 16π **111.** $\pi/2$ **115.** $\left(2, \pm\dfrac{\pi}{3}\right), \dfrac{\pi}{2}$
119. $\pi/2$ **121.** $\pi/4$

Chapter 9 Additional Exercises, pp. 783–786

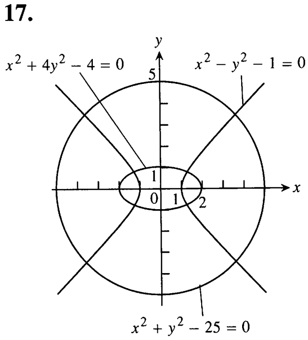
1.



3. $3x^2 + 3y^2 - 8y + 4 = 0$ **5.** $(0, \pm 1)$

7. a) $\dfrac{(y-1)^2}{16} - \dfrac{x^2}{48} = 1$ b) $\dfrac{16\left(y + \dfrac{3}{4}\right)^2}{25} - \dfrac{2x^2}{75} = 1$

17.



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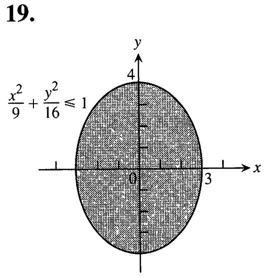

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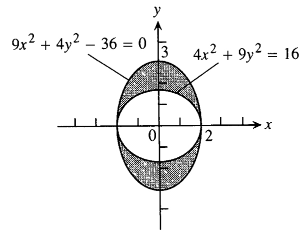
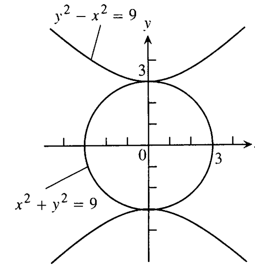


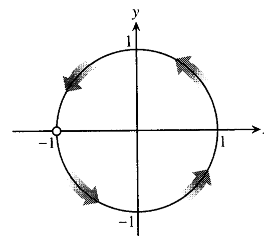
23.



25. $x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\theta\right)$,

$y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\theta\right)$

27. $(-1, 0)$, $t = -1, 0, 1$



29. a) $r = e^{2\theta}$ b) $\dfrac{\sqrt{5}}{2}(e^{4\pi} - 1)$

31. $\dfrac{32\pi - 4\pi\sqrt{2}}{5}$ **33.** $r = \dfrac{4}{1 + 2\cos\theta}$ **35.** $r = \dfrac{2}{2 + \sin\theta}$

37. a) $120°$ **39.** 1×10^7 mi. **41.** $e = \sqrt{2/3}$

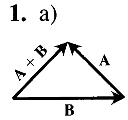
43. Yes, a parabola **45.** a) $r = \dfrac{2a}{1 + \cos\left(\theta - \dfrac{\pi}{4}\right)}$

b) $r = \dfrac{8}{3 - \cos\theta}$ c) $r = \dfrac{3}{1 + 2\sin\theta}$

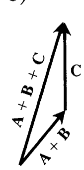
CHAPTER 10

Section 10.1, pp. 794–795

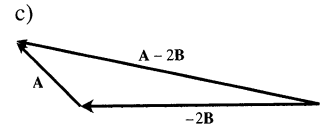
1. a)



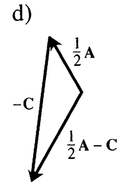
b)



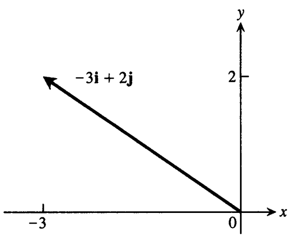
c)



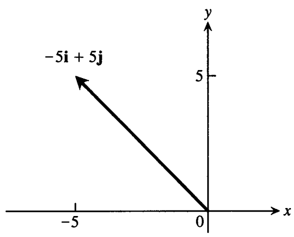
d)



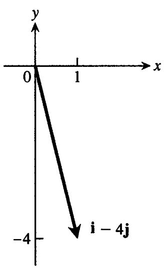
3. $4\mathbf{i}+5\mathbf{j}$ **5.** $(6-(\sqrt{3}/\pi))\mathbf{i}-20\mathbf{j}$
7. a) $\mathbf{w}=\mathbf{v}+\mathbf{u}$ b) $\mathbf{v}=\mathbf{w}-\mathbf{u}$
9.



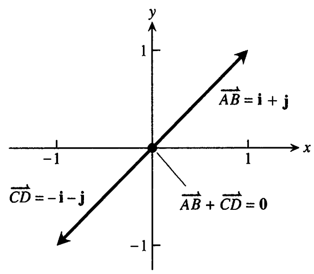
11.



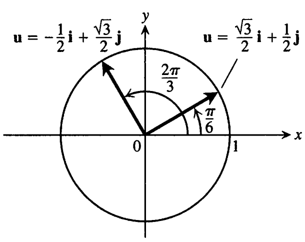
13.



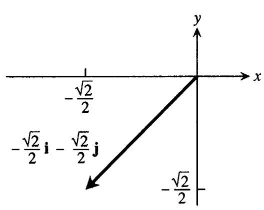
15.



17. $(5, 8)$
19.



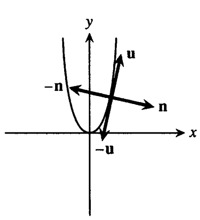
21.



23. $\dfrac{3}{5}\mathbf{i}-\dfrac{4}{5}\mathbf{j}$

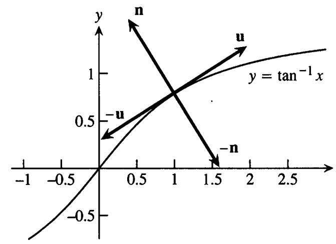
25. $\mathbf{u}=\dfrac{1}{\sqrt{17}}\mathbf{i}+\dfrac{4}{\sqrt{17}}\mathbf{j},\ -\mathbf{u}=-\dfrac{1}{\sqrt{17}}\mathbf{i}-\dfrac{4}{\sqrt{17}}\mathbf{j},$

$\mathbf{n}=\dfrac{4}{\sqrt{17}}\mathbf{i}-\dfrac{1}{\sqrt{17}}\mathbf{j},\ -\mathbf{n}=-\dfrac{4}{\sqrt{17}}\mathbf{i}+\dfrac{1}{\sqrt{17}}\mathbf{j}$



27. $\mathbf{u}=\dfrac{1}{\sqrt{5}}(2\mathbf{i}+\mathbf{j}),\ -\mathbf{u}=\dfrac{1}{\sqrt{5}}(-2\mathbf{i}-\mathbf{j}),\ \mathbf{n}=\dfrac{1}{\sqrt{5}}(-\mathbf{i}+2\mathbf{j}),$

$-\mathbf{n}=\dfrac{1}{\sqrt{5}}(\mathbf{i}-2\mathbf{j})$



29. $\mathbf{u}=\dfrac{\pm1}{5}(-4\mathbf{i}+3\mathbf{j}),\ \mathbf{v}=\dfrac{\pm1}{5}(3\mathbf{i}+4\mathbf{j})$

31. $\mathbf{u}=\dfrac{\pm1}{2}(\mathbf{i}+\sqrt{3}\mathbf{j}),\ \mathbf{v}=\dfrac{\pm1}{2}(-\sqrt{3}\mathbf{i}+\mathbf{j})$

33. $13\left(\dfrac{5}{13}\mathbf{i}+\dfrac{12}{13}\mathbf{j}\right)$

35. $\dfrac{3}{5}\mathbf{i}-\dfrac{4}{5}\mathbf{j}$ and $-\dfrac{3}{5}\mathbf{i}+\dfrac{4}{5}\mathbf{j}$ **39.** $5\sqrt{3}\mathbf{i},\ 5\mathbf{j}$

41. $\alpha=3/2,\ \beta=1/2$

43. a) $(5\cos 60°, 5\sin 60°)=\left(\dfrac{5}{2},\dfrac{5\sqrt{3}}{2}\right)$

b) $(5\cos 60°+10\cos 315°,\ 5\sin 60°+10\sin 315°)$

$=\left(\dfrac{5+\sqrt{2}}{2},\dfrac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

45. The slope of $-\mathbf{v}=-a\mathbf{i}-b\mathbf{j}$ is $(-b)/(-a)=b/a$, the same as the slope of $\mathbf{v}$.

Section 10.2, pp. 804–806

1. The line through the point (2, 3, 0) parallel to the z-axis
3. The x-axis **5.** The circle $x^2+y^2=4$ in the xy-plane
7. The circle $x^2+z^2=4$ in the xz-plane
9. The circle $y^2+z^2=1$ in the yz-plane
11. The circle $x^2+y^2=16$ in the xy-plane
13. a) The first quadrant of the xy-plane
b) The fourth quadrant of the xy-plane
15. a) The ball of radius 1 centered at the origin
b) All points greater than 1 unit from the origin
17. a) The upper hemisphere of radius 1 centered at the origin
b) The solid upper hemisphere of radius 1 centered at the origin
19. a) $x=3$ b) $y=-1$ c) $z=-2$ **21.** a) $z=1$ b) $x=3$
c) $y=-1$ **23.** a) $x^2+(y-2)^2=4,\ z=0$
b) $(y-2)^2+z^2=4,\ x=0$ c) $x^2+z^2=4,\ y=2$
25. a) $y=3,\ z=-1$ b) $x=1,\ z=-1$ c) $x=1,\ y=3$
27. $x^2+y^2+z^2=25,\ z=3$ **29.** $0\le z\le 1$ **31.** $z\le 0$
33. a) $(x-1)^2+(y-1)^2+(z-1)^2<1$

b) $(x-1)^2+(y-1)^2+(z-1)^2>1$ **35.** $3\left(\dfrac{2}{3}\mathbf{i}+\dfrac{1}{3}\mathbf{j}-\dfrac{2}{3}\mathbf{k}\right)$

37. $9\left(\dfrac{1}{9}\mathbf{i}+\dfrac{4}{9}\mathbf{j}-\dfrac{8}{9}\mathbf{k}\right)$ **39.** $5(\mathbf{k})$ **41.** $1\left(\dfrac{3}{5}\mathbf{i}+\dfrac{4}{5}\mathbf{k}\right)$

43. $\sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right)$ **45.** a) $2\mathbf{i}$ b) $-\sqrt{3}\,\mathbf{k}$
c) $\frac{3}{10}\mathbf{j}+\frac{2}{5}\mathbf{k}$ d) $6\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ **47.** $\frac{7}{13}(12\mathbf{i}-5\mathbf{k})$
49. $-\frac{10}{7}\mathbf{i}+\frac{15}{7}\mathbf{j}-\frac{30}{7}\mathbf{k}$ **51.** a) 3 b) $\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$
c) $(2, 2, 1/2)$ **53.** a) 7 b) $\frac{3}{7}\mathbf{i}-\frac{6}{7}\mathbf{j}+\frac{2}{7}\mathbf{k}$ c) $(5/2, 1, 6)$
55. a) $2\sqrt{3}$ b) $\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$ c) $(1, -1, -1)$
57. $A(4, -3, 5)$ **59.** $C(-2, 0, 2)$, $a = 2\sqrt{2}$
61. $C(\sqrt{2}, \sqrt{2}, -\sqrt{2})$, $a = \sqrt{2}$
63. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$
65. $(x+2)^2 + y^2 + z^2 = 3$ **67.** $C(-2, 0, 2)$, $a = \sqrt{8}$
69. $C\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$, $a = \frac{5\sqrt{3}}{4}$
71. a) $\sqrt{y^2+z^2}$ b) $\sqrt{x^2+z^2}$ c) $\sqrt{x^2+y^2}$
73. a) $\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}$ b) $\mathbf{i}+\mathbf{j}-2\mathbf{k}$ c) $(2, 2, 1)$

Section 10.3, pp. 812–814

1. a) -25, 5, 5 b) -1 c) -5 d) $-2\mathbf{i}+4\mathbf{j}-\sqrt{5}\,\mathbf{k}$
3. a) 25, 15, 5 b) 1/3 c) 5/3 d) $\frac{1}{9}(10\mathbf{i}+11\mathbf{j}-2\mathbf{k})$
5. a) 0, $\sqrt{53}$, 1 b) 0 c) 0 d) 0 **7.** a) 2, $\sqrt{34}$, $\sqrt{3}$
b) $\frac{2}{\sqrt{3}\sqrt{34}}$ c) $\frac{2}{\sqrt{34}}$ d) $\frac{1}{17}(5\mathbf{j}-3\mathbf{k})$ **9.** a) $\sqrt{3}-\sqrt{2}$, $\sqrt{2}$, 3
b) $\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$ c) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ d) $\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$
11. $\left(\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}\right)+\left(-\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}+4\mathbf{k}\right)$
13. $\left(\frac{14}{3}\mathbf{i}+\frac{28}{3}\mathbf{j}-\frac{14}{3}\mathbf{k}\right)+\left(\frac{10}{3}\mathbf{i}-\frac{16}{3}\mathbf{j}-\frac{22}{3}\mathbf{k}\right)$
15. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation

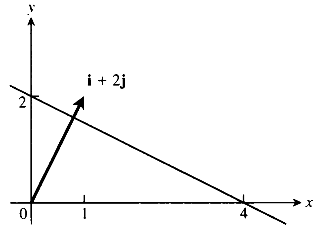
$$(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1+\mathbf{v}_2\cdot\mathbf{v}_1-\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2\cdot\mathbf{v}_2$$
$$= |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2.$$

21. $\tan^{-1}\sqrt{2}$ **23.** 0.75 rad **25.** 1.77 rad
27. $\angle A \approx 1.24$ rad, $\angle B \approx 0.66$ rad, $\angle C \approx 1.24$ rad **29.** 0.62 rad
31. a) Since $|\cos\theta| \le 1$, we have

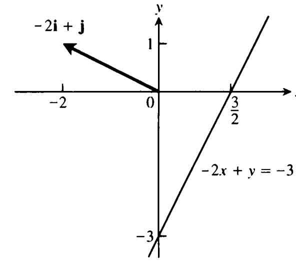
$$|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta| \le |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|.$$

b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ are $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.
33. a **35.** a) $\sqrt{70}$ b) $\sqrt{568}$ **37.** 5 J **39.** 3464.10 J

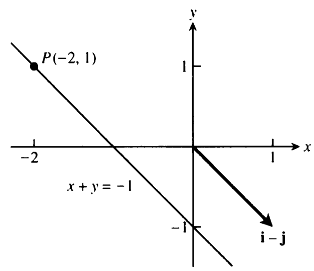
43. $x + 2y = 4$



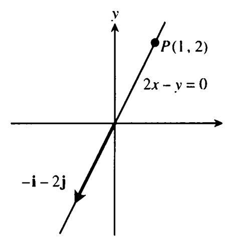
45. $-2x + y = -3$



47. $x + y = -1$



49. $2x - y = 0$

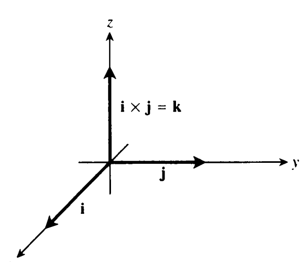


51. $\pi/4$ **53.** $\pi/6$ **55.** 0.14 **57.** $\pi/3$ and $2\pi/3$ at each point
59. At (0, 0): $\pi/2$, at (1, 1): $\pi/4$ and $3\pi/4$

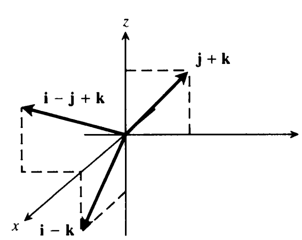
Section 10.4, pp. 820–821

1. $|\mathbf{A}\times\mathbf{B}| = 3$, direction is $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$; $|\mathbf{B}\times\mathbf{A}| = 3$, direction is $-\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$
3. $|\mathbf{A}\times\mathbf{B}| = 0$, no direction; $|\mathbf{B}\times\mathbf{A}| = 0$, no direction
5. $|\mathbf{A}\times\mathbf{B}| = 6$, direction is $-\mathbf{k}$; $|\mathbf{B}\times\mathbf{A}| = 6$, direction is $\mathbf{k}$
7. $|\mathbf{A}\times\mathbf{B}| = 6\sqrt{5}$, direction is $\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{k}$; $|\mathbf{B}\times\mathbf{A}| = 6\sqrt{5}$, direction is $-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k}$

9.



11.



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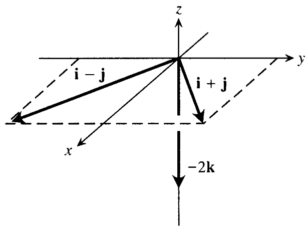


15. a) $2\sqrt{6}$ b) $\pm\dfrac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ **17.** a) $\dfrac{\sqrt{2}}{2}$ b) $\pm\dfrac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$
19. a) None b) **A** and **C** **21.** $10\sqrt{3}$ ft · lb **23.** 8 **25.** 7
27. a) True b) Not always true c) True d) True
e) Not always true f) True g) True h) True
29. a) $\text{proj}_{\mathbf{B}}\,\mathbf{A} = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}\mathbf{B}$ b) $\pm\mathbf{A}\times\mathbf{B}$ c) $\pm(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$
d) $|(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}|$ **31.** a) Yes b) No c) Yes d) No
33. No, **B** need not equal **C**. For example, $\mathbf{i}+\mathbf{j}\neq-\mathbf{i}+\mathbf{j}$, but

$$\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{i}\times\mathbf{i}+\mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$$

$$\mathbf{i}\times(-\mathbf{i}+\mathbf{j}) = -\mathbf{i}\times\mathbf{i}+\mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$$

35. 2 **37.** 13 **39.** 11/2 **41.** 25/2
43. If $\mathbf{A} = a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i}+b_2\mathbf{j}$, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

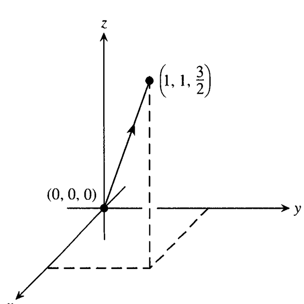
and the triangle's area is

$$\frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

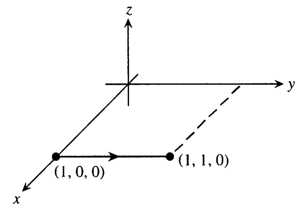
The applicable sign is (+) if the acute angle from **A** to **B** runs counterclockwise in the xy-plane, and (−) if it runs clockwise.

