CHAPTER 9

Conic Sections, Parametrized Curves, and Polar Coordinates

OVERVIEW The study of motion has been important since ancient times, and calculus provides the mathematics we need to describe it. In this chapter, we extend our ability to analyze motion by showing how to track the position of a moving body as a function of time. We begin with equations for conic sections, since these are the paths traveled by planets, satellites, and other bodies (even electrons) whose motions are driven by inverse square forces. As we will see in Chapter 11, once we know that the path of a moving body is a conic section, we immediately have information about the body's velocity and the force that drives it. Planetary motion is best described with the help of polar coordinates (another of Newton's inventions, although James-Jakob-Jacques Bernoulli (1655–1705) usually gets the credit), so we also investigate curves, derivatives, and integrals in this new coordinate system.

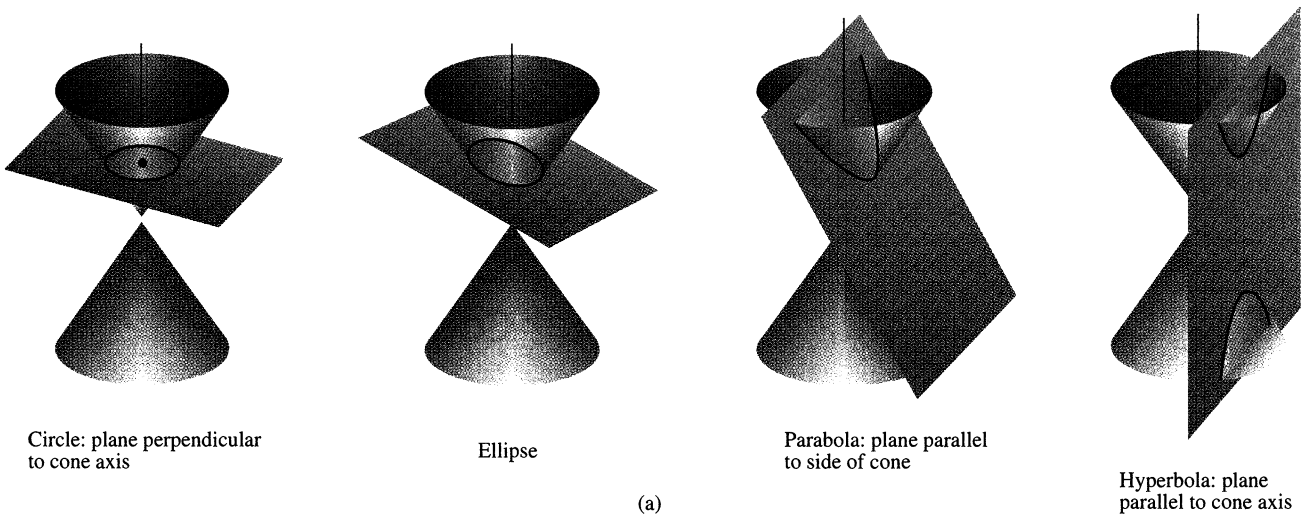
9.1 Conic Sections and Quadratic Equations

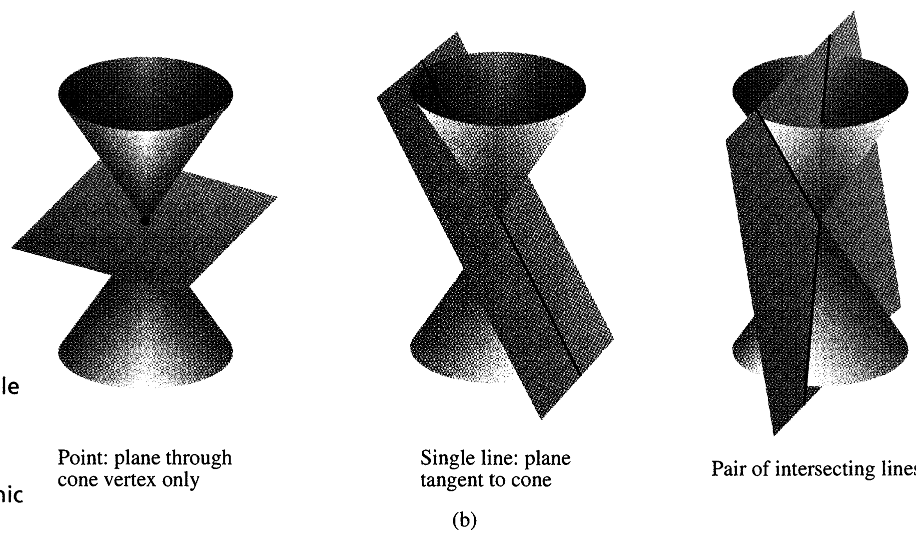
This section shows how the conic sections from Greek geometry are described today as the graphs of quadratic equations in the coordinate plane. The Greeks of Plato's time described these curves as the curves formed by cutting a double cone with a plane (Fig. 9.1, on the following page); hence the name *conic section.*

Circles

Definitions

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant. The fixed point is the **center** of the circle; the constant distance is the **radius.**



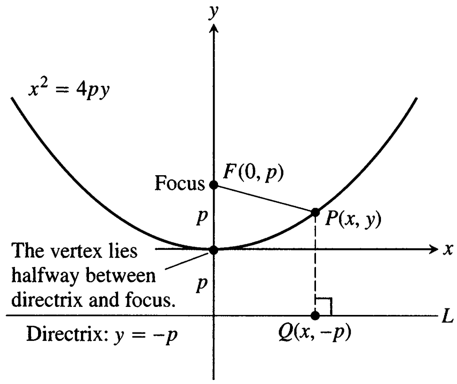


9.1 The standard conic sections (a) are the curves in which a plane cuts a double cone. Hyperbolas come in two parts, called *branches.* The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

The standard-form equations for circles, derived in Preliminaries, Section 4, from the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, are these:

Circles

Circle of radius a centered at the origin:	Circle of radius a centered at the point (h, k):
$x^2 + y^2 = a^2$	$(x - h)^2 + (y - k)^2 = a^2$



9.2 The parabola $x^2 = 4py$.

Parabolas

Definitions

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola.** The fixed point is the **focus** of the parabola. The fixed line is the **directrix.**

If the focus F lies on the directrix L, the parabola is the line through F perpendicular to L. We consider this to be a degenerate case and assume henceforth that F does not lie on L.

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive y-axis and that the directrix is the line $y = -p$ (Fig. 9.2). In the notation of the figure, a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

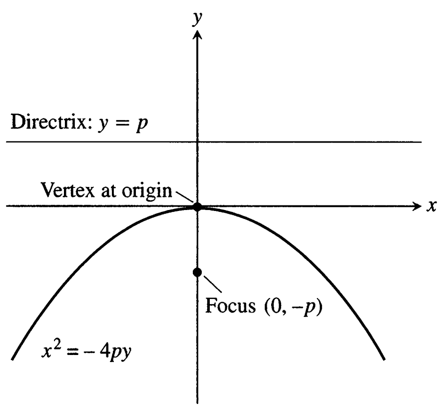
$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py. \qquad \text{Standard form} \qquad (1)$$

These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the **axis** of the parabola (short for "axis of symmetry").

The point where a parabola crosses its axis is the **vertex.** The vertex of the parabola $x^2 = 4py$ lies at the origin (Fig. 9.2). The positive number p is the parabola's **focal length.**



9.3 The parabola $x^2 = -4py$.

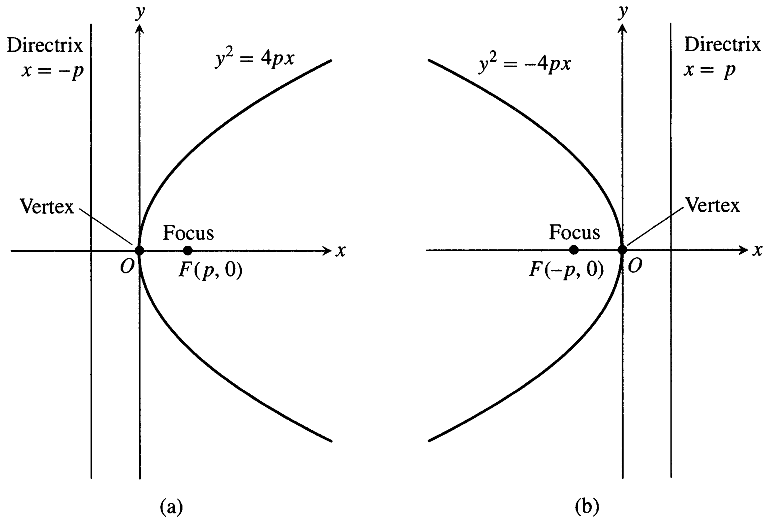
If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Eqs. (1) become

$$y = -\frac{x^2}{4p} \quad \text{and} \quad x^2 = -4py$$

(Fig. 9.3). We obtain similar equations for parabolas opening to the right or to the left (Fig. 9.4, on the following page, and Table 9.1).

Table 9.1 Standard-form equations for parabolas with vertices at the origin ($p > 0$)

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0, p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0, -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left



9.4 (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 = 10x$.

Solution We find the value of p in the standard equation $y^2 = 4px$:

$$4p = 10, \qquad \text{so} \qquad p = \frac{10}{4} = \frac{5}{2}.$$

Then we find the focus and directrix for this value of p:

$$\text{Focus:} \qquad (p, 0) = \left(\frac{5}{2}, 0\right)$$

$$\text{Directrix:} \qquad x = -p \quad \text{or} \quad x = -\frac{5}{2}.$$

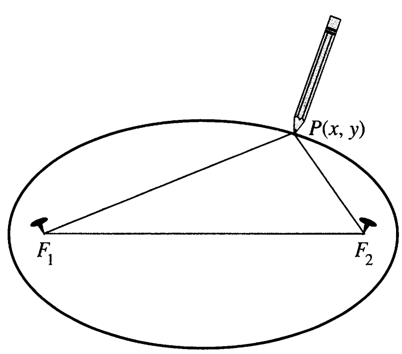
❑

The horizontal and vertical shift formulas in Preliminaries, Section 4, can be applied to the equations in Table 9.1 to give equations for a variety of parabolas in other locations (see Exercises 39, 40, and 45–48).

Ellipses

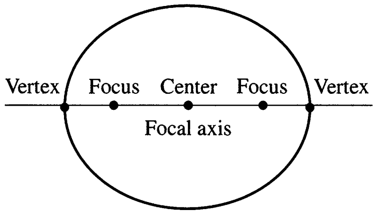
Definitions

An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.



9.5 How to draw an ellipse.

The quickest way to construct an ellipse uses the definition. Put a loop of string around two tacks F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace a closed curve (Fig. 9.5). The curve is an ellipse because the sum $PF_1 + PF_2$, being the length of the loop minus the distance between the tacks, remains constant. The ellipse's foci lie at F_1 and F_2.



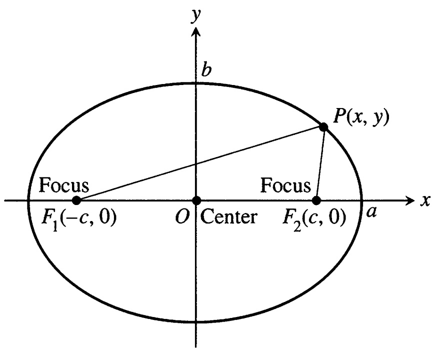
9.6 Points on the focal axis of an ellipse.

Definitions

The line through the foci of an ellipse is the ellipse's **focal axis.** The point on the axis halfway between the foci is the **center.** The points where the focal axis and ellipse cross are the ellipse's **vertices** (Fig. 9.6).

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Fig. 9.7), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$



9.7 The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$.

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{2}$$

Since $PF_1 + PF_2$ is greater than the length F_1F_2 (triangle inequality for triangle PF_1F_2), the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Eq. (2) is positive.

The algebraic steps leading to Eq. (2) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Eq. (2).

If

$$b = \sqrt{a^2 - c^2}, \tag{3}$$

then $a^2 - c^2 = b^2$ and Eq. (2) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

Equation (4) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \qquad \text{Obtained from Eq. (4) by implicit differentiation}$$

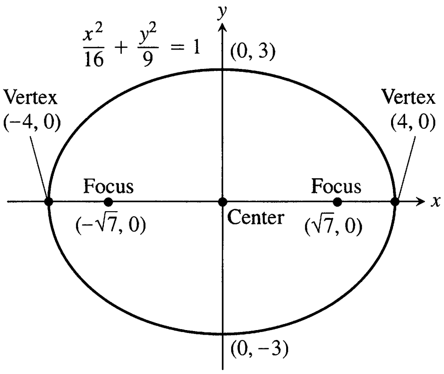
is zero if $x = 0$ and infinite if $y = 0$.

The Major and Minor Axes of an Ellipse

The **major axis** of the ellipse in Eq. (4) is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number a itself is the **semimajor axis,** the number b the **semiminor axis.** The number c, found from Eq. (3) as

$$c = \sqrt{a^2 - b^2},$$

is the **center-to-focus distance** of the ellipse.



9.8 Major axis horizontal (Example 2).

EXAMPLE 2 *Major axis horizontal*

The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{5}$$

(Fig. 9.8) has

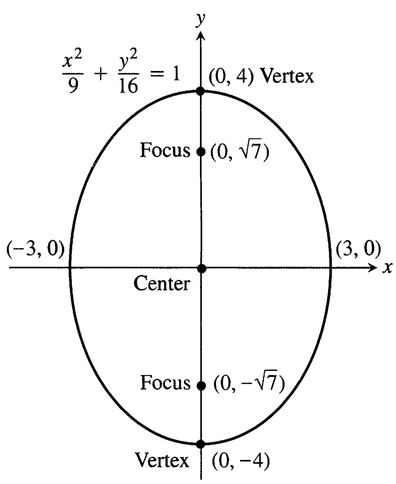
Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: $(0, 0)$. ❑



9.9 Major axis vertical (Example 3).

EXAMPLE 3 *Major axis vertical*

The ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \tag{6}$$

obtained by interchanging x and y in Eq. (5), has its major axis vertical instead of horizontal (Fig. 9.9). With a^2 still equal to 16 and b^2 equal to 9, we have

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(0, \pm c) = (0, \pm\sqrt{7})$

Vertices: $(0, \pm a) = (0, \pm 4)$

Center: $(0, 0)$. ❑

There is never any cause for confusion in analyzing equations like (5) and (6). We simply find the intercepts on the coordinate axes; then we know which way the major axis runs because it is the longer of the two axes. The center always lies at the origin and the foci lie on the major axis.

Standard-Form Equations for Ellipses Centered at the Origin

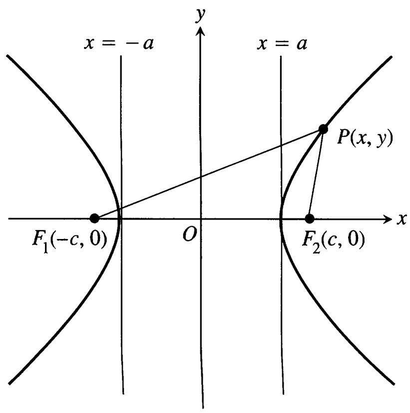
Foci on the x-axis: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$
Foci: $(\pm c, 0)$
Vertices: $(\pm a, 0)$

Foci on the y-axis: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.



9.10 Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$.

Hyperbolas

Definitions

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Fig. 9.10) and the constant difference is $2a$, then a point (x, y) lies on the hyperbola if and only if

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a. \tag{7}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{8}$$

So far, this looks just like the equation for an ellipse. But now $a^2 - c^2$ is negative because $2a$, being the difference of two sides of triangle PF_1F_2, is less than $2c$, the third side.

The algebraic steps leading to Eq. (8) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Eq. (7). A point therefore lies on the hyperbola if and only if its coordinates satisfy Eq. (8).

If we let b denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}, \tag{9}$$

then $a^2 - c^2 = -b^2$ and Eq. (8) takes the more compact form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{10}$$

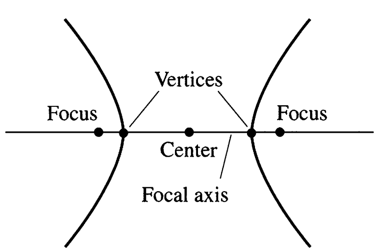
The differences between Eq. (10) and the equation for an ellipse (Eq. 4) are the minus sign and the new relation

$$c^2 = a^2 + b^2. \qquad \text{From Eq. (9)}$$

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the x-axis at the points $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} \qquad \text{Obtained from Eq. (10) by implicit differentiation}$$

is infinite when $y = 0$. The hyperbola has no y-intercepts; in fact, no part of the curve lies between the lines $x = -a$ and $x = a$.



9.11 Points on the focal axis of a hyperbola.

Definitions

The line through the foci of a hyperbola is the **focal axis.** The point on the axis halfway between the foci is the hyperbola's **center.** The points where the focal axis and hyperbola cross are the **vertices** (Fig. 9.11).

Asymptotes of Hyperbolas—Graphing

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{11}$$

has two asymptotes, the lines

$$y = \pm \frac{b}{a} x.$$

The asymptotes give us the guidance we need to graph hyperbolas quickly. (See the drawing lesson.) The fastest way to find the equations of the asymptotes is to replace the 1 in Eq. (11) by 0 and solve the new equation for y:

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \Rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}} \Rightarrow \underbrace{y = \pm \frac{b}{a} x.}_{\text{asymptotes}}$$

Standard-Form Equations for Hyperbolas Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$
Foci: $(\pm c, 0)$
Vertices: $(\pm a, 0)$
Asymptotes: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ or $y = \pm \dfrac{b}{a} x$

Foci on the y-axis: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm a)$
Asymptotes: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 0$ or $y = \pm \dfrac{a}{b} x$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

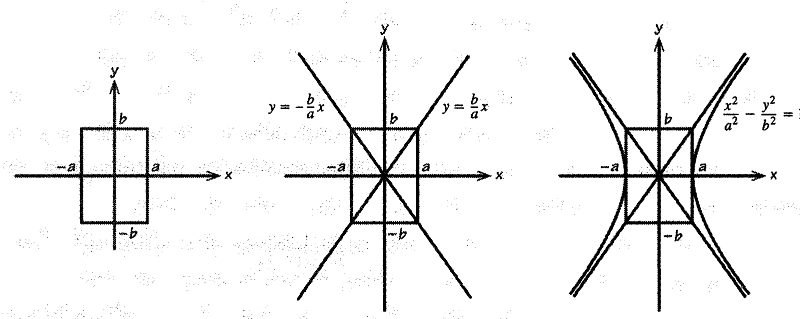
DRAWING LESSON

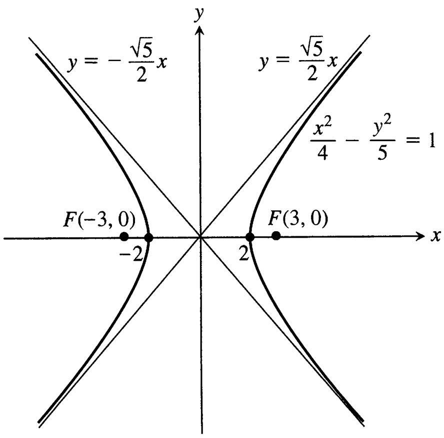
How to Graph the Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

1 **Mark the points $(\pm a, 0)$ and $(0, \pm b)$ with line segments and complete the rectangle they determine.**

2 **Sketch the asymptotes by extending the rectangle's diagonals.**

3 **Use the rectangle and asymptotes to guide your drawing.**





9.12 The hyperbola in Example 4.

EXAMPLE 4 *Foci on the x-axis*

The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \tag{12}$$

is Eq. (10) with $a^2 = 4$ and $b^2 = 5$ (Fig. 9.12). We have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4+5} = 3$

Foci: $(\pm c, 0) = (\pm 3, 0)$, Vertices: $(\pm a, 0) = (\pm 2, 0)$

Center: $(0, 0)$

Asymptotes: $\frac{x^2}{4} - \frac{y^2}{5} = 0$ or $y = \pm \frac{\sqrt{5}}{2} x.$ ❑

EXAMPLE 5 *Foci on the y-axis*

The hyperbola

$$\frac{y^2}{4} - \frac{x^2}{5} = 1,$$

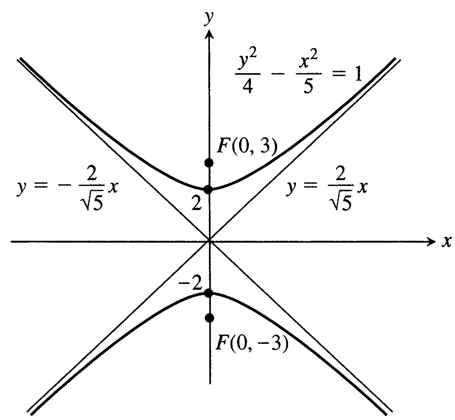
obtained by interchanging x and y in Eq. (12), has its vertices on the y-axis instead of the x-axis (Fig. 9.13). With a^2 still equal to 4 and b^2 equal to 5, we have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4+5} = 3$

Foci: $(0, \pm c) = (0, \pm 3)$, Vertices: $(0, \pm a) = (0, \pm 2)$

Center: $(0, 0)$

Asymptotes: $\frac{y^2}{4} - \frac{x^2}{5} = 0$ or $y = \pm \frac{2}{\sqrt{5}} x.$

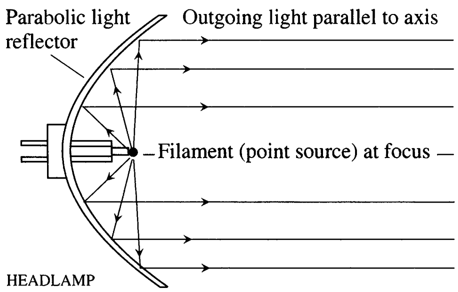


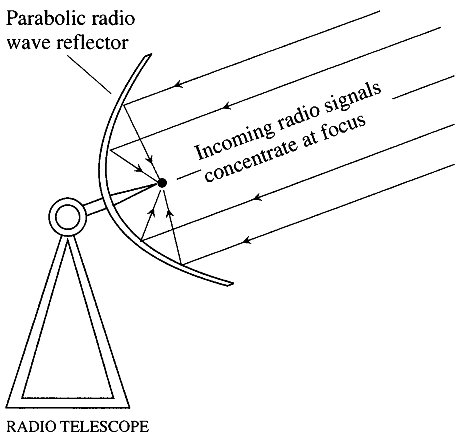
9.13 The hyperbola in Example 5.

❑

Reflective Properties

The chief applications of parabolas involve their use as reflectors of light and radio waves. Rays originating at a parabola's focus are reflected out of the parabola parallel to the parabola's axis (Fig. 9.14, on the following page, and Exercise 90). This property is used by flashlight, headlight, and spotlight reflectors and by microwave broadcast antennas to direct radiation from point sources into narrow beams. Conversely, electromagnetic waves arriving parallel to a parabolic reflector's axis are directed

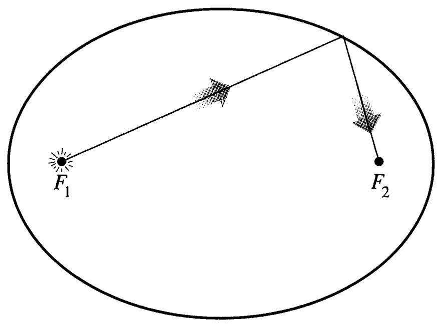




9.14 Two of the many uses of parabolic reflectors.

toward the reflector's focus. This property is used to intensify signals picked up by radio telescopes and television satellite dishes, to focus arriving light in telescopes, and to concentrate sunlight in solar heaters.

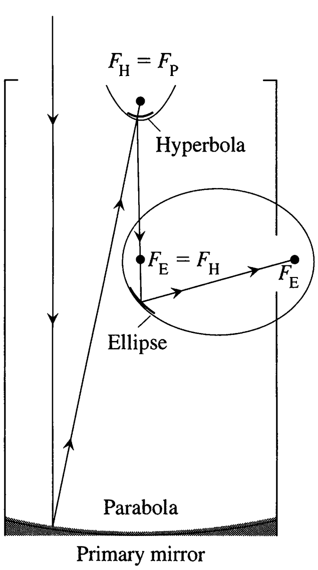
If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*) and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus (Fig. 9.15). Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries,* rooms in which a person standing at one focus can hear a whisper from the other focus. Statuary Hall in the U.S. Capitol building is a whispering gallery. Ellipsoids also appear in instruments used to study aircraft noise in wind tunnels (sound at one focus can be received at the other focus with relatively little interference from other sources).



9.15 An elliptical mirror (shown here in profile) reflects light from one focus to the other.

Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus. This property of hyperbolas is combined with the reflective properties of parabolas and ellipses in designing modern telescopes. In Fig. 9.16 starlight reflects off a primary parabolic mirror toward the mirror's focus F_P. It is then reflected by a small hyperbolic mirror, whose focus is $F_H = F_P$, toward the second focus of the hyperbola, $F_E = F_H$. Since this focus is shared by an ellipse, the light is reflected by the elliptical mirror to the ellipse's second focus to be seen by an observer.

As recent experience with NASA's Hubble space telescope shows, the mirrors have to be nearly perfect to focus properly. The aberration that caused the malfunction in Hubble's primary mirror (now corrected with additional mirrors) amounted to about half a wavelength of visible light, no more than 1/50 the width of a human hair.



9.16 Schematic drawing of a reflecting telescope.

Other Applications

Water pipes are sometimes designed with elliptical cross sections to allow for expansion when the water freezes. The triggering mechanisms in some lasers are elliptical, and stones on a beach become more and more elliptical as they are ground down by waves. There are also applications of ellipses to fossil formation. The ellipsolith, once thought to be a separate species, is now known to be an elliptically deformed nautilus.

Hyperbolic paths arise in Einstein's theory of relativity and form the basis for the (unrelated) LORAN radio navigation system. (LORAN is short for "long range navigation.") Hyperbolas also form the basis for a new system the Burlington Northern Railroad developed for using synchronized electronic signals from satellites to track freight trains. Computers aboard Burlington Northern locomotives in Minnesota have been able to track trains to within one mile per hour of their speed and to within 150 feet of their actual location.

Exercises 9.1

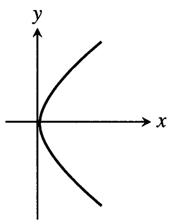
Identifying Graphs

Match the parabolas in Exercises 1–4 with the following equations:

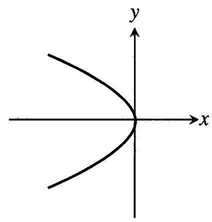
$$x^2 = 2y, \quad x^2 = -6y, \quad y^2 = 8x, \quad y^2 = -4x.$$

Then find the parabola's focus and directrix.

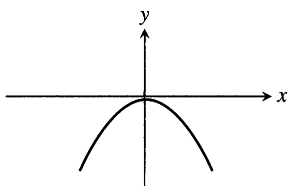
1.



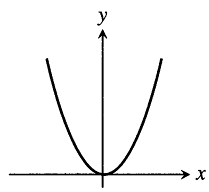
2.



3.



4.



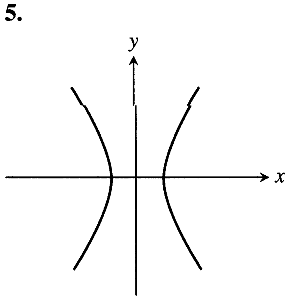
Match each conic section in Exercises 5–8 with one of these equations:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \quad \frac{x^2}{2} + y^2 = 1,$$

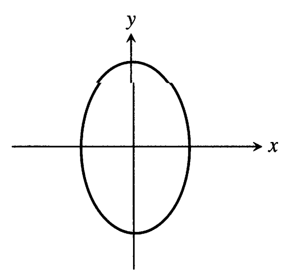
$$\frac{y^2}{4} - x^2 = 1, \quad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

Then find the conic section's foci and vertices. If the conic section is a hyperbola, find its asymptotes as well.

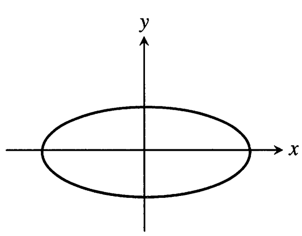
5.



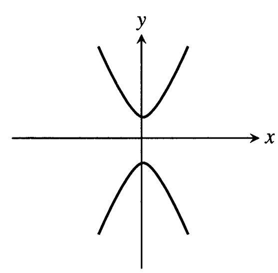
6.



7.



8.



Parabolas

Exercises 9–16 give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

9. $y^2 = 12x$ **10.** $x^2 = 6y$ **11.** $x^2 = -8y$

12. $y^2 = -2x$ **13.** $y = 4x^2$ **14.** $y = -8x^2$

15. $x = -3y^2$ **16.** $x = 2y^2$

Ellipses

Exercises 17–24 give equations for ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

17. $16x^2 + 25y^2 = 400$ **18.** $7x^2 + 16y^2 = 112$

19. $2x^2 + y^2 = 2$ **20.** $2x^2 + y^2 = 4$

21. $3x^2 + 2y^2 = 6$ **22.** $9x^2 + 10y^2 = 90$

23. $6x^2 + 9y^2 = 54$ **24.** $169x^2 + 25y^2 = 4225$

Exercises 25 and 26 give information about the foci and vertices of ellipses centered at the origin of the xy-plane. In each case, find the ellipse's standard-form equation from the given information.

25. Foci: $(\pm\sqrt{2}, 0)$
Vertices: $(\pm 2, 0)$

26. Foci: $(0, \pm 4)$
Vertices: $(0, \pm 5)$

Hyperbolas

Exercises 27–34 give equations for hyperbolas. Put each equation in standard form and find the hyperbola's asymptotes. Then sketch the hyperbola. Include the asymptotes and foci in your sketch.

27. $x^2 - y^2 = 1$ **28.** $9x^2 - 16y^2 = 144$

29. $y^2 - x^2 = 8$ **30.** $y^2 - x^2 = 4$

31. $8x^2 - 2y^2 = 16$ **32.** $y^2 - 3x^2 = 3$

33. $8y^2 - 2x^2 = 16$ **34.** $64x^2 - 36y^2 = 2304$

Exercises 35–38 give information about the foci, vertices, and asymptotes of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's standard-form equation from the information given.

35. Foci: $(0, \pm\sqrt{2})$
Asymptotes: $y = \pm x$

36. Foci: $(\pm 2, 0)$
Asymptotes: $y = \pm \frac{1}{\sqrt{3}} x$

37. Vertices: $(\pm 3, 0)$
Asymptotes: $y = \pm \frac{4}{3} x$

38. Vertices: $(0, \pm 2)$
Asymptotes: $y = \pm \frac{1}{2} x$

Shifting Conic Sections

39. The parabola $y^2 = 8x$ is shifted down 2 units and right 1 unit to generate the parabola $(y + 2)^2 = 8(x - 1)$. (a) Find the new parabola's vertex, focus, and directrix. (b) Plot the new vertex, focus, and directrix, and sketch in the parabola.

40. The parabola $x^2 = -4y$ is shifted left 1 unit and up 3 units to generate the parabola $(x+1)^2 = -4(y-3)$. (a) Find the new parabola's vertex, focus, and directrix. (b) Plot the new vertex, focus, and directrix, and sketch in the parabola.

41. The ellipse $(x^2/16) + (y^2/9) = 1$ is shifted 4 units to the right and 3 units up to generate the ellipse

$$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1.$$

(a) Find the foci, vertices, and center of the new ellipse. (b) Plot the new foci, vertices, and center, and sketch in the new ellipse.

42. The ellipse $(x^2/9) + (y^2/25) = 1$ is shifted 3 units to the left and 2 units down to generate the ellipse

$$\frac{(x+3)^2}{9} + \frac{(y+2)^2}{25} = 1.$$

(a) Find the foci, vertices, and center of the new ellipse. (b) Plot the new foci, vertices, and center, and sketch in the new ellipse.

43. The hyperbola $(x^2/16) - (y^2/9) = 1$ is shifted 2 units to the right to generate the hyperbola

$$\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1.$$

(a) Find the center, foci, vertices, and asymptotes of the new hyperbola. (b) Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

44. The hyperbola $(y^2/4) - (x^2/5) = 1$ is shifted 2 units down to generate the hyperbola

$$\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1.$$

(a) Find the center, foci, vertices, and asymptotes of the new hyperbola. (b) Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

Exercises 45–48 give equations for parabolas and tell how many units up or down and to the right or left each parabola is to be shifted. Find an equation for the new parabola, and find the new vertex, focus, and directrix.

45. $y^2 = 4x$, left 2, down 3

46. $y^2 = -12x$, right 4, up 3

47. $x^2 = 8y$, right 1, down 7

48. $x^2 = 6y$, left 3, down 2

Exercises 49–52 give equations for ellipses and tell how many units up or down and to the right or left each ellipse is to be shifted. Find an equation for the new ellipse, and find the new foci, vertices, and center.

49. $\dfrac{x^2}{6} + \dfrac{y^2}{9} = 1$, left 2, down 1

50. $\dfrac{x^2}{2} + y^2 = 1$, right 3, up 4

51. $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$, right 2, up 3

52. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, left 4, down 5

Exercises 53–56 give equations for hyperbolas and tell how many units up or down and to the right or left each hyperbola is to be shifted. Find an equation for the new hyperbola, and find the new center, foci, vertices, and asymptotes.

53. $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$, right 2, up 2

54. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, left 5, down 1

55. $y^2 - x^2 = 1$, left 1, down 1

56. $\dfrac{y^2}{3} - x^2 = 1$, right 1, up 3

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections in Exercises 57–68.

57. $x^2 + 4x + y^2 = 12$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

59. $x^2 + 2x + 4y - 3 = 0$

60. $y^2 - 4y - 8x - 12 = 0$

61. $x^2 + 5y^2 + 4x = 1$

62. $9x^2 + 6y^2 + 36y = 0$

63. $x^2 + 2y^2 - 2x - 4y = -1$

64. $4x^2 + y^2 + 8x - 2y = -1$

65. $x^2 - y^2 - 2x + 4y = 4$

66. $x^2 - y^2 + 4x - 6y = 6$

67. $2x^2 - y^2 + 6y = 3$

68. $y^2 - 4x^2 + 16x = 24$

Inequalities

Sketch the regions in the xy-plane whose coordinates satisfy the inequalities or pairs of inequalities in Exercises 69–74.

69. $9x^2 + 16y^2 \le 144$

70. $x^2 + y^2 \ge 1$ and $4x^2 + y^2 \le 4$

71. $x^2 + 4y^2 \ge 4$ and $4x^2 + 9y^2 \le 36$

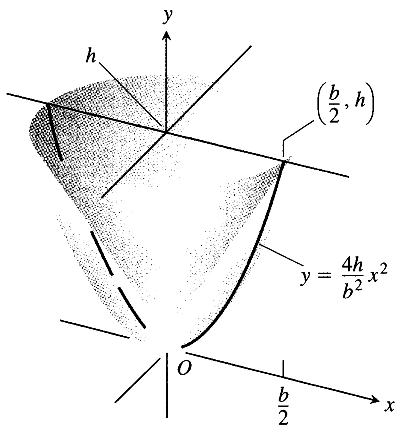
72. $(x^2 + y^2 - 4)(x^2 + 9y^2 - 9) \le 0$

73. $4y^2 - x^2 \ge 4$

74. $|x^2 - y^2| \le 1$

Theory and Examples

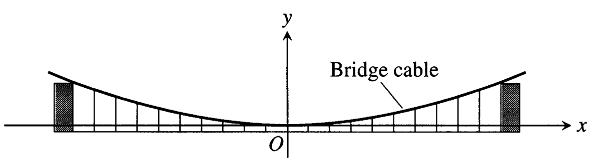
75. *Archimedes' formula for the volume of a parabolic solid.* The region enclosed by the parabola $y = (4h/b^2)x^2$ and the line $y = h$ is revolved about the y-axis to generate the solid shown here. Show that the volume of the solid is 3/2 the volume of the corresponding cone.



76. *Suspension bridge cables hang in parabolas.* The suspension bridge cable shown here supports a uniform load of w pounds per horizontal foot. It can be shown that if H is the horizontal tension of the cable at the origin, then the curve of the cable satisfies the equation

$$\frac{dy}{dx} = \frac{w}{H}x.$$

Show that the cable hangs in a parabola by solving this differential equation subject to the initial condition that $y = 0$ when $x = 0$.



77. Find an equation for the circle through the points (1, 0), (0, 1), and (2, 2).

78. Find an equation for the circle through the points (2, 3), (3, 2), and (−4, 3).

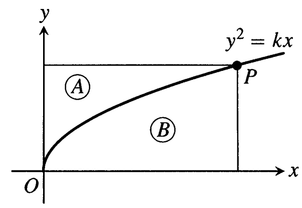
79. Find an equation for the circle centered at (−2, 1) that passes through the point (1, 3). Is the point (1.1, 2.8) inside, outside, or on the circle?

80. Find equations for the tangents to the circle $(x - 2)^2 + (y - 1)^2 = 5$ at the points where the circle crosses the coordinate axes. (*Hint:* Use implicit differentiation.)

81. If lines are drawn parallel to the coordinate axes through a point P on the parabola $y^2 = kx, k > 0$, the parabola partitions the rectangular region bounded by these lines and the coordinate axes into two smaller regions, A and B.

a) If the two smaller regions are revolved about the y-axis, show that they generate solids whose volumes have the ratio 4:1.

b) What is the ratio of the volumes generated by revolving the regions about the x-axis?



82. Show that the tangents to the curve $y^2 = 4px$ from any point on the line $x = -p$ are perpendicular.

83. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $x^2 + 4y^2 = 4$ with its sides parallel to the coordinate axes. What is the area of the rectangle?

84. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about the (a) x-axis, (b) y-axis.

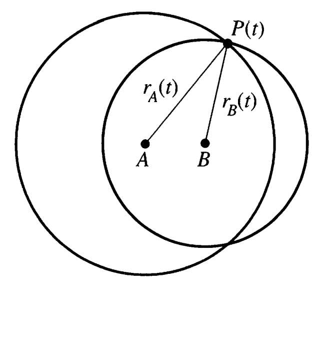
85. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

86. The region bounded on the left by the y-axis, on the right by the hyperbola $x^2 - y^2 = 1$, and above and below by the lines $y = \pm 3$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

87. Find the centroid of the region that is bounded below by the x-axis and above by the ellipse $(x^2/9) + (y^2/16) = 1$.

88. The curve $y = \sqrt{x^2 + 1}$, $0 \le x \le \sqrt{2}$, which is part of the upper branch of the hyperbola $y^2 - x^2 = 1$, is revolved about the x-axis to generate a surface. Find the area of the surface.

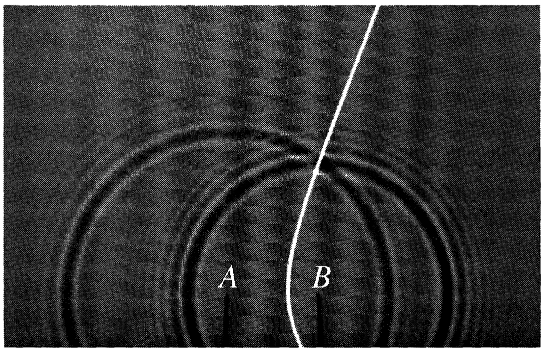
89. The circular waves in the photograph here were made by touching the surface of a ripple tank, first at A and then at B. As the waves expanded, their point of intersection appeared to trace a hyperbola. Did it really do that? To find out, we can model the waves with circles centered at A and B.



At time t, the point P is $r_A(t)$ units from A and $r_B(t)$ units from B. Since the radii of the circles increase at a constant rate, the rate at which the waves are traveling is

$$\frac{dr_A}{dt} = \frac{dr_B}{dt}.$$

Conclude from this equation that $r_A - r_B$ has a constant value, so that P must lie on a hyperbola with foci at A and B.



The expanding waves in Exercise 89.

90. *The reflective property of parabolas.* The figure here shows a typical point $P(x_0, y_0)$ on the parabola $y^2 = 4px$. The line L is tangent to the parabola at P. The parabola's focus lies at $F(p, 0)$. The ray L' extending from P to the right is parallel to the x-axis. We show that light from F to P will be reflected out along L' by showing that β equals α. Establish this equality by taking the following steps.

a) Show that $\tan\beta = 2p/y_0$.

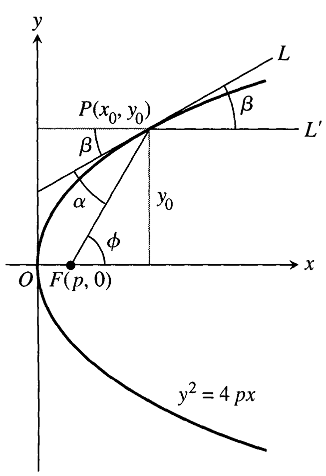
b) Show that $\tan\phi = y_0/(x_0 - p)$.

c) Use the identity

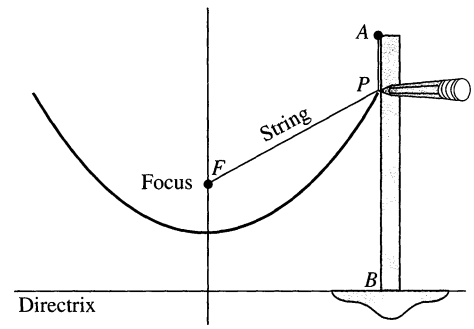
$$\tan\alpha = \frac{\tan\phi - \tan\beta}{1 + \tan\phi\tan\beta}$$

to show that $\tan\alpha = 2p/y_0$.

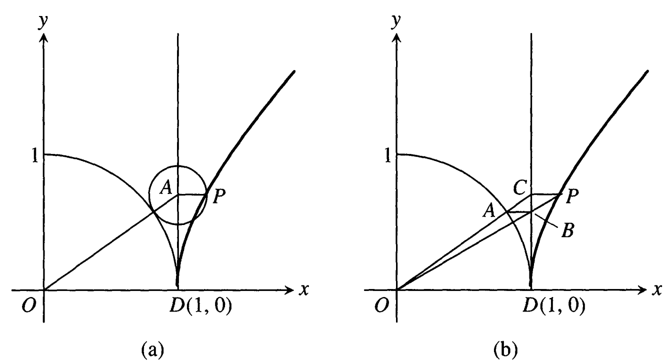
Since α and β are both acute, $\tan\beta = \tan\alpha$ implies $\beta = \alpha$.



91. *How the astronomer Kepler used string to draw parabolas.* Kepler's method for drawing a parabola (with more modern tools) requires a string the length of a T square and a table whose edge can serve as the parabola's directrix. Pin one end of the string to the point where you want the focus to be and the other end to the upper end of the T square. Then, holding the string taut against the T square with a pencil, slide the T square along the table's edge. As the T square moves, the pencil will trace a parabola. Why?



92. *Construction of a hyperbola.* The following diagrams appeared (unlabeled) in Ernest J. Eckert, "Constructions Without Words," *Mathematics Magazine,* Vol. 66, No. 2, April 1993, p. 113. Explain the constructions.



93. *The width of a parabola at the focus.* Show that the number $4p$ is the **width** of the parabola $x^2 = 4py$ $(p > 0)$ at the focus by showing that the line $y = p$ cuts the parabola at points that are $4p$ units apart.

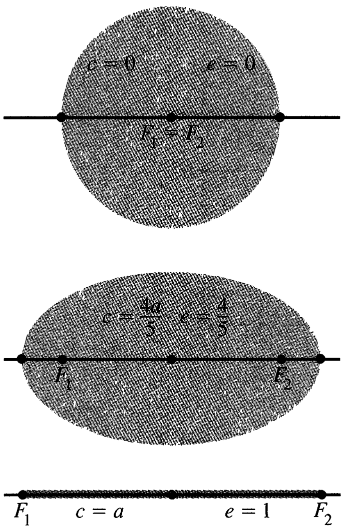
94. *The asymptotes of $(x^2/a^2) - (y^2/b^2) = 1$.* Show that the vertical distance between the line $y = (b/a)x$ and the upper half of the right-hand branch $y = (b/a)\sqrt{x^2 - a^2}$ of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ approaches 0 by showing that

$$\lim_{x\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim_{x\to\infty}\left(x - \sqrt{x^2 - a^2}\right) = 0.$$

Similar results hold for the remaining portions of the hyperbola and the lines $y = \pm(b/a)x$.

9.2 Classifying Conic Sections by Eccentricity

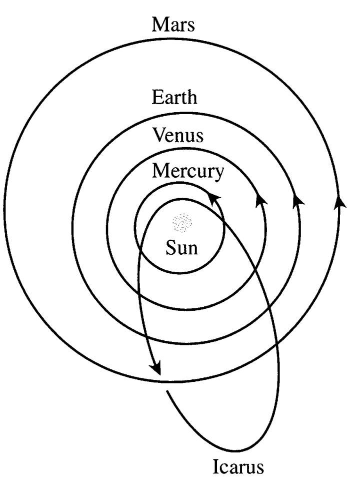
We now show how to associate with each conic section a number called the conic section's eccentricity. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and, in the case of ellipses and hyperbolas, describes the conic section's general proportions.



9.17 The ellipse changes from a circle to a line segment as c increases from 0 to *a*.

Table 9.2 Eccentricities of planetary orbits

Mercury	0.21	Saturn	0.06
Venus	0.01	Uranus	0.05
Earth	0.02	Neptune	0.01
Mars	0.09	Pluto	0.25
Jupiter	0.05		



9.18 The orbit of the asteroid Icarus is highly eccentric. Earth's orbit is so nearly circular that its foci lie inside the sun.

Eccentricity

Although the center-to-focus distance c does not appear in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (a > b)$$

for an ellipse, we can still determine c from the equation $c = \sqrt{a^2 - b^2}$. If we fix a and vary c over the interval $0 \le c \le a$, the resulting ellipses will vary in shape (Fig. 9.17). They are circles if $c = 0$ (so that $a = b$) and flatten as c increases. If $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment.

We use the ratio of c to a to describe the various shapes the ellipse can take. We call this ratio the ellipse's eccentricity.

Definition

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The planets in the solar system revolve around the sun in elliptical orbits with the sun at one focus. Most of the orbits are nearly circular, as can be seen from the eccentricities in Table 9.2. Pluto has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric. Icarus, an asteroid about 1 mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83 (Fig. 9.18).

EXAMPLE 1 The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical units wide. (One *astronomical unit* [AU] is 149,597,870 km, the semimajor axis of Earth's orbit.) Its eccentricity is

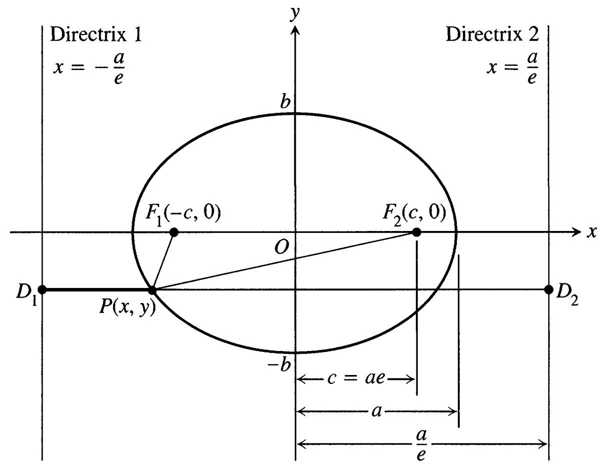
$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{(36.18/2)^2 - (9.12/2)^2}}{(1/2)(36.18)} = \frac{\sqrt{(18.09)^2 - (4.56)^2}}{18.09} \approx 0.97. \quad \square$$

Whereas a parabola has one focus and one directrix, each ellipse has two foci and two directrices. These are the lines perpendicular to the major axis at distances $\pm a/e$ from the center. The parabola has the property that

$$PF = 1 \cdot PD \tag{1}$$

for any point P on it, where F is the focus and D is the point nearest P on the directrix. For an ellipse, it can be shown that the equations that replace (1) are

$$PF_1 = e \cdot PD_1, \qquad PF_2 = e \cdot PD_2. \tag{2}$$



9.19 The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus F_1, and directrix 2 to focus F_2.

Here, e is the eccentricity, P is any point on the ellipse, F_1 and F_2 are the foci, and D_1 and D_2 are the points on the directrices nearest P (Fig. 9.19).

In each equation in (2) the directrix and focus must correspond; that is, if we use the distance from P to F_1, we must also use the distance from P to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

The eccentricity of a hyperbola is also $e = c/a$, only in this case c equals $\sqrt{a^2 + b^2}$ instead of $\sqrt{a^2 - b^2}$. In contrast to the eccentricity of an ellipse, the eccentricity of a hyperbola is always greater than 1.

Halley's comet

Edmund Halley (1656–1742; pronounced "*haw*-ley"), British biologist, geologist, sea captain, pirate, spy, Antarctic voyager, astronomer, adviser on fortifications, company founder and director, and the author of the first actuarial mortality tables, was also the mathematician who pushed and harried Newton into writing his *Principia.* Despite his accomplishments, Halley is known today chiefly as the man who calculated the orbit of the great comet of 1682: "wherefore if according to what we have already said [the comet] should return again about the year 1758, candid posterity will not refuse to acknowledge that this was first discovered by an Englishman." Indeed, candid posterity did not refuse—ever since the comet's return in 1758, it has been known as Halley's comet.

Last seen rounding the sun during the winter and spring of 1985–86, the comet is due to return in the year 2062. A recent study indicates that the comet has made about 2000 cycles so far with about the same number to go before the sun erodes it away completely.

Definition

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

In both ellipse and hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

EXAMPLE 2 Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 7)$.

Solution Since $e = c/a$, the vertices are the points $(0, \pm a)$ where

$$a = \frac{c}{e} = \frac{7}{0.8} = 8.75,$$

or $(0, \pm 8.75)$. ❑

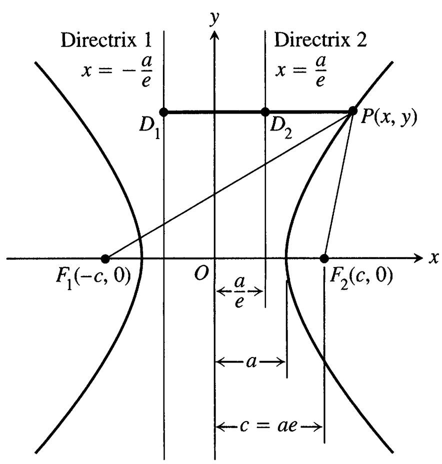
EXAMPLE 3 Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

Solution We divide both sides of the hyperbola's equation by 144 to put it in standard form, obtaining

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1 \qquad \text{and} \qquad \frac{x^2}{16} - \frac{y^2}{9} = 1.$$

With $a^2 = 16$ and $b^2 = 9$, we find that $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$, so

$$e = \frac{c}{a} = \frac{5}{4}.$$ ❑



9.20 The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as directrices for the hyperbola and that

$$PF_1 = e \cdot PD_1 \qquad \text{and} \qquad PF_2 = e \cdot PD_2. \tag{3}$$

Here P is any point on the hyperbola, F_1 and F_2 are the foci, and D_1 and D_2 are the points nearest P on the directrices (Fig. 9.20).

To complete the picture, we define the eccentricity of a parabola to be $e = 1$. Equations (1) – (3) then have the common form $PF = e \cdot PD$.

Definition

The **eccentricity** of a parabola is $e = 1$.

The "focus–directrix" equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F(the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$\boxed{PF = e \cdot PD, \tag{4}}$$

where e is the constant of proportionality. Then the path traced by P is

a) a *parabola* if $e = 1$,
b) an *ellipse* of eccentricity e if $e < 1$, and
c) a *hyperbola* of eccentricity e if $e > 1$.

Equation (4) may not look like much to get excited about. There are no coordinates in it and when we try to translate it into coordinate form it translates in different ways, depending on the size of e. At least, that is what happens in Cartesian coordinates. However, in polar coordinates, as we will see in Section 9.8, the equation $PF = e \cdot PD$ translates into a single equation regardless of the value of e, an equation so simple that it has been the equation of choice of astronomers and space scientists for nearly 300 years.

Given the focus and corresponding directrix of a hyperbola centered at the origin and with foci on the x-axis, we can use the dimensions shown in Fig. 9.20 to find e. Knowing e, we can derive a Cartesian equation for the hyperbola from the equation $PF = e \cdot PD$, as in the next example. We can find equations for ellipses centered at the origin and with foci on the x-axis in a similar way, using the dimensions shown in Fig. 9.19.

EXAMPLE 4 Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.

Solution We first use the dimensions shown in Fig. 9.20 to find the hyperbola's eccentricity. The focus is

$$(c, 0) = (3, 0), \qquad \text{so} \qquad c = 3.$$

The directrix is the line

$$x = \frac{a}{e} = 1, \qquad \text{so} \qquad a = e.$$

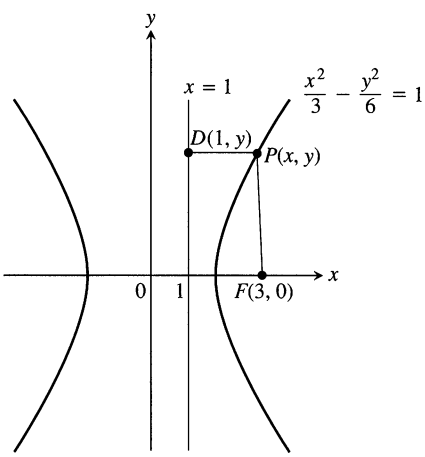
When combined with the equation $e = c/a$ that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \qquad \text{so} \qquad e^2 = 3 \quad \text{and} \quad e = \sqrt{3}.$$

Knowing e, we can now derive the equation we want from the equation $PF = e \cdot PD$. In the notation of Fig. 9.21, we have

$$\begin{aligned} PF &= e \cdot PD && \text{Eq. (4)} \\ \sqrt{(x-3)^2 + (y-0)^2} &= \sqrt{3}\,|x-1| && e = \sqrt{3} \\ x^2 - 6x + 9 + y^2 &= 3(x^2 - 2x + 1) \\ 2x^2 - y^2 &= 6 \\ \frac{x^2}{3} - \frac{y^2}{6} &= 1. \end{aligned}$$

❑



9.21 The hyperbola in Example 4.

Exercises 9.2

Ellipses

In Exercises 1–8, find the eccentricity of the ellipse. Then find and graph the ellipse's foci and directrices.

1. $16x^2 + 25y^2 = 400$

2. $7x^2 + 16y^2 = 112$

3. $2x^2 + y^2 = 2$

4. $2x^2 + y^2 = 4$

5. $3x^2 + 2y^2 = 6$

6. $9x^2 + 10y^2 = 90$

7. $6x^2 + 9y^2 = 54$

8. $169x^2 + 25y^2 = 4225$

Exercises 9–12 give the foci or vertices and the eccentricities of ellipses centered at the origin of the xy-plane. In each case, find the ellipse's standard-form equation.

9. Foci: $(0, \pm 3)$
Eccentricity: 0.5

10. Foci: $(\pm 8, 0)$
Eccentricity: 0.2

11. Vertices: $(0, \pm 70)$
Eccentricity: 0.1

12. Vertices: $(\pm 10, 0)$
Eccentricity: 0.24

Exercises 13–16 give foci and corresponding directrices of ellipses centered at the origin of the xy-plane. In each case, use the dimensions in Fig. 9.19 to find the eccentricity of the ellipse. Then find the ellipse's standard-form equation.

13. Focus: $(\sqrt{5}, 0)$
Directrix: $x = \dfrac{9}{\sqrt{5}}$

14. Focus: $(4, 0)$
Directrix: $x = \dfrac{16}{3}$

15. Focus: $(-4, 0)$
Directrix: $x = -16$

16. Focus: $(-\sqrt{2}, 0)$
Directrix: $x = -2\sqrt{2}$

17. Draw an ellipse of eccentricity 4/5. Explain your procedure.

18. Draw the orbit of Pluto (eccentricity 0.25) to scale. Explain your procedure.

19. The endpoints of the major and minor axes of an ellipse are (1, 1), (3, 4), (1, 7), and (−1, 4). Sketch the ellipse, give its equation in standard form, and find its foci, eccentricity, and directrices.

20. Find an equation for the ellipse of eccentricity 2/3 that has the line $x = 9$ as a directrix and the point (4, 0) as the corresponding focus.

21. What values of the constants a, b, and c make the ellipse

$$4x^2 + y^2 + ax + by + c = 0$$

lie tangent to the x-axis at the origin and pass through the point $(-1, 2)$? What is the eccentricity of the ellipse?

22. *The reflective property of ellipses.* An ellipse is revolved about its major axis to generate an ellipsoid. The inner surface of the ellipsoid is silvered to make a mirror. Show that a ray of light emanating from one focus will be reflected to the other focus. Sound waves also follow such paths, and this property is used in constructing "whispering galleries." (*Hint:* Place the ellipse in standard position in the xy-plane and show that the lines from a point P on the ellipse to the two foci make congruent angles with the tangent to the ellipse at P.)

Hyperbolas

In Exercises 23–30, find the eccentricity of the hyperbola. Then find and graph the hyperbola's foci and directrices.

23. $x^2 - y^2 = 1$

24. $9x^2 - 16y^2 = 144$

25. $y^2 - x^2 = 8$

26. $y^2 - x^2 = 4$

27. $8x^2 - 2y^2 = 16$

28. $y^2 - 3x^2 = 3$

29. $8y^2 - 2x^2 = 16$

30. $64x^2 - 36y^2 = 2304$

Exercises 31–34 give the eccentricities and the vertices or foci of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's standard-form equation.

31. Eccentricity: 3
Vertices: $(0, \pm 1)$

32. Eccentricity: 2
Vertices: $(\pm 2, 0)$

33. Eccentricity: 3
Foci: $(\pm 3, 0)$

34. Eccentricity: 1.25
Foci: $(0, \pm 5)$

Exercises 35–38 give foci and corresponding directrices of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's eccentricity. Then find the hyperbola's standard-form equation.

35. Focus: $(4, 0)$
Directrix: $x = 2$

36. Focus: $(\sqrt{10}, 0)$
Directrix: $x = \sqrt{2}$

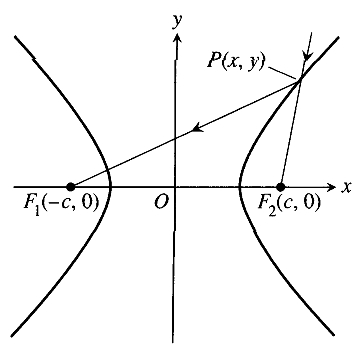
37. Focus: $(-2, 0)$
Directrix: $x = -\dfrac{1}{2}$

38. Focus: $(-6, 0)$
Directrix: $x = -2$

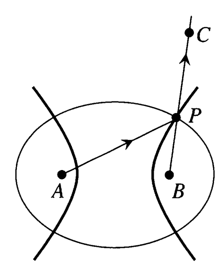
39. A hyperbola of eccentricity 3/2 has one focus at $(1, -3)$. The corresponding directrix is the line $y = 2$. Find an equation for the hyperbola.

40. *The effect of eccentricity on a hyperbola's shape.* What happens to the graph of a hyperbola as its eccentricity increases? To find out, rewrite the equation $(x^2/a^2) - (y^2/b^2) = 1$ in terms of a and e instead of a and b. Graph the hyperbola for various values of e and describe what you find.

41. *The reflective property of hyperbolas.* Show that a ray of light directed toward one focus of a hyperbolic mirror, as in the accompanying figure, is reflected toward the other focus. (*Hint:* Show that the tangent to the hyperbola at P bisects the angle made by segments PF_1 and PF_2.)



42. *A confocal ellipse and hyperbola.* Show that an ellipse and a hyperbola that have the same foci A and B, as in the accompanying figure, cross at right angles at their point of intersection. (*Hint:* A ray of light from focus A that met the hyperbola at P would be reflected from the hyperbola as if it came directly from B (Exercise 41). The same ray would be reflected off the ellipse to pass through B (Exercise 22).)



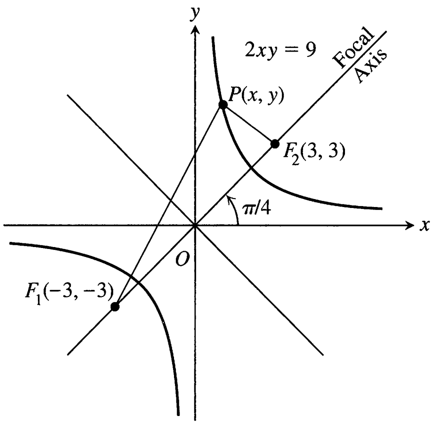
9.3

Quadratic Equations and Rotations

In this section, we examine one of the most amazing results in analytic geometry, which is that the Cartesian graph of any equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

in which A, B, and C are not all zero, is nearly always a conic section. The exceptions are the cases in which there is no graph at all or the graph consists of two parallel lines. It is conventional to call all graphs of Eq. (1), curved or not, **quadratic curves.**



9.22 The focal axis of the hyperbola $2xy = 9$ makes an angle of $\pi/4$ radians with the positive x-axis.

The Cross Product Term

You may have noticed that the term Bxy did not appear in the equations for the conic sections in Section 9.1. This happened because the axes of the conic sections ran parallel to (in fact, coincided with) the coordinate axes.

To see what happens when the parallelism is absent, let us write an equation for a hyperbola with $a = 3$ and foci at $F_1\,(-3, -3)$ and $F_2\,(3, 3)$ (Fig. 9.22). The equation $|PF_1 - PF_2| = 2a$ becomes $|PF_1 - PF_2| = 2(3) = 6$ and

$$\sqrt{(x+3)^2 + (y+3)^2} - \sqrt{(x-3)^2 + (y-3)^2} = \pm 6.$$

When we transpose one radical, square, solve for the radical that still appears, and square again, the equation reduces to

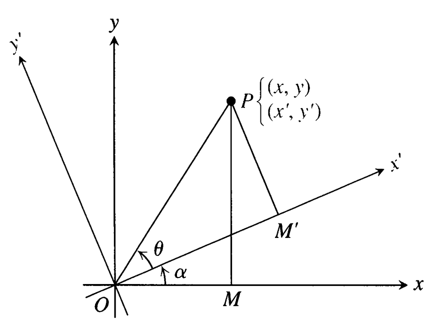
$$2xy = 9, \tag{2}$$

a case of Eq. (1) in which the cross-product term is present. The asymptotes of the hyperbola in Eq. (2) are the x- and y-axes, and the focal axis makes an angle of $\pi/4$ radians with the positive x-axis. As in this example, the cross product term is present in Eq. (1) only when the axes of the conic are tilted.

Rotating the Coordinate Axes to Eliminate the Cross Product Term

To eliminate the xy-term from the equation of a conic, we rotate the coordinate axes to eliminate the "tilt" in the axes of the conic. The equations for the rotations we use are derived in the following way. In the notation of Fig. 9.23, which shows a counterclockwise rotation about the origin through an angle α,

$$\begin{aligned} x &= OM = OP\,\cos(\theta + \alpha) = OP\,\cos\theta\,\cos\alpha - OP\,\sin\theta\,\sin\alpha \\ y &= MP = OP\,\sin(\theta + \alpha) = OP\,\cos\theta\,\sin\alpha + OP\,\sin\theta\,\cos\alpha. \end{aligned} \tag{3}$$



9.23 A counterclockwise rotation through angle α about the origin.

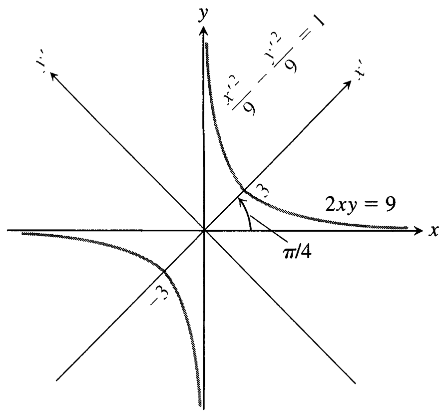
Since

$$OP\,\cos\theta = OM' = x'$$

and

$$OP\,\sin\theta = M'P = y',$$

the equations in (3) reduce to the following.



9.24 The hyperbola in Example 1 (x' and y' are the new coordinates).

Equations for Rotating Coordinate Axes

$$x = x' \cos \alpha - y' \sin \alpha$$
$$y = x' \sin \alpha + y' \cos \alpha \tag{4}$$

EXAMPLE 1 The x- and y-axes are rotated through an angle of $\pi/4$ radians about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

Solution Since $\cos \pi/4 = \sin \pi/4 = 1/\sqrt{2}$, we substitute

$$x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}}$$

from Eqs. (4) into the equation $2xy = 9$ and obtain

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) = 9$$
$$x'^2 - y'^2 = 9$$
$$\frac{x'^2}{9} - \frac{y'^2}{9} = 1.$$

See Fig. 9.24. ❑

If we apply Eqs. (4) to the quadratic equation (1), we obtain a new quadratic equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0. \tag{5}$$

The new and old coefficients are related by the equations

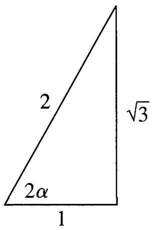
$$\begin{aligned} A' &= A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha \\ B' &= B \cos 2\alpha + (C - A) \sin 2\alpha \\ C' &= A \sin^2 \alpha - B \sin \alpha \cos \alpha + C \cos^2 \alpha \\ D' &= D \cos \alpha + E \sin \alpha \\ E' &= -D \sin \alpha + E \cos \alpha \\ F' &= F. \end{aligned} \tag{6}$$

These equations show, among other things, that if we start with an equation for a curve in which the cross product term is present ($B \neq 0$), we can find a rotation angle α that produces an equation in which no cross product term appears ($B' = 0$). To find α, we set $B' = 0$ in the second equation in (6) and solve the resulting equation,

$$B \cos 2\alpha + (C - A) \sin 2\alpha = 0,$$

for α. In practice, this means determining α from one of the two equations

$$\cot 2\alpha = \frac{A - C}{B} \quad \text{or} \quad \tan 2\alpha = \frac{B}{A - C}. \tag{7}$$



9.25 This triangle identifies $2\alpha = \cot^{-1}(1/\sqrt{3})$ as $\pi/3$ (Example 2).

EXAMPLE 2 The coordinate axes are to be rotated through an angle α to produce an equation for the curve

$$2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$$

that has no cross product term. Find α and the new equation. Identify the curve.

Solution The equation $2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$ has $A = 2$, $B = \sqrt{3}$, and $C = 1$. We substitute these values into Eq. (7) to find α:

$$\cot 2\alpha = \frac{A-C}{B} = \frac{2-1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

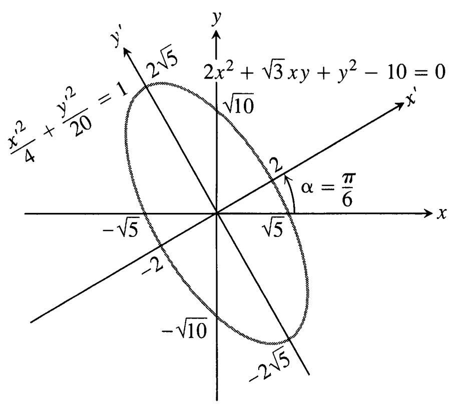
From the right triangle in Fig. 9.25, we see that one appropriate choice of angle is $2\alpha = \pi/3$, so we take $\alpha = \pi/6$. Substituting $\alpha = \pi/6$, $A = 2$, $B = \sqrt{3}$, $C = 1$, $D = E = 0$, and $F = -10$ into Eqs. (6) gives

$$A' = \frac{5}{2}, \qquad B' = 0, \qquad C' = \frac{1}{2}, \qquad D' = E' = 0, \qquad F' = -10.$$

Equation (5) then gives

$$\frac{5}{2}x'^2 + \frac{1}{2}y'^2 - 10 = 0, \qquad \text{or} \qquad \frac{x'^2}{4} + \frac{y'^2}{20} = 1.$$

The curve is an ellipse with foci on the new y'-axis (Fig. 9.26). ❑



9.26 The conic section in Example 2.

Possible Graphs of Quadratic Equations

We now return to the graph of the general quadratic equation.

Since axes can always be rotated to eliminate the cross product term, there is no loss of generality in assuming that this has been done and that our equation has the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{8}$$

Equation (8) represents

a) a *circle* if $A = C \neq 0$ (special cases: the graph is a point or there is no graph at all);
b) a *parabola* if Eq. (8) is quadratic in one variable and linear in the other;
c) an *ellipse* if A and C are both positive or both negative (special cases: circles, a single point or no graph at all);
d) a *hyperbola* if A and C have opposite signs (special case: a pair of intersecting lines);
e) a *straight line* if A and C are zero and at least one of D and E is different from zero;
f) *one or two straight lines* if the left-hand side of Eq. (8) can be factored into the product of two linear factors.

See Table 9.3 (on page 732) for examples.

The Discriminant Test

We do not need to eliminate the xy-term from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{9}$$

to tell what kind of conic section the equation represents. If this is the only information we want, we can apply the following test instead.

As we have seen, if $B \neq 0$, then rotating the coordinate axes through an angle α that satisfies the equation

$$\cot 2\alpha = \frac{A - C}{B} \tag{10}$$

will change Eq. (9) into an equivalent form

$$A' x'^2 + C' y'^2 + D' x' + E' y' + F' = 0 \tag{11}$$

without a cross product term.

Now, the graph of Eq. (11) is a (real or degenerate)

a) *parabola* if A' or $C' = 0$; that is, if $A' C' = 0$;
b) *ellipse* if A' and C' have the same sign; that is, if $A' C' > 0$;
c) *hyperbola* if A' and C' have opposite signs; that is, if $A' C' < 0$.

It can also be verified from Eqs. (6) that for any rotation of axes,

$$B^2 - 4AC = B'^2 - 4A' C'. \tag{12}$$

This means that the quantity $B^2 - 4AC$ is not changed by a rotation. But when we rotate through the angle α given by Eq. (10), B' becomes zero, so

$$B^2 - 4AC = -4A' C'.$$

Since the curve is a parabola if $A' C' = 0$, an ellipse if $A' C' > 0$, and a hyperbola if $A' C' < 0$, the curve must be a parabola if $B^2 - 4AC = 0$, an ellipse if $B^2 - 4AC < 0$, and a hyperbola if $B^2 - 4AC > 0$. The number $B^2 - 4AC$ is called the **discriminant** of Eq. (9).

The Discriminant Test

With the understanding that occasional degenerate cases may arise, the quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

a) a **parabola** if $B^2 - 4AC = 0$,
b) an **ellipse** if $B^2 - 4AC < 0$,
c) a **hyperbola** if $B^2 - 4AC > 0$.

EXAMPLE 3

a) $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$ represents a parabola because

$$B^2 - 4AC = (-6)^2 - 4 \cdot 3 \cdot 3 = 36 - 36 = 0.$$

b) $x^2 + xy + y^2 - 1 = 0$ represents an ellipse because

$$B^2 - 4AC = (1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0.$$

c) $xy - y^2 - 5y + 1 = 0$ represents a hyperbola because

$$B^2 - 4AC = (1)^2 - 4(0)(-1) = 1 > 0.$$

❑

Table 9.3 Examples of quadratic curves

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

	A	B	C	D	E	F	Equation	Remarks
Circle	1		1			−4	$x^2 + y^2 = 4$	$A = C$; $F < 0$
Parabola			1	−9			$y^2 = 9x$	Quadratic in y, linear in x
Ellipse	4		9			−36	$4x^2 + 9y^2 = 36$	A, C have same sign, $A \neq C$; $F < 0$
Hyperbola	1		−1			−1	$x^2 - y^2 = 1$	A, C have opposite signs
One line (still a conic section)	1						$x^2 = 0$	y-axis
Intersecting lines (still a conic section)		1		1	−1	−1	$xy + x - y - 1 = 0$	Factors to $(x-1)(y+1) = 0$, so $x = 1$, $y = -1$
Parallel lines (not a conic section)	1			−3		2	$x^2 - 3x + 2 = 0$	Factors to $(x-1)(x-2) = 0$, so $x = 1$, $x = 2$
Point	1		1				$x^2 + y^2 = 0$	The origin
No graph	1					1	$x^2 = -1$	No graph

Technology *How Calculators Use Rotations to Evaluate Sines and Cosines* Some calculators use rotations to calculate sines and cosines of arbitrary angles. The procedure goes something like this: The calculator has, stored,

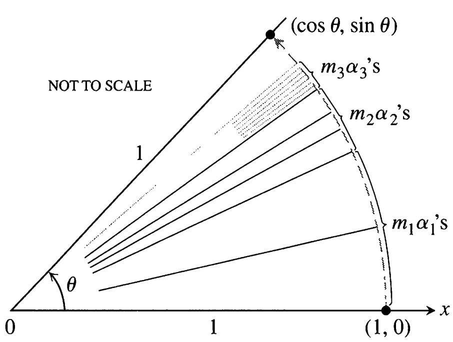
1. ten angles or so, say
$$\alpha_1 = \sin^{-1}(10^{-1}), \qquad \alpha_2 = \sin^{-1}(10^{-2}), \qquad \ldots, \qquad \alpha_{10} = \sin^{-1}(10^{-10}),$$
and

2. twenty numbers, the sines and cosines of the angles $\alpha_1, \alpha_2, \ldots, \alpha_{10}$.

To calculate the sine and cosine of an arbitrary angle θ, we enter θ (in radians) into the calculator. The calculator subtracts or adds multiples of 2π to θ to replace θ by the angle between 0 and 2π that has the same sine and cosine as θ (we continue to call the angle θ). The calculator then "writes" θ as a sum of multiples of α_1 (as many as possible without overshooting) plus multiples of α_2 (again, as many as possible), and so on, working its way to α_{10}. This gives

$$\theta \approx m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{10}\alpha_{10}.$$

The calculator then rotates the point (1, 0) through m_1 copies of α_1 (through α_1, m_1 times in succession), plus m_2 copies of α_2, and so on, finishing off with m_{10} copies of α_{10} (Fig. 9.27). The coordinates of the final position of (1, 0) on the unit circle are the values the calculator gives for ($\cos\theta$, $\sin\theta$).



9.27 To calculate the sine and cosine of an angle θ between 0 and 2π, the calculator rotates the point (1, 0) to an appropriate location on the unit circle and displays the resulting coordinates.

Exercises 9.3

Using the Discriminant

Use the discriminant $B^2 - 4AC$ to decide whether the equations in Exercises 1–16 represent parabolas, ellipses, or hyperbolas.

1. $x^2 - 3xy + y^2 - x = 0$
2. $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$
3. $3x^2 - 7xy + \sqrt{17}\,y^2 = 1$
4. $2x^2 - \sqrt{15}\,xy + 2y^2 + x + y = 0$
5. $x^2 + 2xy + y^2 + 2x - y + 2 = 0$
6. $2x^2 - y^2 + 4xy - 2x + 3y = 6$
7. $x^2 + 4xy + 4y^2 - 3x = 6$
8. $x^2 + y^2 + 3x - 2y = 10$
9. $xy + y^2 - 3x = 5$
10. $3x^2 + 6xy + 3y^2 - 4x + 5y = 12$
11. $3x^2 - 5xy + 2y^2 - 7x - 14y = -1$
12. $2x^2 - 4.9xy + 3y^2 - 4x = 7$
13. $x^2 - 3xy + 3y^2 + 6y = 7$
14. $25x^2 + 21xy + 4y^2 - 350x = 0$
15. $6x^2 + 3xy + 2y^2 + 17y + 2 = 0$
16. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$

Rotating Coordinate Axes

In Exercises 17–26, rotate the coordinate axes to change the given equation into an equation that has no cross product (xy) term. Then identify the graph of the equation. (The new equations will vary with the size and direction of the rotation you use.)

17. $xy = 2$
18. $x^2 + xy + y^2 = 1$
19. $3x^2 + 2\sqrt{3}\,xy + y^2 - 8x + 8\sqrt{3}\,y = 0$
20. $x^2 - \sqrt{3}\,xy + 2y^2 = 1$
21. $x^2 - 2xy + y^2 = 2$
22. $3x^2 - 2\sqrt{3}\,xy + y^2 = 1$
23. $\sqrt{2}\,x^2 + 2\sqrt{2}\,xy + \sqrt{2}\,y^2 - 8x + 8y = 0$
24. $xy - y - x + 1 = 0$
25. $3x^2 + 2xy + 3y^2 = 19$
26. $3x^2 + 4\sqrt{3}\,xy - y^2 = 7$
27. Find the sine and cosine of an angle through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$14x^2 + 16xy + 2y^2 - 10x + 26,370\,y - 17 = 0.$$

Do not carry out the rotation.

28. Find the sine and cosine of an angle through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$4x^2 - 4xy + y^2 - 8\sqrt{5}\,x - 16\sqrt{5}\,y = 0.$$

Do not carry out the rotation.

Calculator

The conic sections in Exercises 17–26 were chosen to have rotation angles that were "nice" in the sense that once we knew cot 2α or tan 2α we could identify 2α and find sin α and cos α from familiar triangles. The conic sections encountered in practice may not have such nice rotation angles, and we may have to use a calculator to determine α from the value of cot 2α or tan 2α.

In Exercises 29–34, use a calculator to find an angle α through which the coordinate axes can be rotated to change the given equation into a quadratic equation that has no cross product term. Then find sin α and cos α to 2 decimal places and use Eqs. (6) to find the coefficients of the new equation to the nearest decimal place. In each case, say whether the conic section is an ellipse, a hyperbola, or a parabola.

29. $x^2 - xy + 3y^2 + x - y - 3 = 0$
30. $2x^2 + xy - 3y^2 + 3x - 7 = 0$
31. $x^2 - 4xy + 4y^2 - 5 = 0$
32. $2x^2 - 12xy + 18y^2 - 49 = 0$
33. $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$
34. $2x^2 + 7xy + 9y^2 + 20x - 86 = 0$

Theory and Examples

35. What effect does a 90° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.
   a) The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$
   b) The hyperbola $(x^2/a^2) - (y^2/b^2) = 1$
   c) The circle $x^2 + y^2 = a^2$
   d) The line $y = mx$
   e) The line $y = mx + b$
36. What effect does a 180° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.
   a) The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$
   b) The hyperbola $(x^2/a^2) - (y^2/b^2) = 1$
   c) The circle $x^2 + y^2 = a^2$
   d) The line $y = mx$
   e) The line $y = mx + b$

37. *The Hyperbola $xy = a$.* The hyperbola $xy = 1$ is one of many hyperbolas of the form $xy = a$ that appear in science and mathematics.

a) Rotate the coordinate axes through an angle of 45° to change the equation $xy = 1$ into an equation with no xy-term. What is the new equation?
b) Do the same for the equation $xy = a$.

38. Find the eccentricity of the hyperbola $xy = 2$.

39. Can anything be said about the graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ if $AC < 0$? Give reasons for your answer.

40. Does any nondegenerate conic section $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ have all of the following properties?

a) It is symmetric with respect to the origin.
b) It passes through the point (1, 0).
c) It is tangent to the line $y = 1$ at the point $(-2, 1)$.

Give reasons for your answer.

41. Show that the equation $x^2 + y^2 = a^2$ becomes $x'^2 + y'^2 = a^2$ for every choice of the angle α in the rotation equations (4).

42. Show that rotating the axes through an angle of $\pi/4$ radians will eliminate the xy-term from Eq. (1) whenever $A = C$.

43. a) Decide whether the equation

$$x^2 + 4xy + 4y^2 + 6x + 12y + 9 = 0$$

represents an ellipse, a parabola, or a hyperbola.
b) Show that the graph of the equation in (a) is the line $2y = -x - 3$.

44. a) Decide whether the conic section with equation

$$9x^2 + 6xy + y^2 - 12x - 4y + 4 = 0$$

represents a parabola, an ellipse, or a hyperbola.
b) Show that the graph of the equation in (a) is the line $y = -3x + 2$.

45. a) What kind of conic section is the curve $xy + 2x - y = 0$?
b) Solve the equation $xy + 2x - y = 0$ for y and sketch the curve as the graph of a rational function of x.
c) Find equations for the lines parallel to the line $y = -2x$ that are normal to the curve. Add the lines to your sketch.

46. Prove or find counterexamples to the following statements about the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

a) If $AC > 0$, the graph is an ellipse.
b) If $AC > 0$, the graph is a hyperbola.
c) If $AC < 0$, the graph is a hyperbola.

47. *A nice area formula for ellipses.* When $B^2 - 4AC$ is negative, the equation

$$Ax^2 + Bxy + Cy^2 = 1$$

represents an ellipse. If the ellipse's semi-axes are a and b, its area is πab (a standard formula). Show that the area is also given by the formula $2\pi/\sqrt{4AC - B^2}$. (*Hint:* Rotate the coordinate axes to eliminate the xy-term and apply Eq. (12) to the new equation.)

48. *Other invariants.* We describe the fact that $B'^2 - 4A'C'$ equals $B^2 - 4AC$ after a rotation about the origin by saying that the discriminant of a quadratic equation is an **invariant** of the equation. Use Eqs. (6) to show that the numbers (a) $A + C$ and (b) $D^2 + E^2$ are also invariants, in the sense that

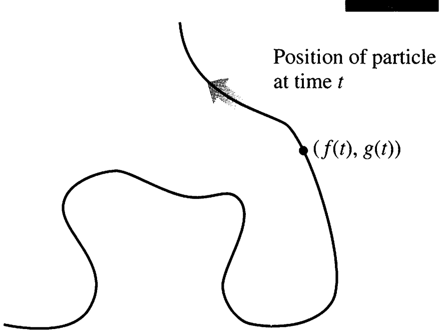
$$A' + C' = A + C \quad \text{and} \quad D'^2 + E'^2 = D^2 + E^2.$$

We can use these equalities to check against numerical errors when we rotate axes. They can also be helpful in shortening the work required to find values for the new coefficients.

49. *A proof that $B'^2 - 4A'C' = B^2 - 4AC$.* Use Eqs. (6) to show that $B'^2 - 4A'C' = B^2 - 4AC$ for any rotation of axes about the origin. The calculation works out nicely but requires patience.

9.4 Parametrizations of Plane Curves

When the path of a particle moving in the plane looks like the curve in Fig. 9.28, we cannot hope to describe it with a Cartesian formula that expresses y directly in terms of x or x directly in terms of y. Instead, we express each of the particle's coordinates as a function of time t and describe the path with a pair of equations, $x = f(t)$ and $y = g(t)$. For studying motion, equations like these are preferable to a Cartesian formula because they tell us the particle's position at any time t.



9.28 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

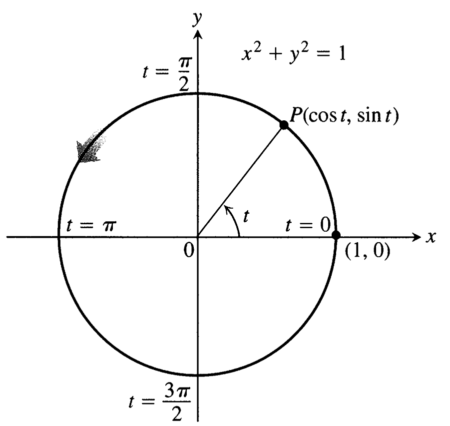
Definitions

If x and y are given as continuous functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **curve** in the coordinate plane. The equations are **parametric equations** for the curve. The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point** of the curve. When we give parametric equations and a parameter interval for a curve in the plane, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization** of the curve.

In many applications t denotes time, but it might instead denote an angle (as in some of the following examples) or the distance a particle has traveled along its path from its starting point (as it sometimes will when we later study motion).



9.29 The equations $x = \cos t, y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 1).

EXAMPLE 1 *The circle $x^2 + y^2 = 1$*

The equations and parameter interval

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi,$$

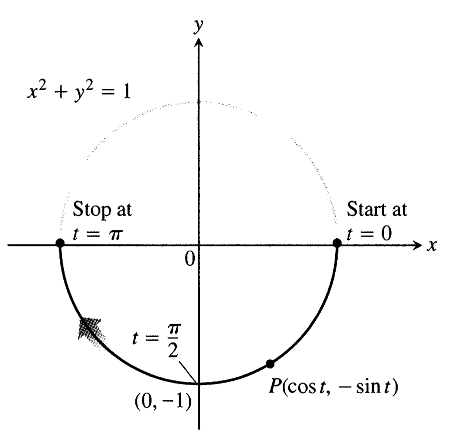
describe the position $P(x, y)$ of a particle that moves counterclockwise around the circle $x^2 + y^2 = 1$ as t increases (Fig. 9.29).

We know that the point lies on this circle for every value of t because

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

But how much of the circle does the point $P(x, y)$ actually traverse?

To find out, we track the motion as t runs from 0 to 2π. The parameter t is the radian measure of the angle that radius OP makes with the positive x-axis. The particle starts at (1, 0), moves up and to the left as t approaches $\pi/2$, and continues around the circle to stop again at (1, 0) when $t = 2\pi$. The particle traces the circle exactly once. ❑



9.30 The point $P(\cos t, -\sin t)$ moves clockwise as t increases from 0 to π (Example 2).

EXAMPLE 2 *A semicircle*

The equations and parameter interval

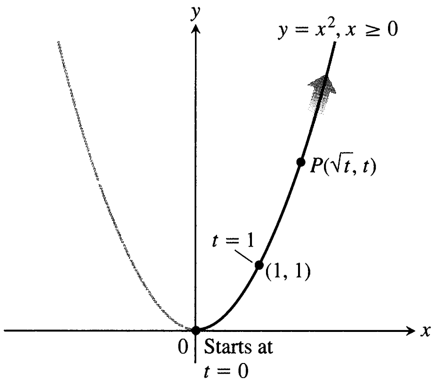
$$x = \cos t, \qquad y = -\sin t, \qquad 0 \le t \le \pi,$$

describe the position $P(x, y)$ of a particle that moves clockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π.

We know that the point P lies on this circle for all t because its coordinates satisfy the circle's equation. How much of the circle does the particle traverse? To find out, we track the motion as t runs from 0 to π. As in Example 1, the particle starts at (1, 0). But now as t increases, y becomes negative, decreasing to -1 when $t = \pi/2$ and then increasing back to 0 as t approaches π. The motion stops at $t = \pi$ with only the lower half of the circle covered (Fig. 9.30). ❑

EXAMPLE 3 *Half a parabola*

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations



9.31 The equations $x = \sqrt{t}, y = t$ and interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 3).

and parameter interval

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

Solution We try to identify the path by eliminating t between the equations $x = \sqrt{t}$ and $y = t$. With any luck, this will produce a recognizable algebraic relation between x and y. We find that

$$y = t = \left(\sqrt{t}\right)^2 = x^2.$$

This means that the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

It would be a mistake, however, to conclude that the particle's path is the entire parabola $y = x^2$— it is only half the parabola. The particle's x-coordinate is never negative. The particle starts at (0, 0) when $t = 0$ and rises into the first quadrant as t increases (Fig. 9.31). ❑

EXAMPLE 4 *An entire parabola*

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = t, \qquad y = t^2, \qquad -\infty < t < \infty.$$

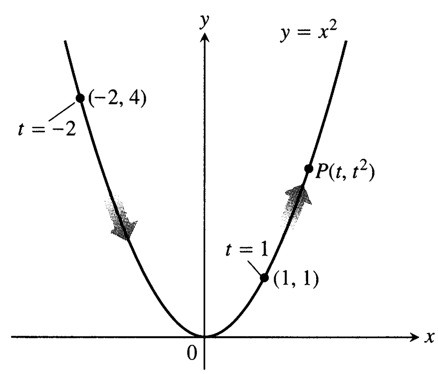
Identify the particle's path and describe the motion.