Section 10.5, pp. 827–829

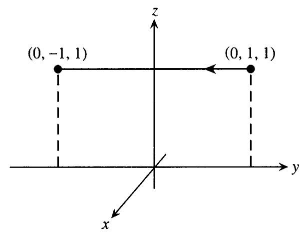
1. $x = 3+t,\ y = -4+t,\ z = -1+t$
3. $x = -2+5t,\ y = 5t,\ z = 3-5t$ **5.** $x = 0,\ y = 2t,\ z = t$
7. $x = 1,\ y = 1,\ z = 1+t$ **9.** $x = t,\ y = -7+2t,\ z = 2t$
11. $x = t,\ y = 0,\ z = 0$
13. $x = t,\ y = t,\ z = \dfrac{3}{2}t,\ 0 \le t \le 1$



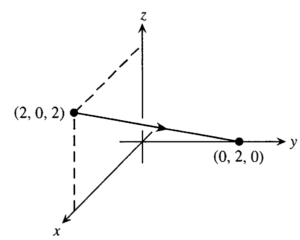
15. $x = 1,\ y = 1+t,\ z = 0,\ -1 \le t \le 0$



17. $x = 0,\ y = 1-2t,\ z = 1,\ 0 \le t \le 1$



19. $x = 2-2t,\ y = 2t,\ z = 2-2t,\ 0 \le t \le 1$

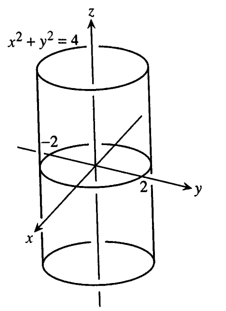


21. $3x-2y-z = -3$ **23.** $7x-5y-4z = 6$
25. $x+3y+4z = 34$ **27.** $(1, 2, 3)$, $-20x+12y+z = 7$
29. $y+z = 3$ **31.** $x-y+z = 0$ **33.** $2\sqrt{30}$ **35.** 0
37. $\dfrac{9\sqrt{42}}{7}$ **39.** 3 **41.** 19/5 **43.** 5/3 **45.** $9/\sqrt{41}$
47. $\pi/4$ **49.** 1.76 rad **51.** 0.82 rad **53.** $\left(\dfrac{3}{2}, -\dfrac{3}{2}, \dfrac{1}{2}\right)$
55. $(1, 1, 0)$ **57.** $x = 1-t,\ y = 1+t,\ z = -1$
59. $x = 4,\ y = 3+6t,\ z = 1+3t$
61. $L1$ intersects $L2$; $L2$ is parallel to $L3$; $L1$ and $L3$ are skew.
63. $x = 2+2t,\ y = -4-t,\ z = 7+3t$; $x = -2-t$, $y = -2+(1/2)t,\ z = 1-(3/2)t$
65. $\left(0, -\dfrac{1}{2}, -\dfrac{3}{2}\right)$, $(-1, 0, -3)$, $(1, -1, 0)$
69. Many possible answers. One possibility: $x+y = 3$ and $2y+z = 7$
71. $(x/a)+(y/b)+(z/c) = 1$ describes all planes *except* those through the origin or parallel to a coordinate axis.

Section 10.6, pp. 839–841

1. d, ellipsoid **3.** a, cylinder **5.** l, hyperbolic paraboloid
7. b, cylinder **9.** k, hyperbolic paraboloid **11.** h, cone

13.



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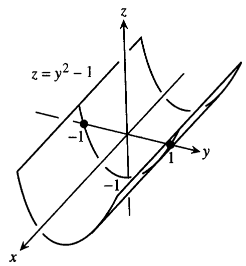

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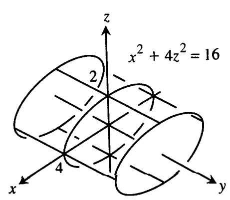

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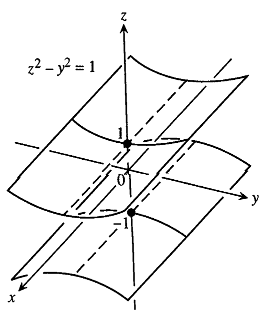

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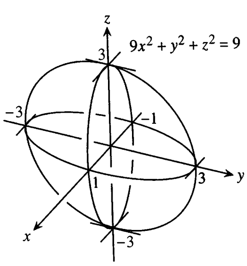

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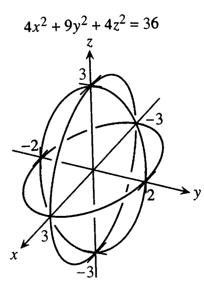

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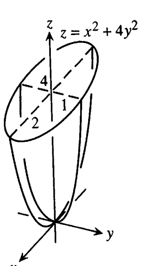

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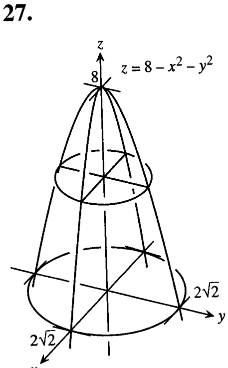

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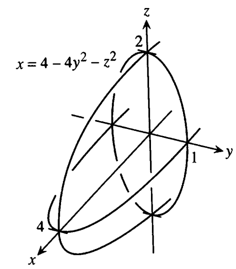

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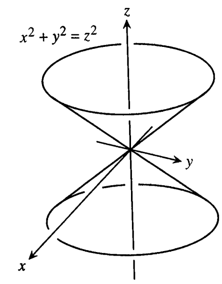

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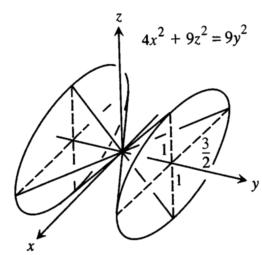

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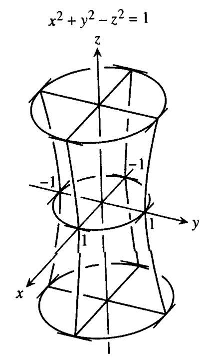

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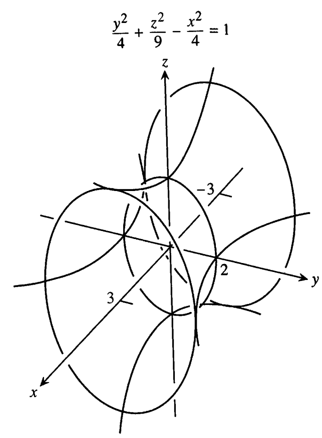

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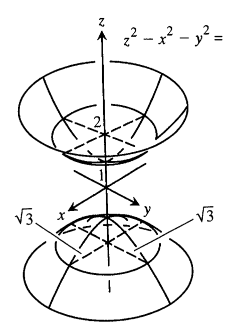

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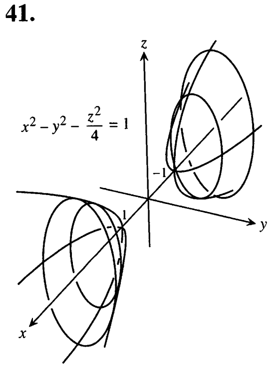

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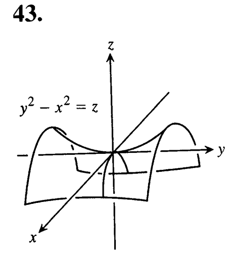

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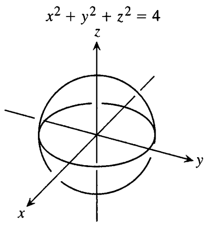

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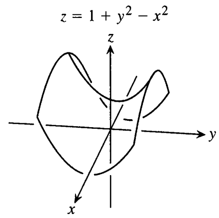

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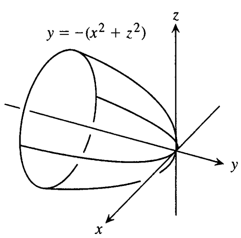

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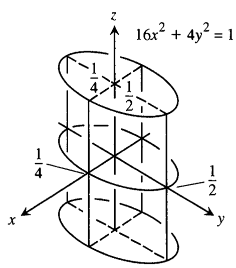

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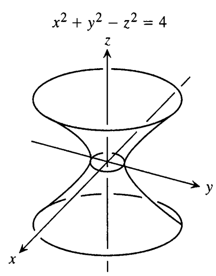

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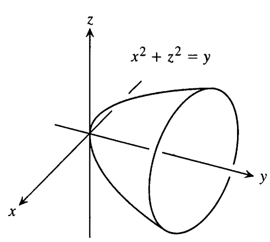

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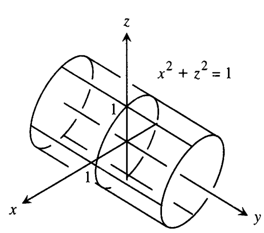

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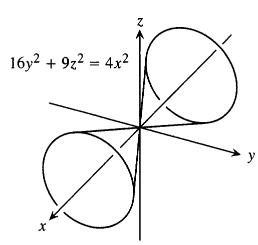

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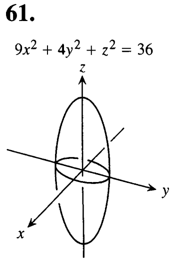

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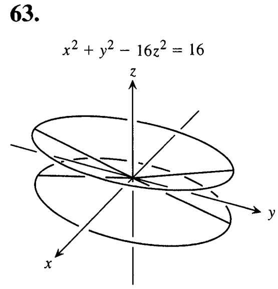

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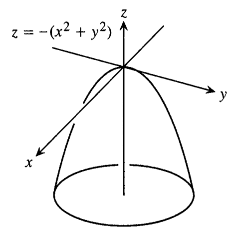

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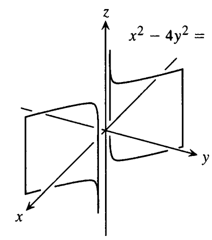

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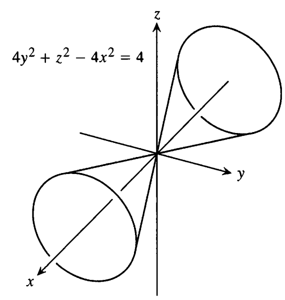

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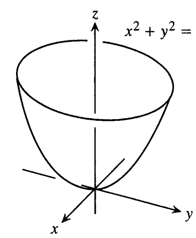

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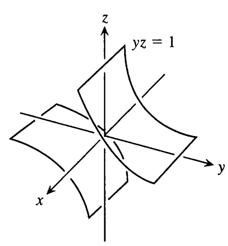

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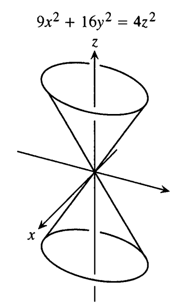


77. a) $\dfrac{2\pi(9 - c^2)}{9}$ b) 8π c) $\dfrac{4\pi abc}{3}$

81. Vertex $(0, y_1, cy_1^2/b^2)$, focus $(0, y_1, c(y_1^2/b^2) - a^2/(4c))$

Section 10.7, pp. 846–847

	Rectangular	Cylindrical	Spherical
1.	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
3.	$(0, 1, 0)$	$(1, \pi/2, 0)$	$(1, \pi/2, \pi/2)$
5.	$(1, 0, 0)$	$(1, 0, 0)$	$(1, \pi/2, 0)$
7.	$(0, 1, 1)$	$(1, \pi/2, 1)$	$(\sqrt{2}, \pi/4, \pi/2)$
9.	$(0, -2\sqrt{2}, 0)$	$(2\sqrt{2}, 3\pi/2, 0)$	$(2\sqrt{2}, \pi/2, 3\pi/2)$

11. $x^2 + y^2 = 0$, $\theta = 0$ or $\theta = \pi$, the z-axis

13. $z = 0$, $\phi = \pi/2$, the xy-plane

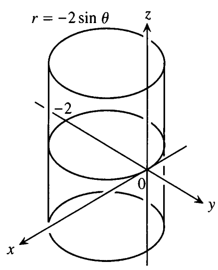
15. $z = r$, $0 \le r \le 1$; $\phi = \pi/4$, $0 \le \rho \le \sqrt{2}$; a (finite) cone

17. $x = 0$, $\theta = \pi/2$, the yz-plane

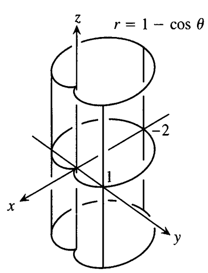
19. $r^2 + z^2 = 4$, $\rho = 2$, sphere of radius 2 centered at the origin

21. $x^2 + y^2 + \left(z - \dfrac{5}{2}\right)^2 = \dfrac{25}{4}$, $r^2 + z^2 = 5z$, sphere of radius 5/2 centered at $(0, 0, 5/2)$ (rectangular)

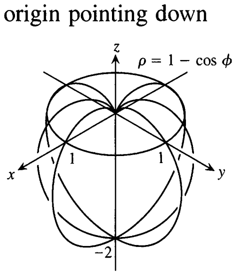
23. $y = 1$, $\rho \sin \phi \sin \theta = 1$, the plane $y = 1$
25. $z = \sqrt{2}$, the plane $z = \sqrt{2}$
27. $r^2 + z^2 = 2z$, $z \le 1$; $\rho = 2 \cos \phi$, $\pi/4 \le \phi \le \pi/2$; lower half (hemisphere) of the sphere of radius 1 centered at $(0, 0, 1)$ (rectangular)
29. $x^2 + y^2 + z^2 = 9$, $-3/2 \le z \le 3/2$; $r^2 + z^2 = 9$, $-3/2 \le z \le 3/2$; the portion of the sphere of radius 3 centered at the origin between the planes $z = -3/2$ and $z = 3/2$
31. $z = 4 - 4(x^2 + y^2)$, $0 \le z \le 4$; $\rho \cos \phi = 4 - 4\rho^2 \sin^2 \phi$, $0 \le \phi \le \pi/2$; the upper portion cut from the paraboloid $z = 4 - 4(x^2 + y^2)$ by the xy-plane
33. $z = -\sqrt{x^2 + y^2}$, $-1 \le z \le 0$; $z = -r$, $0 \le r \le 1$; cone, vertex at origin, base the circle $x^2 + y^2 = 1$ in the plane $z = -1$
35. $z + x^2 - y^2 = 0$ or $z = y^2 - x^2$, $\cos \phi + \rho \sin^2 \phi \cos 2\theta = 0$, hyperbolic paraboloid
37. (2, 3, 1)
39. Right circular cylinder parallel to the z-axis generated by the circle $r = -2 \sin \theta$ in the $r\theta$-plane



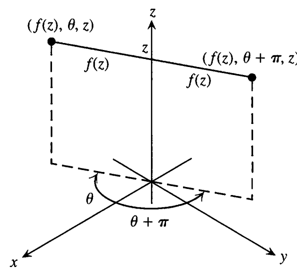
41. Cylinder of lines parallel to the z-axis generated by the cardioid $r = 1 - \cos \theta$ in the $r\theta$-plane



43. Cardioid of revolution symmetric about the y-axis, cusp at the origin pointing down

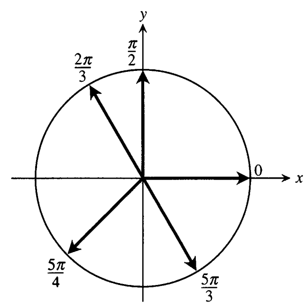


45. b) $\phi = \pi/2$
49. The surface's equation $r = f(z)$ tells us that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ. In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ lies on the surface, so the surface is symmetric with respect to the z-axis.



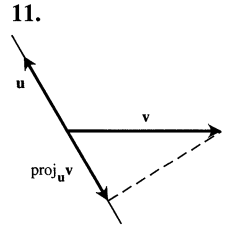
Chapter 10 Practice Exercises, pp. 848–851

1.



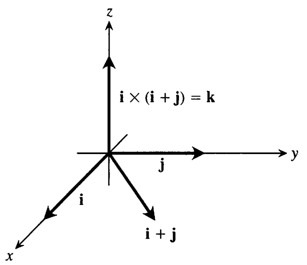
3. $2 \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right)$ **5.** $7 \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$

7. $\frac{8}{\sqrt{33}}\mathbf{i} - \frac{2}{\sqrt{33}}\mathbf{j} + \frac{8z}{\sqrt{33}}\mathbf{k}$
9. a) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$
b) $\overrightarrow{AP} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AD})$
11.



13. $|\mathbf{A}| = \sqrt{2}$, $|\mathbf{B}| = 3$, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = 3$, $\mathbf{A} \times \mathbf{B} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{B} \times \mathbf{A} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $|\mathbf{A} \times \mathbf{B}| = 3$, $\theta = \pi/4$, $|\mathbf{B}| \cos \theta = 3/\sqrt{2}$, $\text{proj}_{\mathbf{A}} \mathbf{B} = (3/2)(\mathbf{i} + \mathbf{j})$
15. $\frac{4}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k}) - \frac{1}{3}(5\mathbf{i} + \mathbf{j} + 11\mathbf{k})$

17.



19. unit tangents $\pm\left(\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}\right)$, unit normals $\pm\left(-\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right)$

23. $2\sqrt{7}$ **25.** a) $A = \sqrt{14}$ b) $V = 1$ **29.** $\sqrt{78}/3$

31. $x = 1 - 3t,\ y = 2,\ z = 3 + 7t$ **33.** $\sqrt{2}$

35. $2x + y - z = 3$ **37.** $-9x + y + 7z = 4$

39. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right)$, $(-1, 0, -3)$, $(1, -1, 0)$

41. $\pi/3$ **43.** $x = -5 + 5t,\ y = 3 - t,\ z = -3t$

45. b) $x = -12t,\ y = 19/12 + 15t,\ z = 1/6 + 6t$

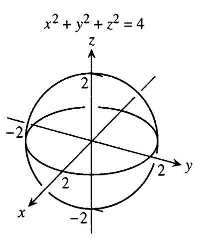
47. Yes; $\mathbf{v}$ is parallel to the plane. **49.** 3 **51.** $-3\,\mathbf{j} + 3\,\mathbf{k}$

53. $\frac{2}{\sqrt{35}}(5\,\mathbf{i} - \mathbf{j} - 3\,\mathbf{k})$ **55.** $\left(\frac{11}{9}, \frac{26}{9}, \frac{7}{9}\right)$

57. $(1, -2, -1)$; $x = 1 - 5t,\ y = -2 + 3t,\ z = -1 + 4t$

59. $2x + 7y + 2z + 10 = 0$ **61.** a) No b) no c) no d) no e) yes **63.** $11/\sqrt{107}$

65.



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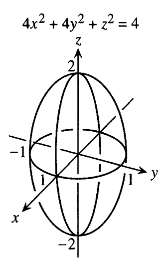

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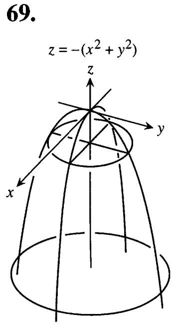

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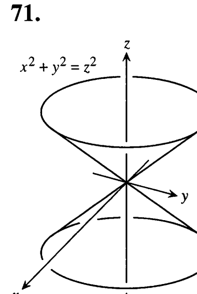

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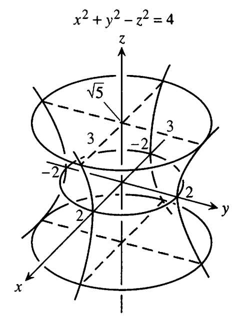

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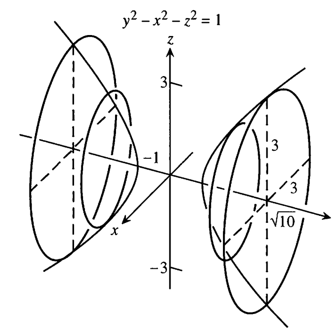


77. The y-axis in the xy-plane; the yz-plane in three dimensional space

79. The circle centered at $(0, 0)$ with radius 2 in the xy-plane; the cylinder parallel to the z-axis in three dimensional space with the circle as a generating curve

81. The parabola $x = y^2$ in the xy-plane; the cylinder parallel to the z-axis in three dimensional space with the parabola as a generating curve

83. A cardioid in the $r\theta$-plane; a cylinder parallel with the z-axis in three dimensional space with the cardioid as a generating curve

85. A horizontal lemniscate of length $2\sqrt{2}$ in the $r\theta$-plane; the cylinder parallel to the z-axis in three dimensional space with the lemniscate as a generating curve

87. The sphere of radius 2 centered at the origin

89. The upper nappe of the cone having its vertex at the origin and making a $\pi/6$ angle with the z-axis

91. The upper hemisphere of the sphere of radius 1 centered at the origin

	Rectangular	Cylindrical	Spherical
93.	$(1, 0, 0)$	$(1, 0, 0)$	$(1, \pi/2, 0)$
95.	$(0, 1, 1)$	$(1, \pi/2, 1)$	$(\sqrt{2}, \pi/4, \pi/2)$
97.	$(-1, 0, -1)$	$(1, \pi, -1)$	$(\sqrt{2}, 3\pi/4, \pi)$

99. Cylindrical: $z = 2$, spherical: $\rho \cos\phi = 2$, a plane parallel with the xy-plane

101. Cylindrical: $r^2 + z^2 = -2z$, spherical: $\rho = -2\cos\phi$, sphere of radius 1 centered at $(0, 0, -1)$ (rectangular)

103. Rectangular: $z = x^2 + y^2$, spherical: $\rho = 0$ or $\rho = \frac{\cos\phi}{\sin^2\phi}$ when $0 < \phi < \pi/2$, a paraboloid symmetric to the z-axis, opening upward, vertex at the origin

105. Rectangular: $x^2 + (y - 7/2)^2 = 49/4$, spherical: $\rho \sin\phi = 7\sin\theta$, cylinder parallel to the z-axis generated by the circle

107. Rectangular: $x^2 + y^2 + z^2 = 16$, cylindrical: $r^2 + z^2 = 16$, sphere of radius 4 centered at the origin

109. Rectangular: $-\sqrt{x^2 + y^2} = z$, cylindrical: $z = -r$, $r \geq 0$, single cone making an angle of $3\pi/4$ with the positive z-axis, vertex at the origin

Chapter 10 Additional Exercises, pp. 851–853

1. $(26, 23, -1/3)$ **3.** $\approx 34{,}641$ J

17. b) $6/\sqrt{14}$ c) $2x - y + 2z = 8$
d) $x - 2y + z = 3 + 5\sqrt{6}$ and $x - 2y + z = 3 - 5\sqrt{6}$

23. $\mathbf{v} - 2\dfrac{\mathbf{v} \cdot \mathbf{z}}{|\mathbf{z}|^2}\mathbf{z}$ **25.** a) $|\mathbf{F}| = \dfrac{GMm}{d^2}\left(1 + \sum_{i=1}^{n} \dfrac{2}{(i^2+1)^{3/2}}\right)$
b) Yes

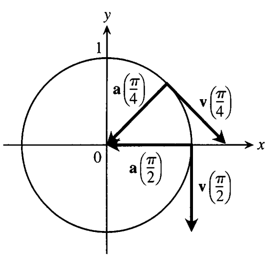
CHAPTER 11

Section 11.1, pp. 865–868

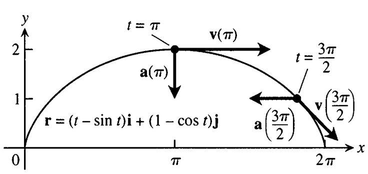
1. $y = x^2 - 2x,\ \mathbf{v} = \mathbf{i} + 2\mathbf{j},\ \mathbf{a} = 2\mathbf{j}$

3. $y = \dfrac{2}{9}x^2,\ \mathbf{v} = 3\mathbf{i} + 4\mathbf{j},\ \mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$

5. $t = \dfrac{\pi}{4} : \mathbf{v} = \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j},\ \mathbf{a} = \dfrac{-\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j};$
$t = \pi/2 : \mathbf{v} = -\mathbf{j},\ \mathbf{a} = -\mathbf{i}$



7. $t = \pi : \mathbf{v} = 2\mathbf{i},\ \mathbf{a} = -\mathbf{j};\ t = \dfrac{3\pi}{2} : \mathbf{v} = \mathbf{i} - \mathbf{j},\ \mathbf{a} = -\mathbf{i}$



9. $\mathbf{v} = \mathbf{i} + 2t\,\mathbf{j} + 2\mathbf{k};\ \mathbf{a} = 2\mathbf{j};$ speed: 3; direction: $\dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k};$
$\mathbf{v}(1) = 3\left(\dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}\right)$

11. $\mathbf{v} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k};$
$\mathbf{a} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j};$ speed: $2\sqrt{5};$
direction: $(-1/\sqrt{5})\mathbf{i} + (2/\sqrt{5})\mathbf{k};$
$\mathbf{v}(\pi/2) = 2\sqrt{5}[(-1/\sqrt{5})\mathbf{i} + (2/\sqrt{5})\mathbf{k}]$

13. $\mathbf{v} = \left(\dfrac{2}{t+1}\right)\mathbf{i} + 2t\,\mathbf{j} + t\,\mathbf{k};\ \mathbf{a} = \left(\dfrac{-2}{(t+1)^2}\right)\mathbf{i} + 2\mathbf{j} + \mathbf{k};$
speed: $\sqrt{6};$ direction: $\dfrac{1}{\sqrt{6}}\mathbf{i} + \dfrac{2}{\sqrt{6}}\mathbf{j} + \dfrac{1}{\sqrt{6}}\mathbf{k};$
$\mathbf{v}(1) = \sqrt{6}\left(\dfrac{1}{\sqrt{6}}\mathbf{i} + \dfrac{2}{\sqrt{6}}\mathbf{j} + \dfrac{1}{\sqrt{6}}\mathbf{k}\right)$

15. $\pi/2$ **17.** $\pi/2$ **19.** $t = 0,\ \pi,\ 2\pi$

21. $(1/4)\mathbf{i} + 7\mathbf{j} + (3/2)\mathbf{k}$ **23.** $\left(\dfrac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + 2\mathbf{k}$

25. $(\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$

27. $\mathbf{r}(t) = \left(\dfrac{-t^2}{2} + 1\right)\mathbf{i} + \left(\dfrac{-t^2}{2} + 2\right)\mathbf{j} + \left(\dfrac{-t^2}{2} + 3\right)\mathbf{k}$

29. $\mathbf{r}(t) = ((t+1)^{3/2} - 1)\mathbf{i} + (-e^{-t} + 1)\mathbf{j} + (\ln(t+1) + 1)\mathbf{k}$

31. $\mathbf{r}(t) = 8t\,\mathbf{i} + 8t\,\mathbf{j} + (-16t^2 + 100)\mathbf{k}$

33. $x = t,\ y = -1,\ z = 1 + t$

35. $x = at,\ y = a,\ z = 2\pi b + bt$

37. a) (i): It has constant speed 1 (ii): Yes (iii): Counterclockwise (iv): Yes b) (i): It has constant speed 2 (ii): Yes (iii): Counterclockwise (iv): Yes c) (i): It has constant speed 1 (ii): Yes (iii): Counterclockwise (iv): It starts at $(0, -1)$ instead of $(1, 0)$ d) (i): It has constant speed 1 (ii): Yes (iii): Clockwise (iv): Yes e) (i): It has variable speed (ii): No (iii): Counterclockwise (iv): Yes

39. $\mathbf{r}(t) = \left(\dfrac{3}{2}t^2 + \dfrac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\dfrac{1}{2}t^2 + \dfrac{2}{\sqrt{11}}t - 2\right)\mathbf{j} +$
$\left(\dfrac{1}{2}t^2 + \dfrac{2}{\sqrt{11}}t + 3\right)\mathbf{k} = \left(\dfrac{1}{2}t^2 + \dfrac{2t}{\sqrt{11}}\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) +$
$(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

41. $\mathbf{v} = 2\sqrt{5}\,\mathbf{i} + \sqrt{5}\,\mathbf{j}$

43. $\max|\mathbf{v}| = 3,\ \min|\mathbf{v}| = 2,\ \max|\mathbf{a}| = 3,\ \min|\mathbf{a}| = 2$

Section 11.2, pp. 873–876

1. 50 sec **3.** a) 72.2 sec, 25,510 m b) 4020 m c) 6378 m

5. $t \approx 2.135$ sec, $x \approx 66.42$ ft

7. $v_0 = 9.9$ m/sec, $\alpha = 18.4°$ or $71.6°$ **9.** 190 mph

11. The golf ball will clip the leaves at the top. **13.** 46.6 ft/sec

17. 141% **21.** 1.92 sec, 73.7 ft (approx.)

25. $\mathbf{v}(t) = -gt\,\mathbf{k} + \mathbf{v}_0,\ \mathbf{r}(t) = -\dfrac{1}{2}gt^2\,\mathbf{k} + \mathbf{v}_0 t$

Section 11.3, pp. 880–881

1. $\mathbf{T} = \left(-\dfrac{2}{3}\sin t\right)\mathbf{i} + \left(\dfrac{2}{3}\cos t\right)\mathbf{j} + \dfrac{\sqrt{5}}{3}\mathbf{k},\ 3\pi$

3. $\mathbf{T} = \dfrac{1}{\sqrt{1+t}}\mathbf{i} + \dfrac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k},\ \dfrac{52}{3}$ **5.** $\mathbf{T} = -\cos t\,\mathbf{j} + \sin t\,\mathbf{k},\ \dfrac{3}{2}$

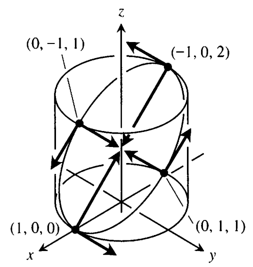
7. $\mathbf{T} = \left(\dfrac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\dfrac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\dfrac{\sqrt{2}\,t^{1/2}}{t+1}\right)\mathbf{k},$
$\dfrac{\pi^2}{2} + \pi$

9. $(0, 5, 24\pi)$ **11.** $s(t) = 5t,\ L = \dfrac{5\pi}{2}$

13. $s(t) = \sqrt{3}e^t - \sqrt{3},\ L = \dfrac{3\sqrt{3}}{4}$ **15.** $\sqrt{2} + \ln(1 + \sqrt{2})$

17. a) Cylinder is $x^2 + y^2 = 1$, plane is $x + z = 1$

b) and c)



d) $L = \int_0^{2\pi} \sqrt{1 + \sin^2 t}\, dt$ e) $L \approx 7.64$

Section 11.4, pp. 890–893

1. $\mathbf{T} = (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j},\ \mathbf{N} = (-\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j},\ \kappa = \cos t$

3. $\mathbf{T} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j},\ \mathbf{N} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j},$

$\kappa = \frac{1}{2\left(\sqrt{1+t^2}\right)^3}$

5. $\mathbf{a} = \frac{2t}{\sqrt{1+t^2}}\mathbf{T} + \frac{2}{\sqrt{1+t^2}}\mathbf{N}$ **7.** b) $\cos x$

9. b) $\mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$

c) $\mathbf{N} = -\frac{1}{2}\left(\sqrt{4-t^2}\,\mathbf{i} + t\,\mathbf{j}\right)$

11. $\mathbf{T} = \frac{3\cos t}{5}\mathbf{i} - \frac{3\sin t}{5}\mathbf{j} + \frac{4}{5}\mathbf{k},\ \mathbf{N} = (-\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j},$

$\mathbf{B} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k},\ \kappa = \frac{3}{25},\ \tau = -\frac{4}{25}$

13. $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t + \sin t}{\sqrt{2}}\right)\mathbf{j},$

$\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j},$

$\mathbf{B} = \mathbf{k},\ \kappa = \frac{1}{e^t\sqrt{2}},\ \tau = 0$

15. $\mathbf{T} = \frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j},\ \mathbf{N} = \frac{\mathbf{i}}{\sqrt{t^2+1}} - \frac{t\,\mathbf{j}}{\sqrt{t^2+1}},$

$\mathbf{B} = -\mathbf{k},\ \kappa = \frac{1}{t(t^2+1)^{3/2}},\ \tau = 0$

17. $\mathbf{T} = \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i} + \left(\tanh\frac{t}{a}\right)\mathbf{j},$

$\mathbf{N} = \left(-\tanh\frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j},$

$\mathbf{B} = \mathbf{k},\ \kappa = \frac{1}{a}\operatorname{sech}^2\frac{t}{a},\ \tau = 0$

19. $\mathbf{a} = |a|\,\mathbf{N}$ **21.** $\mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$ **23.** $\mathbf{a}(0) = 2\,\mathbf{N}$

25. $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k},\ \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j},$

$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j},\ \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$; osculating plane: $z = -1$;
normal plane: $-x + y = 0$; rectifying plane: $x + y = \sqrt{2}$

27. Yes. If the car is moving on a curved path ($\kappa \neq 0$), then $a_{\mathbf{N}} = \kappa|\mathbf{v}|^2 \neq 0$ and $\mathbf{a} \neq \mathbf{0}$.