Solution We identify the path by eliminating t between the equations $x = t$ and $y = t^2$, obtaining

$$y = (t)^2 = x^2.$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

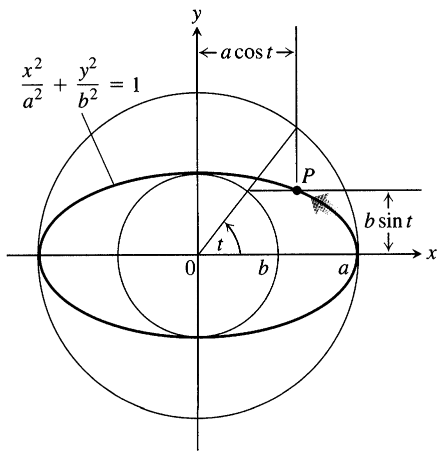
In contrast to Example 3, the particle now traverses the entire parabola. As t increases from $-\infty$ to ∞, the particle comes down the left-hand side, passes through the origin, and moves up the right-hand side (Fig. 9.32).



9.32 The path defined by $x = t$, $y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 4).

❑

As Example 4 illustrates, any curve $y = f(x)$ has the parametrization $x = t$, $y = f(t)$. This is so simple we usually do not use it, but the point of view is occasionally helpful.



9.33 The ellipse in Example 5, drawn for $a > b$. The coordinates of P are $x = a \cos t$, $y = b \sin t$.

EXAMPLE 5 *A parametrization of the ellipse $x^2/a^2 + y^2/b^2 = 1$*

Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$$x = a \cos t, \qquad y = b \sin t, \qquad 0 \le t \le 2\pi.$$

Solution We find a Cartesian equation for the particle's coordinates by eliminating t between the equations

$$\cos t = \frac{x}{a}, \qquad \sin t = \frac{y}{b}.$$

We accomplish this with the identity $\cos^2 t + \sin^2 t = 1$, which yields

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The particle's coordinates (x, y) satisfy the equation $(x^2/a^2) + (y^2/b^2) = 1$, so the particle moves along this ellipse. When $t = 0$, the particle's coordinates are

$$x = a \cos(0) = a, \qquad y = b \sin(0) = 0,$$

so the motion starts at $(a, 0)$. As t increases, the particle rises and moves toward the left, moving counterclockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at time $t = 2\pi$ (Fig. 9.33). ❑

EXAMPLE 6 *A parametrization of the circle $x^2 + y^2 = a^2$*

The equations and parameter interval

$$x = a \cos t, \qquad y = a \sin t, \qquad 0 \le t \le 2\pi,$$

obtained by taking $b = a$ in Example 5, describe the circle $x^2 + y^2 = a^2$. ❑

EXAMPLE 7 *A parametrization of the right-hand branch of the hyperbola $x^2 - y^2 = 1$*

Describe the motion of the particle whose position $P(x, y)$ at time t is given by

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

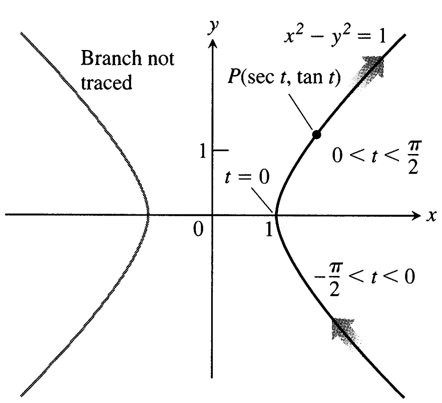
Solution We find a Cartesian equation for the coordinates of P by eliminating t between the equations

$$\sec t = x, \qquad \tan t = y.$$

We accomplish this with the identity $\sec^2 t - \tan^2 t = 1$, which yields

$$x^2 - y^2 = 1.$$

Since the particle's coordinates (x, y) satisfy the equation $x^2 - y^2 = 1$, the motion takes place somewhere on this hyperbola. As t runs between $-\pi/2$ and $\pi/2$, $x = \sec t$ remains positive and $y = \tan t$ runs between $-\infty$ and ∞, so P traverses the hyperbola's right-hand branch. It comes in along the branch's lower half as $t \to 0^-$, reaches $(1, 0)$ at $t = 0$, and moves out into the first quadrant as t increases toward $\pi/2$ (Fig. 9.34). ❑

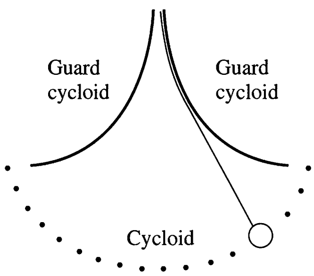


9.34 The equations $x = \sec t$, $y = \tan t$ and interval $-\pi/2 < t < \pi/2$ describe the right-hand branch of the hyperbola $x^2 - y^2 = 1$ (Example 7).

Huygen's clock

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center.

This does not happen if the bob can be made to swing in a cycloid. In 1673, Christiaan Huygens (1629–1695), the Dutch mathematician, physicist, and astronomer who discovered the rings of Saturn, driven by a need to make accurate determinations of longitude at sea, designed a pendulum clock whose bob would swing in a cycloid. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center. How were the guards shaped? They were cycloids, too.

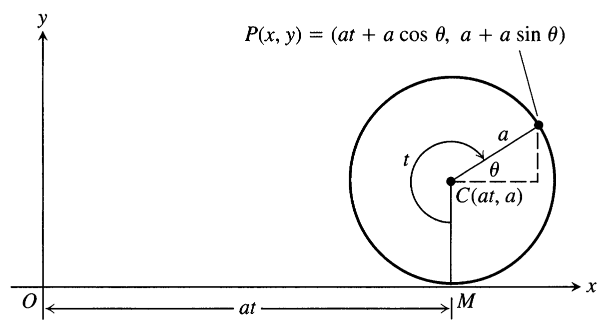


EXAMPLE 8 *Cycloids*

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid.**

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in radians. Figure 9.35 shows the wheel a short while later, when its base lies at units from the origin. The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$



9.35 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$, so that

$$\theta = \frac{3\pi}{2} - t.$$

This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \qquad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t.$$

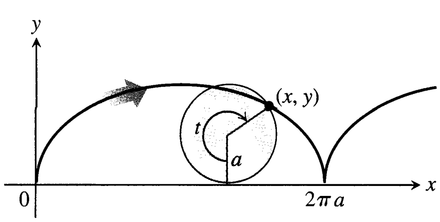
The equations we seek are

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

These are usually written with the a factored out:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t). \tag{1}$$

Figure 9.36 shows the first arch of the cycloid and part of the next. ❑



9.36 The cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, for $t \geq 0$.

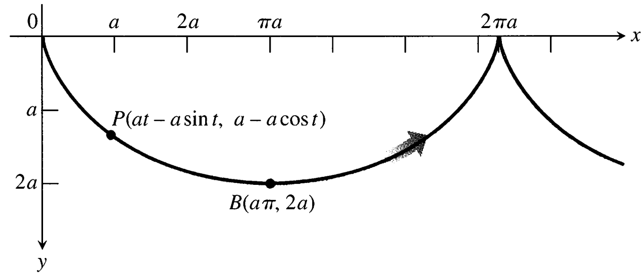
⁜ Brachistochrones and Tautochrones

If we turn Fig. 9.36 upside down, Eqs. (1) still apply and the resulting curve (Fig. 9.37) has two interesting physical properties. The first relates to the origin O and the point B at the bottom of the first arch. Among all smooth curves joining these points, the cycloid is the curve along which a frictionless bead, subject only to the force of gravity, will slide from O to B the fastest. This makes the cycloid a **brachistochrone** ("brah-*kiss*-toe-krone"), or shortest time curve for these points. The second property is that even if you start the bead partway down the curve

The witch of Agnesi

Although l'Hôpital wrote the first text on differential calculus, the first text to include differential and integral calculus along with analytic geometry, infinite series, and differential equations was written in the 1740s by the Italian mathematician Maria Gaetana Agnesi (1718–1799). Agnesi, a gifted scholar and linguist whose Latin essay defending higher education for women was published when she was only nine years old, was a well-published scientist by age 20 and an honorary faculty member of the University of Bologna by age 30.

Today, Agnesi is remembered chiefly for a bell-shaped curve called *the witch of Agnesi.* This name, found only in English texts, is the result of a mistranslation. Agnesi's own name for the curve was *versiera* or "turning curve." John Colson, a noted Cambridge mathematician who felt Agnesi's text so important that he learned Italian to translate it "for the benefit of British youth" (he particularly had in mind young women, for whom he hoped Agnesi would be a role model), probably confused versiera with *avversiera,* which means "wife of the devil" and translates into "witch." You can find out more about the witch by doing Exercise 29.



9.37 To study motion along an upside-down cycloid under the influence of gravity, we turn Fig. 9.36 upside down. This points the y-axis in the direction of the gravitational force and makes the downward y-coordinates positive. The equations and parameter interval for the cycloid are still

$$x = a(t - \sin t),$$
$$y = a(1 - \cos t), \quad t \geq 0.$$

The arrow shows the direction of increasing t.

toward B, it will still take the bead the same amount of time to reach B. This makes the cycloid a **tautochrone** ("*taw*-toe-krone"), or same-time curve for O and B.

Are there any other brachistochrones joining O and B, or is the cycloid the only one? We can formulate this as a mathematical question in the following way. At the start, the kinetic energy of the bead is zero, since its velocity is zero. The work done by gravity in moving the bead from (0, 0) to any other point (x, y) in the plane is mgy, and this must equal the change in kinetic energy. That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the velocity of the bead when it reaches (x, y) has to be

$$v = \sqrt{2gy}.$$

That is,

$$\frac{ds}{dt} = \sqrt{2gy}$$

ds is the arc length differential along the bead's path.

or

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\,dx}{\sqrt{2gy}}.$$

The time T_f it takes the bead to slide along a particular path $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\,dx. \tag{2}$$

What curves $y = f(x)$, if any, minimize the value of this integral?

At first sight, we might guess that the straight line joining O and B would give the shortest time, but perhaps not. There might be some advantage in having the bead fall vertically at first to build up its velocity faster. With a higher velocity, the bead could travel a longer path and still reach B first. Indeed, this is the right idea.

The solution, from a branch of mathematics known as the calculus of variations, is that the original cycloid from O to B is the one and only brachistochrone for O and B.

While the solution of the brachistrochrone problem is beyond our present reach, we can still show why the cycloid is a tautochrone. For the cycloid, Eq. (2) takes the form

$$\begin{aligned} T_{\text{cycloid}} &= \int_{x=0}^{x=a\pi} \sqrt{\frac{dx^2+dy^2}{2gy}} \\ &= \int_{t=0}^{t=\pi} \sqrt{\frac{a^2(2-2\cos t)}{2ga(1-\cos t)}}\,dt \\ &= \int_0^{\pi} \sqrt{\frac{a}{g}}\,dt = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

From Eqs. (1), $dx = a(1-\cos t)\,dt$, $dy = a\sin t\,dt$, and $y = a(1-\cos t)$

Thus, the amount of time it takes the frictionless bead to slide down the cycloid to B after it is released from rest at O is $\pi\sqrt{a/g}$.

Suppose that instead of starting the bead at O we start it at some lower point on the cycloid, a point (x_0, y_0) corresponding to the parameter value $t_0 > 0$. The bead's velocity at any later point (x, y) on the cycloid is

$$v = \sqrt{2g\,(y-y_0)} = \sqrt{2ga\,(\cos t_0 - \cos t)}.$$

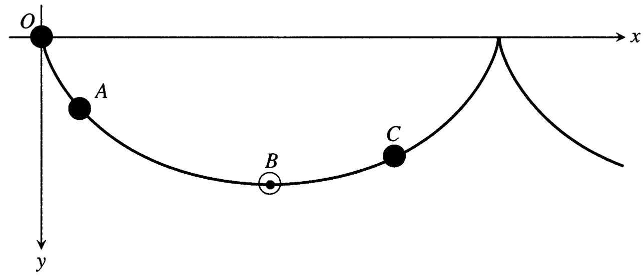
$y = a(1-\cos t)$

Accordingly, the time required for the bead to slide from (x_0, y_0) down to B is

$$\begin{aligned} T &= \int_{t_0}^{\pi} \sqrt{\frac{a^2(2-2\cos t)}{2ga\,(\cos t_0 - \cos t)}}\,dt = \sqrt{\frac{a}{g}}\int_{t_0}^{\pi}\sqrt{\frac{1-\cos t}{\cos t_0 - \cos t}}\,dt \\ &= \sqrt{\frac{a}{g}}\int_{t_0}^{\pi}\sqrt{\frac{2\sin^2(t/2)}{(2\cos^2(t_0/2)-1)-(2\cos^2(t/2)-1)}}\,dt \\ &= \sqrt{\frac{a}{g}}\int_{t_0}^{\pi}\frac{\sin(t/2)\,dt}{\sqrt{\cos^2(t_0/2)-\cos^2(t/2)}} \\ &= \sqrt{\frac{a}{g}}\int_{t=t_0}^{t=\pi}\frac{-2\,du}{\sqrt{a^2-u^2}} \\ &= 2\sqrt{\frac{a}{g}}\left[-\sin^{-1}\frac{u}{c}\right]_{t=t_0}^{t=\pi} \\ &= 2\sqrt{\frac{a}{g}}\left[-\sin^{-1}\frac{\cos(t/2)}{\cos(t_0/2)}\right]_{t_0}^{\pi} \\ &= 2\sqrt{\frac{a}{g}}(-\sin^{-1}0+\sin^{-1}1) = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

$u = \cos(t/2)$
$-2du = \sin(t/2)\,dt$
$c = \cos(t_0/2)$

This is precisely the time it takes the bead to slide to B from O. It takes the bead the same amount of time to reach B no matter where it starts. Beads starting simultaneously from O, A, and C in Fig. 9.38, for instance, will all reach B at the same time.



9.38 Beads released simultaneously on the cycloid at O, A, and C will reach B at the same time.

Standard Parametrizations

Circle $x^2 + y^2 = a^2$:

$$x = a \cos t \qquad y = a \sin t \qquad 0 \le t \le 2\pi$$

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$x = a \cos t \qquad y = b \sin t \qquad 0 \le t \le 2\pi$$

Cycloid generated by a circle of radius a:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t)$$

Exercises 9.4

Finding Cartesian Equations from Parametric Equations

Exercises 1–24 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$
2. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$
3. $x = \sin(2\pi(1-t)), \quad y = \cos(2\pi(1-t)), \quad 0 \le t \le 1$
4. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$
5. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$
6. $x = 4 \sin t, \quad y = 2 \cos t, \quad 0 \le t \le \pi$
7. $x = 4 \cos t, \quad y = 5 \sin t, \quad 0 \le t \le \pi$
8. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \le t \le 2\pi$
9. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$
10. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$
11. $x = t, \quad y = \sqrt{t}, \quad t \ge 0$
12. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
13. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
14. $x = \csc t, \quad y = \cot t, \quad 0 < t < \pi$
15. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$
16. $x = 1 - t, \quad y = 1 + t, \quad -\infty < t < \infty$
17. $x = t, \quad y = 1 - t, \quad 0 \le t \le 1$
18. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$
19. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \le t \le 0$
20. $x = t, \quad y = \sqrt{4 - t^2}, \quad 0 \le t \le 2$
21. $x = t^2, \quad y = \sqrt{t^4 + 1}, \quad t \ge 0$
22. $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \ge 0$
23. $x = -\cosh t, \quad y = \sinh t, \quad -\infty < t < \infty$
24. $x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$

Determining Parametric Equations

25. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$

a) once clockwise, **b)** once counterclockwise,
c) twice clockwise, **d)** twice counterclockwise.

(There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)

26. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

a) once clockwise, **b)** once counterclockwise,
c) twice clockwise, **d)** twice counterclockwise.

(As in Exercise 25, there are many correct answers.)

27. Find parametric equations for the semicircle

$$x^2 + y^2 = a^2, \quad y > 0,$$

using as parameter the slope $t = dy/dx$ of the tangent to the curve at (x, y).

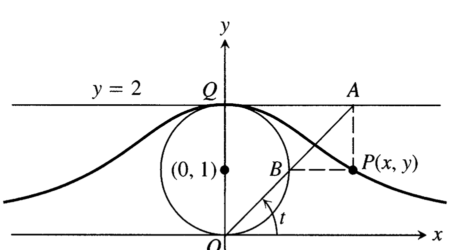
28. Find parametric equations for the circle

$$x^2 + y^2 = a^2,$$

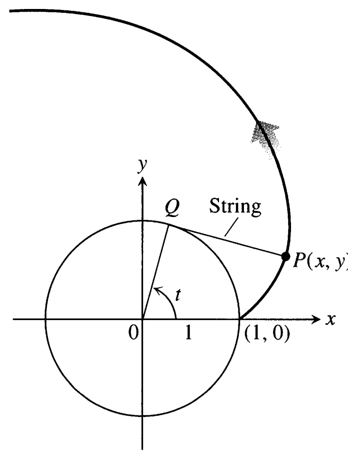
using as parameter the arc length s measured counterclockwise from the point $(a, 0)$ to the point (x, y).

29. *The witch of Maria Agnesi.* The bell-shaped witch of Maria Agnesi can be constructed in the following way. Start with a circle of radius 1, centered at the point (0, 1), as shown in the accompanying figure. Choose a point A on the line $y = 2$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$. Find parametric equations and a parameter interval for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help.

a) $x = AQ$ **b)** $y = 2 - AB \sin t$
c) $AB \cdot OA = (AQ)^2$



30. *The involute of a circle.* If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an *involute* of the circle. In Fig. 9.39, the circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at $(1, 0)$. The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the positive x-axis to segment OQ. Derive parametric equations for the involute by expressing the coordinates x and y of P in terms of t for $t \geq 0$.



9.39 The involute of a circle of radius 1 (Exercise30.)

31. *Parametrizations of lines in the plane* (Fig. 9.40).

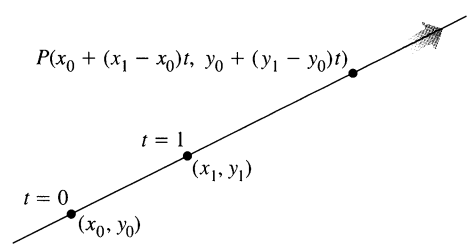
a) Show that the equations and parameter interval

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad -\infty < t < \infty,$$

describe the line through the points (x_0, y_0) and (x_1, y_1).

b) Using the same parameter interval, write parametric equations for the line through a point (x_1, y_1) and the origin.

c) Using the same parameter interval, write parametric equations for the line through $(-1, 0)$ and $(0, 1)$.

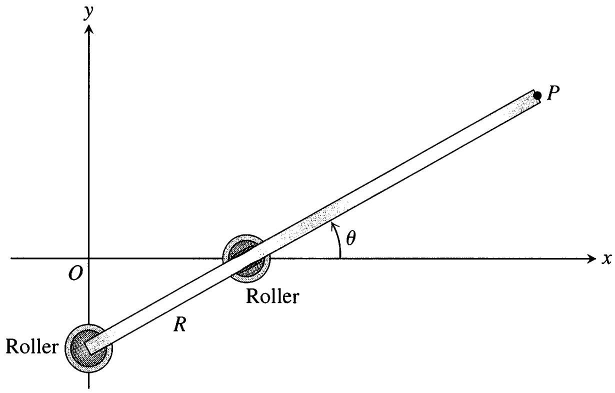


9.40 The line in Exercise 31. The arrow shows the direction of increasing t.

32. *The trammel of Archimedes.* The mechanical system pictured here is called the trammel of Archimedes. It consists of a rigid bar of length L, one end attached to a roller that rolls along the y-axis. At a fixed distance R from this end, the bar is attached to a second roller on the x-axis. Let P be the point at the free end of the bar and let θ be the angle the bar makes with the positive x-axis.

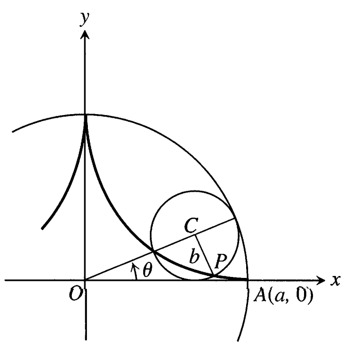
a) Find parametric equations for the path of P in terms of the parameter θ.

b) Find an equation in x and y whose graph is the path of P, and identify this path.

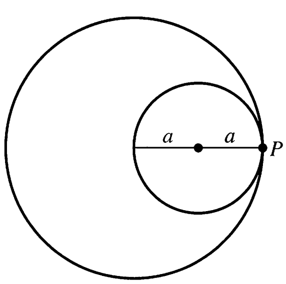


33. *Hypocycloids.* When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid.* Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if $b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

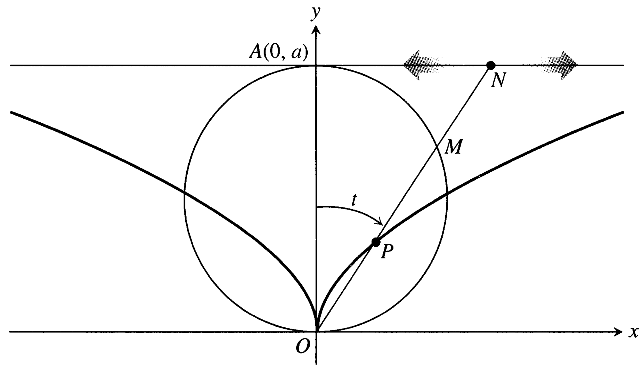
$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$



34. *More about hypocycloids.* The accompanying figure shows a circle of radius a tangent to the inside of a circle of radius $2a$. The point P, shown as the point of tangency in the figure, is attached to the smaller circle. What path does P trace as the smaller circle rolls around the inside of the larger circle?



35. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



36. *Trochoids.* A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a **trochoid,** which is a cycloid when $b = a$.

Distance Using Parametric Equations

37. Find the point on the parabola $x = t$, $y = t^2$, $-\infty < t < \infty$, closest to the point (2, 1/2). (*Hint:* Minimize the square of the distance as a function of t.)

38. Find the point on the ellipse $x = 2 \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ closest to the point (3/4, 0). (*Hint:* Minimize the square of the distance as a function of t.)

Grapher Explorations

If you have a parametric equation grapher, graph the following equations over the given intervals.

39. *Ellipse.* $x = 4 \cos t$, $y = 2 \sin t$, over

a) $0 \le t \le 2\pi$ **b)** $0 \le t \le \pi$
c) $-\pi/2 \le t \le \pi/2$.

40. *Hyperbola branch.* $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over

a) $-1.5 \le t \le 1.5$ **b)** $-0.5 \le t \le 0.5$
c) $-0.1 \le t \le 0.1$.

41. *Parabola.* $x = 2t + 3$, $y = t^2 - 1$, $-2 \le t \le 2$

42. *Cycloid.* $x = t - \sin t$, $y = 1 - \cos t$, over

a) $0 \le t \le 2\pi$ **b)** $0 \le t \le 4\pi$
c) $\pi \le t \le 3\pi$.

43. *Astroid.* $x = \cos^3 t$, $y = \sin^3 t$, over

a) $0 \le t \le 2\pi$ **b)** $-\pi/2 \le t \le \pi/2$.

44. *A nice curve (a deltoid)*

$$x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t, \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with -2 in the equations for x and y? Graph the new equations and find out.

45. *An even nicer curve*

$$x = 3\cos t + \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with -3 in the equations for x and y? Graph the new equations and find out.

46. *Projectile motion.* Graph

$$x = (64\cos\alpha)\,t, \quad y = -16t^2 + (64\sin\alpha)\,t, \quad 0 \le t \le 4\sin\alpha$$

for the following firing angles.

a) $\alpha = \pi/4$ **b)** $\alpha = \pi/6$ **c)** $\alpha = \pi/3$
d) $\alpha = \pi/2$ (watch out—here it comes!)

47. *Three beautiful curves*

a) *Epicycloid:*

$$x = 9\cos t - \cos 9t, \quad y = 9\sin t - \sin 9t, \quad 0 \le t \le 2\pi$$

b) *Hypocycloid:*

$$x = 8\cos t + 2\cos 4t, \quad y = 8\sin t - 2\sin 4t, \quad 0 \le t \le 2\pi$$

c) *Hypotrochoid:*

$$x = \cos t + 5\cos 3t, \quad y = 6\cos t - 5\sin 3t, \quad 0 \le t \le 2\pi$$

48. *More beautiful curves*

a) $x = 6\cos t + 5\cos 3t, \quad y = 6\sin t - 5\sin 3t,$
$0 \le t \le 2\pi$

b) $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 2t - 5\sin 6t,$
$0 \le t \le \pi$

c) $x = 6\cos t + 5\cos 3t, \quad y = 6\sin 2t - 5\sin 3t,$
$0 \le t \le 2\pi$

d) $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 4t - 5\sin 6t,$
$0 \le t \le \pi$

9.5 Calculus with Parametrized Curves

This section shows how to find slopes, lengths, and surface areas associated with parametrized curves.

Slopes of Parametrized Curves

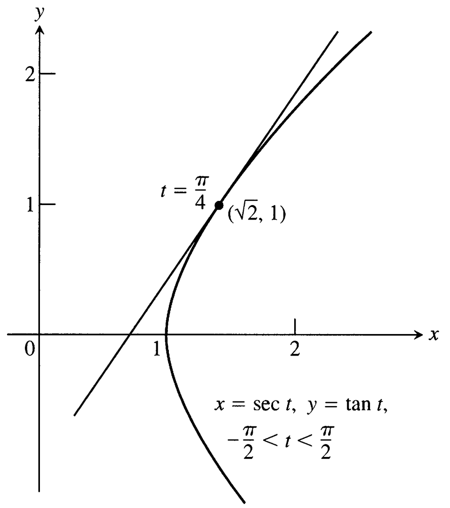
Definitions

A parametrized curve $x = f(t)$, $y = g(t)$ is **differentiable at $t = t_0$** if f and g are differentiable at $t = t_0$. The curve is **differentiable** if it is differentiable at every parameter value. The curve is **smooth** if f' and g' are continuous and not simultaneously zero.

At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dx/dt, dy/dt, and dy/dx are related by the Chain Rule equation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.



9.41 The hyperbola branch in Example 1.

Formula for Finding *dy/dx* from *dy/dt* and *dx/dt* **(*dx/dt* ≠ 0)**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{1}$$

EXAMPLE 1 Find the tangent to the right-hand hyperbola branch

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Fig. 9.41).

Solution The slope of the curve at t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}. \qquad \text{Eq. (1)}$$

Setting t equal to $\pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec(\pi/4)}{\tan(\pi/4)}$$

$$= \frac{\sqrt{2}}{1} = \sqrt{2}.$$

The point–slope equation of the tangent is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{2}(x - \sqrt{2})$$
$$y = \sqrt{2}x - 2 + 1$$
$$y = \sqrt{2}x - 1.$$

The Parametric Formula for d^2y/dx^2

If the parametric equations for a curve define y as a twice-differentiable function of x, we may calculate d^2y/dx^2 as a function of t in the following way:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}. \qquad \text{Eq. (1) with } y \text{ replaced by } y'$$

Notice the lack of symmetry in Eq. (2). To find d^2y/dx^2, we divide the derivative of y' by the derivative of x, not by the derivative of x'.

Formula for Finding d^2y/dx^2 from $y' = dy/dx$ and dx/dt **($dx/dt \neq 0$)**

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \tag{2}$$

How to Express d^2y/dx^2 in Terms of t

Step 1: Express $y' = dy/dx$ in terms of t.

Step 2: Find dy'/dt.

Step 3: Divide dy'/dt by dx/dt. The quotient is d^2y/dx^2.

EXAMPLE 2 Find d^2y/dx^2 if $x = t - t^2$ and $y = t - t^3$.

Solution

Step 1: *Express y' in terms of t:*

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t} \qquad \text{Eq. (1) with } x = t - t^2,\ y = t - t^3$$

Step 2: *Differentiate y' with respect to t:*

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$$

Step 3: *Divide dy'/dt by dx/dt.* Since

$$\frac{dx}{dt} = \frac{d}{dt}(t-t^2) = 1-2t, \qquad x = t - t^2$$

we have

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \qquad \text{Eq. (2)}$$

$$= \frac{2-6t+6t^2}{(1-2t)^2} \cdot \frac{1}{1-2t}$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}.$$

❑

Lengths of Parametrized Curves. Centroids

We find an integral for the length of a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, by rewriting the integral $L = \int ds$ from Section 5.5 in the following way:

$$L = \int_{t=a}^{t=b} ds = \int_a^b \sqrt{dx^2 + dy^2}$$

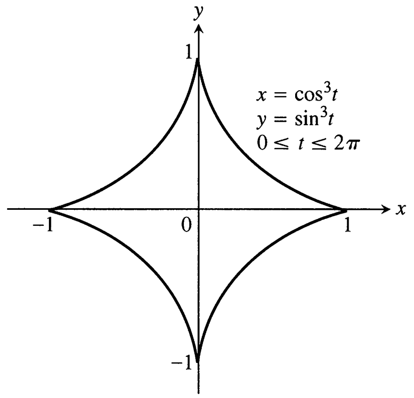
$$= \int_a^b \sqrt{\left(\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}\right) dt^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The only requirement besides the continuity of the integrand is that the point $P(x, y) = P(f(t), g(t))$ not trace any portion of the curve more than once as t moves from a to b.

Length

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, the curve's length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3}$$



9.42 The astroid in Example 3.

The length formulas in Section 5.5 are special cases of Eq. (3) (Exercises 35 and 36).

What if there are two different parametrizations for a curve whose length we want to find—does it matter which one we use? The answer, from advanced calculus, is no, as long as the parametrization we choose meets the conditions preceding Eq. (3).

EXAMPLE 3 Find the length of the astroid (Fig. 9.42)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{9\cos^2 t \sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}} \\ &= \sqrt{9\cos^2 t \sin^2 t} \\ &= 3|\cos t \sin t| \\ &= 3\cos t \sin t. \qquad \cos t \sin t \ge 0 \text{ for } 0 \le t \le \pi/2\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Length of first-quadrant portion} &= \int_0^{\pi/2} 3\cos t \sin t\, dt \\ &= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt \qquad \begin{matrix}\cos t \sin t = \\ (1/2)\sin 2t\end{matrix} \\ &= -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}.\end{aligned}$$

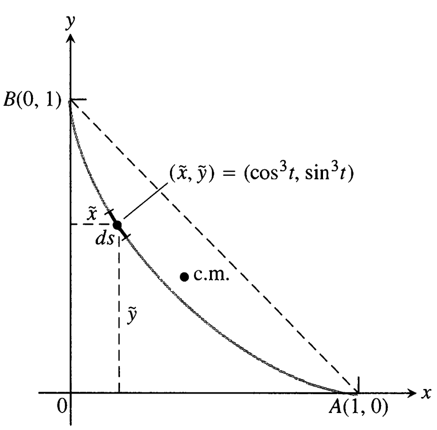
The length of the astroid is four times this: $4(3/2) = 6$. ❑

EXAMPLE 4 Find the centroid of the first-quadrant arc of the astroid in Example 3.

Solution We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes as we did at the end of Section 5.7.

The distribution of mass is symmetric about the line $y = x$, so $\overline{x} = \overline{y}$. A typical segment of the curve (Fig. 9.43) has mass

$$dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3\cos t \sin t\, dt. \qquad \begin{matrix}\text{From} \\ \text{Example 3}\end{matrix}$$



9.43 A typical segment of the arc in Example 4. The centroid (c.m.) of the curve lies about a third of the way toward chord *AB*.

The curve's mass is

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3\cos t \sin t\, dt = \frac{3}{2}. \qquad \text{Again from Example 3}$$

The curve's moment about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t\, dt$$

$$= 3\int_0^{\pi/2} \sin^4 t \cos t\, dt = 3 \cdot \frac{\sin^5 t}{5}\bigg]_0^{\pi/2} = \frac{3}{5}.$$

Hence,

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$

The centroid is the point (2/5, 2/5) (Fig. 9.43). ❑

The Area of a Surface of Revolution

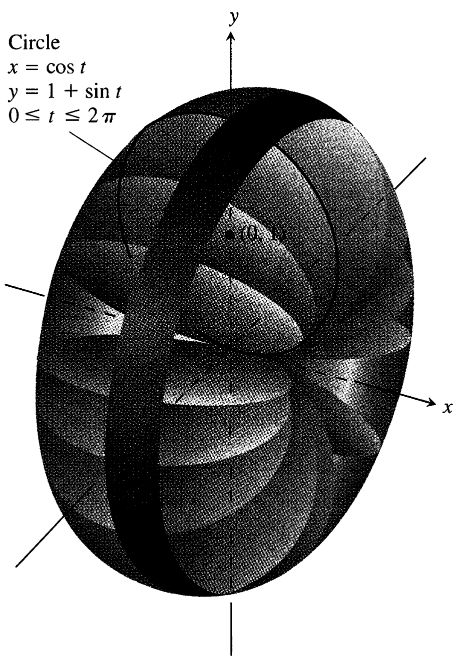
For smooth parametrized curves, the length formula in Eq. (3) leads to the following formulas for surfaces of revolution. The derivations are similar to the derivations of the Cartesian formulas in Section 5.6.

Surface Area

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):
$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{4}$$

2. Revolution about the y-axis ($x \ge 0$):
$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$



9.44 The surface in Example 5.

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

EXAMPLE 5 The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Fig. 9.44).

Solution We evaluate the formula

$$S = \int_a^b 2\pi\, y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Eq. (4) for revolution about the } x\text{-axis}$$

$$= \int_0^{2\pi} 2\pi\,(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi \Big[t - \cos t \Big]_0^{2\pi} = 4\pi^2.$$

❑

Exercises 9.5

Tangents to Parametrized Curves

In Exercises 1–12, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

1. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

2. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$

3. $x = 4\sin t, \quad y = 2\cos t, \quad t = \pi/4$

4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$

5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$

6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

7. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$

8. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$

9. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

10. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$

11. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

12. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 13–16 define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

13. $x^2 - 2tx + 2t^2 = 4, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

14. $x = \sqrt{5 - \sqrt{t}}, \quad y(t-1) = \ln y, \quad t = 1$

15. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

16. $x\sin t + 2x = t, \quad t\sin t - 2t = y, \quad t = \pi$

Lengths of Curves

Find the lengths of the curves in Exercises 17–22.

17. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

18. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$

19. $x = t^2/2, \quad y = (2t+1)^{3/2}/3, \quad 0 \le t \le 4$

20. $x = (2t+3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$

21. $x = 8\cos t + 8t\sin t$
$y = 8\sin t - 8t\cos t,$
$0 \le t \le \pi/2$

22. $x = \ln(\sec t + \tan t) - \sin t$
$y = \cos t, \quad 0 \le t \le \pi/3$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 23–26 about the indicated axes.

23. $x = \cos t, \quad y = 2 + \sin t, \quad 0 \le t \le 2\pi;$ x-axis

24. $x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3};$ y-axis

25. $x = t + \sqrt{2}, \quad y = (t^2/2) + \sqrt{2}\,t, \quad -\sqrt{2} \le t \le \sqrt{2};$ y-axis

26. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \pi/3;$ x-axis

27. *A cone frustum.* The line segment joining the points (0, 1) and (2, 2) is revolved about the x-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t$, $y = t + 1$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi\,(r_1 + r_2)$(slant height).

28. *A cone.* The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht$, $y = rt$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

Centroids

29. a) Find the coordinates of the centroid of the curve

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad 0 \le t \le \pi/2.$$

b) CALCULATOR The curve is a portion of the involute in Fig. 9.39. Sketch the curve. Find the centroid's coordinates to the nearest tenth and add the centroid to your sketch.

30. a) Find the coordinates of the centroid of the curve

$$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi.$$

b) CALCULATOR Sketch the curve. Find the centroid's coordinates to the nearest tenth and add the centroid to your sketch.

31. a) Find the coordinates of the centroid of the curve

$$x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi.$$

b) Sketch the curve and add the centroid to your sketch.

32. INTEGRAL EVALUATOR Most centroid calculations for curves are done with a calculator or computer that has an integral evaluation program. As a case in point, find, to the nearest hundredth, the coordinates of the centroid of the curve

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}.$$

Theory and Examples

33. *Length is independent of parametrization.* To illustrate the fact that the numbers we get for length do not depend on the way we parametrize our curves (except for the mild restrictions mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

a) $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2$

b) $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2$

34. *Elliptic integrals.* The length of the ellipse

$$x = a \cos t, \quad y = b \sin t, \quad 0 \le t \le 2\pi$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t}\, dt,$$

where e is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral,* is nonelementary except when $e = 0$ or 1.

a) CALCULATOR Use the trapezoidal rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b) Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in (a).

35. As mentioned in Section 9.4, the graph of a function $y = f(x)$ over an interval $[a, b]$ automatically has the parametrization

$$x = x, \quad y = f(x), \quad a \le x \le b.$$

The parameter, in this case, is x itself.

Show that for this parametrization the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

reduces to the Cartesian formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

derived in Section 5.5. This will show that the Cartesian formula is a special case of the parametric formula.

36. (*Continuation of Exercise 35.*) Show that the Cartesian formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y)$, $c \le y \le d$ (Section 5.5, Eq. 3), is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

37. Find the area under one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

(*Hint:* Use $dx = (dx/d\theta)\, d\theta$.)

38. Find the length of one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

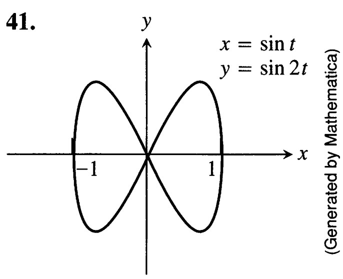
39. Find the area of the surface generated by revolving one arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ about the x-axis.

40. Find the volume swept out by revolving the region bounded by the x-axis and one arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ about the x-axis. (*Hint:* $dV = \pi y^2\, dx = \pi y^2 (dx/d\theta)\, d\theta$.)

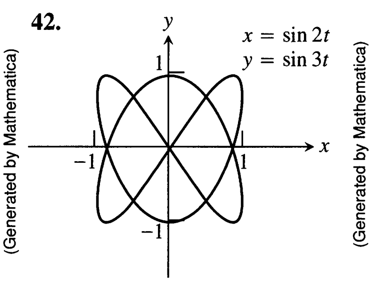
Grapher Explorations

The curves in Exercises 41 and 42 are called *Bowditch curves* or *Lissajous figures.* In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

41.



42.



Graph the parametric curves in Exercises 43–49 over parameter intervals of your choice. The curves are Bowditch curves (Lissajous figures), the general formula being

$$x = a \sin(mt + d), \quad y = b \sin nt,$$

with m and n integers.

43. $x = \sin 2t, \quad y = \sin t$

44. $x = \sin 3t, \quad y = \sin 4t$

45. $x = \sin t, \quad y = \sin 4t$

46. $x = \sin t, \quad y = \sin 5t$

47. $x = \sin 3t, \quad y = \sin 5t$

48. $x = \sin(3t + \pi/2), \quad y = \sin 5t$

49. $x = \sin(3t + \pi/4), \quad y = \sin 5t$

CAS Explorations and Projects

Use a CAS to perform the following steps on the parametrized curves in Exercises 50–55.

a) Plot the curve for the given interval of t values.
b) Find dy/dx and d^2y/dx^2 at the point t_0.
c) Find an equation for the tangent line to the curve at the point defined by the given value t_0. Plot the curve together with the tangent line on a single graph.
d) Find the length of the curve over the interval.

50. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1, \quad t_0 = 1/2$

51. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3, \quad 0 \le t \le 6, \quad t_0 = 3/2$

52. $x = e^t - t^2, \quad y = t + e^{-t}, \quad -1 \le t \le 2, \quad t_0 = 1$

53. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi, \quad t_0 = \pi/4$

54. $x = e^t + \sin 2t, \quad y = e^t + \cos(t^2), \quad -\sqrt{2}\pi \le t \le \pi/4, \quad t_0 = -\pi/4$

55. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi, \quad t_0 = \pi/2$

The equations in Exercises 56 and 57 define x and y implicitly as differentiable functions of t. Use a CAS to perform the following steps:

a) Solve the first equation for x and the second equation for y to find $x = f(t)$ and $y = g(t)$.
b) Find the slope of the curve $x = f(t)$ and $y = g(t)$ at t_0.
c) Find an equation for the tangent line to the curve at the point defined by t_0.
d) Plot the curve together with the tangent line over the specified interval of t-values.

56. $x^2 - 2tx + 3t^2 = 4, \quad y^3 - 2t^2 = 7, \quad -1 \le t \le 2, \quad t_0 = 1$

57. $x^2 \cos t + 2x = t, \quad t \sin t + 2\sqrt{y} = y, \quad -2\pi \le t \le 2\pi, \quad t_0 = -\pi/4$

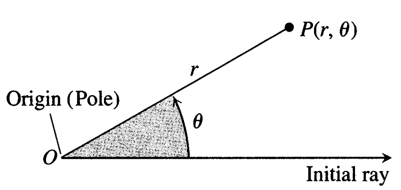
9.6

Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Fig. 9.45). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.



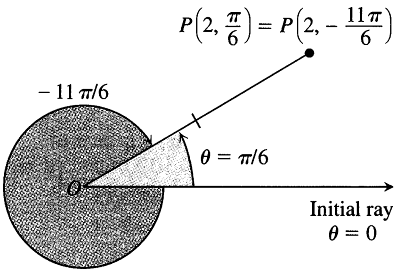
9.45 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Polar Coordinates

$$P(r, \theta) \tag{1}$$

Directed distance from O to P (r); Directed angle from initial ray to OP (θ)

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique.

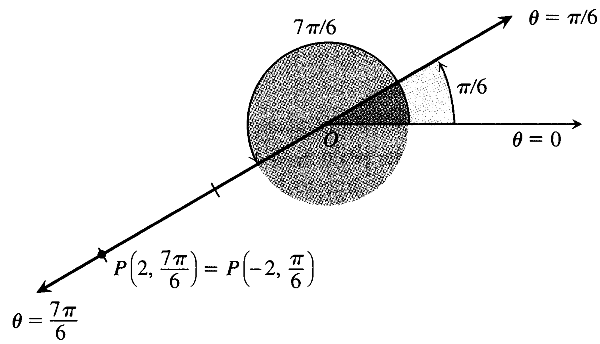


9.46 Polar coordinates are not unique.

For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2, \theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Fig. 9.46).

Negative Values of *r*

There are occasions when we wish to allow r to be negative. That is why we use directed distance in (1). The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ rad counterclockwise from the initial ray and going forward 2 units (Fig. 9.47). It can also be reached by turning $\pi/6$ rad counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.



9.47 Polar coordinates can have negative *r*-values.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

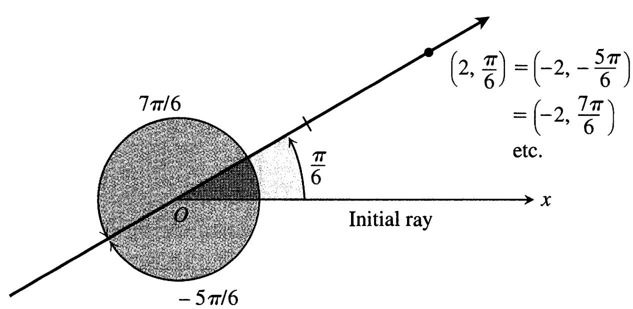
Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ rad with the initial ray, and mark the point $(2, \pi/6)$ (Fig. 9.48). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

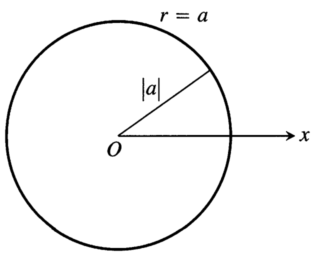
$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots .$$



9.48 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs.



9.49 The polar equation for this circle is $r = a$.

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ❑

Elementary Coordinate Equations and Inequalities

If we hold r fixed at a constant value $r = a \neq 0$, the point $P\,(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Fig. 9.49).

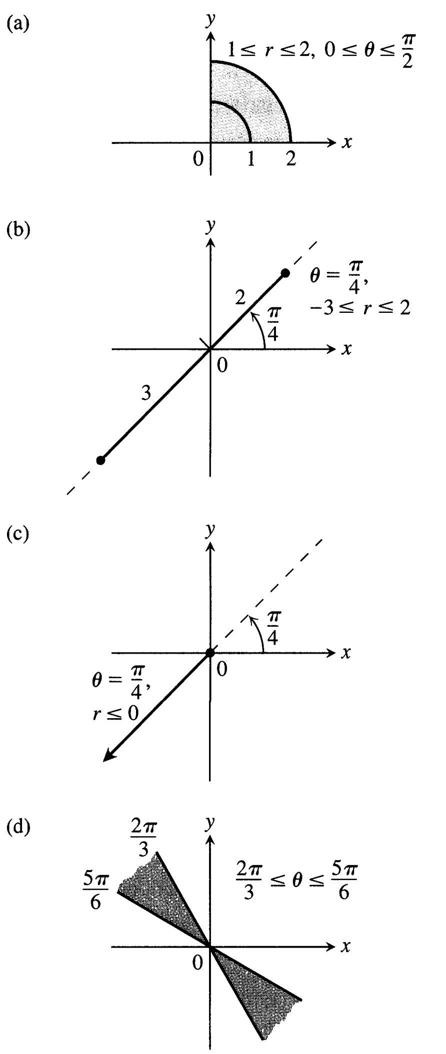
If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P\,(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

Equation	**Graph**
$r = a$	Circle of radius $\lvert a\rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

EXAMPLE 2

a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.
b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Fig. 9.48. ❑

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.



9.50 The graphs of typical inequalities in r and θ (Example 3).

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy the following conditions.

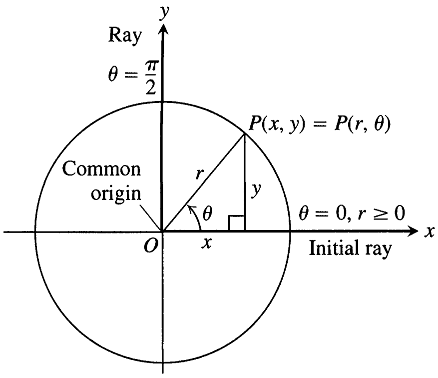
a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \dfrac{\pi}{2}$

b) $-3 \leq r \leq 2$ and $\theta = \dfrac{\pi}{4}$

c) $r \leq 0$ and $\theta = \dfrac{\pi}{4}$

d) $\dfrac{2\pi}{3} \leq \theta \leq \dfrac{5\pi}{6}$ (no restriction on r)

Solution The graphs are shown in Fig. 9.50. ❑



9.51 The usual way to relate polar and Cartesian coordinates.

Cartesian Versus Polar Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$, becomes the positive y-axis (Fig. 9.51). The two coordinate systems are then related by the following equations.

Equations Relating Polar and Cartesian Coordinates

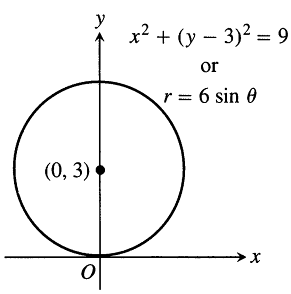
$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad x^2 + y^2 = r^2, \qquad \frac{y}{x} = \tan \theta \qquad (2)$$

We use Eqs. (2) to rewrite polar equations in Cartesian form and vice versa.

EXAMPLE 4

Polar equation	Cartesian equivalent
$r \cos \theta = 2$	$x = 2$
$r^2 \cos \theta \sin \theta = 4$	$xy = 4$
$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r \cos \theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos \theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ❑



9.52 The circle in Example 5.

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Fig. 9.52).

Solution

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{The 9's cancel.}$$

$$r^2 - 6r \sin \theta = 0 \qquad x^2 + y^2 = r^2$$

$$r = 0 \quad \text{or} \quad r - 6 \sin \theta = 0$$

$$r = 6 \sin \theta \qquad \text{Includes both possibilities}$$

We will say more about polar equations of conic sections in Section 9.8. ❑

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

a) $r \cos \theta = -4$
b) $r^2 = 4r \cos \theta$
c) $r = \dfrac{4}{2 \cos \theta - \sin \theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$

The graph: Vertical line through $x = -4$ on the x-axis

b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$\begin{aligned} r^2 &= 4r\cos\theta \\ x^2 + y^2 &= 4x \\ x^2 - 4x + y^2 &= 0 \\ x^2 - 4x + 4 + y^2 &= 4 \qquad \text{Completing the square} \\ (x-2)^2 + y^2 &= 4 \end{aligned}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$\begin{aligned} r(2\cos\theta - \sin\theta) &= 4 \\ 2r\cos\theta - r\sin\theta &= 4 \\ 2x - y &= 4 \\ y &= 2x - 4 \end{aligned}$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ❑

Exercises 9.6

Polar Coordinate Pairs

1. Which polar coordinate pairs label the same point?

a) $(3, 0)$ **b)** $(-3, 0)$ **c)** $(2, 2\pi/3)$
d) $(2, 7\pi/3)$ **e)** $(-3, \pi)$ **f)** $(2, \pi/3)$
g) $(-3, 2\pi)$ **h)** $(-2, -\pi/3)$

2. Which polar coordinate pairs label the same point?

a) $(-2, \pi/3)$ **b)** $(2, -\pi/3)$ **c)** (r, θ)
d) $(r, \theta + \pi)$ **e)** $(-r, \theta)$ **f)** $(2, -2\pi/3)$
g) $(-r, \theta + \pi)$ **h)** $(-2, 2\pi/3)$

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a) $(2, \pi/2)$ **b)** $(2, 0)$
c) $(-2, \pi/2)$ **d)** $(-2, 0)$

4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a) $(3, \pi/4)$ **b)** $(-3, \pi/4)$
c) $(3, -\pi/4)$ **d)** $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.

6. Find the Cartesian coordinates of the following points (given in polar coordinates).

a) $\left(\sqrt{2}, \pi/4\right)$ **b)** $(1, 0)$
c) $(0, \pi/2)$ **d)** $\left(-\sqrt{2}, \pi/4\right)$
e) $(-3, 5\pi/6)$ **f)** $(5, \tan^{-1}(4/3))$
g) $(-1, 7\pi)$ **h)** $\left(2\sqrt{3}, 2\pi/3\right)$

Graphing Polar Equations and Inequalities

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 7–22.

7. $r = 2$ **8.** $0 \le r \le 2$

9. $r \ge 1$ **10.** $1 \le r \le 2$

11. $0 \le \theta \le \pi/6, \quad r \ge 0$ **12.** $\theta = 2\pi/3, \quad r \le -2$

13. $\theta = \pi/3, \quad -1 \le r \le 3$

14. $\theta = 11\pi/4, \quad r \geq -1$

15. $\theta = \pi/2, \quad r \geq 0$

16. $\theta = \pi/2, \quad r \leq 0$

17. $0 \leq \theta \leq \pi, \quad r = 1$

18. $0 \leq \theta \leq \pi, \quad r = -1$

19. $\pi/4 \leq \theta \leq 3\pi/4, \quad 0 \leq r \leq 1$

20. $-\pi/4 \leq \theta \leq \pi/4, \quad -1 \leq r \leq 1$

21. $-\pi/2 \leq \theta \leq \pi/2, \quad 1 \leq r \leq 2$

22. $0 \leq \theta \leq \pi/2, \quad 1 \leq |r| \leq 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 23–48 by equivalent Cartesian equations. Then describe or identify the graph.

23. $r \cos\theta = 2$

24. $r \sin\theta = -1$

25. $r \sin\theta = 0$

26. $r \cos\theta = 0$

27. $r = 4 \csc\theta$

28. $r = -3 \sec\theta$

29. $r \cos\theta + r \sin\theta = 1$

30. $r \sin\theta = r \cos\theta$

31. $r^2 = 1$

32. $r^2 = 4r \sin\theta$

33. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

34. $r^2 \sin 2\theta = 2$

35. $r = \cot\theta \csc\theta$

36. $r = 4 \tan\theta \sec\theta$

37. $r = \csc\theta\, e^{r\cos\theta}$

38. $r \sin\theta = \ln r + \ln \cos\theta$

39. $r^2 + 2r^2 \cos\theta \sin\theta = 1$

40. $\cos^2\theta = \sin^2\theta$

41. $r^2 = -4r \cos\theta$

42. $r^2 = -6r \sin\theta$

43. $r = 8 \sin\theta$

44. $r = 3 \cos\theta$

45. $r = 2\cos\theta + 2\sin\theta$

46. $r = 2\cos\theta - \sin\theta$

47. $r \sin\left(\theta + \dfrac{\pi}{6}\right) = 2$

48. $r \sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 49–62 by equivalent polar equations.

49. $x = 7$

50. $y = 1$

51. $x = y$

52. $x - y = 3$

53. $x^2 + y^2 = 4$

54. $x^2 - y^2 = 1$

55. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

56. $xy = 2$

57. $y^2 = 4x$

58. $x^2 + xy + y^2 = 1$

59. $x^2 + (y-2)^2 = 4$

60. $(x-5)^2 + y^2 = 25$

61. $(x-3)^2 + (y+1)^2 = 4$

62. $(x+2)^2 + (y-5)^2 = 16$

Theory and Examples

63. Find all polar coordinates of the origin.

64. *Vertical and horizontal lines*

a) Show that every vertical line in the xy-plane has a polar equation of the form $r = a \sec\theta$.

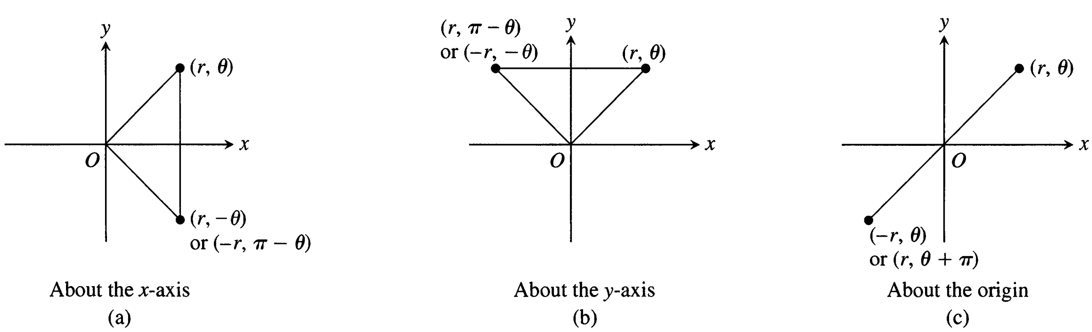
b) Find the analogous polar equation for horizontal lines in the xy-plane.

9.7 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 9.53 illustrates the standard polar coordinate tests for symmetry.



9.53 Three tests for symmetry.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Fig. 9.53a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Fig. 9.53b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Fig. 9.53c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r \cos \theta = f(\theta) \cos \theta, \qquad y = r \sin \theta = f(\theta) \sin \theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Section 9.5, Eq. (1) with $t = \theta$

$$= \frac{\dfrac{d}{d\theta}(f(\theta) \cdot \sin \theta)}{\dfrac{d}{d\theta}(f(\theta) \cdot \cos \theta)}$$

$$= \frac{\dfrac{df}{d\theta} \sin \theta + f(\theta) \cos \theta}{\dfrac{df}{d\theta} \cos \theta - f(\theta) \sin \theta}$$

Product Rule for Derivatives

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}, \tag{1}$$

provided $dx/d\theta \neq 0$ at (r, θ).

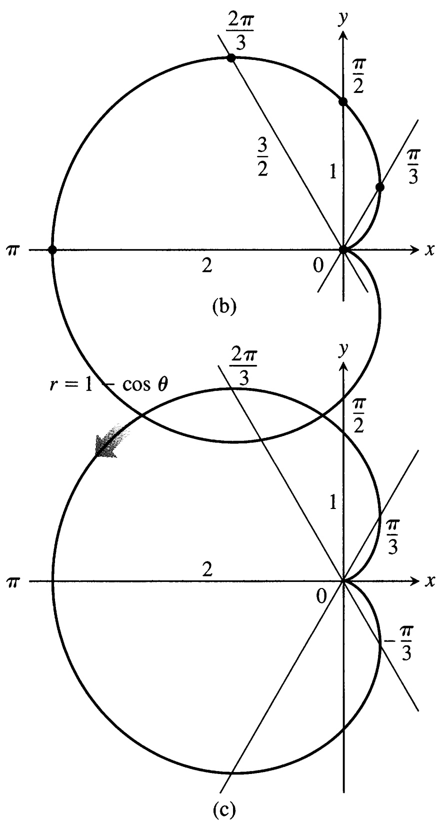
If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and Eq. (1) gives

$$\left.\frac{dy}{dx}\right|_{(0,\,\theta_0)} = \frac{f'(\theta_0) \sin \theta_0}{f'(\theta_0) \cos \theta_0} = \tan \theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan \theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin more than once, with different slopes at different θ-values. This is not the case in our first example, however.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



9.54 **The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.**

EXAMPLE 1 *A cardioid*

Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$\begin{aligned} (r,\theta) \text{ on the graph} \quad &\Rightarrow \quad r = 1 - \cos\theta \\ &\Rightarrow \quad r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow \quad (r, -\theta) \text{ on the graph.} \end{aligned}$$

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope tan $(0) = 0$ and returns to the origin with slope tan $(2\pi) = 0$.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph (Fig. 9.54). The curve is called a *cardioid* because of its heart shape. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennae. ❑

EXAMPLE 2 Graph the curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned} (r,\theta) \text{ on the graph} \quad &\Rightarrow \quad r^2 = 4\cos\theta \\ &\Rightarrow \quad r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow \quad (r, -\theta) \text{ on the graph.} \end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned} (r,\theta) \text{ on the graph} \quad &\Rightarrow \quad r^2 = 4\cos\theta \\ &\Rightarrow \quad (-r)^2 = 4\cos\theta \\ &\Rightarrow \quad (-r, \theta) \text{ on the graph.} \end{aligned}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because tan θ is infinite.

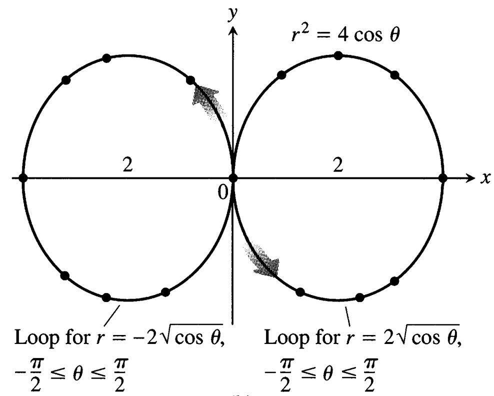
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Fig. 9.55). ❑

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	± 1.9
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	± 1.7
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	± 1.4
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

9.55 The graph of $r^2 = 4\cos\theta$ (Example 2). The arrows show the direction of increasing θ. The values of r in the table are rounded.

Steps for Faster Graphing

Step 1: First graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plane (that is, plot the values of θ on a horizontal axis and the corresponding values of r along a vertical axis).

Step 2: Then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

Faster Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) values, plot the corresponding points, and connect them in order of increasing θ. This can work well if there are enough points to reveal all the loops and dimples in the graph. Here we describe another method of graphing that is usually quicker and more reliable. The steps are listed at left.

This method is better than simple point plotting because the Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing. As examples, we graph $r = 1 + \cos(\theta/2)$ and $r^2 = \sin 2\theta$.

EXAMPLE 3 Graph the curve

$$r = 1 + \cos\frac{\theta}{2}.$$

Solution We first graph r as a function of θ in the Cartesian $r\theta$-plane. Since the cosine has period 2π, we must let θ run from 0 to 4π to produce the entire graph (Fig. 9.56a, on the following page). The arrows from the θ-axis to the curve give radii for graphing $r = 1 + \cos(\theta/2)$ in the polar plane (Fig. 9.56b, on the following page). ❑

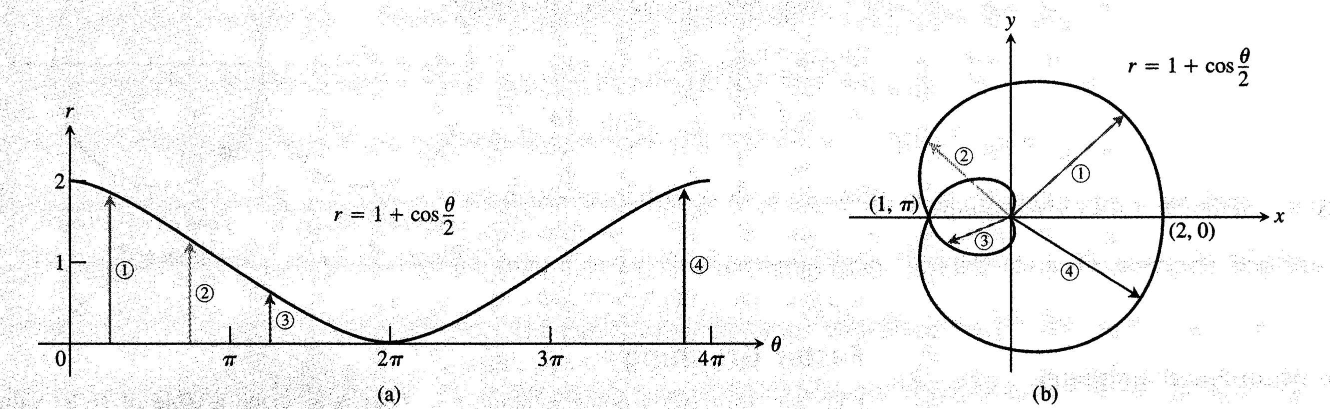
EXAMPLE 4 *A lemniscate*

Graph the curve $r^2 = \sin 2\theta$.

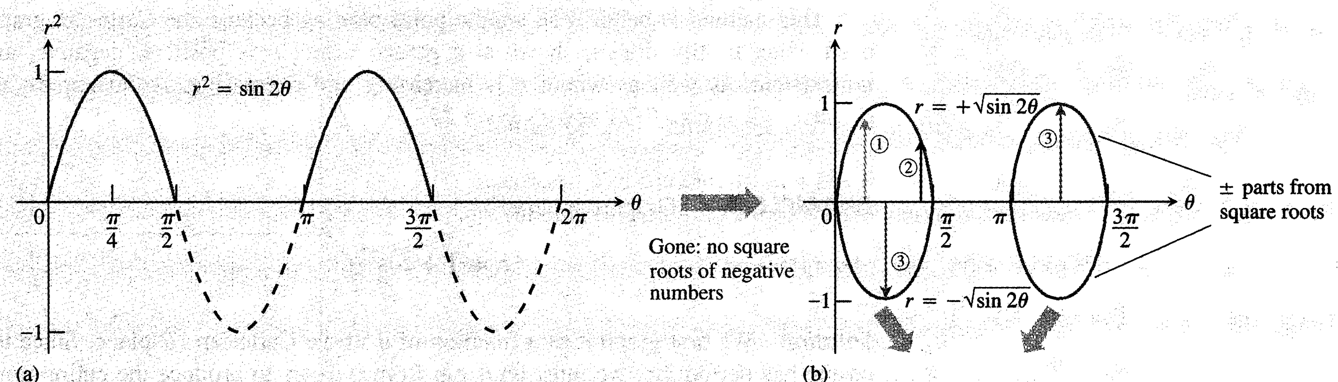
Solution Here we begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane, treating r^2 as a variable that may have negative as well as positive values (Fig. 9.57a, on the following page). We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Fig. 9.57b, on the following page) and then draw the polar graph (Fig. 9.57c, on the following page). The graph in Fig. 9.57(b) "covers" the final polar graph in Fig. 9.57(c) twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we learn a little more about the behavior of the function this way. ❑

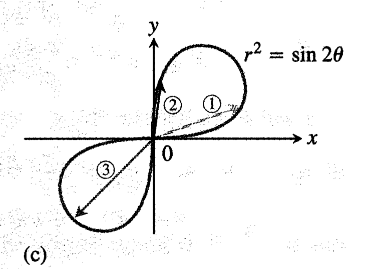
Drawing Lesson

How to Use Cartesian Graphs to Draw Polar Graphs



9.56 (a) The graph of $r = 1 + \cos(\theta/2)$ in the Cartesian $r\theta$-plane gives the radii for the graph in the polar $r\theta$-plane (b).





9.57 (a) The graph of $r^2 = \sin 2\theta$ in the Cartesian $r^2\theta$-plane includes negative values of the dependent variable r^2 as well as positive values. (b) When we graph r vs. θ in the Cartesian $r\theta$-plane, we ignore the points where r is imaginary but plot + and − parts from the points where r^2 is positive. (c) In the polar $r\theta$-plane, the radii from the previous sketch cover the final graph twice.

Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve.

Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. The only sure way to identify all the points of intersection is to graph the equations.

EXAMPLE 5 *Deceptive coordinates*

Show that the point $(2, \pi/2)$ lies on the curve $r = 2 \cos 2\theta$.

Solution It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2 \cos 2\left(\frac{\pi}{2}\right) = 2 \cos \pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the given point in which r is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2 \cos 2\theta$, we find

$$-2 = 2 \cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve. ❑

EXAMPLE 6 *Elusive intersection points*

Find the points of intersection of the curves

$$r^2 = 4 \cos \theta \qquad \text{and} \qquad r = 1 - \cos \theta.$$

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

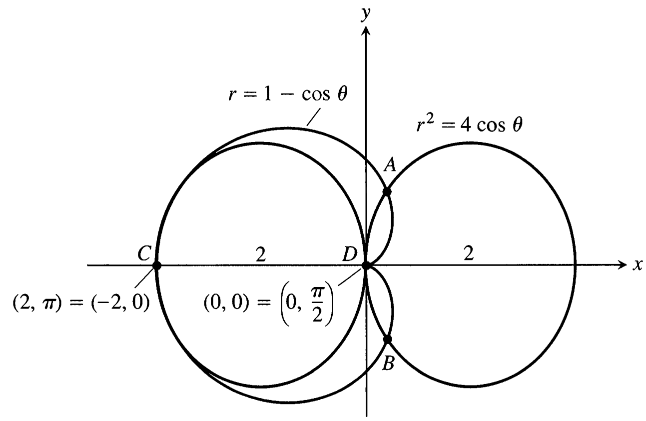
If we substitute $\cos \theta = r^2/4$ in the equation $r = 1 - \cos \theta$, we get

$$\begin{aligned} r &= 1 - \cos \theta = 1 - \frac{r^2}{4} \\ 4r &= 4 - r^2 \\ r^2 + 4r - 4 &= 0 \\ r &= -2 \pm 2\sqrt{2}. && \text{Quadratic formula} \end{aligned}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\begin{aligned} \theta &= \cos^{-1}(1 - r) && \text{From } r = 1 - \cos\theta \\ &= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) && \text{Set } r = 2\sqrt{2} - 2. \\ &= \cos^{-1}\left(3 - 2\sqrt{2}\right) \\ &= \pm 80^\circ. && \text{Rounded to the nearest degree} \end{aligned}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80^\circ)$.



9.58 The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example 6). Only *A* and *B* were found by simultaneous solution. The other two were disclosed by graphing.

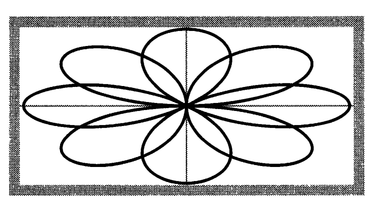
If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Fig. 9.58), as we can now do by combining the graphs in Figs. 9.54 and 9.55, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points (0, 0) and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$. ❑

Technology *Finding Intersections* The *simultaneous mode* of a graphing utility gives new meaning to the *simultaneous solution* of a pair of polar coordinate equations. A simultaneous solution occurs only where the two graphs "collide" while they are being drawn simultaneously and not where one graph intersects the other at a point that had been illuminated earlier. The distinction is particularly important in the areas of traffic control or missile defense. For example, in traffic control the only issue is whether two aircraft are in the same place at the same time. The question of whether the curves the craft follow intersect is unimportant.

To illustrate, graph the polar equations

$$r = \cos 2\theta \qquad \text{and} \qquad r = \sin 2\theta$$

in simultaneous mode with $0 \le \theta < 2\pi$, θ Step $= 0.1$, and view dimensions [xmin, xmax] $= [-1, 1]$ by [ymin, ymax] $= [-1, 1]$. *While the graphs are being drawn on the screen,* count the number of times the two graphs illuminate a single pixel simultaneously. Explain why these points of intersection of the two graphs correspond to simultaneous solutions of the equations. (You may find it helpful to slow down the graphing by making θ Step smaller, say 0.05, for example.) In how many points total do the graphs actually intersect?



$r_1 = \sin 2\theta$ and $r_2 = \cos 2\theta$ graphed together.

Exercises 9.7

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves.

1. $r = 1 + \cos\theta$
2. $r = 2 - 2\cos\theta$
3. $r = 1 - \sin\theta$
4. $r = 1 + \sin\theta$
5. $r = 2 + \sin\theta$
6. $r = 1 + 2\sin\theta$
7. $r = \sin(\theta/2)$
8. $r = \cos(\theta/2)$
9. $r^2 = \cos\theta$
10. $r^2 = \sin\theta$
11. $r^2 = -\sin\theta$
12. $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$
14. $r^2 = 4\sin 2\theta$
15. $r^2 = -\sin 2\theta$
16. $r^2 = -\cos 2\theta$

Slopes of Polar Curves

Use Eq. (1) to find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. *Cardioid.* $r = -1 + \cos\theta;\quad \theta = \pm\pi/2$
18. *Cardioid.* $r = -1 + \sin\theta;\quad \theta = 0,\ \pi$
19. *Four-leaved rose.* $r = \sin 2\theta;\quad \theta = \pm\pi/4,\ \pm 3\pi/4$
20. *Four-leaved rose.* $r = \cos 2\theta;\quad \theta = 0,\ \pm\pi/2,\ \pi$

Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. *Limaçons with an inner loop*
a) $r = \dfrac{1}{2} + \cos\theta$ **b)** $r = \dfrac{1}{2} + \sin\theta$

22. *Cardioids*
a) $r = 1 - \cos\theta$ **b)** $r = -1 + \sin\theta$

23. *Dimpled limaçons*
a) $r = \dfrac{3}{2} + \cos\theta$ **b)** $r = \dfrac{3}{2} - \sin\theta$

24. *Oval limaçons*
a) $r = 2 + \cos\theta$ **b)** $r = -2 + \sin\theta$

Graphing Polar Inequalities

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$
28. $0 \le r^2 \le \cos\theta$

Intersections

29. Show that the point $(2, 3\pi/4)$ lies on the curve $r = 2\sin 2\theta$.

30. Show that $(1/2, 3\pi/2)$ lies on the curve $r = -\sin(\theta/3)$.

Find the points of intersection of the pairs of curves in Exercises 31–38.

31. $r = 1 + \cos\theta,\quad r = 1 - \cos\theta$
32. $r = 1 + \sin\theta,\quad r = 1 - \sin\theta$
33. $r = 2\sin\theta,\quad r = 2\sin 2\theta$
34. $r = \cos\theta,\quad r = 1 - \cos\theta$
35. $r = \sqrt{2},\quad r^2 = 4\sin\theta$
36. $r^2 = \sqrt{2}\sin\theta,\quad r^2 = \sqrt{2}\cos\theta$
37. $r = 1,\quad r^2 = 2\sin 2\theta$
38. $r^2 = \sqrt{2}\cos 2\theta,\quad r^2 = \sqrt{2}\sin 2\theta$

GRAPHER Find the points of intersection of the pairs of curves in Exercises 39–42.

39. $r^2 = \sin 2\theta,\quad r^2 = \cos 2\theta$
40. $r = 1 + \cos\dfrac{\theta}{2},\quad r = 1 - \sin\dfrac{\theta}{2}$
41. $r = 1,\quad r = 2\sin 2\theta$
42. $r = 1,\quad r^2 = 2\sin 2\theta$

Grapher Explorations

43. Which of the following has the same graph as $r = 1 - \cos\theta$?
a) $r = -1 - \cos\theta$
b) $r = 1 + \cos\theta$

Confirm your answer with algebra.

44. Which of the following has the same graph as $r = \cos 2\theta$?
a) $r = -\sin(2\theta + \pi/2)$
b) $r = -\cos(\theta/2)$

Confirm your answer with algebra.

45. *A rose within a rose.* Graph the equation $r = 1 - 2\sin 3\theta$.

46. *The nephroid of Freeth.* Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

47. *Roses.* Graph the roses $r = \cos m\theta$ for $m = 1/3$, 2, 3, and 7.

48. *Spirals.* Polar coordinates are just the thing for defining spirals. Graph the following spirals.

a) $r = \theta$
b) $r = -\theta$
c) *A logarithmic spiral:* $r = e^{\theta/10}$
d) *A hyperbolic spiral:* $r = 8/\theta$
e) *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

Theory and Examples

49. (*Continuation of Example 6.*) The simultaneous solution of the equations

$$r^2 = 4 \cos \theta \tag{2}$$

$$r = 1 - \cos \theta \tag{3}$$

in the text did not reveal the points (0, 0) and $(2, \pi)$ in which their graphs intersected.

a) We could have found the point $(2, \pi)$, however, by replacing the (r, θ) in Eq. (2) by the equivalent $(-r, \theta + \pi)$ to obtain

$$\begin{aligned} r^2 &= 4 \cos \theta \\ (-r)^2 &= 4 \cos (\theta + \pi) \\ r^2 &= -4 \cos \theta . \end{aligned} \tag{4}$$

Solve Eqs. (3) and (4) simultaneously to show that $(2, \pi)$ is a common solution. (This will still not reveal that the graphs intersect at (0, 0).)

b) The origin is still a special case. (It often is.) Here is one way to handle it: Set $r = 0$ in Eqs. (2) and (3) and solve each equation for a corresponding value of θ. Since $(0, \theta)$ is the origin for *any* θ, this will show that both curves pass through the origin even if they do so for different θ-values.

50. If a curve has any two of the symmetries listed at the beginning of the section, can anything be said about its having or not having the third symmetry? Give reasons for your answer.

***51.** Find the maximum width of the petal of the four-leaved rose $r = \cos 2\theta$, which lies along the x-axis.

***52.** Find the maximum height above the x-axis of the cardioid $r = 2(1 + \cos \theta)$.

9.8 Polar Equations for Conic Sections

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets move can all be described with a single relatively simple coordinate equation. We develop that equation here.

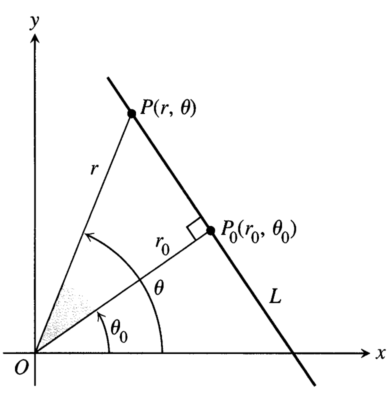
Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Fig. 9.59). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$\frac{r_0}{r} = \cos(\theta - \theta_0)$$

or

$$r \cos(\theta - \theta_0) = r_0 .$$

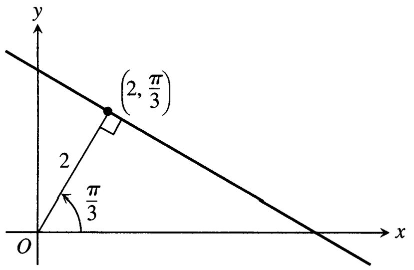


9.59 We can obtain a polar equation for line L by reading the relation $r_0/r = \cos(\theta - \theta_0)$ from triangle OP_0P.

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r \cos(\theta - \theta_0) = r_0 .$$



9.60 The standard polar equation of this line is

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

(Example 1).

EXAMPLE 1 Use the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$ to find a Cartesian equation for the line in Fig. 9.60.

Solution

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 2$$

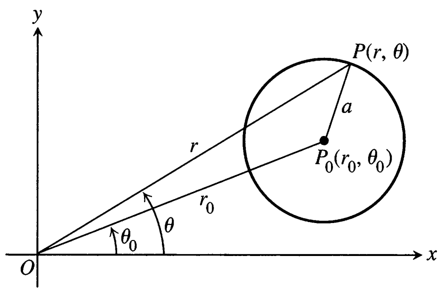
$$x + \sqrt{3}\,y = 4$$

❑

Circles

To find a polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle OP_0P (Fig. 9.61). This gives

$$a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta - \theta_0). \tag{1}$$



9.61 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P.

If the circle passes through the origin, then $r_0 = a$ and Eq. (1) simplifies to

$$a^2 = a^2 + r^2 - 2ar\cos(\theta - \theta_0) \qquad \text{Eq. (1) with } r_0 = a$$

$$r^2 = 2ar\cos(\theta - \theta_0)$$

$$r = 2a\cos(\theta - \theta_0). \tag{2}$$

If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and Eq. (2) becomes

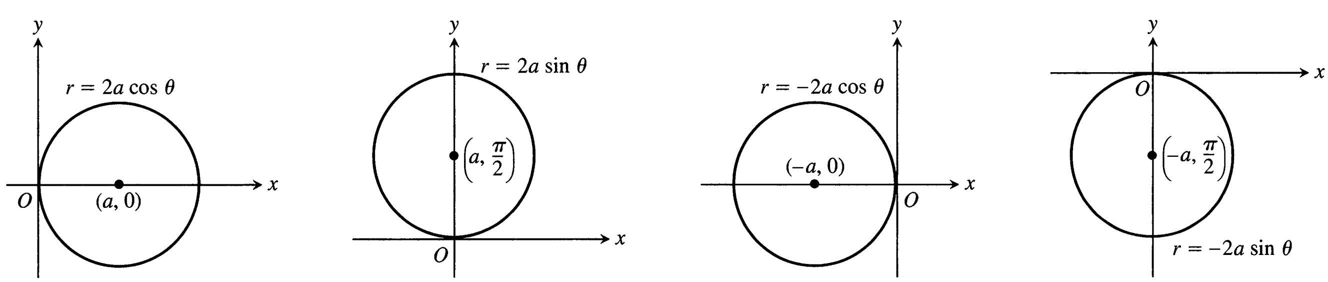
$$r = 2a\cos\theta. \tag{3}$$

If the center lies on the positive y-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and Eq. (2) becomes

$$r = 2a\sin\theta. \tag{4}$$

Equations for circles through the origin centered on the negative x- and y-axes can be obtained from Eqs. (3) and (4) by replacing r with $-r$.

Polar Equations for Circles Through the Origin Centered on the x- and y-axes, Radius a

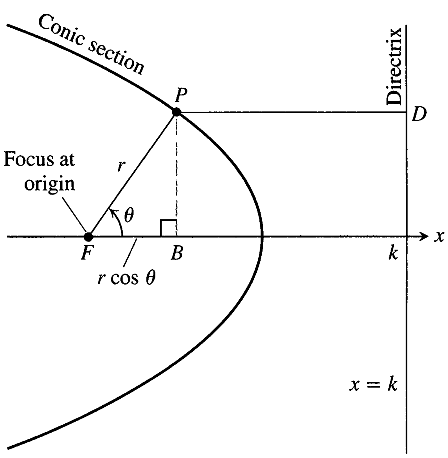


EXAMPLE 2 *Circles through the origin*

Radius	Center (polar coordinates)	Equation
3	$(3, 0)$	$r = 6 \cos \theta$
2	$(2, \pi/2)$	$r = 4 \sin \theta$
1/2	$(-1/2, 0)$	$r = -\cos \theta$
1	$(-1, \pi/2)$	$r = -2 \sin \theta$

❑

Ellipses, Parabolas, and Hyperbolas Unified



9.62 If a conic section is put in this position, then $PF = r$ and $PD = k - r \cos \theta$.

To find polar equations for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ (Fig. 9.62). This makes

$$PF = r$$

and

$$PD = k - FB = k - r \cos \theta.$$

The conic's focus–directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r \cos \theta),$$

which can be solved for r to obtain

$$\boxed{r = \frac{ke}{1 + e \cos \theta}. \qquad (5)}$$

This equation represents an ellipse if $0 < e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. And there we have it—ellipses, parabolas, and hyperbolas all with the same basic equation.

EXAMPLE 3 *Typical conics from Eq. (5)*

$$e = \frac{1}{2}: \quad \text{ellipse} \quad r = \frac{k}{2 + \cos \theta}$$

$$e = 1: \quad \text{parabola} \quad r = \frac{k}{1 + \cos \theta}$$

$$e = 2: \quad \text{hyperbola} \quad r = \frac{2k}{1 + 2 \cos \theta}$$

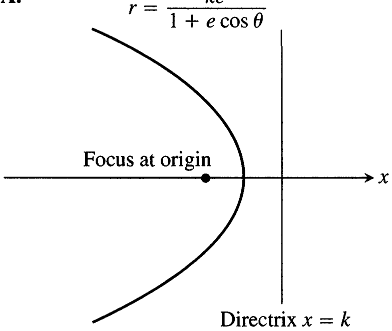
❑

You may see variations of Eq. (5) from time to time, depending on the location of the directrix. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace Eq. (5) by

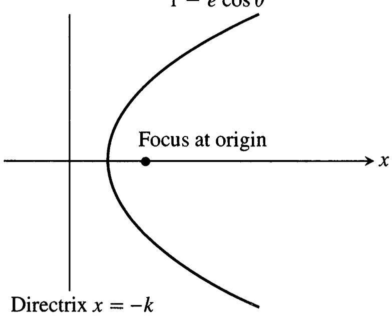
$$r = \frac{ke}{1 - e \cos \theta}.$$

Table 9.4 Equations for conic sections ($e > 0$)

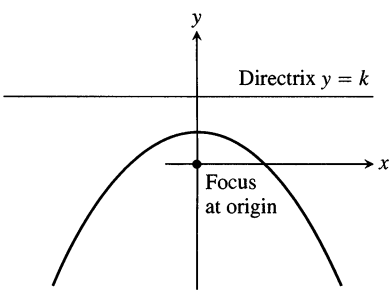
A. $r = \dfrac{ke}{1 + e\cos\theta}$



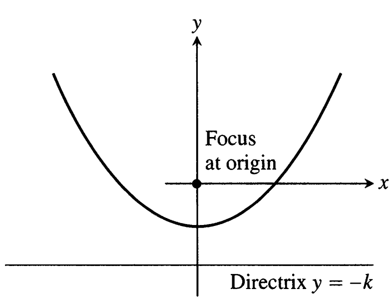
B. $r = \dfrac{ke}{1 - e\cos\theta}$



C. $r = \dfrac{ke}{1 + e\sin\theta}$



D. $r = \dfrac{ke}{1 - e\sin\theta}$



The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations we get have sines in them instead of cosines, as shown in Table 9.4.

EXAMPLE 4 Find an equation for the hyperbola with eccentricity 3/2 and directrix $x = 2$.

Solution We use Eq. (A) in Table 9.4 with $k = 2$ and $e = 3/2$ to get

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$

❑

EXAMPLE 5 Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}.$$

Solution We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

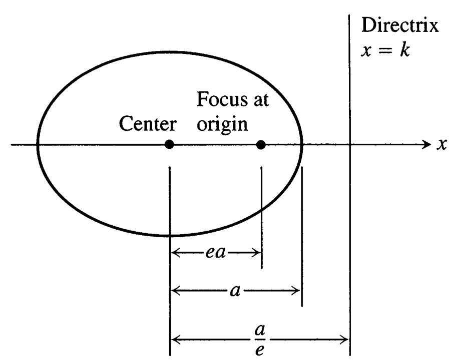
This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$. ❑

From the ellipse diagram in Fig. 9.63, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea. \tag{6}$$



9.63 In an ellipse with semimajor axis a, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

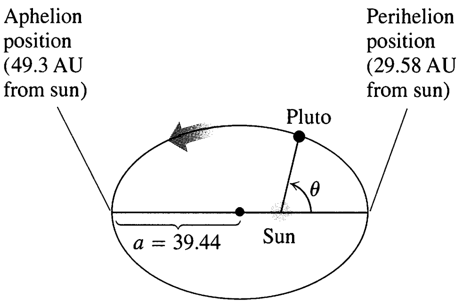
From this, we find that $ke = a(1 - e^2)$. Replacing ke in Eq. (5) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

Ellipse with Eccentricity e and Semimajor Axis a

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{7}$$

Notice that when $e = 0$, Eq. (7) becomes $r = a$, which represents a circle.

Equation (7) is the starting point for calculating planetary orbits.



9.64 The orbit of Pluto (Example 6).

EXAMPLE 6 Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Solution We use Eq. (7) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1-(0.25)^2)}{1+0.25\cos\theta} = \frac{147.9}{4+\cos\theta}.$$

At its point of closest approach (perihelion), Pluto is

$$r = \frac{147.9}{4+1} = 29.58 \text{ AU}$$

from the sun. At its most distant point (aphelion), Pluto is

$$r = \frac{147.9}{4-1} = 49.3 \text{ AU}$$

from the sun (Fig. 9.64). ❑

EXAMPLE 7 Find the distance from one focus of the ellipse in Example 6 to the associated directrix.

Solution We use Eq. (6) with $a = 39.44$ and $e = 0.25$ to find

$$k = 39.44\left(\frac{1}{0.25} - 0.25\right) = 147.9 \text{ AU}.$$

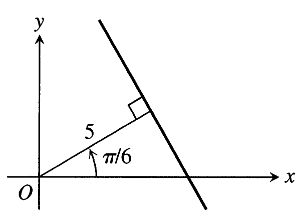
❑

Exercises 9.8

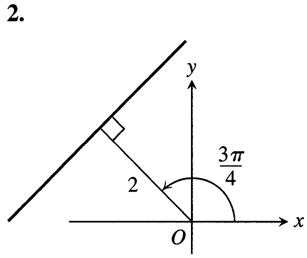
Lines

Find polar and Cartesian equations for the lines in Exercises 1–4.

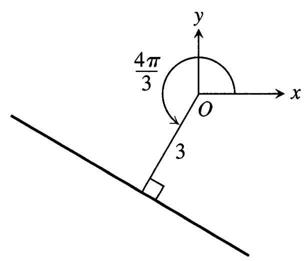
1.