31. $|\mathbf{F}| = \kappa\left(m\left(\frac{ds}{dt}\right)^2\right)$ **35.** $1/(2b)$ **39.** a) $b - a$ b) π

45. $\kappa(x) = 2/(1+4x^2)^{3/2}$ **47.** $\kappa(x) = |\sin x|/(1+\cos^2 x)^{3/2}$

57. Components of $\mathbf{v}$: $-1.8701, 0.7089, 1.0000$
Components of $\mathbf{a}$: $-1.6960,\ -2.0307,\ 0$
Speed: 2.2361; Components of $\mathbf{T}$: $-0.8364, 0.3170, 0.4472$
Components of $\mathbf{N}$: $-0.4143, -0.8998, -0.1369$
Components of $\mathbf{B}$: $0.3590, -0.2998, 0.8839$; Curvature: 0.5060
Torsion: 0.2813; Tangential component of acceleration: 0.7746
Normal component of acceleration: 2.5298

59. Components of $\mathbf{v}$: $2.0000, 0, 0.1629$
Components of $\mathbf{a}$: $0, -1.0000, 0.0086$; Speed: 2.0066
Components of $\mathbf{T}$: $0.9967, 0, 0.0812$
Components of $\mathbf{N}$: $-0.0007, -1.0000, 0.0086$
Components of $\mathbf{B}$: $0.0812, -0.0086, -0.9967$; Curvature: 0.2484
Torsion: -0.0411; Tangential component of acceleration: 0.0007
Normal component of acceleration: 1.0000

Section 11.5, pp. 901–902

1. $T = 93.2$ min **3.** $a = 6763$ km **5.** $T = 1655$ min

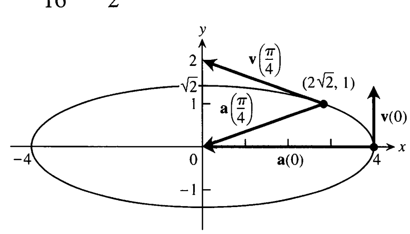
7. $a = 20{,}430$ km **9.** $|v| = 1.9966 \times 10^7\, r^{-1/2}$ m/sec

11. Circle: $v_0 = \sqrt{\frac{GM}{r_0}}$; ellipse: $\sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$;
parabola: $v_0 = \sqrt{\frac{2GM}{r_0}}$; hyperbola: $v_0 > \sqrt{\frac{2GM}{r_0}}$

15. a) $x(t) = 2 + (3 - 4\cos(\pi t))\cos(\pi t),\ y(t) = (3 - 4\cos(\pi t))\sin(\pi t)$

Chapter 11 Practice Exercises, pp. 902–905

1. $\frac{x^2}{16} + \frac{y^2}{2} = 1$



At $t = 0$: $a_{\mathbf{T}} = 0,\ a_{\mathbf{N}} = 4,\ \kappa = 2$;

At $t = \dfrac{\pi}{4} : a_{\mathbf{T}} = \dfrac{7}{3},\ a_{\mathbf{N}} = \dfrac{4\sqrt{2}}{3},\ \kappa = \dfrac{4\sqrt{2}}{27}$

3. $|\mathbf{v}|_{\max} = 1$ **5.** $\kappa = 1/5$ **7.** $dy/dt = -x$; clockwise

11. Shot put is on the ground, about 66 ft, 5 in. from the stopboard.

15. a) 59.19 ft/sec b) 74.58 ft/sec **19.** $\kappa = \pi s$

21. Length $= \dfrac{\pi}{4}\sqrt{1+\dfrac{\pi^2}{16}} + \ln\left(\dfrac{\pi}{4} + \sqrt{1+\dfrac{\pi^2}{16}}\right)$

23. $\mathbf{T}(0) = \dfrac{2}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$; $\mathbf{N}(0) = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$;

$\mathbf{B}(0) = -\dfrac{1}{3\sqrt{2}}\mathbf{i} + \dfrac{1}{3\sqrt{2}}\mathbf{j} + \dfrac{4}{3\sqrt{2}}\mathbf{k}$; $\kappa = \dfrac{\sqrt{2}}{3}$; $\tau = \dfrac{1}{6}$

25. $\mathbf{T}(\ln 2) = \dfrac{1}{\sqrt{17}}\mathbf{i} + \dfrac{4}{\sqrt{17}}\mathbf{j}$; $\mathbf{N}(\ln 2) = -\dfrac{4}{\sqrt{17}}\mathbf{i} + \dfrac{1}{\sqrt{17}}\mathbf{j}$;

$\mathbf{B}(\ln 2) = \mathbf{k}$; $\kappa = \dfrac{8}{17\sqrt{17}}$; $\tau = 0$

27. $\mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$

29. $\mathbf{T} = \left(\dfrac{1}{\sqrt{2}}\cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\dfrac{1}{\sqrt{2}}\cos t\right)\mathbf{k}$;

$\mathbf{N} = \left(-\dfrac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\dfrac{1}{\sqrt{2}}\sin t\right)\mathbf{k}$;

$\mathbf{B} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{k}$; $\kappa = \dfrac{1}{\sqrt{2}}$; $\tau = 0$

31. $\pi/3$ **33.** $x = 1+t,\ y = t,\ z = -t$

35. 5971 km, 1.639×10^7 km^2, 3.21% visible

Chapter 11 Additional Exercises, pp. 905–907

1. a) $\mathbf{r}(t) = \left(-\dfrac{8}{15}t^3 + 4t^2\right)\mathbf{i} + (-20t + 100)\mathbf{j}$; b) $\dfrac{100}{3}$ m

3. $\mathbf{v} = \dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$ **5.** a) $\left.\dfrac{d\theta}{dt}\right|_{\theta=2\pi} = 2\sqrt{\dfrac{\pi gb}{a^2+b^2}}$

b) $\theta = \dfrac{gbt^2}{2(a^2+b^2)},\ z = \dfrac{gb^2t^2}{2(a^2+b^2)}$

c) $\mathbf{v}(t) = \dfrac{gbt}{\sqrt{a^2+b^2}}\mathbf{T}$; $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{bg}{\sqrt{a^2+b^2}}\mathbf{T} + a\left(\dfrac{bgt}{a^2+b^2}\right)^2\mathbf{N}$

There is no component in the direction of **B**.

9. a) $\dfrac{dx}{dt} = \dot r\cos\theta - r\dot\theta\sin\theta,\ \dfrac{dy}{dt} = \dot r\sin\theta + r\dot\theta\cos\theta$

b) $\dfrac{dr}{dt} = \dot x\cos\theta + \dot y\sin\theta,\ r\dfrac{d\theta}{dt} = -\dot x\sin\theta + \dot y\cos\theta$

11. a) $\mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta,\ \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$ b) 6.5 in.

13. c) $\mathbf{v} = \dot r\,\mathbf{u}_r + r\dot\theta\,\mathbf{u}_\theta + \dot z\,\mathbf{k},\ \mathbf{a} = (\ddot r - r\dot\theta^2)\,\mathbf{u}_r + (r\ddot\theta + 2\dot r\dot\theta)\,\mathbf{u}_\theta + \ddot z\,\mathbf{k}$

15. a) $\mathbf{u}_\rho = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$,
$\mathbf{u}_\phi = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}$,
$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

CHAPTER 12

Section 12.1, pp. 914–917

1. a) All points in the xy-plane b) All reals c) The lines $y - x = c$ d) No boundary points e) Both open and closed f) Unbounded

3. a) All points in the xy-plane b) $z \ge 0$ c) For $f(x, y) = 0$, the origin; for $f(x, y) \ne 0$, ellipses with the center $(0, 0)$, and major and minor axes, along the x- and y-axes, respectively d) No boundary points e) Both open and closed f) Unbounded

5. a) All points in the xy-plane b) All reals c) For $f(x, y) = 0$, the x- and y-axes; for $f(x, y) \ne 0$, hyperbolas with the x- and y-axes as asymptotes d) No boundary points e) Both open and closed f) Unbounded

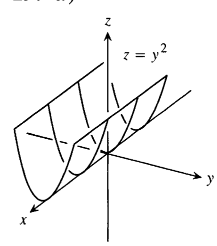
7. a) All (x, y) satisfying $x^2 + y^2 < 16$ b) $z \ge 1/4$ c) Circles centered at the origin with radii $r < 4$ d) Boundary is the circle $x^2 + y^2 = 16$ e) Open f) Bounded

9. a) $(x, y) \ne (0, 0)$ b) All reals c) The circles with center $(0, 0)$ and radii $r > 0$ d) Boundary is the single point $(0, 0)$ e) Open f) Unbounded

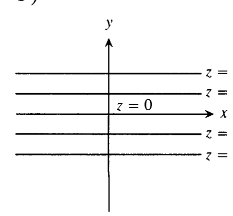
11. a) All (x, y) satisfying $-1 \le y - x \le 1$ b) $-\pi/2 \le z \le \pi/2$ c) Straight lines of the form $y - x = c$ where $-1 \le c \le 1$ d) Boundary is two straight lines $y = 1 + x$ and $y = -1 + x$ e) Closed f) Unbounded

13. f **15.** a **17.** d

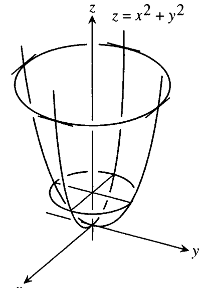
19. a)



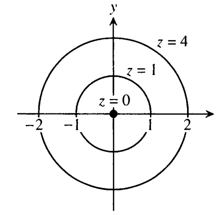
b)



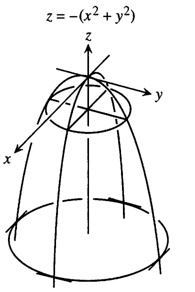
21. a)



b)



23. a)



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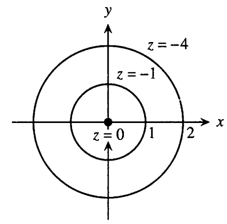
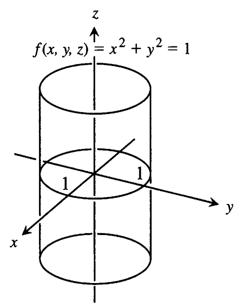


33.



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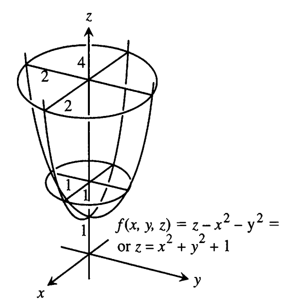
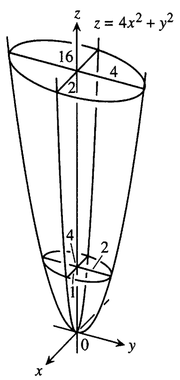


37. $x^2 + y^2 = 10$ **39.** $\tan^{-1} y - \tan^{-1} x = 2 \tan^{-1} \sqrt{2}$

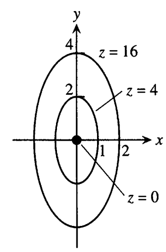
41. $\sqrt{x-y} - \ln z = 2$ **43.** $\dfrac{x+y}{z} = \ln 2$ **45.** Yes, 2000

47. 63 km

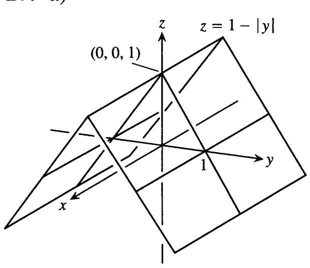
25. a)



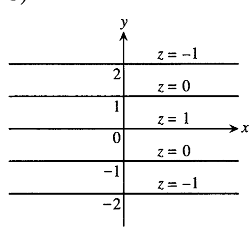
b)



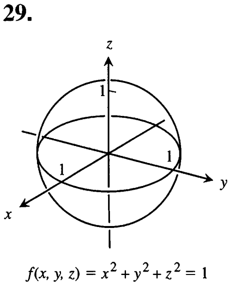
27. a)



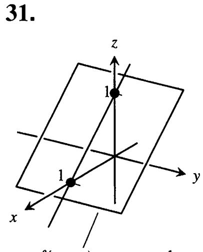
b)



29.



31.



Section 12.2, pp. 921–923

1. 5/2 **3.** $2\sqrt{6}$ **5.** 1 **7.** 1/2 **9.** 1 **11.** 0 **13.** 0
15. −1 **17.** 2 **19.** 1/4 **21.** 19/12 **23.** 2 **25.** 3
27. a) All (x, y) b) All (x, y) except $(0, 0)$
29. a) All (x, y) except where $x = 0$ or $y = 0$ b) All (x, y)
31. a) All (x, y, z) b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$
33. a) All (x, y, z) with $z \neq 0$ b) All (x, y, z) with $x^2 + z^2 \neq 1$
35. Consider paths along $y = x, x > 0$, and along $y = x, x < 0$
37. Consider the paths $y = kx^2$, k a constant
39. Consider the paths $y = mx$, m a constant, $m \neq -1$
41. Consider the paths $y = kx^2$, k a constant, $k \neq 0$ **43.** No
45. The limit is 1 **47.** The limit is 0
49. a) $f(x, y)|_{y=mx} = \sin 2\theta$ where $\tan \theta = m$ **51.** 0
53. Does not exist **55.** $\pi/2$ **57.** $f(0, 0) = \ln 3$ **61.** $\delta = 0.1$
63. $\delta = 0.005$ **65.** $\delta = \sqrt{0.015}$ **67.** $\delta = 0.005$

Section 12.3, pp. 931–933

1. $\dfrac{\partial f}{\partial x} = 4x,\ \dfrac{\partial f}{\partial y} = -3$ **3.** $\dfrac{\partial f}{\partial x} = 2x(y+2),\ \dfrac{\partial f}{\partial y} = x^2 - 1$

5. $\dfrac{\partial f}{\partial x} = 2y(xy-1),\ \dfrac{\partial f}{\partial y} = 2x(xy-1)$

7. $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}},\ \dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

9. $\dfrac{\partial f}{\partial x} = \dfrac{-1}{(x+y)^2},\ \dfrac{\partial f}{\partial y} = \dfrac{-1}{(x+y)^2}$

11. $\dfrac{\partial f}{\partial x} = \dfrac{-y^2-1}{(xy-1)^2},\ \dfrac{\partial f}{\partial y} = \dfrac{-x^2-1}{(xy-1)^2}$

13. $\dfrac{\partial f}{\partial x} = e^{x+y+1},\ \dfrac{\partial f}{\partial y} = e^{x+y+1}$ **15.** $\dfrac{\partial f}{\partial x} = \dfrac{1}{x+y},\ \dfrac{\partial f}{\partial y} = \dfrac{1}{x+y}$

17. $\dfrac{\partial f}{\partial x} = 2\sin(x-3y)\cos(x-3y)$,

$\dfrac{\partial f}{\partial y} = -6\sin(x-3y)\cos(x-3y)$

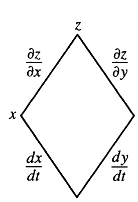
19. $\dfrac{\partial f}{\partial x} = yx^{y-1}$, $\dfrac{\partial f}{\partial y} = x^y \ln x$ 21. $\dfrac{\partial f}{\partial x} = -g(x)$, $\dfrac{\partial f}{\partial y} = g(y)$
23. $f_x = y^2$, $f_y = 2xy$, $f_z = -4z$
25. $f_x = 1$, $f_y = -y(y^2+z^2)^{-1/2}$, $f_z = -z(y^2+z^2)^{-1/2}$
27. $f_x = \dfrac{yz}{\sqrt{1-x^2y^2z^2}}$, $f_y = \dfrac{xz}{\sqrt{1-x^2y^2z^2}}$, $f_z = \dfrac{xy}{\sqrt{1-x^2y^2z^2}}$
29. $f_x = \dfrac{1}{x+2y+3z}$, $f_y = \dfrac{2}{x+2y+3z}$, $f_z = \dfrac{3}{x+2y+3z}$
31. $f_x = -2xe^{-(x^2+y^2+z^2)}$, $f_y = -2ye^{-(x^2+y^2+z^2)}$, $f_z = -2ze^{-(x^2+y^2+z^2)}$
33. $f_x = \operatorname{sech}^2(x+2y+3z)$, $f_y = 2\operatorname{sech}^2(x+2y+3z)$, $f_z = 3\operatorname{sech}^2(x+2y+3z)$
35. $\dfrac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)$, $\dfrac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$
37. $\dfrac{\partial h}{\partial \rho} = \sin\phi\cos\theta$, $\dfrac{\partial h}{\partial \phi} = \rho\cos\phi\cos\theta$, $\dfrac{\partial h}{\partial \theta} = -\rho\sin\phi\sin\theta$
39. $W_P(P, V, \delta, v, g) = V$, $W_V(P, V, \delta, v, g) = P + \dfrac{\delta v^2}{2g}$, $W_P(P, V, \delta, v, g) = \dfrac{Vv^2}{2g}$, $W_v(P, V, \delta, v, g) = \dfrac{V\delta v}{g}$, $W_g(P, V, \delta, v, g) = -\dfrac{V\delta v^2}{2g^2}$
41. $\dfrac{\partial f}{\partial x} = 1+y$, $\dfrac{\partial f}{\partial y} = 1+x$, $\dfrac{\partial^2 f}{\partial x^2} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = 0$, $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial^2 f}{\partial x\,\partial y} = 1$
43. $\dfrac{\partial g}{\partial x} = 2xy + y\cos x$, $\dfrac{\partial g}{\partial y} = x^2 - \sin y + \sin x$, $\dfrac{\partial^2 g}{\partial x^2} = 2y - y\sin x$, $\dfrac{\partial^2 g}{\partial y^2} = -\cos y$, $\dfrac{\partial^2 g}{\partial y\,\partial x} = \dfrac{\partial^2 g}{\partial x\,\partial y} = 2x + \cos x$
45. $\dfrac{\partial r}{\partial x} = \dfrac{1}{x+y}$, $\dfrac{\partial r}{\partial y} = \dfrac{1}{x+y}$, $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{-1}{(x+y)^2}$, $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{-1}{(x+y)^2}$, $\dfrac{\partial^2 r}{\partial y\,\partial x} = \dfrac{\partial^2 r}{\partial x\,\partial y} = \dfrac{-1}{(x+y)^2}$
47. $\dfrac{\partial w}{\partial x} = \dfrac{2}{2x+3y}$, $\dfrac{\partial w}{\partial y} = \dfrac{3}{2x+3y}$, $\dfrac{\partial^2 w}{\partial y\,\partial x} = \dfrac{-6}{(2x+3y)^2}$, $\dfrac{\partial^2 w}{\partial x\,\partial y} = \dfrac{-6}{(2x+3y)^2}$
49. $\dfrac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\dfrac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\dfrac{\partial^2 w}{\partial y\,\partial x} = 2y + 6xy^2 + 12x^2y^3$, $\dfrac{\partial^2 w}{\partial x\,\partial y} = 2y + 6xy^2 + 12x^2y^3$
51. a) x first b) y first c) x first d) x first e) y first f) y first 53. $f_x(1,2) = -13$, $f_y(1,2) = -2$ 55. 12
57. -2 59. $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$, $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$
61. $v_x = \dfrac{\ln v}{(\ln u)(\ln v) - 1}$

Section 12.4, pp. 942–944

1. a) $L(x, y) = 1$ b) $L(x, y) = 2x + 2y - 1$
3. a) $L(x, y) = 3x - 4y + 5$ b) $L(x, y) = 3x - 4y + 5$
5. a) $L(x, y) = 1 + x$ b) $L(x, y) = -y + (\pi/2)$
7. $L(x, y) = 7 + x - 6y$; 0.06 9. $L(x, y) = x + y + 1$; 0.08
11. $L(x, y) = 1 + x$; 0.0222
13. Pay more attention to the smaller of the two dimensions. It will generate the larger partial derivative.
15. Maximum error (estimate) ≤ 0.31 in magnitude
17. Maximum percentage error $= \pm 4.83\%$
19. Let $|x - 1| \le 0.014$, $|y - 1| \le 0.014$ 21. $\approx 0.1\%$
23. a) $L(x, y, z) = 2x + 2y + 2z - 3$ b) $L(x, y, z) = y + z$ c) $L(x, y, z) = 0$ 25. a) $L(x, y, z) = x$ b) $L(x, y, z) = \dfrac{1}{\sqrt{2}}x + \dfrac{1}{\sqrt{2}}y$ c) $L(x, y, z) = \dfrac{1}{3}x + \dfrac{2}{3}y + \dfrac{2}{3}z$
27. a) $L(x, y, z) = 2 + x$ b) $L(x, y, z) = x - y - z + \dfrac{\pi}{2} + 1$ c) $L(x, y, z) = x - y - z + \dfrac{\pi}{2} + 1$
29. $L(x, y, z) = 2x - 6y - 2z + 6$, 0.0024
31. $L(x, y, z) = x + y - z - 1$, 0.00135
33. a) $S_0\left(\dfrac{1}{100}\,dp + dx - 5dw - 30\,dh\right)$ b) More sensitive to a change in height
35. f is most sensitive to a change in d. 37. $(47/24)$ ft^3
39. Magnitude of possible error ≤ 4.8 41. Yes

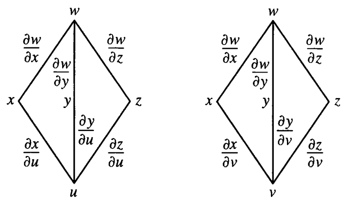
Section 12.5, pp. 950–952

1. $\dfrac{dw}{dt} = 0$, $\dfrac{dw}{dt}(\pi) = 0$ 3. $\dfrac{dw}{dt} = 1$, $\dfrac{dw}{dt}(3) = 1$
5. $\dfrac{dw}{dt} = 4t\tan^{-1}t + 1$, $\dfrac{dw}{dt}(1) = \pi + 1$
7. a) $\dfrac{\partial z}{\partial r} = 4\cos\theta\ln(r\sin\theta) + 4\cos\theta$, $\dfrac{\partial z}{\partial \theta} = -4r\sin\theta\ln(r\sin\theta) + \dfrac{4r\cos^2\theta}{\sin\theta}$ b) $\dfrac{\partial z}{\partial r} = \sqrt{2}(\ln 2 + 2)$, $\dfrac{\partial z}{\partial \theta} = -2\sqrt{2}\ln 2 + 4\sqrt{2}$
9. a) $\dfrac{\partial w}{\partial u} = 2u + 4uv$, $\dfrac{\partial w}{\partial v} = -2v + 2u^2$ b) $\dfrac{\partial w}{\partial u} = 3$, $\dfrac{\partial w}{\partial v} = -\dfrac{3}{2}$
11. a) $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = \dfrac{z}{(z-y)^2}$, $\dfrac{\partial u}{\partial z} = \dfrac{-y}{(z-y)^2}$ b) $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 1$, $\dfrac{\partial u}{\partial z} = -2$
13. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

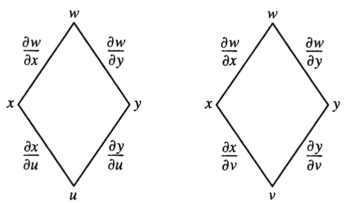


15. $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$,

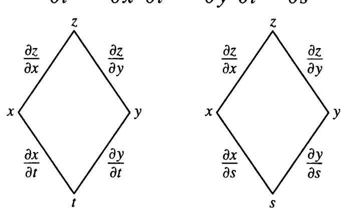
$\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$



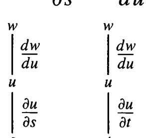
17. $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u}$, $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v}$



19. $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$

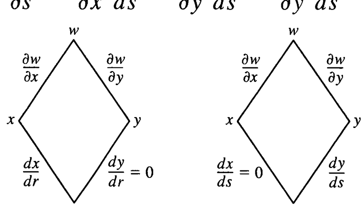


21. $\dfrac{\partial w}{\partial s} = \dfrac{dw}{du}\dfrac{\partial u}{\partial s}$, $\dfrac{\partial w}{\partial t} = \dfrac{dw}{du}\dfrac{\partial u}{\partial t}$



23. $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dr} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dr} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dr}$ since $\dfrac{dy}{dr} = 0$,

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial w}{\partial y}\dfrac{dy}{ds} = \dfrac{\partial w}{\partial y}\dfrac{dy}{ds}$ since $\dfrac{dx}{ds} = 0$



25. $4/3$ **27.** $-4/5$ **29.** $\dfrac{\partial z}{\partial x} = \dfrac{1}{4}$, $\dfrac{\partial z}{\partial y} = -\dfrac{3}{4}$

31. $\dfrac{\partial z}{\partial x} = -1$, $\dfrac{\partial z}{\partial y} = -1$ **33.** 12 **35.** -7

37. $\dfrac{\partial z}{\partial u} = 2$, $\dfrac{\partial z}{\partial v} = 1$ **39.** -0.00005 amps/sec

45. $(\cos 1, \sin 1, 1)$ and $(\cos(-2), \sin(-2), -2)$

47. a) Maximum at $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$, minimum at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ and $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$

b) max $= 6$, min $= 2$ **49.** $2x\sqrt{x^8 + x^3} + \displaystyle\int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}}\,dt$

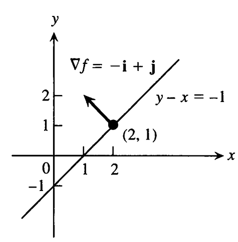
Section 12.6, pp. 956–957

1. a) 0 b) $1 + 2z$ c) $1 + 2z$ **3.** a) $\dfrac{\partial U}{\partial P} + \dfrac{\partial U}{\partial T}\left(\dfrac{V}{nR}\right)$

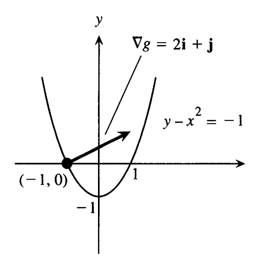
b) $\dfrac{\partial U}{\partial P}\left(\dfrac{nR}{V}\right) + \dfrac{\partial U}{\partial T}$ **5.** a) 5 b) 5 **7.** $\dfrac{x}{\sqrt{x^2 + y^2}}$

Section 12.7, pp. 967–969

1.



3.



5. $\nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ **7.** $\nabla f = -\dfrac{26}{27}\mathbf{i} + \dfrac{23}{54}\mathbf{j} - \dfrac{23}{54}\mathbf{k}$ **9.** -4

11. $31/13$ **13.** 3 **15.** 2

17. $\mathbf{u} = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$, $(D_{\mathbf{u}}f)_{P_0} = \sqrt{2}$; $-\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$, $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$

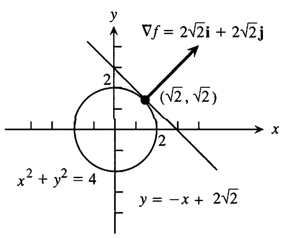
19. $\mathbf{u} = \dfrac{1}{3\sqrt{3}}\mathbf{i} - \dfrac{5}{3\sqrt{3}}\mathbf{j} - \dfrac{1}{3\sqrt{3}}\mathbf{k}$, $(D_{\mathbf{u}}f)_{P_0} = 3\sqrt{3}$;

$-\mathbf{u} = -\dfrac{1}{3\sqrt{3}}\mathbf{i} + \dfrac{5}{3\sqrt{3}}\mathbf{j} + \dfrac{1}{3\sqrt{3}}\mathbf{k}$, $(D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$

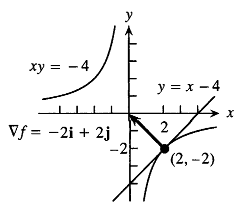
21. $\mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(D_{\mathbf{u}}f)_{P_0} = 2\sqrt{3}$; $-\mathbf{u} = -\dfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$

23. $df = \dfrac{9}{910} \approx 0.01$ **25.** $dg = 0$

27. Tangent: $x+y+z=3$, normal line: $x=1+2t,\ y=1+2t,$ $z=1+2t$
29. Tangent: $2x-z-2=0$, normal line: $x=2-4t,\ y=0,$ $z=2+2t$
31. Tangent: $2x+2y+z-4=0$, normal line: $x=2t,$ $y=1+2t,\ z=2+t$
33. Tangent: $x+y+z-1=0$, normal line: $x=t,\ y=1+t,$ $z=t$
35. $2x-z-2=0$ **37.** $x-y+2z-1=0$
39.



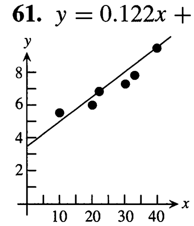
41.



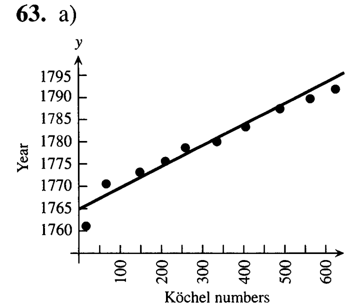
43. $x=1,\ y=1+2t,\ z=1-2t$
45. $x=1-2t,\ y=1,\ z=\dfrac{1}{2}+2t$
47. $x=1+90t,\ y=1-90t,\ z=3$
49. $\mathbf{u}=\dfrac{7}{\sqrt{53}}\mathbf{i}-\dfrac{2}{\sqrt{53}}\mathbf{j},\ -\mathbf{u}=-\dfrac{7}{\sqrt{53}}\mathbf{i}+\dfrac{2}{\sqrt{53}}\mathbf{j}$
51. No, the maximum rate of change is $\sqrt{185}<14$. **53.** $-\dfrac{7}{\sqrt{5}}$
55. a) $\dfrac{\sqrt{3}}{2}\sin\sqrt{3}-\dfrac{1}{2}\cos\sqrt{3}\approx 0.935°\text{C/ft}$
b) $\sqrt{3}\sin\sqrt{3}-\cos\sqrt{3}\approx 1.87°\text{C/sec}$
57. At $-\dfrac{\pi}{4},\ -\dfrac{\pi}{2\sqrt{2}}$; at 0, 0; at $\dfrac{\pi}{4},\ \dfrac{\pi}{2\sqrt{2}}$

Section 12.8, pp. 975–979

1. $f(-3,3)=-5$, local minimum
3. $f\left(\dfrac{2}{3},\dfrac{4}{3}\right)=0$, local maximum **5.** $f(-2,1)$, saddle point
7. $f\left(\dfrac{6}{5},\dfrac{69}{25}\right)$, saddle point **9.** $f(2,1)$, saddle point
11. $f(2,-1)=-6$, local minimum **13.** $f(1,2)$, saddle point
15. $f(0,0)$, saddle point
17. $f(0,0)$, saddle point; $f\left(-\dfrac{2}{3},\dfrac{2}{3}\right)=\dfrac{170}{27}$, local maximum
19. $f(0,0)=0$, local minimum; $f(1,-1)$, saddle point
21. $f(0,0)$, saddle point; $f\left(\dfrac{4}{9},\dfrac{4}{3}\right)=-\dfrac{64}{81}$, local minimum
23. $f(0,0)$, saddle point; $f(0,2)=-12$, local minimum; $f(-2,0)=-4$, local maximum; $f(-2,2)$, saddle point
25. $f(0,0)$, saddle point; $f(1,1)=2,\ f(-1,-1)=2$, local maxima
27. $f(0,0)=-1$, local maximum
29. $f(n\pi,0)$, saddle point; $f(n\pi,0)=0$ for every n
31. Absolute maximum: 1 at (0, 0); absolute minimum: −5 at (1, 2)
33. Absolute maximum: 4 at (0, 2); absolute minimum: 0 at (0, 0)
35. Absolute maximum: 11 at (0, −3); absolute minimum: −10 at (4, −2)
37. Absolute maximum: 4 at (2, 0); absolute minimum: $\dfrac{3\sqrt{2}}{2}$ at $\left(3,-\dfrac{\pi}{4}\right),\ \left(3,\dfrac{\pi}{4}\right),\ \left(1,-\dfrac{\pi}{4}\right)$, and $\left(1,\dfrac{\pi}{4}\right)$ **39.** $a=-3, b=2$
41. Hottest: $2\dfrac{1}{4}^{\circ}$ at $\left(-\dfrac{1}{2},\dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{1}{2},-\dfrac{\sqrt{3}}{2}\right)$; coldest: $-\dfrac{1}{4}^{\circ}$ at $\left(\dfrac{1}{2},0\right)$
43. a) $f(0,0)$, saddle point b) $f(1,2)$, local minimum
c) $f(1,-2)$, local minimum; $f(-1,-2)$, saddle point
49. $\left(\dfrac{1}{6},\dfrac{1}{3},\dfrac{355}{36}\right)$
53. a) On the semicircle, max $f=2\sqrt{2}$ at $t=\pi/4$, min $f=-2$ at $t=\pi$. On the quarter circle, max $f=2\sqrt{2}$ at $t=\pi/4$, min $f=2$ at $t=0,\pi/2$.
b) On the semicircle, max $g=2$ at $t=\pi/4$, min $g=-2$ at $t=3\pi/4$. On the quarter circle, max $g=2$ at $t=\pi/4$, min $g=0$ at $t=0,\pi/2$.
c) On the semicircle, max $h=8$ at $t=0,\pi$; min $h=4$ at $t=\pi/2$. On the quarter circle, max $h=8$ at $t=0$, min $h=4$ at $t=\pi/2$.
55. i) min $f=-1/2$ at $t=-1/2$; no max ii) max $f=0$ at $t=-1,\ 0$; min $f=-1/2$ at $t=-1/2$ iii) max $f=4$ at $t=1$; min $f=0$ at $t=0$
57. $y=-\dfrac{20}{13}x+\dfrac{9}{13},\ y|_{x=4}=-\dfrac{71}{13}$
59. $y=\dfrac{3}{2}x+\dfrac{1}{6},\ y|_{x=4}=\dfrac{37}{6}$
61. $y=0.122x+3.58$



63. a)



b) $y=0.0427\text{K}+1764.8$ c) 1780

Section 12.9, pp. 987–989

1. $\left(\pm\dfrac{1}{\sqrt{2}},\dfrac{1}{2}\right),\ \left(\pm\dfrac{1}{\sqrt{2}},-\dfrac{1}{2}\right)$ **3.** 39 **5.** $(3,\pm3\sqrt{2})$ **7.** a) 8
b) 64 **9.** $r=2$ cm, $h=4$ cm **11.** $l=4\sqrt{2},\ w=3\sqrt{2}$
13. $f(0,0)=0$ is minimum, $f(2,4)=20$ is maximum

15. Minimum = 0°, maximum = 125° **17.** $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ **19.** 1
21. $(0, 0, 2), (0, 0, -2)$
23. $f(1, -2, 5) = 30$ is maximum, $f(-1, 2, -5) = -30$ is minimum
25. 3, 3, 3 **27.** $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ units **29.** $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$
31. $U(8, 14) = \$128$ **33.** $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ **35.** $(2, 4, 4)$
37. Maximum is $1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$, minimum is $1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$
39. Maximum is 4 at $(0, 0, \pm 2)$, minimum is 2 at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$

Section 12.10, p. 993

1. Quadratic: $x + xy$; cubic: $x + xy + \frac{1}{2}xy^2$

3. Quadratic: xy; cubic: xy

5. Quadratic: $y + \frac{1}{2}(2xy - y^2)$;
cubic: $y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3)$

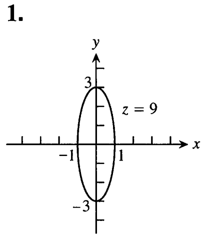
7. Quadratic: $\frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2$; cubic: $x^2 + y^2$

9. Quadratic: $1 + (x + y) + (x + y)^2$;
cubic: $1 + (x + y) + (x + y)^2 + (x + y)^3$

11. Quadratic: $1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, $E(x, y) \le 0.00134$

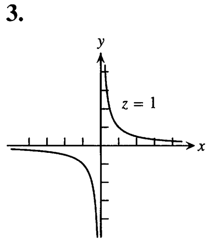
Chapter 12 Practice Exercises, pp. 994–998

1.



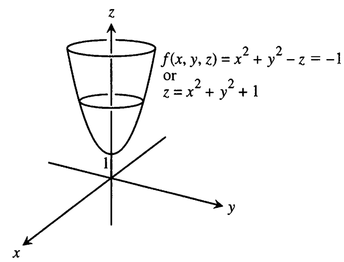
Domain: all points in the xy-plane; range: $z \ge 0$. Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.

3.



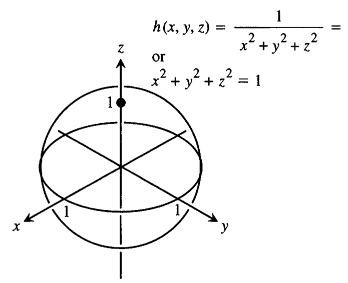
Domain: all (x, y) such that $x \ne 0$ and $y \ne 0$; range: $z \ne 0$. Level curves are hyperbolas with the x- and y-axes as asymptotes.

5.