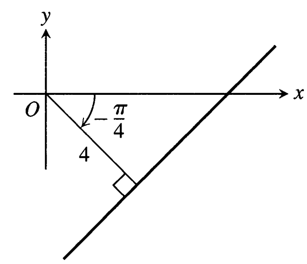
2.



3.



4.



Sketch the lines in Exercises 5–8 and find Cartesian equations for them.

5. $r \cos\left(\theta - \dfrac{\pi}{4}\right) = \sqrt{2}$ **6.** $r \cos\left(\theta + \dfrac{3\pi}{4}\right) = 1$

7. $r \cos\left(\theta - \dfrac{2\pi}{3}\right) = 3$ **8.** $r \cos\left(\theta + \dfrac{\pi}{3}\right) = 2$

Find a polar equation in the form $r \cos(\theta - \theta_0) = r_0$ for each of the lines in Exercises 9–12.

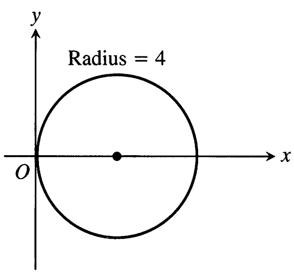
9. $\sqrt{2}x + \sqrt{2}y = 6$ **10.** $\sqrt{3}x - y = 1$

11. $y = -5$ **12.** $x = -4$

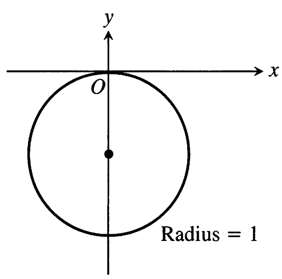
Circles

Find polar equations for the circles in Exercises 13–16.

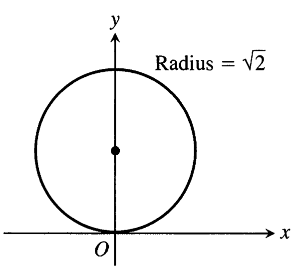
13.



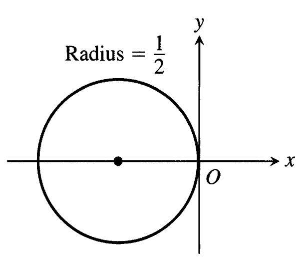
14.



15.



16.



Sketch the circles in Exercises 17–20. Give polar coordinates for their centers and identify their radii.

17. $r = 4 \cos\theta$ **18.** $r = 6 \sin\theta$

19. $r = -2 \cos\theta$ **20.** $r = -8 \sin\theta$

Find polar equations for the circles in Exercises 21–28. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

21. $(x - 6)^2 + y^2 = 36$ **22.** $(x + 2)^2 + y^2 = 4$

23. $x^2 + (y - 5)^2 = 25$ **24.** $x^2 + (y + 7)^2 = 49$

25. $x^2 + 2x + y^2 = 0$ **26.** $x^2 - 16x + y^2 = 0$

27. $x^2 + y^2 + y = 0$ **28.** $x^2 + y^2 - \dfrac{4}{3}y = 0$

Conic Sections from Eccentricities and Directrices

Exercises 29–36 give the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

29. $e = 1, \quad x = 2$ **30.** $e = 1, \quad y = 2$

31. $e = 5, \quad y = -6$ **32.** $e = 2, \quad x = 4$

33. $e = 1/2, \quad x = 1$ **34.** $e = 1/4, \quad x = -2$

35. $e = 1/5, \quad y = -10$ **36.** $e = 1/3, \quad y = 6$

Parabolas and Ellipses

Sketch the parabolas and ellipses in Exercises 37–44. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

37. $r = \dfrac{1}{1 + \cos\theta}$ **38.** $r = \dfrac{6}{2 + \cos\theta}$

39. $r = \dfrac{25}{10 - 5\cos\theta}$ **40.** $r = \dfrac{4}{2 - 2\cos\theta}$

41. $r = \dfrac{400}{16 + 8\sin\theta}$ **42.** $r = \dfrac{12}{3 + 3\sin\theta}$

43. $r = \dfrac{8}{2 - 2\sin\theta}$ **44.** $r = \dfrac{4}{2 - \sin\theta}$

Graphing Inequalities

Sketch the regions defined by the inequalities in Exercises 45 and 46.

45. $0 \le r \le 2\cos\theta$ **46.** $-3\cos\theta \le r \le 0$

Grapher Explorations

Graph the lines and conic sections in Exercises 47–56.

47. $r = 3\sec(\theta - \pi/3)$ **48.** $r = 4\sec(\theta + \pi/6)$

49. $r = 4\sin\theta$ **50.** $r = -2\cos\theta$

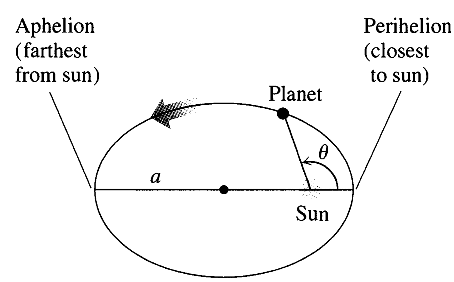
51. $r = 8/(4 + \cos\theta)$ **52.** $r = 8/(4 + \sin\theta)$

53. $r = 1/(1 - \sin\theta)$ **54.** $r = 1/(1 + \cos\theta)$

55. $r = 1/(1 + 2\sin\theta)$ **56.** $r = 1/(1 + 2\cos\theta)$

Theory and Examples

57. *Perihelion and aphelion.* A planet travels about its sun in an ellipse whose semimajor axis has length a.



a) Show that $r = a(1 - e)$ when the planet is closest to the sun and that $r = a(1 + e)$ when the planet is farthest from the sun.

b) Use the data in the table below to find how close each planet in our solar system comes to the sun and how far away each planet gets from the sun.

58. *Planetary orbits.* In Example 6, we found a polar equation for the orbit of Pluto. Use the data in the table below to find polar equations for the orbits of the other planets.

Planet	Semimajor axis (astronomical units)	Eccentricity
Mercury	0.3871	0.2056
Venus	0.7233	0.0068
Earth	1.000	0.0167
Mars	1.524	0.0934
Jupiter	5.203	0.0484
Saturn	9.539	0.0543
Uranus	19.18	0.0460
Neptune	30.06	0.0082
Pluto	39.44	0.2481

59. a) Find Cartesian equations for the curves $r = 4 \sin \theta$ and $r = \sqrt{3} \sec \theta$.

b) Sketch the curves together and label their points of intersection in both Cartesian and polar coordinates.

60. Repeat Exercise 59 for $r = 8 \cos \theta$ and $r = 2 \sec \theta$.

61. Find a polar equation for the parabola with focus (0, 0) and directrix $r \cos \theta = 4$.

62. Find a polar equation for the parabola with focus $(0, 0)$ and directrix $r \cos(\theta - \pi/2) = 2$.

63. a) *The space engineer's formula for eccentricity.* The space engineer's formula for the eccentricity of an elliptical orbit is

$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}},$$

where r is the distance from the space vehicle to the attracting focus of the ellipse along which it travels. Why does the formula work?

b) *Drawing ellipses with string.* You have a string with a knot in each end that can be pinned to a drawing board. The string is 10 in. long from the center of one knot to the center of the other. How far apart should the pins be to use the method illustrated in Fig. 9.5 (Section 9.1) to draw an ellipse of eccentricity 0.2? The resulting ellipse would resemble the orbit of Mercury.

64. *Halley's comet* (See Section 9.2, Example 1.)

a) Write an equation for the orbit of Halley's comet in a coordinate system in which the sun lies at the origin and the other focus lies on the negative x-axis, scaled in astronomical units.

b) How close does the comet come to the sun in astronomical units? in kilometers?

c) What is the farthest the comet gets from the sun in astronomical units? in kilometers?

In Exercises 65–68, find a polar equation for the given curve. In each case, sketch a typical curve.

65. $x^2 + y^2 - 2ay = 0$

66. $y^2 = 4ax + 4a^2$

67. $x \cos \alpha + y \sin \alpha = p \quad (\alpha, p \text{ constant})$

68. $(x^2 + y^2)^2 + 2ax(x^2 + y^2) - a^2y^2 = 0$

CAS Explorations and Projects

69. Use a CAS to plot the polar equation

$$r = \frac{ke}{1 + e \cos \theta}$$

for various values of k and e, $-\pi \le \theta \le \pi$. Answer the following questions.

a) Take $k = -2$. Describe what happens to the plots as you take e to be 3/4, 1, and 5/4. Repeat for $k = 2$.

b) Take $k = -1$. Describe what happens to the plots as you take e to be 7/6, 5/4, 4/3, 3/2, 2, 3, 5, 10, and 20. Repeat for $e =$ 1/2, 1/3, 1/4, 1/10, and 1/20.

c) Now keep $e > 0$ fixed and describe what happens as you take k to be $-1, -2, -3, -4$, and -5. Be sure to look at graphs for parabolas, ellipses, and hyperbolas.

70. Use a CAS to plot the polar ellipse

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

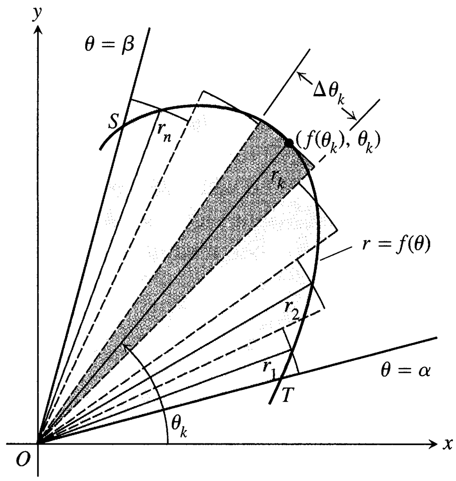
for various values of $a > 0$ and $0 < e < 1$, $-\pi \le \theta \le \pi$.

a) Take $e = 9/10$. Describe what happens to the plots as you let a equal 1, 3/2, 2, 3, 5, and 10. Repeat with $e = 1/4$.

b) Take $a = 2$. Describe what happens as you take e to be 9/10, 8/10, 7/10, ..., 1/10, 1/20, and 1/50.

9.9 Integration in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.



9.65 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

Area in the Plane

The region OTS in Fig. 9.65 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2}\big(f(\theta_k)\big)^2 \, \Delta\theta_k.$$

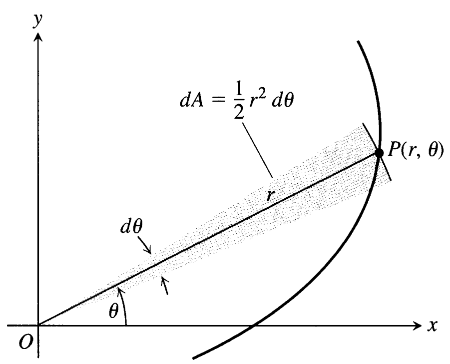
The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} \big(f(\theta_k)\big)^2 \, \Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as $\| P \| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2} \big(f(\theta_k)\big)^2 \, \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)\big)^2 \, d\theta.$$



9.66 The area differential *dA*.

Area of the Fan-shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Fig. 9.66)

$$dA = \frac{1}{2} r^2 \, d\theta.$$

EXAMPLE 1 Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Fig. 9.67) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

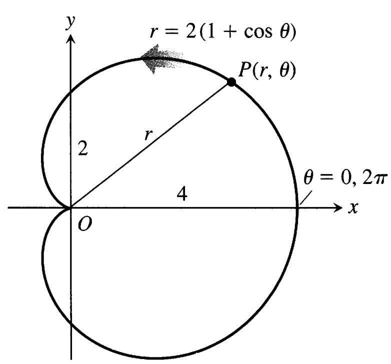
$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

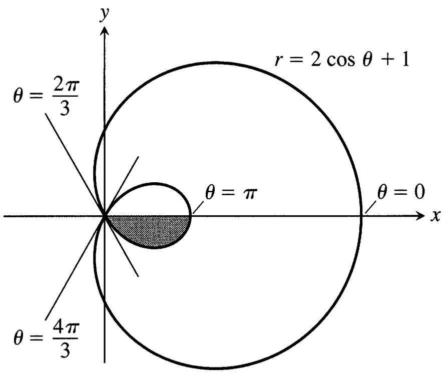
$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi. \qquad \square$$



9.67 The cardioid in Example 1.



9.68 The limaçon in Example 2.

EXAMPLE 2 Find the area inside the smaller loop of the limaçon

$$r = 2 \cos \theta + 1.$$

Solution After sketching the curve (Fig. 9.68), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2 \int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta.$$

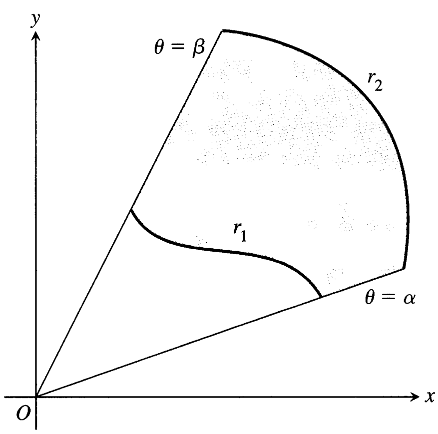
Since

$$\begin{aligned} r^2 &= (2 \cos \theta + 1)^2 = 4 \cos^2 \theta + 4 \cos \theta + 1 \\ &= 4 \cdot \frac{1 + \cos 2\theta}{2} + 4 \cos \theta + 1 \\ &= 2 + 2 \cos 2\theta + 4 \cos \theta + 1 \\ &= 3 + 2 \cos 2\theta + 4 \cos \theta, \end{aligned}$$

we have

$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2 \cos 2\theta + 4 \cos \theta) \, d\theta \\ &= \Big[3\theta + \sin 2\theta + 4 \sin \theta \Big]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4 \cdot \frac{\sqrt{3}}{2} \right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

❑



9.69 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

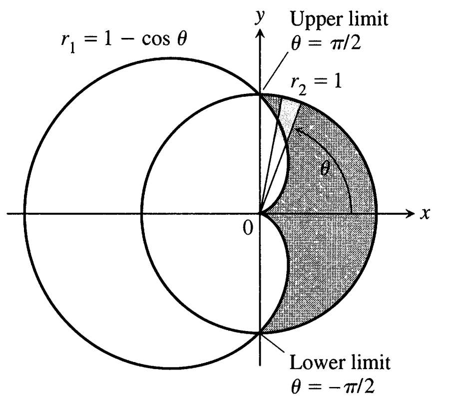
To find the area of a region like the one in Fig. 9.69, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2){r_1}^2 \, d\theta$ from the integral of $(1/2){r_2}^2 \, d\theta$. This leads to the following formula.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \qquad (1)$$

EXAMPLE 3 Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos \theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Fig. 9.70). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos \theta$,



9.70 The region and limits of integration in Example 3.

and θ runs from $-\pi/2$ to $\pi/2$. The area, from Eq. (1), is

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta \\ &= 2\int_0^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta \qquad \text{Symmetry} \\ &= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\,d\theta \\ &= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\,d\theta = \int_0^{\pi/2} \left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$

❑

The Length of a Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \quad (2)$$

The parametric length formula, Eq. (3) from Section 9.5, then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

This equation becomes

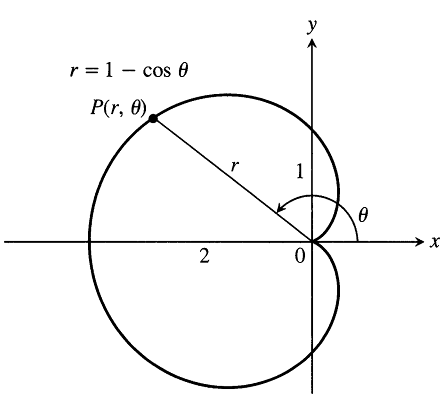
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

when Eqs. (2) are substituted for x and y (Exercise 33).

Length of a Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta. \quad (3)$$



9.71 Example 4 calculates the length of this cardioid.

EXAMPLE 4 Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration (Fig. 9.71). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8. \end{aligned}$$

❑

The Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with Eqs. (2) and apply the surface area equations in Section 9.5.

Area of a Surface of Revolution

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

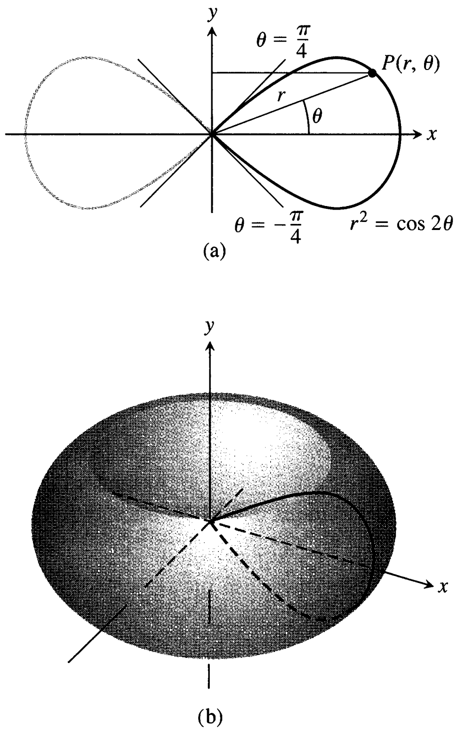
1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_\alpha^\beta 2\pi r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \textbf{(4)}$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \textbf{(5)}$$

EXAMPLE 5 Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.



9.72 The right-hand half of a lemniscate (a) is revolved about the y-axis to generate a surface (b), whose area is calculated in Example 5.

Solution We sketch the loop to determine the limits of integration (Fig. 9.72). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Eq. (4) in stages. First,

$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}. \tag{6}$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$

Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side of Eq. (6) simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

All together, we have

$$\begin{aligned} S &= \int_\alpha^\beta 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \text{Eq. (4)} \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos\theta \cdot (1)\, d\theta \\ &= 2\pi \Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

❑

Exercises 9.9

Areas Inside Polar Curves

Find the areas of the regions in Exercises 1–6.

1. Inside the oval limaçon $r = 4 + 2\cos\theta$
2. Inside the cardioid $r = a(1 + \cos\theta)$, $a > 0$
3. Inside one leaf of the four-leaved rose $r = \cos 2\theta$
4. Inside the lemniscate $r^2 = 2a^2 \cos 2\theta$, $a > 0$
5. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$
6. Inside the six-leaved rose $r^2 = 2\sin 3\theta$

Areas Shared by Polar Regions

Find the areas of the regions in Exercises 7–16.

7. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
8. Shared by the circles $r = 1$ and $r = 2\sin\theta$
9. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$
10. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$
11. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$

12. Inside the circle $r = 3a \cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$, $a > 0$

13. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$

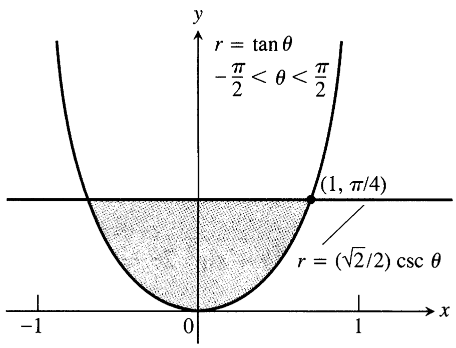
14. a) Inside the outer loop of the limaçon $r = 2\cos\theta + 1$ (See Fig. 9.68.)

b) Inside the outer loop and outside the inner loop of the limaçon $r = 2\cos\theta + 1$

15. Inside the circle $r = 6$ above the line $r = 3\csc\theta$

16. Inside the lemniscate $r^2 = 6\cos 2\theta$ to the right of the line $r = (3/2)\sec\theta$

17. a) Find the area of the shaded region in Fig. 9.73.



9.73 The region in Exercise 17.

b) It looks as if the graph of $r = \tan\theta$, $-\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.

18. The area of the region that lies inside the cardioid curve $r = \cos\theta + 1$ and outside the circle $r = \cos\theta$ is not

$$\frac{1}{2}\int_0^{2\pi} [(\cos\theta + 1)^2 - \cos^2\theta]\, d\theta = \pi.$$

Why not? What *is* the area? Give reasons for your answers.

Lengths of Polar Curves

Find the lengths of the curves in Exercises 19–27.

19. The spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

20. The spiral $r = e^\theta/\sqrt{2}$, $0 \le \theta \le \pi$

21. The cardioid $r = 1 + \cos\theta$

22. The curve $r = a\sin^2(\theta/2)$, $0 \le \theta \le \pi$, $a > 0$

23. The parabolic segment $r = 6/(1 + \cos\theta)$, $0 \le \theta \le \pi/2$

24. The parabolic segment $r = 2/(1 - \cos\theta)$, $\pi/2 \le \theta \le \pi$

25. The curve $r = \cos^3(\theta/3)$, $0 \le \theta \le \pi/4$

26. The curve $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

27. The curve $r = \sqrt{1 + \cos 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

28. *Circumferences of circles.* As usual, when faced with a new formula, it is a good idea to try it on familiar objects to be sure it gives results consistent with past experience. Use the length formula in Eq. (3) to calculate the circumferences of the following circles ($a > 0$):

a) $r = a$ **b)** $r = a\cos\theta$ **c)** $r = a\sin\theta$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 29–32 about the indicated axes.

29. $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \pi/4$, y-axis

30. $r = \sqrt{2}e^{\theta/2}$, $0 \le \theta \le \pi/2$, x-axis

31. $r^2 = \cos 2\theta$, x-axis

32. $r = 2a\cos\theta, a > 0$, y-axis

Theory and Examples

33. *The length of the curve $r = f(\theta), \alpha \le \theta \le \beta$.* Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

(Eqs. 2 in the text) transform

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

into

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

34. *Average value.* If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta), \alpha \le \theta \le \beta$, with respect to θ is given by the formula

$$r_{\text{av}} = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(\theta)\, d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves ($a > 0$).

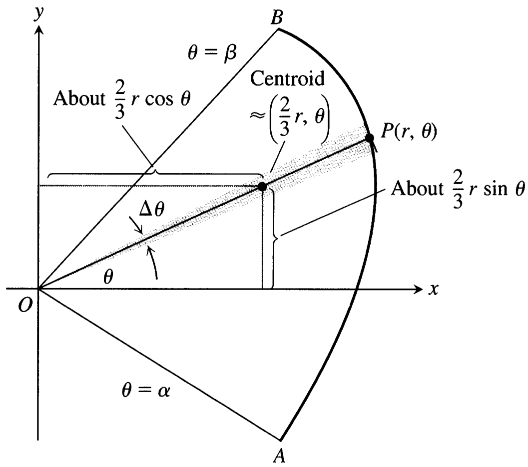
a) The cardioid $r = a(1 - \cos\theta)$
b) The circle $r = a$
c) The circle $r = a\cos\theta$, $-\pi/2 \le \theta \le \pi/2$

35. *$r = f(\theta)$ vs. $r = 2f(\theta)$.* Can anything be said about the relative lengths of the curves $r = f(\theta), \alpha \le \theta \le \beta$, and $r = 2f(\theta), \alpha \le \theta \le \beta$? Give reasons for your answer.

36. *$r = f(\theta)$ vs. $r = 2f(\theta)$.* The curves $r = f(\theta), \alpha \le \theta \le \beta$, and $r = 2f(\theta), \alpha \le \theta \le \beta$, are revolved about the x-axis to generate surfaces. Can anything be said about the relative areas of these surfaces? Give reasons for your answer.

Centroids of Fan-Shaped Regions

Since the centroid of a triangle is located on each median, two-thirds of the way from the vertex to the opposite base, the lever arm for the moment about the x-axis of the thin triangular region in Fig. 9.74 is about $(2/3)r\sin\theta$. Similarly, the lever arm for the moment of the triangular region about the y-axis is about $(2/3)r\cos\theta$. These approximations improve as $\Delta\theta \to 0$ and lead to the following formulas



9.74 The moment of the thin triangular sector about the x-axis is approximately $\frac{2}{3}r\sin\theta\,dA = \frac{2}{3}r\sin\theta \cdot \frac{1}{2}r^2 d\theta = \frac{1}{3}r^3\sin\theta\,d\theta$.

for the coordinates of the centroid of region AOB:

$$\bar{x} = \frac{\int \frac{2}{3}r\cos\theta \cdot \frac{1}{2}r^2\,d\theta}{\int \frac{1}{2}r^2\,d\theta} = \frac{\frac{2}{3}\int r^3\cos\theta\,d\theta}{\int r^2\,d\theta},$$

$$\bar{y} = \frac{\int \frac{2}{3}r\sin\theta \cdot \frac{1}{2}r^2\,d\theta}{\int \frac{1}{2}r^2\,d\theta} = \frac{\frac{2}{3}\int r^3\sin\theta\,d\theta}{\int r^2\,d\theta},$$

with limits $\theta = \alpha$ to $\theta = \beta$ on all integrals.

37. Find the centroid of the region enclosed by the cardioid $r = a(1 + \cos\theta)$.

38. Find the centroid of the semicircular region $0 \le r \le a$, $0 \le \theta \le \pi$.

CHAPTER 9 QUESTIONS TO GUIDE YOUR REVIEW

1. What is a parabola? What are the Cartesian equations for parabolas whose vertices lie at the origin and whose foci lie on the coordinate axes? How can you find the focus and directrix of such a parabola from its equation?
2. What is an ellipse? What are the Cartesian equations for ellipses centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
3. What is a hyperbola? What are the Cartesian equations for hyperbolas centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
4. What is the eccentricity of a conic section? How can you classify conic sections by eccentricity? How are an ellipse's shape and eccentricity related?
5. Explain the equation $PF = e \cdot PD$.
6. What is a quadratic curve in the xy-plane? Give examples of degenerate and nondegenerate quadratic curves.
7. How can you find a Cartesian coordinate system in which the new equation for a conic section in the plane has no xy-term? Give an example.
8. How can you tell what kind of graph to expect from a quadratic equation in x and y? Give examples.
9. What is a parametrized curve in the xy-plane? If you find a Cartesian equation for the path of a particle whose motion in the plane is described parametrically, what kind of match can you expect between the Cartesian equation's graph and the path of motion? Give examples.
10. What are some typical parametrizations for conic sections?
11. What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids?
12. What is the formula for the slope dy/dx of a parametrized curve $x = f(t)$, $y = g(t)$? When does the formula apply? When can you expect to be able to find d^2y/dx^2 as well? Give examples.
13. How do you find the length of a smooth parametrized curve $x = f(t)$, $y = g(t)$, $a \le t \le b$? What does smoothness have to do with length? What else do you need to know about the parametrization in order to find the curve's length? Give examples.
14. Under what conditions can you find the area of the surface generated by revolving a curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, about the x-axis? the y-axis? Give examples.
15. How do you find the centroid of a smooth parametrized curve $x = f(t)$, $y = g(t)$, $a \le t \le b$? Give an example.

16. What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?

17. What consequence does the lack of uniqueness of polar coordinates have for graphing? Give an example.

18. How do you graph equations in polar coordinates? Include in your discussion symmetry, slope, behavior at the origin, and the use of Cartesian graphs. Give examples.

19. What are the standard equations for lines and conic sections in polar coordinates? Give examples.

20. How do you find the area of a region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give examples.

21. Under what conditions can you find the length of a curve $r = f(\theta), \alpha \le \theta \le \beta$, in the polar coordinate plane? Give an example of a typical calculation.

22. Under what conditions can you find the area of the surface generated by revolving a curve $r = f(\theta), \alpha \le \theta \le \beta$, about the x-axis? the y-axis? Give examples of typical calculations.

CHAPTER 9 PRACTICE EXERCISES

Graphing Conic Sections

Sketch the parabolas in Exercises 1–4. Include the focus and directrix in each sketch.

1. $x^2 = -4y$
2. $x^2 = 2y$
3. $y^2 = 3x$
4. $y^2 = -(8/3)x$

Find the eccentricities of the ellipses and hyperbolas in Exercises 5–8. Sketch each conic section. Include the foci, vertices, and asymptotes (as appropriate) in your sketch.

5. $16x^2 + 7y^2 = 112$
6. $x^2 + 2y^2 = 4$
7. $3x^2 - y^2 = 3$
8. $5y^2 - 4x^2 = 20$

Shifting Conic Sections

Exercises 9–14 give equations for conic sections and tell how many units up or down and to the right or left each curve is to be shifted. Find an equation for the new conic section and find the new foci, vertices, centers, and asymptotes, as appropriate. If the curve is a parabola, find the new directrix as well.

9. $x^2 = -12y$, right 2, up 3
10. $y^2 = 10x$, left 1/2, down 1
11. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$, left 3, down 5
12. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$, right 5, up 12
13. $\dfrac{y^2}{8} - \dfrac{x^2}{2} = 1$, right 2, up $2\sqrt{2}$
14. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$, left 10, down 3

Identifying Conic Sections

Identify the conic sections in Exercises 15–22 and find their foci, vertices, centers, and asymptotes (as appropriate). If the curve is a parabola, find its directrix as well.

15. $x^2 - 4x - 4y^2 = 0$
16. $4x^2 - y^2 + 4y = 8$
17. $y^2 - 2y + 16x = -49$
18. $x^2 - 2x + 8y = -17$
19. $9x^2 + 16y^2 + 54x - 64y = -1$
20. $25x^2 + 9y^2 - 100x + 54y = 44$
21. $x^2 + y^2 - 2x - 2y = 0$
22. $x^2 + y^2 + 4x + 2y = 1$

Using the Discriminant

What conic sections or degenerate cases do the equations in Exercises 23–28 represent? Give a reason for your answer in each case.

23. $x^2 + xy + y^2 + x + y + 1 = 0$
24. $x^2 + 4xy + 4y^2 + x + y + 1 = 0$
25. $x^2 + 3xy + 2y^2 + x + y + 1 = 0$
26. $x^2 + 2xy - 2y^2 + x + y + 1 = 0$
27. $x^2 - 2xy + y^2 = 0$
28. $x^2 - 3xy + 4y^2 = 0$

Rotating Conic Sections

Identify the conic sections in Exercises 29–32. Then rotate the coordinate axes to find a new equation for the conic section that has no cross product term. (The new equations will vary with the size and direction of the rotations used.)

29. $2x^2 + xy + 2y^2 - 15 = 0$
30. $3x^2 + 2xy + 3y^2 = 19$
31. $x^2 + 2\sqrt{3}xy - y^2 + 4 = 0$
32. $x^2 - 3xy + y^2 = 5$

Identifying Parametric Equations in the Plane

Exercises 33–38 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion and the portion traced by the particle.

33. $x = t/2, \quad y = t + 1; \quad -\infty < t < \infty$

34. $x = \sqrt{t}, \quad y = 1 - \sqrt{t}; \quad t \geq 0$

35. $x = (1/2) \tan t, \quad y = (1/2) \sec t; \quad -\pi/2 < t < \pi/2$

36. $x = -2 \cos t, \quad y = 2 \sin t; \quad 0 \leq t \leq \pi$

37. $x = -\cos t, \quad y = \cos^2 t; \quad 0 \leq t \leq \pi$

38. $x = 4 \cos t, \quad y = 9 \sin t; \quad 0 \leq t \leq 2\pi$

Finding Parametric Equations and Tangent Lines

39. Find parametric equations and a parameter interval for the motion of a particle in the xy-plane that traces the ellipse $16x^2 + 9y^2 = 144$ once counterclockwise. (There are many ways to do this, so your answer may not be the same as the one in the back of the book.)

40. Find parametric equations and a parameter interval for the motion of a particle that starts at the point $(-2, 0)$ in the xy-plane and traces the circle $x^2 + y^2 = 4$ three times clockwise. (There are many ways to do this.)

In Exercises 41 and 42, find an equation for the line in the xy-plane that is tangent to the curve at the point corresponding to the given value of t. Also, find the value of d^2y/dx^2 at this point.

41. $x = (1/2) \tan t, \quad y = (1/2) \sec t; \quad t = \pi/3$

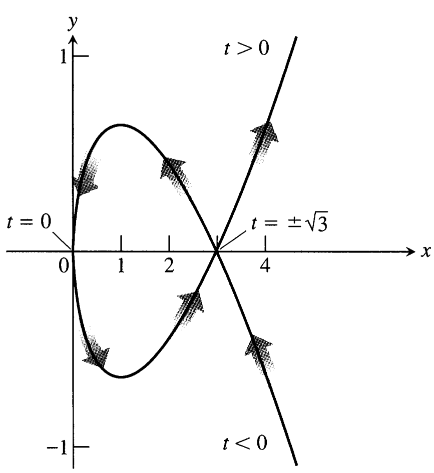
42. $x = 1 + 1/t^2, \quad y = 1 - 3/t; \quad t = 2$

Lengths of Parametrized Curves

Find the lengths of the curves in Exercises 43 and 44.

43. $x = e^{2t} - \dfrac{t}{8}, \quad y = e^t; \quad 0 \leq t \leq \ln 2$

44. The enclosed loop in Fig. 9.75.



9.75 Exercise 44 refers to the curve $x = t^2$, $y = (t^3/3) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.

Surface Areas

Find the areas of the surfaces generated by revolving the curves in Exercises 45 and 46 about the indicated axes.

45. $x = t^2/2, \quad y = 2t, \quad 0 \leq t \leq \sqrt{5}; \quad x\text{-axis}$

46. $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \leq t \leq 1; \quad y\text{-axis}$

Graphs in the Polar Plane

Sketch the regions defined by the polar coordinate inequalities in Exercises 47 and 48.

47. $0 \leq r \leq 6 \cos \theta$

48. $-4 \sin \theta \leq r \leq 0$

Match each graph in Exercises 49–56 with the appropriate equation (a)–(l). There are more equations than graphs, so some equations will not be matched.

a) $r = \cos 2\theta$

b) $r \cos \theta = 1$

c) $r = \dfrac{6}{1 - 2\cos\theta}$

d) $r = \sin 2\theta$

e) $r = \theta$

f) $r^2 = \cos 2\theta$

g) $r = 1 + \cos \theta$

h) $r = 1 - \sin \theta$

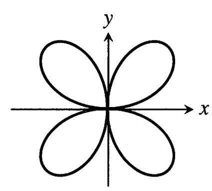
i) $r = \dfrac{2}{1 - \cos\theta}$

j) $r^2 = \sin 2\theta$

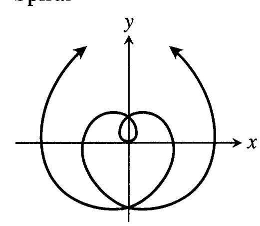
k) $r = -\sin \theta$

l) $r = 2\cos\theta + 1$

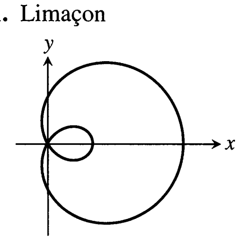
49. Four-leaved rose



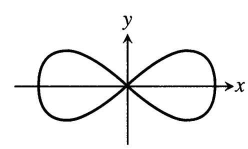
50. Spiral



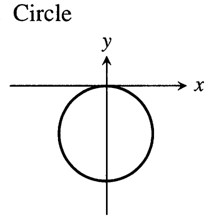
51. Limaçon



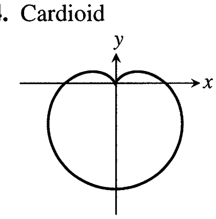
52. Lemniscate



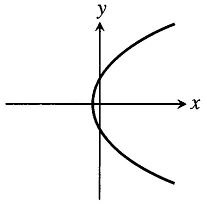
53. Circle



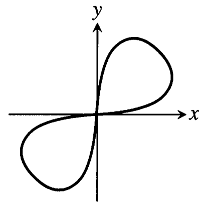
54. Cardioid



55. Parabola



56. Lemniscate



Intersections of Graphs in the Polar Plane

Find the points of intersection of the curves given by the polar coordinate equations in Exercises 57–64.

57. $r = \sin\theta, \quad r = 1 + \sin\theta$

58. $r = \cos\theta, \quad r = 1 - \cos\theta$

59. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

60. $r = 1 + \sin\theta, \quad r = 1 - \sin\theta$

61. $r = 1 + \sin\theta, \quad r = -1 + \sin\theta$

62. $r = 1 + \cos\theta, \quad r = -1 + \cos\theta$

63. $r = \sec\theta, \quad r = 2\sin\theta$

64. $r = -2\csc\theta, \quad r = -4\cos\theta$

Tangent Lines in the Polar Plane

In Exercises 65 and 66, find equations for the lines that are tangent to the polar coordinate curves at the origin.

65. The lemniscate $r^2 = \cos 2\theta$

66. The limaçon $r = 2\cos\theta + 1$

67. Find polar coordinate equations for the lines that are tangent to the tips of the petals of the four-leaved rose $r = \sin 2\theta$.

68. Find polar coordinate equations for the lines that are tangent to the cardioid $r = 1 + \sin\theta$ at the points where it crosses the x-axis.

Polar to Cartesian Equations

Sketch the lines in Exercises 69–74. Also, find a Cartesian equation for each line.

69. $r\cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3}$

70. $r\cos\left(\theta - \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

71. $r = 2\sec\theta$

72. $r = -\sqrt{2}\sec\theta$

73. $r = -(3/2)\csc\theta$

74. $r = \left(3\sqrt{3}\right)\csc\theta$

Find Cartesian equations for the circles in Exercises 75–78. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

75. $r = -4\sin\theta$

76. $r = 3\sqrt{3}\sin\theta$

77. $r = 2\sqrt{2}\cos\theta$

78. $r = -6\cos\theta$

Cartesian to Polar Equations

Find polar equations for the circles in Exercises 79–82. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

79. $x^2 + y^2 + 5y = 0$

80. $x^2 + y^2 - 2y = 0$

81. $x^2 + y^2 - 3x = 0$

82. $x^2 + y^2 + 4x = 0$

Conic Sections in Polar Coordinates

Sketch the conic sections whose polar coordinate equations are given in Exercises 83–86. Give polar coordinates for the vertices and, in the case of ellipses, for the centers as well.

83. $r = \dfrac{2}{1 + \cos\theta}$

84. $r = \dfrac{8}{2 + \cos\theta}$

85. $r = \dfrac{6}{1 - 2\cos\theta}$

86. $r = \dfrac{12}{3 + \sin\theta}$

Exercises 87–90 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

87. $e = 2, \quad r\cos\theta = 2$

88. $e = 1, \quad r\cos\theta = -4$

89. $e = 1/2, \quad r\sin\theta = 2$

90. $e = 1/3, \quad r\sin\theta = -6$

Area, Length, and Surface Area in the Polar Plane

Find the areas of the regions in the polar coordinate plane described in Exercises 91–94.

91. Enclosed by the limaçon $r = 2 - \cos\theta$

92. Enclosed by one leaf of the three-leaved rose $r = \sin 3\theta$

93. Inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

94. Inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$

Find the lengths of the curves given by the polar coordinate equations in Exercises 95–98.

95. $r = -1 + \cos\theta$

96. $r = 2\sin\theta + 2\cos\theta, \quad 0 \le \theta \le \pi/2$

97. $r = 8 \sin^3 (\theta/3), \quad 0 \le \theta \le \pi/4$

98. $r = \sqrt{1 + \cos 2\theta}, \quad -\pi/2 \le \theta \le \pi/2$

Find the areas of the surfaces generated by revolving the polar coordinate curves in Exercises 99 and 100 about the indicated axes.

99. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad x\text{-axis}$

100. $r^2 = \sin 2\theta, \quad y\text{-axis}$

Theory and Examples

101. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about (a) the x-axis, (b) the y-axis.

102. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

103. A ripple tank is made by bending a strip of tin around the perimeter of an ellipse for the wall of the tank and soldering a flat bottom onto this. An inch or two of water is put in the tank and you drop a marble into it, right at one focus of the ellipse. Ripples radiate outward through the water, reflect from the strip around the edge of the tank, and a few seconds later a drop of water spurts up at the second focus. Why?

104. *LORAN.* A radio signal was sent simultaneously from towers A and B, located several hundred miles apart on the northern California coast. A ship offshore received the signal from A 1400 microseconds before receiving the signal from B. Assuming that the signals traveled at the rate of 980 ft/microsecond, what can be said about the location of the ship relative to the two towers?

105. On a level plane, at the same instant, you hear the sound of a rifle and that of the bullet hitting the target. What can be said about your location relative to the rifle and target?

106. *Archimedes spirals.* The graph of an equation of the form $r = a\theta$, where a is a nonzero constant, is called an **Archimedes spiral.** Is there anything special about the widths between the successive turns of such a spiral?

107. a) Show that the equations $x = r \cos \theta$, $y = r \sin \theta$ transform the polar equation

$$r = \frac{k}{1 + e \cos \theta}$$

into the Cartesian equation

$$(1 - e^2)\, x^2 + y^2 + 2kex - k^2 = 0.$$

b) Then apply the criteria of Section 9.3 to show that

$$\begin{aligned} e = 0 \quad &\Rightarrow \quad \text{circle} \\ 0 < e < 1 \quad &\Rightarrow \quad \text{ellipse} \\ e = 1 \quad &\Rightarrow \quad \text{parabola} \\ e > 1 \quad &\Rightarrow \quad \text{hyperbola.} \end{aligned}$$

108. *A satellite orbit.* A satellite is in an orbit that passes over the North and South Poles of the earth. When it is over the South Pole it is at the highest point of its orbit, 1000 miles above the earth's surface. Above the North Pole it is at the lowest point of its orbit, 300 miles above the earth's surface.

a) Assuming that the orbit is an ellipse with one focus at the center of the earth, find its eccentricity. (Take the diameter of the earth to be 8000 miles.)

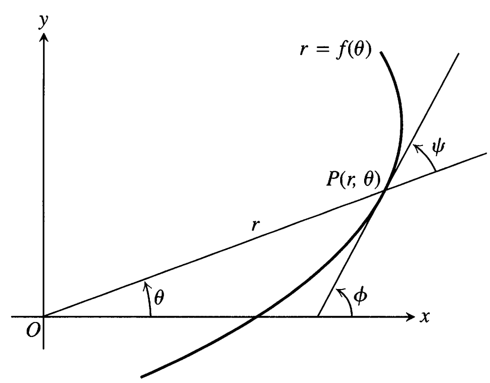
b) Using the north–south axis of the earth as the x-axis and the center of the earth as origin, find a polar equation for the orbit.

The Angle Between the Radius Vector and the Tangent Line to a Polar Coordinate Curve

In Cartesian coordinates, when we want to discuss the direction of a curve at a point, we use the angle ϕ measured counterclockwise from the positive x-axis to the tangent line. In polar coordinates, it is more convenient to calculate the angle ψ from the *radius vector* to the tangent line (Fig. 9.76). The angle ϕ can then be calculated from the relation

$$\phi = \theta + \psi, \tag{1}$$

which comes from applying the exterior angle theorem to the triangle in Fig. 9.76.



9.76 The angle ψ between the tangent line and the radius vector.

Suppose the equation of the curve is given in the form $r = f(\theta)$, where $f(\theta)$ is a differentiable function of θ. Then

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \tag{2}$$

are differentiable functions of θ with

$$\begin{aligned} \frac{dx}{d\theta} &= -r \sin \theta + \cos \theta \frac{dr}{d\theta}, \\ \frac{dy}{d\theta} &= r \cos \theta + \sin \theta \frac{dr}{d\theta}. \end{aligned} \tag{3}$$

Since $\psi = \phi - \theta$ from (1),

$$\tan \psi = \tan (\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta}.$$

Furthermore,

$$\tan\phi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

because $\tan\phi$ is the slope of the curve at P. Also,

$$\tan\theta = \frac{y}{x}.$$

Hence

$$\tan\psi = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \dfrac{y}{x}}{1 + \dfrac{y}{x}\dfrac{dy/d\theta}{dx/d\theta}} = \frac{x\dfrac{dy}{d\theta} - y\dfrac{dx}{d\theta}}{x\dfrac{dx}{d\theta} + y\dfrac{dy}{d\theta}}. \tag{4}$$

The numerator in the last expression in Eq. (4) is found from Eqs. (2) and (3) to be

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = r^2.$$

Similarily, the denominator is

$$x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = r\frac{dr}{d\theta}.$$

When we substitute these into Eq. (4), we obtain

$$\tan\psi = \frac{r}{dr/d\theta}. \tag{5}$$

This is the equation we use for finding ψ as a function of θ.

109. Show, by reference to a figure, that the angle β between the tangents to two curves at a point of intersection may be found from the formula

$$\tan\beta = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2 \tan\psi_1}. \tag{6}$$

When will the two curves intersect at right angles?

110. Find the value of $\tan\psi$ for the curve $r = \sin^4(\theta/4)$.

111. Find the angle between the radius vector to the curve $r = 2a\sin 3\theta$ and its tangent when $\theta = \pi/6$.

112. **a)** GRAPHER Graph the hyperbolic spiral $r\theta = 1$. What appears to happen to ψ as the spiral winds in around the origin?

b) Confirm your finding in (a) analytically.

113. The circles $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$ intersect at the point $(\sqrt{3}/2, \pi/3)$. Show that their tangents are perpendicular there.

114. Sketch the cardioid $r = a(1 + \cos\theta)$ and circle $r = 3a\cos\theta$ in one diagram and find the angle between their tangents at the point of intersection that lies in the first quadrant.

115. Find the points of intersection of the parabolas

$$r = \frac{1}{1 - \cos\theta} \quad \text{and} \quad r = \frac{3}{1 + \cos\theta}$$

and the angles between their tangents at these points.

116. Find points on the cardioid $r = a(1 + \cos\theta)$ where the tangent line is (a) horizontal, (b) vertical.

117. Show that parabolas $r = a/(1 + \cos\theta)$ and $r = b/(1 - \cos\theta)$ are orthogonal at each point of intersection ($ab \neq 0$).

118. Find the angle at which the cardioid $r = a(1 - \cos\theta)$ crosses the ray $\theta = \pi/2$.

119. Find the angle between the line $r = 3\sec\theta$ and the cardioid $r = 4(1 + \cos\theta)$ at one of their intersections.

120. Find the slope of the tangent line to the curve $r = a\tan(\theta/2)$ at $\theta = \pi/2$.

121. Find the angle at which the parabolas $r = 1/(1 - \cos\theta)$ and $r = 1/(1 - \sin\theta)$ intersect in the first quadrant.

122. The equation $r^2 = 2\csc 2\theta$ represents a curve in polar coordinates.

a) Sketch the curve.
b) Find an equivalent Cartesian equation for the curve.
c) Find the angle at which the curve intersects the ray $\theta = \pi/4$.

123. Suppose that the angle ψ from the radius vector to the tangent line of the curve $r = f(\theta)$ has the constant value α.

a) Show that the area bounded by the curve and two rays $\theta = \theta_1, \theta = \theta_2$, is proportional to $r_2^2 - r_1^2$, where (r_1, θ_1) and (r_2, θ_2) are polar coordinates of the ends of the arc of the curve between these rays. Find the factor of proportionality.
b) Show that the length of the arc of the curve in part (a) is proportional to $r_2 - r_1$, and find the proportionality constant.

124. Let P be a point on the hyperbola $r^2\sin 2\theta = 2a^2$. Show that the triangle formed by OP, the tangent at P, and the initial line is isosceles.

Chapter 9 Additional Exercises–Theory, Examples, Applications

Finding Conic Sections

1. Find an equation for the parabola with focus $(4, 0)$ and directrix $x = 3$. Sketch the parabola together with its vertex, focus, and directrix.

2. Find the vertex, focus, and directrix of the parabola
$$x^2 - 6x - 12y + 9 = 0.$$

3. Find an equation for the curve traced by the point $P(x, y)$ if the distance from P to the vertex of the parabola $x^2 = 4y$ is twice the distance from P to the focus. Identify the curve.

4. A line segment of length $a + b$ runs from the x-axis to the y-axis. The point P on the segment lies a units from one end and b units from the other end. Show that P traces an ellipse as the ends of the segment slide along the axes.

5. The vertices of an ellipse of eccentricity 0.5 lie at the points $(0, \pm 2)$. Where do the foci lie?

6. Find an equation for the ellipse of eccentricity 2/3 that has the line $x = 2$ as a directrix and the point $(4, 0)$ as the corresponding focus.

7. One focus of a hyperbola lies at the point $(0, -7)$ and the corresponding directrix is the line $y = -1$. Find an equation for the hyperbola if its eccentricity is (a) 2, (b) 5.

8. Find an equation for the hyperbola with foci $(0, -2)$ and $(0, 2)$ that passes through the point $(12, 7)$.

Orthogonal Curves

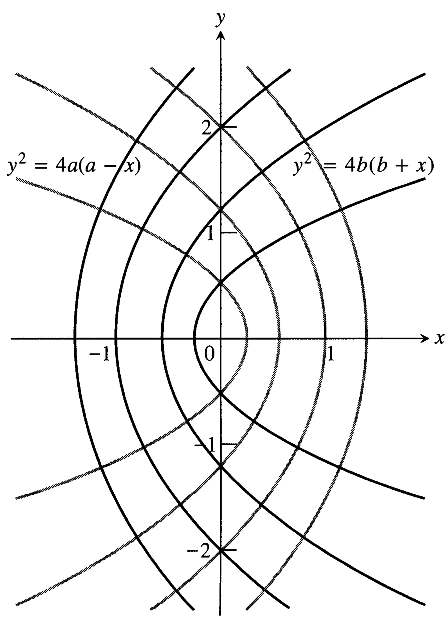
Two curves are said to be **orthogonal** if their tangents cross at right angles at every point where the curves intersect. Exercises 9–12 are about orthogonal conic sections.

9. Sketch the curves $xy = 2$ and $x^2 - y^2 = 3$ together and show that they are orthogonal.

10. Sketch the curves $y^2 = 4x + 4$ and $y^2 = 64 - 16x$ together and show that they are orthogonal.

11. Show that the curves $2x^2 + 3y^2 = a^2$ and $ky^2 = x^3$ are orthogonal for all values of the constants a and k ($a \neq 0, k \neq 0$). Sketch the four curves corresponding to $a = 2, a = 4, k = 1/2, k = -2$ in one diagram.

12. Show that the parabolas
$$y^2 = 4a(a - x), \quad a > 0$$
and
$$y^2 = 4b(b + x), \quad b > 0$$
have a common focus, the same for any a and b. Show that the parabolas intersect at the points $(a - b, \pm 2\sqrt{ab})$ and that each a-parabola is orthogonal to every b-parabola. By varying a and b, we obtain two families of confocal parabolas. Each family is said to be a set of **orthogonal trajectories** of the other family (Fig. 9.77).



9.77 Confocal parabolas in Exercise 12.

Tangents to Conic Sections

13. *Constructing tangents to parabolas.* Show that the tangent to the parabola $y^2 = 4px$ at the point $P(x_1, y_1) \neq (0, 0)$ on the parabola meets the axis of symmetry x_1 units to the left of the vertex. This provides an accurate way to construct a tangent to the parabola at any point other than the origin (where we already have the y-axis): Mark the point $P(x_1, y_1)$ in question, drop a perpendicular from P to the x-axis, measure $2x_1$ units to the left, mark that point, and draw a line from there through P.

14. Show that no tangent can be drawn from the origin to the hyperbola $x^2 - y^2 = 1$. (*Hint:* If the tangent to a curve at a point $P(x, y)$ on the curve passes through the origin, then the slope of the curve at P is y/x.)

15. Show that any tangent to the hyperbola $xy = a^2$ makes a triangle of area $2a^2$ with the hyperbola's asymptotes.

16. a) Show that the line

$$b^2xx_1 + a^2yy_1 - a^2b^2 = 0$$

is tangent to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the ellipse.

b) Show that the line

$$b^2xx_1 - a^2yy_1 - a^2b^2 = 0$$

is tangent to the hyperbola $b^2x^2 - a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the hyperbola.

c) Show that the tangent to the conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

at a point (x_1, y_1) on it has an equation that may be written in the form

$$Axx_1 + B\left(\frac{x_1y + xy_1}{2}\right) + Cyy_1 + D\left(\frac{x + x_1}{2}\right)$$

$$+ E\left(\frac{y + y_1}{2}\right) + F = 0.$$

Equations and Inequalities

What points in the xy-plane satisfy the equations and inequalities in Exercises 17–24? Draw a figure for each exercise.

17. $(x^2 - y^2 - 1)(x^2 + y^2 - 25)(x^2 + 4y^2 - 4) = 0$

18. $(x + y)(x^2 + y^2 - 1) = 0$

19. $(x^2/9) + (y^2/16) \le 1$

20. $(x^2/9) - (y^2/16) \le 1$

21. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$

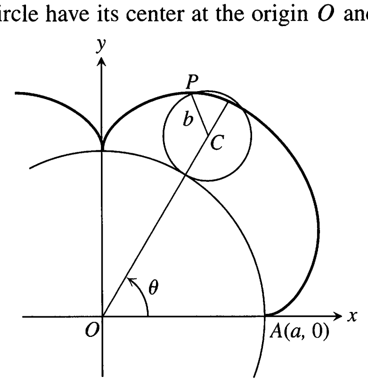
22. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$

23. $x^4 - (y^2 - 9)^2 = 0$

24. $x^2 + xy + y^2 < 3$

Parametric Equations

25. *Epicycloids.* When a circle rolls externally along the circumference of a second, fixed circle, any point P on the circumference of the rolling circle describes an *epicycloid,* as shown here. Let the fixed circle have its center at the origin O and have radius a.



Let the radius of the rolling circle be b and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the epicycloid, using as the parameter the angle θ from the positive x-axis to the line through the circles' centers.

26. Find parametric equations and a Cartesian equation for the curve traced by the point $P(x, y)$ if its coordinates satisfy the differential equations

$$\frac{dx}{dt} = -2y, \quad \frac{dy}{dt} = \cos t,$$

subject to the conditions that $x = 3$ and $y = 0$ when $t = 0$. Identify the curve.

27. *Pythagorean triples.* Suppose that the coordinates of a particle $P(x, y)$ moving in the plane are

$$x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad y = \frac{2t}{1 + t^2}$$

for $-\infty < t < \infty$. Show that $x^2 + y^2 = 1$ and hence that the motion takes place on the unit circle. What one point of the circle is not covered by the motion? Sketch the circle and indicate the direction of motion for increasing t. For what values of t does $(x, y) = (0, -1)$? $(1, 0)$? $(0, 1)$?

From $x^2 + y^2 = 1$, we obtain

$$(t^2 - 1)^2 + (2t)^2 = (t^2 + 1)^2,$$

an equation of interest in number theory because it generates *Pythagorean triples* of integers. When t is an integer greater than 1, $a = t^2 - 1$, $b = 2t$, and $c = t^2 + 1$ are positive integers that satisfy the equation $a^2 + b^2 = c^2$.

28. a) Find the centroid of the region enclosed by the x-axis and the cycloid arch

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad 0 \le t \le 2\pi.$$

b) Find the first moments about the coordinate axes of the curve

$$x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}.$$

Polar Coordinates

29. a) Find an equation in polar coordinates for the curve

$$x = e^{2t}\cos t, \quad y = e^{2t}\sin t, \quad -\infty < t < \infty.$$

b) Find the length of the curve from $t = 0$ to $t = 2\pi$.

30. Find the length of the curve $r = 2\sin^3(\theta/3)$, $0 \le \theta \le 3\pi$, in the polar coordinate plane.

31. Find the area of the surface generated by revolving the first-quadrant portion of the cardioid $r = 1 + \cos\theta$ about the x-axis. (*Hint:* Use the identities $1 + \cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ to simplify the integral.)

32. Sketch the regions enclosed by the curves $r = 2a\cos^2(\theta/2)$ and $r = 2a\sin^2(\theta/2)$, $a > 0$, in the polar coordinate plane and find the area of the portion of the plane they have in common.

Exercises 33–36 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

33. $e = 2, \quad r\cos\theta = 2$

34. $e = 1, \quad r\cos\theta = -4$

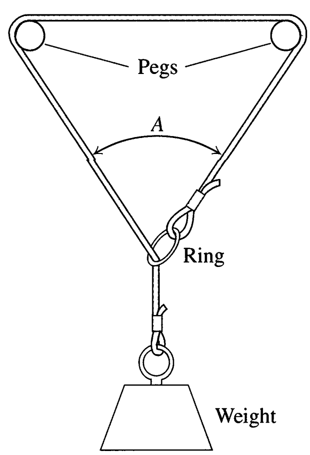
35. $e = 1/2, \quad r\sin\theta = 2$

36. $e = 1/3, \quad r\sin\theta = -6$

Theory and Examples

37. A rope with a ring in one end is looped over two pegs in a horizontal line. The free end, after being passed through the ring, has a weight suspended from it to make the rope hang taut. If the rope slips freely over the pegs and through the ring, the weight will descend as far as possible. Assume that the length of the rope is at least four times as great as the distance between the pegs and that the configuration of the rope is symmetric with respect to the line of the vertical part of the rope.

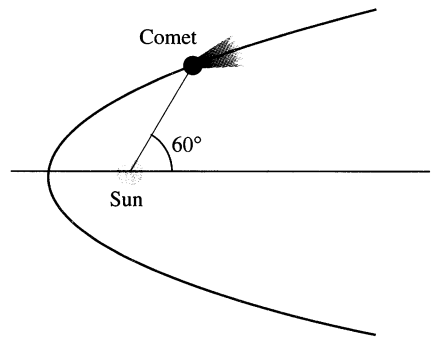
a) Find the angle formed at the bottom of the loop (Fig. 9.78).
b) Show that for each fixed position of the ring on the rope, the possible locations of the ring in space lie on an ellipse with foci at the pegs.
c) Justify the original symmetry assumption by combining the result in (b) with the assumption that the rope and weight will take a rest position of minimal potential energy.



9.78 Exercise 37 asks how large the angle *A* will be when the frictionless rope shown here is pulled tight by the weight.

38. Two radar stations lie 20 km apart along an east–west line. A low-flying plane traveling from west to east is known to have a speed of v_0 km/sec. At $t = 0$ a signal is sent from the station at $(-10, 0)$, bounces off the plane, and is received at $(10, 0)$ $30/c$ seconds later (c is the velocity of the signal). When $t = 10/v_0$, another signal is sent out from the station at $(-10, 0)$, reflects off the plane, and is once again received $30/c$ seconds later by the other station. Find the position of the plane when it reflects the second signal under the assumption that v_0 is much less than c.

39. A comet moves in a parabolic orbit with the sun at the focus. When the comet is 4×10^7 miles from the sun, the line from the comet to the sun makes a 60° angle with the orbit's axis, as shown here. How close will the comet come to the sun?



40. Find the points on the parabola $x = 2t$, $y = t^2$, $-\infty < t < \infty$, closest to the point $(0, 3)$.

41. Find the eccentricity of the ellipse $x^2 + xy + y^2 = 1$ to the nearest hundredth.

42. Find the eccentricity of the hyperbola $xy = 1$.

43. Is the curve $\sqrt{x} + \sqrt{y} = 1$ part of a conic section? If so, what kind of conic section? If not, why not?

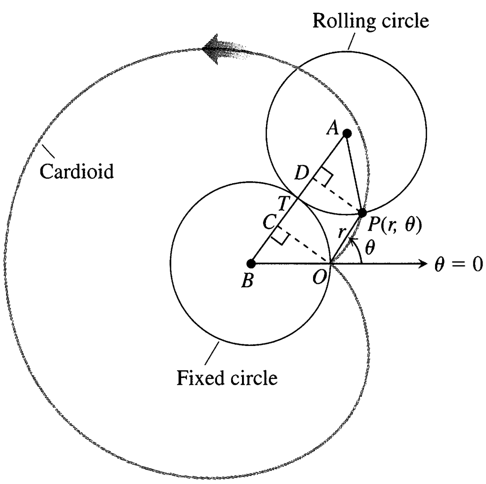
44. Show that the curve $2xy - \sqrt{2}y + 2 = 0$ is a hyperbola. Find the hyperbola's center, vertices, foci, axes, and asymptotes.

45. Find a polar coordinate equation for

a) the parabola with focus at the origin and vertex at $(a, \pi/4)$;
b) the ellipse with foci at the origin and (2, 0) and one vertex at (4, 0);
c) the hyperbola with one focus at the origin, center at $(2, \pi/2)$, and a vertex at $(1, \pi/2)$.

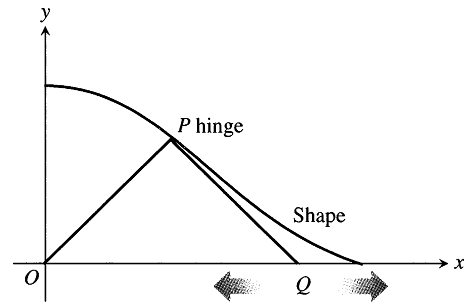
46. Any line through the origin will intersect the ellipse $r = 3/(2 + \cos\theta)$ in two points P_1 and P_2. Let d_1 be the distance between P_1 and the origin and let d_2 be the distance between P_2 and the origin. Compute $(1/d_1) + (1/d_2)$.

47. *Generating a cardioid with circles.* Cardioids are special epicycloids (Exercise 25). Show that if you roll a circle of radius a about another circle of radius a in the polar coordinate plane, as in Fig. 9.79, the original point of contact P will trace a cardioid. (*Hint:* Start by showing that angles OBC and PAD both have measure θ.)



9.79 As the circle centered at *A* rolls around the circle centered at *B*, the point *P* traces a cardioid (Exercise 47).

48. *A bifold closet door.* A bifold closet door consists of two one-foot-wide panels, hinged at point P. The outside bottom corner of one panel rests on a pivot at O (see the accompanying figure). The outside bottom corner of the other panel, denoted by Q, slides along a straight track, shown in the figure as a portion of the x-axis. Assume that as Q moves back and forth, the bottom of the door rubs against a thick carpet. What shape will the door sweep out on the surface of the carpet?



49. GRAPHER EXPLORATION Graph the curve $r = \cos 5\theta + n \cos\theta$, $0 \leq \theta \leq \pi$ for integers $n = -5$ (heart) to $n = 5$ (bell). (Source: *The College Mathematics Journal,* Vol. 25, No. 1, Jan. 1994.)

CHAPTER 10

Vectors and Analytic Geometry in Space

OVERVIEW This chapter introduces vectors and three-dimensional coordinate systems. Just as the coordinate plane is the natural place to study functions of a single variable, coordinate space is the place to study functions of two variables (or more). We establish coordinates in space by adding a third axis that measures distance above and below the xy-plane. This builds on what we already know without forcing us to start over again.

10.1 Vectors in the Plane

Some of the things we measure are determined by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. But we need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moves as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going.

Quantities that have direction as well as magnitude are usually represented by arrows that point in the direction of the action and whose lengths give the magnitude of the action in terms of a suitably chosen unit.

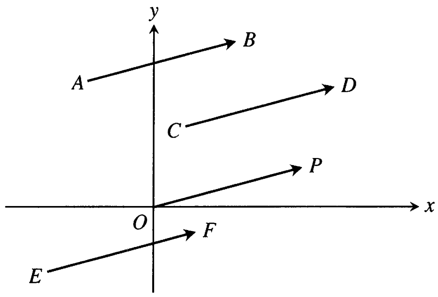
When we discuss these arrows abstractly, we think of them as directed line segments and we call them *vectors.*

Definitions

A **vector** in the plane is a directed line segment. Two vectors are **equal** or **the same** if they have the same length and direction.

Thus, the arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction.

In print, vectors are usually described with single boldface roman letters, as in $\mathbf{v}$ ("vector vee"). The vector defined by the directed line segment from point A to point B is written as $\overrightarrow{AB}$ ("vector ab").



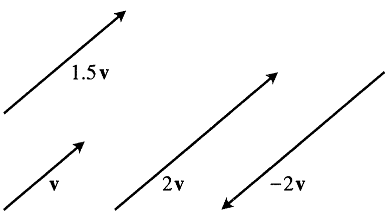
10.1 Arrows with the same length and direction represent the same vector (Example 1).

EXAMPLE 1 The four arrows in Fig. 10.1 have the same length and direction. They therefore represent the same vector, and we write

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}. \qquad \square$$

Scalars and Scalar Multiples

We multiply a vector by a positive real number by multiplying its length by the number (Fig. 10.2). To multiply a vector by 2, we double its length. To multiply a vector by 1.5, we increase its length by 50%, and so on. We multiply a vector by a negative number by reversing the vector's direction and multiplying the length by the number's absolute value.



10.2 Scalar multiples of $\mathbf{v}$.

If c is a nonzero real number and $\mathbf{v}$ a vector, the direction of $c\mathbf{v}$ agrees with that of $\mathbf{v}$ if c is positive and is opposite to that of $\mathbf{v}$ if c is negative. Since real numbers work like scaling factors in this context, we call them **scalars** and call multiples like $c\mathbf{v}$ **scalar multiples** of $\mathbf{v}$.

To include multiplication by zero, we adopt the convention that multiplying a vector by zero produces the **zero vector 0,** consisting of points that are degenerate line segments of zero length. Unlike other vectors, the vector $\mathbf{0}$ has no direction.

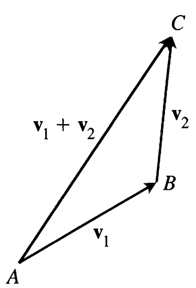
Geometric Addition: The Parallelogram Law

Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be added geometrically by drawing a representative of $\mathbf{v}_1$, say from A to B as in Fig. 10.3, and then a representative of $\mathbf{v}_2$ starting from the terminal point B of $\mathbf{v}_1$. In Fig. 10.3, $\mathbf{v}_2 = \overrightarrow{BC}$. The sum $\mathbf{v}_1 + \mathbf{v}_2$ is then the vector represented by the arrow from the initial point A of $\mathbf{v}_1$ to the terminal point C of $\mathbf{v}_2$. That is, if

$$\mathbf{v}_1 = \overrightarrow{AB} \qquad \text{and} \qquad \mathbf{v}_2 = \overrightarrow{BC},$$

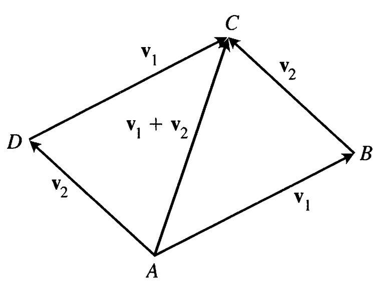
then

$$\mathbf{v}_1 + \mathbf{v}_2 = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$



10.3 The sum of $\mathbf{v}_1$ and $\mathbf{v}_2$.

This description of addition is sometimes called the **Parallelogram Law** of addition because $\mathbf{v}_1 + \mathbf{v}_2$ is given by the diagonal of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ (Fig. 10.4).



10.4 The Parallelogram Law of Addition. Quadrilateral *ABCD* is a parallelogram because opposite sides have equal lengths. The law was used by Aristotle to describe the combined action of two forces.

Components

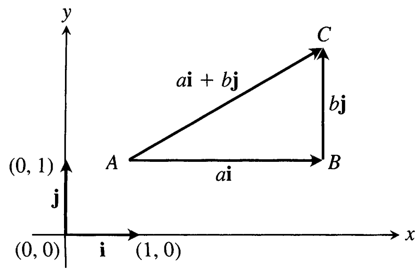
Two vectors are said to be **parallel** if they are nonzero scalar multiples of one another or, equivalently, if the line segments representing them are parallel.

Whenever a vector $\mathbf{v}$ can be written as a sum

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

of two *nonparallel* vectors, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are said to be **components** of $\mathbf{v}$. We also say that we have **represented** or **resolved** $\mathbf{v}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The most common algebra of vectors is based on representing each vector in terms of components parallel to the Cartesian coordinate axes and writing each



10.5 The basic vectors **i** and **j**. Any vector $\overrightarrow{AC}$ in the plane can be expressed as a scalar multiple of **i** plus a scalar multiple of **j**.

component as an appropriate multiple of a **basic** vector of length 1. The basic vector in the positive x-direction is the vector **i** determined by the directed line segment that runs from (0, 0) to (1, 0). The basic vector in the positive y-direction is the vector **j** determined by the directed line segment from (0, 0) to (0, 1). Then $a\,\mathbf{i}$, a being a scalar, represents a vector of length $|a|$ parallel to the x-axis, pointing to the right if $a > 0$ and to the left if $a < 0$. Similarly, $b\,\mathbf{j}$ is a vector of length $|b|$ parallel to the y-axis, pointing up if $b > 0$ and down if $b < 0$. Figure 10.5 shows a vector $\mathbf{v} = \overrightarrow{AC}$ resolved into its **i**- and **j**-components as the sum

$$\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}.$$

Definitions

If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$, the vectors $a\,\mathbf{i}$ and $b\,\mathbf{j}$ are the **vector components of v in the directions of i and j.** The numbers a and b are the **scalar components of v in the directions of i and j.**

Components enable us to define the equality of vectors algebraically.

Definition

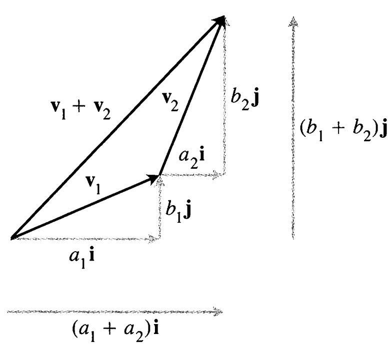
Equality of Vectors (Algebraic Definition)

$$a\,\mathbf{i} + b\,\mathbf{j} = a'\,\mathbf{i} + b'\,\mathbf{j} \quad \Leftrightarrow \quad a = a' \text{ and } b = b' \tag{1}$$

Two vectors are equal if and only if their scalar components in the directions of **i** and **j** are identical.

Algebraic Addition

Vectors may be added algebraically by adding their corresponding scalar components, as shown in Fig. 10.6.



10.6 If $\mathbf{v}_1 = a_1\,\mathbf{i} + b_1\,\mathbf{j}$ and $\mathbf{v}_2 = a_2\,\mathbf{i} + b_2\,\mathbf{j}$, then $\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2)\,\mathbf{i} + (b_1 + b_2)\,\mathbf{j}$.

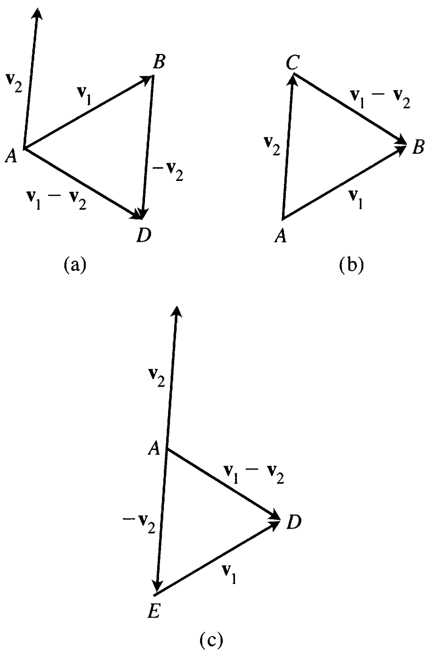
> If $\mathbf{v}_1 = a_1\,\mathbf{i} + b_1\,\mathbf{j}$ and $\mathbf{v}_2 = a_2\,\mathbf{i} + b_2\,\mathbf{j}$, then
>
> $$\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2)\,\mathbf{i} + (b_1 + b_2)\,\mathbf{j}. \tag{2}$$

EXAMPLE 2

$$(2\,\mathbf{i} - 4\,\mathbf{j}) + (5\,\mathbf{i} + 3\,\mathbf{j}) = (2 + 5)\,\mathbf{i} + (-4 + 3)\,\mathbf{j} = 7\,\mathbf{i} - \mathbf{j} \qquad \square$$

Subtraction

The negative of a vector **v** is the vector $-\mathbf{v} = (-1)\,\mathbf{v}$. It has the same length as **v** but points in the opposite direction. To subtract a vector $\mathbf{v}_2$ from a vector $\mathbf{v}_1$, we add $-\mathbf{v}_2$ to $\mathbf{v}_1$. This can be done geometrically by drawing $-\mathbf{v}_2$ from the tip of $\mathbf{v}_1$



10.7 Three ways to draw $\mathbf{v}_1 - \mathbf{v}_2$ (there are others): (a) as $\mathbf{v}_1 + (-\mathbf{v}_2)$; (b) as the vector from the tip of $\mathbf{v}_2$ to the tip of $\mathbf{v}_1$; and (c) as $-\mathbf{v}_2 + \mathbf{v}_1$.

and then drawing the vector from the initial point of $\mathbf{v}_1$ to the tip of $-\mathbf{v}_2$, as shown in Fig. 10.7(a), where

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \mathbf{v}_1 + (-\mathbf{v}_2) = \mathbf{v}_1 - \mathbf{v}_2.$$

Another way to draw $\mathbf{v}_1 - \mathbf{v}_2$ is to draw $\mathbf{v}_1$ and $\mathbf{v}_2$ with a common initial point and then draw $\mathbf{v}_1 - \mathbf{v}_2$ as the vector from the tip of $\mathbf{v}_2$ to the tip of $\mathbf{v}_1$. This is illustrated in Fig. 10.7(b), where

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\mathbf{v}_2 + \mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_2.$$

Still another way is to draw $\mathbf{v}_1$ from the tip of $-\mathbf{v}_2$ (Fig. 10.7c).

In terms of components, vector subtraction follows the algebraic law

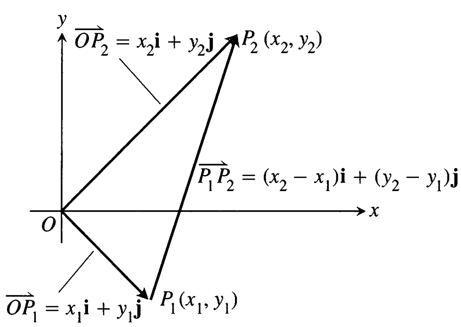
$$\mathbf{v}_1 - \mathbf{v}_2 = (a_1 - a_2)\,\mathbf{i} + (b_1 - b_2)\,\mathbf{j}, \qquad (3)$$

which says that corresponding scalar components are subtracted.

EXAMPLE 3

$$(6\,\mathbf{i} + 2\,\mathbf{j}) - (3\,\mathbf{i} - 5\,\mathbf{j}) = (6 - 3)\,\mathbf{i} + (2 - (-5))\,\mathbf{j} = 3\,\mathbf{i} + 7\,\mathbf{j} \qquad \square$$

We find the components of the vector from a point $P_1(x_1, y_1)$ to a point $P_2(x_2, y_2)$ by subtracting the components of $\overrightarrow{OP}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ from the components of $\overrightarrow{OP}_2 = x_2\,\mathbf{i} + y_2\,\mathbf{j}$ (Fig. 10.8).



10.8 $\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$

The vector from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

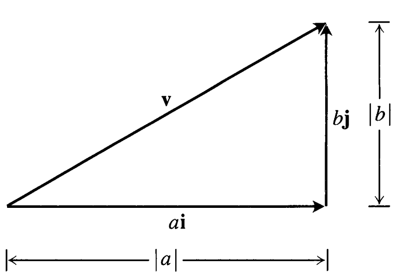
$$\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}. \qquad (4)$$

EXAMPLE 4 The vector from $P_1(3, 4)$ to $P_2(5, 1)$ is

$$\overrightarrow{P_1P_2} = (5 - 3)\,\mathbf{i} + (1 - 4)\,\mathbf{j} = 2\,\mathbf{i} - 3\,\mathbf{j}. \qquad \square$$

Magnitude

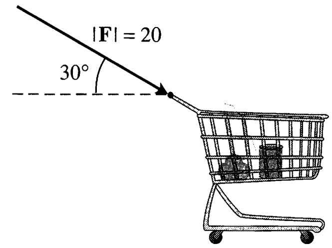
The **magnitude** or **length** of $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. We arrive at this number by applying the Pythagorean theorem to the right triangle determined by $\mathbf{v}$ and its two vector components (Fig. 10.9). The bars in $|\mathbf{v}|$ (read "the magnitude of $\mathbf{v}$" or "the length of $\mathbf{v}$") are the same bars we use for absolute values.

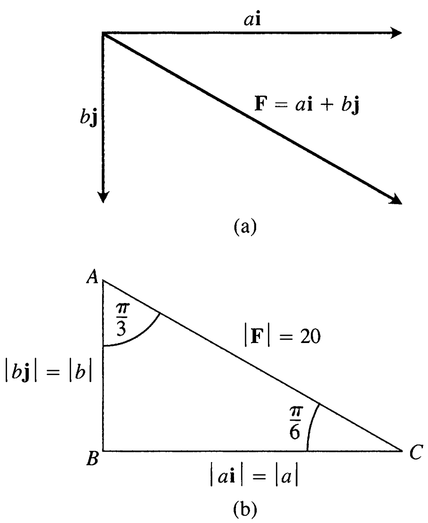


10.9 The length of $\mathbf{v}$ is $\sqrt{|a|^2 + |b|^2} = \sqrt{a^2 + b^2}$.

The magnitude or length of $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. $\qquad (5)$

EXAMPLE 5 You push a loaded supermarket cart by applying a 20-lb force $\mathbf{F}$ that makes a 30° angle with the horizontal. Resolve $\mathbf{F}$ into its horizontal and vertical components. (The horizontal component is the effective force in the direction of motion. The vertical component just adds weight to the cart.)





10.10 (a) The horizontal and vertical components of the vector **F** in Example 5. (b) The right triangle determined by **F** and its components.

Solution We draw a vector triangle for $\mathbf{F} = a\,\mathbf{i} + b\,\mathbf{j}$ and its vector components along with the right triangle determined by their magnitudes (Fig. 10.10). The triangle is a 30-60-90 triangle, so $|a| = 10\sqrt{3}$ and $|b| = 10$. The horizontal component of $\mathbf{F}$ is $10\sqrt{3}\,\mathbf{i}$. The vertical component is $-10\,\mathbf{j}$ (negative because it points down). That is, $\mathbf{F} = 10\sqrt{3}\,\mathbf{i} - 10\,\mathbf{j}$. ❑

Scalar Multiplication

Scalar multiplication can be accomplished component by component.

> If c is a scalar and $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$ is a vector, then
> $$c\,\mathbf{v} = c(a\,\mathbf{i} + b\,\mathbf{j}) = (ca)\,\mathbf{i} + (cb)\,\mathbf{j}. \tag{6}$$

The length of $c\,\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$:

$$\begin{aligned}
|c\,\mathbf{v}| &= |(ca)\,\mathbf{i} + (cb)\,\mathbf{j}| && \text{Eq. (6)} \\
&= \sqrt{(ca)^2 + (cb)^2} && \text{Eq. (5) with } ca \text{ and } cb \text{ in place of } a \text{ and } b \\
&= \sqrt{c^2(a^2 + b^2)} \\
&= \sqrt{c^2}\sqrt{a^2 + b^2} \\
&= |c||\mathbf{v}|.
\end{aligned}$$

> If c is a scalar and $\mathbf{v}$ is a vector, then $|c\,\mathbf{v}| = |c||\mathbf{v}|$.

EXAMPLE 6 If $c = -2$ and $\mathbf{v} = -3\,\mathbf{i} + 4\,\mathbf{j}$, then

$$|\mathbf{v}| = |-3\,\mathbf{i} + 4\,\mathbf{j}| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\begin{aligned}
|-2\,\mathbf{v}| &= |(-2)(-3\,\mathbf{i} + 4\,\mathbf{j})| = |6\,\mathbf{i} - 8\,\mathbf{j}| = \sqrt{(6)^2 + (-8)^2} = \sqrt{36 + 64} \\
&= \sqrt{100} = 10 = |-2|5 = |c||\mathbf{v}|.
\end{aligned}$$
❑

The Zero Vector

In terms of components, the zero vector is the vector

$$\mathbf{0} = 0\,\mathbf{i} + 0\,\mathbf{j}.$$

It is the only vector whose length is zero, as we can see from the fact that

$$|a\,\mathbf{i} + b\,\mathbf{j}| = \sqrt{a^2 + b^2} = 0 \qquad \Leftrightarrow \qquad a = b = 0.$$

In handwritten work it is common to denote unit vectors with small "hats," as in $\hat{u}$ (pronounced "u hat"). In hat notation, **i** and **j** become $\hat{\imath}$ and $\hat{\jmath}$.

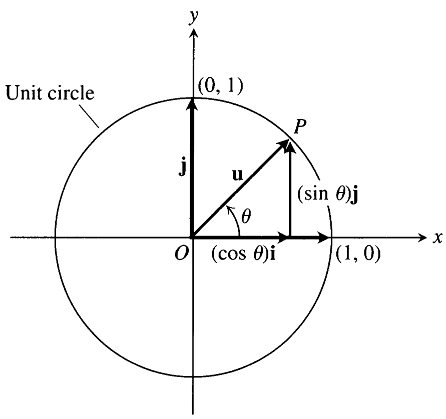
Unit Vectors

Any vector whose length is 1 is a **unit vector.** The vectors **i** and **j** are unit vectors.

$$|\mathbf{i}| = |1\,\mathbf{i} + 0\,\mathbf{j}| = \sqrt{1^2 + 0^2} = 1, \qquad |\mathbf{j}| = |0\,\mathbf{i} + 1\,\mathbf{j}| = \sqrt{0^2 + 1^2} = 1$$

If **u** is the unit vector obtained by rotating **i** through an angle θ in the positive direction, then **u** has a horizontal component $\cos\theta$ and vertical component $\sin\theta$ (Fig. 10.11), so that

$$\mathbf{u} = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}. \tag{7}$$



10.11 The unit vector that makes an angle of measure θ with the positive x-axis. Every unit vector in the plane has the form

$$\mathbf{u} = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}$$

for some θ.

As θ varies from 0 to 2π, the point P in Fig. 10.11 traces the circle $x^2 + y^2 = 1$ counterclockwise. This takes in all possible directions, so *Eq. (7) gives every unit vector in the plane.*

Length vs. Direction

If $\mathbf{v} \neq \mathbf{0}$, then

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1,$$

so $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of **v**. We can therefore express **v** in terms of its two important features, length and direction, by writing $\mathbf{v} = |\mathbf{v}|(\mathbf{v}/|\mathbf{v}|)$.

> If $\mathbf{v} \neq \mathbf{0}$, then
>
> **1.** $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of **v**;
>
> **2.** the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses **v** in terms of its length and direction.

It is common in applied fields that use vectors a great deal to refer to the vector $\mathbf{v}/|\mathbf{v}|$ itself as the direction of **v**. The equation $\mathbf{v} = |\mathbf{v}|(\mathbf{v}/|\mathbf{v}|)$ is then said to express **v** as a *product* of its length and direction.

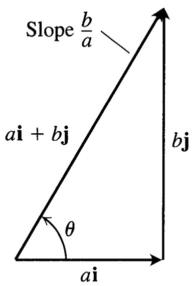
EXAMPLE 7 Express $\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}$ as a product of its length and direction.

Solution Length of **v**: $\quad |\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5$

Direction of **v**: $\quad \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3\,\mathbf{i} - 4\,\mathbf{j}}{5} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

$$\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j} = \underbrace{5}_{\text{length}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{direction}}$$

❑



10.12 If $a \neq 0$, the vector $a\,\mathbf{i} + b\,\mathbf{j}$ has slope $b/a = \tan\theta$.

Slopes, Tangents, and Normals

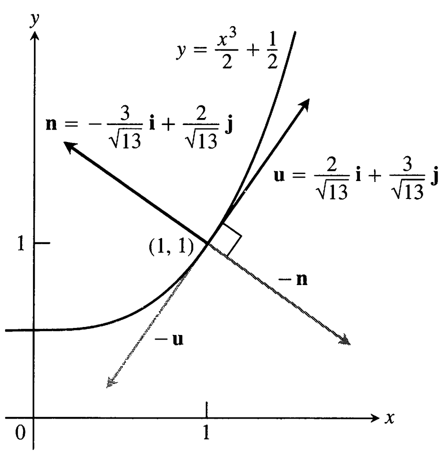
A vector is **parallel** to a line if the segments that represent the vector are parallel to the line. The **slope** of a vector that is not vertical is the slope shared by the lines parallel to the vector. Thus, if $a \neq 0$, the vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$ has a well-defined slope, which can be calculated from the components of $\mathbf{v}$ as the number b/a (Fig. 10.12).

A vector is **tangent** or **normal** to a curve at a point if it is parallel or normal to the line that is tangent to the curve at that point. The next example shows how to find such vectors.

EXAMPLE 8 Find unit vectors tangent and normal to the curve

$$y = \frac{x^3}{2} + \frac{1}{2}$$

at the point (1, 1).



10.13 The unit tangent and normal vectors at the point (1, 1) on the curve $y = (x^3/2) + 1/2$.

Solution We find the unit vectors that are parallel and normal to the curve's tangent line at (1, 1) (Fig. 10.13).

The slope of the line tangent to the curve at (1, 1) is

$$y' = \frac{3x^2}{2}\bigg|_{x=1} = \frac{3}{2}.$$

We look for a unit vector with this slope. The vector $\mathbf{v} = 2\,\mathbf{i} + 3\,\mathbf{j}$ has slope 3/2, as does every nonzero multiple of $\mathbf{v}$. To find a multiple of $\mathbf{v}$ that is a unit vector, we divide $\mathbf{v}$ by

$$|\mathbf{v}| = \sqrt{2^2 + 3^2} = \sqrt{13},$$

obtaining

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}.$$

The vector $\mathbf{u}$ is tangent to the curve at (1, 1) because it has the same direction as $\mathbf{v}$. Of course,

$$-\mathbf{u} = -\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j},$$

which points in the opposite direction, is also tangent to the curve at (1, 1). Without some additional requirement, there is no reason to prefer one of these vectors to the other.

To find unit vectors normal to the curve at (1, 1), we look for unit vectors whose slopes are the negative reciprocal of the slope of $\mathbf{u}$. This is quickly done by interchanging the scalar components of $\mathbf{u}$ and changing the sign of one of them. We obtain

$$\mathbf{n} = -\frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}.$$

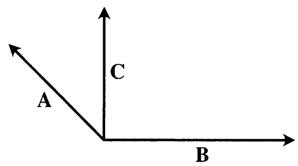
Again, either one will do. The vectors have opposite directions but both are normal to the curve at (1, 1). ❑

If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$, then $\mathbf{p} = -b\,\mathbf{i} + a\,\mathbf{j}$ and $\mathbf{q} = b\,\mathbf{i} - a\,\mathbf{j}$ are perpendicular to $\mathbf{v}$ because their slopes are both $-a/b$, the negative reciprocal of $\mathbf{v}$'s slope.