Domain: all (x, y, z) such that $(x, y, z) \ne (0, 0, 0)$; range: all real numbers. Level surfaces are paraboloids of revolution with the z-axis as axis.

7.



Domain: all (x, y, z) such that $(x, y, z) \ne (0, 0, 0)$; range: positive real numbers. Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.

9. -2 **11.** $1/2$ **13.** 1 **15.** Let $y = kx^2$, $k \ne 1$
17. a) Does not exist b) Not continuous at $(0, 0)$
19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$
21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$
23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$
25. $\frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y\,\partial x} = \frac{\partial^2 g}{\partial x\,\partial y} = -\frac{1}{y^2}$
27. $\frac{\partial^2 f}{\partial x^2} = -30x + \frac{2 - 2x^2}{(x^2 + 1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = 1$
29. Answers will depend on the upper bound used for $|f_{xx}|$, $|f_{xy}|$, $|f_{yy}|$. With $M = \sqrt{2}/2$, $|E| \le 0.0142$. With $M = 1$, $|E| \le 0.02$.
31. $L(x, y, z) = y - 3z$, $L(x, y, z) = x + y - z - 1$
33. Be more careful with the diameter.
35. $dl = 0.038$, % change in $V = -4.17\%$, % change in $R = -20\%$, % change in $l = 15.83\%$
37. a) 5% **39.** $\left.\frac{dw}{dt}\right|_{t=0} = -1$
41. $\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(\pi,0)} = 2$, $\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(\pi,0)} = 2 - \pi$
43. $\left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)\sin 1 + (\cos 1 + \cos 2)\cos 1 - 2(\sin 1 + \cos 1)\sin 2$

45. $\left.\dfrac{dy}{dx}\right|_{(x,y)=(0,1)} = -1$

47. a) $(2y + x^2 z)e^{yz}$ b) $x^2 e^{yz}\left(y - \dfrac{z}{2y}\right)$ c) $(1 + x^2 y)e^{yz}$

49. Increases most rapidly in the direction $\mathbf{u} = -\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$;

decreases most rapidly in the direction $-\mathbf{u} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$;

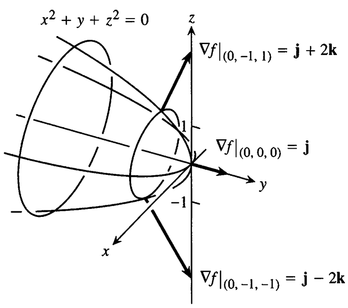
$D_{\mathbf{u}} f = \dfrac{\sqrt{2}}{2},\ D_{-\mathbf{u}} f = -\dfrac{\sqrt{2}}{2},\ D_{\mathbf{u}_1} f = -\dfrac{7}{10}$

51. Increases most rapidly in the direction $\mathbf{u} = \dfrac{2}{7}\mathbf{i} + \dfrac{3}{7}\mathbf{j} + \dfrac{6}{7}\mathbf{k}$;

decreases most rapidly in the direction $-\mathbf{u} = -\dfrac{2}{7}\mathbf{i} - \dfrac{3}{7}\mathbf{j} - \dfrac{6}{7}\mathbf{k}$;
$D_{\mathbf{u}} f = 7, D_{-\mathbf{u}} f = -7, D_{\mathbf{u}_1} f = 7$

53. $\pi/\sqrt{2}$ **55.** a) $f_x(1,2) = f_y(1,2) = 2$ b) $14/5$

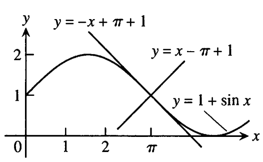
57.



59. Tangent: $4x - y - 5z = 4$, normal line: $x = 2 + 4t$, $y = -1 - t,\ z = 1 - 5t$

61. $2y - z - 2 = 0$

63. Tangent: $x + y = \pi + 1$, normal line: $y = x - \pi + 1$



65. $x = 1 - 2t,\ y = 1,\ z = \dfrac{1}{2} + 2t$

67. Local minimum of -8 at $(-2, -2)$

69. Saddle point at $(0,0)$, $f(0,0) = 0$; local maximum of $1/4$ at $\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$

71. Saddle point at $(0,0)$, $f(0,0) = 0$; local minimum of -4 at $(0,2)$; local maximum of 4 at $(-2,0)$; saddle point at $(-2,2)$, $f(-2,2) = 0$

73. Absolute maximum: 28 at $(0,4)$, absolute minimum: $-9/4$ at $(3/2, 0)$

75. Absolute maximum: 18 at $(2,-2)$, absolute minimum: $-17/4$ at $(-2, 1/2)$

77. Absolute maximum: 8 at $(-2,0)$, absolute minimum: -1 at $(1,0)$

79. Absolute maximum: 4 at $(1,0)$, absolute minimum: -4 at $(0,-1)$

81. Absolute maximum: 1 at $(0, \pm 1)$ and $(1,0)$, absolute minimum: -1 at $(-1,0)$

83. Maximum: 5 at $(0,1)$, minimum: $-1/3$ at $(0, -1/3)$

85. Maximum: $\sqrt{3}$ at $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, minimum: $-\sqrt{3}$ at $(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$

87. Width $= \left(\dfrac{c^2 V}{ab}\right)^{1/3}$, depth $= \left(\dfrac{b^2 V}{ac}\right)^{1/3}$, height $= \left(\dfrac{a^2 V}{bc}\right)^{1/3}$

89. Maximum $= 3/2$ at $(1/\sqrt{2}, 1/\sqrt{2}, \sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2}, -\sqrt{2})$, minimum $= 1/2$ at $(-1/\sqrt{2}, 1/\sqrt{2}, -\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2}, \sqrt{2})$

91. $\dfrac{\partial w}{\partial x} = \cos\theta\,\dfrac{\partial w}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial w}{\partial \theta},\ \dfrac{\partial w}{\partial y} = \sin\theta\,\dfrac{\partial w}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial w}{\partial \theta}$

97. $(t, -t \pm 4, t)$, t a real number

Chapter 12 Additional Exercises, pp. 998–1000

1. $f_{xy}(0,0) = -1,\ f_{yx}(0,0) = 1$ **7.** c) $r^2 = \dfrac{1}{2}(x^2 + y^2 + z^2)$

15. $V = \dfrac{\sqrt{3}abc}{2}$ **19.** $f(x,y) = \dfrac{y}{2} + 4,\ g(x,y) = \dfrac{x}{2} + \dfrac{9}{2}$

21. $y = 2\ln|\sin x| + \ln 2$ **23.** a) $\dfrac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$

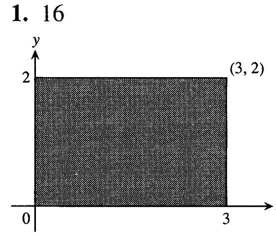
b) $\dfrac{-1}{\sqrt{29097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$ **25.** $w = e^{-c^2\pi^2 t}\sin \pi x$

27. 0.213%

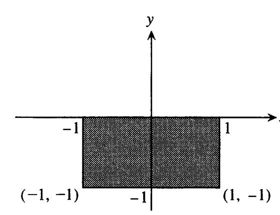
CHAPTER 13

Section 13.1, pp. 1010–1011

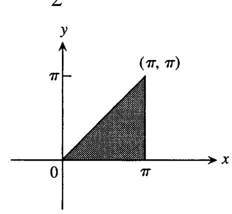
1. 16



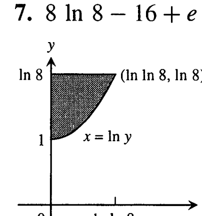
3. 1



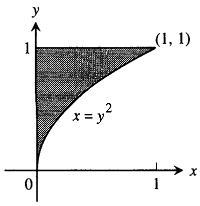
5. $\dfrac{\pi^2}{2} + 2$



7. $8\ln 8 - 16 + e$

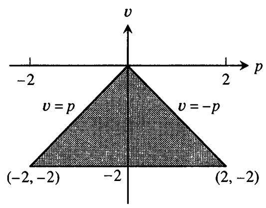


9. $e-2$

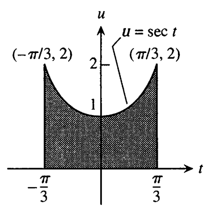


11. $\dfrac{3}{2}\ln 2$ **13.** $1/6$ **15.** $-1/10$

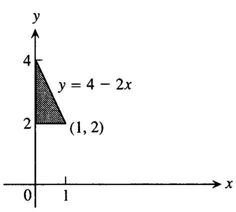
17. 8



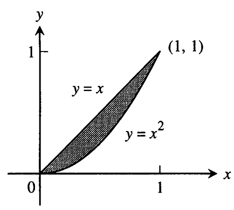
19. 2π



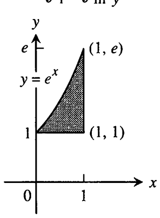
21. $\displaystyle\int_2^4 \int_0^{(4-y)/2} dx\,dy$



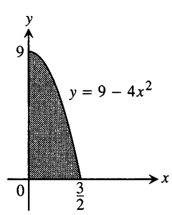
23. $\displaystyle\int_0^1 \int_{x^2}^{x} dy\,dx$



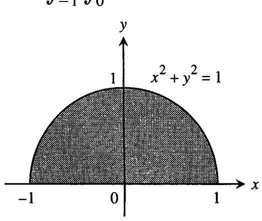
25. $\displaystyle\int_1^e \int_{\ln y}^{1} dx\,dy$



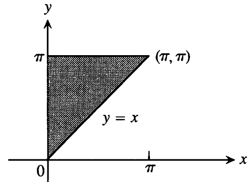
27. $\displaystyle\int_0^9 \int_0^{(\sqrt{9-y})/2} 16x\,dx\,dy$



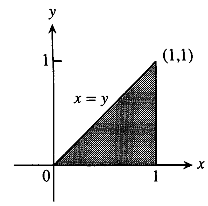
29. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\,dy\,dx$



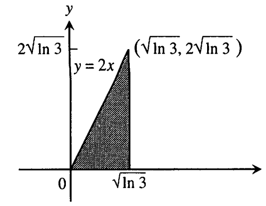
31. 2



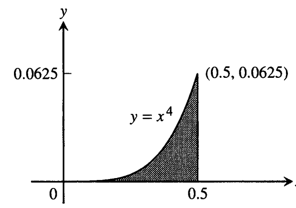
33. $\dfrac{e-2}{2}$



35. 2

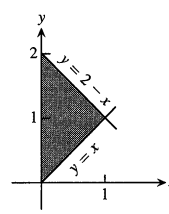


37. $\dfrac{1}{80\pi}$



39. $-2/3$ **41.** $4/3$ **43.** $625/12$ **45.** 16 **47.** 20
49. $2(1+\ln 2)$ **51.** 1 **53.** π^2 **55.** $-1/4$ **57.** $20\sqrt{3}/9$

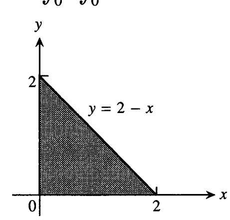
59. $\displaystyle\int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx = \frac{4}{3}$



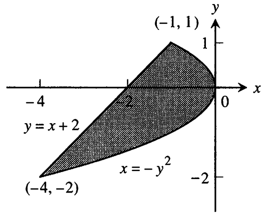
67. 0.603 **69.** 0.233

Section 13.2, pp. 1018–1020

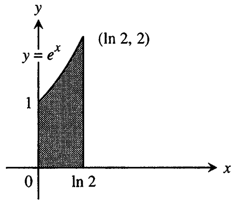
1. $\displaystyle\int_0^2 \int_0^{2-x} dy\,dx = 2$ or $\displaystyle\int_0^2 \int_0^{2-y} dx\,dy = 2$



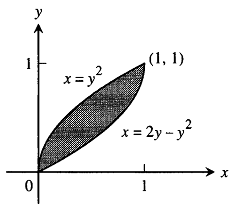
3. $\int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy = 9/2$



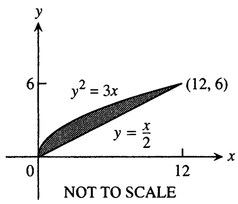
5. $\int_{0}^{\ln 2}\int_{0}^{e^x} dy\,dx = 1$



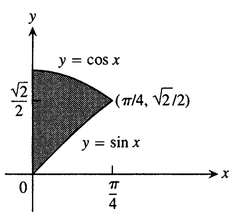
7. $\int_{0}^{1}\int_{y^2}^{2y-y^2} dx\,dy = 1/3$



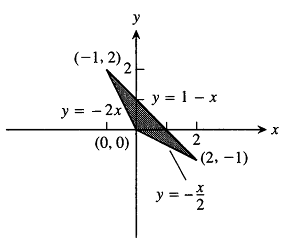
9. 12



11. $\sqrt{2} - 1$



13. 3/2



15. a) 0 b) $4/\pi^2$ **17.** 8/3 **19.** $\bar{x} = \dfrac{5}{14},\ \bar{y} = \dfrac{38}{35}$

21. $\bar{x} = \dfrac{64}{35},\ \bar{y} = \dfrac{5}{7}$ **23.** $\bar{x} = 0,\ \bar{y} = \dfrac{4}{3\pi}$ **25.** $\bar{x} = \bar{y} = \dfrac{4a}{3\pi}$

27. $\bar{x} = \dfrac{\pi}{2},\ \bar{y} = \dfrac{\pi}{8}$ **29.** $\bar{x} = -1,\ \bar{y} = \dfrac{1}{4}$

31. $I_x = \dfrac{64}{105},\ R_x = 2\sqrt{\dfrac{2}{7}}$ **33.** $\bar{x} = \dfrac{3}{8},\ \bar{y} = \dfrac{17}{16}$

35. $\bar{x} = \dfrac{11}{3},\ \bar{y} = \dfrac{14}{27},\ I_y = 432,\ R_y = 4$

37. $\bar{x} = 0,\ \bar{y} = \dfrac{13}{31},\ I_y = \dfrac{7}{5},\ R_y = \sqrt{\dfrac{21}{31}}$

39. $\bar{x} = 0,\ \bar{y} = 7/10;\ I_x = 9/10,\ I_y = 3/10,\ I_0 = 6/5;$
$R_x = \dfrac{3\sqrt{6}}{10},\ R_y = \dfrac{3\sqrt{2}}{10},\ R_0 = \dfrac{3\sqrt{2}}{5}$

41. $40{,}000(1 - e^{-2})\ln\left(\dfrac{7}{2}\right) \approx 43{,}329$

43. If $0 < a \le 5/2$, then the appliance will have to be tipped more than 45° to fall over.

45. $(\bar{x}, \bar{y}) = (2/\pi, 0)$ **47.** a) 3/2 b) They are the same.

53. a) $\left(\dfrac{7}{5}, \dfrac{31}{10}\right)$ b) $\left(\dfrac{19}{7}, \dfrac{18}{7}\right)$ c) $\left(\dfrac{9}{2}, \dfrac{19}{8}\right)$ d) $\left(\dfrac{11}{4}, \dfrac{43}{16}\right)$

55. In order for c.m. to be on the common boundary, $h = a\sqrt{2}$. In order for c.m. to be inside T, $h > a\sqrt{2}$.

Section 13.3, pp. 1024–1026

1. $\pi/2$ **3.** $\pi/8$ **5.** πa^2 **7.** 36 **9.** $(1 - \ln 2)\pi$

11. $(2\ln 2 - 1)(\pi/2)$ **13.** $\dfrac{\pi}{2} + 1$ **15.** $\pi(\ln(4) - 1)$

17. $2(\pi - 1)$ **19.** 12π **21.** $\dfrac{3\pi}{8} + 1$ **23.** 4 **25.** $6\sqrt{3} - 2\pi$

27. $\bar{x} = 5/6,\ \bar{y} = 0$ **29.** $2a/3$ **31.** $2a/3$ **33.** 2π

35. $\dfrac{4}{3} + \dfrac{5\pi}{8}$ **37.** a) $\sqrt{\pi}/2$ b) 1 **39.** $\pi\ln 4$, no

41. $\dfrac{1}{2}(a^2 + 2h^2)$

Section 13.4, pp. 1031–1034

1. 1

3. $\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx,\quad \int_0^2\int_0^{1-y/2}\int_0^{3-3x-3y/2} dz\,dx\,dy,$

$\int_0^1\int_0^{3-3x}\int_0^{2-2x-2z/3} dy\,dz\,dx,\quad \int_0^3\int_0^{1-z/3}\int_0^{2-2x-2z/3} dy\,dx\,dz,$

$\int_0^2\int_0^{3-3y/2}\int_0^{1-y/2-z/3} dx\,dz\,dy,\quad \int_0^3\int_0^{2-2z/3}\int_0^{1-y/2-z/3} dx\,dy\,dz.$

The value of all six integrals is 1.

5. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx,$

$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dx\,dy,$

$\int_{-2}^{2}\int_{4}^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1\,dx\,dz\,dy + \int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1\,dx\,dz\,dy,$

$\int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1\,dx\,dy\,dz + \int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1\,dx\,dy\,dz,$

$\int_{-2}^{2}\int_{4}^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1\,dy\,dz\,dx + \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1\,dy\,dz\,dx,$

$\int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1\,dy\,dx\,dz + \int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1\,dy\,dx\,dz.$

The value of all six integrals is 16π.