Exercises 10.1

Geometry and Calculation

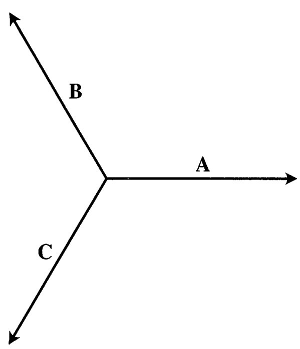
1. The vectors **A, B**, and **C** in the figure here lie in a plane. Copy them on a sheet of paper. Then, by arranging vectors head to tail, as in Figs. 10.3 and 10.6, sketch



a) $\mathbf{A}+\mathbf{B}$ b) $\mathbf{A}+\mathbf{B}+\mathbf{C}$

c) $\mathbf{A}-2\mathbf{B}$ d) $\frac{1}{2}\mathbf{A}-\mathbf{C}$

2. The vectors **A, B**, and **C** in the figure here lie in a plane. Copy them on a sheet of paper. Then, by arranging vectors head to tail, as in Figs. 10.3 and 10.6, sketch



a) $\mathbf{A}-\mathbf{B}$ b) $\mathbf{A}+\mathbf{B}+\mathbf{C}$

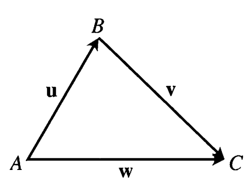
c) $2\mathbf{A}-\frac{1}{2}\mathbf{B}$ d) $\mathbf{A}-(\mathbf{B}-\mathbf{C})$

Let $\mathbf{A}=2\mathbf{i}-7\mathbf{j}$, $\mathbf{B}=\mathbf{i}+6\mathbf{j}$, and $\mathbf{C}=\sqrt{3}\,\mathbf{i}-\pi\,\mathbf{j}$. Write each of the vectors in Exercises 3–6 in the form $a\,\mathbf{i}+b\,\mathbf{j}$.

3. $\mathbf{A}+2\mathbf{B}$ 4. $\mathbf{A}+\mathbf{B}-\mathbf{C}$

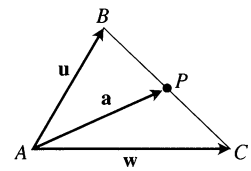
5. $3\mathbf{A}-\frac{1}{\pi}\mathbf{C}$ 6. $2\mathbf{A}-3\mathbf{B}+32\mathbf{j}$

7. Vectors **u, v**, and **w** are determined by the sides of triangle ABC as shown.



a) Express **w** in terms of **u** and **v**.
b) Express **v** in terms of **u** and **w**.

8. Vectors **u** and **w** are determined by the sides of triangle ABC as shown, and P is the midpoint of side BC. Express **a** in terms of **u** and **w**.



Express the vectors in Exercises 9–16 in the form $a\,\mathbf{i}+b\,\mathbf{j}$ and sketch them as arrows in the coordinate plane beginning at the origin.

9. $\overrightarrow{P_1P_2}$ if P_1 is the point (5, 7) and P_2 is the point (2, 9)
10. $\overrightarrow{P_1P_2}$ if P_1 is the point (1, 2) and P_2 is the point (−3, 5)
11. $\overrightarrow{AB}$ if A is the point (−5, 3) and B is the point (−10, 8)
12. $\overrightarrow{AB}$ if A is the point (−7, −8) and B is the point (6, 11)
13. $\overrightarrow{P_1P_2}$ if P_1 is the point (1, 3) and P_2 is the point (2, −1)
14. $\overrightarrow{P_3P_4}$ if P_3 is the point (1, 3) and P_4 is the midpoint of the line segment P_1P_2 joining $P_1(2,-1)$ and $P_2(-4,3)$
15. The sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, given the four points $A(1,-1)$, $B(2,0)$, $C(-1,3)$, and $D(-2,2)$
16. The vector from the point A to the origin where $\overrightarrow{AB}=4\mathbf{i}-2\mathbf{j}$ and B is the point (−2, 5)
17. Given the vector $\overrightarrow{AB}=3\mathbf{i}-\mathbf{j}$ and A is the point (2, 9), find the point B.
18. Given the vector $\overrightarrow{PQ}=-6\mathbf{i}-4\mathbf{j}$ and Q is the point (3, 3), find the point P.

Unit Vectors

Sketch the vectors in Exercises 19–22 and express each vector in the form $a\,\mathbf{i}+b\,\mathbf{j}$.

19. The unit vectors $\mathbf{u}=(\cos\theta)\,\mathbf{i}+(\sin\theta)\,\mathbf{j}$ for $\theta=\pi/6$ and $\theta=2\pi/3$. Include the circle $x^2+y^2=1$ in your sketch.
20. The unit vectors $\mathbf{u}=(\cos\theta)\,\mathbf{i}+(\sin\theta)\,\mathbf{j}$ for $\theta=-\pi/4$ and $\theta=-3\pi/4$. Include the circle $x^2+y^2=1$ in your sketch.
21. The unit vector obtained by rotating **j** counterclockwise $3\pi/4$ rad about the origin
22. The unit vector obtained by rotating **j** clockwise $2\pi/3$ rad about the origin

For the vectors in Exercises 23 and 24, find unit vectors $\mathbf{u}=(\cos\theta)\,\mathbf{i}+(\sin\theta)\,\mathbf{j}$ in the same direction.

23. $6\mathbf{i}-8\mathbf{j}$ 24. $-\mathbf{i}+3\mathbf{j}$

In Exercises 25–28, find the unit vectors that are tangent and normal to the curve at the given point (four vectors in all). Then sketch the vectors and curve together.

25. $y = x^2$, $(2, 4)$

26. $x^2 + 2y^2 = 6$, $(2, 1)$

27. $y = \tan^{-1}x$, $(1, \pi/4)$

28. $y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $(0, 1)$

In Exercises 29–32, find the unit vectors that are tangent and normal to the curve at the given point (four vectors in all).

29. $3x^2 + 8xy + 2y^2 - 3 = 0$, $(1, 0)$

30. $x^2 - 6xy + 8y^2 - 2x - 1 = 0$, $(1, 1)$

31. $y = \int_0^x \sqrt{3 + t^4}\, dt$, $(0, 0)$

32. $y = \int_e^x \ln(\ln t)\, dt$, $(e, 0)$

Length and Direction

In Exercises 33 and 34, express each vector as a product of its length and direction.

33. $5\mathbf{i} + 12\mathbf{j}$

34. $2\mathbf{i} - 3\mathbf{j}$

35. Find the unit vectors that are parallel to the vector $3\mathbf{i} - 4\mathbf{j}$ (two vectors in all).

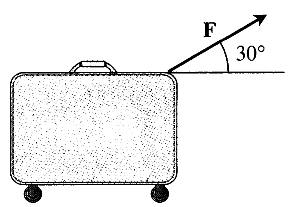
36. Find a vector of length 2 whose direction is the opposite of the direction of the vector $\mathbf{A} = -\mathbf{i} + 2\mathbf{j}$. How many such vectors are there?

37. Show that $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} - 2\mathbf{j}$ have opposite directions. Sketch $\mathbf{A}$ and $\mathbf{B}$ together.

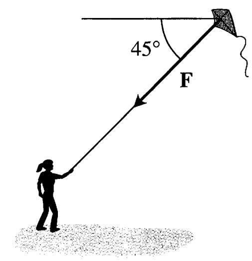
38. Show that $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j}$ and $\mathbf{B} = (1/2)\mathbf{i} + \mathbf{j}$ have the same direction.

Theory and Applications

39. You are pulling on a suitcase with a force $\mathbf{F}$ (pictured here) whose magnitude is $|\mathbf{F}| = 10$ lb. Find the x- and y-components of $\mathbf{F}$.



40. A kite string exerts a 12-lb pull ($|\mathbf{F}| = 12$) on a kite and makes a 45° angle with the horizontal. Find the horizontal and vertical components of $\mathbf{F}$.



41. Let $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{j}$, and $\mathbf{C} = \mathbf{i} - \mathbf{j}$. Find scalars α, β, such that $\mathbf{A} = \alpha\,\mathbf{B} + \beta\,\mathbf{C}$.

42. Let $\mathbf{A} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ where $\mathbf{A}_1$ is parallel to $\mathbf{B}$ and $\mathbf{A}_2$ is parallel to $\mathbf{C}$. (See Exercise 41.)

43. A bird flies from its nest 5 km in the direction 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast to land atop a telephone pole. Place an xy-coordinate system so that the origin is the bird's nest, the x-axis points east, and the y-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

44. A bird flies from its nest 7 km in the direction northeast, where it stops to rest on a tree. It then flies 8 km in the direction 30° south of west to land atop a telephone pole. Place an xy-coordinate system so that the origin is the bird's nest, the x-axis points east, and the y-axis points north.

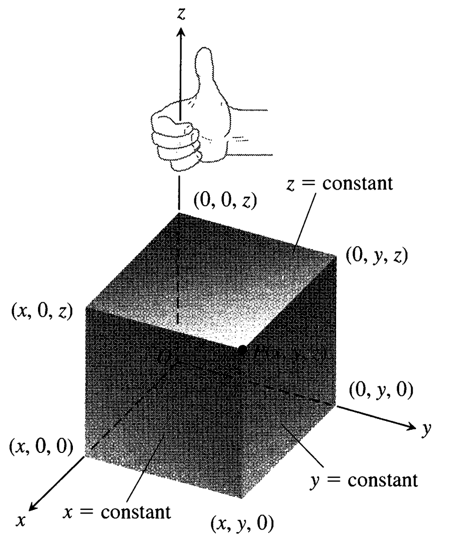
a) At what point is the tree located?

b) At what point is the telephone pole?

45. Let $\mathbf{v}$ be a vector in the plane not parallel to the y-axis. How is the slope of $-\mathbf{v}$ related to the slope of $\mathbf{v}$? Give reasons for your answer.

10.2 Cartesian (Rectangular) Coordinates and Vectors in Space

Our goal now is to describe the three-dimensional Cartesian coordinate system and learn our way around in space. This means defining distance, practicing with the arithmetic of vectors in space (the rules are the same as in the plane but with an extra term), and making connections between sets of points and equations and inequalities.



10.14 The Cartesian coordinate system is right-handed.

Cartesian Coordinates

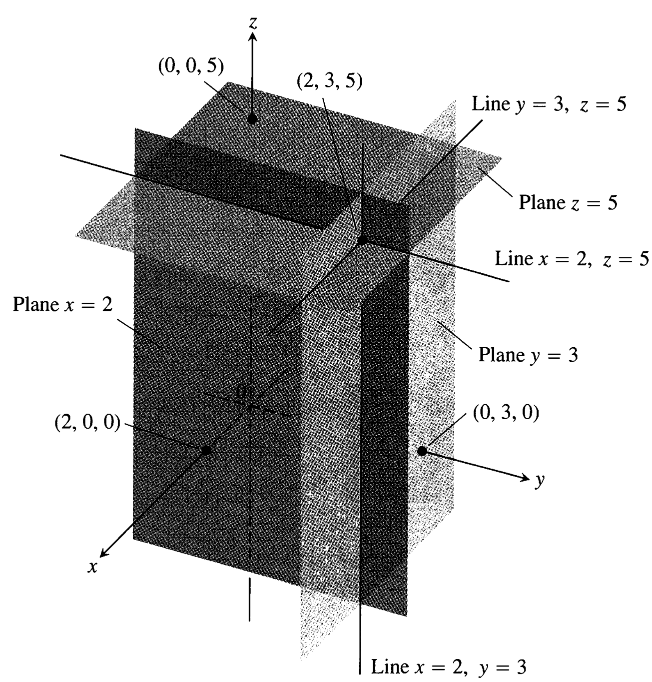
To locate points in space, we use three mutually perpendicular coordinate axes, arranged as in Fig. 10.14. The axes Ox, Oy, and Oz shown there make a *right-handed* coordinate frame. When you hold your right hand so that the fingers curl from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis.

The Cartesian coordinates (x, y, z) of a point P in space are the numbers at which the planes through P perpendicular to the axes cut the axes.

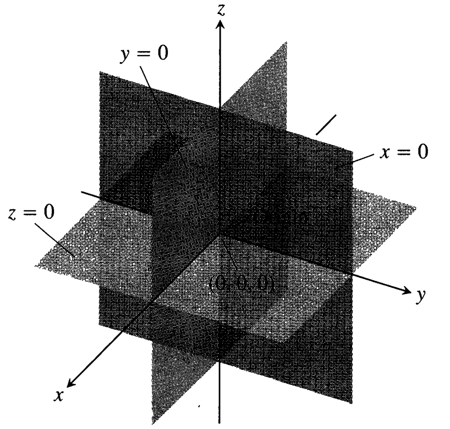
Points on the x-axis have y- and z-coordinates equal to zero. That is, they have coordinates of the form $(x, 0, 0)$. Similarly, points on the y-axis have coordinates of the form $(0, y, 0)$. Points on the z-axis have coordinates of the form $(0, 0, z)$.

The points in a plane perpendicular to the x-axis all have the same x-coordinate, this being the number at which that plane cuts the x-axis. The y- and z-coordinates can be any numbers. Similarly, the points in a plane perpendicular to the y-axis have a common y-coordinate and the points in a plane perpendicular to the z-axis have a common z-coordinate. To write equations for these planes, we name the common coordinate's value. The plane $x = 2$ is the plane perpendicular to the x-axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the y-axis at $y = 3$. The plane $z = 5$ is the plane perpendicular to the z-axis at $z = 5$. Figure 10.15 shows the planes $x = 2$, $y = 3$, and $z = 5$, together with their intersection point $(2, 3, 5)$.

The planes $x = 2$ and $y = 3$ in Fig. 10.15 intersect in a line parallel to the z-axis. This line is described by the *pair* of equations $x = 2$, $y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$. Similarly, the line of intersection of the planes $y = 3$ and $z = 5$ is described by the equation pair $y = 3, z = 5$. This line runs parallel to the x-axis. The line of intersection of the planes $x = 2$ and $z = 5$, parallel to the y-axis, is described by the equation pair $x = 2, z = 5$.



10.15 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.



10.16 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

The planes determined by the coordinate axes are the ***xy*-plane,** whose standard equation is $z = 0$; the ***yz*-plane,** whose standard equation is $x = 0$; and the ***xz*-plane,** whose standard equation is $y = 0$. They meet at the **origin** (0, 0, 0) (Fig. 10.16).

The three **coordinate planes** $x = 0$, $y = 0$, and $z = 0$ divide space into eight cells called **octants.** The octant in which the point coordinates are all positive is called the **first octant;** there is no conventional numbering for the other seven octants.

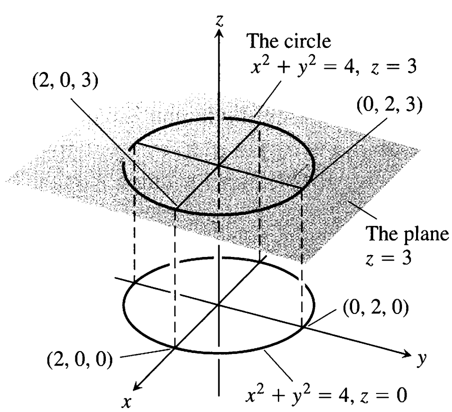
Cartesian coordinates for space are also called **rectangular coordinates** because the axes that define them meet at right angles.

In the following examples, we match coordinate equations and inequalities with the sets of points they define in space.

EXAMPLE 1

Defining equations and inequalities	Verbal description
$z \geq 0$	The half-space consisting of the points on and above the xy-plane.
$x = -3$	The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.
$z = 0, x \leq 0, y \geq 0$	The second quadrant of the xy-plane.
$x \geq 0, y \geq 0, z \geq 0$	The first octant.
$-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included).
$y = -2, z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x-axis.

❑



10.17 The circle $x^2 + y^2 = 4$, $z = 3$.

EXAMPLE 2 What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \qquad \text{and} \qquad z = 3?$$

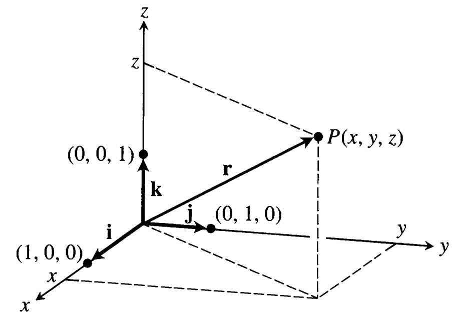
Solution The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points "the circle $x^2 + y^2 = 4$ in the plane $z = 3$" or, more simply, "the circle $x^2 + y^2 = 4$, $z = 3$" (Fig. 10.17). ❑

Vectors in Space

The sets of equivalent directed line segments that we use to represent forces, displacements, and velocities in space are called vectors, just as in the plane. The same rules of addition, subtraction, and scalar multiplication apply.

The vectors represented by the directed line segments from the origin to the points (1, 0, 0), (0, 1, 0), and (0, 0, 1) are the **basic vectors** (Fig. 10.18, on the following page). We denote them by **i, j,** and **k.** The **position vector r** from the origin O to the typical point $P(x, y, z)$ is

$$\mathbf{r} = \overrightarrow{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}. \tag{1}$$



10.18 The position vector of a point in space.

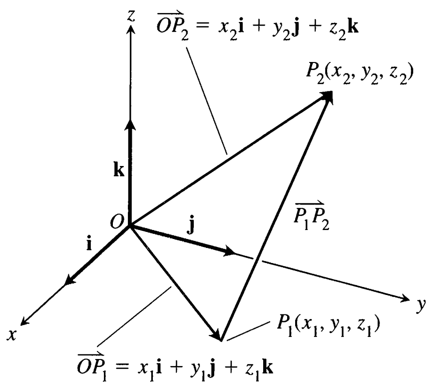
Definition
Addition and Subtraction for Vectors in Space
For any vectors $\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$,

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k}$$
$$\mathbf{A} - \mathbf{B} = (a_1 - b_1)\,\mathbf{i} + (a_2 - b_2)\,\mathbf{j} + (a_3 - b_3)\,\mathbf{k}.$$

The Vector Between Two Points

We can express the vector $\overrightarrow{P_1P_2}$ from the point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ in terms of the coordinates of P_1 and P_2 because (Fig. 10.19)

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1}\\ &= (x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}) - (x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}) \qquad (2)\\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}.\end{aligned}$$



10.19 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$.

The vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}. \qquad (3)$$

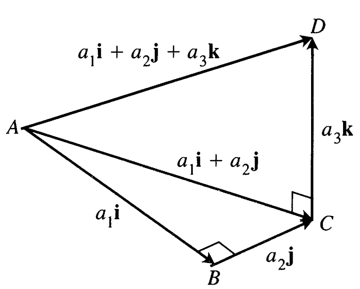
Magnitude

As always, the important features of a vector are its magnitude and direction. We find a formula for the magnitude (length) of $a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ by applying the Pythagorean theorem to the right triangles in Fig. 10.20. From triangle *ABC*,

$$|\overrightarrow{AC}| = \sqrt{a_1^2 + a_2^2}$$

and from triangle *ACD*,

$$|a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}| = |\overrightarrow{AD}| = \sqrt{|\overrightarrow{AC}|^2 + |\overrightarrow{CD}|^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$



10.20 We find the length of $\overrightarrow{AD}$ by applying the Pythagorean theorem to the right triangles *ABC* and *ACD*.

The **magnitude (length)** of $\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ is

$$|\mathbf{A}| = |a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \qquad (4)$$

Scalar Multiplication

Definition

If c is a scalar and $\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ is a vector, then

$$c\,\mathbf{A} = (ca_1)\,\mathbf{i} + (ca_2)\,\mathbf{j} + (ca_3)\,\mathbf{k}.$$

EXAMPLE 3 The length of $\mathbf{A} = \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ is

$$|\mathbf{A}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}. \quad \square$$

If we multiply $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ by a scalar c, the length of $c\mathbf{A}$ is $|c|$ times the length of $\mathbf{A}$, as in the plane. The reason is the same, as well:

$$\begin{aligned} c\,\mathbf{A} &= ca_1\,\mathbf{i} + ca_2\,\mathbf{j} + ca_3\,\mathbf{k}, \\ |c\,\mathbf{A}| &= \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = \sqrt{c^2{a_1}^2 + c^2{a_2}^2 + c^2{a_3}^2} \\ &= |c|\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} = |c||\mathbf{A}|. \end{aligned} \tag{5}$$

EXAMPLE 4 If $\mathbf{A}$ is the vector of Example 3, then the length of

$$2\,\mathbf{A} = 2(\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) = 2\,\mathbf{i} - 4\,\mathbf{j} + 6\,\mathbf{k}$$

is

$$\begin{aligned} \sqrt{(2)^2 + (-4)^2 + (6)^2} &= \sqrt{4+16+36} = \sqrt{56} \\ &= \sqrt{4 \cdot 14} = 2\sqrt{14} = 2|\mathbf{A}|. \end{aligned} \quad \square$$

The Zero Vector

The **zero vector** in space is the vector $\mathbf{0} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$. As in the plane, $\mathbf{0}$ has zero length, and no direction.

Unit Vectors

A **unit vector** in space is a vector of length 1. The basic vectors are unit vectors because

$$\begin{aligned} |\mathbf{i}| &= |1\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}| = \sqrt{1^2 + 0^2 + 0^2} = 1, \\ |\mathbf{j}| &= |0\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k}| = \sqrt{0^2 + 1^2 + 0^2} = 1, \\ |\mathbf{k}| &= |0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}| = \sqrt{0^2 + 0^2 + 1^2} = 1. \end{aligned}$$

Magnitude and Direction

If $\mathbf{A} \neq \mathbf{0}$, then $\mathbf{A}/|\mathbf{A}|$ is a unit vector in the direction of $\mathbf{A}$ and we can use the equation

$$\mathbf{A} = |\mathbf{A}|\,\frac{\mathbf{A}}{|\mathbf{A}|} \tag{6}$$

to express $\mathbf{A}$ as a product of its magnitude and direction.

EXAMPLE 5 Express $\mathbf{A} = \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ as a product of its magnitude and direction.

Solution

$$\mathbf{A} = |\mathbf{A}| \cdot \frac{\mathbf{A}}{|\mathbf{A}|} \qquad \text{Eq. (6)}$$

$$= \sqrt{14} \cdot \frac{\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}}{\sqrt{14}} \qquad \text{From Example 3}$$

$$= \sqrt{14}\left(\frac{1}{\sqrt{14}}\,\mathbf{i} - \frac{2}{\sqrt{14}}\,\mathbf{j} + \frac{3}{\sqrt{14}}\,\mathbf{k}\right) = (\text{length of } \mathbf{A}) \cdot (\text{direction of } \mathbf{A})$$ ❑

EXAMPLE 6 Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3-1)\,\mathbf{i} + (2-0)\,\mathbf{j} + (0-1)\,\mathbf{k} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} - \frac{1}{3}\,\mathbf{k}.$$ ❑

EXAMPLE 7 Find a vector 6 units long in the direction of $\mathbf{A} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

Solution The vector we want is

$$6\,\frac{\mathbf{A}}{|\mathbf{A}|} = 6\,\frac{2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\,\frac{2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = 4\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}.$$ ❑

Distance in Space

The distance between two points P_1 and P_2 in space is the length of $\overrightarrow{P_1P_2}$.

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

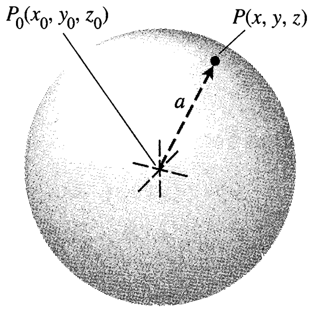
$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{7}$$

EXAMPLE 8 The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$|\overrightarrow{P_1P_2}| = \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2}$$

$$= \sqrt{16 + 4 + 25}$$

$$= \sqrt{45} = 3\sqrt{5}.$$ ❑



10.21 The standard equation of the sphere of radius a centered at (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Spheres

We use Eq. (7) to write equations for spheres in space (Fig. 10.21). A point $P(x, y, z)$ lies on the sphere of radius a centered at $P_0(x_0, y_0, z_0)$ precisely when $|\overrightarrow{P_0P}| = a$ or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2 \qquad (8)$$

EXAMPLE 9 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x-, y-, and z-terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x \quad) + y^2 + (z^2 - 4z \quad) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

This is Eq. (8) with $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

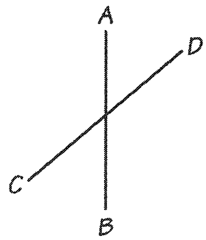
EXAMPLE 10 *Sets bounded by spheres or portions of spheres*

Defining equations and inequalities	Description
a) $x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
b) $x^2 + y^2 + z^2 \le 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
c) $x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
d) $x^2 + y^2 + z^2 = 4$, $z \le 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

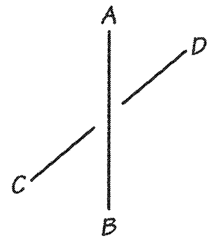
DRAWING LESSON

How to Draw Three-dimensional Objects to Look Three-dimensional

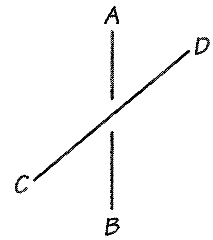
1 Break lines. When one line passes behind another, break it to show that it doesn't touch and that part of it is hidden.



Intersecting

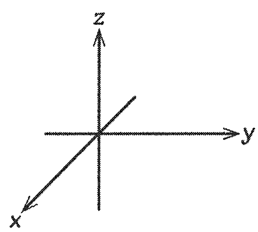


CD behind *AB*

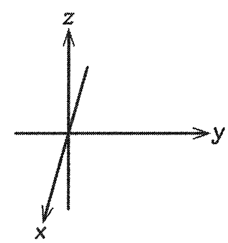


AB behind *CD*

2 Make the angle between the positive *x*-axis and the positive *y*-axis large enough.

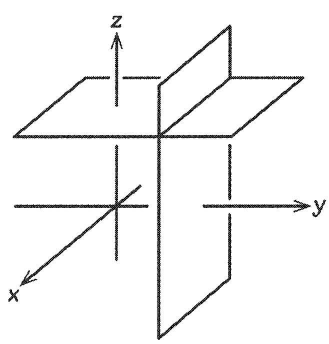


This

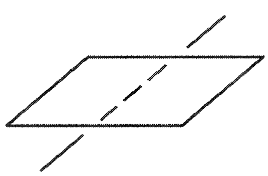


Not this

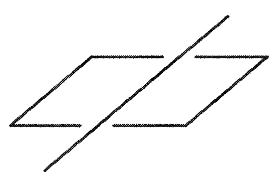
3 Draw planes parallel to the coordinate planes as if they were rectangles with sides parallel to the coordinate axes.



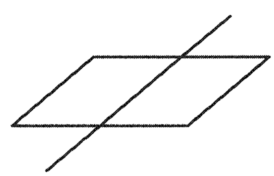
4 Dash or omit hidden portions of lines. Don't let the line touch the boundary of the parallelogram that represents the plane, unless the line lies in the plane.



Line below plane

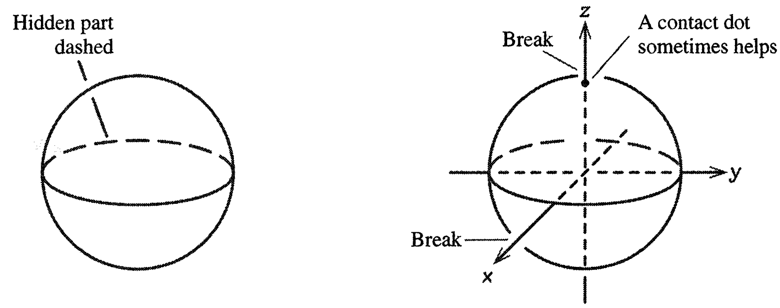


Line above plane



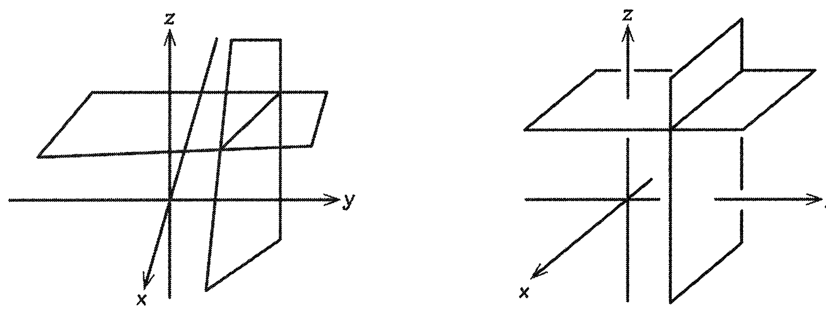
Line *in* plane

5 **Spheres: Draw the sphere first (outline and equator); draw axes, if any, later. Use line breaks and dashed lines.**



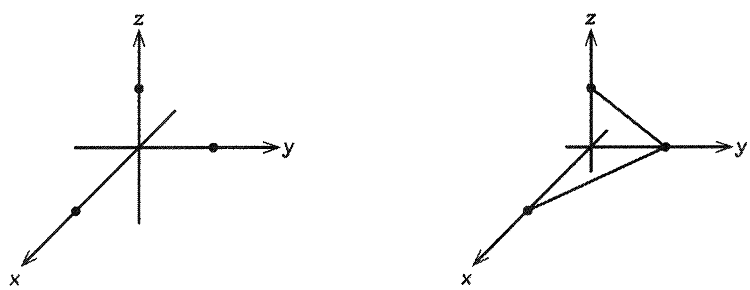
Sphere first Axes later

6 **A general rule for perspective: Draw the object as if it lies some distance away, below, and to your left.**

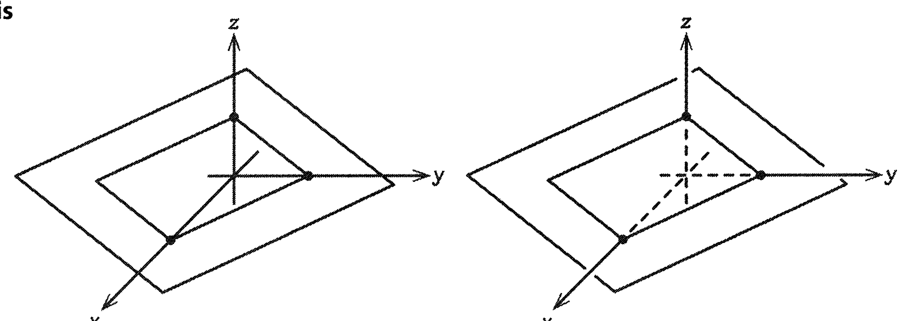


Advice ignored Advice followed

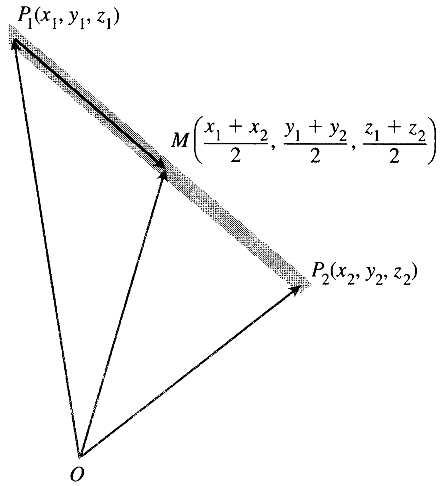
7 **To draw a plane that crosses all three coordinate axes, follow the steps shown here:**
(a) Sketch the axes and mark the intercepts. (b) Connect the intercepts to form two sides of a parallelogram. (c) Complete the parallelogram and enlarge it by drawing lines parallel to its sides. (d) Darken the exposed parts, break hidden lines, and, if desired, dash hidden portions of the axes. You may wish to erase the smaller parallelogram at this point.



(a) (b)



(c) (d)



10.22 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

Midpoints

The coordinates of the midpoint of a line segment are found by averaging.

> The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point
> $$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

To see why, observe (Fig. 10.22) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1+x_2}{2}\mathbf{i} + \frac{y_1+y_2}{2}\mathbf{j} + \frac{z_1+z_2}{2}\mathbf{k}.\end{aligned}$$

EXAMPLE 11 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5, 1, 2).$$

❑

Exercises 10.2

Sets, Equations, and Inequalities

In Exercises 1–12, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

1. $x = 2, \quad y = 3$

2. $x = -1, \quad z = 0$

3. $y = 0, \quad z = 0$

4. $x = 1, \quad y = 0$

5. $x^2 + y^2 = 4, \quad z = 0$

6. $x^2 + y^2 = 4, \quad z = -2$

7. $x^2 + z^2 = 4, \quad y = 0$

8. $y^2 + z^2 = 1, \quad x = 0$

9. $x^2 + y^2 + z^2 = 1, \quad x = 0$

10. $x^2 + y^2 + z^2 = 25, \quad y = -4$

11. $x^2 + y^2 + (z+3)^2 = 25, \quad z = 0$

12. $x^2 + (y-1)^2 + z^2 = 4, \quad y = 0$

In Exercises 13–18, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

13. a) $x \geq 0, \quad y \geq 0, \quad z = 0$
b) $x \geq 0, \quad y \leq 0, \quad z = 0$

14. a) $0 \leq x \leq 1$
b) $0 \leq x \leq 1, \quad 0 \leq y \leq 1$
c) $0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$

15. a) $x^2 + y^2 + z^2 \leq 1$
b) $x^2 + y^2 + z^2 > 1$

16. a) $x^2 + y^2 \leq 1, \quad z = 0$
b) $x^2 + y^2 \leq 1, \quad z = 3$
c) $x^2 + y^2 \leq 1$, no restriction on z

17. a) $x^2 + y^2 + z^2 = 1, \quad z \geq 0$
b) $x^2 + y^2 + z^2 \leq 1, \quad z \geq 0$

18. a) $x = y, \quad z = 0$
b) $x = y$, no restriction on z

In Exercises 19–28, describe the given set with a single equation or with a pair of equations.

19. The plane perpendicular to the
a) x-axis at $(3, 0, 0)$
b) y-axis at $(0, -1, 0)$
c) z-axis at $(0, 0, -2)$

20. The plane through the point $(3, -1, 2)$ perpendicular to the

a) x-axis **b)** y-axis **c)** z-axis

21. The plane through the point $(3, -1, 1)$ parallel to the

a) xy-plane **b)** yz-plane **c)** xz-plane

22. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the

a) xy-plane **b)** yz-plane **c)** xz-plane

23. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the

a) xy-plane **b)** yz-plane **c)** plane $y = 2$

24. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the

a) xy-plane **b)** yz-plane **c)** xz-plane

25. The line through the point $(1, 3, -1)$ parallel to the

a) x-axis **b)** y-axis **c)** z-axis

26. The set of points in space equidistant from the origin and the point $(0, 2, 0)$

27. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z-axis meets the sphere of radius 5 centered at the origin

28. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

Write inequalities to describe the sets in Exercises 29–34.

29. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included)

30. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

31. The half-space consisting of the points on and below the xy-plane

32. The upper hemisphere of the sphere of radius 1 centered at the origin

33. The (a) interior and (b) exterior of the sphere of radius 1 centered at the point $(1, 1, 1)$

34. The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin. (*Closed* means the spheres are to be included. Had we wanted the spheres left out, we would have asked for the *open* region bounded by the spheres. This is analogous to the way we use *closed* and *open* to describe intervals: *closed* means endpoints included, *open* means endpoints left out. Closed sets include boundaries; open sets leave them out.)

Length and Direction

In Exercises 35–44, express each vector as a product of its length and direction.

35. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

36. $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$

37. $\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$

38. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$

39. $5\mathbf{k}$

40. $-4\mathbf{j}$

41. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

42. $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$

43. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

44. $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

45. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

	Length	Direction
a)	2	$\mathbf{i}$
b)	$\sqrt{3}$	$-\mathbf{k}$
c)	$\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$
d)	7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

46. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

	Length	Direction
a)	7	$-\mathbf{j}$
b)	$\sqrt{2}$	$-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
c)	$\frac{13}{12}$	$\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
d)	$a > 0$	$\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

47. Find a vector of magnitude 7 in the direction of $\mathbf{A} = 12\mathbf{i} - 5\mathbf{k}$.

48. Find a vector $\sqrt{5}$ units long in the direction of $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

49. Find a vector 5 units long in the direction opposite to the direction of $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

50. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{A} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.

Vectors Determined by Points; Midpoints and Distance

In Exercises 51–56, find

a) the distance between points P_1 and P_2,
b) the direction of $\overrightarrow{P_1P_2}$,
c) the midpoint of line segment P_1P_2.

51. $P_1(1, 1, 1)$, $P_2(3, 3, 0)$

52. $P_1(-1, 1, 5)$, $P_2(2, 5, 0)$

53. $P_1(1, 4, 5)$, $P_2(4, -2, 7)$

54. $P_1(3, 4, 5)$, $P_2(2, 3, 4)$

55. $P_1(0, 0, 0)$, $P_2(2, -2, -2)$

56. $P_1(5, 3, -2)$, $P_2(0, 0, 0)$

57. If $\overrightarrow{AB} = \mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$ and B is the point (5, 1, 3), find A.

58. If $\overrightarrow{AB} = -7\,\mathbf{i} + 3\,\mathbf{j} + 8\,\mathbf{k}$ and A is the point $(-2, -3, 6)$, find B.

Spheres

Find the centers and radii of the spheres in Exercises 59–62.

59. $(x+2)^2 + y^2 + (z-2)^2 = 8$

60. $\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 + \left(z+\frac{1}{2}\right)^2 = \frac{21}{4}$

61. $\left(x-\sqrt{2}\right)^2 + \left(y-\sqrt{2}\right)^2 + \left(z+\sqrt{2}\right)^2 = 2$

62. $x^2 + \left(y+\frac{1}{3}\right)^2 + \left(z-\frac{1}{3}\right)^2 = \frac{29}{9}$

Find equations for the spheres whose centers and radii are given in Exercises 63–66.

	Center	Radius
63.	(1, 2, 3)	$\sqrt{14}$
64.	$(0, -1, 5)$	2
65.	$(-2, 0, 0)$	$\sqrt{3}$
66.	$(0, -7, 0)$	7

Find the centers and radii of the spheres in Exercises 67–70.

67. $x^2 + y^2 + z^2 + 4x - 4z = 0$

68. $x^2 + y^2 + z^2 - 6y + 8z = 0$

69. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

70. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$

71. Find a formula for the distance from the point $P(x, y, z)$ to the

a) x-axis **b)** y-axis **c)** z-axis

72. Find a formula for the distance from the point $P(x, y, z)$ to the

a) xy-plane **b)** yz-plane **c)** xz-plane

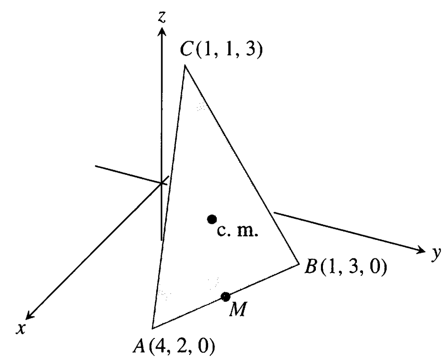
Geometry with Vectors

73. Suppose that A, B, and C are the corner points of the thin triangular plate of constant density shown here.

a) Find the vector from C to the midpoint M of side AB.

b) Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c) Find the coordinates of the point in which the medians of ΔABC intersect. According to Exercise 31, Section 5.7, this point is the plate's center of mass.



74. Find the vector from the origin to the point of intersection of the medians of the triangle whose vertices are

$$A(1, -1, 2), \quad B(2, 1, 3), \quad \text{and} \quad C(-1, 2, -1).$$

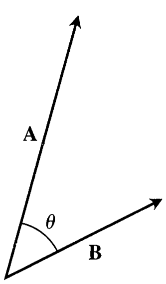
75. Let $ABCD$ be a general, not necessarily planar, quadrilateral in space. Show that the two segments joining the midpoints of opposite sides of $ABCD$ bisect each other. (*Hint:* Show that the segments have the same midpoint.)

76. Vectors are drawn from the center of a regular n-sided polygon in the plane to the vertices of the polygon. Show that the sum of the vectors is zero. (*Hint:* What happens to the sum if you rotate the polygon about its center?)

77. Suppose that A, B, and C are vertices of a triangle and that a, b, and c are, respectively, the midpoints of the opposite sides. Show that $\overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = 0$.

10.3 Dot Products

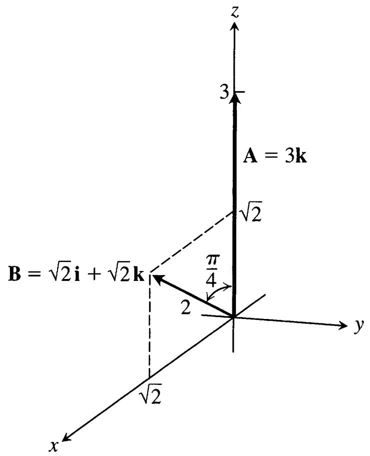
We now introduce the dot product, the first of two methods we will learn for multiplying vectors together. Dot products are also called *scalar* products because the multiplication results in a scalar, not a vector. The products in the next section are vectors.



10.23 The angle between **A** and **B**.

Scalar Products

When two nonzero vectors **A** and **B** are placed so their initial points coincide, they form an angle θ of measure $0 \le \theta \le \pi$ (Fig. 10.23). This angle is called the **angle** between **A** and **B**.



10.24 The vectors in Example 1.

Definition

The **scalar product (dot product)** $\mathbf{A} \cdot \mathbf{B}$ ("**A** dot **B**") of vectors **A** and **B** is the number

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta, \tag{1}$$

where θ is the angle between **A** and **B.**

In words, $\mathbf{A} \cdot \mathbf{B}$ is the length of **A** times the length of **B** times the cosine of the angle between **A** and **B.**

EXAMPLE 1 If $\mathbf{A} = 3\,\mathbf{k}$ and $\mathbf{B} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{k}$ (Fig. 10.24), then

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta = (3)(2) \cos\frac{\pi}{4} = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}.$$ ❑

Since the sign of $\mathbf{A} \cdot \mathbf{B}$ is determined by $\cos\theta$, the scalar product is positive if the angle between the vectors is acute, negative if the angle is obtuse. (We look at right angles in a moment.)

Since the angle a vector **A** makes with itself is zero, and $\cos 0 = 1$,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos 0 = |\mathbf{A}||\mathbf{A}|(1) = |\mathbf{A}|^2, \qquad \text{or} \qquad |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}. \tag{2}$$

Calculation

To calculate $\mathbf{A} \cdot \mathbf{B}$ from the components of **A** and **B** in a Cartesian coordinate system with unit vectors **i, j,** and **k,** we let

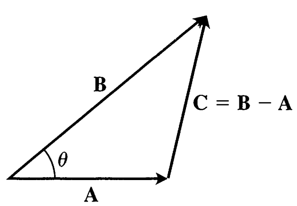
$$\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k},$$
$$\mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k},$$

and

$$\mathbf{C} = \mathbf{B} - \mathbf{A} = (b_1 - a_1)\,\mathbf{i} + (b_2 - a_2)\,\mathbf{j} + (b_3 - a_3)\,\mathbf{k}.$$

The law of cosines for the triangle whose sides represent **A, B,** and **C** (Fig. 10.25) is

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta,$$
$$|\mathbf{A}||\mathbf{B}| \cos\theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{C}|^2}{2}.$$



10.25 We obtain Eq. (3) by applying the law of cosines to a triangle whose sides represent **A, B,** and $\mathbf{C} = \mathbf{B} - \mathbf{A}$.

The left side of this equation is $\mathbf{A} \cdot \mathbf{B}$. We evalute the right side by squaring the components of **A, B,** and **C** (Eq. 4, Section 10.2). The resulting cancellations give

$$\boxed{\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{3}}$$

Thus, to find the scalar product of two given vectors we multiply their corresponding **i-, j-,** and **k-** components and add the results.

Solving Eq. (1) for θ gives a formula for finding angles between vectors.

> The angle between two nonzero vectors **A** and **B** is
> $$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right). \tag{4}$$

Since the values of the arc cosine lie in $[0, \pi]$, Eq. (4) automatically gives the angle made by **A** and **B.**

EXAMPLE 2 Find the angle between $\mathbf{A} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use Eq. (4):

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4 \\
|\mathbf{A}| &= \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
|\mathbf{B}| &= \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7 \\
\theta &= \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) && \text{Eq. (4)} \\
&= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ rad} && \text{About } 101
\end{aligned}$$

❑

Laws of the Dot Product

From the equation $\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$, we can see right away that

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}. \tag{5}$$

In other words, the dot product is commutative. We can also see from Eq. (3) that if c is any number (or scalar), then

$$(c\,\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(c\,\mathbf{B}) = c\,(\mathbf{A}\cdot\mathbf{B}). \tag{6}$$

If $\mathbf{C} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is any third vector, then

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}.
\end{aligned}$$

Hence dot products obey the distributive law:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{7}$$

If we combine this with the commutative law, Eq. (5), it is also evident that

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}. \tag{8}$$

Equations (7) and (8) together permit us to multiply sums of vectors by the familiar laws of algebra. For example,

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{C} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{D}. \tag{9}$$

Perpendicular (Orthogonal) Vectors

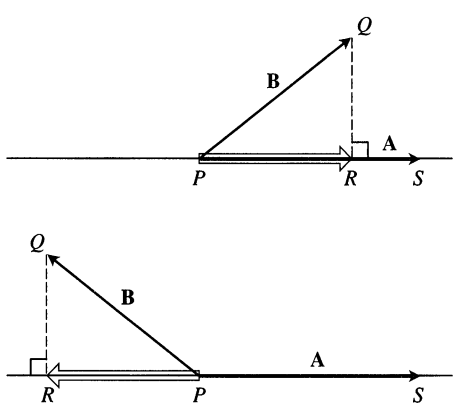
Two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular or **orthogonal** if the angle between them is $\pi/2$. For such vectors, we automatically have $\mathbf{A} \cdot \mathbf{B} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors with $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

Nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular (orthogonal) if and only if $\mathbf{A} \cdot \mathbf{B} = 0$.

EXAMPLE 3 $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{A} \cdot \mathbf{B} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

Vector Projections

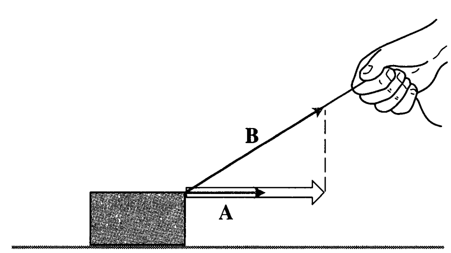


10.26 The vector projection of **B** onto **A**.

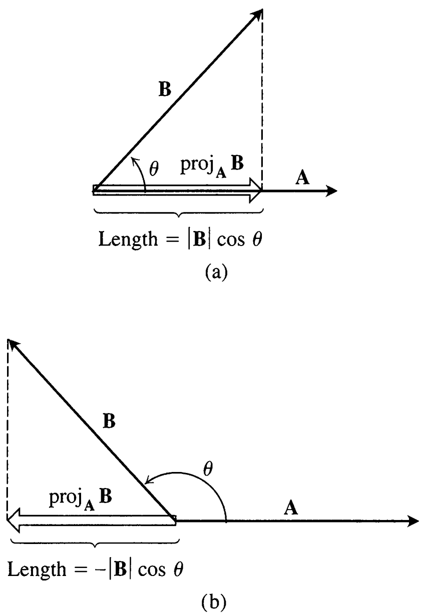
The **vector projection** of $\mathbf{B} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{A} = \overrightarrow{PS}$ (Fig. 10.26) is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS. The notation for this vector is

$$\text{proj}_{\mathbf{A}}\, \mathbf{B} \qquad (\text{"the vector projection of } \mathbf{B} \text{ onto } \mathbf{A}\text{"}).$$

If $\mathbf{B}$ represents a force, then $\text{proj}_{\mathbf{A}}\, \mathbf{B}$ represents the effective force in the direction of $\mathbf{A}$ (Fig. 10.27).



10.27 If we pull on a box with force **B**, the effective force in the direction of **A** is the vector projection of **B** onto **A**.



10.28 The length of $\text{proj}_{\mathbf{A}}\,\mathbf{B}$ is (a) $|\mathbf{B}|\cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{B}|\cos\theta$ if $\cos\theta < 0$.

If the angle θ between **A** and **B** is acute, $\text{proj}_{\mathbf{A}}\,\mathbf{B}$ has length $|\mathbf{B}|\cos\theta$ and direction $\mathbf{A}/|\mathbf{A}|$(Fig. 10.28). If θ is obtuse, $\cos\theta < 0$ and $\text{proj}_{\mathbf{A}}\,\mathbf{B}$ has length $-|\mathbf{B}|\cos\theta$ and direction $-\mathbf{A}/|\mathbf{A}|$. In any case,

$$\begin{aligned}\text{proj}_{\mathbf{A}}\,\mathbf{B} &= (|\mathbf{B}|\cos\theta)\frac{\mathbf{A}}{|\mathbf{A}|}\\ &= \left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}\right)\frac{\mathbf{A}}{|\mathbf{A}|}\\ &= \left(\mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|}\right)\frac{\mathbf{A}}{|\mathbf{A}|}.\end{aligned}$$

$$|\mathbf{B}|\cos\theta = \frac{|\mathbf{A}||\mathbf{B}|\cos\theta}{|\mathbf{A}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}$$

$$\boxed{\text{proj}_{\mathbf{A}}\,\mathbf{B} = \left(\mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|}\right)\frac{\mathbf{A}}{|\mathbf{A}|} = \left(\frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}} \tag{10}$$

The number $|\mathbf{B}|\cos\theta$ is called the **scalar component of B in the direction of A.** Since

$$|\mathbf{B}|\cos\theta = \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|}, \tag{11}$$

we can find the scalar component by "dotting" **B** with the direction of **A.** Equation (10) says that the vector projection of **B** onto **A** is the scalar component of **B** in the direction of **A** times the direction of **A.**

While the first part of Eq. (10) describes the effect of **B** in the direction of **A,** the second part is better for calculation because it avoids square roots.

EXAMPLE 4 Find the vector projection of $\mathbf{B} = 6\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$ onto $\mathbf{A} = \mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}$ and the scalar component of **B** in the direction of **A.**

Solution We find $\text{proj}_{\mathbf{A}}\,\mathbf{B}$ from Eq. (10):

$$\begin{aligned}\text{proj}_{\mathbf{A}}\,\mathbf{B} &= \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A} = \frac{6-6-4}{1+4+4}(\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k})\\ &= -\frac{4}{9}(\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

We find the scalar component of **B** in the direction of **A** from Eq. (11):

$$\begin{aligned}|\mathbf{B}|\cos\theta &= \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|} = (6\,\mathbf{i} + 3\,\mathbf{j} + 2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)\\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

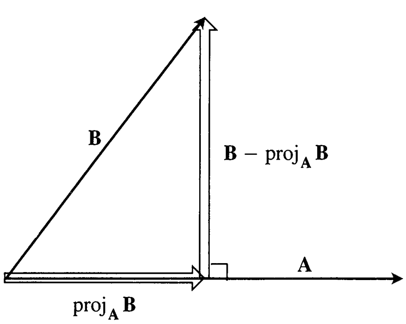
❑

Writing a Vector as a Sum of Orthogonal Vectors

In mechanics, we often need to express a vector **B** as a sum of a vector parallel to a vector **A** and a vector orthogonal to **A.** We can accomplish this with the equation

$$\mathbf{B} = \text{proj}_{\mathbf{A}}\,\mathbf{B} + (\mathbf{B} - \text{proj}_{\mathbf{A}}\,\mathbf{B}), \tag{12}$$

shown in Fig. 10.29.



10.29 Writing **B** as the sum of vectors parallel and orthogonal to **A.**

Where vectors came from

Although Aristotle used vectors to describe the effects of forces, the idea of resolving vectors into geometric components parallel to the coordinate axes came from Descartes. The algebra of vectors we use today was developed simultaneously and independently in the 1870s by Josiah Willard Gibbs (1839–1903), a mathematical physicist at Yale University, and by the English mathematical physicist Oliver Heaviside (1850–1925), the Heaviside of Heaviside layer fame. The works of Gibbs and Heaviside grew out of more complicated mathematical theories developed some years earlier by the Irish mathematician William Hamilton (1805–1865) and the German linguist, physicist, and geometer Hermann Grassman (1809–1877).

How to Write B as a Vector Parallel to A Plus a Vector Orthogonal to A

$$\begin{aligned}\mathbf{B} &= \operatorname{proj}_{\mathbf{A}}\mathbf{B} + (\mathbf{B} - \operatorname{proj}_{\mathbf{A}}\mathbf{B}) \\ &= \underbrace{\left(\frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}}_{\text{parallel to }\mathbf{A}} + \underbrace{\left(\mathbf{B} - \left(\frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}\right)}_{\text{orthogonal to }\mathbf{A}}\end{aligned} \tag{13}$$

EXAMPLE 5 Express $\mathbf{B} = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ as the sum of a vector parallel to $\mathbf{A} = 3\,\mathbf{i} - \mathbf{j}$ and a vector orthogonal to $\mathbf{A}$.

Solution We use Eq. (13). With

$$\mathbf{A}\cdot\mathbf{B} = 6 - 1 = 5 \qquad \text{and} \qquad \mathbf{A}\cdot\mathbf{A} = 9 + 1 = 10,$$

Eq. (13) gives

$$\begin{aligned}\mathbf{B} = \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A} + \left(\mathbf{B} - \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A}\right) &= \frac{5}{10}(3\,\mathbf{i} - \mathbf{j}) + \left(2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k} - \frac{5}{10}(3\,\mathbf{i} - \mathbf{j})\right) \\ &= \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) + \left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\,\mathbf{k}\right).\end{aligned}$$

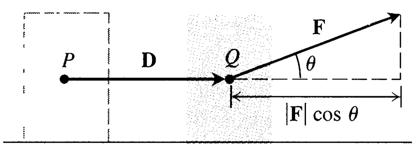
Check: The first vector in the sum is parallel to **A** because it is (1/2) **A**. The second vector in the sum is orthogonal to **A** because

$$\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\,\mathbf{k}\right)\cdot(3\,\mathbf{i} - \mathbf{j}) = \frac{3}{2} - \frac{3}{2} = 0.$$

❑

Work

In Section 5.8, we calculated the work done by a constant force of magnitude F in moving an object through a distance d as $W = Fd$. That formula holds only if the force is directed along the line of motion. If a force **F** moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of **F** in the direction of **D**. If θ is the angle between **F** and **D** (Fig. 10.30), then



10.30 The work done by a constant force **F** during a displacement **D** is $(|\mathbf{F}|\cos\theta)|\mathbf{D}|$.

$$\begin{aligned}\text{Work} &= \begin{pmatrix}\text{scalar component of }\mathbf{F} \\ \text{in the direction of }\mathbf{D}\end{pmatrix}\left(\text{length of }\mathbf{D}\right) \\ &= (|\mathbf{F}|\cos\theta)\ |\mathbf{D}| \\ &= \mathbf{F}\cdot\mathbf{D}\end{aligned}$$

Definition

The **work** done by a constant force **F** acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta, \tag{14}$$

where θ is the angle between **F** and **D**.

Work

The standard units of work are the foot-pound and newton-meter, both force-distance units. The newton-meter is usually called a *joule.* For more about the notion of work and the units involved, read the first few paragraphs of Section 5.8.

EXAMPLE 6 If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60°$, the work done by $\mathbf{F}$ in acting from P to Q is

$$\begin{aligned} \text{Work} &= |\mathbf{F}||\mathbf{D}|\cos\theta && \text{Eq. (14)} \\ &= (40)(3)\cos 60° && \text{Given values} \\ &= (120)(1/2) \\ &= 60 \text{ J (joules)} \end{aligned}$$

We will encounter more interesting work problems in Chapter 14 when we can find the work done by a variable force along a path in space.

Exercises 10.3

Calculations

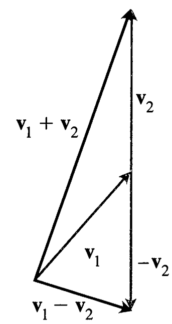
In Exercises 1–10, find

a) $\mathbf{A} \cdot \mathbf{B}$, $|\mathbf{A}|$, $|\mathbf{B}|$;
b) the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$;
c) the scalar component of $\mathbf{B}$ in the direction of $\mathbf{A}$;
d) the vector $\text{proj}_{\mathbf{A}}\mathbf{B}$.

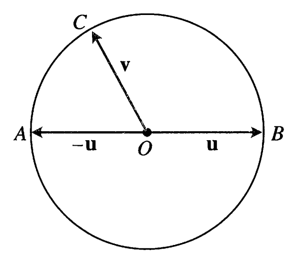
1. $\mathbf{A} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$, $\mathbf{B} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2. $\mathbf{A} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$, $\mathbf{B} = 5\mathbf{i} + 12\mathbf{j}$
3. $\mathbf{A} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{B} = 3\mathbf{j} + 4\mathbf{k}$
4. $\mathbf{A} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
5. $\mathbf{A} = -2\mathbf{i} + 7\mathbf{j}$, $\mathbf{B} = \mathbf{k}$
6. $\mathbf{A} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$, $\mathbf{B} = \frac{1}{\sqrt{2}}\mathbf{j} - \mathbf{k}$
7. $\mathbf{A} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
8. $\mathbf{A} = \mathbf{i} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
9. $\mathbf{A} = -\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$
10. $\mathbf{A} = -5\mathbf{i} + \mathbf{j}$, $\mathbf{B} = 2\mathbf{i} + \sqrt{17}\mathbf{j} + 10\mathbf{k}$
11. Write $\mathbf{B} = 3\mathbf{j} + 4\mathbf{k}$ as the sum of a vector parallel to $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and a vector orthogonal to $\mathbf{A}$.
12. Write $\mathbf{B} = \mathbf{j} + \mathbf{k}$ as the sum of a vector parallel to $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and a vector orthogonal to $\mathbf{A}$.
13. Write $\mathbf{B} = 8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$ as the sum of a vector parallel to $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and a vector orthogonal to $\mathbf{A}$.
14. $\mathbf{B} = \mathbf{i} + (\mathbf{j} + \mathbf{k})$ is already the sum of a vector parallel to $\mathbf{i}$ and a vector orthogonal to $\mathbf{i}$. If you use Eq. (13) with $\mathbf{A} = \mathbf{i}$, do you get $\mathbf{B}_1 = \mathbf{i}$ and $\mathbf{B}_2 = \mathbf{j} + \mathbf{k}$? Try it and find out.

Geometry

15. *Sums and differences.* In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.

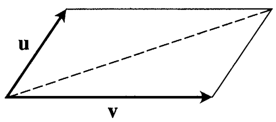


16. Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

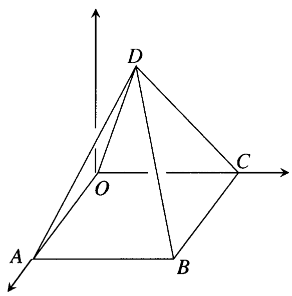


17. Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.
18. Show that squares are the only rectangles with perpendicular diagonals.
19. Prove the fact, often exploited by carpenters, that a parallelogram is a rectangle if and only if its diagonals are equal in length.

20. Show that the indicated diagonal of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$ if $|\mathbf{u}| = |\mathbf{v}|$.



21. The accompanying figure shows a pyramid *OABCD* with a square base whose sides are 1 unit long. The pyramid's height is also 1 unit, and the point *D* stands directly above the midpoint of the diagonal *OB*. Find the angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.



22. *Direction angles and direction cosines.* The direction angles α, β, and γ of a vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$,

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$,

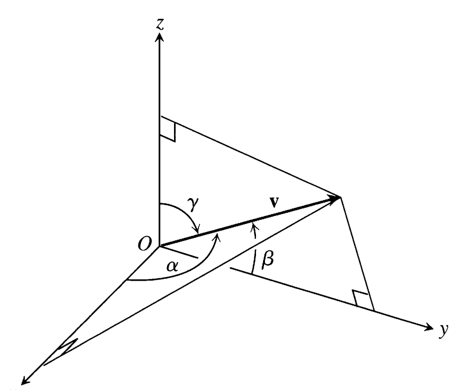
γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.

a) Show that

$$\cos\alpha = \frac{a}{|\mathbf{v}|}, \quad \cos\beta = \frac{b}{|\mathbf{v}|}, \quad \cos\gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\mathbf{v}$.

b) *Unit vectors are built from direction cosines.* Show that if $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is a unit vector then *a, b,* and *c are* the direction cosines of $\mathbf{v}$.



Angles Between Vectors

CALCULATOR Find the angles between the vectors in Exercises 23–26 to the nearest hundredth of a radian.

23. $\mathbf{A} = 2\,\mathbf{i} + \mathbf{j}, \quad \mathbf{B} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$

24. $\mathbf{A} = 2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = 3\,\mathbf{i} + 4\,\mathbf{k}$

25. $\mathbf{A} = \sqrt{3}\,\mathbf{i} - 7\,\mathbf{j}, \quad \mathbf{B} = \sqrt{3}\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$

26. $\mathbf{A} = \mathbf{i} + \sqrt{2}\,\mathbf{j} - \sqrt{2}\,\mathbf{k}, \quad \mathbf{B} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

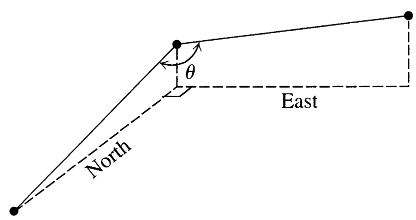
CALCULATOR Find the angles in Exercises 27–29 to the nearest hundredth of a radian.

27. The interior angles of the triangle *ABC* whose vertices are $A(-1, 0, 2)$, $B(2, 1, -1)$, and $C(1, -2, 2)$

28. The angle between $\mathbf{A} = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\,\mathbf{i} + 10\,\mathbf{j} - 11\,\mathbf{k}$

29. The angle between the diagonal of a cube and the diagonal of one of its faces. (*Hint:* Use a cube whose edges represent **i, j,** and **k.**)

30. CALCULATOR A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east. (See Preliminaries, Section 2, Exercise 46.)

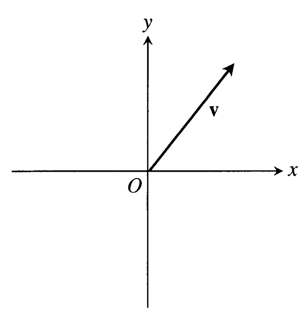


Theory and Examples

31. a) Use the fact that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ to show that the inequality $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

b) Under what circumstances, if any, does $|\mathbf{u} \cdot \mathbf{v}|$ equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.

32. Copy the axes and vector shown here. Then shade in the points (x, y) for which $(x\,\mathbf{i} + y\,\mathbf{j}) \cdot \mathbf{v} \le 0$. Justify your answer.



33. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\,\mathbf{u}_1 + b\,\mathbf{u}_2$, find $\mathbf{v} \cdot \mathbf{u}_1$.

34. *Cancellation in dot products.* In real-number multiplication, if $ab_1 = ab_2$ and a is not zero, we can cancel the a and conclude that $b_1 = b_2$. Does the same rule hold for vector multiplication: If $\mathbf{A} \cdot \mathbf{B}_1 = \mathbf{A} \cdot \mathbf{B}_2$ and $\mathbf{A} \neq 0$, can you conclude that $\mathbf{B}_1 = \mathbf{B}_2$? Give reasons for your answer.

35. Suppose **A**, **B**, and **C** are mutually orthogonal vectors. Let $\mathbf{D} = 5\mathbf{A} - 6\mathbf{B} + 3\mathbf{C}$.

a) If **A**, **B**, and **C** are unit vectors, find $|\mathbf{D}|$, the magnitude of **D**.

b) If $|\mathbf{A}| = 2$, $|\mathbf{B}| = 3$, and $|\mathbf{C}| = 4$, find $|\mathbf{D}|$.

36. Suppose **A**, **B**, and **C** are mutually orthogonal unit vectors. If **D** is a vector such that $\mathbf{D} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$ where α, β, and γ are scalars, prove that $\alpha = \mathbf{D} \cdot \mathbf{A}$, $\beta = \mathbf{D} \cdot \mathbf{B}$, and $\gamma = \mathbf{D} \cdot \mathbf{C}$.

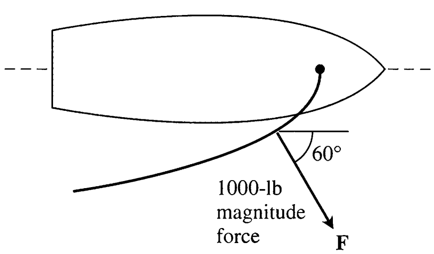
Work

37. Find the work done by a force $\mathbf{F} = 5\mathbf{k}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1, 1) (distance in meters).

38. The union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?

39. How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200-N force at an angle of 30° from the horizontal?

40. The wind passing over a boat's sail exerted a 1000-lb magnitude force **F** as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Equations for Lines in the Plane

41. Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of **v** is the negative reciprocal of the slope of the given line.

42. Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing **v** is the same as the slope of the given line.

In Exercises 43–46, use the result of Exercise 41 to find an equation for the line through P perpendicular to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

43. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

44. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$

45. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$

46. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 47–50, use the result of Exercise 42 to find an equation for the line through P parallel to **v**. Then sketch the line. Include **v** in your sketch *as a vector starting at the origin.*

47. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

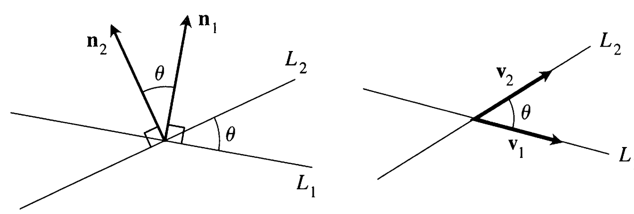
48. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

49. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$

50. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

Angles Between Lines in the Plane

The acute angle between intersecting lines that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



Use this fact and the results of Exercise 41 or 42 to find the acute angles between the lines in Exercises 51–54.

51. $3x + y = 5$, $2x - y = 4$

52. $y = \sqrt{3}x - 1$, $y = -\sqrt{3}x + 2$

53. $\sqrt{3}x - y = -2$, $x - \sqrt{3}y = 1$

54. $x + \sqrt{3}y = 1$, $(1 - \sqrt{3})x + (1 + \sqrt{3})y = 8$

CALCULATOR In Exercises 55 and 56, find the acute angle between the lines to the nearest hundredth of a radian.

55. $3x - 4y = 3$, $x - y = 7$

56. $12x + 5y = 1$, $2x - 2y = 3$

Angles Between Differentiable Curves

The angles between two differentiable curves at a point of intersection are the angles between the curves' tangent lines at these points. Find the angles between the curves in Exercises 57–60. (You will not need a calculator.)

57. $y = (3/2) - x^2$, $y = x^2$ (two points of intersection)

58. $x = (3/4) - y^2$, $x = y^2 - (3/4)$ (two points of intersection)

59. $y = x^3$, $x = y^2$ (two points of intersection)

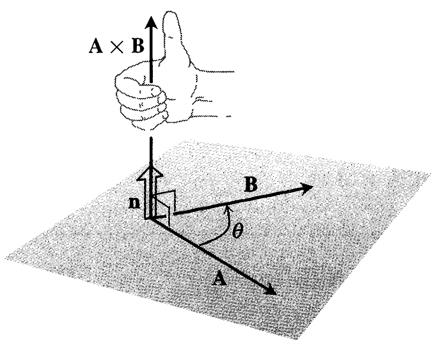
60. $y = -x^2$, $y = \sqrt[3]{x}$ (two points of intersection)

10.4 Cross Products

In studying lines in the plane, we needed to describe how a line was tilting, and we did so with the notions of slope and angle of inclination. In space, we need to be able to describe how a plane is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane. The direction of this third vector tells us the "inclination" of the plane. The product we use to multiply the vectors together is the *vector* or *cross product,* the second of the two vector multiplication methods we study in calculus.

Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics. This section presents the mathematical properties that account for the use of cross products in these fields.

The Cross Product of Two Vectors in Space



10.31 The construction of **A** × **B**.

We start with two nonzero vectors **A** and **B** in space. If **A** and **B** are not parallel, they determine a plane. We select a unit vector **n** perpendicular to the plane by the **right-hand rule.** This means we choose **n** to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from **A** to **B** (Fig. 10.31). We then define the **vector product A** × **B** ("**A** cross **B**") to be the *vector*

Definition

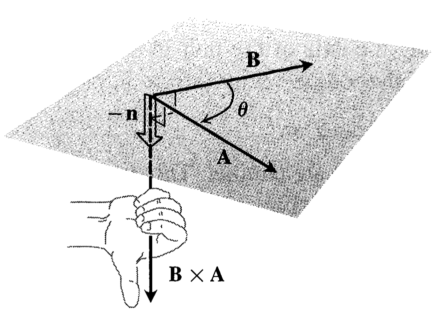
$$\mathbf{A} \times \mathbf{B} = (|\mathbf{A}||\mathbf{B}| \sin\theta)\, \mathbf{n} \qquad (1)$$

The vector **A** × **B** is orthogonal to both **A** and **B** because it is a scalar multiple of **n.** The vector product of **A** and **B** is often called the **cross product** of **A** and **B** because of the cross in the notation **A** × **B.**

Since the sines of 0 and π are both zero in Eq. (1), it makes sense to define the cross product of two parallel nonzero vectors to be **0.**

If one or both of **A** and **B** are zero, we also define **A** × **B** to be zero. This way, the cross product of two vectors **A** and **B** is zero if and only if **A** and **B** are parallel or one or both of them are zero.

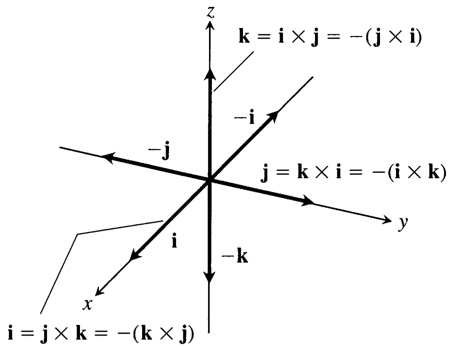
> Nonzero vectors **A** and **B** are parallel if and only if **A** × **B** = **0.**



10.32 The construction of **B** × **A**.

A x B vs. B x A

Reversing the order of the factors in a nonzero cross product reverses the direction of the product. When the fingers of our right hand curl through the angle θ from **B** to **A,** our thumb points the opposite way and the unit vector we choose in forming **B** × **A** is the negative of the one we choose in forming **A** × **B** (Fig. 10.32). Thus,



10.33 The pairwise cross products of **i, j,** and **k.**

for all vectors **A** and **B,**

$$\boxed{\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}). \qquad (2)}$$

Unlike the dot product, the cross product is not commutative.

When we apply the definition to calculate the pairwise cross products of **i, j,** and **k,** we find (Fig. 10.33)

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \end{aligned} \qquad (3)$$

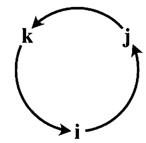


Diagram for recalling these products

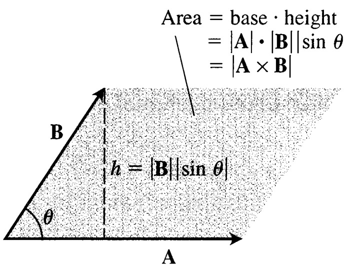
and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

| A x B | Is the Area of a Parallelogram

Because **n** is a unit vector, the magnitude of $\mathbf{A} \times \mathbf{B}$ is

$$\boxed{|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}||\sin\theta||\mathbf{n}| = |\mathbf{A}||\mathbf{B}|\sin\theta. \qquad (4)}$$



10.34 The parallelogram determined by **A** and **B.**

This is the area of the parallelogram determined by **A** and **B** (Fig. 10.34), $|\mathbf{A}|$ being the base of the parallelogram and $|\mathbf{B}||\sin\theta|$ the height.

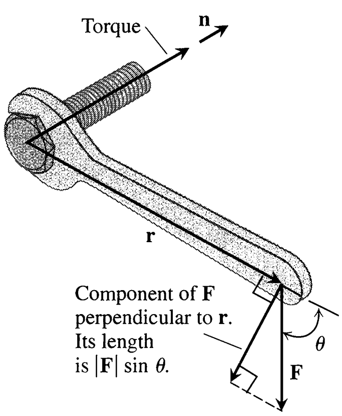
Torque

When we turn a bolt by applying a force **F** to a wrench (Fig. 10.35), the torque we produce acts along the axis of the bolt to drive the bolt forward. The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm **r** and the scalar component of **F** perpendicular to **r.** In the notation of Fig. 10.35,

$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}|\sin\theta,$$

or $|\mathbf{r} \times \mathbf{F}|$. If we let **n** be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r} \times \mathbf{F}$, or

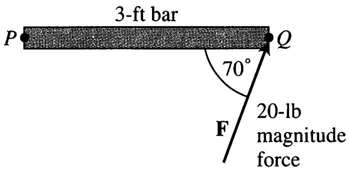
$$\text{Torque vector} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$$



10.35 The torque vector describes the tendency of the force **F** to drive the bolt forward.

Recall that we defined $\mathbf{A} \times \mathbf{B}$ to be $\mathbf{0}$ when **A** and **B** are parallel. This is consistent with the torque interpretation as well. If the force **F** in Fig. 10.35 is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

EXAMPLE 1 The magnitude of the torque exerted by force **F** about the pivot



10.36 The magnitude of the torque exerted by **F** and P is about 56.4 ft-lb (Example 1).

point P in Fig. 10.36 is

$$\begin{aligned} |\overrightarrow{PQ} \times \mathbf{F}| &= |\overrightarrow{PQ}||\mathbf{F}| \sin 70^\circ \qquad \text{Eq. (4)} \\ &\approx (3)(20)(0.94) \\ &\approx 56.4 \text{ ft-lb.} \end{aligned}$$

❑

The Associative and Distributive Laws

As a rule, cross-product multiplication is *not associative* because $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ lies in the plane of **A** and **B** whereas $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ lies in the plane of **B** and **C.** However, the following laws do hold.

Scalar Distributive Law

$$(r\mathbf{A}) \times (s\mathbf{B}) = (rs)(\mathbf{A} \times \mathbf{B}) \qquad (5)$$

Vector Distributive Laws

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad (6)$$

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A} \qquad (7)$$

As a special case of Eq. (5) we also have

$$(-\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (-\mathbf{B}) = -(\mathbf{A} \times \mathbf{B}). \qquad (8)$$

The Scalar Distributive Law can be verified by applying Eq. (1) to the products on both sides of Eq. (5) and comparing the results. The Vector Distributive Law in Eq. (6) is not easy to prove. We will assume it here and leave the proof to Appendix 7. Equation (7) follows from Eq. (6): Multiply both sides of Eq. (6) by -1 and reverse the orders of the products.

The Determinant Formula for A x B

Our next objective is to calculate $\mathbf{A} \times \mathbf{B}$ from the components of **A** and **B** relative to a Cartesian coordinate system.

Suppose

$$\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \qquad \mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}.$$

Then the distributive laws and the rules for multiplying **i, j,** and **k** tell us that

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) \times (b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \\ &= a_1b_1\,\mathbf{i} \times \mathbf{i} + a_1b_2\,\mathbf{i} \times \mathbf{j} + a_1b_3\,\mathbf{i} \times \mathbf{k} \\ &\quad + a_2b_1\,\mathbf{j} \times \mathbf{i} + a_2b_2\,\mathbf{j} \times \mathbf{j} + a_2b_3\,\mathbf{j} \times \mathbf{k} \\ &\quad + a_3b_1\,\mathbf{k} \times \mathbf{i} + a_3b_2\,\mathbf{k} \times \mathbf{j} + a_3b_3\,\mathbf{k} \times \mathbf{k} \\ &= (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}. \end{aligned}$$

Determinants

(For more information, see Appendix 8.)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

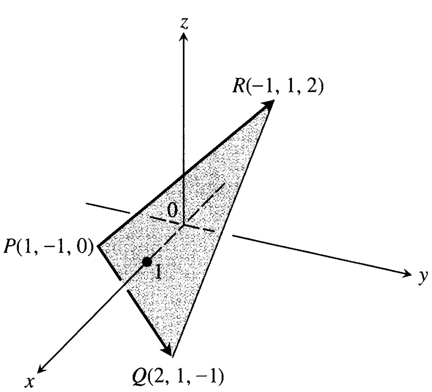
EXAMPLE

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (1)(-4)$$
$$= 6 + 4 = 10 \qquad \square$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE

$$\begin{vmatrix} -5 & 3 & 1 \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$
$$= (-5)\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - (3)\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}$$
$$+ (1)\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$
$$= -5(1-3) - 3(2+4) + 1(6+4)$$
$$= 10 - 18 + 10 = 2 \qquad \square$$



10.37 The area of triangle PQR is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ (Example 4).

The terms in the last line are the same as the terms in the expansion of the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

We therefore have the following rule.

> If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then
>
> $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (9)$$

EXAMPLE 2 Find $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ if

$$\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \mathbf{B} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

Solution

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k} \qquad \text{Eq. (9)}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \qquad \text{Eq. (2)}$$

$\square$

EXAMPLE 3 Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}. \qquad \square$$

EXAMPLE 4 Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Fig. 10.37).

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| \qquad \text{Values from Example 3}$$
$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$. ❑

EXAMPLE 5 Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane. Taking values from Examples 3 and 4, we have

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$ ❑

The Triple Scalar or Box Product

The product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ is called the **triple scalar product** of **A, B,** and **C** (in that order). As you can see from the formula

$$|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| = |\mathbf{A} \times \mathbf{B}||\mathbf{C}||\cos\theta|, \tag{10}$$

the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determined by **A, B,** and **C** (Fig. 10.38). The number $|\mathbf{A} \times \mathbf{B}|$ is the area of the base parallelogram. The number $|\mathbf{C}||\cos\theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ is also called the **box product** of **A, B,** and **C.**

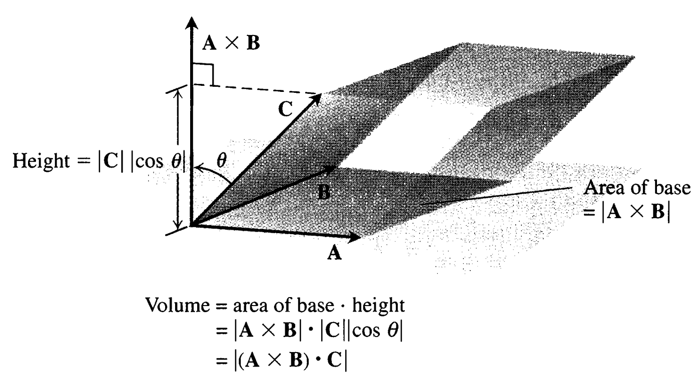
The dot and cross may be interchanged in a triple scalar product without altering its value.

By treating the planes of **B** and **C** and of **C** and **A** as the base planes of the parallelepiped determined by **A, B,** and **C,** we see that

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}. \tag{11}$$

Since the dot product is commutative, Eq. (11) also gives

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \tag{12}$$



10.38 The number $|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}|$ is the volume of a parallelepiped.

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\cdot\left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix}\mathbf{i}-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix}\mathbf{j}+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}\mathbf{k}\right] \\ &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}\end{aligned}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} \tag{13}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{A}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{B}=-2\mathbf{i}+3\mathbf{k}$, and $\mathbf{C}=7\mathbf{j}-4\mathbf{k}$.

Solution

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4\end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 7 & -4\end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4\end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7\end{vmatrix} \\ &= -21-16+14=-23\end{aligned}$$

The volume is $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = 23$. ❑

Exercises 10.4

Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{A}\times\mathbf{B}$ and $\mathbf{B}\times\mathbf{A}$.

1. $\mathbf{A}=2\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $\mathbf{B}=\mathbf{i}-\mathbf{k}$
2. $\mathbf{A}=2\mathbf{i}+3\mathbf{j}$, $\mathbf{B}=-\mathbf{i}+\mathbf{j}$
3. $\mathbf{A}=2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$, $\mathbf{B}=-\mathbf{i}+\mathbf{j}-2\mathbf{k}$
4. $\mathbf{A}=\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{B}=\mathbf{0}$
5. $\mathbf{A}=2\mathbf{i}$, $\mathbf{B}=-3\mathbf{j}$
6. $\mathbf{A}=\mathbf{i}\times\mathbf{j}$, $\mathbf{B}=\mathbf{j}\times\mathbf{k}$
7. $\mathbf{A}=-8\mathbf{i}-2\mathbf{j}-4\mathbf{k}$, $\mathbf{B}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$
8. $\mathbf{A}=\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+\mathbf{k}$, $\mathbf{B}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}\times\mathbf{B}$ as vectors starting at the origin.

9. $\mathbf{A}=\mathbf{i}$, $\mathbf{B}=\mathbf{j}$
10. $\mathbf{A}=\mathbf{i}-\mathbf{k}$, $\mathbf{B}=\mathbf{j}$
11. $\mathbf{A}=\mathbf{i}-\mathbf{k}$, $\mathbf{B}=\mathbf{j}+\mathbf{k}$
12. $\mathbf{A}=2\mathbf{i}-\mathbf{j}$, $\mathbf{B}=\mathbf{i}+2\mathbf{j}$
13. $\mathbf{A}=\mathbf{i}+\mathbf{j}$, $\mathbf{B}=\mathbf{i}-\mathbf{j}$
14. $\mathbf{A}=\mathbf{j}+2\mathbf{k}$, $\mathbf{B}=\mathbf{i}$

In Exercises 15–18:

a) Find the area of the triangle determined by the points P, Q, and R.
b) Find a unit vector perpendicular to plane PQR.

15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$

16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$

17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$

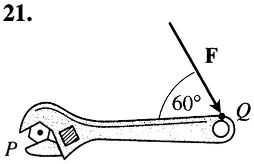
18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

19. Let $\mathbf{A} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{C} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular, (b) parallel? Give reasons for your answers.

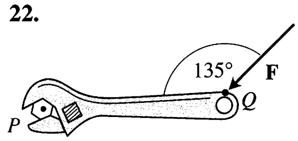
20. Let $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{B} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{C} = \mathbf{i} + \mathbf{k}$, $\mathbf{D} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are (a) perpendicular, (b) parallel? Give reasons for your answers.

In Exercises 21 and 22, find the magnitude of the torque exerted by $\mathbf{F}$ on the bolt at P if $|\overrightarrow{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

21.



22.



In Exercises 23–26, verify that $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$ and find the volume of the parallelepiped (box) determined by **A**, **B**, and **C**.

	A	B	C
23.	$2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
24.	$\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
25.	$2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
26.	$\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Theory and Examples

27. Which of the following are *always true* and which are *not always true?* Give reasons for your answers.

a) $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$
b) $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|$
c) $\mathbf{A} \times \mathbf{0} = \mathbf{0} \times \mathbf{A} = \mathbf{0}$
d) $\mathbf{A} \times (-\mathbf{A}) = \mathbf{0}$
e) $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$
f) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
g) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = 0$
h) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

28. Which of the following are *always true* and which are *not always true?* Give reasons for your answers.

a) $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
b) $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$
c) $(-\mathbf{A}) \times \mathbf{B} = -(\mathbf{A} \times \mathbf{B})$
d) $(c\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (c\mathbf{B}) = c(\mathbf{A} \cdot \mathbf{B})$ (any number c)
e) $c(\mathbf{A} \times \mathbf{B}) = (c\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (c\mathbf{B})$ (any number c)
f) $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$
g) $(\mathbf{A} \times \mathbf{A}) \cdot \mathbf{A} = 0$
h) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B})$

29. Given nonzero vectors **A**, **B**, and **C**, use dot-product and cross-product notation, as appropriate, to describe the following.

a) The vector projection of **A** onto **B**
b) A vector orthogonal to **A** and **B**
c) A vector orthogonal to $\mathbf{A} \times \mathbf{B}$ and **C**
d) The volume of the parallelepiped determined by **A**, **B**, and **C**

30. Given nonzero vectors **A**, **B**, and **C**, use dot-product and cross-product notation to describe the following.

a) A vector orthogonal to $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \times \mathbf{C}$
b) A vector orthogonal to $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$
c) A vector of length $|\mathbf{A}|$ in the direction of **B**
d) The area of the parallelogram determined by **A** and **C**

31. Let **A**, **B**, and **C** be vectors. Which of the following make sense, and which do not? Give reasons for your answers.

a) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
b) $\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$
c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
d) $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$

32. Show that except in degenerate cases $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ lies in the plane of **A** and **B** while $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ lies in the plane of **B** and **C**. What *are* the degenerate cases?

33. *Cancellation in cross products.* If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ and $\mathbf{A} \neq \mathbf{0}$, then does $\mathbf{B} = \mathbf{C}$? Give reasons for your answer.

34. *Double cancellation.* If $\mathbf{A} \neq \mathbf{0}$, and if $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ and $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$, then does $\mathbf{B} = \mathbf{C}$? Give reasons for your answers.

Area in the Plane

Find the areas of the parallelograms whose vertices are given in Exercises 35–38.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$

36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$

37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$

38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$

Find the areas of the triangles whose vertices are given in Exercises 39–42.

39. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$

40. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$

41. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$

42. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$

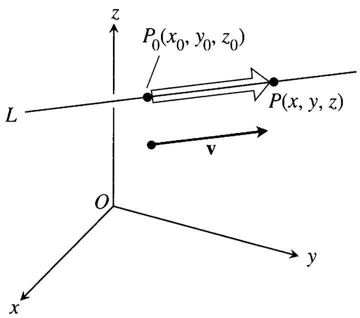
43. Find a formula for the area of the triangle in the xy-plane with vertices at (0, 0), (a_1, a_2), and (b_1, b_2). Explain your work.

44. Find a concise formula for the area of a triangle with vertices (a_1, a_2), (b_1, b_2), and (c_1, c_2).

10.5

Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.



10.39 A point P lies on the line through P_0 parallel to **v** if and only if $\overrightarrow{P_0P}$ is a scalar multiple of **v**.

Lines and Line Segments in Space

Suppose L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to **v** (Fig. 10.39). That is, P lies on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of **v**.

Vector Equation for the Line Through $P_0(x_0, y_0, z_0)$ Parallel to v

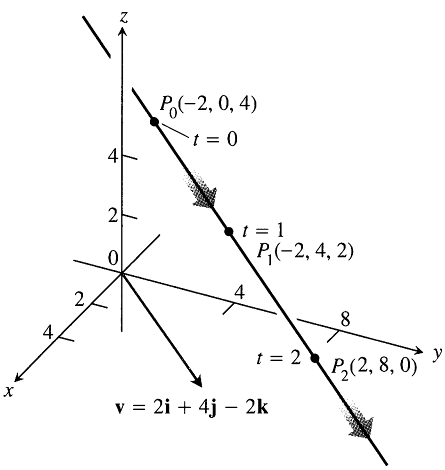
$$\overrightarrow{P_0P} = t\,\mathbf{v}, \qquad -\infty < t < \infty \tag{1}$$

Equating the corresponding components of the two sides of Eq. (1) gives three scalar equations involving the parameter t:

$$(x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k} = t(A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}), \qquad \text{Eq. (1) expanded}$$

$$x - x_0 = tA, \qquad y - y_0 = tB, \qquad z - z_0 = tC. \qquad \text{Components equated}$$

When rearranged, these equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$.