7. 1 **9.** 1 **11.** $\dfrac{\pi^3}{2}(1 - \cos 1)$ **13.** 18 **15.** 7/6 **17.** 0

19. $\dfrac{1}{2} - \dfrac{\pi}{8}$ **21.** a) $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$

b) $\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$ c) $\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dy\,dz\,dx$

d) $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$ e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz\,dx\,dy$ **23.** 2/3

25. 20/3 **27.** 1 **29.** 16/3 **31.** $8\pi - \frac{32}{3}$ **33.** 2 **35.** 4π

37. 31/3 **39.** 1 **41.** $2\sin 4$ **43.** 4 **45.** $a = 3$ or $a = \frac{13}{3}$

Section 13.5, pp. 1036–1039

1. $R_x = \sqrt{\frac{b^2+c^2}{12}},\ R_y = \sqrt{\frac{a^2+c^2}{12}},\ R_z = \sqrt{\frac{a^2+b^2}{12}}$

3. $I_x = \frac{M}{3}(b^2+c^2),\ I_y = \frac{M}{3}(a^2+c^2),\ I_z = \frac{M}{3}(a^2+b^2)$

5. $\bar{x} = \bar{y} = 0,\ \bar{z} = \frac{12}{5},\ I_x = \frac{7904}{105} \approx 75.28,\ I_y = \frac{4832}{63} \approx 76.70,$
$I_z = \frac{256}{45} \approx 5.69$

7. a) $\bar{x} = \bar{y} = 0,\ \bar{z} = \frac{8}{3}$ b) $c = 2\sqrt{2}$

9. $I_L = 1386,\ R_L = \sqrt{\frac{77}{2}}$ **11.** $I_L = \frac{40}{3},\ R_L = \sqrt{\frac{5}{3}}$ **13.** a) $\frac{4}{3}$
b) $\bar{x} = \frac{4}{5},\ \bar{y} = \bar{z} = \frac{2}{5}$ **15.** a) $\frac{5}{2}$ b) $\bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$
c) $I_x = I_y = I_z = \frac{11}{6}$ d) $R_x = R_y = R_z = \sqrt{\frac{11}{15}}$ **17.** 3

19. a) $\frac{4}{3}g$ b) $\frac{4}{3}g$

23. a) $I_{\text{c.m.}} = \frac{abc(a^2+b^2)}{12},\ R_{\text{c.m.}} = \sqrt{\frac{a^2+b^2}{12}}$
b) $I_L = \frac{abc(a^2+7b^2)}{3},\ R_L = \sqrt{\frac{a^2+7b^2}{3}}$

27. a) $h = a\sqrt{3}$ b) $h = a\sqrt{2}$

Section 13.6, pp. 1044–1047

1. $4\pi(\sqrt{2}-1)/3$ **3.** $17\pi/5$ **5.** $\pi(6\sqrt{2}-8)$ **7.** $3\pi/10$

9. $\pi/3$ **11.** a) $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$
b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r\,dr\,dz\,d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$
c) $\int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r\,d\theta\,dz\,dr$

13. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} \int_0^{3r^2} f(r,\theta,z)\,r\,dz\,dr\,d\theta$

15. $\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4-r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$

17. $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} \int_0^4 f(r,\theta,z)\,dz\,r\,dr\,d\theta$

19. $\int_0^{\pi/4} \int_0^{\sec\theta} \int_0^{2-r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$ **21.** π^2 **23.** $\pi/3$

25. 5π **27.** 2π **29.** $\left(\frac{8-5\sqrt{2}}{2}\right)\pi$

31. a) $\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta +$
$\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$
b) $\int_0^{2\pi} \int_1^2 \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta +$
$\int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta$

33. $\int_0^{2\pi} \int_0^{\pi/2} \int_{\cos\phi}^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{31\pi}{6}$

35. $\int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{8\pi}{3}$

37. $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{3}$

39. a) $8\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$
b) $8\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$
c) $8\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$

41. a) $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$
b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$
c) $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$ d) $\frac{5\pi}{3}$ **43.** $8\pi/3$

45. 9/4 **47.** $(3\pi-4)/18$ **49.** $\frac{2\pi a^3}{3}$ **51.** $5\pi/3$ **53.** $\pi/2$

55. $\frac{4(2\sqrt{2}-1)\pi}{3}$ **57.** 16π **59.** $5\pi/2$ **61.** $\frac{4\pi(8-3\sqrt{3})}{3}$

63. 2/3 **65.** 3/4 **67.** $\bar{x} = \bar{y} = 0,\ \bar{z} = 3/8$
69. $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3/8)$ **71.** $\bar{x} = \bar{y} = 0,\ \bar{z} = 5/6$

73. $I_z = 30\pi,\ R_z = \sqrt{\frac{5}{2}}$ **75.** $I_x = \pi/4$ **77.** $\frac{a^4 h\pi}{10}$

79. a) $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{4}{5}\right),\ I_z = \frac{\pi}{12},\ R_z = \sqrt{\frac{1}{3}}$

b) $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{5}{6}\right),\ I_z = \frac{\pi}{14},\ R_z = \sqrt{\frac{5}{14}}$

83. $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{2h^2 + 3h}{3h + 6}\right),\ I_z = \frac{\pi a^4(h^2 + 2h)}{4},\ R_z = \frac{a}{\sqrt{2}}$

85. $\frac{3M}{\pi R^3}$

Section 13.7, pp. 1054–1055

1. a) $x = \frac{u+v}{3},\ y = \frac{v - 2u}{3};\ \frac{1}{3}$

b) Triangular region with boundaries $u = 0,\ v = 0,$ and $u + v = 3$

3. a) $x = \frac{1}{5}(2u - v),\ y = \frac{1}{10}(3v - u);\ \frac{1}{10}$

b) Triangular region with boundaries $3v = u,\ v = 2u,$ and $3u + v = 10$

5. a) $\begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

b) $\begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$

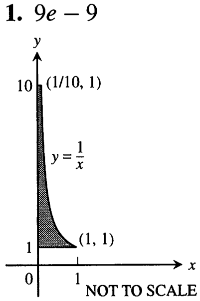
9. $64/5$ **11.** $\int_1^2 \int_1^3 (u + v)\frac{2u}{v}\, du\, dv = 8 + \frac{52}{3} \ln 2$

13. $\frac{\pi ab(a^2 + b^2)}{4}$ **15.** $\frac{1}{3}\left(1 + \frac{3}{e^2}\right) \approx 0.4687$ **19.** $\frac{4\pi\, abc}{3}$

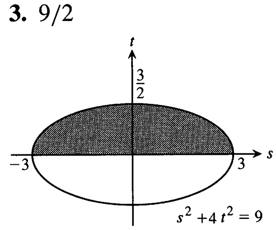
21. $\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8$

Chapter 13 Practice Exercises, pp. 1056–1058

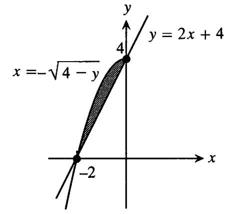
1. $9e - 9$



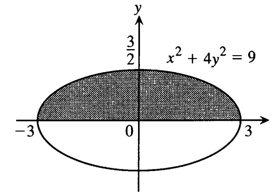
3. $9/2$



5. $\int_{-2}^{0} \int_{2x+4}^{4-x^2} dy\, dx = \frac{4}{3}$



7. $\int_{-3}^{3} \int_{0}^{(1/2)\sqrt{9-x^2}} y\, dy\, dx = \frac{9}{2}$



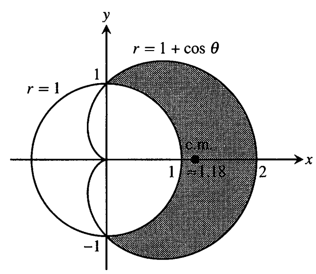
9. $\sin 4$ **11.** $\frac{\ln 17}{4}$ **13.** $4/3$ **15.** $4/3$ **17.** $1/4$

19. $\bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$ **21.** $I_0 = 104$ **23.** $I_x = 2\delta,\ R_x = \sqrt{\frac{2}{3}}$

25. $M = 4,\ M_x = 0,\ M_y = 0$ **27.** π **29.** $\bar{x} = \frac{3\sqrt{3}}{\pi},\ \bar{y} = 0$

31. a) $\bar{x} = \frac{15\pi + 32}{6\pi + 48},\ \bar{y} = 0$

b)



33. $\frac{\pi - 2}{4}$ **35.** 0 **37.** $8/35$ **39.** $\pi/2$ **41.** $\frac{2(31 - 3^{5/2})}{3}$

43. a) $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\, dz\, dx\, dy$

b) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} 3\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ c) $2\pi(8 - 4\sqrt{2})$

45. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{\pi}{3}$

47. $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 xy\, dz\, dy\, dx +$

$\int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 xy\, dz\, dy\, dx$

49. a) $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$

b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$

c) $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

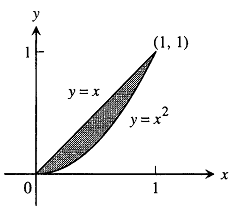
51. a) $\dfrac{8\pi(4\sqrt{2}-5)}{3}$ b) $\dfrac{8\pi(4\sqrt{2}-5)}{3}$ **53.** $I_z = \dfrac{8\pi\,\delta(b^5-a^5)}{15}$

Chapter 13 Additional Exercises, pp. 1058–1060

1. a) $\int_{-3}^{2}\int_{x}^{6-x^2} x^2 \, dy \, dx$ b) $\int_{-3}^{2}\int_{x}^{6-x^2}\int_{0}^{x^2} dz \, dy \, dx$ c) $\dfrac{125}{4}$

3. 2π **5.** $3\pi/2$ **7.** a) Hole radius $= 1$, sphere radius $= 2$
b) $4\sqrt{3}\pi$ **9.** $\pi/4$

11. $\int_0^1\int_y^{\sqrt{y}} f(x, y) \, dx \, dy$



13. $\ln\left(\dfrac{b}{a}\right)$ **17.** $1/\sqrt[4]{3}$ **19.** $\bar{x} = \dfrac{15\pi+32}{6\pi+48}$, $\bar{y} = 0$

21. Mass $= a^2\cos^{-1}\left(\dfrac{b}{a}\right) - b\sqrt{a^2-b^2}$,

$I_0 = \dfrac{a^4}{2}\cos^{-1}\left(\dfrac{b}{a}\right) - \dfrac{b^3}{2}\sqrt{a^2-b^2} - \dfrac{b^3}{6}(a^2-b^2)^{3/2}$

23. a) $\bar{x} = \bar{y} = 0$, $\bar{z} = 1/2$; $I_z = \dfrac{\pi}{8}$, $R_z = \dfrac{\sqrt{3}}{2}$

b) $\bar{x} = \bar{y} = 0$, $\bar{z} = 5/14$; $I_z = \dfrac{2\pi}{7}$, $R_z = \sqrt{\dfrac{5}{7}}$

25. $\bar{x} = \bar{y} = 0$, $\bar{z} = \dfrac{3a}{8}$ **27.** $\dfrac{1}{ab}\left(e^{a^2b^2}-1\right)$ **29.** b) 1 c) 0

33. $h = \sqrt{20}$ in., $h = \sqrt{60}$ in. **37.** $\dfrac{1}{2}\pi^2$

CHAPTER 14

Section 14.1, pp. 1065–1067

1. c **3.** g **5.** d **7.** f **9.** $\sqrt{2}$ **11.** $13/2$ **13.** $3\sqrt{14}$

15. $(1/6)(5\sqrt{5}+9)$ **17.** $\sqrt{3}\ln(b/a)$ **19.** $\dfrac{10\sqrt{5}-2}{3}$ **21.** 8

23. $2\sqrt{2}-1$ **25.** a) $4\sqrt{2}-2$ b) $\sqrt{2}+\ln(1+\sqrt{2})$
27. $I_z = 2\pi\delta a^3$, $R_z = a$ **29.** a) $I_z = 2\pi\sqrt{2}\,\delta$, $R_z = 1$
b) $I_z = 4\pi\sqrt{2}\,\delta$, $R_z = 1$ **31.** $I_x = 2\pi - 2$, $R_x = 1$

Section 14.2, pp. 1074–1076

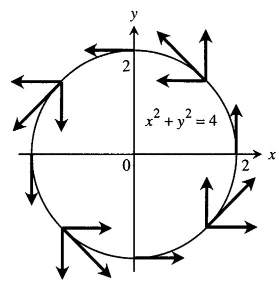
1. $\nabla f = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})(x^2+y^2+z^2)^{-3/2}$
3. $\nabla g = -(2x/(x^2+y^2))\mathbf{i} - (2y/(x^2+y^2))\mathbf{j} + e^z\mathbf{k}$

5. $\mathbf{F} = -\dfrac{kx}{(x^2+y^2)^{3/2}}\mathbf{i} - \dfrac{ky}{(x^2+y^2)^{3/2}}\mathbf{j}$, any $k > 0$

7. a) $\dfrac{9}{2}$ b) $\dfrac{13}{3}$ c) $\dfrac{9}{2}$ **9.** a) $\dfrac{1}{3}$ b) $-\dfrac{1}{5}$ c) 0 **11.** a) 2

b) $\dfrac{3}{2}$ c) $\dfrac{1}{2}$ **13.** $1/2$ **15.** $-\pi$ **17.** $207/12$ **19.** $-39/2$

21. $25/6$ **23.** a) $\text{Circ}_1 = 0$, $\text{circ}_2 = 2\pi$, $\text{flux}_1 = 2\pi$, $\text{flux}_2 = 0$
b) $\text{Circ}_1 = 0$, $\text{circ}_2 = 8\pi$, $\text{flux}_1 = 8\pi$, $\text{flux}_2 = 0$
25. Circ $= 0$, flux $= a^2\pi$ **27.** Circ $= a^2\pi$, flux $= 0$
29. a) $-\pi/2$ b) 0 c) 1
31.



33. a) $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$ b) $\mathbf{G} = \sqrt{x^2+y^2}\,\mathbf{F}$
35. $\mathbf{F} = -(x\mathbf{i}+y\mathbf{j})/\sqrt{x^2+y^2}$ **37.** 48 **39.** π **41.** 0
43. $1/2$

Section 14.3, pp. 1083–1084

1. Conservative **3.** Not conservative **5.** Not conservative

7. $f(x, y, z) = x^2 + \dfrac{3y^2}{2} + 2z^2 + C$ **9.** $f(x, y, z) = xe^{y+2z} + C$

11. $f(x, y, z) = x\ln x - x + \tan(x+y) + \dfrac{1}{2}\ln(y^2+z^2) + C$

13. 49 **15.** -16 **17.** 1 **19.** $9\ln 2$ **21.** 0 **23.** -3

27. $\mathbf{F} = \nabla\left(\dfrac{x^2-1}{y}\right)$ **29.** a) 1 b) 1 c) 1 **31.** a) 2 b) 2

33. $f(x, y, z) = \dfrac{Gm\,M}{(x^2+y^2+z^2)^{1/2}}$ **35.** a) $c = b = 2a$
b) $c = b = 2$
37. It does not matter what path you use. The work will be the same on any path because the field is conservative.

Section 14.4, pp. 1093–1095

1. Flux $= 0$, circ $= 2\pi a^2$ **3.** Flux $= -\pi a^2$, circ $= 0$
5. Flux $= 2$, circ $= 0$ **7.** Flux $= -9$, circ $= 9$
9. Flux $= 1/2$, circ $= 1/2$ **11.** Flux $= 1/5$, circ $= -1/12$

13. 0 **15.** $2/33$ **17.** 0 **19.** -16π **21.** πa^2 **23.** $\dfrac{3}{8}\pi$

25. a) 0 b) $(h-k)$(area of the region) **35.** a) 0

Section 14.5, pp. 1103–1105

1. $\frac{13}{3}\pi$ **3.** 4 **5.** $6\sqrt{6}-2\sqrt{2}$ **7.** $\pi\sqrt{c^2+1}$

9. $\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$ **11.** $3+2\ln 2$ **13.** $9a^3$

15. $\frac{abc}{4}(ab+ac+bc)$ **17.** 2 **19.** 18 **21.** $\pi a^3/6$

23. $\pi a^2/4$ **25.** $\pi a^3/2$ **27.** -32 **29.** -4 **31.** $3a^4$

33. $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

35. $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right)$, $I_z = \frac{15\pi\sqrt{2}}{2}\delta$, $R_z = \frac{\sqrt{10}}{2}$

37. a) $\frac{8\pi}{3}a^4\delta$ b) $\frac{20\pi}{3}a^4\delta$ **39.** $\frac{\pi}{6}(13\sqrt{13}-1)$ **41.** $5\pi\sqrt{2}$

43. $\frac{2}{3}(5\sqrt{5}-1)$

Section 14.6, pp. 1112–1114

1. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

3. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (r/2)\mathbf{k}$, $0 \le r \le 6$, $0 \le \theta \le \pi/2$

5. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{9-r^2}\,\mathbf{k}$, $0 \le r \le 3\sqrt{2}/2$, $0 \le \theta \le 2\pi$; Also: $\mathbf{r}(\phi,\theta) = (3\sin\phi\cos\theta)\mathbf{i} + (3\sin\phi\sin\theta)\mathbf{j} + (3\cos\phi)\mathbf{k}$, $0 \le \phi \le \pi/4$, $0 \le \theta \le 2\pi$

7. $\mathbf{r}(\phi,\theta) = (\sqrt{3}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{3}\sin\phi\sin\theta)\mathbf{j} + (\sqrt{3}\cos\phi)\mathbf{k}$, $\pi/3 \le \phi \le 2\pi/3$, $0 \le \theta \le 2\pi$

9. $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-y^2)\mathbf{k}$, $0 \le x \le 2$, $-2 \le y \le 2$

11. $\mathbf{r}(u,v) = u\mathbf{i} + (3\cos v)\mathbf{j} + (3\sin v)\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le 2\pi$

13. a) $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r\cos\theta - r\sin\theta)\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$
b) $\mathbf{r}(u,v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le 2\pi$

15. $\mathbf{r}(u,v) = (4\cos^2 v)\mathbf{i} + u\mathbf{j} + (4\cos v\sin v)\mathbf{k}$, $0 \le u \le 3$, $-(\pi/2) \le v \le (\pi/2)$; Another way: $\mathbf{r}(u,v) = (2+2\cos v)\mathbf{i} + u\mathbf{j} + (2\sin v)\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le 2\pi$

17. $\int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2} r\,dr\,d\theta = \frac{\pi\sqrt{5}}{2}$

19. $\int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta = 8\pi\sqrt{5}$

21. $\int_0^{2\pi}\int_1^4 1\,du\,dv = 6\pi$

23. $\int_0^{2\pi}\int_0^1 u\sqrt{4u^2+1}\,du\,dv = \frac{(5\sqrt{5}-1)}{6}\pi$

25. $\int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\,d\phi\,d\theta = (4+2\sqrt{2})\pi$

27. $\iint_S x\,d\sigma = \int_0^3\int_0^2 u\sqrt{4u^2+1}\,du\,dv = \frac{17\sqrt{17}-1}{4}$

29. $\iint_S x^2\,d\sigma = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\,d\phi\,d\theta = \frac{4\pi}{3}$

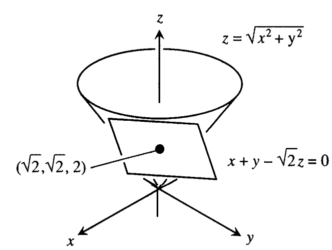
31. $\iint_S z\,d\sigma = \int_0^1\int_0^1 (4-u-v)\sqrt{3}\,dv\,du = 3\sqrt{3}$ (for $x = u$, $y = v$)

33. $\iint_S x^2\sqrt{5-4z}\,d\sigma = \int_0^1\int_0^{2\pi} u^2\cos^2 v \cdot \sqrt{4u^2+1} \cdot u\sqrt{4u^2+1}\,dv\,du = \int_0^1\int_0^{2\pi} u^3(4u^2+1)\cos^2 v\,dv\,du = \frac{11\pi}{12}$

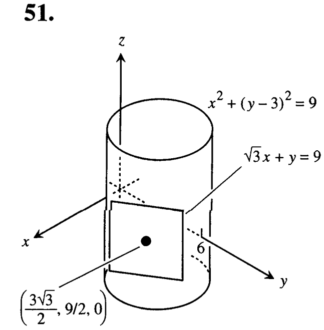
35. -32 **37.** $\pi a^3/6$ **39.** $13a^4/6$ **41.** $2\pi/3$ **43.** $-73\pi/6$

45. $(a/2, a/2, a/2)$ **47.** $8\delta\pi a^4/3$

49.



51.



55. b) $A = \int_0^{2\pi}\int_0^{\pi} [a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^4\phi\cos^2\theta + a^2c^2\cos^4\phi\sin^2\theta]^{1/2}\,d\phi\,d\theta$

57. $x_0x + y_0y = 25$

Section 14.7, pp. 1122–1123

1. 4π **3.** $-5/6$ **5.** 0 **7.** -6π **9.** $2\pi a^2$ **13.** 12π

15. $-\frac{\pi}{4}$ **17.** -15π **25.** $16\,I_y + 16\,I_x$

Section 14.8, pp. 1132–1134

1. 0 **3.** 0 **5.** -16 **7.** -8π **9.** 3π **11.** $-40/3$

13. 12π **15.** $12\pi(4\sqrt{2}-1)$

21. The integral's value never exceeds the surface area of S.

Chapter 14 Practice Exercises, pp. 1134–1137

1. Path 1: $2\sqrt{3}$, Path 2: $1+3\sqrt{2}$ **3.** $4a^2$ **5.** 0 **7.** 0

9. 0 **11.** $\pi\sqrt{3}$ **13.** $2\pi\left(1-\frac{1}{\sqrt{2}}\right)$ **15.** $\frac{abc}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

17. 50

19. $\mathbf{r}(\phi,\theta) = (6\sin\phi\cos\theta)\mathbf{i} + (6\sin\phi\sin\theta)\mathbf{j} + (6\cos\phi)\mathbf{k}$, $(\pi/6) \le \phi \le 2\pi/3$, $0 \le \theta \le 2\pi$

21. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1+r)\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

23. $\mathbf{r}(u, v) = (u \cos v)\,\mathbf{i} + 2u^2\,\mathbf{j} + (u \sin v)\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$
25. $\sqrt{6}$ **27.** $\pi[\sqrt{2} + \ln(1+\sqrt{2})]$ **29.** Conservative
31. Not conservative **33.** $f(x, y, z) = y^2 + yz + 2x + z$
35. Path 1: 2, Path 2: 8/3 **37.** a) $1 - e^{-2\pi}$ b) $1 - e^{-2\pi}$
39. a) $-\pi/2$ b) 0 c) 1 **41.** 0 **43.** a) $4\sqrt{2} - 2$
b) $\sqrt{2} + \ln(1+\sqrt{2})$
45. $(\bar{x}, \bar{y}, \bar{z}) = \left(1, \dfrac{16}{15}, \dfrac{2}{3}\right)$; $I_x = \dfrac{232}{45}$, $I_y = \dfrac{64}{15}$, $I_z = \dfrac{56}{9}$;
$R_x = \sqrt{\dfrac{116}{45}}$, $R_y = \sqrt{\dfrac{32}{15}}$, $R_z = \sqrt{\dfrac{28}{9}}$
47. $\bar{z} = \dfrac{3}{2}$, $I_z = \dfrac{7\sqrt{3}}{3}$, $R_z = \sqrt{\dfrac{7}{3}}$
49. $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 49/12)$, $I_z = 640\pi$, $R_z = 2\sqrt{2}$
51. Flux: 3/2, Circ: $-1/2$ **55.** 3 **57.** $\dfrac{2\pi}{3}(7 - 8\sqrt{2})$ **59.** 0
61. π

Chapter 14 Additional Exercises, pp. 1137–1139

1. 6π **3.** 2/3 **5.** a) $\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$
b) $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{k}$ c) $\mathbf{F}(x, y, z) = z\,\mathbf{i}$ **7.** $\dfrac{16\pi R^3}{3}$
9. $a = 2, b = 1$. The minimum flux is -4. **11.** b) $\dfrac{16}{3}g$
c) Work $= \left(\int_C g\,xy\,ds\right)\bar{y} = g\int_C xy^2\,ds$ **13.** c) $\dfrac{4}{3}\pi w$
19. False if $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$

APPENDICES

Appendix A.3, pp. A-16–A-17

1. a) (14, 8) b) $(-1, 8)$ c) $(0, -5)$
3. a) By reflecting z across the real axis b) By reflecting z across the imaginary axis c) By reflecting z in the origin d) By reflecting z in the real axis and then multiplying the length of the vector by $1/|z|^2$
5. a) Points on the circle $x^2 + y^2 = 4$ b) points inside the circle $x^2 + y^2 = 4$ c) points outside the circle $x^2 + y^2 = 4$
7. Points on a circle of radius 1, center $(-1, 0)$
9. Points on the line $y = -x$ **11.** $4e^{2\pi i/3}$ **13.** $1e^{2\pi i/3}$
21. $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ **23.** 1, $-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$
25. $2i$, $-\sqrt{3} - i$, $\sqrt{3} - i$ **27.** $\dfrac{\sqrt{6}}{2} \pm \dfrac{\sqrt{2}}{2}i$, $-\dfrac{\sqrt{6}}{2} \pm \dfrac{\sqrt{2}}{2}i$
29. $1 \pm \sqrt{3}i$, $-1 \pm \sqrt{3}i$

Appendix A.8, p. A-29

1. -5 **3.** 1 **5.** -7 **7.** 38 **9.** $x = -4$, $y = 1$
11. $x = 3$, $y = 2$ **13.** $x = 3$, $y = -2$, $z = 2$
15. $x = 2$, $y = 0$, $z = -1$ **17.** a) $h = 6$, $k = 4$
b) $h = 6$, $k \neq 4$

Index

Note: Numbers in parentheses refer to exercises on the pages indicated.

A Brief Table of Integrals

1. $\int u\,dv = uv - \int v\,du$

2. $\int a^u\,du = \frac{a^u}{\ln a} + C, \quad a \neq 1, \quad a > 0$

3. $\int \cos u\,du = \sin u + C$

4. $\int \sin u\,du = -\cos u + C$

5. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$

6. $\int (ax+b)^{-1}\,dx = \frac{1}{a} \ln|ax+b| + C$

7. $\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \quad n \neq -1, -2$

8. $\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2} \ln|ax+b| + C$

9. $\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

10. $\int \frac{dx}{x(ax+b)} = \frac{1}{b} \ln\left|\frac{x}{ax+b}\right| + C$

11. $\int (\sqrt{ax+b}\,)^n\,dx = \frac{2}{a}\frac{(\sqrt{ax+b}\,)^{n+2}}{n+2} + C, \quad n \neq -2$

12. $\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

13. a) $\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1}\sqrt{\frac{ax-b}{b}} + C$

b) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

14. $\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$

15. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$

17. $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \tan^{-1}\frac{x}{a} + C$

18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C$

19. $\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3} \ln\left|\frac{x+a}{x-a}\right| + C$

20. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln(x+\sqrt{a^2+x^2}\,) + C$

21. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \ln(x+\sqrt{a^2+x^2}\,) + C$

22. $\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8} \ln(x+\sqrt{a^2+x^2}\,) + C$

23. $\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a \ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

24. $\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = \ln(x+\sqrt{a^2+x^2}\,) - \frac{\sqrt{a^2+x^2}}{x} + C$

25. $\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = -\frac{a^2}{2} \ln(x+\sqrt{a^2+x^2}\,) + \frac{x\sqrt{a^2+x^2}}{2} + C$

26. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

27. $\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$

28. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

29. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$

30. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}(a^2-2x^2) + C$

31. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

32. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C$

33. $\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C$

34. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

35. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

36. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

37. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

38. $\int (\sqrt{x^2-a^2})^n\,dx = \frac{x(\sqrt{x^2-a^2})^n}{n+1} - \frac{na^2}{n+1}\int(\sqrt{x^2-a^2})^{n-2}dx, \quad n \neq -1$

39. $\int \frac{dx}{(\sqrt{x^2-a^2})^n} = \frac{x(\sqrt{x^2-a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int\frac{dx}{(\sqrt{x^2-a^2})^{n-2}}, \quad n \neq 2$

40. $\int x(\sqrt{x^2-a^2})^n\,dx = \frac{(\sqrt{x^2-a^2})^{n+2}}{n+2} + C, \quad n \neq -2$

41. $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

42. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

43. $\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| - \frac{\sqrt{x^2-a^2}}{x} + C$

44. $\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{x}{2}\sqrt{x^2-a^2} + C$

45. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$

46. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

47. $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

48. $\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

49. $\int (\sqrt{2ax-x^2})^n dx = \frac{(x-a)(\sqrt{2ax-x^2})^n}{n+1} + \frac{na^2}{n+1}\int(\sqrt{2ax-x^2})^{n-2}\,dx$

50. $\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{(x-a)(\sqrt{2ax-x^2})^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2}\int\frac{dx}{(\sqrt{2ax-x^2})^{n-2}}$

51. $\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

52. $\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$

53. $\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

54. $\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} + C$

55. $\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$

56. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

57. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

58. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

59. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

60. $\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$

61. $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

62. a) $\int \sin ax \cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$

b) $\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$

c) $\int \cos ax \cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$

63. $\int \sin ax \cos ax\,dx = -\frac{\cos 2ax}{4a} + C$

64. $\int \sin^n ax \cos ax\,dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$

65. $\int \frac{\cos ax}{\sin ax}\,dx = \frac{1}{a}\ln|\sin ax| + C$

66. $\int \cos^n ax \sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$

67. $\int \frac{\sin ax}{\cos ax}\,dx = -\frac{1}{a}\ln|\cos ax| + C$

68. $\int \sin^n ax \cos^m ax\,dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n}\int \sin^{n-2} ax \cos^m ax\,dx, \quad n \neq -m \quad (\text{reduces } \sin^n ax)$

69. $\int \sin^n ax \cos^m ax\,dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^n ax \cos^{m-2} ax\,dx, \quad m \neq -n \quad (\text{reduces } \cos^m ax)$

70. $\int \frac{dx}{b + c\sin ax} = \frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right] + C, \quad b^2 > c^2$

71. $\int \frac{dx}{b + c\sin ax} = \frac{-1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c + b\sin ax + \sqrt{c^2-b^2}\cos ax}{b + c\sin ax}\right| + C, \quad b^2 < c^2$

72. $\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

73. $\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + C$

74. $\int \frac{dx}{b + c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\frac{ax}{2}\right] + C, \quad b^2 > c^2$

75. $\int \frac{dx}{b + c\cos ax} = \frac{1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c + b\cos ax + \sqrt{c^2-b^2}\sin ax}{b + c\cos ax}\right| + C, \quad b^2 < c^2$

76. $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

77. $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

78. $\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$

79. $\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$

80. $\int x^n \sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$

81. $\int x^n \cos ax\,dx = \frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx$

82. $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$

83. $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$

84. $\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x + C$

85. $\int \cot^2 ax\,dx = -\frac{1}{a}\cot ax - x + C$

86. $\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax\,dx, \quad n \neq 1$

87. $\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax\,dx, \quad n \neq 1$

88. $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

89. $\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

90. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

91. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

92. $\int \sec^n ax\,dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx, \quad n \neq 1$

93. $\int \csc^n ax\,dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2} ax\,dx, \quad n \neq 1$

94. $\int \sec^n ax \tan ax\,dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$

95. $\int \csc^n ax \cot ax\,dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$

96. $\int \sin^{-1} ax\,dx = x\sin^{-1} ax + \frac{1}{a}\sqrt{1 - a^2x^2} + C$

97. $\int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{1}{a}\sqrt{1 - a^2x^2} + C$

98. $\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln(1 + a^2x^2) + C$

99. $\int x^n \sin^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1 - a^2x^2}}, \quad n \neq -1$

100. $\int x^n \cos^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1 - a^2x^2}}, \quad n \neq -1$

101. $\int x^n \tan^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1 + a^2x^2}, \quad n \neq -1$

102. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

103. $\int b^{ax}\,dx = \frac{1}{a}\frac{b^{ax}}{\ln b} + C, \quad b > 0,\ b \neq 1$

104. $\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1) + C$

105. $\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

106. $\int x^n b^{ax}\,dx = \frac{x^n b^{ax}}{a\ln b} - \frac{n}{a\ln b}\int x^{n-1}b^{ax}\,dx, \quad b > 0, b \neq 1$

107. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$

108. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$

109. $\int \ln ax\,dx = x\ln ax - x + C$

110. $\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}dx, \quad n \neq -1$

111. $\int x^{-1}(\ln ax)^m\,dx = \frac{(\ln ax)^{m+1}}{m+1} + C, \quad m \neq -1$

112. $\int \frac{dx}{x\ln ax} = \ln|\ln ax| + C$

113. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

114. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$

115. $\int \sinh^2 ax\,dx = \frac{\sinh 2ax}{4a} - \frac{x}{2} + C$

116. $\int \cosh^2 ax\,dx = \frac{\sinh 2ax}{4a} + \frac{x}{2} + C$

117. $\int \sinh^n ax\,dx = \frac{\sinh^{n-1} ax \cosh ax}{na} - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx, \quad n \neq 0$

118. $\int \cosh^n ax\,dx = \frac{\cosh^{n-1} ax \sinh ax}{na} + \frac{n-1}{n}\int \cosh^{n-2} ax\,dx, \quad n \neq 0$

119. $\int x \sinh ax\,dx = \frac{x}{a}\cosh ax - \frac{1}{a^2}\sinh ax + C$

120. $\int x \cosh ax\,dx = \frac{x}{a}\sinh ax - \frac{1}{a^2}\cosh ax + C$

121. $\int x^n \sinh ax\,dx = \frac{x^n}{a}\cosh ax - \frac{n}{a}\int x^{n-1}\cosh ax\,dx$

122. $\int x^n \cosh ax\,dx = \frac{x^n}{a}\sinh ax - \frac{n}{a}\int x^{n-1}\sinh ax\,dx$

123. $\int \tanh ax\,dx = \frac{1}{a}\ln(\cosh ax) + C$

124. $\int \coth ax\,dx = \frac{1}{a}\ln|\sinh ax| + C$

125. $\int \tanh^2 ax\,dx = x - \frac{1}{a}\tanh ax + C$

126. $\int \coth^2 ax\,dx = x - \frac{1}{a}\coth ax + C$

127. $\int \tanh^n ax\,dx = -\frac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax\,dx, \quad n \neq 1$

128. $\int \coth^n ax\,dx = -\frac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax\,dx, \quad n \neq 1$

129. $\int \operatorname{sech} ax\,dx = \frac{1}{a}\sin^{-1}(\tanh ax) + C$

130. $\int \operatorname{csch} ax\,dx = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right| + C$

131. $\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$

132. $\int \operatorname{csch}^2 ax\,dx = -\frac{1}{a}\coth ax + C$

133. $\int \operatorname{sech}^n ax\,dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{(n-1)a} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\,dx, \quad n \neq 1$

134. $\int \operatorname{csch}^n ax\,dx = -\frac{\operatorname{csch}^{n-2} ax \coth ax}{(n-1)a} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax\,dx, \quad n \neq 1$

135. $\int \operatorname{sech}^n ax \tanh ax\,dx = -\frac{\operatorname{sech}^n ax}{na} + C, \quad n \neq 0$

136. $\int \operatorname{csch}^n ax \coth ax\,dx = -\frac{\operatorname{csch}^n ax}{na} + C, \quad n \neq 0$

137. $\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$

138. $\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} + \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$

139. $\int_0^\infty x^{n-1}e^{-x}\,dx = \Gamma(n) = (n-1)!, \quad n > 0$

140. $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0$

141. $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is an even integer} \geq 2 \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}, & \text{if } n \text{ is an odd integer} \geq 3 \end{cases}$$