10.40 Selected points and parameter values on the line $x = -2 + 2t$, $y = 4t$, $z = 4 - 2t$. The arrows show the direction of increasing t (Example 1).

Standard Parametrization of the Line Through $P_0(x_0, y_0, z_0)$ Parallel to $\mathbf{v} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$

$$x = x_0 + tA, \qquad y = y_0 + tB, \qquad z = z_0 + tC, \qquad -\infty < t < \infty \tag{2}$$

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$ (Fig. 10.40).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ equal to $2\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$, Eqs. (2) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t. \qquad \square$$

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\,\mathbf{i} + (-1 - 2)\,\mathbf{j} + (4 - (-3))\,\mathbf{k} \\ &= 4\,\mathbf{i} - 3\,\mathbf{j} + 7\,\mathbf{k}\end{aligned}$$

is parallel to the line, and Eqs. (2) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

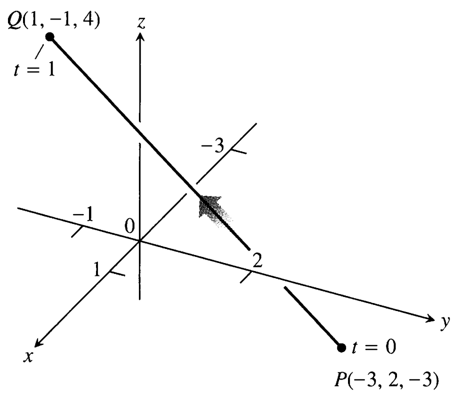
We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

Notice that parametrizations are not unique.

These equations serve as well as the first; they simply place you at a different point for a given value of t. ❑

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the t-values for the endpoints and restrict t to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.



10.41 Example 3 derives a parametrization of line segment *PQ*. The arrow shows the direction of increasing *t*.

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Fig. 10.41).

Solution We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t. \tag{3}$$

We observe that the point

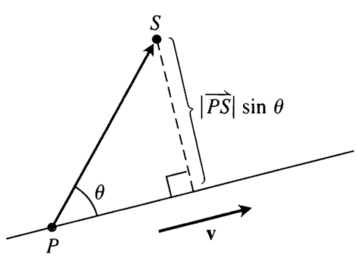
$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \le t \le 1$ to Eqs. (3) to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \le t \le 1.$$ ❑

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the length of the component of $\overrightarrow{PS}$ normal to the line (Fig. 10.42). In the notation of the figure, the length is $|\overrightarrow{PS}| \sin\theta$, which is $|\overrightarrow{PS} \times \mathbf{v}|/|\mathbf{v}|$.



10.42 The distance from *S* to the line through *P* parallel to **v** is $|\overrightarrow{PS}| \sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and **v**.

Distance from a Point *S* to a Line Through *P* Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{4}$$

EXAMPLE 4 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \qquad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

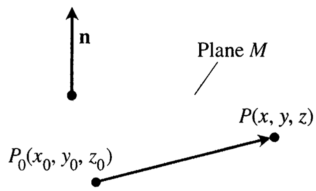
$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Eq. (4) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

❑

Equations for Planes in Space



10.43 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Suppose plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal (perpendicular) to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (Fig. 10.43). That is, P lies on M if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Plane Through $P_0(x_0, y_0, z_0)$ Normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$	(5)
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$	(6)

EXAMPLE 5 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0 \qquad \text{Eq. (5) with the given } \mathbf{n} \text{ and } P_0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22$$

❑

Notice in Example 5 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x, y, and z in the equation $5x + 2y - z = -22$. This was no accident, because Eq. (6), when rearranged, takes the form

$$Ax + By + Cz = D,$$

where $D = Ax_0 + By_0 + Cz_0$.

$A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 6 *A plane determined by three points*

Find the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of the point $(0, 0, 1)$ into Eq. (6) to get

$$3(x-0) + 2(y-0) + 6(z-1) = 0,$$
$$3x + 2y + 6z = 6. \qquad \square$$

EXAMPLE 7 *The intersection of a line and a plane*

Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane; that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right). \qquad \square$$

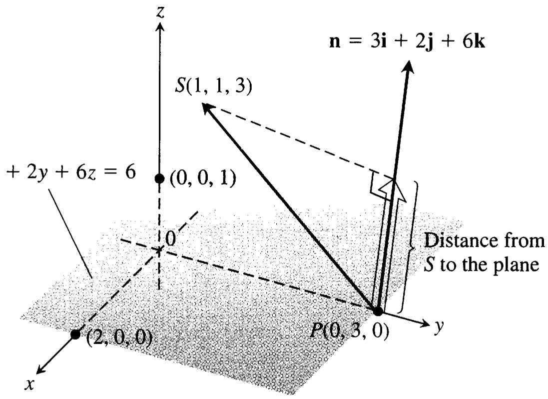
EXAMPLE 8 *The distance from a point to a plane*

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Fig. 10.44, on the following page).

The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$



10.44 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 8).

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept (0, 3, 0), then

$$\begin{aligned}\overrightarrow{PS} &= (1-0)\,\mathbf{i} + (1-3)\,\mathbf{j} + (3-0)\,\mathbf{k} \\ &= \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}, \\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.\end{aligned}$$

The distance from S to the plane is

$$\begin{aligned}d &= \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \qquad \text{length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \\ &= \left|(\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right| \\ &= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.\end{aligned}$$

❑

How to Find the Distance from a Point to a Plane

To find the distance from a point S to a plane $Ax + By + Cz = D$, find

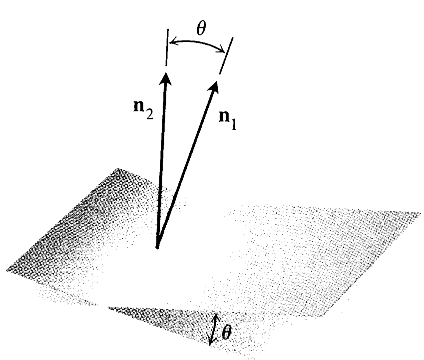
1. a point P on the plane,
2. $\overrightarrow{PS}$,
3. the direction of $\mathbf{n} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$.

Then calculate the distance as

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|.$$

Angles Between Planes; Lines of Intersection

The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors (Fig. 10.45).



10.45 The angle between two planes is obtained from the angle between their normals.

EXAMPLE 9 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

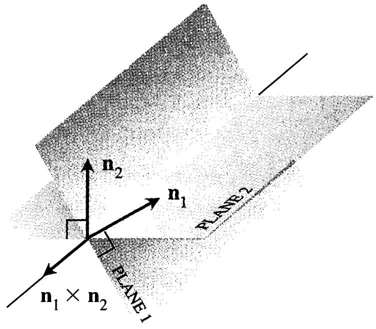
Solution The vectors

$$\mathbf{n}_1 = 3\,\mathbf{i} - 6\,\mathbf{j} - 2\,\mathbf{k}, \qquad \mathbf{n}_2 = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) \qquad \text{Eq. (4), Section 10.3} \\ &= \cos^{-1}\left(\frac{4}{21}\right) \\ &\approx 1.38 \text{ rad} \qquad \text{About 79}\end{aligned}$$

❑



10.46 How the line of intersection of two planes is related to the planes' normal vectors (Example 10).

EXAMPLE 10 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to the planes' normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (Fig. 10.46), and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\,\mathbf{i} + 2\,\mathbf{j} + 15\,\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ❑

EXAMPLE 11 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use Eqs. (2).

Example 10 identifies $\mathbf{v} = 14\,\mathbf{i} + 2\,\mathbf{j} + 15\,\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$ ❑

Exercises 10.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\,\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z-axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$
10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{A} = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{B} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$
11. The x-axis
12. The z-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$
14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$
16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$
18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$
20. $(1, 0, -1)$, $(0, 3, 0)$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$
23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$

25. The plane through $P_0(2, 4, 5)$ perpendicular to the line
$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A

27. Find the point of intersection of the lines $x = 2t + 1, y = 3t + 2, z = 4t + 3$, and $x = s + 2, y = 2s + 4, z = -4s - 1$, and then find the plane determined by these lines.

28. Find the point of intersection of the lines $x = t, y = -t + 2, z = t + 1$, and $x = 2s + 2, y = s + 3, z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane determined by the intersecting lines.

29. $L1$: $x = -1 + t,\ y = 2 + t,\ z = 1 - t,\ -\infty < t < \infty$
$L2$: $x = 1 - 4s,\ y = 1 + 2s,\ z = 2 - 2s,\ -\infty < s < \infty$

30. $L1$: $x = t,\ y = 3 - 3t,\ z = -2 - t,\ -\infty < t < \infty$
$L2$: $x = 1 + s,\ y = 4 + s,\ z = -1 + s,\ -\infty < s < \infty$

31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3, x + 2y + z = 2$.

32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $x = 4t, \quad y = -2t, \quad z = 2t$

34. $(0, 0, 0)$; $x = 5 + 3t, \quad y = 5 + 4t, \quad z = -3 - 5t$

35. $(2, 1, 3)$; $x = 2 + 2t, \quad y = 1 + 6t, \quad z = 3$

36. $(2, 1, -1)$; $x = 2t, \quad y = 1 + 2t, \quad z = 2t$

37. $(3, -1, 4)$; $x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

38. $(-1, 4, 3)$; $x = 10 + 4t, \quad y = -3, \quad z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $\quad x + 2y + 2z = 13$

40. $(0, 0, 0)$, $\quad 3x + 2y + 6z = 6$

41. $(0, 1, 1)$, $\quad 4y + 3z = -12$

42. $(2, 2, 3)$, $\quad 2x + y + 2z = 4$

43. $(0, -1, 0)$, $\quad 2x + y + 2z = 4$

44. $(1, 0, -1)$, $\quad -4x + y + z = 4$

45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48. (You will not need a calculator.)

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

CALCULATOR Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

50. $x + y + z = 1, \quad z = 0$ (the xy-plane)

51. $2x + 2y - z = 3, \quad x + 2y + z = 2$

52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

54. $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

55. $x = 1 + 2t, \quad y = 1 + 5t, \quad z = 3t; \quad x + y + z = 2$

56. $x = -1 + 3t, \quad y = -2, \quad z = 5t; \quad 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, \quad x + y = 2$

58. $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$

59. $x - 2y + 4z = 2, \quad x + y - 2z = 5$

60. $5x - 2y = 11, \quad 4y - 5z = -17$

Given two lines in space, either they are parallel, or they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky). Exercises 61 and 62 each give three lines. In each exercise, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection.

61. $L1$: $x = 3 + 2t, y = -1 + 4t, z = 2 - t, -\infty < t < \infty$
$L2$: $x = 1 + 4s, y = 1 + 2s, z = -3 + 4s, -\infty < s < \infty$
$L3$: $x = 3 + 2r, y = 2 + r, z = -2 + 2r, -\infty < r < \infty$

62. $L1$: $x = 1 + 2t, y = -1 - t, z = 3t, -\infty < t < \infty$
$L2$: $x = 2 - s, y = 3s, z = 1 + s, -\infty < s < \infty$
$L3$: $x = 5 + 2r, y = 1 - r, z = 8 + 3r, -\infty < r < \infty$

Theory and Examples

63. Use Eqs. (2) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.

64. Use Eq. (6) to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} - \sqrt{2}\,\mathbf{k}$.

65. Find the points in which the line $x = 1 + 2t, y = -1 - t, z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.

66. Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with $\mathbf{i}$ and an angle of $\pi/3$ rad with $\mathbf{j}$. Describe the reasoning behind your answer.

67. Is the line $x = 1 - 2t$, $y = 2 + 5t$, $z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.

68. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? perpendicular? Give reasons for your answer.

69. Find two different planes whose intersection is the line $x = 1 + t$, $y = 2 - t$, $z = 3 + 2t$. Write equations for each plane in the form $Ax + By + Cz = D$.

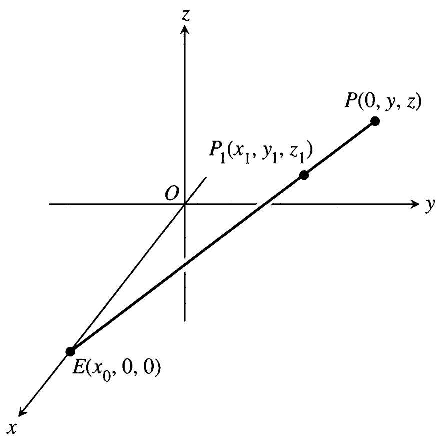
70. Find a plane through the origin that meets the plane M: $2x + 3y + z = 12$ in a right angle. How do you know that your plane is perpendicular to M?

71. For any nonzero numbers a, b, and c, the graph of $(x/a) + (y/b) + (z/c) = 1$ is a plane. Which planes have an equation of this form?

72. Suppose L_1 and L_2 are disjoint (nonintersecting) nonparallel lines. Is it possible for a nonzero vector to be perpendicular to both L_1 and L_2? Give reasons for your answer.

Computer Graphics

73. *Perspective in computer graphics.* In computer graphics and perspective drawing we need to represent objects seen by the eye in space as images on a two-dimensional plane. Suppose the eye is at $E(x_0, 0, 0)$ as shown here and that we want to represent a point $P_1(x_1, y_1, z_1)$ as a point on the yz–plane. We do this by projecting P_1 onto the plane with a ray from E. The point P_1 will be portrayed as the point $P(0, y, z)$. The problem for us as graphics designers is to find y and z given E and P_1.



a) Write a vector equation that holds between $\overrightarrow{EP}$ and $\overrightarrow{EP_1}$. Use the equation to express y and z in terms of x_0, x_1, y_1, and z_1.

b) Test the formulas obtained for y and z in part (a) by investigating their behavior at $x_1 = 0$ and $x_1 = x_0$ and by seeing what happens as $x_0 \to \infty$.

74. *Hidden lines.* Here is another typical problem in computer graphics. Your eye is at $(4, 0, 0)$. You are looking at a triangular plate whose vertices are at $(1, 0, 1)$, $(1, 1, 0)$, and $(-2, 2, 2)$. The line segment from $(1, 0, 0)$ to $(0, 2, 2)$ passes through the plate. What portion of the line segment is hidden from your view by the plate? (This is an exercise in finding intersections of lines and planes.)

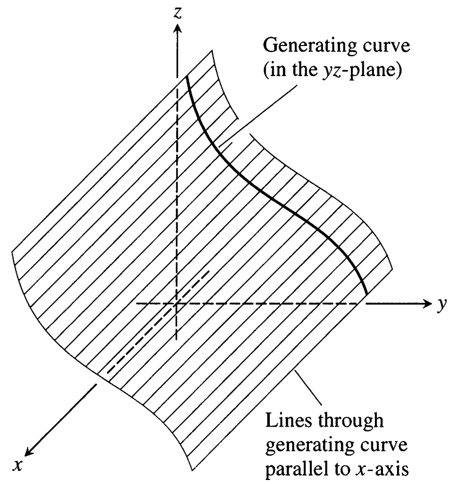
10.6 Cylinders and Quadric Surfaces

In the calculus of functions of a single variable, we began with lines and used our knowledge of lines to study curves in the plane. We investigated tangents and found that, when highly magnified, differentiable curves were effectively linear. Among the curves of special interest were the conic sections, curves defined by second-degree equations in x and y.

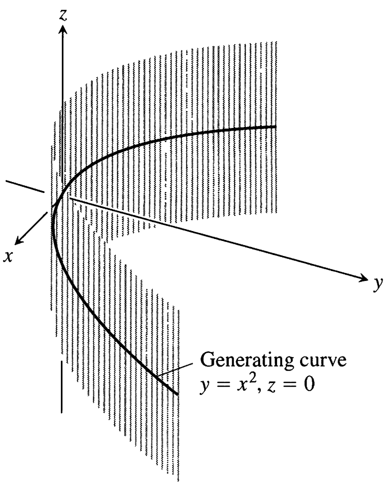
To study the calculus of functions of more than one variable, we begin much the same way. We start with planes and use our knowledge of planes to study surfaces in space. Among the surfaces of special interest are cylinders and quadric surfaces, surfaces defined by second-degree equations in x, y, and z. We described the planes in Section 10.5. In this section, we describe the surfaces.

Cylinders

A **cylinder** is the surface composed of all the lines that (1) lie parallel to a given line in space and (2) pass through a given plane curve. The curve is a **generating curve** for the cylinder (Fig. 10.47, on the following page). In solid geometry, where *cylinder* means *circular cylinder,* the generating curves are circles, but now we allow generating curves of any kind. The cylinder in our first example is generated by a parabola.



10.47 A cylinder and generating curve.



10.48 The cylinder of lines passing through the parabola $y = x^2$ in the xy-plane parallel to the z-axis (Example 1).

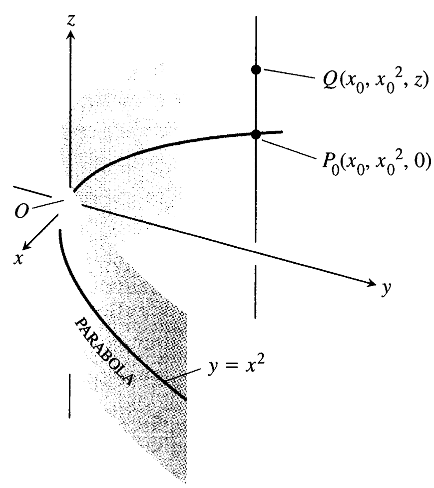
When graphing a cylinder or other surface by hand or analyzing one generated by a computer, it helps to look at the curves formed by intersecting the surface with planes parallel to the coordinate planes. These curves are called **cross sections** or **traces.**

EXAMPLE 1 *The parabolic cylinder* $y = x^2$

Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$ (Fig. 10.48).

Solution Suppose that the point $P_0(x_0, x_0{}^2, 0)$ lies on the parabola $y = x^2$ in the xy-plane. Then, for any value of z, the point $Q(x_0, x_0{}^2, z)$ will lie on the cylinder because it lies on the line $x = x_0$, $y = x_0{}^2$ through P_0 parallel to the z-axis. Conversely, any point $Q(x_0, x_0{}^2, z)$ whose y-coordinate is the square of its x-coordinate lies on the cylinder because it lies on the line $x = x_0$, $y = x_0{}^2$ through P_0 parallel to the z-axis (Fig. 10.49).

Regardless of the value of z, therefore, the points on the surface are the points whose coordinates satisfy the equation $y = x^2$. This makes $y = x^2$ an equation for the cylinder. Because of this, we call the cylinder "the cylinder $y = x^2$."



10.49 Every point of the cylinder in Fig. 10.48 has coordinates of the form $(x_0, x_0{}^2, z)$. We call the cylinder "the cylinder $y = x^2$."

❑

DRAWING LESSON

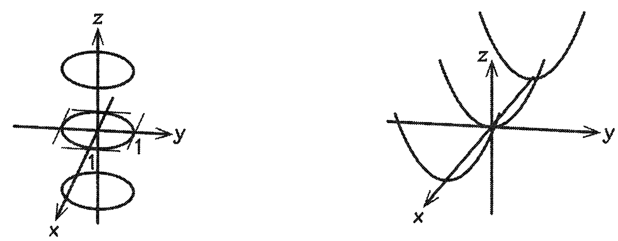
How to Draw Cylinders Parallel to the Coordinate Axes

$x^2 + y^2 = 1$ $\qquad$ $z = y^2$

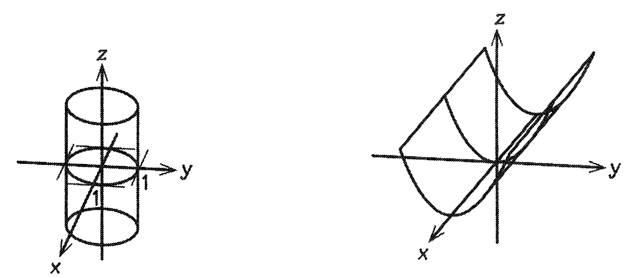
1 Sketch all three coordinate axes *very lightly.*

2 Sketch the trace of the cylinder in the coordinate plane of the two variables that appear in the cylinder's equation. Sketch *very lightly.*

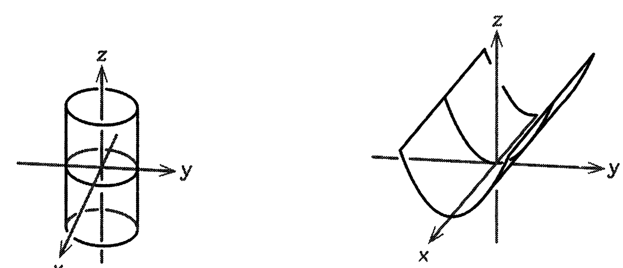
3 Sketch traces in parallel planes on either side (again, *lightly*).

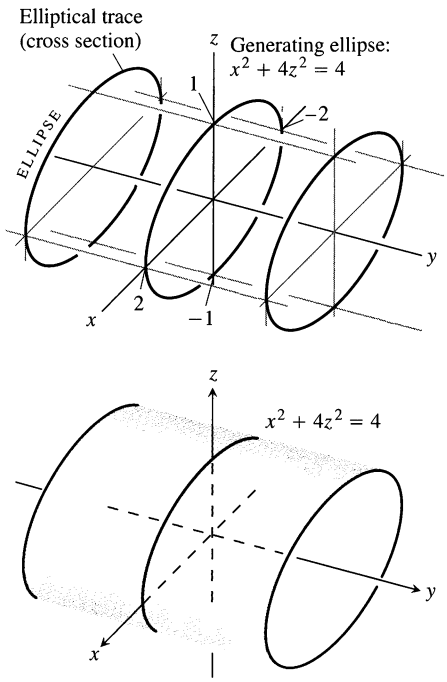


4 Add parallel outer edges to give the shape definition.



5 If more definition is required, darken the parts of the lines that are exposed to view. Leave the hidden parts light. Use line breaks when you can.

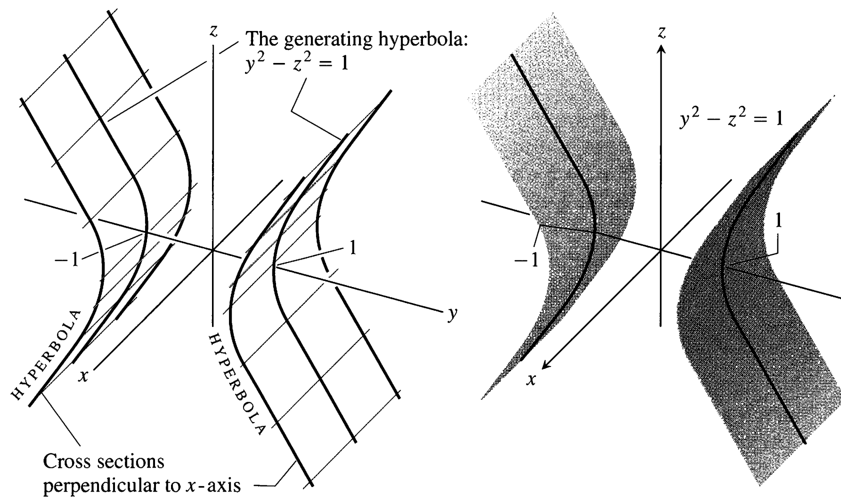




10.50 The elliptic cylinder $x^2 + 4z^2 = 4$ is made of lines parallel to the y-axis and passing through the ellipse $x^2 + 4z^2 = 4$ in the xz-plane. The cross sections or "traces" of the cylinder in planes perpendicular to the y-axis are ellipses congruent to the generating ellipse. The cylinder extends along the entire y-axis.

As Example 1 suggests, any curve $f(x, y) = c$ in the xy-plane defines a cylinder parallel to the z-axis whose equation is also $f(x, y) = c$. The equation $x^2 + y^2 = 1$ defines the circular cylinder made by the lines parallel to the z-axis that pass through the circle $x^2 + y^2 = 1$ in the xy-plane. The equation $x^2 + 4y^2 = 9$ defines the elliptical cylinder made by the lines parallel to the z-axis that pass through the ellipse $x^2 + 4y^2 = 9$ in the xy-plane.

In a similar way, any curve $g(x, z) = c$ in the xz-plane defines a cylinder parallel to the y-axis whose space equation is also $g(x, z) = c$ (Fig. 10.50). Any curve $h(y, z) = c$ defines a cylinder parallel to the x-axis whose space equation is also $h(y, z) = c$ (Fig. 10.51).



10.51 The hyperbolic cylinder $y^2 - z^2 = 1$ is made of lines parallel to the x-axis and passing through the hyperbola $y^2 - z^2 = 1$ in the yz-plane. The cross sections of the cylinder in planes perpendicular to the x-axis are hyperbolas congruent to the generating hyperbola.

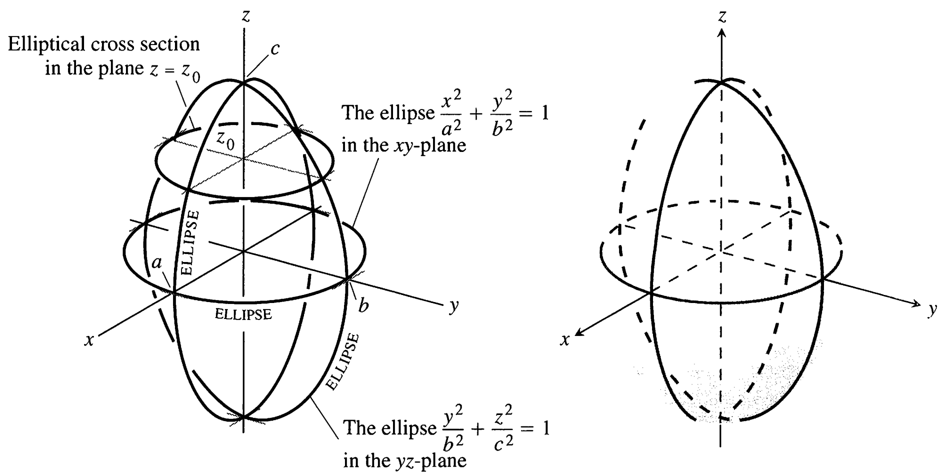
> An equation in any two of the three Cartesian coordinates defines a cylinder parallel to the axis of the third coordinate.

Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in x, y, and z. The most general form is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0,$$

where A, B, C, and so on are constants, but the equation can be simplified by translation and rotation, as in the two-dimensional case in Section 9.3. We will study only the simpler equations. Although the definition did not require it, the cylinders in Figs. 10.49–10.51 were also examples of quadric surfaces. We now examine ellipsoids (these include spheres), paraboloids, cones, and hyperboloids.



10.52 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2.

EXAMPLE 2 The **ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

(Fig. 10.52) cuts the coordinate axes at $(\pm\ a, 0, 0)$, $(0, \pm\ b, 0)$, and $(0, 0, \pm\ c)$. It lies within the rectangular box defined by the inequalities $|x| \leq a$, $|y| \leq b$, and $|z| \leq c$. The surface is symmetric with respect to each of the coordinate planes because the variables in the defining equation are squared.

The curves in which the three coordinate planes cut the surface are ellipses. For example,

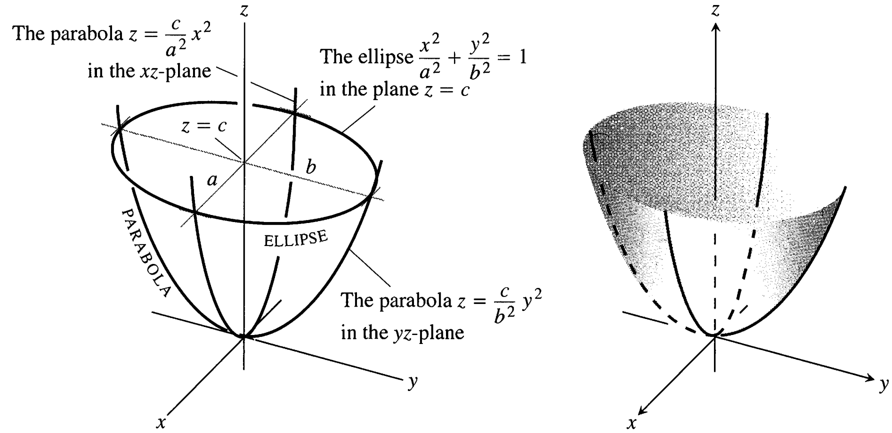
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{when} \qquad z = 0.$$

The section cut from the surface by the plane $z = z_0$, $|z_0| < c$, is the ellipse

$$\frac{x^2}{a^2(1 - (z_0/c)^2)} + \frac{y^2}{b^2(1 - (z_0/c)^2)} = 1. \tag{2}$$

If any two of the semiaxes a, b, and c are equal, the surface is an **ellipsoid of revolution.** If all three are equal, the surface is a sphere. ❑

Technology *Visualizing in Space* A Computer Algebra System (CAS) or other computer graphing utility can help in visualizing surfaces in space. It can draw traces in different planes with far more patience than most people can muster. Many computer graphing systems can rotate the figure so you can see it as if it were a physical model you could turn in your hand. Hidden-line algorithms (see Exercise 74, Section 10.5) are used to block out portions of the surface that you would not see from your current viewing angle. Often a CAS will require surfaces to be entered in parametric form, as discussed in Section 14.6 (see also CAS Exercises 57–60 in Section 12.1). Sometimes you may have to manipulate the grid mesh to see all portions of a surface.



10.53 The elliptic paraboloid $(x^2/a^2) + (y^2/b^2) = z/c$ in Example 3, shown for $c > 0$. The cross sections perpendicular to the z-axis above the xy-plane are ellipses. The cross sections in the planes that contain the z-axis are parabolas.

EXAMPLE 3 The **elliptic paraboloid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \tag{3}$$

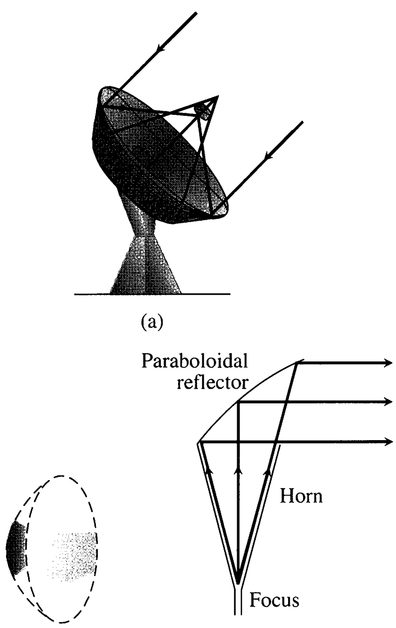
is symmetric with respect to the planes $x = 0$ and $y = 0$ (Fig. 10.53). The only intercept on the axes is the origin. Except for this point, the surface lies above or entirely below the xy-plane, depending on the sign of c. The sections cut by the coordinate planes are

$$\begin{aligned} x = 0: &\quad \text{the parabola } z = \frac{c}{b^2}y^2 \\ y = 0: &\quad \text{the parabola } z = \frac{c}{a^2}x^2 \\ z = 0: &\quad \text{the point } (0, 0, 0). \end{aligned} \tag{4}$$

Each plane $z = z_0$ above the xy-plane cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0}{c}.$$

❑



10.54 Many antennas are shaped like pieces of paraboloids of revolution. (a) Radio telescopes use the same principles as optical telescopes. (b) A "rectangular-cut" radar reflector. (c) The profile of a horn antenna in a microwave radio link.

EXAMPLE 4 The **circular paraboloid** or **paraboloid of revolution**

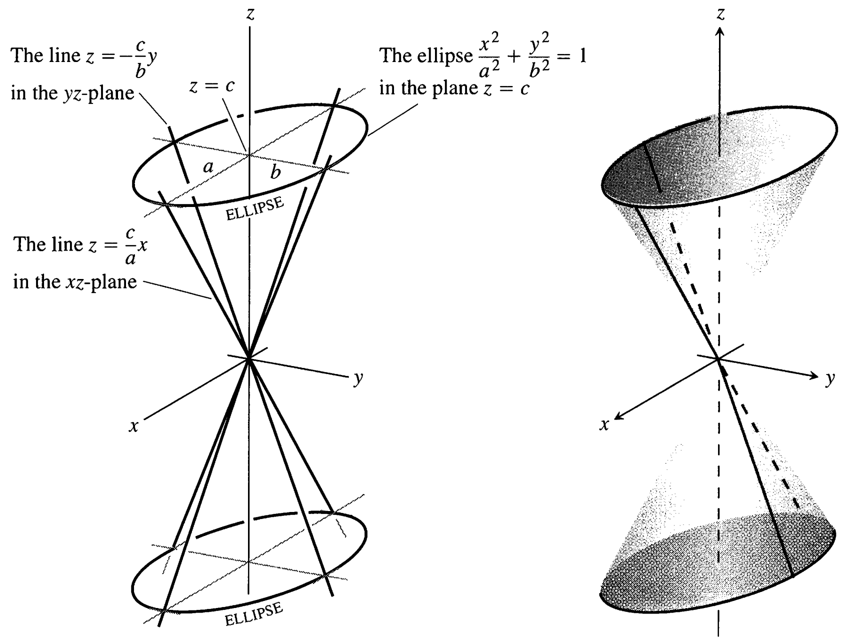
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z}{c} \tag{5}$$

is obtained by taking $b = a$ in Eq. (3) for the elliptic paraboloid. The cross sections of the surface by planes perpendicular to the z-axis are circles centered on the z-axis. The cross sections by planes containing the z-axis are congruent parabolas with a common focus at the point $(0, 0, a^2/4c)$.

Shapes cut from circular paraboloids are used for antennas in radio telescopes, satellite trackers, and microwave radio links (Fig. 10.54). ❑

EXAMPLE 5 The **elliptic cone**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \tag{6}$$



10.55 The elliptic cone $(x^2/a^2) + (y^2/b^2) = (z^2/c^2)$ in Example 5. Planes perpendicular to the z-axis cut the cone in ellipses above and below the xy-plane. Vertical planes that contain the z-axis cut it in pairs of intersecting lines.

is symmetric with respect to the three coordinate planes (Fig. 10.55). The sections cut by the coordinate planes are

$$x = 0: \quad \text{the lines } z = \pm \frac{c}{b} y \tag{7}$$

$$y = 0: \quad \text{the lines } z = \pm \frac{c}{a} x \tag{8}$$

$$z = 0: \quad \text{the point } (0, 0, 0).$$

The sections cut by planes $z = z_0$ above and below the xy-plane are ellipses whose centers lie on the z-axis and whose vertices lie on the lines in Eqs. (7) and (8).

If $a = b$, the cone is a right circular cone. ❑

EXAMPLE 6 The **hyperboloid of one sheet**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{9}$$

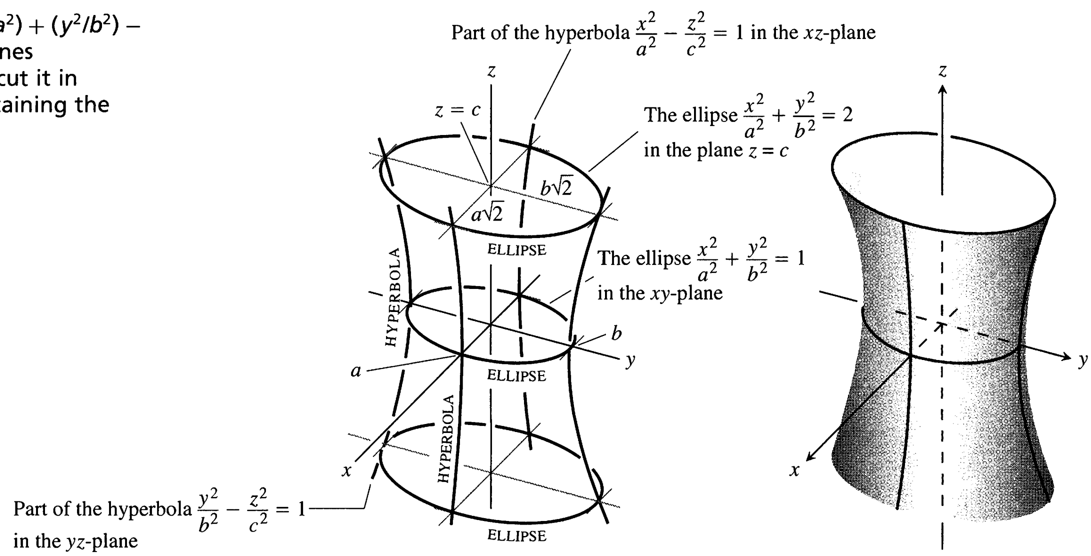
is symmetric with respect to each of the three coordinate planes (Fig. 10.56, on the following page). The sections cut out by the coordinate planes are

$$x = 0: \quad \text{the hyperbola } \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$y = 0: \quad \text{the hyperbola } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \tag{10}$$

$$z = 0: \quad \text{the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The plane $z = z_0$ cuts the surface in an ellipse with center on the z-axis and vertices on one of the hyperbolas in (10).



10.56 The hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ in Example 6. Planes perpendicular to the z-axis cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.

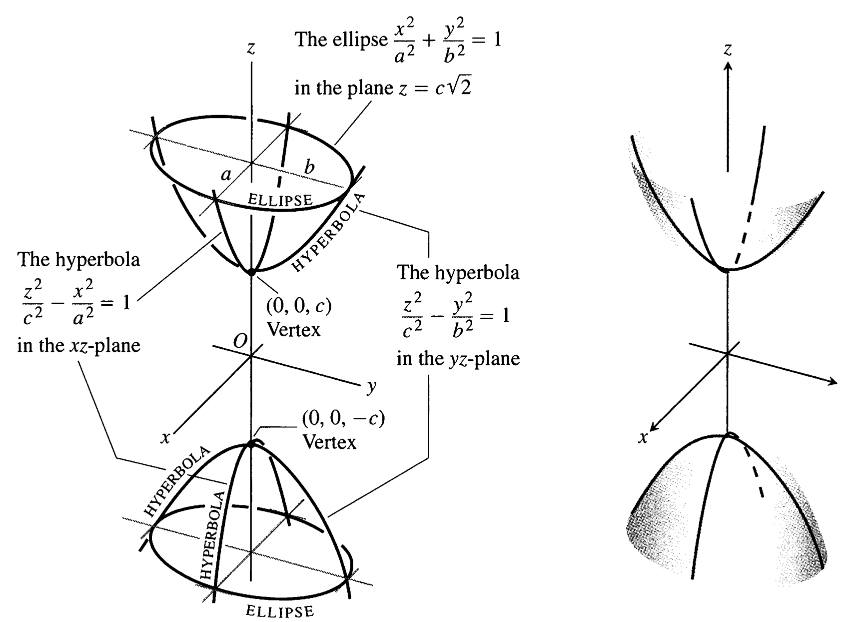
The surface is connected, meaning that it is possible to travel from one point on it to any other without leaving the surface. For this reason, it is said to have *one* sheet, in contrast to the hyperboloid in the next example, which has two sheets.

If $a = b$, the hyperboloid is a surface of revolution. ❑

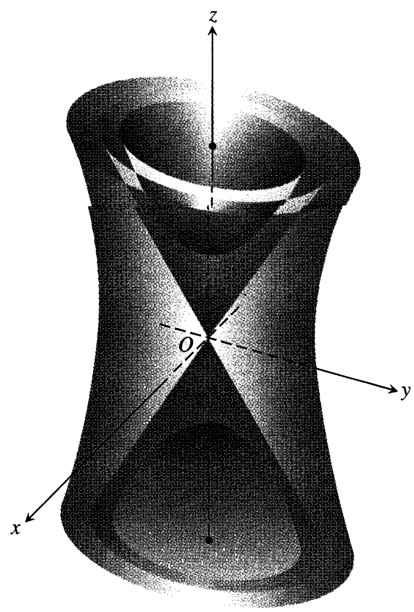
EXAMPLE 7 The **hyperboloid of two sheets**

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{11}$$

is symmetric with respect to the three coordinate planes (Fig. 10.57). The plane $z = 0$ does not intersect the surface; in fact, for a horizontal plane to intersect the



10.57 The hyperboloid $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$ in Example 7. Planes perpendicular to the z-axis above and below the vertices cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.



10.58 Both hyperboloids are asymptotic to the cone (Example 7).

surface, we must have $|z| \geq c$. The hyperbolic sections

$$x = 0: \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1, \qquad y = 0: \quad \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1,$$

have their vertices and foci on the z-axis. The surface is separated into two portions, one above the plane $z = c$ and the other below the plane $z = -c$. This accounts for it's name.

Equations (9) and (11) have different numbers of negative terms. The number in each case is the same as the number of sheets of the hyperboloid. If we replace the 1 on the right side of either Eq. (9) or Eq. (11) by 0, we obtain the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

for an elliptic cone (Eq. 6). The hyperboloids are asymptotic to this cone (Fig. 10.58) in the same way that the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$$

are asymptotic to the lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

in the xy-plane. ❑

EXAMPLE 8 The **hyperbolic paraboloid**

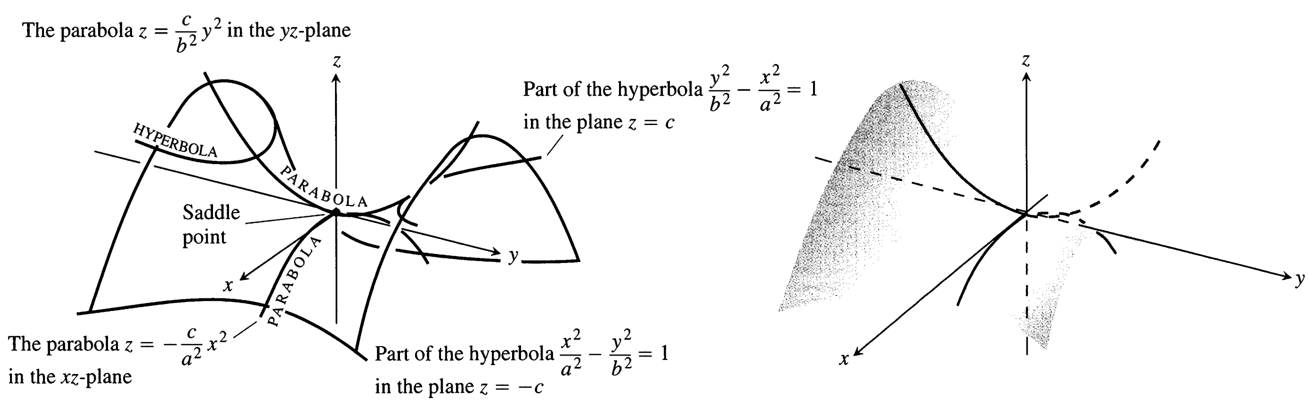
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \qquad c > 0 \tag{12}$$

has symmetry with respect to the planes $x = 0$ and $y = 0$ (Fig. 10.59). The sections in these planes are

$$x = 0: \quad \text{the parabola } z = \frac{c}{b^2}\,y^2, \tag{13}$$

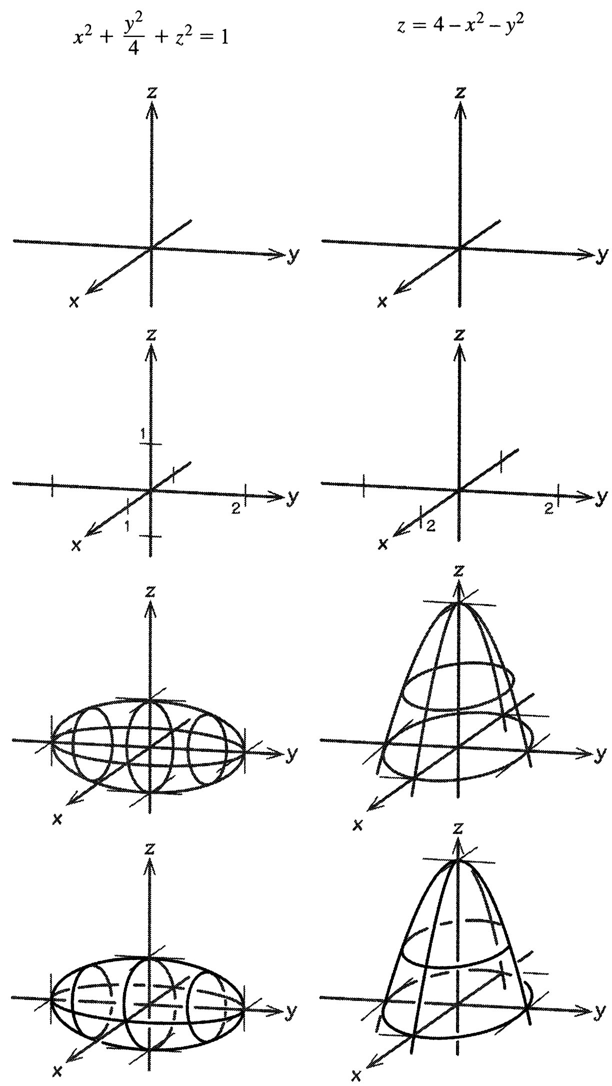
$$y = 0: \quad \text{the parabola } z = -\frac{c}{a^2}\,x^2. \tag{14}$$

10.59 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$ in Example 8. The cross sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross sections in planes perpendicular to the other axes are parabolas.



DRAWING LESSON

How to Draw Quadric Surfaces



1 **Lightly sketch the three coordinate axes.**

2 **Decide on a scale and mark the intercepts on the axes.**

3 **Sketch cross sections in the coordinate planes and in a few parallel planes, but don't clutter the picture. Use tangent lines as guides.**

4 **If more is required, darken the parts exposed to view. Leave the rest light. Use line breaks when you can.**

In the plane $x = 0$, the parabola opens upward from the origin. The parabola in the plane $y = 0$ opens downward.

If we cut the surface by a plane $z = z_0 > 0$, the section is a hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c}, \tag{15}$$

with its focal axis parallel to the y-axis and its vertices on the parabola in (13). If z_0 is negative, the focal axis is parallel to the x-axis and the vertices lie on the parabola in (14).

Near the origin, the surface is shaped like a saddle. To a person traveling along the surface in the yz-plane, the origin looks like a minimum. To a person traveling in the xz-plane, the origin looks like a maximum. Such a point is called a **minimax** or **saddle point** of a surface. ❑

Liquid Mirror Telescopes

When a circular pan of liquid is rotated about its vertical axis, the surface of the liquid does not stay flat. Instead, it assumes the shape of a paraboloid of revolution, exactly what is needed for the primary mirror of a reflecting telescope. At the turn of the century, attempts to make reliable mirrors with revolving mercury failed because of surface ripples and focus losses caused by variations in the speed of rotation. Today, these difficulties can be overcome with synchronous motors driven by oscillator-stabilized power supplies and checked against constant clocks.

Using the same idea, astronomers at the Steward Observatory's Mirror Laboratory in Tucson, Arizona, have used a large heated spinning turntable to cast borosilicate glass blanks for lightweight mirrors. Spincast mirrors cost less than traditionally cast mirrors and can be made larger. Their shorter focal lengths also allow them to be installed in compact telescope frames that are less expensive to house and less likely to flex in strong winds.

Exercises 10.6

Matching Equations with Surfaces

In Exercises 1–12, match the equation with the surface it defines. Also, identify each surface by type (paraboloid, ellipsoid, etc.) The surfaces are labeled (a)–(l).

1. $x^2+y^2+4z^2=10$

2. $z^2+4y^2-4x^2=4$

3. $9y^2+z^2=16$

4. $y^2+z^2=x^2$

5. $x=y^2-z^2$

6. $x=-y^2-z^2$

7. $x^2+2z^2=8$

8. $z^2+x^2-y^2=1$

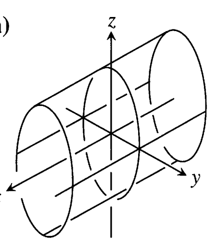
9. $x=z^2-y^2$

10. $z=-4x^2-y^2$

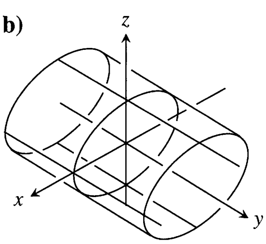
11. $x^2+4z^2=y^2$

12. $9x^2+4y^2+2z^2=36$

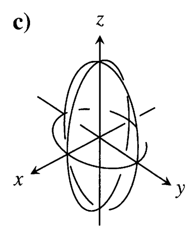
a)



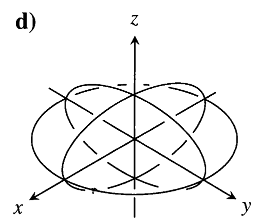
b)



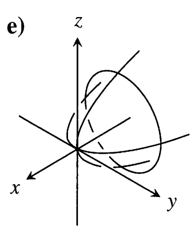
c)



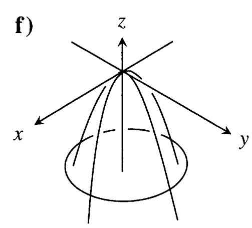
d)



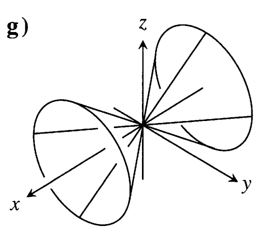
e)



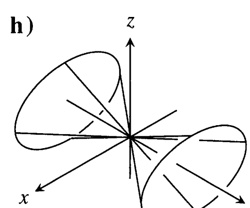
f)



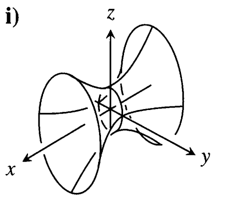
g)



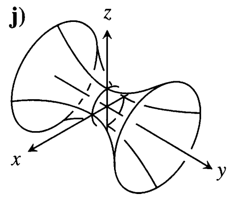
h)



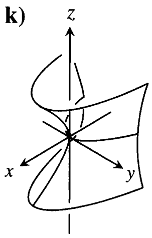
i)



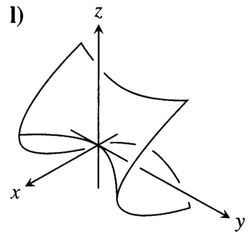
j)



k)



l)



Drawing

Sketch the surfaces in Exercises 13–76.

CYLINDERS

13. $x^2 + y^2 = 4$ **14.** $x^2 + z^2 = 4$

15. $z = y^2 - 1$ **16.** $x = y^2$

17. $x^2 + 4z^2 = 16$ **18.** $4x^2 + y^2 = 36$

19. $z^2 - y^2 = 1$ **20.** $yz = 1$

ELLIPSOIDS

21. $9x^2 + y^2 + z^2 = 9$ **22.** $4x^2 + 4y^2 + z^2 = 16$

23. $4x^2 + 9y^2 + 4z^2 = 36$ **24.** $9x^2 + 4y^2 + 36z^2 = 36$

PARABOLOIDS

25. $z = x^2 + 4y^2$ **26.** $z = x^2 + 9y^2$

27. $z = 8 - x^2 - y^2$ **28.** $z = 18 - x^2 - 9y^2$

29. $x = 4 - 4y^2 - z^2$ **30.** $y = 1 - x^2 - z^2$

CONES

31. $x^2 + y^2 = z^2$ **32.** $y^2 + z^2 = x^2$

33. $4x^2 + 9z^2 = 9y^2$ **34.** $9x^2 + 4y^2 = 36z^2$

HYPERBOLOIDS

35. $x^2 + y^2 - z^2 = 1$ **36.** $y^2 + z^2 - x^2 = 1$

37. $(y^2/4) + (z^2/9) - (x^2/4) = 1$

38. $(x^2/4) + (y^2/4) - (z^2/9) = 1$

39. $z^2 - x^2 - y^2 = 1$ **40.** $(y^2/4) - (x^2/4) - z^2 = 1$

41. $x^2 - y^2 - (z^2/4) = 1$ **42.** $(x^2/4) - y^2 - (z^2/4) = 1$

HYPERBOLIC PARABOLOIDS

43. $y^2 - x^2 = z$ **44.** $x^2 - y^2 = z$

ASSORTED

45. $x^2 + y^2 + z^2 = 4$ **46.** $4x^2 + 4y^2 = z^2$

47. $z = 1 + y^2 - x^2$ **48.** $y^2 - z^2 = 4$

49. $y = -(x^2 + z^2)$ **50.** $z^2 - 4x^2 - 4y^2 = 4$

51. $16x^2 + 4y^2 = 1$ **52.** $z = x^2 + y^2 + 1$

53. $x^2 + y^2 - z^2 = 4$ **54.** $x = 4 - y^2$

55. $x^2 + z^2 = y$ **56.** $z^2 - (x^2/4) - y^2 = 1$

57. $x^2 + z^2 = 1$ **58.** $4x^2 + 4y^2 + z^2 = 4$

59. $16y^2 + 9z^2 = 4x^2$ **60.** $z = x^2 - y^2 - 1$

61. $9x^2 + 4y^2 + z^2 = 36$ **62.** $4x^2 + 9z^2 = y^2$

63. $x^2 + y^2 - 16z^2 = 16$ **64.** $z^2 + 4y^2 = 9$

65. $z = -(x^2 + y^2)$ **66.** $y^2 - x^2 - z^2 = 1$

67. $x^2 - 4y^2 = 1$ **68.** $z = 4x^2 + y^2 - 4$

69. $4y^2 + z^2 - 4x^2 = 4$ **70.** $z = 1 - x^2$

71. $x^2 + y^2 = z$ **72.** $(x^2/4) + y^2 - z^2 = 1$

73. $yz = 1$ **74.** $36x^2 + 9y^2 + 4z^2 = 36$

75. $9x^2 + 16y^2 = 4z^2$ **76.** $4z^2 - x^2 - y^2 = 4$

Theory and Examples

77. a) Express the area A of the cross section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by the plane $z = c$ as a function of c. (The area of an ellipse with semiaxes a and b is πab.)

b) Use slices perpendicular to the z-axis to find the volume of the ellipsoid in (a).

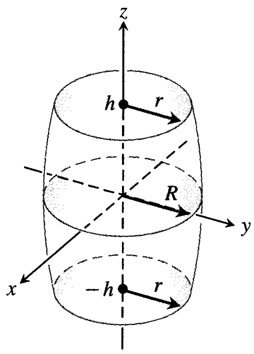
c) Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Does your formula give the volume of a sphere of radius a if $a = b = c$?

78. The barrel shown here is shaped like an ellipsoid with equal pieces cut from the ends by planes perpendicular to the z-axis. The cross sections perpendicular to the z-axis are circular. The barrel is $2h$ units high, its midsection radius is R, and its end radii are both r. Find a formula for the barrel's volume. Then check two things. First, suppose the sides of the barrel are straightened to turn the barrel into a cylinder of radius R and height $2h$. Does your formula give the cylinder's volume? Second, suppose $r = 0$

and $h = R$ so the barrel is a sphere. Does your formula give the sphere's volume?



79. Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude. (Figure 10.53 shows the segment for the special case $h = c$.)

80. **a)** Find the volume of the solid bounded by the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the planes $z = 0$ and $z = h, h > 0$.

b) Express your answer in (a) in terms of h and the areas A_0 and A_h of the regions cut by the hyperboloid from the planes $z = 0$ and $z = h$.

c) Show that the volume in (a) is also given by the formula

$$V = \frac{h}{6}\left(A_0 + 4A_m + A_h\right),$$

where A_m is the area of the region cut by the hyperboloid from the plane $z = h/2$.

81. If the hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$ is cut by the plane $y = y_1$, the resulting curve is a parabola. Find its vertex and focus.

82. Suppose you set $z = 0$ in the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$$

to obtain a curve in the xy-plane. What will the curve be like? Give reasons for your answer.

83. Every time we found the trace of a quadric surface in a plane parallel to one of the coordinate planes, it turned out to be a conic section. Was this mere coincidence? Did it have to happen? Give reasons for your answer.

84. Suppose you intersect a quadric surface with a plane that is *not* parallel to one of the coordinate planes. What will the trace in the plane be like? Give reasons for your answer.

Computer Grapher Explorations

Plot the surfaces in Exercises 85–88 over the indicated domains. If you can, rotate the surface into different viewing positions.

85. $z = y^2, \quad -2 \le x \le 2, \quad -0.5 \le y \le 2$

86. $z = 1 - y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

87. $z = x^2 + y^2, \quad -3 \le x \le 3, \quad -3 \le y \le 3$

88. $z = x^2 + 2y^2$ over

a) $-3 \le x \le 3, \quad -3 \le y \le 3$
b) $-1 \le x \le 1, \quad -2 \le y \le 3$
c) $-2 \le x \le 2, \quad -2 \le y \le 2$
d) $-2 \le x \le 2, \quad -1 \le y \le 1$

CAS Explorations and Projects

Use a CAS to plot the surfaces in Exercises 89–94. Identify the type of quadric surface from your graph.

89. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1 - \dfrac{z^2}{25}$

90. $\dfrac{x^2}{9} - \dfrac{z^2}{9} = 1 - \dfrac{y^2}{16}$

91. $5x^2 = z^2 - 3y^2$

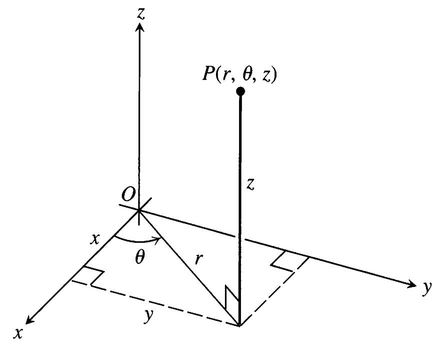
92. $\dfrac{y^2}{16} = 1 - \dfrac{x^2}{9} + z$

93. $\dfrac{x^2}{9} - 1 = \dfrac{y^2}{16} + \dfrac{z^2}{2}$

94. $y - \sqrt{4 - z^2} = 0$

10.7 Cylindrical and Spherical Coordinates

This section introduces two new coordinate systems for space: the cylindrical coordinate system and the spherical coordinate system. Cylindrical coordinates simplify the equations of cylinders. Spherical coordinates simplify the equations of spheres and cones. We will use cylindrical coordinates to study planetary motion in Section 11.5.



10.60 The cylindrical coordinates of a point in space are r, θ, and z.

Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Fig. 10.60.

Definition

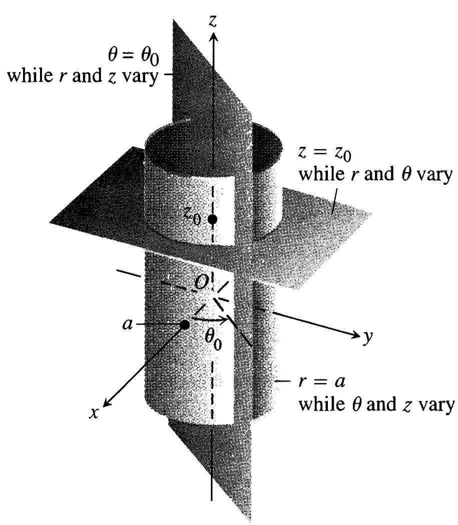
Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane,
2. z is the rectangular vertical coordinate.

The values of $x, y, r,$ and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x \tag{1}$$

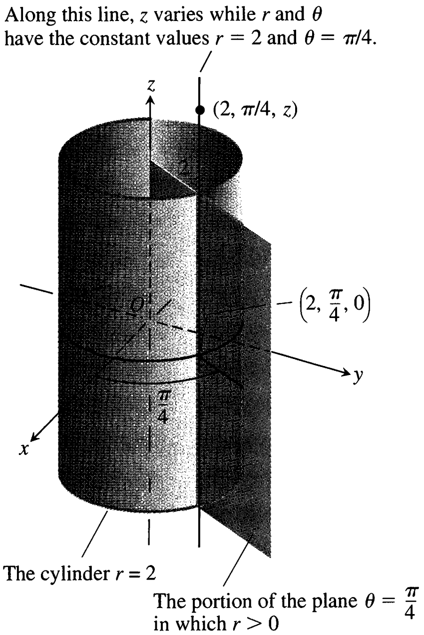


10.61 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Fig. 10.61). The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

EXAMPLE 1 What points satisfy the equations

$$r = 2, \qquad \theta = \frac{\pi}{4}?$$



10.62 The points whose first two cylindrical coordinates are $r = 2$ and $\theta = \pi/4$ form a line parallel to the z-axis (Example 1).

Solution These points make up the line in which the cylinder $r = 2$ cuts the portion of the plane $\theta = \pi/4$ where r is positive (Fig. 10.62). This is the line through the point $(2, \pi/4, 0)$ parallel to the z-axis. ❑

EXAMPLE 2 Sketch the surface $r = 1 + \cos\theta$.

Solution The equation involves only r and θ; the coordinate variable z is missing. Therefore, the surface is a cylinder of lines that pass through the cardioid $r = 1 + \cos\theta$ in the $r\theta$-plane and lie parallel to the z-axis. The rules for sketching the cylinder are the same as always: sketch the x-, y-, and z-axes, draw a few perpendicular cross sections, connect the cross sections with parallel lines, and darken the exposed parts (Fig. 10.63). ❑

EXAMPLE 3 Find a Cartesian equation for the surface $z = r^2$ and identify the surface.

Solution From Eqs. (1) we have $z = r^2 = x^2 + y^2$. The surface is the circular paraboloid $x^2 + y^2 = z$. ❑

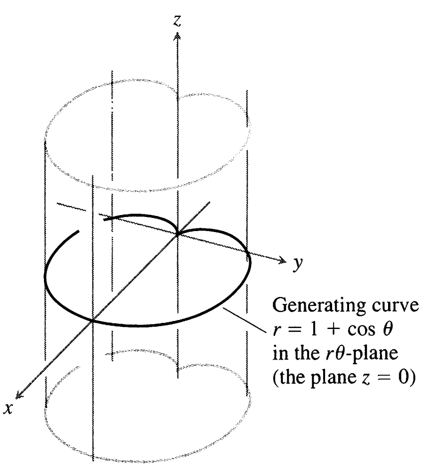
EXAMPLE 4 Find an equation for the circular cylinder $4x^2 + 4y^2 = 9$ in cylindrical coordinates.

Solution The cylinder consists of the points whose distance from the z-axis is $\sqrt{x^2 + y^2} = 3/2$. The corresponding equation in cylindrical coordinates is $r = 3/2$. ❑

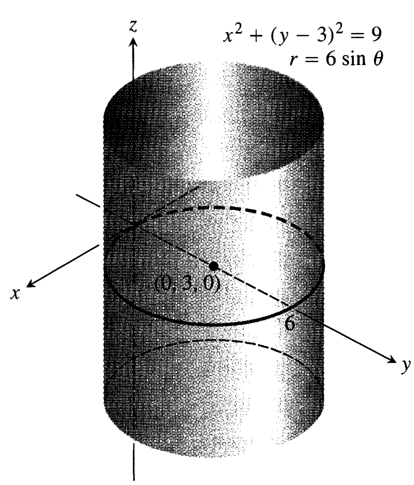
EXAMPLE 5 Find an equation for the cylinder $x^2 + (y - 3)^2 = 9$ in cylindrical coordinates (Fig. 10.64).

Solution The equation for the cylinder in cylindrical coordinates is the same as the polar equation for the cylinder's base in the xy-plane:

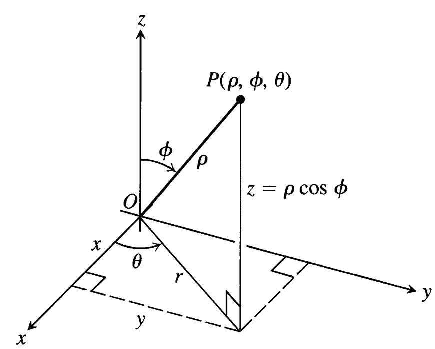
$$r = 6\sin\theta. \qquad ❑$$



10.63 The cylindrical coordinate equation $r = 1 + \cos\theta$ defines a cylinder in space whose cross sections perpendicular to the z-axis are cardioids (Example 2).



10.64 The cylinder in Example 5.



10.65 The spherical coordinates ρ, ϕ, and θ and their relation to *x, y, z,* and *r.*

A few books give spherical coordinates in the order (ρ, θ, ϕ), with θ and ϕ reversed. In some cases, you may also find r being used for ρ. Watch out for this when you read elsewhere.

Spherical Coordinates

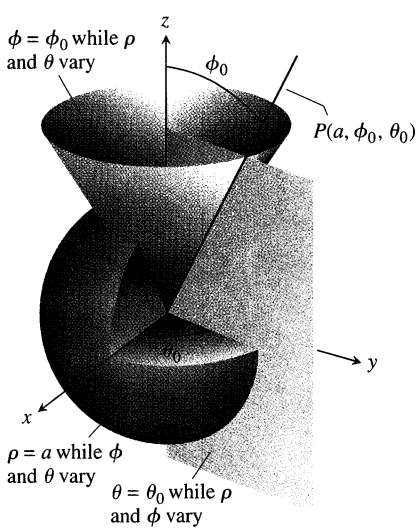
Spherical coordinates locate points in space with angles and a distance, as shown in Fig. 10.65.

The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike r, *the variable* ρ *is never negative.* The second coordinate, ϕ, is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

Definition

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin,
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis ($0 \le \phi \le \pi$),
3. θ is the angle from cylindrical coordinates.



10.66 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

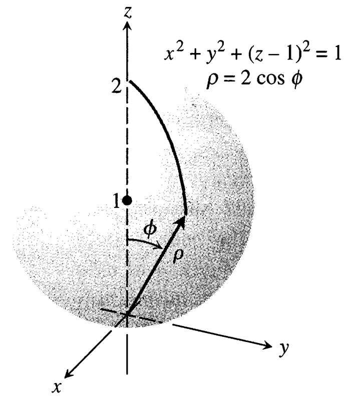
The equation $\rho = a$ describes the sphere of radius a centered at the origin (Fig. 10.66). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho\sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \qquad (2)$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

EXAMPLE 6 Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z - 1)^2 = 1.$$



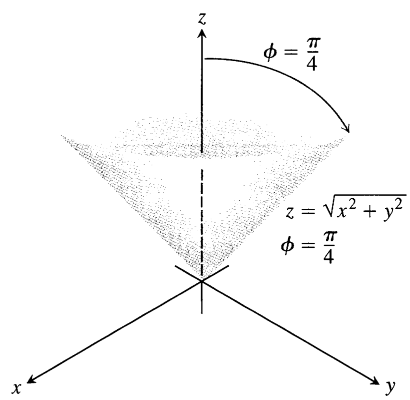
10.67 The sphere in Example 6.

Solution We use Eqs. (2) to substitute for x, y, and z:

$$\begin{aligned}
x^2 + y^2 + (z-1)^2 &= 1 \\
\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 &= 1 && \text{Eqs. (2)} \\
\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 \\
\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} &= 2\rho\cos\phi \\
\rho^2 &= 2\rho\cos\phi \\
\rho &= 2\cos\phi
\end{aligned}$$

See Fig. 10.67. ❑

EXAMPLE 7 Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ (Fig. 10.68).



10.68 The cone in Example 7.

Solution 1 Use geometry. The cone is symmetric with respect to the z-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive z-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$.

Solution 2 Use algebra. If we use Eqs. (2) to substitute for x, y, and z we obtain the same result:

$$\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
\rho\cos\phi &= \sqrt{\rho^2 \sin^2\phi} && \text{Example 6} \\
\rho\cos\phi &= \rho\sin\phi && \rho \ge 0,\ \sin\phi \ge 0 \\
\cos\phi &= \sin\phi \\
\phi &= \frac{\pi}{4} && 0 \le \phi \le \pi
\end{aligned}$$

❑

Exercises 10.7

Converting Point Coordinates

The following table gives the coordinates of specific points in space in one of three coordinate systems. In Exercises 1–10, find coordinates for each point in the other two systems. There may be more than one right answer because points in cylindrical and spherical coordinates may have more than one coordinate triple.

	Rectangular (x, y, z)	Cylindrical (r, θ, z)	Spherical (ρ, ϕ, θ)
1.	(0, 0, 0)		
2.	(1, 0, 0)		
3.	(0, 1, 0)		
4.	(0, 0, 1)		
5.		(1, 0, 0)	
6.		$(\sqrt{2}, 0, 1)$	
7.		$(1, \pi/2, 1)$	
8.			$(\sqrt{3}, \pi/3, -\pi/2)$
9.			$(2\sqrt{2}, \pi/2, 3\pi/2)$
10.			$(\sqrt{2}, \pi, 2\pi/2)$

Converting Equations and Inequalities; Associated Figures

In Exercises 11–36, translate the equations and inequalities from the given coordinate system (rectangular, cylindrical, spherical) into equations and inequalities in the other two systems. Also, identify the figure being defined.

11. $r = 0$

12. $x^2 + y^2 = 5$

13. $z = 0$

14. $z = -2$

15. $z = \sqrt{x^2 + y^2}, \quad z \le 1$

16. $z = \sqrt{x^2 + y^2}, \quad 1 \le z \le 2$

17. $\rho \sin\phi \cos\theta = 0$

18. $\tan^2\phi = 1$

19. $x^2 + y^2 + z^2 = 4$

20. $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$

21. $\rho = 5\cos\phi$

22. $\rho = -6\cos\phi$

23. $r = \csc\theta$

24. $r = -3\sec\theta$

25. $\rho = \sqrt{2}\sec\phi$

26. $\rho = 9\csc\phi$

27. $x^2 + y^2 + (z-1)^2 = 1, \quad z \le 1$

28. $r^2 + z^2 = 4, \quad z \le -\sqrt{2}$

29. $\rho = 3, \quad \pi/3 \le \phi \le 2\pi/3$

30. $x^2 + y^2 + z^2 = 3, \quad 0 \le z \le \sqrt{3}/2$

31. $z = 4 - 4r^2, \quad 0 \le r \le 1$

32. $z = 4 - r, \quad 0 \le r \le 4$

33. $\phi = 3\pi/4, \quad 0 \le \rho \le \sqrt{2}$

34. $\phi = \pi/2, \quad 0 \le \rho \le \sqrt{7}$

35. $z + r^2\cos 2\theta = 0$

36. $z^2 - r^2 = 1$

37. Find the rectangular coordinates of the center of the sphere
$$r^2 + z^2 = 4r\cos\theta + 6r\sin\theta + 2z.$$

38. Find the rectangular coordinates of the center of the sphere
$$\rho = 2\sin\phi \quad (\cos\theta - 2\sin\theta).$$

Sets Defined by Equations

In Exercises 39–42, describe the set of points whose cylindrical coordinates satisfy the given equation. Sketch each set.

39. $r = -2\sin\theta$

40. $r = 2\cos\theta$

41. $r = 1 - \cos\theta$

42. $r = 1 + \sin\theta$

In Exercises 43 and 44, describe the set of points in space whose spherical coordinates satisfy the given equation. Sketch each set.

43. $\rho = 1 - \cos\phi$

44. $\rho = 1 + \cos\phi$

Theory and Examples

45. *Horizontal planes in cylindrical and spherical coordinates.* (a) Show that the plane whose equation is $z = c \quad (c \neq 0)$ in rectangular and cylindrical coordinates has the equation $\rho = c\sec\phi$ in spherical coordinates. (b) Find an equation for the xy-plane in spherical coordinates.

46. *Vertical circular cylinders in spherical coordinates.* Find an equation of the form $\rho = f(\phi)$ for the cylinder $x^2 + y^2 = a^2$.

47. *Vertical planes in cylindrical coordinates.* (a) Show that planes perpendicular to the x-axis have equations of the form $r = a\sec\theta$ in cylindrical coordinates. (b) Show that planes perpendicular to the y-axis have equations of the form $r = b\csc\theta$.

48. *(Continuation of Exercise 47.)* Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c, c \neq 0$.

49. *Symmetry.* What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

50. *Symmetry.* What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

CAS Explorations and Projects

Use a CAS to perform the following steps in Exercises 51 and 52.

a) Solve the given equation for r (cylindrical coordinates) or ρ (spherical coordinates). Choose the expression with the positive square root and simplify it.
b) Plot r as a function of θ and z, or ρ as a function of θ and ϕ, as appropriate. Use the specified ranges of the variables for your plots. (Other ranges can present difficulties for a computer grapher in producing the complete surface. Experiment with other ranges later if you wish.)
c) From your plot in (b), estimate the center and radius of the sphere you graphed to the nearest integer value.
d) Convert the given equation from cylindrical or spherical coordinates to rectangular coordinates. Simplify the results. You may need to complete the final simplification by hand to obtain an equation for the sphere in the factored form $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$. Compare this result against your estimates from (c).
e) Plot the implicit equation obtained in (d). How does it compare with the plot produced in (b)? Can you explain any discrepancies from the way in which a computer grapher plots surfaces?

51. $r^2 + z^2 = 2r(\cos\theta + \sin\theta) + 2, \quad \dfrac{\pi}{4} \le \theta \le \dfrac{9\pi}{4}, \quad -2 \le z \le 2$

52. $\rho^2 = 2\rho(\cos\theta \sin\phi - \cos\phi) + 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

CHAPTER 10 QUESTIONS TO GUIDE YOUR REVIEW

1. When do directed line segments (in the plane or space) represent the same vector?
2. How are vectors added and subtracted geometrically? algebraically?
3. How do you find a vector's magnitude and direction?
4. If a vector is multiplied by a positive scalar, how is the result related to the original vector? What if the scalar is zero? negative?
5. Define the *dot product* (*scalar product*) of two vectors. Which algebraic laws (commutative, associative, distributive, cancellation) are satisfied by dot products, and which, if any, are not? Give examples. When is the dot product of two vectors equal to zero?
6. What geometric or physical interpretations do dot products have? Give examples.
7. What is the vector projection of a vector **B** onto a vector **A**? How do you write **B** as the sum of a vector parallel to **A** and a vector orthogonal to **A**?
8. Define the *cross product* (*vector product*) of two vectors. Which algebraic laws (commutative, associative, distributive, cancellation) are satisfied by cross products, and which are not? Give examples. When is the cross product of two vectors equal to zero?
9. What geometric or physical interpretations do cross products have? Give examples.
10. What is the determinant formula for calculating the cross product of two vectors relative to the Cartesian **i, j, k**-coordinate system? Use it in an example.
11. How do you find equations for lines, line segments, and planes in space? Give examples. Can you express a line in space by a single equation? A plane?
12. How do you find the distance from a point to a line in space? From a point to a plane? Give examples.
13. What are box products? What significance do they have? How are they evaluated? Give an example.
14. How do you find equations for spheres in space? Give examples.
15. How do you find the intersection of two lines in space? a line and a plane? two planes? Give examples.
16. What is a cylinder? Give examples of equations that define cylinders in Cartesian coordinates; in cylindrical coordinates. What advice can you give about drawing cylinders?
17. What are quadric surfaces? Give examples of different kinds of ellipsoids, paraboloids, cones, and hyperboloids (equations and sketches). What advice can you give about sketching quadric surfaces?
18. How are cylindrical and spherical coordinates defined? Draw diagrams that show how cylindrical and spherical coordinates are related to Cartesian coordinates.
19. Are cylindrical and spherical coordinates for a point in space unique? Explain.
20. What sets have constant-coordinate equations (like $x = 1$ or $r = 1$ or $\phi = \pi/3$) in the three coordinate systems for space?
21. What equations are available for changing from one space coordinate system to another?

CHAPTER 10 PRACTICE EXERCISES

Vector Calculations

1. Draw the unit vectors $\mathbf{u} = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}$ for $\theta = 0, \pi/2, 2\pi/3, 5\pi/4$, and $5\pi/3$, together with the coordinate axes and unit circle.

2. Find the unit vector obtained by rotating
   a) $\mathbf{i}$ clockwise through an angle $\pi/4$;
   b) $\mathbf{j}$ counterclockwise through an angle $2\pi/3$.

Express the vectors in Exercises 3–6 in terms of their lengths and directions.

3. $\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

4. $-\mathbf{i} - \mathbf{j}$

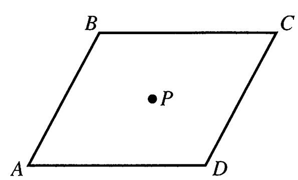
5. $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

6. $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

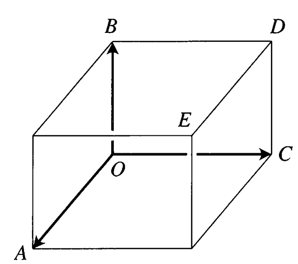
7. Find a vector 2 units long in the direction of $\mathbf{A} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

8. Find a vector 5 units long in the direction opposite to the direction of $\mathbf{A} = (3/5)\,\mathbf{i} + (4/5)\mathbf{k}$.

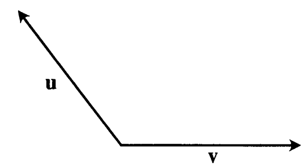
9. The accompanying figure shows parallelogram $ABCD$ and the midpoint P of diagonal BD.
   a) Express $\overrightarrow{BD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.
   b) Express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.
   c) Prove that P is also the midpoint of diagonal AC.



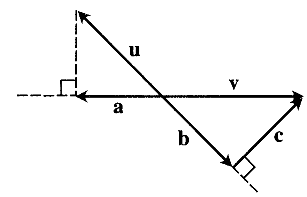
10. The points O, A, B, C, D, and E are vertices of the rectangular box shown here. Express $\overrightarrow{OD}$ and $\overrightarrow{OE}$ in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.



11. Copy the vectors $\mathbf{u}$ and $\mathbf{v}$ and sketch the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.



12. Express vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.



In Exercises 13 and 14, find $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A}\cdot\mathbf{B}$, $\mathbf{B}\cdot\mathbf{A}$, $\mathbf{A}\times\mathbf{B}$, $\mathbf{B}\times\mathbf{A}$, $|\mathbf{A}\times\mathbf{B}|$, the angle between $\mathbf{A}$ and $\mathbf{B}$, the scalar component of $\mathbf{B}$ in the direction of $\mathbf{A}$, and the vector projection of $\mathbf{B}$ onto $\mathbf{A}$.

13. $\mathbf{A} = \mathbf{i} + \mathbf{j}$
    $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

14. $\mathbf{A} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$
    $\mathbf{B} = -\mathbf{i} - \mathbf{k}$

In Exercises 15 and 16, write $\mathbf{B}$ as the sum of a vector parallel to $\mathbf{A}$ and a vector orthogonal to $\mathbf{A}$.

15. $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$,
    $\mathbf{B} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$

16. $\mathbf{A} = \mathbf{i} - 2\mathbf{j}$,
    $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

In Exercises 17 and 18, draw coordinate axes and then sketch $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}\times\mathbf{B}$ as vectors at the origin.

17. $\mathbf{A} = \mathbf{i}$, $\quad \mathbf{B} = \mathbf{i} + \mathbf{j}$

18. $\mathbf{A} = \mathbf{i} - \mathbf{j}$, $\quad \mathbf{B} = \mathbf{i} + \mathbf{j}$

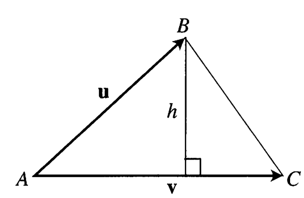
In Exercises 19 and 20, find the unit vectors that are tangent and normal to the curve at point P.

19. $y = \tan x$, $\quad P(\pi/4, 1)$

20. $x^2 + y^2 = 25$, $\quad P(3, 4)$

21. For any vectors $\mathbf{A}$ and $\mathbf{B}$, show that $|\mathbf{A}+\mathbf{B}|^2 + |\mathbf{A}-\mathbf{B}|^2 = 2\,|\mathbf{A}|^2 + 2\,|\mathbf{B}|^2$.

22. Let ABC be the triangle determined by vectors $\mathbf{u}$ and $\mathbf{v}$.
    a) Express the area of ΔABC in terms of $\mathbf{u}$ and $\mathbf{v}$.
    b) Express the triangle's altitude h in terms of $\mathbf{u}$ and $\mathbf{v}$.
    c) Find the area and altitude if $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{k}$.



23. If $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/3$, find $|\mathbf{v} - 2\mathbf{w}|$.

24. For what value or values of a will the vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} - 8\mathbf{j} + a\mathbf{k}$ be parallel?

In Exercises 25 and 26, find (a) the area of the parallelogram determined by vectors **A** and **B,** (b) the volume of the parallelepiped determined by the vectors **A, B,** and **C.**

25. $\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{C} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

26. $\mathbf{A} = \mathbf{i} + \mathbf{j}, \quad \mathbf{B} = \mathbf{j}, \quad \mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Lines, Planes, and Distances

27. Suppose that $\mathbf{n}$ is normal to a plane and that $\mathbf{v}$ is parallel to the plane. Describe how you would find a vector $\mathbf{u}$ that is both perpendicular to $\mathbf{v}$ and parallel to the plane.

28. Find a vector in the plane parallel to the line $ax + by = c$.

In Exercises 29 and 30, find the distance from the point to the line.

29. $(2, 2, 0); \quad x = -t, \quad y = t, \quad z = -1 + t$

30. $(0, 4, 1); \quad x = 2 + t, \quad y = 2 + t, \quad z = t$

31. Parametrize the line that passes through the point $(1, 2, 3)$ parallel to the vector $\mathbf{v} = -3\mathbf{i} + 7\mathbf{k}$.

32. Parametrize the line segment joining the points $P(1, 2, 0)$ and $Q(1, 3, -1)$.

In Exercises 33 and 34, find the distance from the point to the plane.

33. $(6, 0, -6), \quad x - y = 4$

34. $(3, 0, 10), \quad 2x + 3y + z = 2$

35. Find an equation for the plane that passes through the point $(3, -2, 1)$ normal to the vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

36. Find an equation for the plane that passes through the point $(-1, 6, 0)$ perpendicular to the line $x = -1 + t, y = 6 - 2t, z = 3t$.

In Exercises 37 and 38, find an equation for the plane through points P, Q, and R.

37. $P(1, -1, 2), \quad Q(2, 1, 3), \quad R(-1, 2, -1)$

38. $P(1, 0, 0), \quad Q(0, 1, 0), \quad R(0, 0, 1)$

39. Find the points in which the line $x = 1 + 2t, y = -1 - t, z = 3t$ meets the three coordinate planes.

40. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.

41. Find the acute angle between the planes $x = 7$ and $x + y + \sqrt{2}z = -3$.

42. Find the acute angle between the planes $x + y = 1$ and $y + z = 1$.

43. Find parametric equations for the line in which the planes $x + 2y + z = 1$ and $x - y + 2z = -8$ intersect.

44. Show that the line in which the planes

$$x + 2y - 2z = 5 \quad \text{and} \quad 5x - 2y - z = 0$$

intersect is parallel to the line

$$x = -3 + 2t, \quad y = 3t, \quad z = 1 + 4t.$$

45. The planes $3x + 6z = 1$ and $2x + 2y - z = 3$ intersect in a line.
    a) Show that the planes are orthogonal.
    b) Find equations for the line of intersection.

46. Find an equation for the plane that passes through the point $(1, 2, 3)$ parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

47. Is $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ related in any special way to the plane $2x + y = 5$? Give reasons for your answer.

48. The equation $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ represents the plane through P_0 normal to $\mathbf{n}$. What set does the inequality $\mathbf{n} \cdot \overrightarrow{P_0P} > 0$ represent?

49. Find the distance from the point $P(1, 4, 0)$ to the plane through $A(0, 0, 0)$, $B(2, 0, -1)$ and $C(2, -1, 0)$.

50. Find the distance from the point $(2, 2, 3)$ to the plane $2x + 3y + 5z = 0$.

51. Find a vector parallel to the plane $2x - y - z = 4$ and orthogonal to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

52. Find a unit vector orthogonal to **A** in the plane of **B** and **C** if $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

53. Find a vector of magnitude 2 parallel to the line of intersection of the planes $x + 2y + z - 1 = 0$ and $x - y + 2z + 7 = 0$.

54. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.

55. Find the point in which the line through $P(3, 2, 1)$ normal to the plane $2x - y + 2z = -2$ meets the plane.

56. What angle does the line of intersection of the planes $2x + y - z = 0$ and $x + y + 2z = 0$ make with the positive x-axis?

57. The line

$$L: \quad x = 3 + 2t, \quad y = 2t, \quad z = t$$

intersects the plane $x + 3y - z = -4$ in a point P. Find the coordinates of P and find equations for the line through P perpendicular to L.

58. Show that for every real number k the plane

$$x - 2y + z + 3 + k(2x - y - z + 1) = 0$$

contains the line of intersection of the planes

$$x - 2y + z + 3 = 0 \quad \text{and} \quad 2x - y - z + 1 = 0.$$

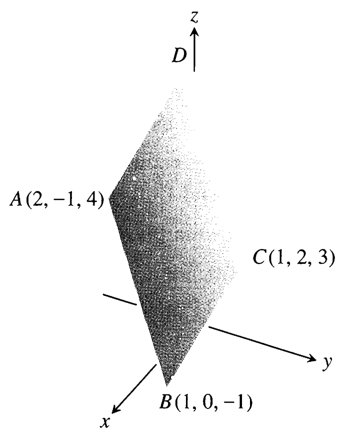
59. Find an equation for the plane through $A(-2, 0, -3)$ and $B(1, -2, 1)$ that lies parallel to the line through $C(-2, -13/5, 26/5)$ and $D(16/5, -13/5, 0)$.

60. Is the line $x = 1 + 2t, y = -2 + 3t, z = -5t$ related in any way to the plane $-4x - 6y + 10z = 9$? Give reasons for your answer.

61. Which of the following are equations for the plane through the points $P(1, 1, -1)$, $Q(3, 0, 2)$, and $R(-2, 1, 0)$?

a) $(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = 0$
b) $x = 3 - t, \quad y = -11t, \quad z = 2 - 3t$
c) $(x + 2) + 11(y - 1) = 3z$
d) $(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \times ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = \mathbf{0}$
e) $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (-3\mathbf{i} + \mathbf{k}) \cdot ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = 0$

62. The parallelogram shown here has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$, and D. Find



a) the coordinates of D,
b) the cosine of the interior angle at B,
c) the vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,
d) the area of the parallelogram,
e) an equation for the plane of the parallelogram,
f) the areas of the orthogonal projections of the parallelogram on the three coordinate planes.

63. *Distance between lines.* Find the distance between the line L_1 through the points $A(1, 0, -1)$ and $B(-1, 1, 0)$ and the line L_2 through the points $C(3, 1, -1)$ and $D(4, 5, -2)$. The distance is to be measured along the line perpendicular to the two lines. First find a vector $\mathbf{n}$ perpendicular to both lines. Then project $\overrightarrow{AC}$ onto $\mathbf{n}$.

64. *(Continuation of Exercise 63.)* Find the distance between the line through $A(4, 0, 2)$ and $B(2, 4, 1)$ and the line through $C(1, 3, 2)$ and $D(2, 2, 4)$.

Quadric Surfaces

Identify and sketch the surfaces in Exercises 65–76.

65. $x^2 + y^2 + z^2 = 4$
66. $x^2 + (y - 1)^2 + z^2 = 1$
67. $4x^2 + 4y^2 + z^2 = 4$
68. $36x^2 + 9y^2 + 4z^2 = 36$
69. $z = -(x^2 + y^2)$
70. $y = -(x^2 + z^2)$
71. $x^2 + y^2 = z^2$
72. $x^2 + z^2 = y^2$
73. $x^2 + y^2 - z^2 = 4$
74. $4y^2 + z^2 - 4x^2 = 4$
75. $y^2 - x^2 - z^2 = 1$
76. $z^2 - x^2 - y^2 = 1$

Coordinate Systems

The equations in Exercises 77–86 define sets both in the plane and in three-dimensional space. Identify both sets for each equation.

RECTANGULAR COORDINATES

77. $x = 0$
78. $x + y = 1$
79. $x^2 + y^2 = 4$
80. $x^2 + 4y^2 = 16$
81. $x = y^2$
82. $y^2 - x^2 = 1$

CYLINDRICAL COORDINATES

83. $r = 1 - \cos\theta$
84. $r = \sin\theta$
85. $r^2 = 2\cos 2\theta$
86. $r = \cos 2\theta$

Describe the sets defined by the spherical coordinate equations and inequalities in Exercises 87–92.

87. $\rho = 2$
88. $\theta = \pi/4$
89. $\phi = \pi/6$
90. $\rho = 1, \quad \phi = \pi/2$
91. $\rho = 1, \quad 0 \le \phi \le \pi/2$
92. $1 \le \rho \le 2$

The following table gives the coordinates of points in space in one of three coordinate systems. In Exercises 93–98, find coordinates for each point in the other two systems. There may be more than one right answer because cylindrical and spherical coordinates are not unique.

	Rectangular (x, y, z)	Cylindrical (r, θ, z)	Spherical (ρ, ϕ, θ)
93.		$(1, 0, 0)$	
94.		$\left(1, \frac{\pi}{2}, 0\right)$	
95.			$\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$
96.			$\left(2, \frac{5\pi}{6}, 0\right)$
97.	$(-1, 0, -1)$		
98.	$(0, -1, 1)$		

In Exercises 99–110, translate the equations from the given coordinate system (rectangular, cylindrical, spherical) into the other two systems. Identify the set of points in space defined by the equation.

RECTANGULAR

99. $z = 2$
100. $z = \sqrt{3x^2 + 3y^2}$
101. $x^2 + y^2 + (z + 1)^2 = 1$
102. $x^2 + y^2 + (z - 3)^2 = 9$

CYLINDRICAL

103. $z = r^2$

104. $z = |r|$

105. $r = 7 \sin\theta$

106. $r = 4\cos\theta$

SPHERICAL

107. $\rho = 4$

108. $\rho = \sqrt{3}\sec\phi$

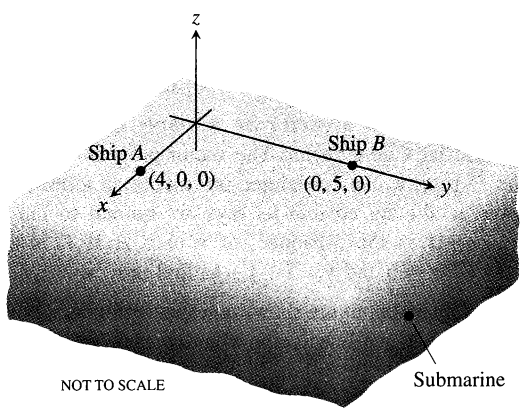
109. $\phi = 3\pi/4$

110. $\rho\cos\phi + \rho^2\sin^2\phi = 1$

CHAPTER 10 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Applications

1. *Submarine hunting.* Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. As shown here, ship A is located at (4, 0, 0) while ship B is located at (0, 5, 0). All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - (1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i} - 6\mathbf{j} - \mathbf{k}$. Four minutes ago, the submarine was located at $(2, -1, -1/3)$. The aircraft is due in 20 min. Assuming the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?



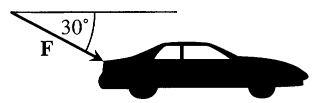
2. *A helicopter rescue.* Two helicopters, H_1 and H_2, are traveling together. At time $t = 0$ hours, they separate and follow different straight-line paths given by

$$H_1: \quad x = 6 + 40t, \quad y = -3 + 10t, \quad z = -3 + 2t$$

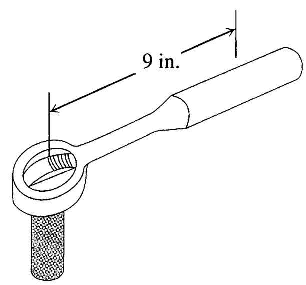
$$H_2: \quad x = 6 + 110t, \quad y = -3 + 4t, \quad z = -3 + t,$$

all coordinates measured in miles. Due to system malfunctions, H_2 stops its flight at (446, 13, 1) and, in a negligible amount of time, lands at (446, 13, 0). Two hours later, H_1 is advised of this fact and heads toward H_2 at 150 mph. How long will it take H_1 to reach H_2?

3. *Work.* Find the work done in pushing a car 250 m with a force of magnitude 160 N directed at an angle of $\pi/6$ rad downward from the horizontal against the back of the car.



4. *Torque.* The operator's manual for the Toro® 21-in. lawnmower says "tighten the spark plug to 15 ft-lb (20.4 N • m)." If you are installing the plug with a 10.5-in. socket wrench that places the center of your hand 9 in. from the axis of the spark plug, about how hard should you pull? Answer in pounds.



Theory and Examples

5. Show that $|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$ for any vectors $\mathbf{A}$ and $\mathbf{B}$.

6. Suppose that vectors $\mathbf{A}$ and $\mathbf{B}$ are not parallel and that $\mathbf{A} = \mathbf{C} + \mathbf{D}$, where $\mathbf{C}$ is parallel to $\mathbf{B}$ and $\mathbf{D}$ is orthogonal to $\mathbf{B}$. Express $\mathbf{C}$ and $\mathbf{D}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.

7. Show that $\mathbf{C} = |\mathbf{B}|\mathbf{A} + |\mathbf{A}|\mathbf{B}$ bisects the angle between $\mathbf{A}$ and $\mathbf{B}$.

8. Show that $|\mathbf{B}|\mathbf{A} + |\mathbf{A}|\mathbf{B}$ and $|\mathbf{B}|\mathbf{A} - |\mathbf{A}|\mathbf{B}$ are orthogonal.

9. *Dot multiplication is positive definite.* Show that dot multiplication of vectors is *positive definite;* that is, show that $\mathbf{A} \cdot \mathbf{A} \ge 0$ for every vector $\mathbf{A}$ and that $\mathbf{A} \cdot \mathbf{A} = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

10. By forming the cross product of two appropriate vectors, derive the trigonometric identity

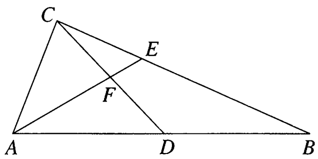
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

11. Use vectors to prove that

$$(a^2 + b^2)(c^2 + d^2) \ge (ac + bd)^2$$

for any four numbers a, b, c, and d. (*Hint:* Let $\mathbf{A} = a\,\mathbf{i} + b\,\mathbf{j}$ and $\mathbf{B} = c\,\mathbf{i} + d\,\mathbf{j}$.)

12. In the figure here, D is the midpoint of side AB of triangle ABC, and E is one-third of the way between C and B. Use vectors to prove that F is the midpoint of line segment CD.



13. a) Show that

$$\begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$$

is an equation for the plane through the three noncollinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

b) What set of points in space is described by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0?$$

14. Show that the lines

$$x = a_1 s + b_1,\ y = a_2 s + b_2,\ z = a_3 s + b_3,\ -\infty < s < \infty,$$

and

$$x = c_1 t + d_1,\ y = c_2 t + d_2,\ z = c_3 t + d_3,\ -\infty < t < \infty,$$

intersect or are parallel if and only if

$$\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0.$$

15. Use vectors to show that the distance from $P_1(x_1, y_1)$ to the line $ax + by = c$ is

$$d = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

16. a) Use vectors to show that the distance from $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz = D$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

b) Find an equation for the sphere that is tangent to the planes $x + y + z = 3$ and $x + y + z = 9$ if the planes $2x - y = 0$ and $3x - z = 0$ pass through the center of the sphere.

17. a) Show that the distance between the parallel planes $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$ is

$$d = \frac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}. \tag{1}$$

b) Use Eq. (1) to find the distance between the planes $2x + 3y - z = 6$ and $2x + 3y - z = 12$.

c) Find an equation for the plane parallel to the plane $2x - y + 2z = -4$ if the point $(3, 2, -1)$ is equidistant from the two planes.

d) Write equations for the planes that lie parallel to and 5 units away from the plane $x - 2y + z = 3$.

18. Prove that four points A, B, C, and D are coplanar (lie in a common plane) if and only if $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

19. *Triple vector products.* The **triple vector products** $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ are usually not equal, although the formulas for evaluating them from components are similar:

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}. \tag{2}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{3}$$

Verify each formula for the following vectors by evaluating its two sides and comparing the results.

	A	**B**	**C**
a)	$2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
b)	$\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
c)	$2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
d)	$\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

20. Show that if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are any vectors, then

a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$,

b) $\mathbf{A} \times \mathbf{B} = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{i})\mathbf{i} + (\mathbf{A} \cdot \mathbf{B} \times \mathbf{j})\mathbf{j} + (\mathbf{A} \cdot \mathbf{B} \times \mathbf{k})\,\mathbf{k}$,

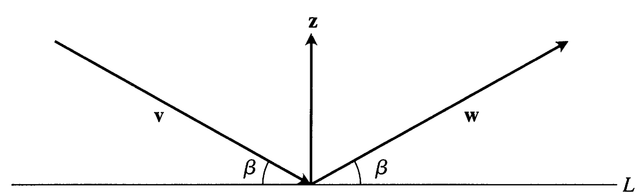
c) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{C} \\ \mathbf{A} \cdot \mathbf{D} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix}$.

21. Prove or disprove the formula

$$\mathbf{A} \times (\mathbf{A} \times (\mathbf{A} \times \mathbf{B})) \cdot \mathbf{C} = -|\mathbf{A}|^2 \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}.$$

22. *The projection of a vector on a plane.* Let P be a plane in space and let $\mathbf{v}$ be a vector. The vector projection of $\mathbf{v}$ onto the plane P, $\text{proj}_P \mathbf{v}$, can be defined informally as follows. Suppose the sun is shining so that its rays are normal to the plane P. Then $\text{proj}_P \mathbf{v}$ is the "shadow" of $\mathbf{v}$ onto P. If P is the plane $x + 2y + 6z = 6$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find $\text{proj}_P \mathbf{v}$.

23. The accompanying figure shows nonzero vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$, with $\mathbf{z}$ orthogonal to the line L, and $\mathbf{v}$ and $\mathbf{w}$ making equal angles β with L. Assuming $|\mathbf{v}| = |\mathbf{w}|$, find $\mathbf{w}$ in terms of $\mathbf{v}$ and $\mathbf{z}$.



24. *The parabolic coordinate system.* This exercise introduces a new coordinate system for space, the **parabolic coordinate system.** A point P is determined by an ordered triple (α, β, γ) in which

i) $\alpha\beta$ is the square of the distance from P to the z-axis,

ii) $|\alpha - \beta|$ is twice the distance from P to the xy-plane, and P lies above the xy-plane if $\alpha - \beta > 0$ and below it if $\alpha - \beta < 0$,

iii) $\gamma = \theta$, where θ has the same meaning as in cylindrical and spherical coordinate systems, except that we restrict γ to lie in $[0, 2\pi]$.

a) What are the equations for changing from parabolic coordinates to Cartesian coordinates?

b) Why is "parabolic coordinate system" an appropriate name?

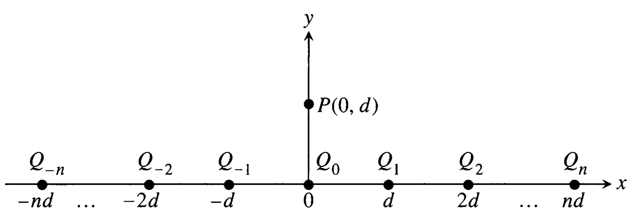
***25.** *Point masses and gravitation.* In physics the law of gravitation says that if P and Q are (point) masses with mass M and m, respectively, then P is attracted to Q by the force

$$\mathbf{F} = \frac{GMm\,\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r}$ is the vector from P to Q and G is a constant (the gravitational constant). Moreover, if $Q_1, \ldots, Q_k$ are (point) masses with mass $m_1, \ldots, m_k$, respectively, then the force on P due to all the Q_i's is

$$\mathbf{F} = \sum_{i=1}^{k} \frac{GMm_i}{|\mathbf{r}_i|^3}\mathbf{r}_i,$$

where $\mathbf{r}_i$ is the vector from P to Q_i.



a) Let point P with mass M be located at the point $(0, d)$, $d > 0$, in the coordinate plane. For $i = -n, -n+1, \ldots, -1, 0, 1, \ldots, n$, let Q_i be located at the point $(id, 0)$ and have mass m. Find the magnitude of the gravitational force on P due to all the Q_i's.

b) Is the limit as $n \to \infty$ of the magnitude of the force on P finite? Why, or why not?

***26.** *Relativistic sums.* Einstein's special theory of relativity roughly says that with respect to a reference frame (coordinate system) no material object can travel as fast as c, the speed of light. So, if $\overrightarrow{x}$ and $\overrightarrow{y}$ are two velocities such that $|\overrightarrow{x}| < c$ and $|\overrightarrow{y}| < c$, then the **relativistic sum** $\overrightarrow{x} \oplus \overrightarrow{y}$ of $\overrightarrow{x}$ and $\overrightarrow{y}$ must have length less than c. Einstein's special theory of relativity says

$$\overrightarrow{x} \oplus \overrightarrow{y} = \frac{\overrightarrow{x} + \overrightarrow{y}}{1 + \dfrac{\overrightarrow{x} \cdot \overrightarrow{y}}{c^2}} + \frac{1}{c^2} \cdot \frac{\gamma_x}{\gamma_x + 1} \cdot \frac{\overrightarrow{x} \times (\overrightarrow{x} \times \overrightarrow{y})}{1 + \dfrac{\overrightarrow{x} \cdot \overrightarrow{y}}{c^2}},$$

where

$$\gamma_x = \frac{1}{\sqrt{1 - \dfrac{\overrightarrow{x} \cdot \overrightarrow{x}}{c^2}}}.$$

It can be shown that if $|\overrightarrow{x}| < c$ and $|\overrightarrow{y}| < c$, then $|\overrightarrow{x} \oplus \overrightarrow{y}| < c$. This exercise deals with two special cases.

a) Prove that if $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal, $|\overrightarrow{x}| < c$, $|\overrightarrow{y}| < c$, then $|\overrightarrow{x} \oplus \overrightarrow{y}| < c$.

b) Prove that if $\overrightarrow{x}$ and $\overrightarrow{y}$ are parallel, $|\overrightarrow{x}| < c$, $|\overrightarrow{y}| < c$, then $|\overrightarrow{x} \oplus \overrightarrow{y}| < c$.

c) Compute $\lim_{c \to \infty} \overrightarrow{x} \oplus \overrightarrow{y}$.

CHAPTER 11

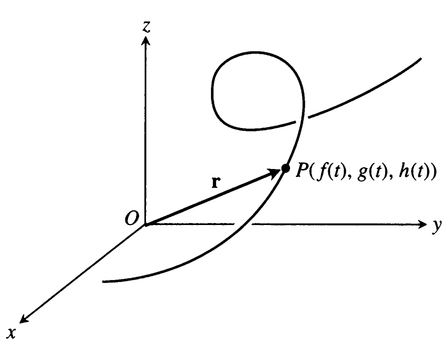
Vector-Valued Functions and Motion in Space

OVERVIEW When a body travels through space, the equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ that give the body's coordinates as functions of time serve as parametric equations for the body's motion and path. With vector notation, we can condense these into a single equation $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ that gives the body's position as a vector function of time.

In this chapter, we show how to use calculus to study the paths, velocities, and accelerations of moving bodies. As we go along, we will see how our work answers the standard questions about the paths and motions of projectiles, planets, and satellites. In the final section, we use our new vector calculus to derive Kepler's laws of planetary motion from Newton's laws of motion and gravitation.

11.1 Vector-Valued Functions and Space Curves

To track a particle moving in space, we run a vector $\mathbf{r}$ from the origin to the particle (Fig. 11.1) and study the changes in $\mathbf{r}$. If the particle's position coordinates are twice-differentiable functions of time, then so is $\mathbf{r}$, and we can find the particle's velocity and acceleration vectors at any time by differentiating $\mathbf{r}$. Conversely, if we know either the particle's velocity vector or acceleration vector as a continuous function of time, and if we have enough information about the particle's initial velocity and position, we can find $\mathbf{r}$ as a function of time by integration.



11.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

Definitions

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{1}$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path.** The equations and interval in (1) **parametrize** the curve. The vector

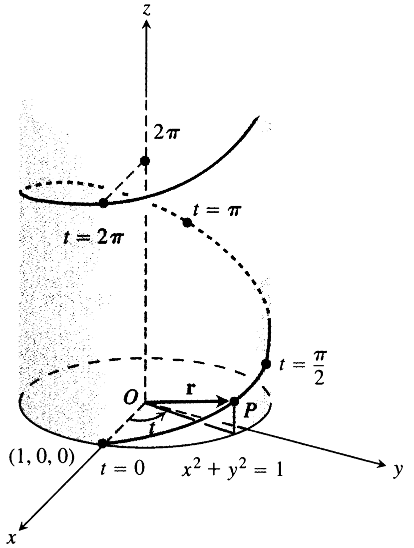
$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t is the particle's **position vector.** The functions f, g, and h are the **component functions**

(components) of the position vector. We think of the particle's path as the **curve traced by r** during the time interval I.

Equation (1) defines **r** as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D. For now, the domains will be intervals of real numbers. Later, in Chapter 14, the domains will be regions in the plane or in space. Vector functions will then be called "vector fields."

We refer to real-valued functions as **scalar functions** to distinguish them from vector functions. The components of **r** are scalar functions of t. When we define a vector-valued function by giving its component functions, we assume the vector function's domain to be the common domain of the components.



11.2 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$.

EXAMPLE 1 *A Helix*

The vector function

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$$

is defined for all real values of t. The curve traced by **r** is a helix (from an old Greek word for "spiral") that winds around the circular cylinder $x^2 + y^2 = 1$ (Fig. 11.2). The curve lies on the cylinder because the **i**- and **j**-components of **r**, being the x- and y-coordinates of the tip of **r**, satisfy the cylinder's equation:

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

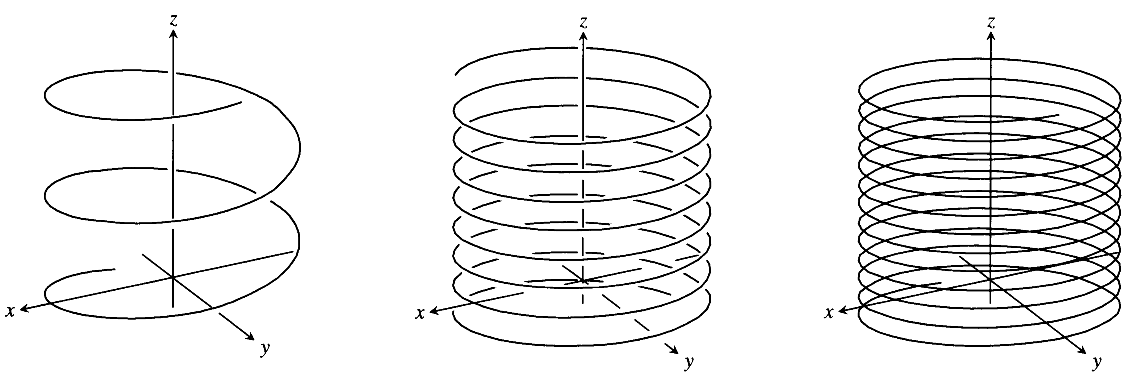
The curve rises as the **k**-component $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The equations

$$x = \cos t, \qquad y = \sin t, \qquad z = t$$

parametrize the helix, the interval $-\infty < t < \infty$ being understood. You will find more helices in Fig. 11.3. ❑

Limits and Continuity

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

(Generated by Mathematica)

11.3 Helices drawn by computer.

Definition

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit L** as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \qquad \Rightarrow \qquad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

If $\mathbf{L} = L_1\,\mathbf{i} + L_2\,\mathbf{j} + L_3\,\mathbf{k}$, then $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ precisely when

$$\lim_{t \to t_0} f(t) = L_1, \qquad \lim_{t \to t_0} g(t) = L_2, \qquad \text{and} \qquad \lim_{t \to t_0} h(t) = L_3.$$

The equation

$$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\mathbf{i} + \left(\lim_{t \to t_0} g(t)\right)\mathbf{j} + \left(\lim_{t \to t_0} h(t)\right)\mathbf{k} \tag{2}$$

provides a practical way to calculate limits of vector functions.

EXAMPLE 2 If $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, then

$$\begin{aligned}\lim_{t \to \pi/4} \mathbf{r}(t) &= \left(\lim_{t \to \pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t \to \pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t \to \pi/4} t\right)\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\,\mathbf{i} + \frac{\sqrt{2}}{2}\,\mathbf{j} + \frac{\pi}{4}\,\mathbf{k}.\end{aligned}$$

❑

We define continuity for vector functions the same way we define continuity for scalar functions.

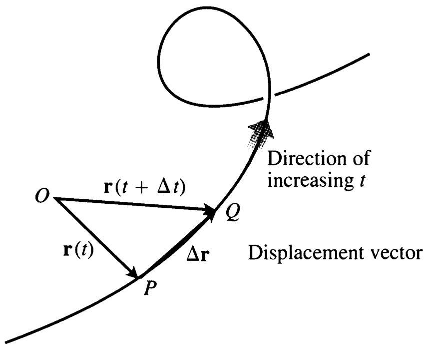
Definition

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Since limits can be expressed in terms of components, we can test vector functions for continuity by examining their components (Exercise 51).

Component Test for Continuity at a Point

The vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is continuous at $t = t_0$ if and only if f, g, and h are continuous at t_0.



11.4 Between time t and time $t+\Delta t$, the particle moving along the path shown here undergoes the displacement $\overrightarrow{PQ} = \Delta\mathbf{r}$. The vector sum $\mathbf{r}(t)+\Delta\mathbf{r}$ gives the new position, $\mathbf{r}(t+\Delta t)$.

EXAMPLE 3

a) The function

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$$

is continuous because $\cos t$, $\sin t$, and t are continuous.

b) The function

$$\mathbf{g}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + \lfloor t \rfloor \mathbf{k}$$

is discontinuous at every integer. ❑

Derivatives and Motion

Suppose that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is the position vector of a particle moving along a curve in space and that f, g, and h are differentiable functions of t. Then the difference between the particle's positions at time t and time $t+\Delta t$ is

$$\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$$

(Fig. 11.4). In terms of components,

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}(t+\Delta t) - \mathbf{r}(t)\\ &= [f(t+\Delta t)\,\mathbf{i} + g(t+\Delta t)\,\mathbf{j} + h(t+\Delta t)\,\mathbf{k}]\\ &\quad -[f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\\ &= [f(t+\Delta t) - f(t)]\,\mathbf{i} + [g(t+\Delta t) - g(t)]\,\mathbf{j} + [h(t+\Delta t) - h(t)]\,\mathbf{k}.\end{aligned}$$

As Δt approaches zero, three things seem to happen simultaneously. First, Q approaches P along the curve. Second, the secant line PQ seems to approach a limiting position tangent to the curve at P. Third, the quotient $\Delta\mathbf{r}/\Delta t$ approaches the limit

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} &= \left[\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\right]\mathbf{i} + \left[\lim_{\Delta t\to 0}\frac{g(t+\Delta t)-g(t)}{\Delta t}\right]\mathbf{j}\\ &\quad + \left[\lim_{\Delta t\to 0}\frac{h(t+\Delta t)-h(t)}{\Delta t}\right]\mathbf{k}\\ &= \left[\frac{df}{dt}\right]\mathbf{i} + \left[\frac{dg}{dt}\right]\mathbf{j} + \left[\frac{dh}{dt}\right]\mathbf{k}.\end{aligned}$$

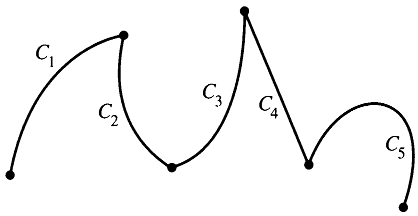
We are therefore led by past experience to the following definitions.

Definitions

The vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is **differentiable at $\mathbf{t} = \mathbf{t_0}$** if f, g, and h are differentiable at t_0. Also, $\mathbf{r}$ is said to be **differentiable** if it is differentiable at every point of its domain. At any point t at which $\mathbf{r}$ is differentiable, its **derivative** is the vector

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, i.e., if f, g, and h have continuous first derivatives that are not simultaneously 0.



11.5 A piecewise smooth curve made up of five smooth curves connected end to end in continuous fashion.

The vector $d\mathbf{r}/dt$, when different from $\mathbf{0}$, is also a vector *tangent* to the curve. The **tangent line** to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is defined to be the line through the point parallel to $d\mathbf{r}/dt$ at $t = t_0$. We require $d\mathbf{r}/dt \neq \mathbf{0}$ for a smooth curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve there are no sharp corners or cusps.

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth** (Fig. 11.5).

Look once again at Fig. 11.4. We drew the figure for Δt positive, so $\Delta\mathbf{r}$ points forward, in the direction of the motion. The vector $\Delta\mathbf{r}/\Delta t$ (not shown), having the same direction as $\Delta\mathbf{r}$, points forward too. Had Δt been negative, $\Delta\mathbf{r}$ would have pointed backward, against the direction of motion. The quotient $\Delta\mathbf{r}/\Delta t$, however, being a negative scalar multiple of $\Delta\mathbf{r}$, would once again have pointed forward. No matter how $\Delta\mathbf{r}$ points, $\Delta\mathbf{r}/\Delta t$ points forward and we expect the vector $d\mathbf{r}/dt = \lim_{\Delta t \to 0} \Delta\mathbf{r}/\Delta t$, when different from $\mathbf{0}$, to do the same. This means that the derivative $d\mathbf{r}/dt$ is just what we want for modeling a particle's velocity. It points in the direction of motion and gives the rate of change of position with respect to time. For a smooth curve, the velocity is never zero; the particle does not stop or reverse direction.

Definitions

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed,** and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector.** In short,

1. Velocity is the derivative of position: $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt}.$
2. Speed is the magnitude of velocity: $\quad \text{Speed} = |\mathbf{v}|.$
3. Acceleration is the derivative of velocity: $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}.$
4. The vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

We can express the velocity of a moving particle as the product of its speed and direction.

$$\text{Velocity} = |\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = (\text{speed})\,(\text{direction})$$

EXAMPLE 4 The vector

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + t^2\,\mathbf{k}$$

gives the position of a moving body at time t. Find the body's speed and direction when $t = 2$. At what times, if any, are the body's velocity and acceleration orthogonal?

Solution

$$\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\,\mathbf{k}$$
$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$$

At $t = 2$, the body's speed and direction are

Speed: $|\mathbf{v}(2)| = \sqrt{(-3\sin 2)^2 + (3\cos 2)^2 + (4)^2} = 5$

Direction: $\dfrac{\mathbf{v}(2)}{|\mathbf{v}(2)|} = -\left(\dfrac{3}{5}\sin 2\right)\mathbf{i} + \left(\dfrac{3}{5}\cos 2\right)\mathbf{j} + \dfrac{4}{5}\mathbf{k}.$

To find the times when $\mathbf{v}$ and $\mathbf{a}$ are orthogonal, we look for values of t for which

$$\mathbf{v}\cdot\mathbf{a} = 9\sin t\cos t - 9\cos t\sin t + 4t = 4t = 0.$$

The only value is $t = 0$. ❑

Differentiation Rules

Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the same form as the rules for differentiating scalar functions.

Differentiation Rules for Vector Functions

Constant Function Rule: $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$ (any constant vector $\mathbf{C}$)

If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions of t, then

Scalar Multiple Rules: $\dfrac{d}{dt}(c\,\mathbf{u}) = c\dfrac{d\mathbf{u}}{dt}$ (any number c)

$\dfrac{d}{dt}(f\,\mathbf{u}) = \dfrac{df}{dt}\mathbf{u} + f\dfrac{d\mathbf{u}}{dt}$ (any differentiable scalar function $f(t)$)

Sum Rule: $\dfrac{d}{dt}(\mathbf{u} + \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}$

Difference Rule: $\dfrac{d}{dt}(\mathbf{u} - \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} - \dfrac{d\mathbf{v}}{dt}$

Dot Product Rule: $\dfrac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}$

Cross Product Rule: $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\dfrac{d\mathbf{v}}{dt}$

When you use the Cross Product Rule, remember to preserve the order of the factors. If $\mathbf{u}$ comes first on the left side of the equation, it must also come first on the right or the signs will be wrong.

As an algebraic convenience, we sometimes write the product of a scalar c and a vector $\mathbf{v}$ as $\mathbf{v}c$ instead of $c\mathbf{v}$. This permits us, for instance, to write the Chain Rule in a familiar form:

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds}.$$

Chain Rule (Short Form): If $\mathbf{r}$ is a differentiable function of t and t is a differentiable function of s, then

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds}.$$

We will prove the product rules and Chain Rule but leave the rules for constants, scalar multiples, sums, and differences as exercises.

Proof of the Dot Product Rule Suppose that

$$\mathbf{u} = u_1(t)\,\mathbf{i} + u_2(t)\,\mathbf{j} + u_3(t)\,\mathbf{k}$$

and

$$\mathbf{v} = v_1(t)\,\mathbf{i} + v_2(t)\,\mathbf{j} + v_3(t)\,\mathbf{k}.$$

Then

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3)\\ &= \underbrace{u_1'v_1 + u_2'v_2 + u_3'v_3}_{\mathbf{u}'\cdot\mathbf{v}} + \underbrace{u_1v_1' + u_2v_2' + u_3v_3'}_{\mathbf{u}\cdot\mathbf{v}'}.\end{aligned}$$

❑

Proof of the Cross Product Rule We model the proof after the proof of the product rule for scalar functions. According to the definition of derivative,

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t)}{h}.$$

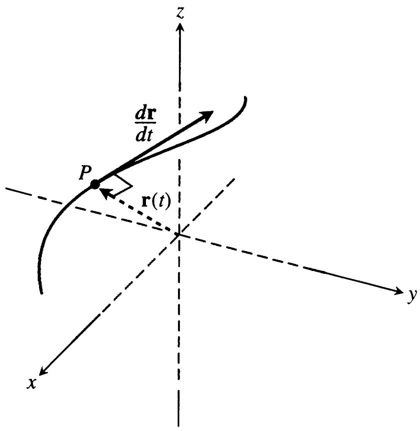
To change this fraction into an equivalent one that contains the difference quotients for the derivatives of $\mathbf{u}$ and $\mathbf{v}$, we subtract and add $\mathbf{u}(t)\times\mathbf{v}(t+h)$ in the numerator. Then

$$\begin{aligned}&\frac{d}{dt}(\mathbf{u}\times\mathbf{v})\\ &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t)}{h}\\ &= \lim_{h\to 0}\left[\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}\right]\\ &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\lim_{h\to 0}\mathbf{v}(t+h) + \lim_{h\to 0}\mathbf{u}(t)\times\lim_{h\to 0}\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}.\end{aligned}$$

The last of these equalities holds because the limit of the cross product of two vector functions is the cross product of their limits if the latter exist (Exercise 52). As h approaches zero, $\mathbf{v}(t+h)$ approaches $\mathbf{v}(t)$ because $\mathbf{v}$, being differentiable at t, is continuous at t (Exercise 53). The two fractions approach the values of $d\mathbf{u}/dt$ and $d\mathbf{v}/dt$ at t. In short,

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt}.$$

❑



11.6 If a particle moves on a sphere in such a way that its position **r** is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

Proof of the Chain Rule Suppose that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is a differentiable vector function of t and that t is a differentiable scalar function of some other variable s. Then f, g, and h are differentiable functions of s, and the Chain Rule for differentiable real-valued functions gives

$$\begin{aligned}\frac{d\mathbf{r}}{ds} &= \frac{df}{ds}\mathbf{i} + \frac{dg}{ds}\mathbf{j} + \frac{dh}{ds}\mathbf{k} \\ &= \frac{df}{dt}\frac{dt}{ds}\mathbf{i} + \frac{dg}{dt}\frac{dt}{ds}\mathbf{j} + \frac{dh}{dt}\frac{dt}{ds}\mathbf{k} \\ &= \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right)\frac{dt}{ds} \\ &= \frac{d\mathbf{r}}{dt}\frac{dt}{ds}.\end{aligned}$$ ❑

Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin (Fig. 11.6), the position vector has a constant length equal to the radius of the sphere. The velocity vector $d\mathbf{r}/dt$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to **r**. This is always the case for a differentiable vector function of constant length: The vector and its first derivative are orthogonal. With the length constant, the change in the function is a change in direction only, and direction changes take place at right angles.

We will use this observation repeatedly in Section 11.4.

> If **u** is a differentiable vector function of t of constant length, then
> $$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0. \tag{3}$$

To see why Eq. (3) holds, suppose that **u** is a differentiable function of t and that $|\mathbf{u}|$ is constant. Then $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ is constant and we may differentiate both sides of this equation to get

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \frac{d}{dt}(\text{constant}) = 0$$

$$\frac{d\mathbf{u}}{dt} \cdot \mathbf{u} + \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0 \qquad \text{Dot Product Rule with } \mathbf{v} = \mathbf{u}$$

$$2\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0 \qquad \text{Dot multiplication is commutative.}$$

$$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0.$$

EXAMPLE 5 Show that

$$\mathbf{u}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \sqrt{3}\,\mathbf{k}$$

has constant length and is orthogonal to its derivative.

Solution

$$\begin{aligned}
\mathbf{u}(t) &= (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \\
|\mathbf{u}(t)| &= \sqrt{(\sin t)^2 + (\cos t)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2 \\
\frac{d\mathbf{u}}{dt} &= (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} \\
\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} &= \sin t \cos t - \sin t \cos t = 0
\end{aligned}$$

❑

Integrals of Vector Functions

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval I if $d\mathbf{R}/dt = \mathbf{r}$ at each point of I. If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on I, it can be shown, working one component at a time, that every antiderivative of $\mathbf{r}$ on I has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$ (Exercise 56). The set of all antiderivatives of $\mathbf{r}$ on I is the **indefinite integral** of $\mathbf{r}$ on I.

Definition

The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

The usual arithmetic rules for indefinite integrals apply.

EXAMPLE 6

$$\begin{aligned}
&\int ((\cos t)\,\mathbf{i} + \mathbf{j} - 2t\,\mathbf{k})\,dt \\
&\qquad = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k} &&(4)\\
&\qquad = (\sin t + C_1)\,\mathbf{i} + (t + C_2)\,\mathbf{j} - (t^2 + C_3)\,\mathbf{k} &&(5)\\
&\qquad = (\sin t)\,\mathbf{i} + t\,\mathbf{j} - t^2\,\mathbf{k} + \mathbf{C} \qquad \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}
\end{aligned}$$

As in the integration of scalar functions, we recommend that you skip the steps in (4) and (5) and go directly to the final form. Find an antiderivative for each component and add a constant vector at the end. ❑

Definite integrals of vector functions are defined in terms of components.

Definition

If the components of $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ are integrable over $[a, b]$ then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

The usual arithmetic rules for definite integrals apply (Exercise 54).

EXAMPLE 7

$$\begin{aligned}\int_0^{\pi} ((\cos t)\,\mathbf{i} + \mathbf{j} - 2t\,\mathbf{k})\,dt &= \left(\int_0^{\pi} \cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi} dt\right)\mathbf{j} - \left(\int_0^{\pi} 2t\,dt\right)\mathbf{k} \\ &= \left[\sin t\right]_0^{\pi}\,\mathbf{i} + \left[t\right]_0^{\pi}\,\mathbf{j} - \left[t^2\right]_0^{\pi}\,\mathbf{k} \\ &= [0-0]\,\mathbf{i} + [\pi - 0]\,\mathbf{j} - [\pi^2 - 0^2]\,\mathbf{k} \\ &= \pi\,\mathbf{j} - \pi^2\,\mathbf{k}\end{aligned}$$

EXAMPLE 8 *Finding a particle's position function from its velocity function and initial position*

The velocity of a particle moving in space is

$$\frac{d\mathbf{r}}{dt} = (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + \mathbf{k}.$$

Find the particle's position as a function of t if $\mathbf{r} = 2\,\mathbf{i} + \mathbf{k}$ when $t = 0$.

Solution Our goal is to solve the initial value problem that consists of

The differential equation: $\dfrac{d\mathbf{r}}{dt} = (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + \mathbf{k}$

The initial condition: $\mathbf{r}(0) = 2\,\mathbf{i} + \mathbf{k}$

Integrating both sides of the differential equation with respect to t gives

$$\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + t\,\mathbf{k} + \mathbf{C}.$$

We then use the initial condition to find the right value for $\mathbf{C}$:

$$\begin{aligned}(\sin 0)\,\mathbf{i} + (\cos 0)\,\mathbf{j} + (0)\,\mathbf{k} + \mathbf{C} &= 2\,\mathbf{i} + \mathbf{k} \qquad \mathbf{r}(0) = 2\,\mathbf{i} + \mathbf{k} \\ \mathbf{j} + \mathbf{C} &= 2\,\mathbf{i} + \mathbf{k} \\ \mathbf{C} &= 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

The particle's position as a function of t is

$$\mathbf{r}(t) = (\sin t + 2)\,\mathbf{i} + (\cos t - 1)\,\mathbf{j} + (t + 1)\,\mathbf{k}.$$

To check (always a good idea), we can see from this formula that

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (\cos t + 0)\,\mathbf{i} + (-\sin t - 0)\,\mathbf{j} + (1 + 0)\,\mathbf{k} \\ &= (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + \mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}(0) &= (\sin 0 + 2)\,\mathbf{i} + (\cos 0 - 1)\,\mathbf{j} + (0 + 1)\,\mathbf{k} \\ &= 2\,\mathbf{i} + \mathbf{k}.\end{aligned}$$

Exercises 11.1

Motion in the *xy*-plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $\mathbf{r}(t) = (t+1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j}, \quad t = 1$
2. $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + (2t - 1)\,\mathbf{j}, \quad t = 1/2$
3. $\mathbf{r}(t) = e^t\,\mathbf{i} + \dfrac{2}{9}e^{2t}\,\mathbf{j}, \quad t = \ln 3$
4. $\mathbf{r}(t) = (\cos 2t)\,\mathbf{i} + (3\sin 2t)\,\mathbf{j}, \quad t = 0$

Exercises 5–8 give the position vectors of particles moving along various curves in the xy-plane. In each case, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

5. *Motion on the circle* $x^2 + y^2 = 1$

   $\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}; \quad t = \pi/4$ and $\pi/2$
6. *Motion on the circle* $x^2 + y^2 = 16$

   $\mathbf{r}(t) = \left(4\cos\dfrac{t}{2}\right)\mathbf{i} + \left(4\sin\dfrac{t}{2}\right)\mathbf{j}; \quad t = \pi$ and $3\pi/2$
7. *Motion on the cycloid* $x = t - \sin t,\ y = 1 - \cos t$

   $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}; \quad t = \pi$ and $3\pi/2$
8. *Motion on the parabola* $y = x^2 + 1$

   $\mathbf{r}(t) = t\,\mathbf{i} + (t^2 + 1)\,\mathbf{j}; \quad t = -1, 0,$ and 1

Velocity and Acceleration in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t+1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + 2t\,\mathbf{k}, \quad t = 1$
10. $\mathbf{r}(t) = (1+t)\,\mathbf{i} + \dfrac{t^2}{\sqrt{2}}\,\mathbf{j} + \dfrac{t^3}{3}\,\mathbf{k}, \quad t = 1$
11. $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + 4t\,\mathbf{k}, \quad t = \pi/2$
12. $\mathbf{r}(t) = (\sec t)\,\mathbf{i} + (\tan t)\,\mathbf{j} + \dfrac{4}{3}t\,\mathbf{k}, \quad t = \pi/6$
13. $\mathbf{r}(t) = (2\ln(t+1))\,\mathbf{i} + t^2\,\mathbf{j} + \dfrac{t^2}{2}\,\mathbf{k}, \quad t = 1$
14. $\mathbf{r}(t) = (e^{-t})\,\mathbf{i} + (2\cos 3t)\,\mathbf{j} + (2\sin 3t)\,\mathbf{k}, \quad t = 0$

In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

15. $\mathbf{r}(t) = (3t+1)\,\mathbf{i} + \sqrt{3}\,t\,\mathbf{j} + t^2\,\mathbf{k}$
16. $\mathbf{r}(t) = \left(\dfrac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\dfrac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$
17. $\mathbf{r}(t) = (\ln(t^2+1))\,\mathbf{i} + (\tan^{-1} t)\,\mathbf{j} + \sqrt{t^2+1}\,\mathbf{k}$
18. $\mathbf{r}(t) = \dfrac{4}{9}(1+t)^{3/2}\,\mathbf{i} + \dfrac{4}{9}(1-t)^{3/2}\,\mathbf{j} + \dfrac{1}{3}t\,\mathbf{k}$

In Exercises 19 and 20, $\mathbf{r}(t)$ is the position vector of a particle in space at time t. Find the time or times in the given time interval when the velocity and acceleration vectors are orthogonal.

19. $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}, \quad 0 \le t \le 2\pi$
20. $\mathbf{r}(t) = (\sin t)\,\mathbf{i} + t\,\mathbf{j} + (\cos t)\,\mathbf{k}, \quad t \ge 0$

Integrating Vector-valued Functions

Evaluate the integrals in Exercises 21–26.

21. $\displaystyle\int_0^1 [t^3\,\mathbf{i} + 7\,\mathbf{j} + (t+1)\,\mathbf{k}]\,dt$
22. $\displaystyle\int_1^2 \left[(6 - 6t)\,\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right] dt$
23. $\displaystyle\int_{-\pi/4}^{\pi/4} [(\sin t)\,\mathbf{i} + (1 + \cos t)\,\mathbf{j} + (\sec^2 t)\,\mathbf{k}]\,dt$
24. $\displaystyle\int_0^{\pi/3} [(\sec t \tan t)\,\mathbf{i} + (\tan t)\,\mathbf{j} + (2\sin t\cos t)\,\mathbf{k}]\,dt$
25. $\displaystyle\int_1^4 \left[\frac{1}{t}\,\mathbf{i} + \frac{1}{5-t}\,\mathbf{j} + \frac{1}{2t}\,\mathbf{k}\right] dt$
26. $\displaystyle\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\,\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\,\mathbf{k}\right] dt$

Initial Value Problems for Vector–valued Functions

Solve the initial value problems in Exercises 27–32 for $\mathbf{r}$ as a vector function of t.

27. Differential equation: $\dfrac{d\mathbf{r}}{dt} = -t\,\mathbf{i} - t\,\mathbf{j} - t\,\mathbf{k}$

    Initial condition: $\mathbf{r}(0) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$
28. Differential equation: $\dfrac{d\mathbf{r}}{dt} = (180t)\,\mathbf{i} + (180t - 16t^2)\,\mathbf{j}$

    Initial condition: $\mathbf{r}(0) = 100\,\mathbf{j}$
29. Differential equation: $\dfrac{d\mathbf{r}}{dt} = \dfrac{3}{2}(t+1)^{1/2}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \dfrac{1}{t+1}\,\mathbf{k}$

    Initial condition: $\mathbf{r}(0) = \mathbf{k}$

30. Differential equation: $\dfrac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

31. Differential equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$

Initial conditions: $\mathbf{r}(0) = 100\mathbf{k}$ and $\left.\dfrac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}$

32. Differential equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Initial conditions: $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$ and $\left.\dfrac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0}$

Tangent Lines to Smooth Curves

As mentioned in the text, the tangent line to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0. In Exercises 33–36, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

33. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

34. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 5t\mathbf{k}, \quad t_0 = 4\pi$

35. $\mathbf{r}(t) = (a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + bt\mathbf{k}, \quad t_0 = 2\pi$

36. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \dfrac{\pi}{2}$

Motion on Circular Paths

37. Each of the following equations (a)–(e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in (a)–(e) is the same, the behavior, or "dynamics," of each particle is different. For each particle, answer the following questions.

- **i)** Does the particle have constant speed? If so, what is its constant speed?
- **ii)** Is the particle's acceleration vector always orthogonal to its velocity vector?
- **iii)** Does the particle move clockwise or counterclockwise around the circle?
- **iv)** Does the particle begin at the point (1, 0)?

- **a)** $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \ge 0$
- **b)** $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}, \quad t \ge 0$
- **c)** $\mathbf{r}(t) = \cos(t - \pi/2)\mathbf{i} + \sin(t - \pi/2)\mathbf{j}, \quad t \ge 0$
- **d)** $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad t \ge 0$
- **e)** $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}, \quad t \ge 0$

38. Show that the vector-valued function

$$\mathbf{r}(t) = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \cos t\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) + \sin t\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$$

describes the motion of a particle moving in the circle of radius 1 centered at the point (2, 2, 1) and lying in the plane $x + y - 2z = 2$.

Motion along a Straight Line

39. At time $t = 0$, a particle is located at the point (1, 2, 3). It travels in a straight line to the point (4, 1, 4), has speed 2 at (1, 2, 3) and constant acceleration $3\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find an equation for the position vector $\mathbf{r}(t)$ of the particle at time t.

40. A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point (3, 0, 3) with constant acceleration $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

Theory and Examples

41. A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point (2, 2).

42. A particle moves on a cycloid in the xy-plane in such a way that its position at time t is

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$$

Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (*Hint:* Find the extreme values of $|\mathbf{v}|^2$ and $|\mathbf{a}|^2$ first and take square roots later.)

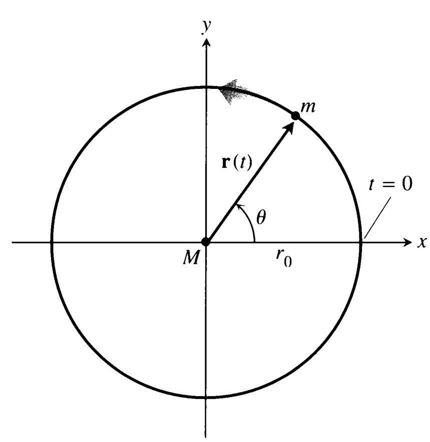
43. A particle moves around the ellipse $(y/3)^2 + (z/2)^2 = 1$ in the yz-plane in such a way that its position at time t is

$$\mathbf{r}(t) = (3\cos t)\mathbf{j} + (2\sin t)\mathbf{k}.$$

Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (See the hint in Exercise 42.)

44. *A satellite in circular orbit.* A satellite of mass m is revolving at a constant speed v around a body of mass M (Earth, for example) in a circular orbit of radius r_0 (measured from the body's center of mass). Determine the satellite's orbital period T (the time to complete one full orbit), as follows:

a) Coordinatize the orbital plane by placing the origin at the body's center of mass, with the satellite on the x-axis at $t = 0$ and moving counterclockwise, as in the accompanying figure.



Let $\mathbf{r}(t)$ be the satellite's position vector at time t. Show that $\theta = vt/r_0$ and hence that

$$\mathbf{r}(t) = \left(r_0 \cos \frac{vt}{r_0}\right)\mathbf{i} + \left(r_0 \sin \frac{vt}{r_0}\right)\mathbf{j}.$$

b) Find the acceleration of the satellite.

c) According to Newton's Law of Gravitation, the gravitational force exerted on the satellite is directed toward M and is given by

$$\mathbf{F} = \left(-\frac{GmM}{r_0^2}\right)\frac{\mathbf{r}}{r_0},$$

where G is the universal constant of gravitation. Using Newton's second law, $\mathbf{F} = m\,\mathbf{a}$, show that $v^2 = GM/r_0$.

d) Show that the orbital period T satisfies $vT = 2\pi r_0$.

e) From parts (c) and (d), deduce that

$$T^2 = \frac{4\pi^2}{GM} r_0^3.$$

That is, the square of the period of a satellite in circular orbit is proportional to the cube of the radius from the orbital center.

45. Let $\mathbf{v}$ be a differentiable vector function of t. Show that if $\mathbf{v} \cdot (d\mathbf{v}/dt) = 0$ for all t, then $|\mathbf{v}|$ is constant.

46. *Derivatives of triple scalar products*

a) Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable vector functions of t, then

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) \tag{6}$$
$$= \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} \times \mathbf{w} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}.$$

b) Show that Eq. (6) is equivalent to

$$\frac{d}{dt}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} \frac{du_1}{dt} & \frac{du_2}{dt} & \frac{du_3}{dt} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \\ w_1 & w_2 & w_3 \end{vmatrix}$$
$$+ \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{dw_1}{dt} & \frac{dw_2}{dt} & \frac{dw_3}{dt} \end{vmatrix}. \tag{7}$$

Equation (7) says that the derivative of a 3 by 3 determinant of differentiable functions is the sum of the three determinants obtained from the original by differentiating one row at a time. The result extends to determinants of any order.

47. *(Continuation of Exercise 46.)* Suppose that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ and that f, g, and h have derivatives through order three. Use Eq. (6) or (7) to show that

$$\frac{d}{dt}\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3}\right). \tag{8}$$

(*Hint:* Differentiate on the left and look for vectors whose products are zero.)

48. *The Constant Function Rule.* Prove that if $\mathbf{u}$ is the vector function with the constant value $\mathbf{C}$, then $d\mathbf{u}/dt = \mathbf{0}$.

49. *The Scalar Multiple Rules*

a) Prove that if $\mathbf{u}$ is a differentiable function of t and c is any real number, then

$$\frac{d(c\mathbf{u})}{dt} = c\frac{d\mathbf{u}}{dt}.$$

b) Prove that if $\mathbf{u}$ is a differentiable function of t and f is a differentiable scalar function of t, then

$$\frac{d}{dt}(f\,\mathbf{u}) = \frac{df}{dt}\mathbf{u} + f\frac{d\mathbf{u}}{dt}.$$

50. *The Sum and Difference Rules.* Prove that if $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t, then

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{u} - \mathbf{v}) = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}.$$

51. *The component test for continuity at a point.* Show that the vector function $\mathbf{r}$ defined by the rule $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $t = t_0$ if and only if f, g, and h are continuous at t_0.

52. *Limits of cross products of vector functions.* Suppose that $\mathbf{r}_1(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, $\mathbf{r}_2(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$, $\lim_{t\to t_0} \mathbf{r}_1(t) = \mathbf{A}$, and $\lim_{t\to t_0} \mathbf{r}_2(t) = \mathbf{B}$. Use the determinant formula for cross products and the Limit Product Rule for scalar functions to show that

$$\lim_{t\to t_0} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{A} \times \mathbf{B}$$

53. *Differentiable vector functions are continuous.* Show that if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is differentiable at $t = t_0$, then it is continuous at t_0 as well.

54. Establish the following properties of integrable vector functions.

a) The *Constant Scalar Multiple Rule:*

$$\int_a^b k\,\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt \quad \text{(any scalar } k\text{)}$$

The *Rule for Negatives,*

$$\int_a^b (-\mathbf{r}(t))\,dt = -\int_a^b \mathbf{r}(t)\,dt,$$

is obtained by taking $k = -1$.

b) The *Sum and Difference Rules:*

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t))\,dt = \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$$

c) The *Constant Vector Multiple Rules:*

$$\int_a^b \mathbf{C} \cdot \mathbf{r}(t)\,dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t)\,dt \quad \text{(any constant vector } \mathbf{C}\text{)}$$

and

$$\int_a^b \mathbf{C} \times \mathbf{r}(t)\,dt = \mathbf{C} \times \int_a^b \mathbf{r}(t)\,dt \quad \text{(any constant vector } \mathbf{C}\text{)}$$

55. *Products of scalar and vector functions.* Suppose that the scalar function $u(t)$ and the vector function $\mathbf{r}(t)$ are both defined for $a \le t \le b$.

a) Show that $u\,\mathbf{r}$ is continuous on $[a, b]$ if u and $\mathbf{r}$ are continuous on $[a, b]$.

b) If u and $\mathbf{r}$ are both differentiable on $[a, b]$, show that $u\,\mathbf{r}$ is differentiable on $[a, b]$ and that

$$\frac{d}{dt}(u\,\mathbf{r}) = u\frac{d\mathbf{r}}{dt} + \mathbf{r}\frac{du}{dt}.$$

56. *Antiderivatives of vector functions*

a) Use Corollary 2 of the Mean Value Theorem for scalar functions to show that if two vector functions $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on an interval I, then the functions differ by a constant vector value throughout I.

b) Use the result in (a) to show that if $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$ on I, then every other antiderivative of $\mathbf{r}$ on I equals $\mathbf{R}(t) + \mathbf{C}$ for some constant vector $\mathbf{C}$.

57. *The Fundamental Theorem of Calculus.* The Fundamental Theorem of Calculus for scalar functions of a real variable holds for vector functions of a real variable as well. Prove this by using the theorem for scalar functions to show first that if a vector function $\mathbf{r}(t)$ is continuous for $a \le t \le b$, then

$$\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$$

at every point t of $[a, b]$. Then use the conclusion in part (b) of Exercise 56 to show that if $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ on $[a, b]$ then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a).$$

CAS Explorations and Projects

Use a CAS to perform the following steps in Exercises 58–61.

a) Plot the space curve traced out by the position vector $\mathbf{r}$.

b) Find the components of the velocity vector $d\mathbf{r}/dt$.

c) Evaluate $d\mathbf{r}/dt$ at the given point t_0 and determine the equation of the tangent line to the curve at $\mathbf{r}(t_0)$.

d) Plot the tangent line together with the curve over the given interval.

58. $\mathbf{r}(t) = (\sin t - t\cos t)\,\mathbf{i} + (\cos t + t\sin t)\,\mathbf{j} + t^2\,\mathbf{k},\quad 0 \le t \le 6\pi,\quad t_0 = 3\pi/2$

59. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k},\quad -2 \le t \le 3,\quad t_0 = 1$

60. $\mathbf{r}(t) = (\sin 2t)\,\mathbf{i} + (\ln(1+t))\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 4\pi,\quad t_0 = \pi/4$

61. $\mathbf{r}(t) = (\ln(t^2+2))\,\mathbf{i} + (\tan^{-1} 3t)\,\mathbf{j} + \sqrt{t^2+1}\,\mathbf{k},\quad -3 \le t \le 5,\quad t_0 = 3$

In Exercises 62 and 63, you will explore graphically the behavior of the helix

$$\mathbf{r}(t) = (\cos at)\,\mathbf{i} + (\sin at)\,\mathbf{j} + bt\,\mathbf{k}$$

as you change the values of the constants a and b. Use a CAS to perform the steps in each exercise.

62. Set $b = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $a = 1, 2, 4,$ and 6 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as a increases through these positive values.

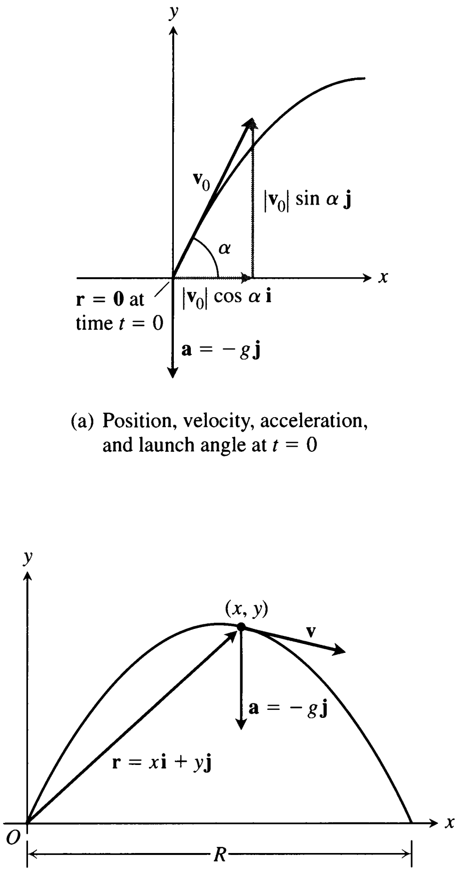
63. Set $a = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $b = 1/4, 1/2, 2,$ and 4 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as b increases through these positive values.

11.2 Modeling Projectile Motion

When we shoot a projectile into the air we usually want to know beforehand how far it will go (will it reach the target?), how high it will rise (will it clear the hill?), and when it will land (when do we get results?). We get this information from the direction and magnitude of the projectile's initial velocity vector, using Newton's Second Law of Motion.

The Vector and Parametric Equations for Ideal Projectile Motion

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and that the only force acting on the projectile during its flight is the constant force of gravity, which always points



(a) Position, velocity, acceleration, and launch angle at $t = 0$

(b) Position, velocity, and acceleration at a later time t

11.7 The flight of an ideal projectile.

straight down. In practice, none of these assumptions really holds. The ground moves beneath the projectile as the earth turns, the air creates a frictional force that varies with the projectile's speed and altitude, and the force of gravity changes as the projectile moves along. All this must be taken into account by applying corrections to the predictions of the *ideal* equations we are about to derive. The corrections, however, are not the subject of this section.

We assume that our projectile is launched from the origin at time $t = 0$ into the first quadrant with an initial velocity $\mathbf{v}_0$ (Fig. 11.7). If $\mathbf{v}_0$ makes an angle α with the horizontal, then

$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha)\,\mathbf{i} + (|\mathbf{v}_0| \sin \alpha)\,\mathbf{j}. \tag{1}$$

If we use the simpler notation v_0 for the initial speed $|\mathbf{v}_0|$, then

$$\mathbf{v}_0 = (v_0 \cos \alpha)\,\mathbf{i} + (v_0 \sin \alpha)\,\mathbf{j}. \tag{2}$$

The projectile's initial position is

$$\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j} = \mathbf{0}. \tag{3}$$

Newton's Second Law of Motion says that the force acting on the projectile is its mass m times its acceleration, or $m(d^2\mathbf{r}/dt^2)$ if $\mathbf{r}$ is the projectile's position vector and t is time. If the force is solely the gravitational force $-mg\,\mathbf{j}$, then

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\,\mathbf{j} \qquad \text{and} \qquad \frac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j}. \tag{4}$$

We find $\mathbf{r}$ as a function of t by solving the following initial value problem:

Differential equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j}$

Initial conditions: $\mathbf{r} = \mathbf{r}_0$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{v}_0$ when $t = 0$

The first integration gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\,\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$$

Substituting the values of $\mathbf{v}_0$ and $\mathbf{r}_0$ from Eqs. (2) and (3) gives

$$\mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} + \underbrace{(v_0 \cos \alpha)t\,\mathbf{i} + (v_0 \sin \alpha)t\,\mathbf{j}}_{\mathbf{v}_0 t} + \mathbf{0}$$

or

$$\mathbf{r} = (v_0 \cos \alpha)t\,\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}. \tag{5}$$

Equation (5) is the *vector equation* for ideal projectile motion. The angle α is the projectile's **launch angle (firing angle, angle of elevation)**, and v_0, as we said before, is the projectile's **initial speed.**

Equation (5) is equivalent to a pair of scalar equations,

$$x = (v_0 \cos \alpha)t, \qquad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2. \tag{6}$$

These are known as the *parametric equations* for ideal projectile motion. If time is measured in seconds and distance in meters, g is 9.8 m/sec^2 and Eqs. (6) give x and y in meters. With feet in place of meters, g is 32 ft/sec^2 and Eqs. (6) give x and y in feet.

EXAMPLE 1 A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec at a launch angle of 60°. Where will the projectile be 10 sec later?

Solution We use Eqs. (6) with $v_0 = 500$, $\alpha = 60°$, $g = 9.8$, and $t = 10$ to find the projectile's coordinates to the nearest meter 10 sec after firing:

$$x = (v_0 \cos \alpha)t = 500 \cdot \frac{1}{2} \cdot 10 = 2500 \text{ m}$$

$$\begin{aligned} y &= (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \\ &= 500 \cdot \frac{\sqrt{3}}{2} \cdot 10 - \frac{1}{2} \cdot 9.8 \cdot (10)^2 \\ &= 2500\sqrt{3} - 490 \\ &\approx 3840 \text{ m.} \end{aligned}$$

Ten seconds after firing, the projectile is 3840 m in the air and 2500 m downrange. ❑

Height, Flight Time, and Range

Equations (6) enable us to answer most questions about an ideal projectile fired from the origin.

The projectile reaches its highest point when its vertical velocity component is zero, that is, when

$$\frac{dy}{dt} = v_0 \sin \alpha - gt = 0, \qquad \text{or} \qquad t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t, the value of y is

$$y_{\max} = (v_0 \sin \alpha)\left(\frac{v_0 \sin \alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin \alpha}{g}\right)^2 = \frac{(v_0 \sin \alpha)^2}{2g}. \tag{7}$$

To find when the projectile lands, when fired over horizontal ground, we set y equal to zero in Eqs. (6) and solve for t:

$$\begin{aligned} (v_0 \sin \alpha)t - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0 \sin \alpha - \frac{1}{2}gt\right) &= 0 \\ t = 0, \qquad t &= \frac{2v_0 \sin \alpha}{g}. \end{aligned} \tag{8}$$

Since 0 is the time the projectile is fired, $(2v_0 \sin \alpha)/g$ must be the time when the projectile strikes the ground.

To find the projectile's range R, the distance from the origin to the point of impact on horizontal ground, we find the value of x when $t = (2v_0 \sin \alpha)/g$:

$$\begin{aligned} x &= (v_0 \cos \alpha)t \\ R &= (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha}{g}\right) \\ &= \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g} \sin 2\alpha. \end{aligned} \tag{9}$$

The range is largest when $\sin 2\alpha = 1$ or $\alpha = 45°$.

EXAMPLE 2 Find the maximum height, flight time, and range of a projectile fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60° (same projectile as in Example 1).

Solution

Maximum height (Eq. 7): $$y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{(500 \sin 60°)^2}{2(9.8)} \approx 9566 \text{ m}$$

Flight time (Eq. 8): $$t = \frac{2v_0 \sin \alpha}{g} = \frac{2(500) \sin 60°}{9.8} \approx 88 \text{ sec}$$

Range (Eq. 9): $$R = \frac{v_0^2}{g} \sin 2\alpha = \frac{(500)^2 \sin 120°}{9.8} \approx 22{,}092 \text{ m}$$ ❑

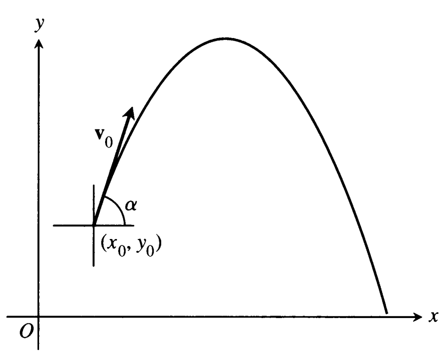
Ideal Trajectories Are Parabolic

It is often claimed that water from a hose traces a parabola in the air, but anyone who looks closely enough will see this is not so. The air slows the water down, and its forward progress is too slow at the end to match the rate at which it falls.

What is really being claimed is that ideal projectiles move along parabolas, and this we can see from Eqs. (6). If we substitute $t = x/(v_0 \cos \alpha)$ from the first equation into the second, we obtain the Cartesian coordinate equation

$$y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right) x^2 + (\tan \alpha)x.$$

This equation has the form $y = ax^2 + bx$, so its graph is a parabola.



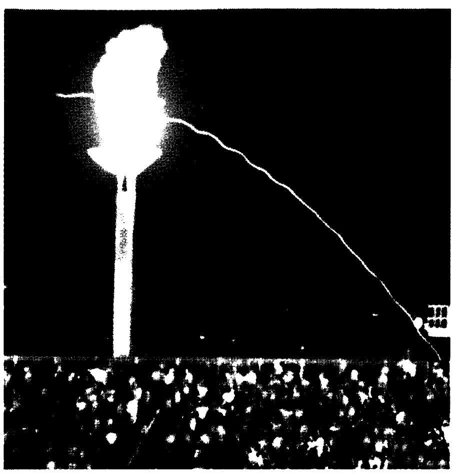
11.8 The path of a projectile fired from (x_0, y_0) with an initial velocity $\mathbf{v}_0$ at an angle of α degrees with the horizontal.

Firing from (x_0, y_0)

If we fire our ideal projectile from the point (x_0, y_0) instead of the origin (Fig. 11.8), the equations that replace Eqs. (6) are

$$x = x_0 + (v_0 \cos \alpha)t, \qquad y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2, \tag{10}$$

as you will be invited to show in Exercise 19.



11.9 Spanish archer Antonio Rebollo lights the Olympic torch in Barcelona with a flaming arrow.

EXAMPLE 3 To open the 1992 Summer Olympics in Barcelona, bronze medalist archer Antonio Rebollo lit the Olympic torch with a flaming arrow (Fig. 11.9).

Suppose that Rebollo wanted the arrow to reach its maximum height exactly 4 ft above the center of the cauldron (Fig. 11.10).

a) If he shot the arrow at a height of 6 ft above ground level 30 yd from the 70-ft-high cauldron, express y_{max} in terms of the initial speed v_0 and firing angle α.

b) If $y_{max} = 74$ ft (Fig. 11.10), use the results of part (a) to find the value of $v_0 \sin \alpha$.

c) When the arrow reaches y_{max}, the horizontal distance traveled to the center of the cauldron is $x = 90$ ft. Use this fact to find the value of $v_0 \cos \alpha$.

d) Find the initial firing angle of the arrow.

Solution

a) We use a coordinate system in which the x-axis lies along the ground toward the left (to match the photograph in Fig. 11.9) and the coordinates of the flaming arrow at $t = 0$ are $x_0 = 0$ and $y_0 = 6$ (Fig. 11.10). We have

$$y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \qquad \text{Eqs. (10)}$$

$$= 6 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2. \qquad y_0 = 6$$

We find the time when the arrow reaches its highest point by setting $dy/dt = 0$ and solving for t, obtaining

$$t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t, the value of y is

$$y_{max} = 6 + (v_0 \sin \alpha)\left(\frac{v_0 \sin \alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin \alpha}{g}\right)^2 = 6 + \frac{(v_0 \sin \alpha)^2}{2g}.$$

b) Using $y_{max} = 74$ and $g = 32$, we see from part (a) that

$$74 = 6 + \frac{(v_0 \sin \alpha)^2}{(2)(32)}$$

or

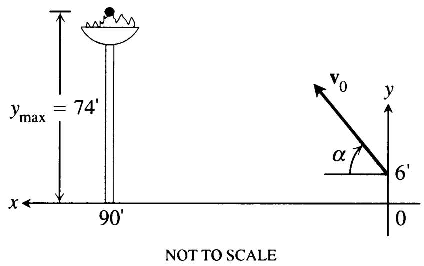
$$v_0 \sin \alpha = \sqrt{(68)(64)}.$$

c) We substitute the time to reach y_{max} from part (a) and the horizontal distance $x = 90$ ft into Eqs. (10) to obtain

$$x = x_0 + (v_0 \cos \alpha)t \qquad \text{Eqs. (10)}$$

$$90 = 0 + (v_0 \cos \alpha)t \qquad x = 90,\quad x_0 = 0$$

$$= (v_0 \cos \alpha)\left(\frac{v_0 \sin \alpha}{g}\right). \qquad t = (v_0 \sin \alpha)/g$$



11.10 Ideal path of the arrow that lit the Olympic torch.

Solving this equation for $v_0 \cos\alpha$ and using the result from part (b), we have

$$v_0 \cos \alpha = \frac{90g}{v_0 \sin \alpha}$$

d) Parts (b) and (c) together tell us that

$$\begin{aligned}\tan \alpha &= \frac{v_0 \sin \alpha}{v_0 \cos \alpha} \\ &= \frac{(v_0 \sin \alpha)^2}{90g} \\ &= \frac{(68)(64)}{(90)(32)} = \frac{68}{45}\end{aligned}$$

or

$$\alpha \approx \tan^{-1}\left(\frac{68}{45}\right) \approx 57^\circ.$$

This is Rebollo's firing angle. ❑

Exercises 11.2

The projectiles in the following exercises are to be treated as ideal projectiles whose behavior is faithfully portrayed by the equations derived in the text. Most of the arithmetic, however, is realistic and is best done with a calculator. All launch angles are assumed to be measured from the horizontal. All projectiles are assumed to be fired from the origin over horizontal ground, unless stated otherwise.

1. A projectile is fired at a speed of 840 m/sec at an angle of 60°. How long will it take to get 21 km downrange?

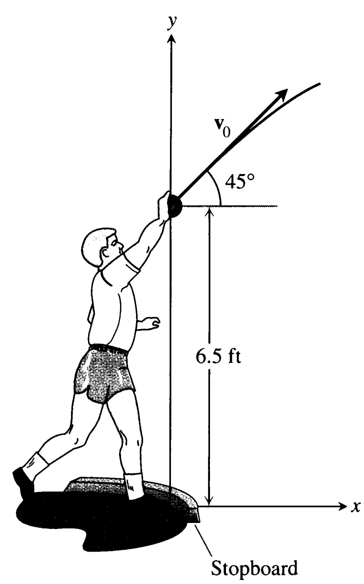
2. Find the muzzle speed of a gun whose maximum range is 24.5 km.

3. A projectile is fired with an initial speed of 500 m/sec at an angle of elevation of 45°.

- **a)** When and how far away will the projectile strike?
- **b)** How high overhead will the projectile be when it is 5 km downrange?
- **c)** What is the highest the projectile will go?

4. A baseball is thrown from the stands 32 ft above the field at an angle of 30° up from horizontal. When and how far away will the ball strike the ground if its initial speed is 32 ft/sec?

5. An athlete throws a 16-lb shot at an angle of 45° to the horizontal from 6.5 ft above the ground at an initial speed of 44 ft/sec. How long after launch and how far from the inner edge of the stopboard does the shot land?



6. Because of its initial evaluation, the shot in Exercise 5 would have gone slightly farther if it had been launched at a 40° angle. How much farther? Answer in inches.

7. A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 m away. What was the ball's initial speed? For the same initial speed, find the two firing angles that make the range 6 m.

8. An electron in a TV tube is beamed horizontally at a speed of 5×10^6 m/sec toward the face of the tube 40 cm away. About how far will the electron drop before it hits?

9. Laboratory tests designed to find how far golf balls of different hardness go when hit with a driver showed that a 100-compression ball hit with a club head speed of 100 mph at a launch angle of 9° carried 248.8 yd. What was the launch speed of the ball? (It was more than 100 mph. At the same time the club head was moving forward, the compressed ball was kicking away from the club face, adding to the ball's forward speed.)

10. A human cannonball is to be fired with an initial speed of $v_0 = 80\sqrt{10}/3$ ft/sec. The circus performer (of the right caliber, naturally) hopes to land on a special cushion located 200 ft downrange. The circus is being held in a large room with a flat ceiling 75 ft high. Can the performer be fired to the cushion without striking the ceiling? If so, what should the cannon's angle of elevation be?

11. A golf ball leaves the ground at a 30° angle at a speed of 90 ft/sec. Will it clear the top of a 30-ft tree 135 ft away?

12. A golf ball is hit from the tee to a green elevated 45 ft above the tee with an initial speed of 116 ft/sec at an angle of elevation of 45°. Assuming that the pin, 369 ft downrange, does not get in the way, where will the ball land in relation to the pin?

13. In Moscow in 1987, Natalya Lisouskaya set a women's world record by putting an 8-lb 13-oz shot 73 ft 10 in. Assuming that she launched the shot at a 40° angle to the horizontal 6.5 ft above the ground, what was the shot's initial speed?

14. A baseball hit by a Boston Red Sox player at a 20° angle from 3 ft above the ground just cleared the left end of the "Green Monster," the left-field wall in Fenway Park (Fig. 11.11). This wall is 37 ft high and 315 ft from home plate. About how fast was the ball going? How long did it take the ball to reach the wall?

15. Show that a projectile fired at an angle of α degrees, $0 < \alpha < 90$, has the same range as a projectile fired at the same speed at an angle of $(90 - \alpha)$ degrees. (In models that take air resistance into account, this symmetry is lost.)

16. What two angles of elevation will enable a projectile to reach a target 16 km downrange on the same level as the gun if the projectile's initial speed is 400 m/sec?

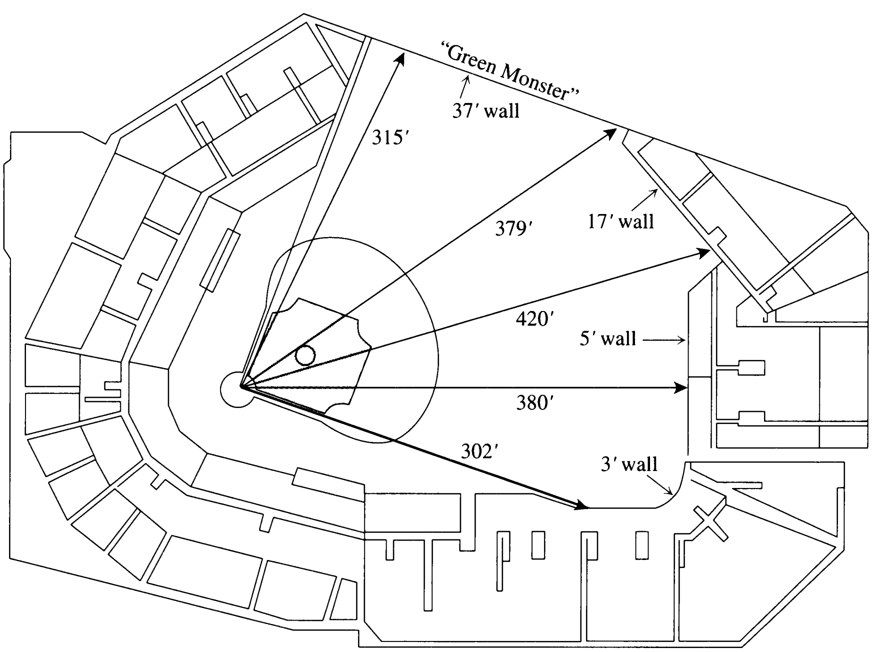
17. Show that doubling a projectile's initial speed at a given launch angle multiplies its range by 4. By about what percentage should you increase the initial speed to double the height and range?

18. Show that a projectile attains three-quarters of its maximum height in half the time it takes to reach the maximum height.

19. Derive the equations

$$x = x_0 + (v_0 \cos \alpha)t$$

$$y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$



11.11 The "Green Monster," the left-field wall at Fenway Park in Boston (Exercise 14).

(Eqs. 10 in the text) by solving the following initial value problem for a vector $\mathbf{r}$ in the plane.

Differential equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j}$

Initial conditions: $\mathbf{r} = x_0\,\mathbf{i} + y_0\,\mathbf{j}$

and

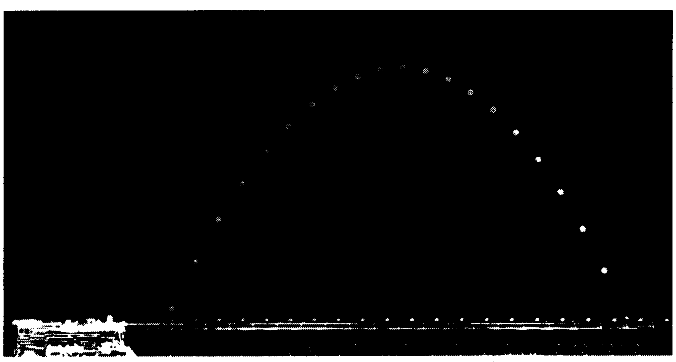
$\dfrac{d\,\mathbf{r}}{dt} = (v_0 \cos \alpha)\,\mathbf{i} + (v_0 \sin \alpha)\,\mathbf{j}$

when $t = 0$

20. Using the firing angle $\alpha = 57°$ in Example 3, find the speed with which the flaming arrow left Rebollo's bow. See Fig. 11.10.

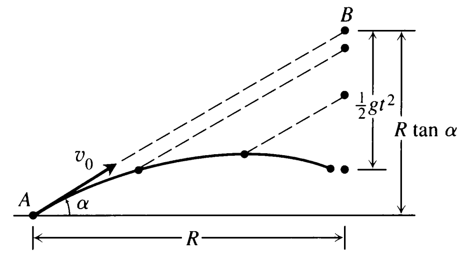
21. The cauldron in Example 3 is 12 ft in diameter. Using Eqs. (10) and Example 3(c), find how long it takes the flaming arrow to cover the horizontal distance to the rim. How high is the arrow at this time?

22. The multiflash photograph shown below shows a model train engine moving at a constant speed on a straight track. As the engine moved along, a marble was fired into the air by a spring in the engine's smokestack. The marble, which continued to move with the same forward speed as the engine, rejoined the engine 1 sec after it was fired. Measure the angle the marble's path made with the horizontal and use the information to find how high the marble went and how fast the engine was moving.



The train in Exercise 22.

23. Figure 11.12 shows an experiment with two marbles. Marble A was launched toward marble B with launch angle α and initial speed v_0. At the same instant, marble B was released to fall from rest $R \tan \alpha$ units directly above a spot R units downrange from A. The marbles were found to collide regardless of the value of v_0. Was this mere coincidence, or must this happen? Give reasons for your answer.

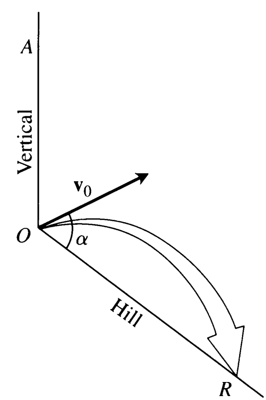


11.12 The marbles in Exercise 23.

24. An ideal projectile is launched straight down an inclined plane, as shown in profile in Fig. 11.13.

a) Show that the greatest downhill range is achieved when the initial velocity vector bisects angle AOR.

b) If the projectile were fired uphill instead of down, what launch angle would maximize its range? Give reasons for your answer.



11.13 Maximum downhill range occurs when the velocity vector bisects angle AOR (Exercise 24).

25. An ideal projectile, launched from the origin into the first octant at time $t = 0$ with initial velocity $\mathbf{v}_0$, experiences a constant downward acceleration $\mathbf{a} = -g\mathbf{k}$ from gravity. Find the projectile's velocity and position as functions of t.

26. *Air resistance proportional to velocity.* If a projectile of mass m launched with initial velocity $\mathbf{v}_0$ encounters an air resistance proportional to its velocity, the total force $m(d^2\mathbf{r}/dt^2)$ on the projectile satisfies the equation

$$m\frac{d^2\,\mathbf{r}}{dt^2} = -mg\,\mathbf{j} - k\frac{d\mathbf{r}}{dt},$$

where k is the proportionality constant. Show that one integration of this equation gives

$$\frac{d\mathbf{r}}{dt} + \frac{k}{m}\,\mathbf{r} = \mathbf{v}_0 - gt\,\mathbf{j}.$$

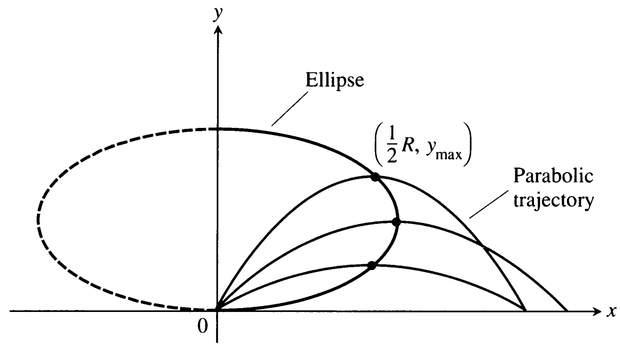
Solve this equation. To do so, multiply both sides of the equation by $e^{(k/m)t}$. The left-hand side will then be the derivative of a product, with the result that both sides of the equation can now be integrated.

The function $e^{(k/m)t}$ is called an *integrating factor* for the differential equation because multiplying the equation by it makes the equation integrable.

27. For a projectile fired from the ground at launch angle α with initial speed v_0, consider α as a variable and v_0 as a fixed constant. For each α, $0 < \alpha \le \pi/2$, we obtain a parabolic trajectory (Fig 11.14). Show that the points in the plane that give the maximum heights of these parabolic trajectories all lie on the ellipse

$$x^2 + 4\left(y - \frac{v_0{}^2}{4g}\right)^2 = \frac{v_0{}^4}{4g^2},$$

where $x \ge 0$.



11.14 The parabolas in Exercise 27.

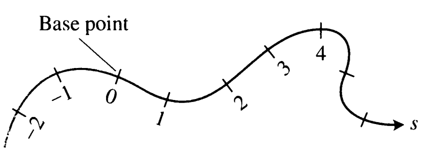
28. GRAPHER If you have access to a parametric equation grapher and have not yet done Exercise 46 in Section 9.4, do it now. It is about ideal projectile motion.

11.3 Arc Length and the Unit Tangent Vector T

As you can imagine, differentiable curves, especially those with continuous first and second derivatives, have been subjects of intense study, for their mathematical interest as well as their applications to motion in space. In this section and the next, we study some of the features that account for the importance of these curves.

Arc Length Along a Curve

One of the special features of smooth space curves is that they have a measurable length. This enables us to locate points along these curves by giving their directed distance s along the curve from some **base point,** the way we locate points on coordinate axes by giving their directed distance from the origin (Fig. 11.15). Time is the natural parameter for describing a moving body's velocity and acceleration, but s is the natural parameter for studying a curve's shape. Both parameters appear in analyses of space flight.



11.15 Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance from a preselected base point.

To measure distance along a smooth curve in space, we add a z-term to the formula we use for curves in the plane.

Definition

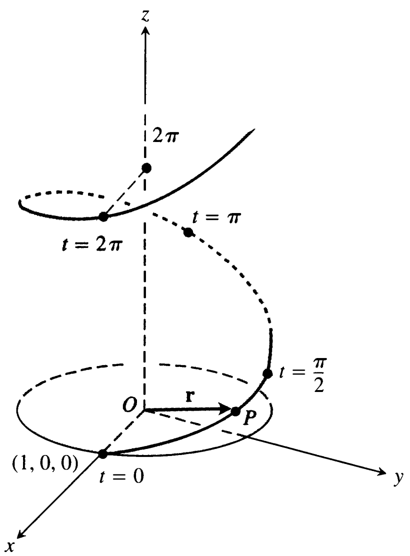
The **length** of a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (1)$$

Just as for plane curves, we can calculate the length of a curve in space from any convenient parametrization that meets the stated conditions. Again, we omit the proof.

The square root in Eq. (1) is $|\mathbf{v}|$, the length of the velocity vector $d\mathbf{r}/dt$. This enables us to write the formula for length a shorter way.



11.16 The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ in Example 1.

Length Formula (Short Form)

$$L = \int_a^b |\mathbf{v}|\, dt \tag{2}$$

EXAMPLE 1 Find the length of one turn of the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

Solution The helix makes one full turn as t runs from 0 to 2π (Fig. 11.16). The length of this portion of the curve is

$$\begin{aligned} L &= \int_a^b |\mathbf{v}|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}. \end{aligned}$$

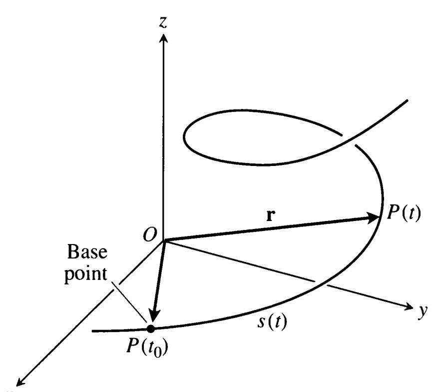
This is $\sqrt{2}$ times the length of the circle in the xy-plane over which the helix stands. ❑

We use the Greek letter τ ("tau") as the variable of integration in Eq. (3) because the letter t is already in use as the upper limit.

If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance"

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau, \tag{3}$$

measured along C from the base point (Fig. 11.17). If $t > t_0$, $s(t)$ is the distance from $P(t_0)$ to $P(t)$. If $t < t_0$, $s(t)$ is the negative of the distance. Each value of s determines a point on C and this parametrizes C with respect to s. We call s an **arc length parameter** for the curve. The parameter's value increases in the direction of increasing t.



11.17 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is

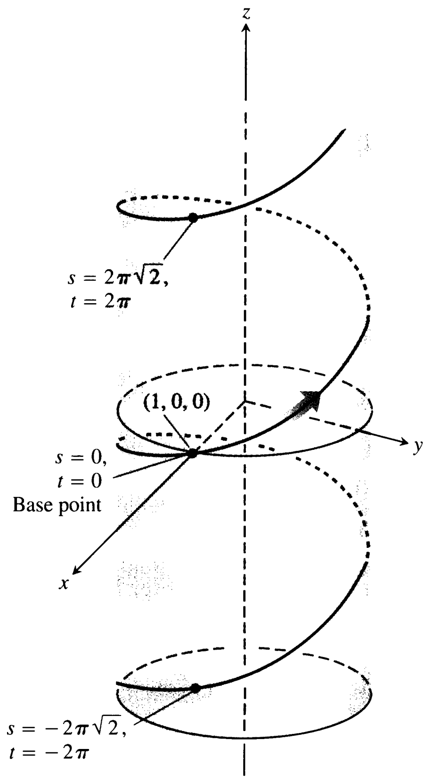
$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau.$$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \tag{4}$$

EXAMPLE 2 If $t_0 = 0$, the arc length parameter along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$



11.18 Arc length parameter values on the helix $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$ (Example 2).

from t_0 to t is

$$
\begin{aligned}
s(t) &= \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau && \text{Eq. (4)} \\
&= \int_{0}^{t} \sqrt{2}\, d\tau && \text{Value from Example 1} \\
&= \sqrt{2}\, t.
\end{aligned}
$$

Thus, $s(2\pi) = 2\pi\sqrt{2}$, $s(-2\pi) = -2\pi\sqrt{2}$, and so on (Fig. 11.18). ❑

EXAMPLE 3 *Distance Along a Line.*

Show that if $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ is a unit vector, then the directed distance along the line

$$\mathbf{r}(t) = (x_0 + tu_1)\,\mathbf{i} + (y_0 + tu_2)\,\mathbf{j} + (z_0 + tu_3)\,\mathbf{k}$$

from the point $P_0(x_0, y_0, z_0)$ where $t = 0$ is t itself.

Solution

$$\mathbf{v} = \frac{d}{dt}(x_0 + tu_1)\,\mathbf{i} + \frac{d}{dt}(y_0 + tu_2)\,\mathbf{j} + \frac{d}{dt}(z_0 + tu_3)\,\mathbf{k} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k} = \mathbf{u},$$

so

$$s(t) = \int_0^t |\mathbf{v}|\, d\tau = \int_0^t |\mathbf{u}|\, d\tau = \int_0^t 1\, d\tau = t.$$ ❑

Speed on a Smooth Curve

Since the derivatives beneath the radical in Eq. (4) are continuous (the curve is smooth), the Fundamental Theorem of Calculus tells us that s is a differentiable function of t with derivative

$$\frac{ds}{dt} = |\mathbf{v}(t)|. \tag{5}$$

As we expect, the speed with which the particle moves along its path is the magnitude of $\mathbf{v}$.

Notice that while the base point $P(t_0)$ plays a role in defining s in Eq. (4), it plays no role in Eq. (5). The rate at which a moving particle covers distance along its path has nothing to do with how far away the base point is.

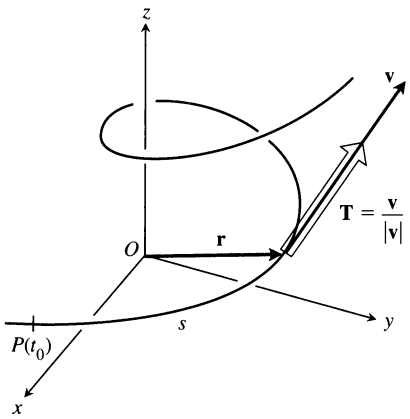
Notice also that $ds/dt > 0$ since, by definition, $|\mathbf{v}|$ is never zero for a smooth curve. We see once again that s is an increasing function of t.

The Unit Tangent Vector T

Since $ds/dt > 0$ for the curves we are considering, s is one-to-one and has an inverse that gives t as a differentiable function of s (Section 6.1). The derivative of the inverse is

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}|}. \tag{6}$$

This makes $\mathbf{r}$ a differentiable function of s whose derivative can be calculated with



11.19 We find the unit tangent vector **T** by dividing **v** by |**v**|.

the Chain Rule to be

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}\frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{7}$$

Equation (7) says that $d\mathbf{r}/ds$ is a unit vector in the direction of **v**. We call $d\mathbf{r}/ds$ the unit tangent vector of the curve traced by **r** and denote it by **T** (Fig. 11.19).

Definition

The **unit tangent vector** of a differentiable curve $\mathbf{r}(t)$ is

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{8}$$

The unit tangent vector **T** is a differentiable function of t whenever **v** is a differentiable function of t. As we will see in the next section, **T** is one of three unit vectors in a traveling reference frame that is used to describe the motion of space vehicles and other bodies moving in three dimensions.

EXAMPLE 4 Find the unit tangent vector of the helix

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}.$$

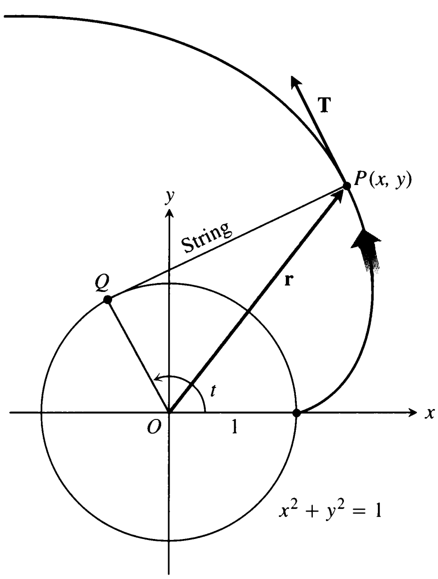
Solution

$$\mathbf{v} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}$$

$$|\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{\sin t}{\sqrt{2}}\,\mathbf{i} + \frac{\cos t}{\sqrt{2}}\,\mathbf{j} + \frac{1}{\sqrt{2}}\,\mathbf{k}$$

❑



11.20 The *involute* of a circle is the path traced by the endpoint *P* of a string unwinding from a circle, here the unit circle in the *xy*-plane.

EXAMPLE 5 *The involute of a circle (Fig. 11.20)*

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}, \qquad t > 0.$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\,\mathbf{i} + (\cos t - \cos t + t\sin t)\,\mathbf{j}$$

$$= (t\cos t)\,\mathbf{i} + (t\sin t)\,\mathbf{j}$$

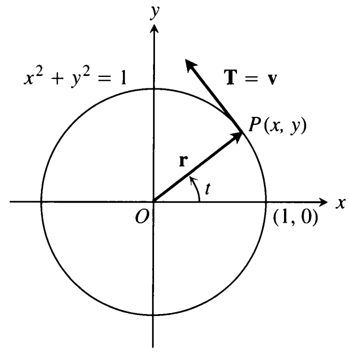
$$|\mathbf{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t \qquad |t| = t \text{ because } t > 0$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{t} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$$

❑

EXAMPLE 6 For the counterclockwise motion

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$$



11.21 The motion $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ (Example 6).

around the unit circle,

$$\mathbf{v} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}$$

is already a unit vector, so $\mathbf{T} = \mathbf{v}$ (Fig. 11.21). ❑

Exercises 11.3

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + \sqrt{5}\,t\,\mathbf{k}, \quad 0 \le t \le \pi$

2. $\mathbf{r}(t) = (6\sin 2t)\,\mathbf{i} + (6\cos 2t)\,\mathbf{j} + 5t\,\mathbf{k}, \quad 0 \le t \le \pi$

3. $\mathbf{r}(t) = t\,\mathbf{i} + (2/3)t^{3/2}\,\mathbf{k}, \quad 0 \le t \le 8$

4. $\mathbf{r}(t) = (2+t)\,\mathbf{i} - (t+1)\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 3$

5. $\mathbf{r}(t) = (\cos^3 t)\,\mathbf{j} + (\sin^3 t)\,\mathbf{k}, \quad 0 \le t \le \pi/2$

6. $\mathbf{r}(t) = 6t^3\,\mathbf{i} - 2t^3\,\mathbf{j} - 3t^3\,\mathbf{k}, \quad 1 \le t \le 2$

7. $\mathbf{r}(t) = (t\cos t)\,\mathbf{i} + (t\sin t)\,\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\,\mathbf{k}, \quad 0 \le t \le \pi$

8. $\mathbf{r}(t) = (t\sin t + \cos t)\,\mathbf{i} + (t\cos t - \sin t)\,\mathbf{j}, \quad \sqrt{2} \le t \le 2$

9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\,\mathbf{i} + (5\cos t)\,\mathbf{j} + 12t\,\mathbf{k}$$

at a distance 26π units along the curve from the origin in the direction of increasing arc length.

10. Find the point on the curve

$$\mathbf{r}(t) = (12\sin t)\,\mathbf{i} - (12\cos t)\,\mathbf{j} + 5t\,\mathbf{k}$$

at a distance 13π units along the curve from the origin in the direction opposite to the direction of increasing arc length.

In Exercises 11–14, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\,d\tau$$

from Eq. (3). Then find the length of the indicated portion of the curve.

11. $\mathbf{r}(t) = (4\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j} + 3t\,\mathbf{k}, \quad 0 \le t \le \pi/2$

12. $\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}, \quad \pi/2 \le t \le \pi$

13. $\mathbf{r}(t) = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + e^t\,\mathbf{k}, \quad -\ln 4 \le t \le 0$

14. $\mathbf{r}(t) = (1+2t)\,\mathbf{i} + (1+3t)\,\mathbf{j} + (6-6t)\,\mathbf{k}, \quad -1 \le t \le 0$

15. Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}\,t)\,\mathbf{i} + (\sqrt{2}\,t)\,\mathbf{j} + (1-t^2)\,\mathbf{k}$$

from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

16. The length $2\pi\sqrt{2}$ of the turn of the helix in Example 1 is also the length of the diagonal of a square 2π units on a side. Show how to obtain this square by cutting away and flattening a portion of the cylinder around which the helix winds.

17. **a)** Show that the curve $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + (1 - \cos t)\,\mathbf{k}$, $0 \le t \le 2\pi$, is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.

b) Sketch the ellipse on the cylinder. Add to your sketch the unit tangent vectors at $t = 0, \pi/2, \pi$, and $3\pi/2$.

c) Show that the acceleration vector always lies parallel to the plane (orthogonal to a vector normal to the plane). Thus, if you draw the acceleration as a vector attached to the ellipse, it will lie in the plane of the ellipse. Add the acceleration vectors for $t = 0, \pi/2, \pi$, and $3\pi/2$ to your sketch.

d) Write an integral for the length of the ellipse. Do not try to evaluate the integral—it is nonelementary.

e) NUMERICAL INTEGRATOR Estimate the length of the ellipse to two decimal places.

18. *Length is independent of parametrization.* To illustrate the fact that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.

a) $\mathbf{r}(t) = (\cos 4t)\,\mathbf{i} + (\sin 4t)\,\mathbf{j} + 4t\,\mathbf{k}, \quad 0 \le t \le \pi/2$

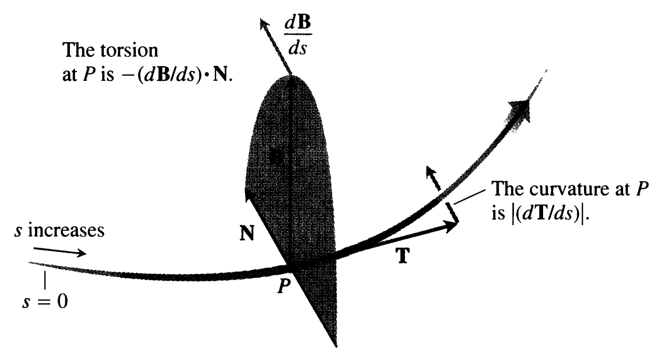
b) $\mathbf{r}(t) = [\cos(t/2)]\,\mathbf{i} + [\sin(t/2)]\,\mathbf{j} + (t/2)\,\mathbf{k}, \quad 0 \le t \le 4\pi$

c) $\mathbf{r}(t) = (\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} - t\,\mathbf{k}, \quad -2\pi \le t \le 0$

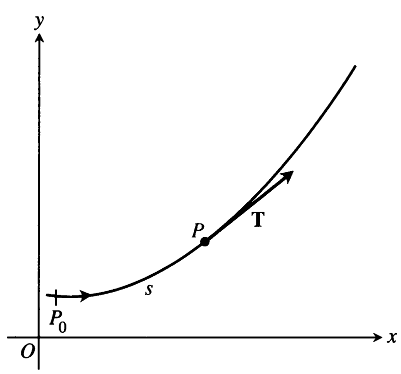
11.4 Curvature, Torsion, and the TNB Frame

In this section we define a frame of mutually orthogonal unit vectors that always travels with a body moving along a curve in space (Fig. 11.22). The frame has three vectors. The first is $\mathbf{T}$, the unit tangent vector. The second is $\mathbf{N}$, the unit vector that gives the direction of $d\mathbf{T}/ds$. The third is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. These vectors and their derivatives, when available, give useful information about a vehicle's orientation in space and about how the vehicle's path turns and twists.

For example, $|d\mathbf{T}/ds|$ tells how much a vehicle's path turns to the left or right as it moves along; it is called the *curvature* of the vehicle's path. The number $-(d\mathbf{B}/ds) \cdot \mathbf{N}$ tells how much a vehicle's path rotates or twists out of its plane of motion as the vehicle moves along; it is called the *torsion* of the vehicle's path. Look at Fig. 11.22 again. If P is a train climbing up a curved track, the rate at which the headlight turns from side to side per unit distance is the curvature of the track. The rate at which the engine tends to twist out of the plane formed by $\mathbf{T}$ and $\mathbf{N}$ is the torsion.



11.22 Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.



11.23 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

The Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\mathbf{T} = d\mathbf{r}/ds$ turns as the curve bends. Since $\mathbf{T}$ is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the *curvature* (Fig. 11.23). The traditional symbol for the curvature function is the Greek letter κ ("kappa").

Definition

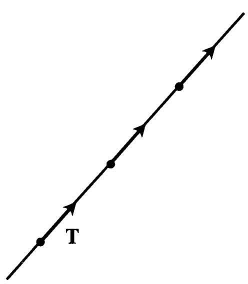
If $\mathbf{T}$ is the unit tangent vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller. Testing the definition, we see in Examples 1 and 2 that the curvature is constant for straight lines and circles.

EXAMPLE 1 *The curvature of a straight line is zero*

On a straight line, the unit tangent vector $\mathbf{T}$ always points in the same direction, so its components are constants. Therefore $|d\mathbf{T}/ds| = |\mathbf{0}| = 0$ (Fig. 11.24). ❑



11.24 Along a straight line, **T** always points in the same direction. The curvature, $|d\mathbf{T}/ds|$, is zero (Example 1).

EXAMPLE 2 *The curvature of a circle of radius a is 1/a*

To see why (Fig. 11.25), start with the parametrization

$$\mathbf{r}(\theta) = (a\cos\theta)\,\mathbf{i} + (a\sin\theta)\,\mathbf{j}$$

and substitute $\theta = s/a$ to parametrize in terms of arc length s.

$$\mathbf{r} = \left(a\cos\frac{s}{a}\right)\mathbf{i} + \left(a\sin\frac{s}{a}\right)\mathbf{j}.$$

Then

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \left(-\sin\frac{s}{a}\right)\mathbf{i} + \left(\cos\frac{s}{a}\right)\mathbf{j}$$

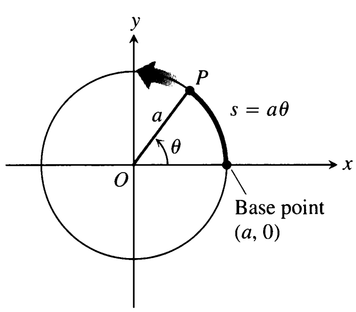
and

$$\frac{d\mathbf{T}}{ds} = \left(-\frac{1}{a}\cos\frac{s}{a}\right)\mathbf{i} - \left(\frac{1}{a}\sin\frac{s}{a}\right)\mathbf{j}.$$

Hence, for any value of s,

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\ &= \sqrt{\frac{1}{a^2}\cos^2\left(\frac{s}{a}\right) + \frac{1}{a^2}\sin^2\left(\frac{s}{a}\right)} \\ &= \frac{1}{\sqrt{a^2}} = \frac{1}{|a|} = \frac{1}{a}. \qquad \text{Since } a > 0,\ |a| = a.\end{aligned}$$

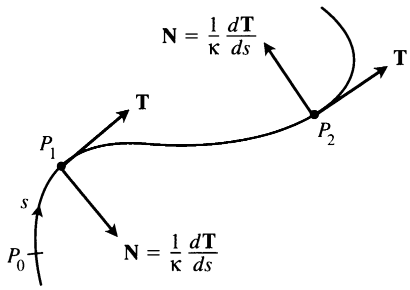
❑



11.25 The point P has coordinates $(a\cos\theta, a\sin\theta) = (a\cos(s/a),\ a\sin(s/a))$ (Example 2).

The Principal Unit Normal Vector for Plane Curves

Since $\mathbf{T}$ has constant length, the vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$ (Section 11.1). Therefore, if we divide $d\mathbf{T}/ds$ by the length κ, we obtain a *unit* vector orthogonal to $\mathbf{T}$ (Fig. 11.26).



11.26 The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which **T** is turning. The vector **N** is the direction of $d\mathbf{T}/ds$.

Definition

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

The vector $d\mathbf{T}/ds$ points in the direction in which $\mathbf{T}$ turns as the curve bends. Therefore, if we face in the direction of increasing arc length, the vector $d\mathbf{T}/ds$ points toward the right if $\mathbf{T}$ turns clockwise and toward the left if $\mathbf{T}$ turns counterclockwise. In other words, the principal normal vector $\mathbf{N}$ will point toward the concave side of the curve (Fig. 11.26). Exercise 10 illustrates what happens when $\kappa = 0$ at a point.

Because the arc length parameter for a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is defined with ds/dt positive, $ds/dt = |ds/dt|$, and the Chain Rule gives

$$\begin{aligned} \mathbf{N} &= \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} \\ &= \frac{(d\mathbf{T}/dt)(dt/ds)}{|d\mathbf{T}/dt||dt/ds|} \\ &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \end{aligned} \tag{1}$$

This formula enables us to find $\mathbf{N}$ without having to find κ and s first.

EXAMPLE 3 Find $\mathbf{T}$ and $\mathbf{N}$ for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

Solution We first find **T**:

$$\begin{aligned} \mathbf{v} &= -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}, \\ |\mathbf{v}| &= \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2, \\ \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}. \end{aligned}$$

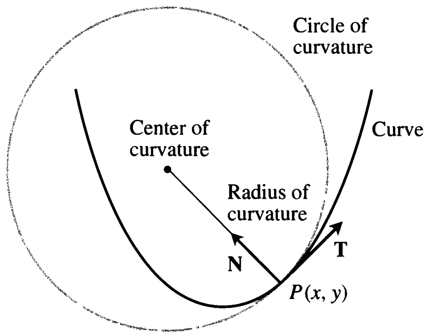
From this we find

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}, \\ \left|\frac{d\mathbf{T}}{dt}\right| &= \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2, \end{aligned}$$

and

$$\begin{aligned} \mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \\ &= -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}. \qquad \text{Eq. (1)} \end{aligned}$$

❑



11.27 The osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Circle of Curvature and Radius of Curvature

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has);
2. has the same curvature the curve has at P; and
3. lies toward the concave or inner side of the curve (as in Fig. 11.27).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which, according to Example 2, is

$$\text{Radius of curvature } = \rho = \frac{1}{\kappa}. \tag{2}$$

To find ρ, we find κ and take the reciprocal. The **center of curvature** of the curve at P is the center of the circle of curvature.

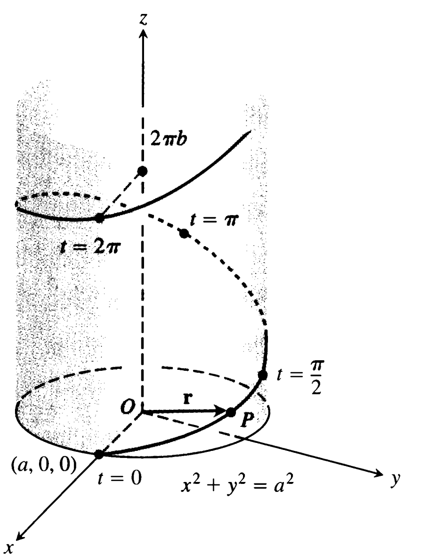
Curvature and Normal Vectors for Space Curves

Just as it does for a curve in the plane, the arc length parameter s gives the unit tangent vector $\mathbf{T} = d\mathbf{r}/ds$ for a smooth curve in space. We again define the curvature to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|. \tag{3}$$

The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$ and we define the principal unit normal to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \tag{4}$$



11.28 The helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}$, drawn with a and b positive and $t \geq 0$ (Example 4).

EXAMPLE 4 Find the curvature for the helix (Fig. 11.28)

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\,\mathbf{k}, \qquad a, b \geq 0,\ a^2 + b^2 \neq 0.$$

Solution We calculate $\mathbf{T}$ from the velocity vector $\mathbf{v}$:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\,\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\,\mathbf{k}].$$

Then, using the Chain Rule, we find $d\mathbf{T}/ds$ as

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} \qquad \text{Chain Rule}$$

$$= \frac{d\mathbf{T}}{dt}\cdot\frac{1}{|\mathbf{v}|} \qquad \frac{ds}{dt} = |\mathbf{v}|, \text{ so } \frac{dt}{ds} = \frac{1}{|\mathbf{v}|}$$

$$= \frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}]\cdot\left(\frac{1}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{a}{a^2+b^2}[-(\cos t)\mathbf{i} - (\sin t)\mathbf{j}].$$

Therefore,

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\ &= \frac{a}{a^2+b^2}\left|-(\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j}\right| \\ &= \frac{a}{a^2+b^2}\sqrt{(\cos t)^2 + (\sin t)^2} = \frac{a}{a^2+b^2}. \qquad (5)\end{aligned}$$

From Eq. (5) we see that increasing b for a fixed a decreases the curvature. Decreasing a for a fixed b eventually decreases the curvature as well. Stretching a spring tends to straighten it.

If $b = 0$, the helix reduces to a circle of radius a and its curvature reduces to $1/a$, as it should. If $a = 0$, the helix becomes the z-axis, and its curvature reduces to 0, again as it should. ❑

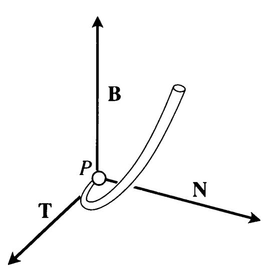
EXAMPLE 5 Find **N** for the helix in Example 4.

Solution We have

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= -\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j}] && \text{Example 4} \\ \left|\frac{d\mathbf{T}}{dt}\right| &= \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}} \\ \mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} && \text{Eq. (4)} \\ &= -\frac{\sqrt{a^2+b^2}}{a}\cdot\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j}] \\ &= -(\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j}.\end{aligned}$$

❑

Torsion and the Binormal Vector



11.29 The vectors **T, N,** and **B** (in that order) make a right-handed frame of mutually orthogonal unit vectors in space. You can call it the **Frenet** ("fre-*nay*") **frame** (after Jean-Frédéric Frenet, 1816–1900), or you can call it the **TNB frame.**

The **binormal vector** of a curve in space is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, a unit vector orthogonal to both **T** and **N** (Fig. 11.29). Together **T, N,** and **B** define a moving right-handed vector frame that plays a significant role in calculating the flight paths of space vehicles.

How does $d\mathbf{B}/ds$ behave in relation to **T, N,** and **B?** From the rule for differentiating a cross product, we have

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$$

Since **N** is the direction of $d\mathbf{T}/ds$, $(d\mathbf{T}/ds)\times\mathbf{N} = \mathbf{0}$ and

$$\frac{d\mathbf{B}}{ds} = \mathbf{0} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}. \qquad (6)$$

From this we see that $d\mathbf{B}/ds$ is orthogonal to **T** since a cross product is orthogonal to its factors.

Since $d\mathbf{B}/ds$ is also orthogonal to **B** (the latter has constant length), it follows that $d\mathbf{B}/ds$ is orthogonal to the plane of **B** and **T.** In other words, $d\mathbf{B}/ds$ is parallel to **N,** so $d\mathbf{B}/ds$ is a scalar multiple of **N.** In symbols,

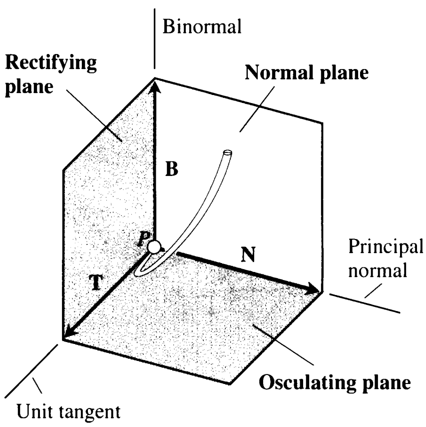
$$\frac{d\mathbf{B}}{ds} = -\tau\,\mathbf{N}.$$

The minus sign in this equation is traditional. The scalar τ is called the torsion along the curve. Notice that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\,\mathbf{N} \cdot \mathbf{N} = -\tau\,(1) = -\tau,$$

so that

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$



11.30 The names of the three planes determined by **T, N,** and **B.**

Definition

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

Unlike the curvature κ, which is never negative, the torsion τ may be positive, negative, or zero.

The three planes determined by **T, N,** and **B** are shown in Fig. 11.30. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along the curve. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about **T** as P moves along the curve. Torsion measures how the curve twists.

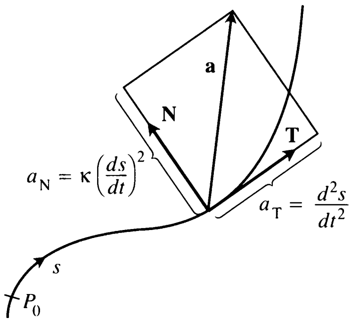
The Tangential and Normal Components of Acceleration

When a body is accelerated by gravity, brakes, a combination of rocket motors, or whatever, we usually want to know how much of the acceleration acts to move the body straight ahead in the direction of motion, in the tangential direction **T.** We can find out if we use the Chain Rule to rewrite **v** as

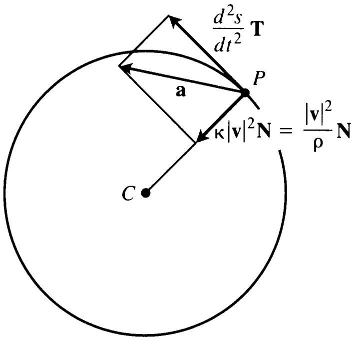
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

and differentiate both ends of this string of equalities to get

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\,\mathbf{N}\frac{ds}{dt}\right)\\ &= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.\end{aligned}$$



11.31 The tangential and normal components of acceleration. The acceleration **a** always lies in the plane of **T** and **N,** orthogonal to **B.**



11.32 The tangential and normal components of the acceleration of a body that is speeding up as it moves counterclockwise around a circle of radius ρ.

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \tag{7}$$

where

$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \qquad \text{and} \qquad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\,|\mathbf{v}|^2 \tag{8}$$

are the **tangential** and **normal** scalar components of acceleration.

Equation (7) is remarkable in that **B** does not appear. No matter how the path of the moving body we are watching may appear to twist and turn in space, the acceleration **a** *always lies in the plane of* **T** and **N** orthogonal to **B.** The equation also tells us exactly how much of the acceleration takes place tangent to the motion (d^2s/dt^2) and how much takes place normal to the motion [$\kappa(ds/dt)^2$] (Fig. 11.31).

What information can we read from Eqs. (8)? By definition, acceleration **a** is the rate of change of velocity **v,** and in general both the length and direction of **v** change as a body moves along its path. The tangential component of acceleration a_{T} measures the rate of change of the *length* of **v** (that is, the change in the speed). The normal component of acceleration a_{N} measures the rate of change of the *direction* of **v.**

Notice that the normal scalar component of the acceleration is the curvature times the *square* of the speed. This explains why you have to hold on when your car makes a sharp (large κ), high-speed (large $|\mathbf{v}|$) turn. If you double the speed of your car, you will experience four times the normal component of acceleration for the same curvature.

If a body moves in a circle at a constant speed, d^2s/dt^2 is zero and all the acceleration points along **N** toward the circle's center. If the body is speeding up or slowing down, **a** has a nonzero tangential component (Fig. 11.32).

To calculate a_{N} we usually use the formula $a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2}$, which comes from solving the equation $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a} = a_{\mathrm{T}}{}^2 + a_{\mathrm{N}}{}^2$ for a_{N}. With this formula we can find a_{N} without having to calculate κ first.

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2} \tag{9}$$

EXAMPLE 6 Without finding **T** and **N,** write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}, \qquad t > 0$$

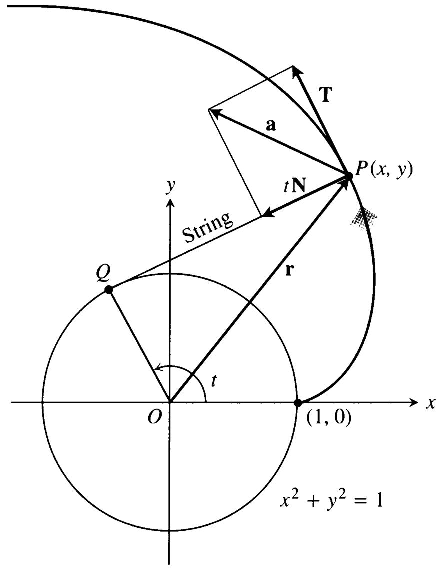
in the form $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$. (The path of the motion is the involute of the circle in Fig. 11.33, on the following page.)

Solution We use the first of Eqs. (8) to find a_{T}:

$$\mathbf{v} = (t\cos t)\,\mathbf{i} + (t\sin t)\,\mathbf{j} \qquad \text{Value from Section 11.3, Example 5}$$

$$|\mathbf{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t \qquad t > 0$$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1. \qquad \text{Eq. (8)}$$



11.33 The tangential and normal components of the acceleration of the motion $\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}$, for $t > 0$ (Example 6).

Knowing a_{T}, we use Eq. (9) to find a_{N}:

$$\begin{aligned}
\mathbf{a} &= (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} \\
|\mathbf{a}|^2 &= t^2 + 1 \qquad \text{After some algebra} \\
a_{\mathrm{N}} &= \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2} \\
&= \sqrt{(t^2+1) - (1)} = \sqrt{t^2} = t.
\end{aligned}$$

We then use Eq. (7) to find **a**:

$$\mathbf{a} = a_{\mathrm{T}}\,\mathbf{T} + a_{\mathrm{N}}\,\mathbf{N} = (1)\,\mathbf{T} + (t)\,\mathbf{N} = \mathbf{T} + t\,\mathbf{N}.$$

See Fig. 11.33. ❑

Formulas for Computing Curvature and Torsion

We now give some easy-to-use formulas for computing the curvature and torsion of a smooth curve. From Eq. (7), we have

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{N}\right] \qquad \text{From Section 11.3, Eq. (8), } \mathbf{v} = d\mathbf{r}/dt = (ds/dt)\,\mathbf{T} \\
&= \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3 (\mathbf{T} \times \mathbf{N}) \\
&= \kappa\left(\frac{ds}{dt}\right)^3 \mathbf{B}. \qquad \mathbf{T} \times \mathbf{T} = \mathbf{0} \text{ and } \mathbf{T} \times \mathbf{N} = \mathbf{B}
\end{aligned}$$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left|\frac{ds}{dt}\right|^3 |\mathbf{B}| = \kappa\,|\mathbf{v}|^3. \qquad \frac{ds}{dt} = |\mathbf{v}| \text{ and } |\mathbf{B}| = 1$$

Solving for κ gives the following formula.

A Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \tag{10}$$

Equation (10) calculates the curvature, a geometric property of the curve, from the velocity and acceleration of any vector representation of the curve in which $|\mathbf{v}|$ is different from zero. Take a moment to think about how remarkable this really is: From any formula for motion along a curve, no matter how variable the motion may be (as long as $\mathbf{v}$ is never zero), we can calculate a physical property of the curve that seems to have nothing to do with the way the curve is traversed.

The most widely used formula for torsion, derived in more advanced texts, is

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}). \tag{11}$$

Newton's dot notation for derivatives

The dots in Eq. (11) denote differentiation with respect to t, one derivative for each dot. Thus, $\dot{x}$ ("x dot") means dx/dt, $\ddot{x}$ ("x double dot") means d^2x/dt^2, and $\dddot{x}$ ("x triple dot") means d^3x/dt^3. Similarly, $\dot{y} = dy/dt$ and so on.

This formula calculates the torsion directly from the derivatives of the component functions $x = f(t), y = g(t), z = h(t)$ that make up $\mathbf{r}$. The determinant's first row comes from $\mathbf{v}$, the second row comes from $\mathbf{a}$, and the third row comes from $\dot{\mathbf{a}} = d\mathbf{a}/dt$.

EXAMPLE 7 Use Eqs. (10) and (11) to find κ and τ for the helix

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j} + bt\,\mathbf{k}, \qquad a, b \geq 0,\ a^2 + b^2 \neq 0.$$

Solution We calculate the curvature with Eq. (10):

$$\mathbf{v} = -(a\sin t)\,\mathbf{i} + (a\cos t)\,\mathbf{j} + b\,\mathbf{k},$$

$$\mathbf{a} = -(a\cos t)\,\mathbf{i} - (a\sin t)\,\mathbf{j},$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (ab\sin t)\,\mathbf{i} - (ab\cos t)\,\mathbf{j} + a^2\,\mathbf{k},$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2+b^2)^{3/2}} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}. \tag{12}$$

Notice that Eq. (12) agrees with Eq. (5) in Example 4, where we calculated the curvature directly from its definition.

To evaluate Eq. (11) for the torsion, we find the entries in the determinant by differentiating $\mathbf{r}$ with respect to t. We already have $\mathbf{v}$ and $\mathbf{a}$, and

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a\sin t)\,\mathbf{i} - (a\cos t)\,\mathbf{j}.$$

Hence,

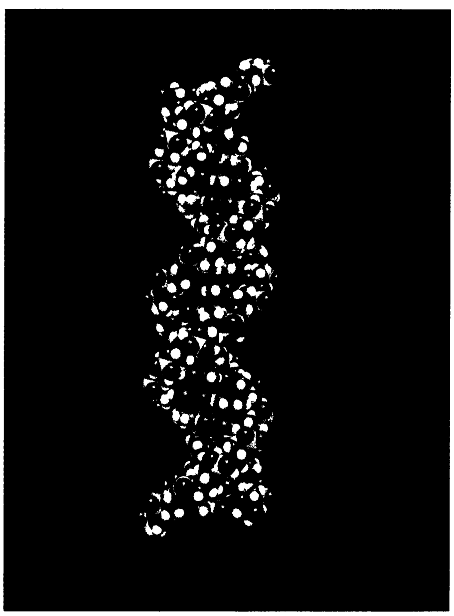
$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{(a\sqrt{a^2+b^2}\,)^2}$$

Value of $|\mathbf{v}\times\mathbf{a}|$ from Eq. (12)

$$= \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2+b^2)}$$

$$= \frac{b}{a^2+b^2}. \tag{13}$$

❏



11.34 The helical shape of a DNA molecule is characterized by its constant curvature and torsion.

From Eq. (13), we see that the torsion of a helix about a circular cylinder is constant. In fact, constant curvature and constant torsion characterize the helix among all curves in space.

The DNA molecule, the basic building block of life forms, is designed in the form of two helices winding around each other, a little like the rungs and sides of a twisted rope ladder (Fig. 11.34). Not only is the space occupied by the DNA molecule very much smaller than it would be if it were unraveled, but when the molecule is damaged the imperfect piece can be snipped out by a kind of molecular scissors (because the curvature and torsion functions are constant) and the DNA made right again.

Formulas for Curves in Space

Unit tangent vector:	$\mathbf{T} = \dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$
Principal unit normal vector:	$\mathbf{N} = \dfrac{d\mathbf{T}/dt}{\lvert d\mathbf{T}/dt\rvert}$
Binormal vector:	$\mathbf{B} = \mathbf{T} \times \mathbf{N}$
Curvature:	$\kappa = \left\lvert \dfrac{d\mathbf{T}}{ds} \right\rvert = \dfrac{\lvert\mathbf{v} \times \mathbf{a}\rvert}{\lvert\mathbf{v}\rvert^3}$
Torsion:	$\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\lvert\mathbf{v} \times \mathbf{a}\rvert^2}$
Tangential and normal scalar components of acceleration:	$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$ $a_{\mathrm{T}} = \dfrac{d}{dt}\lvert\mathbf{v}\rvert$ $a_{\mathrm{N}} = \kappa\lvert\mathbf{v}\rvert^2 = \sqrt{\lvert\mathbf{a}\rvert^2 - a_{\mathrm{T}}{}^2}$

Exercises 11.4

Plane Curves

Find **T**, **N**, and κ for the plane curves in Exercises 1–4.

1. $\mathbf{r}(t) = t\,\mathbf{i} + (\ln \cos t)\,\mathbf{j}, \quad -\pi/2 < t < \pi/2$

2. $\mathbf{r}(t) = (\ln \sec t)\,\mathbf{i} + t\,\mathbf{j}, \quad -\pi/2 < t < \pi/2$

3. $\mathbf{r}(t) = (2t + 3)\,\mathbf{i} + (5 - t^2)\,\mathbf{j}$

4. $\mathbf{r}(t) = (\cos t + t \sin t)\,\mathbf{i} + (\sin t - t \cos t)\,\mathbf{j}, \quad t > 0$

In Exercises 5 and 6, write **a** in the form $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$ without finding **T** and **N**.

5. $\mathbf{r}(t) = (2t + 3)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j}$

6. $\mathbf{r}(t) = \ln(t^2 + 1)\,\mathbf{i} + (t - 2\tan^{-1} t)\,\mathbf{j}$

7. *A formula for the curvature of the graph of a function in the xy-plane*

a) The graph $y = f(x)$ in the xy-plane automatically has the parametrization $x = x$, $y = f(x)$, and the vector formula $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$. Use this formula to show that if f is a twice-differentiable function of x, then

$$\kappa(x) = \frac{\lvert f''(x)\rvert}{\left[1 + (f'(x))^2\right]^{3/2}}.$$

b) Use the formula for κ in (a) to find the curvature of $y = \ln(\cos x)$, $-\pi/2 < x < \pi/2$. Compare your answer with the answer in Exercise 1.

c) Show that the curvature is zero at a point of inflection.

8. *A formula for the curvature of a parametrized plane curve*

a) Show that the curvature of a smooth curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula

$$\kappa = \frac{\lvert \dot{x}\ddot{y} - \dot{y}\ddot{x}\rvert}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Apply the formula to find the curvatures of the following curves.

b) $\mathbf{r}(t) = t\,\mathbf{i} + (\ln \sin t)\,\mathbf{j}, \quad 0 < t < \pi$

c) $\mathbf{r}(t) = [\tan^{-1}(\sinh t)]\,\mathbf{i} + (\ln \cosh t)\,\mathbf{j}.$

9. *Normals to plane curves*

a) Show that $\mathbf{n}(t) = -g'(t)\,\mathbf{i} + f'(t)\,\mathbf{j}$ and $-\mathbf{n}(t) = g'(t)\,\mathbf{i} - f'(t)\,\mathbf{j}$ are both normal to the curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ at the point $(f(t), g(t))$.

To obtain **N** for a particular plane curve, we can choose the one of **n** or $-\mathbf{n}$ from part (a) that points toward the concave side of the curve, and make it into a unit vector. (See Fig. 11.26.) Apply this method to find $\mathbf{N}$ for the following curves.

b) $\mathbf{r}(t) = t\,\mathbf{i} + e^{2t}\,\mathbf{j}$

c) $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t\,\mathbf{j}, \quad -2 \le t \le 2$

10. *(Continuation of Exercise 9.)*

a) Use the method of Exercise 9 to find $\mathbf{N}$ for the curve $\mathbf{r}(t) = t\,\mathbf{i} + (1/3)\,t^3\,\mathbf{j}$ when $t < 0$; when $t > 0$.

b) Calculate

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},\ t \neq 0,$$

for the curve in part (a). Does $\mathbf{N}$ exist at $t = 0$? Graph the curve and explain what is happening to $\mathbf{N}$ as t passes from negative to positive values.

Space Curves

Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ for the space curves in Exercises 11–18.

11. $\mathbf{r}(t) = (3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + 4t\,\mathbf{k}$

12. $\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + 3\,\mathbf{k}$

13. $\mathbf{r}(t) = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$

14. $\mathbf{r}(t) = (6\sin 2t)\,\mathbf{i} + (6\cos 2t)\,\mathbf{j} + 5t\,\mathbf{k}$

15. $\mathbf{r}(t) = (t^3/3)\,\mathbf{i} + (t^2/2)\,\mathbf{j}, \quad t > 0$

16. $\mathbf{r}(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 < t < \pi/2$

17. $\mathbf{r}(t) = t\,\mathbf{i} + (a\cosh(t/a))\,\mathbf{j}, \quad a > 0$

18. $\mathbf{r}(t) = (\cosh t)\,\mathbf{i} - (\sinh t)\,\mathbf{j} + t\,\mathbf{k}$

In Exercises 19 and 20, write $\mathbf{a}$ in the form $a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$.

19. $\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j} + bt\,\mathbf{k}$

20. $\mathbf{r}(t) = (1+3t)\,\mathbf{i} + (t-2)\,\mathbf{j} - 3t\,\mathbf{k}$

In Exercises 21–24, write $\mathbf{a}$ in the form $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$ at the given value of t without finding $\mathbf{T}$ and $\mathbf{N}$.

21. $\mathbf{r}(t) = (t+1)\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}, \quad t = 1$

22. $\mathbf{r}(t) = (t\cos t)\,\mathbf{i} + (t\sin t)\,\mathbf{j} + t^2\,\mathbf{k}, \quad t = 0$

23. $\mathbf{r}(t) = t^2\,\mathbf{i} + (t + (1/3)t^3)\,\mathbf{j} + (t - (1/3)t^3)\,\mathbf{k}, \quad t = 0$

24. $\mathbf{r}(t) = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + \sqrt{2}e^t\,\mathbf{k}, \quad t = 0$

In Exercises 25 and 26, find $\mathbf{r}$, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t.

25. $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} - \mathbf{k}, \quad t = \pi/4$

26. $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}, \quad t = 0$

Physical Applications

27. The speedometer on your car reads a steady 35 mph. Could you be accelerating? Explain.

28. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

29. Can anything be said about the speed of a particle whose acceleration is always orthogonal to its velocity? Give reasons for your answer.

30. An object of mass m travels along the parabola $y = x^2$ with a constant speed of 10 units/sec. What is the force on the object due to its acceleration at (0, 0)? at $(2^{1/2}, 2)$? Write your answers in terms of $\mathbf{i}$ and $\mathbf{j}$. (Remember Newton's law, $\mathbf{F} = m\,\mathbf{a}$.)

31. The following is a quotation from an article in *The American Mathematical Monthly,* titled "Curvature in the Eighties" by Robert Osserman (October 1990, page 731):

> Curvature also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectories.

Explain mathematically why the second sentence of the quotation is true.

32. Show that a moving particle will move in a straight line if the normal component of its acceleration is zero.

More on Curvature

33. Show that the parabola $y = ax^2, a \neq 0$, has its largest curvature at its vertex and has no minimum curvature. (*Note:* Since the curvature of a curve remains the same if the curve is translated or rotated, this result is true for any parabola.)

34. Show that the ellipse $x = a\cos t$, $y = b\sin t$, $a > b > 0$, has its largest curvature on its major axis and its smallest curvature on its minor axis. (As in Exercise 33, the same is true for any ellipse.)

35. *Maximizing the curvature of a helix.* In Example 4, we found the curvature of the helix $\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j} + bt\,\mathbf{k}$ $(a, b \ge 0)$ to be $\kappa = a/(a^2+b^2)$. What is the largest value κ can have for a given value of b? Give reasons for your answer.

36. *A sometime shortcut to curvature.* If you already know $|a_{\mathrm{N}}|$ and $|\mathbf{v}|$, then the formula $a_{\mathrm{N}} = \kappa\,|\mathbf{v}|^2$ gives a convenient way to find the curvature. Use it to find the curvature and radius of curvature of the curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}, \quad t > 0.$$

(Take a_{N} and $|\mathbf{v}|$ from Example 6.)

37. Show that κ and τ are both zero for the line

$$\mathbf{r}(t) = (x_0 + At)\,\mathbf{i} + (y_0 + Bt)\,\mathbf{j} + (z_0 + Ct)\,\mathbf{k}.$$

38. *Total curvature.* We find the **total curvature** of the portion of a smooth curve that runs from $s = s_0$ to $s = s_1 > s_0$ by integrating κ from s_0 to s_1. If the curve has some other parameter, say t, then the total curvature is

$$K = \int_{s_0}^{s_1} \kappa\,ds = \int_{t_0}^{t_1} \kappa\,\frac{ds}{dt}\,dt = \int_{t_0}^{t_1} \kappa|\mathbf{v}|dt,$$

where t_0 and t_1 correspond to s_0 and s_1. Find the total curvature of the portion of the helix $\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 4\pi$.

39. *(Continuation of Exercise 38.)* Find the total curvatures of the following curves.

a) The involute of the unit circle: $\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}$, $a \le t \le b\ (a > 0)$. (Exercise 36 gives a convenient way to find κ. Use values from Example 6.)

b) The parabola $y = x^2$, $-\infty < x < \infty$

40. a) Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = t\,\mathbf{i} + (\sin t)\,\mathbf{j}$ at the point $(\pi/2, 1)$. (The curve parametrizes the graph of $y = \sin x$ in the *xy*-plane.)

b) Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = (2\ln t)\,\mathbf{i} - [t + (1/t)]\,\mathbf{j}$, $e^{-2} \le t \le e^2$, at the point $(0, -2)$, where $t = 1$.

Theory and Examples

41. What can be said about the torsion of a (sufficiently differentiable) plane curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$? Give reasons for your answer.

42. *The torsion of a helix.* In Example 7, we found the torsion of the helix

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j} + bt\,\mathbf{k}, \quad a, b \ge 0$$

to be $\tau = b/(a^2 + b^2)$. What is the largest value τ can have for a given value of a? Give reasons for your answer.

43. *Differentiable curves with zero torsion lie in planes.* That a sufficiently differentiable curve with zero torsion lies in a plane is a special case of the fact that a particle whose velocity remains perpendicular to a fixed vector $\mathbf{C}$ moves in a plane perpendicular to $\mathbf{C}$. This, in turn, can be viewed as the solution of the following problem in calculus.

Suppose $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is twice differentiable for all t in an interval $[a, b]$, that $\mathbf{r} = 0$ when $t = a$, and that $\mathbf{v} \cdot \mathbf{k} = 0$ for all t in $[a, b]$. Then $h(t) = 0$ for all t in $[a, b]$.

Solve this problem. (*Hint:* Start with $\mathbf{a} = d^2\mathbf{r}/dt^2$ and apply the initial conditions in reverse order.)

44. *A formula that calculates τ from* **B** *and* **v**. If we start with the definition $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ and apply the Chain Rule to rewrite $d\mathbf{B}/ds$ as

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\frac{dt}{ds} = \frac{d\mathbf{B}}{dt}\frac{1}{|\mathbf{v}|},$$

we arrive at the formula

$$\tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right).$$

The advantage of this formula over Eq. (11) is that it is easier to derive and state. The disadvantage is that it can take a lot of work to evaluate without a computer. Use the new formula to find the torsion of the helix in Example 7.

Grapher Explorations

The formula

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}},$$

derived in Exercise 7, expresses the curvature $\kappa(x)$ of a twice-differentiable plane curve $y = f(x)$ as a function of x. Find the curvature function of each of the curves in Exercises 45–48. Then graph $f(x)$ together with $\kappa(x)$ over the given interval. You will find some surprises.

45. $y = x^2, \quad -2 \le x \le 2$

46. $y = x^4/4, \quad -2 \le x \le 2$

47. $y = \sin x, \quad 0 \le x \le 2\pi$

48. $y = e^x, \quad -1 \le x \le 2$

CAS Explorations and Projects—Circles of Curvature

In Exercises 49–56 you will use a CAS to explore the osculating circle at a point P on a plane curve where $\kappa \ne 0$. Use a CAS to perform the following steps:

a) Plot the plane curve given in parametric or function form over the specified interval to see what it looks like.

b) Calculate the curvature κ of the curve at the given value t_0 using the appropriate formula from Exercise 7 or 8. Use the parametrization $x = t$ and $y = f(t)$ if the curve is given as a function $y = f(x)$.

c) Find the unit normal vector $\mathbf{N}$ at t_0. Notice that the signs of the components of $\mathbf{N}$ depend on whether the unit tangent vector $\mathbf{T}$ is turning clockwise or counterclockwise at $t = t_0$. (See Exercise 9.)

d) If $\mathbf{C} = a\,\mathbf{i} + b\,\mathbf{j}$ is the vector from the origin to the center (a, b) of the osculating circle, find the center $\mathbf{C}$ from the vector equation

$$\mathbf{C} = \mathbf{r}(t_0) + \frac{1}{\kappa(t_0)}\mathbf{N}(t_0).$$

The point $P(x_0, y_0)$ on the curve is given by the position vector $\mathbf{r}(t_0)$.

e) Plot implicitly the equation $(x - a)^2 + (y - b)^2 = 1/\kappa^2$ of the osculating circle. Then plot the curve and osculating circle together. You may need to experiment with the size of the viewing window, but be sure it is square.

49. $\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (5\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi, \quad t_0 = \pi/4$

50. $\mathbf{r}(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le 2\pi, \quad t_0 = \pi/4$

51. $\mathbf{r}(t) = t^2\,\mathbf{i} + (t^3 - 3t)\,\mathbf{j}, \quad -4 \le t \le 4, \quad t_0 = 3/5$

52. $\mathbf{r}(t) = (t^3 - 2t^2 - t)\,\mathbf{i} + \dfrac{3t}{\sqrt{1 + t^2}}\,\mathbf{j}, \quad -2 \le t \le 5, \quad t_0 = 1$

53. $\mathbf{r}(t) = (2t - \sin t)\,\mathbf{i} + (2 - 2\cos t)\,\mathbf{j}, \quad 0 \le t \le 3\pi, \quad t_0 = 3\pi/2$

54. $\mathbf{r}(t) = (e^{-t}\cos t)\,\mathbf{i} + (e^{-t}\sin t)\,\mathbf{j}, \quad 0 \le t \le 6\pi, \quad t_0 = \pi/4$

55. $y = x^2 - x, \quad -2 \le x \le 5, \quad x_0 = 1$

56. $y = x(1 - x)^{2/5}, \quad -1 \le x \le 2, \quad x_0 = 1/2$

CAS Explorations and Projects—Curvature, Torsion, and the TNB Frame

Rounding the answers to four decimal places, use a CAS to find **v**, **a**, speed, **T**, **N**, **B**, κ, τ, and the tangential and normal components of acceleration for the curves in Exercises 57–60 at the given values of t.

57. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t\mathbf{k}, \quad t = \sqrt{3}$

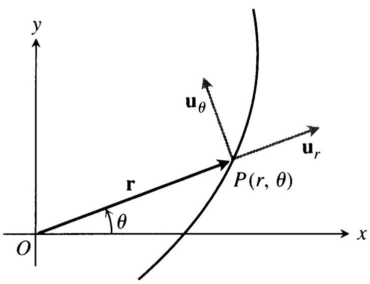
58. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + e^t\mathbf{k}, \quad t = \ln 2$

59. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + \sqrt{-t}\,\mathbf{k}, \quad t = -3\pi$

60. $\mathbf{r}(t) = (3t - t^2)\mathbf{i} + (3t^2)\mathbf{j} + (3t + t^3)\mathbf{k}, \quad t = 1$

11.5 Planetary Motion and Satellites

In this section, we derive Kepler's laws of planetary motion from Newton's laws of motion and gravitation and discuss the orbits of Earth satellites. The derivation of Kepler's laws from Newton's is one of the triumphs of calculus. It draws on almost everything we have studied so far, including the algebra and geometry of vectors in space, the calculus of vector functions, the solutions of differential equations and initial value problems, and the polar coordinate description of conic sections.



11.35 The length of **r** is the positive polar coordinate r of the point P. Thus, $\mathbf{u}_r$, which is $\mathbf{r}/|\mathbf{r}|$, is also $\mathbf{r}/r$. Equations (1) express $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in terms of **i** and **j**.

Vector Equations for Motion in Polar and Cylindrical Coordinates

When a particle moves along a curve in the polar coordinate plane, we express its position, velocity, and acceleration in terms of the moving unit vectors

$$\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \qquad \mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}, \tag{1}$$

shown in Fig. 11.35. The vector $\mathbf{u}_r$ points along the position vector $\overrightarrow{OP}$, so $\mathbf{r} = r\mathbf{u}_r$. The vector $\mathbf{u}_\theta$, orthogonal to $\mathbf{u}_r$, points in the direction of increasing θ.

We find from (1) that

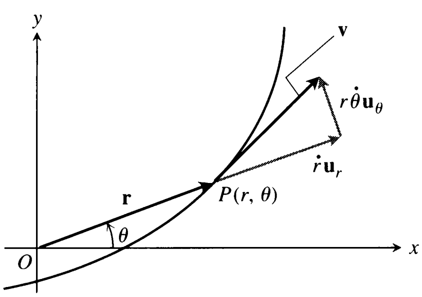
$$\begin{aligned} \frac{d\,\mathbf{u}_r}{d\,\theta} &= -(\sin\,\theta)\,\mathbf{i} + (\cos\,\theta)\,\mathbf{j} = \mathbf{u}_\theta \\ \frac{d\,\mathbf{u}_\theta}{d\,\theta} &= -(\cos\,\theta)\,\mathbf{i} - (\sin\,\theta)\,\mathbf{j} = -\mathbf{u}_r. \end{aligned} \tag{2}$$

When we differentiate $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to t to find how they change with time, the Chain Rule gives

$$\dot{\mathbf{u}}_r = \frac{d\,\mathbf{u}_r}{d\theta}\dot{\theta} = \dot{\theta}\,\mathbf{u}_\theta, \qquad \dot{\mathbf{u}}_\theta = \frac{d\,\mathbf{u}_\theta}{d\theta}\dot{\theta} = -\dot{\theta}\,\mathbf{u}_r. \tag{3}$$

Hence,

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\left(r\,\mathbf{u}_r\right) = \dot{r}\,\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta. \tag{4}$$



11.36 In polar coordinates, the velocity vector is

$$\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$

See Fig. 11.36.

The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\,\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\,\mathbf{u}_\theta + r\ddot{\theta}\,\mathbf{u}_\theta + r\,\dot{\theta}\,\dot{\mathbf{u}}_\theta). \tag{5}$$

As in the previous section, we use Newton's dot notation for time derivatives to keep the formulas as simple as we can: $\dot{\mathbf{u}}_r$ means $d\mathbf{u}_r/dt$, $\dot{\theta}$ means $d\theta/dt$, and so on.

When Eqs. (3) are used to evaluate $\dot{\mathbf{u}}_r$ and $\dot{\mathbf{u}}_\theta$ and the components are separated, the equation for acceleration becomes

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{u}_\theta. \tag{6}$$

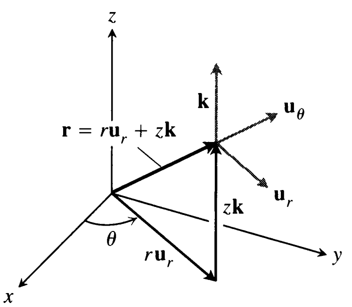
Notice that $|\mathbf{r}| \neq r$ if $z \neq 0$.

To extend these equations of motion to space, we add $z\mathbf{k}$ to the right-hand side of the equation $\mathbf{r} = r\mathbf{u}_r$. Then, in cylindrical coordinates,

$$
\begin{aligned}
\mathbf{r} &= r\,\mathbf{u}_r + z\,\mathbf{k} \\
\mathbf{v} &= \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta + \dot{z}\,\mathbf{k} \qquad (7)\\
\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\,\mathbf{k}.
\end{aligned}
$$

The vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ make a right-handed frame (Fig. 11.37) in which

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \qquad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \qquad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta. \tag{8}$$



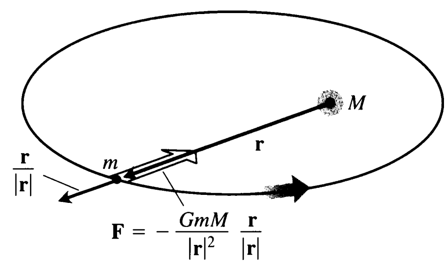
11.37 Position vector and basic unit vectors in cylindrical coordinates.

Planets Move in Planes

Newton's Law of Gravitation says that if $\mathbf{r}$ is the radius vector from the center of a sun of mass M to the center of a planet of mass m, then the force $\mathbf{F}$ of the gravitational attraction between the planet and sun is

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|} \tag{9}$$

(Fig. 11.38). The number G is the (universal) **gravitational constant.** If we measure mass in kilograms, force in newtons, and distance in meters, G is about $6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$.



11.38 The force of gravity is directed along the line joining the centers of mass.

Combining Eq. (9) with Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, for the force acting on the planet gives

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}, \\
\ddot{\mathbf{r}} &= -\frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}. \qquad (10)
\end{aligned}
$$

The planet is accelerated toward the sun's center at all times.

Equation (10) says that $\ddot{\mathbf{r}}$ is a scalar multiple of $\mathbf{r}$, so that

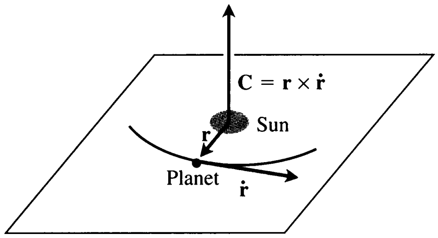
$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}. \tag{11}$$

A routine calculation shows $\mathbf{r} \times \ddot{\mathbf{r}}$ to be the derivative of $\mathbf{r} \times \dot{\mathbf{r}}$:

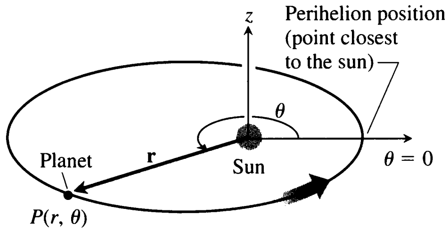
$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \underbrace{\dot{\mathbf{r}} \times \dot{\mathbf{r}}}_{\mathbf{0}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}}. \tag{12}$$

Hence Eq. (11) is equivalent to

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{0}, \tag{13}$$



11.39 A planet that obeys Newton's laws of gravitation and motion travels in the plane through the sun's center of mass perpendicular to $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$.



11.40 The coordinate system for planetary motion. The motion is counterclockwise when viewed from above, as it is here, and $\dot{\theta} > 0$.

which integrates to

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C} \tag{14}$$

for some constant vector **C**.

Equation (14) tells us that $\mathbf{r}$ and $\dot{\mathbf{r}}$ always lie in a plane perpendicular to **C**. Hence, the planet moves in a fixed plane through the center of its sun (Fig. 11.39).

Coordinates and Initial Conditions

We now introduce cylindrical coordinates in a way that places the origin at the sun's center of mass and makes the plane of the planet's motion the polar coordinate plane. This makes $\mathbf{r}$ the planet's polar coordinate position vector and makes $|\mathbf{r}|$ equal to r and $\mathbf{r}/|\mathbf{r}|$ equal to $\mathbf{u}_r$. We also position the z-axis in a way that makes $\mathbf{k}$ the direction of $\mathbf{C}$. Thus, $\mathbf{k}$ has the same right-hand relation to $\mathbf{r} \times \dot{\mathbf{r}}$ that $\mathbf{C}$ does, and the planet's motion is counterclockwise when viewed from the positive z-axis. This makes θ increase with t, so that $\dot{\theta} > 0$ for all t. Finally, we rotate the polar coordinate plane about the z-axis, if necessary, to make the initial ray coincide with the direction $\mathbf{r}$ has when the planet is closest to the sun. This runs the ray through the planet's **perihelion** position (Fig. 11.40).

If we measure time so that $t = 0$ at perihelion, we have the following initial conditions for the planet's motion.

1. $r = r_0$, the minimum radius, when $t = 0$
2. $\dot{r} = 0$ when $t = 0$ (because r has a minimum value then)
3. $\theta = 0$ when $t = 0$
4. $|\mathbf{v}| = v_0$ when $t = 0$

Since

$$\begin{aligned}
v_0 &= |\mathbf{v}|_{t=0} \\
&= \left|\dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta\right|_{t=0} && \text{Eq. (4)} \\
&= \left|r\dot{\theta}\,\mathbf{u}_\theta\right|_{t=0} && \dot{r} = 0 \text{ when } t = 0 \\
&= \left(\left|r\dot{\theta}\right|\left|\mathbf{u}_\theta\right|\right)_{t=0} \\
&= \left|r\dot{\theta}\right|_{t=0} && |\mathbf{u}_\theta| = 1 \\
&= (r\dot{\theta})_{t=0}, && r \text{ and } \dot{\theta} \text{ both positive}
\end{aligned}$$

we also know that

5. $r\dot{\theta} = v_0$ when $t = 0$.

Statement of Kepler's First Law (The Conic Section Law)

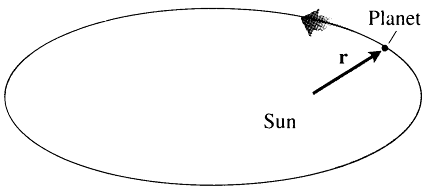
Kepler's first law says that a planet's path is a conic section with the sun at one focus. The eccentricity of the conic is

$$e = \frac{r_0 v_0^2}{GM} - 1 \tag{15}$$

and the polar equation is

$$r = \frac{(1+e)r_0}{1+e\cos\theta}. \tag{16}$$

The derivation uses Kepler's second law, so we will state and prove the second law before proving the first law.



11.41 The line joining a planet to its sun sweeps over equal areas in equal times.

Kepler's Second Law (The Equal Area Law)

Kepler's second law says that the radius vector from the sun to a planet (the vector **r** in our model) sweeps out equal areas in equal times (Fig. 11.41). To derive the law, we use Eq. (4) to evaluate the cross product $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$ from Eq. (14):

$$\begin{aligned}
\mathbf{C} &= \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} \\
&= r\,\mathbf{u}_r \times (\dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta) && \text{Eq. (4)} \\
&= r\dot{r}\,\underbrace{(\mathbf{u}_r \times \mathbf{u}_r)}_{\mathbf{0}} + r(r\dot{\theta})\,\underbrace{(\mathbf{u}_r \times \mathbf{u}_\theta)}_{\mathbf{k}} \\
&= r(r\dot{\theta})\,\mathbf{k}.
\end{aligned} \tag{17}$$

Setting t equal to zero shows that

$$\mathbf{C} = [r(r\dot{\theta})]_{t=0}\,\mathbf{k} = r_0 v_0\,\mathbf{k}. \tag{18}$$

Substituting this value for **C** in Eq. (17) gives

$$r_0 v_0\,\mathbf{k} = r^2\dot{\theta}\,\mathbf{k}, \qquad \text{or} \qquad r^2\dot{\theta} = r_0 v_0. \tag{19}$$

This is where the area comes in. The area differential in polar coordinates is

$$dA = \frac{1}{2}r^2 d\theta$$

(Section 9.9). Accordingly, dA/dt has the constant value

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r_0 v_0, \tag{20}$$

which is Kepler's second law.

For Earth, r_0 is about 150,000,000 km, v_0 is about 30 km/sec, and dA/dt is about 2,250,000,000 km^2/sec. Every time your heart beats, Earth advances 30 km along its orbit, and the radius joining Earth to the sun sweeps out 2,250,000,000 km^2 of area.

The German astronomer, mathematician, and physicist Johannes Kepler (1571–1630) was the first, and until Descartes the only, scientist to demand physical (as opposed to theological) explanations of celestial phenomena. His three laws of motion, the results of a lifetime of work, changed the course of astronomy forever and played a crucial role in the development of Newton's physics.

Proof of Kepler's First Law

To prove that a planet moves along a conic section with one focus at its sun, we need to express the planet's radius r as a function of θ. This requires a long sequence of calculations and some substitutions that are not altogether obvious.

We begin with the equation that comes from equating the coefficients of $\mathbf{u}_r = \mathbf{r}/|\mathbf{r}|$ in Eqs. (6) and (10):

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}. \tag{21}$$

We eliminate $\dot{\theta}$ temporarily by replacing it with $r_0 v_0/r^2$ from Eq. (19) and rearrange the resulting equation to get

$$\ddot{r} = \frac{{r_0}^2{v_0}^2}{r^3} - \frac{GM}{r^2}. \tag{22}$$

We change this into a first order equation by a change of variable. With

$$p = \frac{dr}{dt}, \qquad \frac{d^2r}{dt^2} = \frac{dp}{dt} = \frac{dp}{dr}\frac{dr}{dt} = p\frac{dp}{dr}, \qquad \text{Chain Rule}$$

Eq. (22) becomes

$$p\frac{dp}{dr} = \frac{{r_0}^2{v_0}^2}{r^3} - \frac{GM}{r^2}. \tag{23}$$

Multiplying through by 2 and integrating with respect to r gives

$$p^2 = (\dot{r})^2 = -\frac{{r_0}^2{v_0}^2}{r^2} + \frac{2GM}{r} + C_1. \tag{24}$$

The initial conditions that $r = r_0$ and $\dot{r} = 0$ when $t = 0$ determine the value of C_1 to be

$$C_1 = {v_0}^2 - \frac{2GM}{r_0}.$$

Accordingly, Eq. (24), after a suitable rearrangement, becomes

$$\dot{r}^2 = {v_0}^2\left(1 - \frac{{r_0}^2}{r^2}\right) + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{25}$$

The effect of going from Eq. (21) to Eq. (25) has been to replace a second order differential equation in r by a first order differential equation in r. Our goal is still to express r in terms of θ, so we now bring θ back into the picture. To accomplish this, we divide both sides of Eq. (25) by the squares of the corresponding sides of the equation $r^2\dot{\theta} = r_0v_0$ (Eq. 19) and use the fact that $\dot{r}/\dot{\theta} = (dr/dt)/(d\theta/dt) = dr/d\theta$ to get

$$\begin{aligned}\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 &= \frac{1}{{r_0}^2} - \frac{1}{r^2} + \frac{2GM}{{r_0}^2{v_0}^2}\left(\frac{1}{r} - \frac{1}{r_0}\right) \\ &= \frac{1}{{r_0}^2} - \frac{1}{r^2} + 2h\left(\frac{1}{r} - \frac{1}{r_0}\right). \qquad h = \frac{GM}{{r_0}^2{v_0}^2}\end{aligned} \tag{26}$$

To simplify further, we substitute

$$u = \frac{1}{r}, \qquad u_0 = \frac{1}{r_0}, \qquad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}, \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2,$$

obtaining

$$\left(\frac{du}{d\theta}\right)^2 = {u_0}^2 - u^2 + 2hu - 2hu_0 = (u_0 - h)^2 - (u - h)^2, \tag{27}$$

$$\frac{du}{d\theta} = \pm\sqrt{(u_0 - h)^2 - (u - h)^2}. \tag{28}$$

Which sign do we take? We know that $\dot{\theta} = r_0v_0/r^2$ is positive. Also, r starts from a minimum value at $t = 0$, so it cannot immediately decrease, and $\dot{r} \geq 0$, at least for early positive values of t. Therefore,

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \geq 0 \qquad \text{and} \qquad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \leq 0.$$

The correct sign for Eq. (28) is the negative sign. With this determined, we rearrange Eq. (28) and integrate both sides with respect to θ:

$$\begin{aligned} \frac{-1}{\sqrt{(u_0 - h)^2 - (u - h)^2}}\frac{du}{d\theta} &= 1 \\ \cos^{-1}\left(\frac{u - h}{u_0 - h}\right) &= \theta + C_2. \end{aligned} \tag{29}$$

The constant C_2 is zero because $u = u_0$ when $\theta = 0$ and $\cos^{-1}(1) = 0$. Therefore,

$$\frac{u - h}{u_0 - h} = \cos\theta$$

and

$$\frac{1}{r} = u = h + (u_0 - h)\cos\theta. \tag{30}$$

A few more algebraic maneuvers produce the final equation

$$r = \frac{(1 + e)r_0}{1 + e\cos\theta}, \tag{31}$$

where

$$e = \frac{1}{r_0 h} - 1 = \frac{r_0 v_0{}^2}{GM} - 1. \tag{32}$$

Together, Eqs. (31) and (32) say that the path of the planet is a conic section with one focus at the sun and with eccentricity $(r_0 v_0{}^2/GM) - 1$. This is the modern formulation of Kepler's first law.

Statement of Kepler's Third Law (The Time–Distance Law)

The time T it takes a planet to go around its sun once is the planet's **orbital period.** *Kepler's third law* says that T and the orbit's semimajor axis a are related by the equation

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}. \tag{33}$$

Since the right-hand side of this equation is constant within a given solar system, the ratio of T^2 to a^3 *is the same for every planet in the system.*

Kepler's third law is the starting point for working out the size of our solar system. It allows the semimajor axis of each planetary orbit to be expressed in astronomical units, Earth's semimajor axis being one unit. The distance between any two planets at any time can then be predicted in astronomical units and all that remains is to find one of these distances in kilometers. This can be done by bouncing radar waves off Venus, for example. The astronomical unit is now known, after a series of such measurements, to be 149,597,870 km.

We derive Kepler's third law by combining two formulas for the area enclosed by the planet's elliptical orbit:

Formula 1: $\text{Area} = \pi ab$ — The geometry formula in which a is the semimajor axis and b is the semiminor axis

$$\begin{aligned} \text{Formula 2:} \quad \text{Area} &= \int_0^T dA \\ &= \int_0^T \frac{1}{2} r_0 v_0 \, dt \qquad \text{Eq. (20)} \\ &= \frac{1}{2} T r_0 v_0. \end{aligned}$$

Equating these gives

$$T = \frac{2\pi ab}{r_0 v_0} = \frac{2\pi a^2}{r_0 v_0}\sqrt{1-e^2}. \qquad \text{For any ellipse, } b = a\sqrt{1-e^2} \tag{34}$$

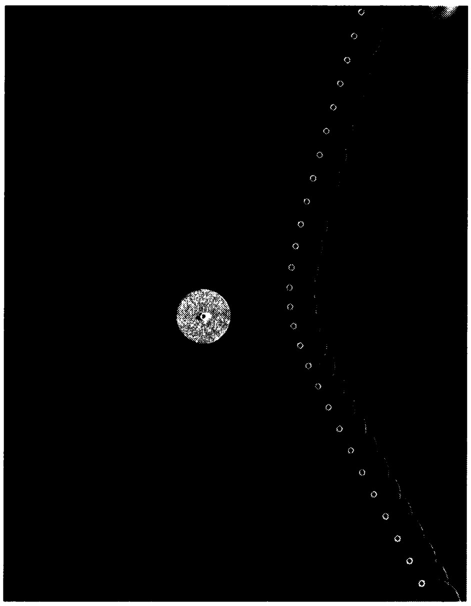
It remains only to express a and e in terms of r_0, v_0, G, and M. Equation (32) does this for e. For a, we observe that setting θ equal to π in Eq. (31) gives

$$r_{\max} = r_0 \frac{1+e}{1-e}.$$

Hence,

$$2a = r_0 + r_{\max} = \frac{2r_0}{1-e} = \frac{2r_0 GM}{2GM - r_0 {v_0}^2}. \tag{35}$$

Squaring both sides of Eq. (34) and substituting the results of Eqs. (32) and (35) now produces Kepler's third law (Exercise 14).

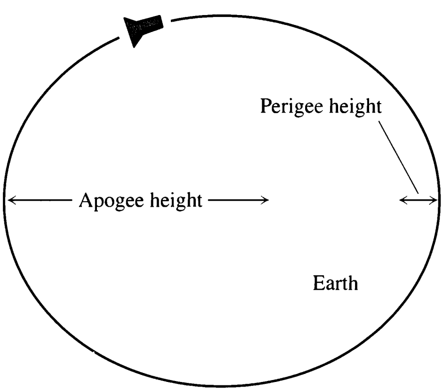


11.42 This multiflash photograph shows a body being deflected by an inverse square law force. It moves along a hyperbola.

Orbit Data

Although Kepler discovered his laws empirically and stated them only for the six planets known at the time, the modern derivations of Kepler's laws show that they apply to any body driven by a force that obeys an inverse square law. They apply to Halley's comet and the asteroid Icarus. They apply to the moon's orbit about Earth, and they applied to the orbit of the spacecraft *Apollo 8* about the moon. They also applied to the air puck shown in Fig. 11.42 being deflected by an inverse square law force—its path is a hyperbola. Charged particles fired at the nuclei of atoms scatter along hyperbolic paths.

Tables 11.1–11.3 give additional data for planetary orbits and for the orbits of seven of Earth's artificial satellites (Fig. 11.43). *Vanguard 1* sent back data that revealed differences between the levels of Earth's oceans and provided the first determination of the precise locations of some of the more isolated Pacific islands. The data also verified that the gravitation of the sun and moon would affect the orbits of Earth's satellites and that solar radiation could exert enough pressure to deform an orbit.



11.43 The orbit of an Earth satellite: $2a$ = diameter of earth + perigee height + apogee height.

Syncom 3 is one of a series of U.S. Department of Defense telecommunications satellites. *Tiros 11* (for "television infrared observation satellite") is one of a series of weather satellites. *GOES 4* (for "geostationary operational environmental satellite") is one of a series of satellites designed to gather information about Earth's atmosphere. Its orbital period, 1436.2 minutes, is nearly the same as Earth's rotational period of 1436.1 minutes, and its orbit is nearly circular ($e = 0.0003$). *Intelsat 5* is a heavy-capacity commercial telecommunications satellite.

Table 11.1 Values of a, e, and T for the major planets

Planet	Semimajor axis $a^{\dagger}$	Eccentricity e	Period T
Mercury	57.95	0.2056	87.967 days
Venus	108.11	0.0068	224.701 days
Earth	149.57	0.0167	365.256 days
Mars	227.84	0.0934	1.8808 years
Jupiter	778.14	0.0484	11.8613 years
Saturn	1427.0	0.0543	29.4568 years
Uranus	2870.3	0.0460	84.0081 years
Neptune	4499.9	0.0082	164.784 years
Pluto	5909	0.2481	248.35 years

† Millions of kilometers

Table 11.2 Data on Earth's satellites

Name	Launch date	Time or expected time aloft	Mass at launch (kg)	Period (min)	Perigee height (km)	Apogee height (km)	Semimajor axis a (km)	Eccentricity
Sputnik 1	Oct. 1957	57.6 days	83.6	96.2	215	939	6,955	0.052
Vanguard 1	March 1958	300 years	1.47	138.5	649	4,340	8,872	0.208
Syncom 3	Aug. 1964	$>10^6$ years	39	1436.2	35,718	35,903	42,189	0.002
Skylab 4	Nov. 1973	84.06 days	13,980	93.11	422	437	6,808	0.001
Tiros 11	Oct. 1978	500 years	734	102.12	850	866	7,236	0.001
GOES 4	Sept. 1980	$>10^6$ years	627	1436.2	35,776	35,800	42,166	0.0003
Intelsat 5	Dec. 1980	$>10^6$ years	1,928	1417.67	35,143	35,707	41,803	0.007

Table 11.3 Numerical data

Gravitational constant: $G = 6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$
(When you use this value of G in a calculation, remember to express force in newtons, distance in meters, mass in kilograms, and time in seconds.)

Sun's mass: 1.99×10^{30} kg

Earth's mass: 5.975×10^{24} kg

Equatorial radius of Earth: 6378.533 km

Polar radius of Earth: 6356.912 km

Earth's rotational period: 1436.1 min

Earth's orbital period: 1 year = 365.256 days

Exercises 11.5

Reminder: When a calculation involves the gravitational constant G, express force in newtons, distance in meters, mass in kilograms, and time in seconds.

Calculator Exercises

1. Since the orbit of *Skylab 4* had a semimajor axis of $a = 6808$ km, Kepler's third law with M equal to the earth's mass should give the period. Calculate it. Compare your result with the value in Table 11.2.

2. Earth's distance from the sun at perihelion is approximately 149,577,000 km, and the eccentricity of the earth's orbit about the sun is 0.0167. Find the velocity v_0 of Earth in its orbit at perihelion. (Use Eq. 15.)

3. In July 1965, the USSR launched *Proton I,* weighing 12,200 kg (at launch), with a perigee height of 183 km, an apogee height of 589 km, and a period of 92.25 min. Using the relevant data for the mass of Earth and the gravitational constant G, find the semimajor axis a of the orbit from Eq. (33). Compare your answer with the number you get by adding the perigee and apogee heights to the diameter of the earth.

4. a) The *Viking 1* orbiter, which surveyed Mars from August 1975 to June 1976, had a period of 1639 min. Use this and the fact that the mass of Mars is 6.418×10^{23} kg to find the semimajor axis of the *Viking 1* orbit.
   b) The *Viking 1* orbiter was 1499 km from the surface of Mars at its closest point and 35,800 km from the surface at its farthest point. Use this information together with the value you obtained in part (a) to estimate the average diameter of Mars.

5. The *Viking 2* orbiter, which surveyed Mars from September 1975 to August 1976, moved in an ellipse whose semimajor axis was 22,030 km. What was the orbital period? (Express your answer in minutes.)

6. *Geosynchronous orbits.* Several satellites in the earth's equatorial plane have nearly circular orbits whose periods are the same as the earth's rotational period. Such orbits are called **geosynchronous** or **geostationary** because they hold the satellite over the same spot on Earth's surface.
   a) Approximately what is the semimajor axis of a geosynchronous orbit? Give reasons for your answer.
   b) About how high is a geosynchronous orbit above the Earth's surface?
   c) Which of the satellites in Table 11.2 have (nearly) geosynchronous orbits?

7. The mass of Mars is 6.418×10^{23} kg. If a satellite revolving about Mars is to hold a stationary orbit (have the same period as the period of Mars's rotation, which is 1477.4 min), what must the semimajor axis of its orbit be? Give reasons for your answer.

8. The period of the moon's rotation about Earth is 2.36055×10^6 sec. About how far away is the moon?

9. A satellite moves around Earth in a circular orbit. Express the satellite's speed as a function of the orbit's radius.

10. If T is measured in seconds and a in meters, what is the value of T^2/a^3 for planets in our solar system? for satellites orbiting Earth? for satellites orbiting the moon? (The moon's mass is 7.354×10^{22} kg.)

Noncalculator Exercises

11. For what values of v_0 in Eq. (15) is the orbit in Eq. (16) a circle? an ellipse? a parabola? a hyperbola?

12. Show that a planet in a circular orbit moves with a constant speed. (*Hint:* This is a consequence of one of Kepler's laws.)

13. Suppose that $\mathbf{r}$ is the position vector of a particle moving along a plane curve and dA/dt is the rate at which the vector sweeps out area. Without introducing coordinates, and assuming the necessary derivatives exist, give a geometric argument based on increments and limits for the validity of the equation
$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}|.$$

14. Complete the derivation of Kepler's third law (the part following Eq. 34).

15. Two planets, planet A and planet B, are orbiting their sun in circular orbits with A being the inner planet and B being farther away from the sun. Suppose the positions of A and B at time t are
$$\mathbf{r}_A(t) = 2\cos(2\pi t)\,\mathbf{i} + 2\sin(2\pi t)\,\mathbf{j}$$
and
$$\mathbf{r}_B(t) = 3\cos(\pi t)\,\mathbf{i} + 3\sin(\pi t)\,\mathbf{j},$$
respectively, where the sun is assumed to be located at the origin and distance is measured in astronomical units. (Notice that planet A moves faster than planet B.)

    The people on planet A regard their planet, not the sun, as the center of their planetary system (their solar system).
    a) Using planet A as the origin of a new coordinate system, give parametric equations for the location of planet B at time t. Write your answer in terms of $\cos(\pi t)$ and $\sin(\pi t)$.
    b) GRAPHER Using planet A as the origin, sketch a graph of the path of planet B.

    This exercise illustrates the difficulty that people before Kepler's time, with an earth-centered (planet A) view of our solar system, had in understanding the motions of the planets (i.e., planet B = Mars). See D. G. Saari's article in *The American Mathematical Monthly,* Vol. 97, Feb. 1990, pp. 105–119.

16. Kepler discovered that the path of the earth around the sun is an ellipse with the sun at one of the foci. Let $\mathbf{r}(t)$ be the position vector from the center of the sun to the center of the earth at time t. Let $\mathbf{w}$ be the vector from the earth's South Pole to North Pole. It is known that $\mathbf{w}$ is constant and not orthogonal to the plane of the ellipse (the earth's axis is tilted). In terms of $\mathbf{r}(t)$ and $\mathbf{w}$, give the mathematical meaning of (i) perihelion, (ii) aphelion, (iii) equinox, (iv) summer solstice, (v) winter solstice.

CHAPTER 11 QUESTIONS TO GUIDE YOUR REVIEW

1. State the rules for differentiating and integrating vector functions. Give examples.
2. How do you define and calculate the velocity, speed, direction of motion, and acceleration of a body moving along a sufficiently differentiable space curve? Give an example.
3. What is special about the derivatives of vector functions of constant length? Give an example.
4. What are the vector and parametric equations for ideal projectile motion? How do you find a projectile's maximum height, flight time, and range? Give examples.
5. How do you define and calculate the length of a segment of a smooth space curve? Give an example. What mathematical assumptions are involved in the definition?
6. How do you measure distance along a smooth curve in space from a preselected base point? Give an example.
7. What is a differentiable curve's unit tangent vector? Give an example.
8. Define curvature, circle of curvature (osculating circle), center of curvature, and radius of curvature for twice-differentiable curves in the plane. Give examples. What curves have zero curvature? constant curvature?
9. What is a plane curve's principal normal vector? When is it defined? Which way does it point? Give an example.
10. How do you define $\mathbf{N}$ and κ for curves in space? How are these quantities related? Give examples.
11. What is a curve's binormal vector? Give an example. How is this vector related to the curve's torsion? Give an example.
12. What formulas are available for writing a moving body's acceleration as a sum of its tangential and normal components? Give an example. Why might one want to write the acceleration this way? What if the body moves at a constant speed? at a constant speed around a circle?
13. State Kepler's laws. To what do they apply?

CHAPTER 11 PRACTICE EXERCISES

Motion in a Cartesian Plane

In Exercises 1 and 2, graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\mathbf{a}$ in the form $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$, and find the value of κ at the given values of t.

1. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (\sqrt{2}\sin t)\mathbf{j}, \quad t = 0$ and $\pi/4$
2. $\mathbf{r}(t) = (\sqrt{3}\sec t)\mathbf{i} + (\sqrt{3}\tan t)\mathbf{j}, \quad t = 0$
3. The position of a particle in the plane at time t is
$$\mathbf{r} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\mathbf{j}.$$
Find the particle's highest speed.
4. Suppose $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j}$. Show that the angle between $\mathbf{r}$ and $\mathbf{a}$ never changes. What *is* the angle?
5. At point P, the velocity and acceleration of a particle moving in the plane are $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j}$. Find the curvature of the particle's path at P.
6. Find the point on the curve $y = e^x$ where the curvature is greatest.
7. A particle moves around the unit circle in the xy-plane. Its position at time t is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t. Find dy/dt if $\mathbf{v}\cdot\mathbf{i} = y$. Is the motion clockwise, or counterclockwise?
8. You send a message through a pneumatic tube that follows the

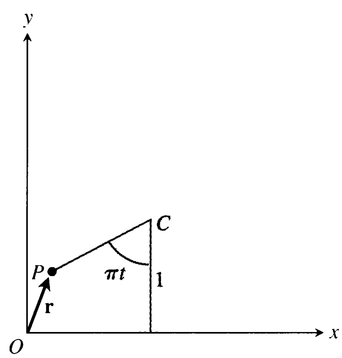
curve $9y = x^3$ (distance in meters). At the point $(3, 3)$, $\mathbf{v} \cdot \mathbf{i} = 4$ and $\mathbf{a} \cdot \mathbf{i} = -2$. Find the values of $\mathbf{v} \cdot \mathbf{j}$ and $\mathbf{a} \cdot \mathbf{j}$ at $(3, 3)$.

9. A particle moves in the plane so that its velocity and position vectors are always orthogonal. Show that the particle moves in a circle centered at the origin.

10. A circular wheel with radius 1 ft and center C rolls to the right along the x-axis at a half-turn per second. (See accompanying figure.) At time t seconds, the position vector of the point P on the wheel's circumference is

$$\mathbf{r} = (\pi t - \sin \pi t)\,\mathbf{i} + (1 - \cos \pi t)\,\mathbf{j}.$$

a) Sketch the curve traced by P during the interval $0 \le t \le 3$.
b) Find $\mathbf{v}$ and $\mathbf{a}$ at $t = 0, 1, 2$, and 3 and add these vectors to your sketch.
c) At any given time, what is the forward speed of the topmost point of the wheel? of C?



Projectile Motion and Motion in a Plane

11. *Shot put.* A shot leaves the thrower's hand 6.5 ft above the ground at a 45° angle at 44 ft/sec. Where is it 3 sec later?

12. *Javelin.* A javelin leaves the thrower's hand 7 ft above the ground at a 45° angle at 80 ft/sec. How high does it go?

13. A golf ball is hit with an initial speed v_0 at an angle α to the horizontal from a point that lies at the foot of a straight-sided hill that is inclined at an angle ϕ to the horizontal, where

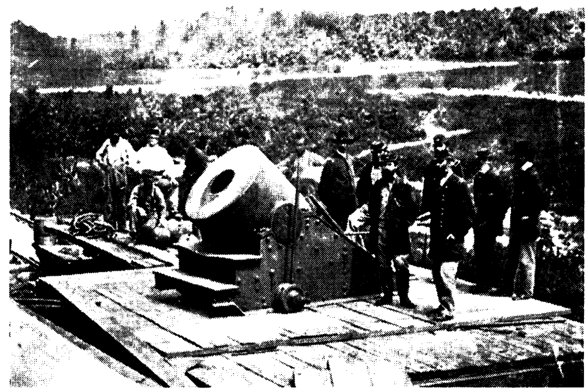
$$0 < \phi < \alpha < \frac{\pi}{2}.$$

Show that the ball lands at a distance

$$\frac{2v_0{}^2 \cos \alpha}{g \cos^2 \phi} \sin(\alpha - \phi),$$

measured up the face of the hill. Hence, show that the greatest range that can be achieved for a given v_0 occurs when $\alpha = (\phi/2) + (\pi/4)$, i.e., when the initial velocity vector bisects the angle between the vertical and the hill.

14. *The Dictator.* The Civil War mortar Dictator weighed so much (17,120 lb) that it had to be mounted on a railroad car. It had a 13-in. bore and used a 20-lb powder charge to fire a 200-lb shell. The mortar was made by Mr. Charles Knapp in his ironworks in Pittsburgh, Pennsylvania, and was used by the Union army in 1864 in the siege of Petersburg, Virginia. How far did it shoot? Here we have a difference of opinion. The ordnance manual claimed 4325 yd, while field officers claimed 4752 yd. Assuming a 45° firing angle, what muzzle speeds are involved here?



15. *The world's record for popping a champagne cork*

a) Until 1988, the world's record for popping a champagne cork was 109 ft. 6 in., once held by Captain Michael Hill of the British Royal Artillery (of course). Assuming Cpt. Hill held the bottle neck at ground level at a 45° angle, and the cork behaved like an ideal projectile, how fast was the cork going as it left the bottle?
b) A new world record of 177 ft. 9 in. was set on June 5, 1988, by Prof. Emeritus Heinrich of Rensselaer Polytechnic Institute, firing from 4 ft. above ground level at the Woodbury Vineyards Winery, New York. Assuming an ideal trajectory, what was the cork's initial speed?

16. *Javelin.* In Potsdam in 1988, Petra Felke of (then) East Germany set a women's world record by throwing a javelin 262 ft 5 in.

a) Assuming that Felke launched the javelin at a 40° angle to the horizontal 6.5 ft above the ground, what was the javelin's initial speed?
b) How high did the javelin go?

17. *Synchronous curves.* By eliminating α from the ideal projectile equations

$$x = (v_0 \cos \alpha)t, \qquad y = (v_0 \sin \alpha)t - \frac{1}{2} g t^2,$$

show that $x^2 + (y + gt^2/2)^2 = v_0{}^2 t^2$. This shows that projectiles launched simultaneously from the origin at the same initial speed will, at any given instant, all lie on the circle of radius $v_0 t$ centered at $(0, -gt^2/2)$, regardless of their launch angle. These circles are the *synchronous curves* of the launching.

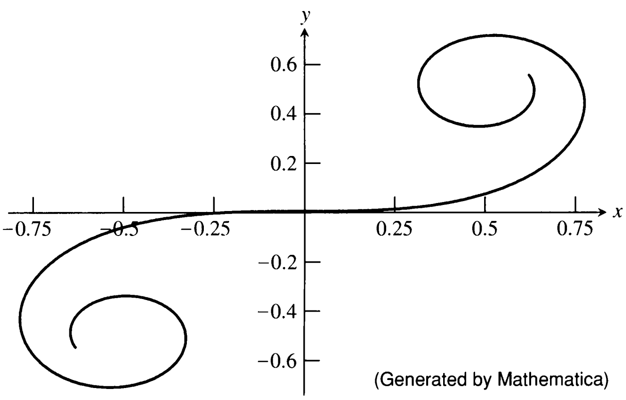
18. Show that the radius of curvature of a twice-differentiable plane curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is given by the formula

$$\rho = \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2}}, \quad \text{where } \ddot{s} = \frac{d}{dt}\sqrt{\dot{x}^2 + \dot{y}^2}.$$

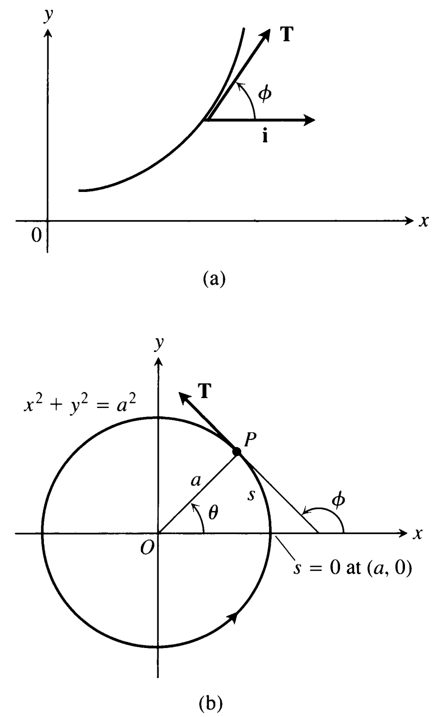
19. Express the curvature of the curve

$$\mathbf{r}(t) = \left(\int_0^t \cos\left(\frac{1}{2}\pi\theta^2\right) d\theta\right)\mathbf{i} + \left(\int_0^t \sin\left(\frac{1}{2}\pi\theta^2\right) d\theta\right)\mathbf{j}$$

as a function of the directed distance s measured along the curve from the origin (Fig. 11.44).



11.44 The curve in Exercise 19.



11.45 Figures for Exercise 20.

20. *An alternative definition of curvature in the plane.* An alternative definition gives the curvature of a sufficiently differentiable plane curve to be $|d\phi/ds|$, where ϕ is the angle between **T** and **i** (Fig. 11.45(a)). Figure 11.45(b) shows the distance s measured counterclockwise around the circle $x^2 + y^2 = a^2$ from the point $(a, 0)$ to a point P, along with the angle ϕ at P. Calculate the circle's curvature using the alternative definition. (*Hint*: $\phi = \theta + \pi/2$.)

Motion in Space

Find the lengths of the curves in Exercises 21 and 22.

21. $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + t^2\,\mathbf{k}, \quad 0 \le t \le \pi/4$

22. $\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + 2t^{3/2}\,\mathbf{k}, \quad 0 \le t \le 3$

In Exercises 23–26, find **T**, **N**, **B**, κ, and τ at the given value of t.

23. $\mathbf{r}(t) = \frac{4}{9}(1+t)^{3/2}\,\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\,\mathbf{j} + \frac{1}{3}t\,\mathbf{k}, \quad t = 0$

24. $\mathbf{r}(t) = (e^t \sin 2t)\,\mathbf{i} + (e^t \cos 2t)\,\mathbf{j} + 2e^t\,\mathbf{k}, \quad t = 0$

25. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}e^{2t}\,\mathbf{j}, \quad t = \ln 2$

26. $\mathbf{r}(t) = (3\cosh 2t)\,\mathbf{i} + (3\sinh 2t)\,\mathbf{j} + 6t\,\mathbf{k}, \quad t = \ln 2$

In Exercises 27 and 28, write **a** in the form $\mathbf{a} = a_{\mathrm{T}}\,\mathbf{T} + a_{\mathrm{N}}\,\mathbf{N}$ at $t = 0$ without finding **T** and **N**.

27. $\mathbf{r}(t) = (2 + 3t + 3t^2)\,\mathbf{i} + (4t + 4t^2)\,\mathbf{j} - (6\cos t)\,\mathbf{k}$

28. $\mathbf{r}(t) = (2 + t)\,\mathbf{i} + (t + 2t^2)\,\mathbf{j} + (1 + t^2)\,\mathbf{k}$

29. Find **T**, **N**, **B**, κ, and τ as functions of t if $\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\sqrt{2}\cos t)\,\mathbf{j} + (\sin t)\,\mathbf{k}$.

30. At what times in the interval $0 \le t \le \pi$ are the velocity and acceleration vectors of the motion $\mathbf{r}(t) = \mathbf{i} + (5\cos t)\,\mathbf{j} + (3\sin t)\,\mathbf{k}$ orthogonal?

31. The position of a particle moving in space at time $t \ge 0$ is

$$\mathbf{r}(t) = 2\,\mathbf{i} + \left(4\sin\frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k}.$$

Find the first time **r** is orthogonal to the vector $\mathbf{i} - \mathbf{j}$.

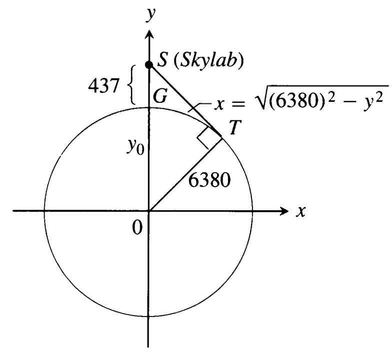
32. Find equations for the osculating, normal, and rectifying planes of the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ at the point $(1, 1, 1)$.

33. Find parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + (\sin t)\,\mathbf{j} + \ln(1 - t)\,\mathbf{k}$ at $t = 0$.

34. Find parametric equations for the line tangent to the helix $\mathbf{r}(t) = (\sqrt{2}\cos t)\,\mathbf{i} + (\sqrt{2}\sin t)\,\mathbf{j} + t\,\mathbf{k}$ at the point where $t = \pi/4$.

35. *The view from* Skylab 4. What percentage of Earth's surface area could the astronauts see when *Skylab 4* was at its apogee height, 437 km above the surface? To find out, model the visible surface as the surface generated by revolving the circular arc GT, shown here, about the y-axis. Then carry out these steps:

1. Use similar triangles in the figure to show that $y_0/6380 = 6380/(6380 + 437)$. Solve for y_0.
2. To four significant digits, calculate the visible area as
$$VA = \int_{y_0}^{6380} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$
3. Express the result as a percentage of Earth's surface area.



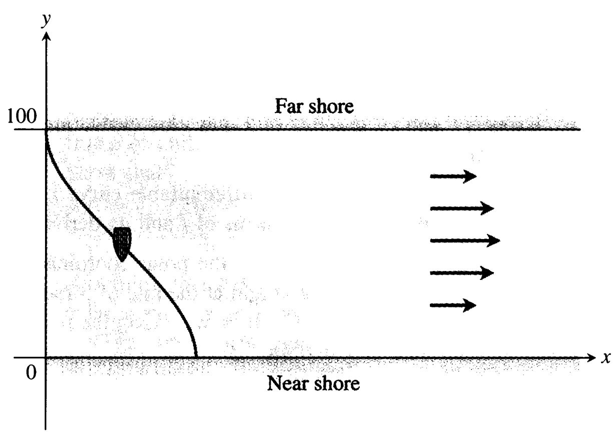
Chapter 11 Additional Exercises–Theory, Examples, Applications

Applications

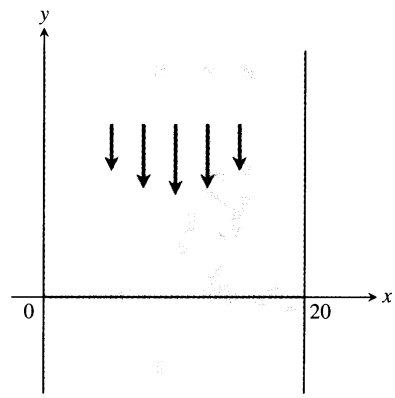
1. A straight river is 100 m wide. A rowboat leaves the far shore at time $t = 0$. The person in the boat rows at a rate of 20 m/min, always toward the near shore. The velocity of the river at (x, y) is $\mathbf{v} = \left(-\frac{1}{250}(y - 50)^2 + 10\right)\mathbf{i}$ m/min, $0 < y < 100$.

   a) Given that $\mathbf{r}(0) = 0\mathbf{i} + 100\mathbf{j}$, what is the position of the boat at time t?

   b) How far downstream will the boat land on the near shore?



2. A straight river is 20 m wide. The velocity of the river at (x, y) is
$$\mathbf{v} = -\frac{3x(20 - x)}{100}\mathbf{j} \text{ m/min, } 0 \le x \le 20.$$
A boat leaves the shore at (0, 0) and travels through the water with a constant velocity. It arrives at the opposite shore at (20, 0). The speed of the boat is always $\sqrt{20}$ m/min.

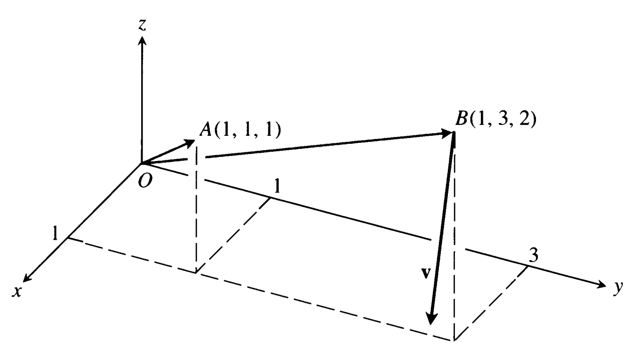


   a) Find the velocity of the boat.

   b) Find the location of the boat at time t.

   c) Sketch the path of the boat.

3. The line through the origin and the point $A(1, 1, 1)$ is the axis of rotation of a rigid body rotating with a constant angular speed of 3/2 rad/sec. The rotation appears to be clockwise when we look toward the origin from A. Find the velocity $\mathbf{v}$ of the point of the body that is at the position B (1, 3, 2).



4. The curve

$$\mathbf{r}(t) = (2\sqrt{t}\cos t)\,\mathbf{i} + (3\sqrt{t}\sin t)\,\mathbf{j} + \sqrt{1-t}\,\mathbf{k}, \quad 0 \le t \le 1,$$

lies on a quadric surface. Describe the surface and find an equation for it.

5. A frictionless particle P, starting from rest at time $t = 0$ at the point $(a, 0, 0)$, slides down the helix

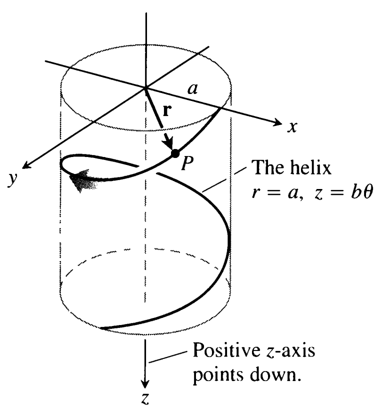
$$\mathbf{r}(\theta) = (a\cos\theta)\,\mathbf{i} + (a\sin\theta)\,\mathbf{j} + b\theta\,\mathbf{k} \quad (a, b > 0)$$

under the influence of gravity, as in Fig. 11.46. The θ in this equation is the cylindrical coordinate θ and the helix is the curve $r = a, z = b\theta, \theta \ge 0$, in cylindrical coordinates. We assume θ to be a differentiable function of t for the motion. The law of conservation of energy tells us that the particle's speed after it has fallen straight down a distance z is $\sqrt{2gz}$, where g is the constant acceleration of gravity.

a) Find the angular velocity $d\theta/dt$ when $\theta = 2\pi$.

b) Express the particle's θ- and z-coordinates as functions of t.

c) Express the tangential and normal components of the velocity $d\mathbf{r}/dt$ and acceleration $d^2\mathbf{r}/dt^2$ as functions of t. Does the acceleration have any nonzero component in the direction of the binormal vector $\mathbf{B}$?



11.46 The circular helix in Exercise 5.

6. Suppose the curve in Exercise 5 is replaced by the conical helix $r = a\theta, z = b\theta$ shown in Fig. 11.47.

a) Express the angular velocity $d\theta/dt$ as a function of θ.

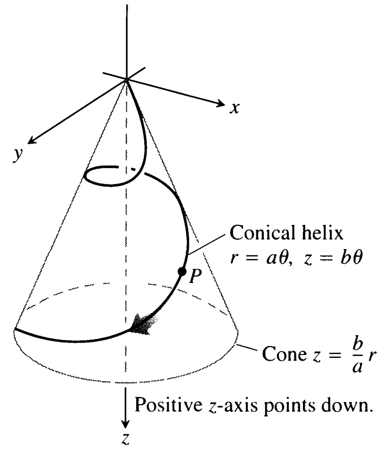
b) Express the distance the particle travels along the helix as a function of θ.

Polar Coordinate Systems and Motion in Space

7. Deduce from the orbit equation

$$r = \frac{(1+e)r_0}{1 + e\cos\theta}$$

that a planet is closest to its sun when $\theta = 0$ and show that $r = r_0$ at that time.



11.47 The conical helix in Exercise 6.

8. *A Kepler equation.* The problem of locating a planet in its orbit at a given time and date eventually leads to solving "Kepler" equations of the form

$$f(x) = x - 1 - \frac{1}{2}\sin x = 0.$$

a) Show that this particular equation has a solution between $x = 0$ and $x = 2$.

b) With your computer or calculator in radian mode, use Newton's method to find the solution to as many places as you can.

9. In Section 11.5, we found the velocity of a particle moving in the plane to be

$$\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta.$$

a) Express $\dot{x}$ and $\dot{y}$ in terms of $\dot{r}$ and $r\dot{\theta}$ by evaluating the dot products $\mathbf{v} \cdot \mathbf{i}$ and $\mathbf{v} \cdot \mathbf{j}$.

b) Express $\dot{r}$ and $r\dot{\theta}$ in terms of $\dot{x}$ and $\dot{y}$ by evaluating the dot products $\mathbf{v} \cdot \mathbf{u}_r$ and $\mathbf{v} \cdot \mathbf{u}_\theta$.

10. Express the curvature of a twice-differentiable curve $r = f(\theta)$ in the polar coordinate plane in terms of f and its derivatives.

11. A slender rod through the origin of the polar coordinate plane rotates (in the plane) about the origin at the rate of 3 rad/min. A beetle starting from the point (2, 0) crawls along the rod toward the origin at the rate of 1 in./min.

a) Find the beetle's acceleration and velocity in polar form when it is halfway to (1 in. from) the origin.

b) CALCULATOR To the nearest tenth of an inch, what will be the length of the path the beetle has traveled by the time it reaches the origin?

12. *Conservation of angular momentum.* Let $\mathbf{r}(t)$ denote the position in space of a moving object at time t. Suppose the force acting on the object at time t is

$$\mathbf{F}(t) = -\frac{c}{|\mathbf{r}(t)|^3}\mathbf{r}(t),$$

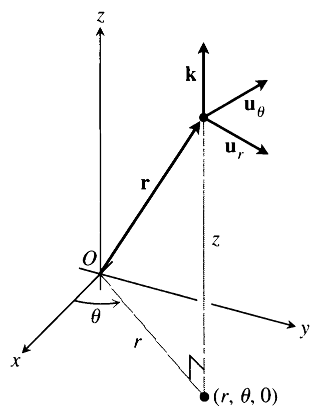
where c is a constant. In physics the **angular momentum** of an object at time t is defined to be $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$, where m is the mass of the object and $\mathbf{v}(t)$ is the velocity. Prove that angular momentum is a conserved quantity; i.e., prove that $\mathbf{L}(t)$ is a constant vector, independent of time. Remember Newton's law $\mathbf{F} = m\mathbf{a}$. (This is a calculus problem, not a physics problem.)

Cylindrical Coordinate Systems

13. *Unit vectors for position and motion in cylindrical coordinates.* When the position of a particle moving in space is given in cylindrical coordinates, the unit vectors we use to describe its position and motion are

$$\mathbf{u}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}, \quad \mathbf{u}_\theta = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j},$$

and $\mathbf{k}$ (Fig. 11.48). The particle's position vector is then $\mathbf{r} = r\,\mathbf{u}_r + z\,\mathbf{k}$, where r is the positive polar distance coordinate of the particle's position.



11.48 The unit vectors for describing motion in cylindrical coordinates (Exercise 13).

a) Show that $\mathbf{u}_r, \mathbf{u}_\theta$, and $\mathbf{k}$, in this order, form a right-handed frame of unit vectors.

b) Show that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \quad \text{and} \quad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r.$$

c) Assuming that the necessary derivatives with respect to t exist, express $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \ddot{\mathbf{r}}$ in terms of $\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{k}, \dot{r}$, and $\dot{\theta}$. (The dots indicate derivatives with respect to t: $\dot{\mathbf{r}}$ means $d\mathbf{r}/dt$, $\ddot{\mathbf{r}}$ means $d^2\mathbf{r}/dt^2$, and so on.) Section 11.5 derives these formulas and shows how the vectors mentioned here are used in describing planetary motion.

14. *Arc length in cylindrical coordinates*

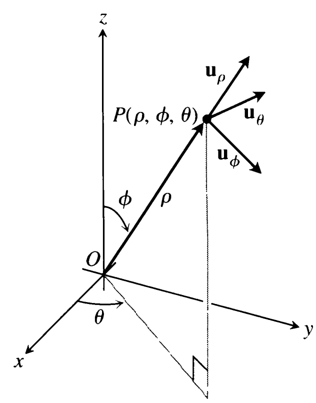
a) Show that when you express $ds^2 = dx^2 + dy^2 + dz^2$ in terms of cylindrical coordinates, you get $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

b) Interpret this result geometrically in terms of the edges and a diagonal of a box. Sketch the box.

c) Use the result in (a) to find the length of the curve $r = e^\theta, z = e^\theta, 0 \le \theta \le \ln 8$.

Spherical Coordinate Systems

15. *Unit vectors for position and motion in spherical coordinates.* Hold two of the three spherical coordinates ρ, ϕ, θ of a point P in space constant while letting the remaining coordinate increase. Let $\mathbf{u}$, with a subscript corresponding to the increasing coordinate, be the unit vector that points in the direction in which P then starts to move. The three resulting unit vectors, $\mathbf{u}_\rho, \mathbf{u}_\phi$, and $\mathbf{u}_\theta$ at P are shown in Fig. 11.49.



11.49 The unit vectors for describing motion in spherical coordinates (Exercise 15).

a) Express $\mathbf{u}_\rho, \mathbf{u}_\phi$, and $\mathbf{u}_\theta$ in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

b) Show that $\mathbf{u}_\rho \cdot \mathbf{u}_\phi = 0$.

c) Show that $\mathbf{u}_\theta = \mathbf{u}_\rho \times \mathbf{u}_\phi$.

d) Show that $\mathbf{u}_\rho, \mathbf{u}_\phi$, and $\mathbf{u}_\theta$, in that order, make a right-handed frame of mutually orthogonal vectors.

16. *Arc length in spherical coordinates*

a) Show that when you express $ds^2 = dx^2 + dy^2 + dz^2$ in terms of spherical coordinates, you get $ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi\, d\theta^2$.

b) Interpret this result geometrically in terms of the edges and a diagonal of a "box" cut from a solid sphere. Sketch the box.

c) Use the result in (a) to find the length of the curve $\rho = 2e^\theta, \phi = \pi/6,\ 0 \le \theta \le \ln 8$.

CHAPTER 12

Multivariable Functions and Partial Derivatives

OVERVIEW Functions with two or more independent variables appear more often in science than functions of a single variable, and their calculus is even richer. Their derivatives are more varied and more interesting because of the different ways in which the variables can interact. Their integrals lead to a greater variety of applications. The studies of probability, statistics, fluid dynamics, and electricity, to mention only a few, all lead in natural ways to functions of more than one variable. The mathematics of these functions is one of the finest achievements in science.

12.1 Functions of Several Variables

Many functions depend on more than one independent variable. The function $V = \pi r^2 h$ calculates the volume of a right circular cylinder from its radius and height. The function $f(x, y) = x^2 + y^2$ calculates the height of the paraboloid $z = x^2 + y^2$ above the point $P(x, y)$ from the two coordinates of P. In this section, we define functions of more than one independent variable and discuss ways to graph them.

Functions and Variables

Real-valued functions of several independent real variables are defined much the way you would imagine from the single-variable case. The domains are sets of ordered pairs (triples, quadruples, whatever) of real numbers, and the ranges are sets of real numbers of the kind we have worked with all along.

Definitions

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain.** The set of w-values taken on by f is the function's **range.** The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x's the function's **input variables** and call w the function's **output variable.**

If f is a function of two independent variables, we usually call the independent variables x and y and picture the domain of f as a region in the xy-plane. If f is a function of three independent variables, we call the variables x, y, and z and picture the domain as a region in space.

In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r, h)$. To be more specific, we might replace the notation $f(r, h)$ by the formula that calculates the value of V from the values of r and h, and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variable of the function.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable.

EXAMPLE 1 The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (3, 0, 4) is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5. \qquad \square$$

Domains

In defining functions of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}$, y cannot be less than x^2. If $f(x, y) = 1/(xy)$, xy cannot be zero. The domains of functions are otherwise assumed to be the largest sets for which the defining rules generate real numbers.

EXAMPLE 2 *Functions of two variables*

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

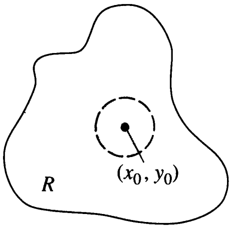
$\square$

EXAMPLE 3 *Functions of three variables*

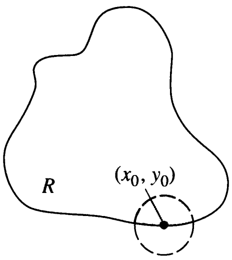
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

$\square$

The domains of functions defined on portions of the plane can have interior points and boundary points just the way the domains of functions defined on intervals of the real line can.

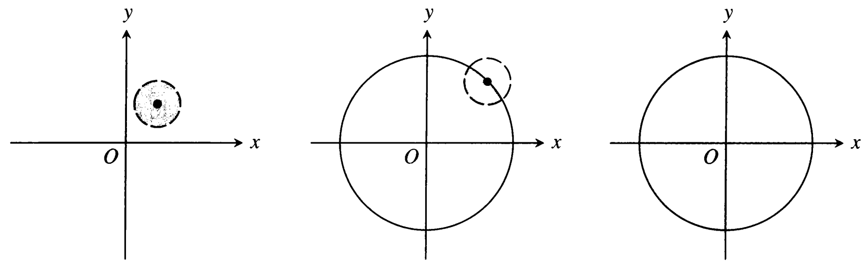


(a) Interior point



(b) Boundary point

12.1 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk. Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk. Contains all boundary points.

12.2 Interior points and boundary points of the unit disk in the plane.

Definitions

A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk that lies entirely in R (Fig. 12.1). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary.** A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all of its boundary points (Fig. 12.2).

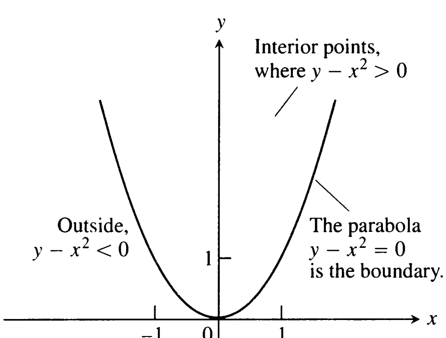
As with intervals of real numbers, some regions in the plane are neither open nor closed. If you start with the open disk in Fig. 12.2 and add to it some but not all of its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

Definitions

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

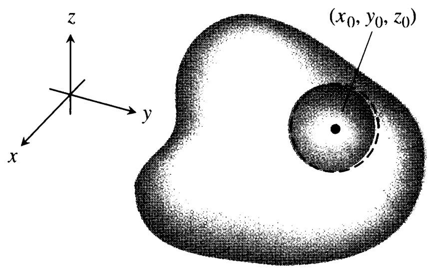
EXAMPLE 4

Bounded sets in the plane:	Line segments, triangles, interiors of triangles, rectangles, disks
Unbounded sets in the plane:	Lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, the plane itself

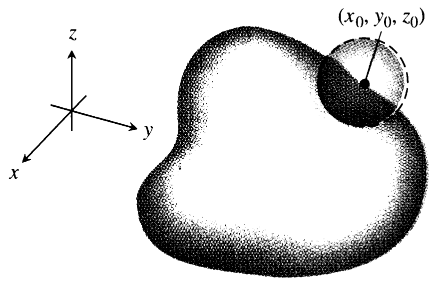


12.3 The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$.

EXAMPLE 5 The domain of the function $f(x, y) = \sqrt{y - x^2}$ is closed and unbounded (Fig. 12.3). The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior.



(a) Interior point



(b) Boundary point

12.4 **Interior points and boundary points of a region in space.**

The definitions of interior, boundary, open, closed, bounded, and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use balls instead of disks. A **closed ball** consists of the region of points inside a sphere together with the sphere. An **open ball** is the region of points inside a sphere without the bounding sphere.

Definitions

A point (x_0, y_0, z_0) in a region D in space is an **interior point** of D if it is the center of a ball that lies entirely in D (Fig. 12.4). A point (x_0, y_0, z_0) is a **boundary point** of D if every sphere centered at (x_0, y_0, z_0) encloses points that lie outside D as well as points that lie inside D. The **interior** of D is the set of interior points of D. The **boundary** of D is the set of boundary points of D.

A region D is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

EXAMPLE 6

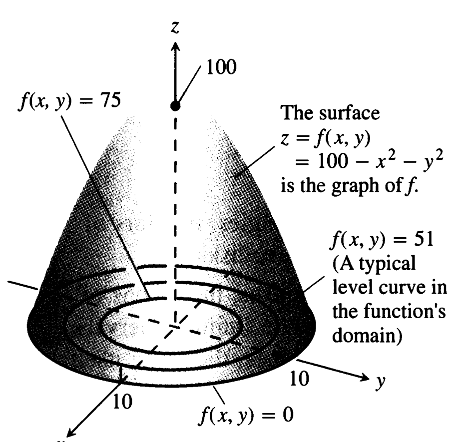
Open sets in space:	Open balls; the open half-space $z > 0$; the first octant (bounding planes absent); space itself
Closed sets in space:	Lines; planes; closed balls; the closed half-space $z \geq 0$; the first octant together with its bounding planes; space itself
Neither open nor closed:	A closed ball with part of its bounding sphere removed; solid cube with a missing face, edge, or corner point ❑

Graphs and Level Curves of Functions of Two Variables

There are two standard ways to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x, y)$ in space.

Definitions

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface $z = f(x,y)$.**



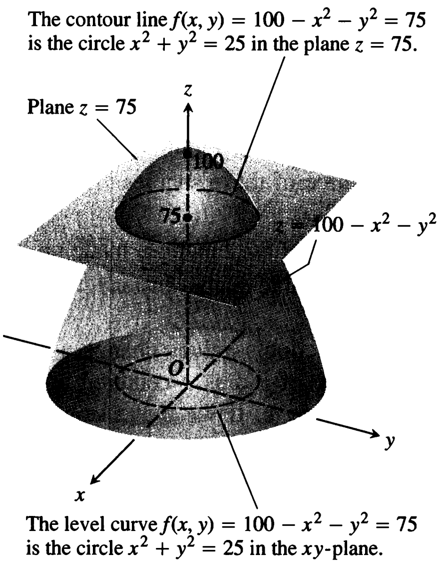
12.5 **The graph and selected level curves of the function $f(x, y) = 100 - x^2 - y^2$.**

EXAMPLE 7 Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, a portion of which is shown in Fig. 12.5.

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$



12.6 The graph of $f(x, y) = 100 - x^2 - y^2$ and its intersection with the plane $z = 75$.

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Fig. 12.5) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \quad \text{or} \quad x^2 + y^2 = 49,$$
$$f(x, y) = 100 - x^2 - y^2 = 75, \quad \text{or} \quad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.) ❑

Contour Lines

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour line** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f. Figure 12.6 shows the contour line $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour line lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

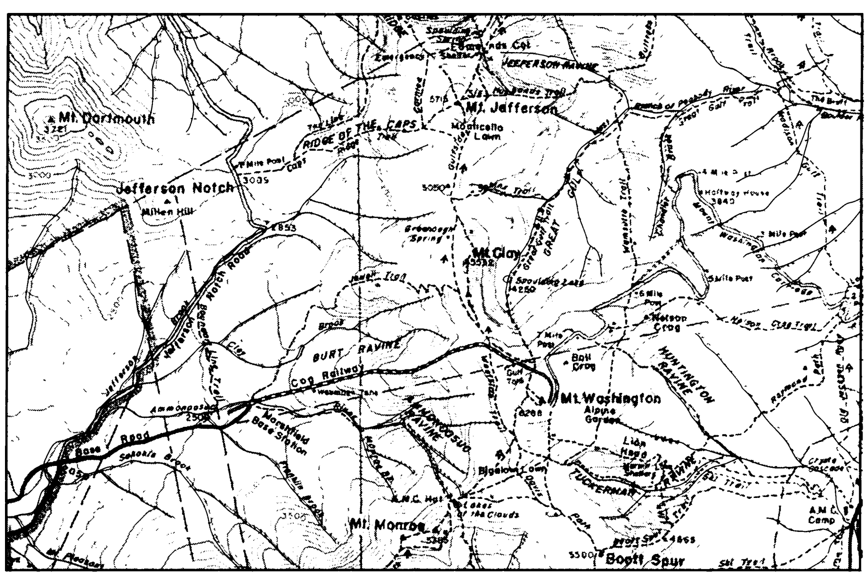
Not everyone makes this distinction, however, and you may wish to call both kinds of curves by a single name and rely on context to convey which one you have in mind. On most maps, for example, the curves that represent constant elevation (height above sea level) are called contours, not level curves (Fig. 12.7).

Level Surfaces of Functions of Three Variables

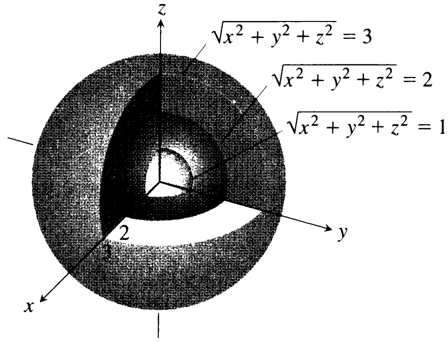
In the plane, the points where a function of two independent variables has a constant value $f(x, y) = c$ make a curve in the function's domain. In space, the points where a function of three independent variables has a constant value $f(x, y, z) = c$ make a surface in the function's domain.

Definition

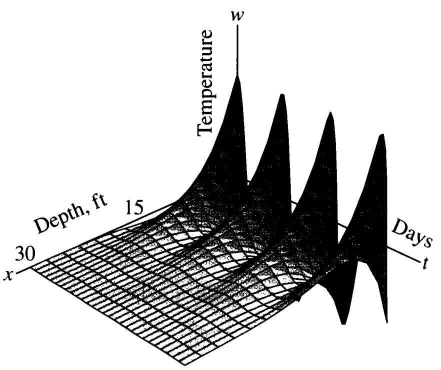
The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.



12.7 Contours on Mt. Washington in north central New Hampshire. The streams, which follow paths of steepest descent, run perpendicular to the contours. So does the Cog Railway.



12.8 The level surfaces of $f(x, y, z) = \sqrt{x^2+y^2+z^2}$ are concentric spheres.



(Generated by Mathematica)

12.9 This computer-generated graph of

$$w = \cos(1.7 \times 10^{-2}t - 0.2x)e^{-0.2x}$$

shows the seasonal variation of the temperature below ground as a fraction of surface temperature. At $x = 15$ ft the variation is only 5% of the variation at the surface. At $x = 30$ ft the variation is less than 0.25% of the surface variation. (Adapted from art provided by Norton Starr for G. C. Berresford's "Differential Equations and Root Cellars," *The UMAP Journal*, Vol. 2, No. 3 [1981], pp. 53–75.)

EXAMPLE 8 Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2+y^2+z^2}.$$

Solution The value of f is the distance from the origin to the point (x, y, z). Each level surface $\sqrt{x^2+y^2+z^2} = c$, $c > 0$, is a sphere of radius c centered at the origin. Figure 12.8 shows a cutaway view of three of these spheres. The level surface $\sqrt{x^2+y^2+z^2} = 0$ consists of the origin alone.

We are not graphing the function here. The graph of the function, made up of the points $(x, y, z, \sqrt{x^2+y^2+z^2})$, lies in a four-variable space. Instead, we are looking at level surfaces in the function's domain.

The function's level surfaces show how the function's values change as we move through its domain. If we remain on a sphere of radius c centered at the origin, the function maintains a constant value, namely c. If we move from one sphere to another, the function's value changes. It increases if we move away from the origin and decreases if we move toward the origin. The way the function's values change depends on the direction we take. The dependence of change on direction is important. We will return to it in Section 12.7. ❑

Computer Graphing

The three-dimensional graphing programs for computers make it possible to graph functions of two variables with only a few keystrokes. We can often get information more quickly from a graph than from a formula.

EXAMPLE 9 Figure 12.9 shows a computer-generated graph of the function $w = \cos(1.7 \times 10^{-2}t - 0.2x)\,e^{-0.2x}$, where t is in days and x is in feet. The graph shows how the temperature beneath the earth's surface varies with time. The variation is given as a fraction of the variation at the surface. At a depth of 15 ft, the variation (change in vertical amplitude in the figure) is about 5 percent of the surface variation. At 30 ft, there is almost no variation during the year.

The graph also shows that the temperature 15 ft below the surface is about half a year out of phase with the surface temperature. When the temperature is lowest on the surface (late January, say) it is at its highest 15 ft below. Fifteen feet below the ground, the seasons are reversed. ❑

Exercises 12.1

Domain, Range, and Level Curves

In Exercises 1–12, (a) find the function's domain, (b) find the function's range, (c) describe the function's level curves, (d) find the boundary of the function's domain, (e) determine if the domain is an open region, a closed region, or neither, and (f) decide if the domain is bounded or unbounded.

1. $f(x, y) = y - x$

2. $f(x, y) = \sqrt{y - x}$

3. $f(x, y) = 4x^2 + 9y^2$

4. $f(x, y) = x^2 - y^2$

5. $f(x, y) = xy$

6. $f(x, y) = y/x^2$

7. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$

8. $f(x, y) = \sqrt{9 - x^2 - y^2}$

9. $f(x, y) = \ln(x^2 + y^2)$

10. $f(x, y) = e^{-(x^2+y^2)}$

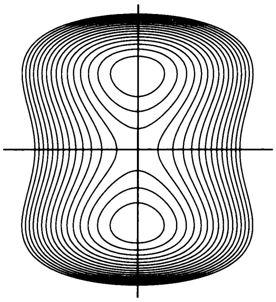
11. $f(x, y) = \sin^{-1}(y - x)$

12. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

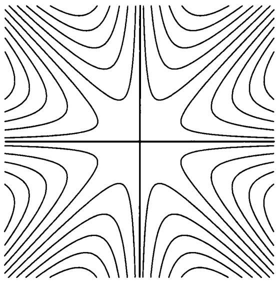
Identifying Surfaces and Level Curves

Exercises 13–18 show level curves for the functions graphed in (a)–(f). Match each set of curves with the appropriate function.

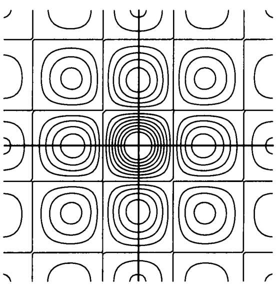
13.



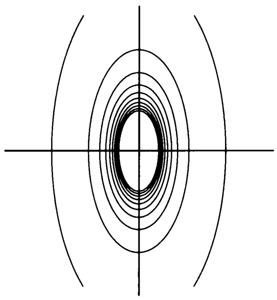
14.



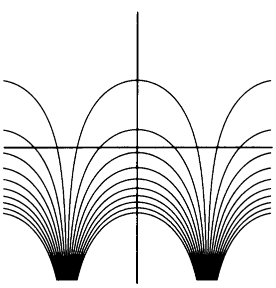
15.



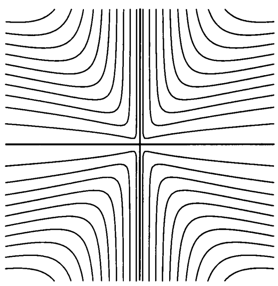
16.



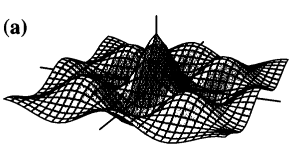
17.



18.



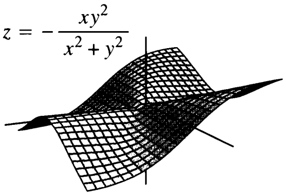
(a)



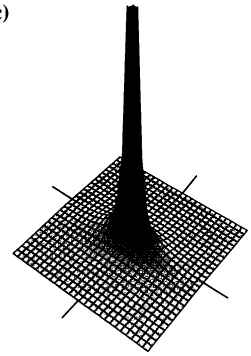
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$

(b)

$z = -\dfrac{xy^2}{x^2 + y^2}$

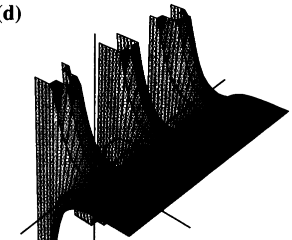


(c)



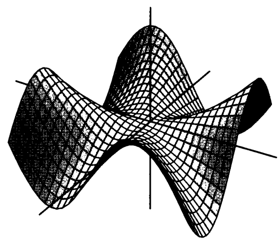
$z = \dfrac{1}{(4x^2 + y^2)}$

(d)



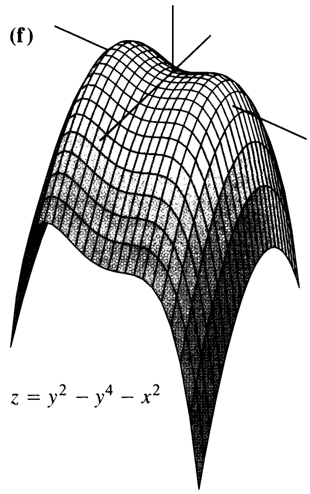
$z = e^{-y} \cos x$

(e)



$z = \dfrac{xy(x^2 - y^2)}{(x^2 + y^2)}$

(f)



$z = y^2 - y^4 - x^2$

Identifying Functions of Two Variables

Display the values of the functions in Exercises 19–28 in two ways: (a) by sketching the surface $z = f(x, y)$ and (b) by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

19. $f(x, y) = y^2$

20. $f(x, y) = 4 - y^2$

21. $f(x, y) = x^2 + y^2$

22. $f(x, y) = \sqrt{x^2 + y^2}$

23. $f(x, y) = -(x^2 + y^2)$

24. $f(x, y) = 4 - x^2 - y^2$

25. $f(x, y) = 4x^2 + y^2$

26. $f(x, y) = 4x^2 + y^2 + 1$

27. $f(x, y) = 1 - |y|$

28. $f(x, y) = 1 - |x| - |y|$

Level Surfaces

In Exercises 29–36, sketch a typical level surface for the function.

29. $f(x, y, z) = x^2 + y^2 + z^2$

30. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

31. $f(x, y, z) = x + z$

32. $f(x, y, z) = z$

33. $f(x, y, z) = x^2 + y^2$

34. $f(x, y, z) = y^2 + z^2$

35. $f(x, y, z) = z - x^2 - y^2$

36. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Finding a Level Curve

In Exercises 37–40, find an equation for the level curve of the function $f(x, y)$ that passes through the given point.

37. $f(x, y) = 16 - x^2 - y^2$, $\left(2\sqrt{2}, \sqrt{2}\right)$

38. $f(x, y) = \sqrt{x^2 - 1}$, $(1, 0)$

39. $f(x, y) = \displaystyle\int_x^y \frac{dt}{1 + t^2}$, $\left(-\sqrt{2}, \sqrt{2}\right)$

40. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$, $(1, 2)$

Finding a Level Surface

In Exercises 41–44, find an equation for the level surface of the function through the given point.

41. $f(x, y, z) = \sqrt{x - y} - \ln z$, $(3, -1, 1)$

42. $f(x, y, z) = \ln(x^2 + y + z^2)$, $(-1, 2, 1)$

43. $g(x, y, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + y)^n}{n!\, z^n}$, $(\ln 2, \ln 4, 3)$

44. $g(x, y, z) = \displaystyle\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2 - 1}}$, $(0, 1/2, 2)$

Theory and Examples

45. *The maximum value of a function on a line in space.* Does the function $f(x, y, z) = xyz$ have a maximum value on the line $x = 20 - t$, $y = t$, $z = 20$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t.)

46. *The minimum value of a function on a line in space.* Does the function $f(x, y, z) = xy - z$ have a minimum value on the line $x = t - 1$, $y = t - 2$, $z = t + 7$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t.)

47. *The Concorde's sonic booms.* The width w of the region in which people on the ground hear the *Concorde's* sonic boom directly, not reflected from a layer in the atmosphere, is a function of

$T =$ air temperature at ground level (in degrees Kelvin),

$h =$ the *Concorde*'s altitude (in km),

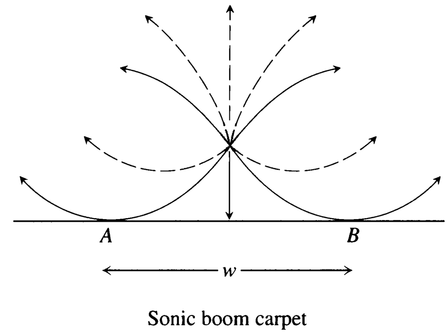
$d =$ the vertical temperature gradient (temperature drop in degrees Kelvin per km).

The formula for w is

$$w = 4(Th/d)^{1/2}.$$

See Fig. 12.10.

The Washington-bound *Concorde* approaches the United States from Europe on a course that takes it south of Nantucket Island at an altitude of 16.8 km. If the surface temperature is 290 K and the vertical temperature gradient is 5 K/km, how many kilometers south of Nantucket must the plane be flown to keep its sonic boom carpet away from the island? (From "Concorde Sonic Booms as an Atmospheric Probe" by N. K. Balachandra, W. L. Donn, and D. H. Rind, *Science,* July 1, 1977, Vol. 197, pp. 47–49).



12.10 Sound waves from the *Concorde* bend as the temperature changes above and below the altitude at which the plane flies. The sonic boom carpet is the region on the ground that receives shock waves directly from the plane, not reflected from the atmosphere or diffracted along the ground. The carpet is determined by the grazing rays striking the ground from the point directly under the plane (Exercise 47).

48. As you know, the graph of a real-valued function of a single real variable is a set in a two-coordinate space. The graph of a real-valued function of two independent real variables is a set in a three-coordinate space. The graph of a real-valued function of three independent real variables is a set in a four-coordinate space. How would you define the graph of a real-valued function $f(x_1, x_2, x_3, x_4)$ of four independent real variables? How would you define the graph of a real-valued function $f(x_1, x_2, x_3, \ldots, x_n)$ of n independent real variables?

CAS Explorations and Projects—Explicit Surfaces

Use a CAS to perform the following steps for each of the functions in Exercises 49–52.

a) Plot the surface over the given rectangle.

b) Plot several level curves in the rectangle.

c) Plot the level curve of f through the given point.

49. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x$, $0 \le x \le 5\pi$, $0 \le y \le 5\pi$, $P(3\pi, 3\pi)$

50. $f(x, y) = (\sin x)(\cos y)\, e^{\sqrt{x^2+y^2}/8}$, $0 \le x \le 5\pi$, $0 \le y \le 5\pi$, $P(4\pi, 4\pi)$

51. $f(x, y) = \sin(x + 2\cos y)$, $-2\pi \le x \le 2\pi$, $-2\pi \le y \le 2\pi$, $P(\pi, \pi)$

52. $f(x, y) = e^{(x^{0\,1} - y)} \sin(x^2 + y^2)$, $0 \le x \le 2\pi$, $-2\pi \le y \le \pi$, $P(\pi, -\pi)$

CAS Explorations and Projects—Implicit Surfaces

Use a CAS to plot the level surfaces in Exercises 53–56.

53. $4\ln(x^2+y^2+z^2)=1$ **54.** $x^2+z^2=1$

55. $x+y^2-3z^2=1$

56. $\sin\left(\dfrac{x}{2}\right)-(\cos y)\sqrt{x^2+z^2}=2$

CAS Explorations and Projects—Parametrized Surfaces

Just as you describe curves in the plane parametrically with a pair of equations $x=f(t)$, $y=g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x=f(u,v)$, $y=g(u,v)$, $z=h(u,v)$ defined on some parameter rectangle $a\le u\le b$, $c\le v\le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode.* (Parametrized surfaces are discussed in detail in Section 14.6.) Use a CAS to plot the surfaces in Exercises 57–60. Also plot several level curves in the xy-plane.

57. $x=u\cos v,\quad y=u\sin v,\quad z=u,\quad 0\le u\le 2,\quad 0\le v\le 2\pi$

58. $x=u\cos v,\quad y=u\sin v,\quad z=v,\quad 0\le u\le 2,\quad 0\le v\le 2\pi$

59. $x=(2+\cos u)\cos v,\quad y=(2+\cos u)\sin v,\quad z=\sin u,$
$0\le u\le 2\pi,\quad 0\le v\le 2\pi$

60. $x=2\cos u\cos v,\quad y=2\cos u\sin v,\quad z=2\sin u,$
$0\le u\le 2\pi,\quad 0\le v\le \pi$

12.2 Limits and Continuity

This section treats limits and continuity for multivariable functions.

Limits

If the values of $f(x,y)$ lie arbitrarily close to a fixed real number L for all points (x,y) sufficiently close to a point (x_0,y_0), we say that f approaches the limit L as (x,y) approaches (x_0,y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (x_0,y_0) lies in the interior of f's domain, (x,y) can approach (x_0,y_0) from any direction. The direction of approach can be an issue, as in some of the examples that follow.

Definition

We say that a function $f(x,y)$ approaches the **limit** L as (x,y) approaches (x_0,y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

if, for every number $\epsilon>0$, there exists a corresponding number $\delta>0$ such that for all (x,y) in the domain of f,

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \quad\Rightarrow\quad |f(x,y)-L|<\epsilon. \qquad (1)$$

The δ-ϵ requirement in the definition of limit is equivalent to the requirement that, given $\epsilon>0$, there exists a corresponding $\delta>0$ such that for all x,

$$0<|x-x_0|<\delta \ \text{ and } \ 0<|y-y_0|<\delta \quad\Rightarrow\quad |f(x,y)-L|<\epsilon \qquad (2)$$

(Exercise 59). Thus, in calculating limits we can think either in terms of distance in the plane or in terms of differences in coordinates.

The definition of limit applies to boundary points (x_0,y_0) as well as interior points of the domain of f. The only requirement is that the point (x,y) remain in the domain at all times.

It can be shown, as for functions of a single variable, that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0$$
$$\lim_{(x,y)\to(x_0,y_0)} y = y_0 \tag{3}$$
$$\lim_{(x,y)\to(x_0,y_0)} k = k. \qquad \text{(Any number } k\text{)}$$

It can also be shown that the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, products, constant multiples, quotients, and powers.

Theorem 1

Properties of Limits of Functions of Two Variables

The following rules hold if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim\,[f(x,y) + g(x,y)] = L + M$
2. *Difference Rule:* $\lim\,[f(x,y) - g(x,y)] = L - M$
3. *Product Rule:* $\lim\, f(x,y)\cdot g(x,y) = L\cdot M$
4. *Constant Multiple Rule:* $\lim\, kf(x,y) = kL$ (Any number k)
5. *Quotient Rule:* $\lim \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ if $M \neq 0$.
6. *Power Rule:* If m and n are integers, then $\lim\,[f(x,y)]^{m/n} = L^{m/n}$, provided $L^{m/n}$ is a real number.

All limits are to be taken as $(x,y)\to(x_0,y_0)$, and L and M are to be real numbers.

When we apply Theorem 1 to the limits in Eqs. (3), we obtain the useful result that the limits of polynomials and rational functions as $(x,y)\to(x_0,y_0)$ can be calculated by evaluating the functions at (x_0,y_0). The only requirement is that the functions be defined at (x_0,y_0).

EXAMPLE 1

a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$ ❑

EXAMPLE 2

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x,y)\to(0,0)$, we cannot use the Quotient Rule from Theorem 1. However, if we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, we produce an equivalent fraction whose limit we *can* find:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{\left(x^2 - xy\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\,(x - y)\left(\sqrt{x} + \sqrt{y}\right)}{x - y} \qquad \text{Algebra}$$

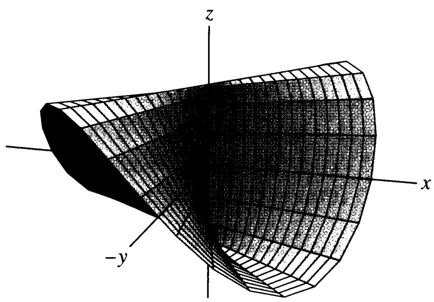
$$= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right) \qquad \text{Cancel the factor } (x - y).$$

$$= 0\left(\sqrt{0} + \sqrt{0}\right) = 0$$

❑

Continuity

As with functions of a single variable, continuity is defined in terms of limits.



(Generated by Mathematica)

(a)

Definitions

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f. The only requirement is that the point (x, y) remain in the domain at all times.

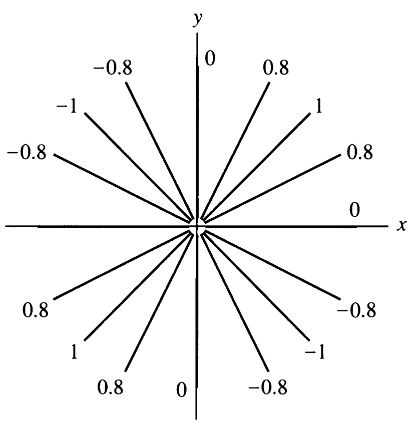
As you may have guessed, one of the consequences of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

If $z = f(x, y)$ is a continuous function of x and y, and $w = g(z)$ is a continuous function of z, then the composite $w = g(f(x, y))$ is continuous. Thus,

$$e^{x-y}, \qquad \cos\frac{xy}{x^2+1}, \qquad \ln\left(1 + x^2y^2\right)$$

are continuous at every point (x, y).

As with functions of a single variable, the general rule is that composites of continuous functions are continuous. The only requirement is that each function be continuous where it is applied.



(b)

12.11 (a) The graph of

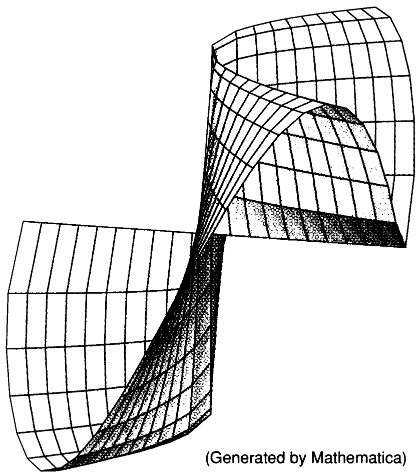
$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

The function is continuous at every point except the origin. (b) The level curves of f.

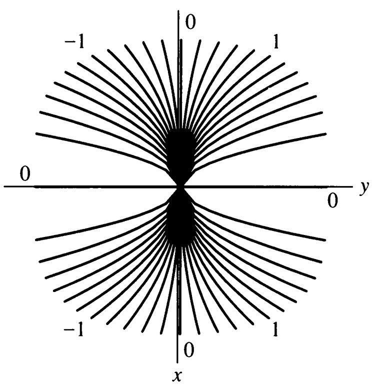
EXAMPLE 3 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Fig. 12.11).



(a)



(b)

12.12 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. As the graph suggests and the level-curve values in (b) confirm, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Solution The function f is continuous at any point $(x, y) \neq (0, 0)$ because its values are then given by a rational function of x and y.

At (0, 0) the value of f is defined, but f, we claim, has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we will now see.

For every value of m, the function f has a constant value on the "punctured" line $y = mx,\ x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches (0, 0) along the line:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous. ❑

Example 3 illustrates an important point about limits of functions of two variables (or even more variables, for that matter). For a limit to exist at a point, the limit must be the same along every approach path. Therefore, if we ever find paths with different limits, we know the function has no limit at the point they approach.

The Two-Path Test for the Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

EXAMPLE 4 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Fig. 12.12) has no limit as (x, y) approaches (0, 0).

Solution Along the curve $y = kx^2, x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches (0, 0) along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches (0, 0) along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches (0, 0).

The language here may seem contradictory. You might well ask, "What do you mean f has no limit as (x, y) approaches the origin—it has lots of limits." But that is the point. There is no single path-independent limit, and therefore, by the definition, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. It is our translating this formal statement into the more colloquial "has no limit" that creates the apparent contradiction. The mathematics is fine. The problem arises in how we talk about it. We need the formality to keep things straight. ❑

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P\to(1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Exercises 12.2

Evaluating Limits

Find the limits in Exercises 1–12.

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\displaystyle\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

3. $\displaystyle\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$

5. $\displaystyle\lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$

6. $\displaystyle\lim_{(x,y)\to(0,0)} \cos\frac{x^2 + y^3}{x + y + 1}$

7. $\displaystyle\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$

8. $\displaystyle\lim_{(x,y)\to(1,1)} \ln|1 + x^2y^2|$

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

10. $\displaystyle\lim_{(x,y)\to(1,1)} \cos\sqrt[3]{|xy| - 1}$

11. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x \sin y}{x^2 + 1}$

12. $\displaystyle\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–20 by rewriting the fractions first.

13. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - 2xy + y^2}{x - y}$

14. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - y^2}{x - y}$

15. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy - y - 2x + 2}{x - 1}$

16. $\displaystyle\lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y + 4}{x^2y - xy + 4x^2 - 4x}$

17. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

18. $\displaystyle\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x + y - 4}{\sqrt{x + y} - 2}$

19. $\displaystyle\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x - y} - 2}{2x - y - 4}$

20. $\displaystyle\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

Limits with Three Variables

Find the limits in Exercises 21–26.

21. $\displaystyle\lim_{P\to(1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

22. $\displaystyle\lim_{P\to(1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}$

23. $\displaystyle\lim_{P\to(3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

24. $\displaystyle\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

25. $\displaystyle\lim_{P\to(\pi,0,3)} ze^{-2y} \cos 2x$

26. $\displaystyle\lim_{P\to(0,-2,0)} \ln\sqrt{x^2 + y^2 + z^2}$

Continuity in the Plane

At what points (x, y) in the plane are the functions in Exercises 27–30 continuous?

27. a) $f(x, y) = \sin(x + y)$ **b)** $f(x, y) = \ln(x^2 + y^2)$

28. a) $f(x, y) = \dfrac{x + y}{x - y}$ **b)** $f(x, y) = \dfrac{y}{x^2 + 1}$

29. a) $g(x, y) = \sin \dfrac{1}{xy}$ **b)** $g(x, y) = \dfrac{x + y}{2 + \cos x}$

30. a) $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ **b)** $g(x, y) = \dfrac{1}{x^2 - y}$

Continuity in Space

At what points (x, y, z) in space are the functions in Exercises 31–34 continuous?

31. a) $f(x, y, z) = x^2 + y^2 - 2z^2$

b) $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

32. a) $f(x, y, z) = \ln xyz$ **b)** $f(x, y, z) = e^{x+y} \cos z$

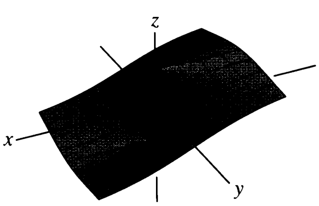
33. a) $h(x, y, z) = xy \sin \dfrac{1}{z}$ **b)** $h(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$

34. a) $h(x, y, z) = \dfrac{1}{|y| + |z|}$ **b)** $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

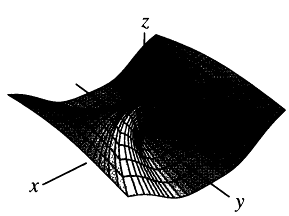
No Limit at a Point

By considering different paths of approach, show that the functions in Exercises 35–42 have no limit as $(x, y) \to (0, 0)$.

35. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$ **36.** $f(x, y) = \dfrac{x^4}{x^4 + y^2}$



(Generated by Mathematica)



(Generated by Mathematica)

37. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$ **38.** $f(x, y) = \dfrac{xy}{|xy|}$

39. $g(x, y) = \dfrac{x - y}{x + y}$ **40.** $g(x, y) = \dfrac{x + y}{x - y}$

41. $h(x, y) = \dfrac{x^2 + y}{y}$ **42.** $h(x, y) = \dfrac{x^2}{x^2 - y}$

Theory and Examples

43. If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, must f be defined at (x_0, y_0)? Give reasons for your answer.

44. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

if f is continuous at (x_0, y_0)? if f is not continuous at (x_0, y_0)? Give reasons for your answer.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \le f(x, y) \le h(x, y)$ for all $(x, y) \ne (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 45–48.

45. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

46. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

47. Does knowing that $|\sin(1/x)| \le 1$ tell you anything about

$$\lim_{(x,y)\to(0,0)} y \sin\frac{1}{x}?$$

Give reasons for your answer.

48. Does knowing that $|\cos(1/y)| \le 1$ tell you anything about

$$\lim_{(x,y)\to(0,0)} x \cos\frac{1}{y}?$$

Give reasons for your answer.

49. (*Continuation of Example 3.*)

a) Reread Example 3. Then substitute $m = \tan\theta$ into the formula

$$f(x, y)\Big|_{y=mx} = \frac{2m}{1 + m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b) Use the formula you obtained in (a) to show that the limit of f as $(x, y) \to (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

50. Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy\,\frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing to Polar Coordinates

If you cannot make any headway with $\lim_{(x,y)\to(0,0)} f(x,y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r\cos\theta$, $y = r\sin\theta$, and investigate the limit of the resulting expression as $r \to 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |f(r,\theta) - L| < \epsilon. \tag{4}$$

If such an L exists, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = L.$$

For instance,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

To verify the last of these equalities, we need to show that (4) is satisfied with $f(r,\theta) = r\cos^3\theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |r\cos^3\theta - 0| < \epsilon.$$

Since

$$|r\cos^3\theta| = |r||\cos^3\theta| \le |r|\cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2+y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x,y)\to(0,0)} x^2/(x^2+y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 4 illustrates this point. In polar coordinates, $f(x,y) = (2x^2y)/(x^4+y^2)$ becomes

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + \sin^2\theta}$$

for $r \ne 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r\sin\theta = r^2\cos^2\theta$ and

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2} \\ &= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1. \end{aligned}$$

In Exercises 51–56, find the limit of f as $(x,y) \to (0,0)$ or show that the limit does not exist.

51. $f(x,y) = \dfrac{x^3 - xy^2}{x^2+y^2}$

52. $f(x,y) = \cos\left(\dfrac{x^3 - y^3}{x^2+y^2}\right)$

53. $f(x,y) = \dfrac{y^2}{x^2+y^2}$

54. $f(x,y) = \dfrac{2x}{x^2+x+y^2}$

55. $f(x,y) = \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right)$

56. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

In Exercises 57 and 58, define $f(0,0)$ in a way that extends f to be continuous at the origin.

57. $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

58. $f(x,y) = \dfrac{2xy^2}{x^2+y^2}$

Using the δ-ϵ Definitions

59. Show that the δ-ϵ requirement in the definition of limit expressed in Eq. (1) is equivalent to the requirement expressed in Eq. (2).

60. Using the formal δ-ϵ definition of limit of a function $f(x,y)$ as $(x,y) \to (x_0,y_0)$ as a guide, state a formal definition for the limit of a function $g(x,y,z)$ as $(x,y,z) \to (x_0,y_0,z_0)$. What would be the analogous definition for a function $h(x,y,z,t)$ of four independent variables?

Each of Exercises 61–64 gives a function $f(x,y)$ and a positive number ϵ. In each exercise, either show that there exists a $\delta > 0$ such that for all (x,y),

$$\sqrt{x^2+y^2} < \delta \quad \Rightarrow \quad |f(x,y) - f(0,0)| < \epsilon$$

or show that there exists a $\delta > 0$ such that for all (x,y),

$$|x| < \delta \ \text{ and } \ |y| < \delta \quad \Rightarrow \quad |f(x,y) - f(0,0)| < \epsilon.$$

Do either one or the other, whichever seems more convenient. There is no need to do both.

61. $f(x,y) = x^2 + y^2, \quad \epsilon = 0.01$

62. $f(x,y) = y/(x^2+1), \quad \epsilon = 0.05$

63. $f(x,y) = (x+y)/(x^2+1), \quad \epsilon = 0.01$

64. $f(x,y) = (x+y)/(2+\cos x), \quad \epsilon = 0.02$

Each of Exercises 65–68 gives a function $f(x,y,z)$ and a positive number ϵ. In each exercise, either show that there exists a $\delta > 0$ such that for all (x,y,z),

$$\sqrt{x^2+y^2+z^2} < \delta \quad \Rightarrow \quad |f(x,y,z) - f(0,0,0)| < \epsilon$$

or show that there exists a $\delta > 0$ such that for all (x,y,z),

$$|x| < \delta, \quad |y| < \delta, \quad \text{and}$$
$$|z| < \delta \quad \Rightarrow \quad |f(x,y,z) - f(0,0,0)| < \epsilon.$$

Do either one or the other, whichever seems more convenient. There is no need to do both.

65. $f(x,y,z) = x^2 + y^2 + z^2, \quad \epsilon = 0.015$

66. $f(x,y,z) = xyz, \quad \epsilon = 0.008$

67. $f(x,y,z) = \dfrac{x+y+z}{x^2+y^2+z^2+1}, \quad \epsilon = 0.015$

68. $f(x,y,z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \epsilon = 0.03$

69. Show that $f(x,y,z) = x + y - z$ is continuous at every point (x_0, y_0, z_0).

70. Show that $f(x,y,z) = x^2 + y^2 + z^2$ is continuous at the origin.

12.3 Partial Derivatives

When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. This section shows how partial derivatives arise and how to calculate partial derivatives by applying the rules for differentiating functions of a single variable.

Definitions and Notation

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Fig. 12.13). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

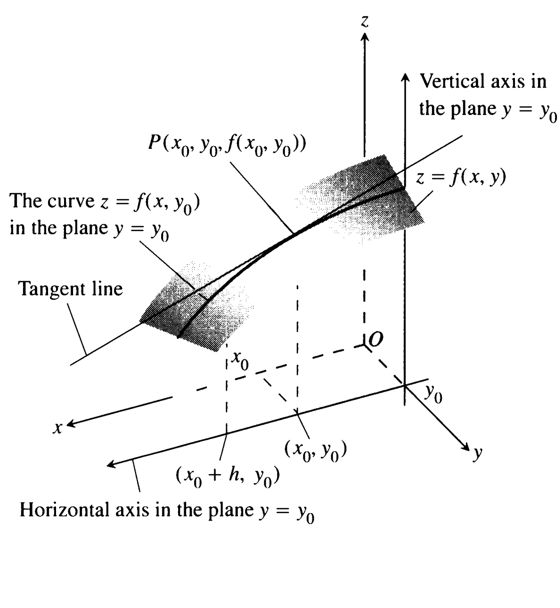
Definition

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

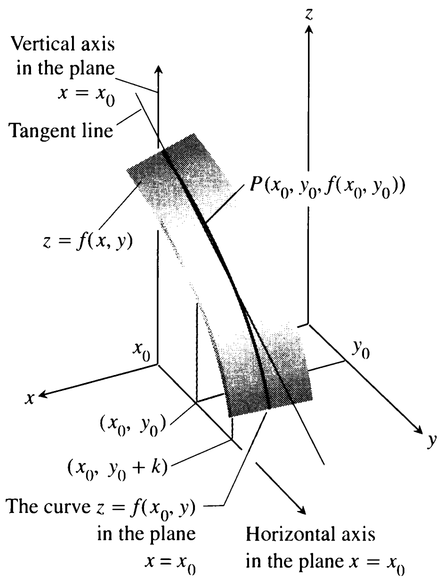
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \tag{1}$$

provided the limit exists. (Think of ∂ as a kind of d.)

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0. This is the rate of change of f in the direction of $\mathbf{i}$ at (x_0, y_0).



12.13 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from a point above the first quadrant of the xy-plane.



12.14 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

The notation for a partial derivative depends on what we want to emphasize:

$\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$	"Partial derivative of f with respect to x at (x_0, y_0)" or "f sub x at (x_0, y_0)." Convenient for stressing the point (x_0, y_0).
$\left.\dfrac{\partial z}{\partial x}\right\vert_{(x_0, y_0)}$	"Partial derivative of z with respect to x at (x_0, y_0)." Common in science and engineering when you are dealing with variables and do not mention the function explicitly.
$f_x,\ \dfrac{\partial f}{\partial x},\ z_x,$ or $\dfrac{\partial z}{\partial x}$	"Partial derivative of f (or z) with respect to x." Convenient when you regard the partial derivative as a function in its own right.

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

Definition

The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\begin{aligned}\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} &= \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} \\ &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},\end{aligned} \tag{2}$$

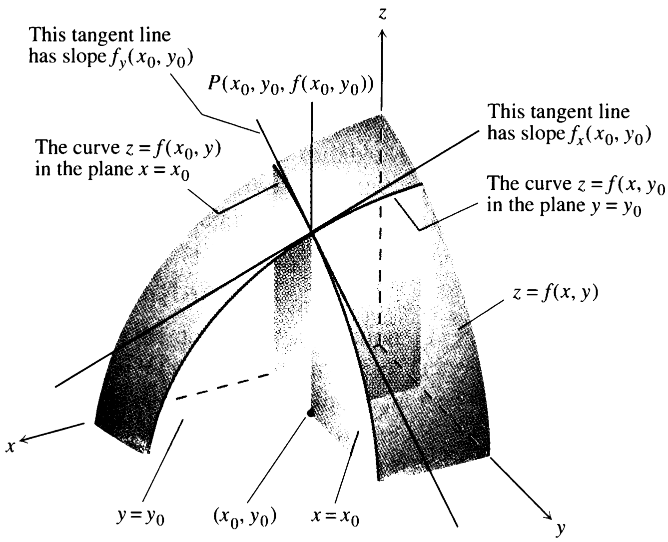
provided the limit exists.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Fig. 12.14) is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0. This is the rate of change of f in the direction of $\mathbf{j}$ at (x_0, y_0).

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Fig. 12.15, on the following page). Is the plane they determine tangent to the surface at P? It would be nice if it were, but we have to learn more about partial derivatives before we can find out.



12.15 Figures 12.13 and 12.14 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Calculations

As Eq. (1) shows, we calculate $\partial f/\partial x$ by differentiating f with respect to x in the usual way while treating y as a constant. As Eq. (2) shows, we can calculate $\partial f/\partial y$ by differentiating f with respect to y in the usual way while holding x constant.

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (4, −5) if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we regard y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at (4, −5) is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we regard x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at (4, −5) is $3(4) + 1 = 13$.

EXAMPLE 2 Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Technology *Partial Differentiation* A simple grapher can support your calculations even in multiple dimensions. If you specify the values of all but one independent variable, the grapher can calculate partial derivatives and can plot traces with respect to that remaining variable. Typically a Computer Algebra System can compute partial derivatives symbolically and numerically as easily as it can compute simple derivatives. Most systems use the same command to differentiate a function, regardless of the number of variables. (Simply specify the variable with which differentiation is to take place.)

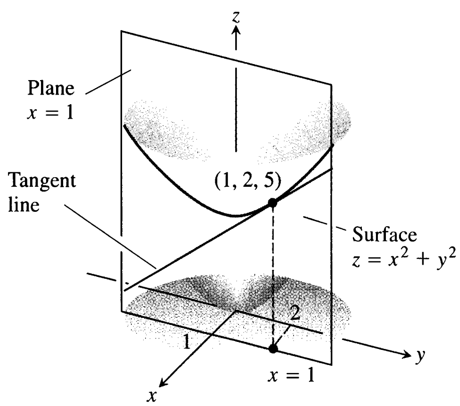
EXAMPLE 3 Find f_x if $f(x, y) = \dfrac{2y}{y + \cos x}$.

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\dfrac{\partial}{\partial x}(2y) - 2y\dfrac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2} \\ &= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}. \end{aligned}$$

❑

EXAMPLE 4 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5) (Fig. 12.16).



12.16 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point (1, 2, 5) (Example 4).

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2+y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1+y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

❑

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives.

EXAMPLE 5 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

With y constant, $\dfrac{\partial}{\partial x}(yz) = y\dfrac{\partial z}{\partial x}$.

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}.$$

❑

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 If x, y, and z are independent variables and

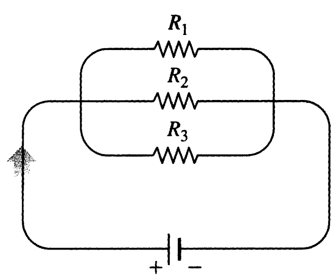
$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z)$$

$$= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).$$

EXAMPLE 7 *Electrical resistors in parallel*



12.17 Resistors arranged this way are said to be connected in parallel (Example 7). Each resistor lets a portion of the current through. Their combined resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{3}$$

(Fig. 12.17). Find the value of $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution To find $\partial R/\partial R_2$, we treat R_1 and R_3 as constants and differentiate both sides of Eq. (3) with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2{}^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

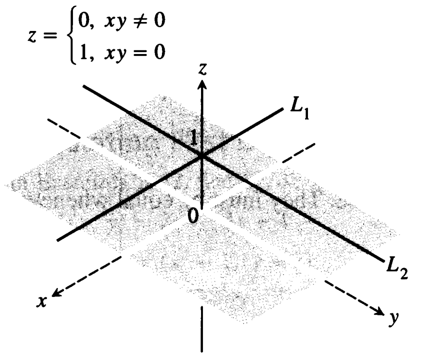
$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

The Relationship Between Continuity and the Existence of Partial Derivatives

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. However, if the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), then f is continuous at (x_0, y_0), as we will see in the next section.



12.18 **The graph of**

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there.

EXAMPLE 8 The function

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Fig. 12.18) is not continuous at (0, 0). The limit of f as (x, y) approaches (0, 0) along the line $y = x$ is 0, but $f(0, 0) = 1$. The partial derivatives f_x and f_y, being the slopes of the horizontal lines L_1 and L_2 in Fig. 12.18, both exist at (0, 0). ┘

Second Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second order derivatives. These derivatives are usually denoted by

$\dfrac{\partial^2 f}{\partial x^2}$	"d squared f d x squared"	or	f_{xx}	"f sub x x"
$\dfrac{\partial^2 f}{\partial y^2}$	"d squared f d y squared"		f_{yy}	"f sub y y"
$\dfrac{\partial^2 f}{\partial x \partial y}$	"d squared f d x d y"		f_{yx}	"f sub y x"
$\dfrac{\partial^2 f}{\partial y \partial x}$	"d squared f d y d x"		f_{xy}	"f sub x y"

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the derivatives are taken:

$\dfrac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, then

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

Also,

$$\frac{\partial f}{\partial y} = -x \sin y + e^x$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

┘

Euler's Theorem

You may have noticed that the "mixed" second order partial derivatives

$$\frac{\partial^2 f}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x\,\partial y}$$

in Example 9 were equal. This was not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous.

Theorem 2

Euler's Theorem (The Mixed Derivative Theorem)

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{4}$$

You can find a proof of Theorem 2 in Appendix 9.

Theorem 2 says that to calculate a mixed second order derivative we may differentiate in either order. This can work to our advantage.

EXAMPLE 10 Find $\partial^2 w/\partial x\,\partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x\,\partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we postpone the differentiation with respect to y and differentiate first with respect to x, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y\,\partial x} = 1.$$

We are in for more work if we differentiate first with respect to y. (Just try it.) ❑

Partial Derivatives of Still Higher Order

Although we will deal mostly with first and second order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus we get third and fourth order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x\,\partial y^2} = f_{yyx},$$

$$\frac{\partial^4 f}{\partial x^2\,\partial y^2} = f_{yyxx},$$

and so on. As with second order derivatives, the order of differentiation is immaterial as long as the derivatives through the order in question are continuous.

Exercises 12.3

Calculating First Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$
2. $f(x, y) = x^2 - xy + y^2$
3. $f(x, y) = (x^2 - 1)(y + 2)$
4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
5. $f(x, y) = (xy - 1)^2$
6. $f(x, y) = (2x - 3y)^3$
7. $f(x, y) = \sqrt{x^2 + y^2}$
8. $f(x, y) = (x^3 + (y/2))^{2/3}$
9. $f(x, y) = 1/(x + y)$
10. $f(x, y) = x/(x^2 + y^2)$
11. $f(x, y) = (x + y)/(xy - 1)$
12. $f(x, y) = \tan^{-1}(y/x)$
13. $f(x, y) = e^{(x+y+1)}$
14. $f(x, y) = e^{-x}\sin(x + y)$
15. $f(x, y) = \ln(x + y)$
16. $f(x, y) = e^{xy}\ln y$
17. $f(x, y) = \sin^2(x - 3y)$
18. $f(x, y) = \cos^2(3x - y^2)$
19. $f(x, y) = x^y$
20. $f(x, y) = \log_y x$
21. $f(x, y) = \int_x^y g(t)\,dt$ (g continuous for all t)
22. $f(x, y) = \sum_{n=0}^{\infty} (xy)^n$ ($|xy| < 1$)

In Exercises 23–34, find f_x, f_y, and f_z.

23. $f(x, y, z) = 1 + xy^2 - 2z^2$
24. $f(x, y, z) = xy + yz + xz$
25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$
26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
27. $f(x, y, z) = \sin^{-1}(xyz)$
28. $f(x, y, z) = \sec^{-1}(x + yz)$
29. $f(x, y, z) = \ln(x + 2y + 3z)$
30. $f(x, y, z) = yz\ln(xy)$
31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$
32. $f(x, y, z) = e^{-xyz}$
33. $f(x, y, z) = \tanh(x + 2y + 3z)$
34. $f(x, y, z) = \sinh(xy - z^2)$

In Exercises 35–40, find the partial derivative of the function with respect to each variable.

35. $f(t, \alpha) = \cos(2\pi t - \alpha)$
36. $g(u, v) = v^2 e^{(2u/v)}$
37. $h(\rho, \phi, \theta) = \rho\sin\phi\cos\theta$
38. $g(r, \theta, z) = r(1 - \cos\theta) - z$
39. *Work done by the heart.* (Section 3.7, Exercise 56)

$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$

40. *Wilson lot size formula.* (Section 3.6, Exercise 57)

$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Calculating Second Order Partial Derivatives

Find all the second order partial derivatives of the functions in Exercises 41–46.

41. $f(x, y) = x + y + xy$
42. $f(x, y) = \sin xy$
43. $g(x, y) = x^2y + \cos y + y\sin x$
44. $h(x, y) = xe^y + y + 1$
45. $r(x, y) = \ln(x + y)$
46. $s(x, y) = \tan^{-1}(y/x)$

Mixed Partial Derivatives

In Exercises 47–50, verify that $w_{xy} = w_{yx}$.

47. $w = \ln(2x + 3y)$
48. $w = e^x + x\ln y + y\ln x$
49. $w = xy^2 + x^2y^3 + x^3y^4$
50. $w = x\sin y + y\sin x + xy$

51. Which order of differentiation will calculate f_{xy} faster: x first, or y first? Try to answer without writing anything down.

- **a)** $f(x, y) = x\sin y + e^y$
- **b)** $f(x, y) = 1/x$
- **c)** $f(x, y) = y + (x/y)$
- **d)** $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$
- **e)** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
- **f)** $f(x, y) = x\ln xy$

52. The fifth order partial derivative $\partial^5 f/\partial x^2\partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x, or y? Try to answer without writing anything down.

- **a)** $f(x, y) = y^2x^4e^x + 2$
- **b)** $f(x, y) = y^2 + y(\sin x - x^4)$
- **c)** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
- **d)** $f(x, y) = xe^{y^2/2}$

Using the Partial Derivative Definition

In Exercises 53 and 54, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

53. $f(x, y) = 1 - x + y - 3x^2y$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(1, 2)$

54. $f(x, y) = 4 + 2x - 3y - xy^2$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 1)$

55. Let $w = f(x, y, z)$ be a function of three independent variables, and write the formal definition of the partial derivative $\partial f/\partial z$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

56. Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial y$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial y$ at $(-1, 0, 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Differentiating Implicitly

57. Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

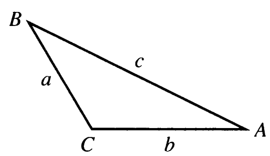
defines z as a function of the two independent variables x and y and the partial derivative exists.

58. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Exercises 59 and 60 are about the triangle shown here.



59. Express A implicitly as a function of a, b, and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$.

60. Express a implicitly as a function of A, b, and B and calculate $\partial a/\partial A$ and $\partial a/\partial B$.

61. Express v_x in terms of u and v if the equations $x = v\ln u$ and $y = u\ln v$ define u and v as functions of the independent variables x and y, and if v_x exists. (*Hint:* Differentiate both equations with respect to x and solve for v_x with Cramer's rule.)

62. Find $\partial x/\partial u$ and $\partial y/\partial u$ if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v, and the partial derivatives exist. (See the hint in Exercise 61.) Then let $s = x^2 + y^2$ and find $\partial s/\partial u$.

Laplace Equations

The three-dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{5}$$

is satisfied by steady-state temperature distributions $T = f(x, y, z)$ in space, by gravitational potentials, and by electrostatic potentials. The *two-dimensional Laplace equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{6}$$

obtained by dropping the $\partial^2 f/\partial z^2$ term from Eq. (5), describes potentials and steady-state temperature distributions in a plane (Fig. 12.19).

Show that each function in Exercises 63–68 satisfies a Laplace equation.

63. $f(x, y, z) = x^2 + y^2 - 2z^2$

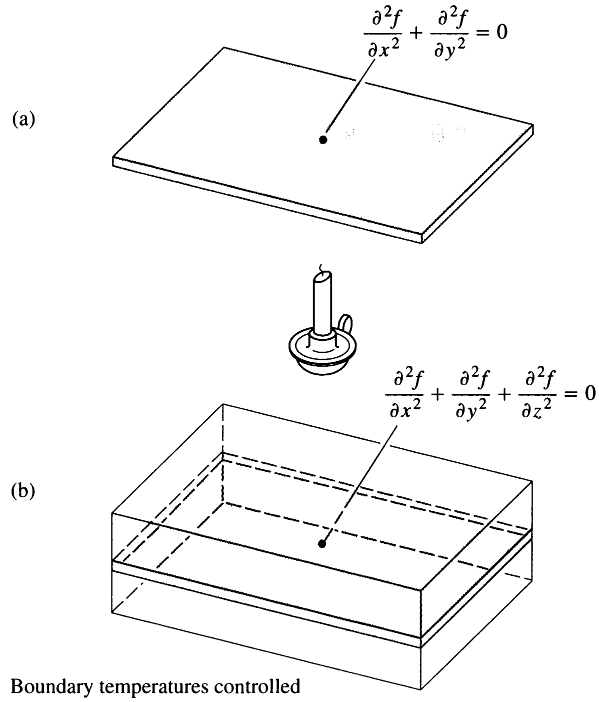
64. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

65. $f(x, y) = e^{-2y}\cos 2x$

66. $f(x, y) = \ln\sqrt{x^2 + y^2}$

67. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

68. $f(x, y, z) = e^{3x+4y}\cos 5z$



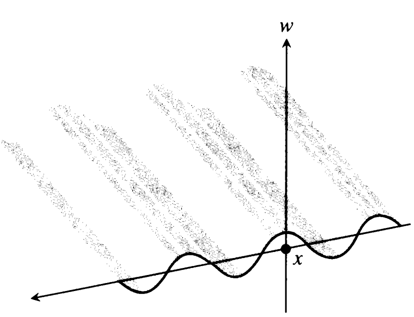
12.19 **Steady-state temperature distributions in planes and solids satisfy Laplace equations. The plane (a) may be treated as a thin slice of the solid (b) perpendicular to the *z*-axis.**

The Wave Equation

If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the *one-dimensional wave equation*

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial x^2}, \tag{7}$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.



In our example, x is the distance across the ocean's surface, but in other applications x might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number c varies with the medium and type of wave.

Show that the functions in Exercises 69–75 are all solutions of the wave equation.

69. $w = \sin(x + ct)$

70. $w = \cos(2x + 2ct)$

71. $w = \sin(x + ct) + \cos(2x + 2ct)$

72. $w = \ln(2x + 2ct)$

73. $w = \tan(2x - 2ct)$

74. $w = 5\cos(3x + 3ct) + e^{x+ct}$

75. $w = f(u)$, where f is a differentiable function of u and $u = a(x + ct)$, where a is a constant.

12.4 Differentiability, Linearization, and Differentials

In this section, we define differentiability and proceed from there to linearizations and differentials. The mathematical results of the section stem from the Increment Theorem. As we will see in the next section, this theorem also underlies the Chain Rule for multivariable functions.

Differentiability

Surprising as it may seem, the starting point for differentiability is not Fermat's difference quotient but rather the idea of increment. You may recall from our work with functions of a single variable that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$. For functions of two variables, the analogous property becomes the definition of differentiability. The Increment Theorem (from advanced calculus) tells us when to expect the property to hold.

Theorem 3

The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

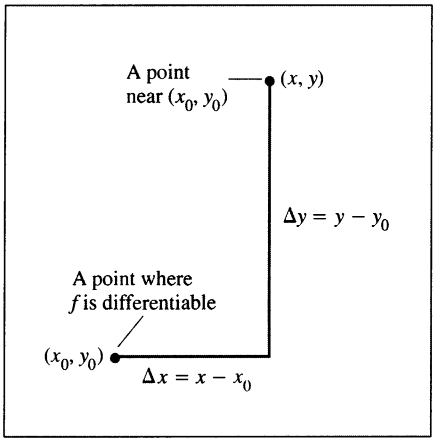
$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \tag{2}$$

in which $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

You will see where the epsilons come from if you read the proof in Appendix 10. You will also see that similar results hold for functions of more than two independent variables.



12.20 If f is differentiable at (x_0, y_0), then the value of f at any point (x, y) nearby is approximately $f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

Definition

A function $f(x, y)$ is **differentiable at (x_0, y_0)** if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Eq. (2) holds for f at (x_0, y_0). We call f **differentiable** if it is differentiable at every point in its domain.

In light of this definition, we have the immediate corollary of Theorem 3 that a function is differentiable if its first partial derivatives are *continuous.*

Corollary of Theorem 3

If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

If we replace the Δz in Eq. (2) by the expression $f(x, y) - f(x_0, y_0)$ and rewrite the equation as

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{3}$$

we see that the right-hand side of the new equation approaches $f(x_0, y_0)$ as Δx and Δy approach 0. This tells us that a function $f(x, y)$ is continuous at every point where it is differentiable.

As we can see from Theorems 3 and 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0). But remember that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Section 12.3, Example 8. Existence alone is not enough.

Theorem 4

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

How to Linearize a Function of Two Variables

Functions of two variables can be complicated, and we sometimes need to replace them with simpler ones that give the accuracy required for specific applications without being so hard to work with. We do this in a way that is similar to the way we find linear replacements for functions of a single variable (Section 3.7).

Suppose the function we wish to replace is $z = f(x, y)$ and that we want the replacement to be effective near a point (x_0, y_0) at which we know the values of f, f_x, and f_y and at which f is differentiable. Since f is differentiable, Eq. (3) holds for f at (x_0, y_0). Therefore, if we move from (x_0, y_0) to any point (x, y) by increments $\Delta x = x - x_0$ and $\Delta y = y - y_0$ (Fig. 12.20), the new value of f will be

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) \\ & + f_y(x_0, y_0)(y - y_0) + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \end{aligned}$$

Eq. (3), with $\Delta x = x - x_0$ and $\Delta y = y - y_0$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. If the increments Δx and Δy are small, the products $\epsilon_1 \Delta x$ and $\epsilon_2 \Delta y$ will eventually be smaller still and we will have

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x,y)}.$$

In other words, as long as Δx and Δy are small, f will have approximately the same value as the linear function L. If f is hard to use, and our work can tolerate the error involved, we may safely replace f by L.

Definitions

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{4}$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

In Section 12.8 we will see that the plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0). Thus, the linearization of a function of two variables is a tangent-*plane* approximation in the same way that the linearization of a function of a single variable is a tangent-*line* approximation.

EXAMPLE 1 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

Solution We evaluate Eq. (4) with

$$f(x_0, y_0) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = 8,$$

$$f_x(x_0, y_0) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (2x - y)_{(3,2)} = 4,$$

$$f_y(x_0, y_0) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (-x + y)_{(3,2)} = -1,$$

getting

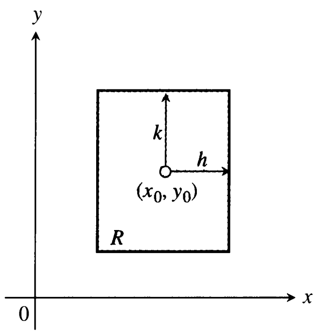
$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \qquad \text{Eq. (4)} \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at (3, 2) is $L(x, y) = 4x - y - 2$. ❑

How Accurate Is the Standard Linear Approximation?

To find the error in the approximation $f(x, y) \approx L(x, y)$, we use the second order partial derivatives of f. Suppose that the first and second order partial derivatives of f are continuous throughout an open set containing a closed rectangular region R centered at (x_0, y_0) and given by the inequalities

$$|x - x_0| \le h, \qquad |y - y_0| \le k$$



12.21 The rectangular region R: $|x - x_0| \le h$, $|y - y_0| \le k$ in the xy-plane. On this kind of region, we can find useful error bounds for our approximations.

(Fig. 12.21). Since R is closed and bounded, the second partial derivatives all take on absolute maximum values on R. If B is the largest of these values, then, as explained in Section 12.10, the error $E(x, y) = f(x, y) - L(x, y)$ in the standard linear approximation satisfies the inequality

$$|E(x, y)| \le \frac{1}{2} B\,(|x - x_0| + |y - y_0|)^2$$

throughout R.

When we use this inequality to estimate E, we usually cannot find the values of f_{xx}, f_{yy}, and f_{xy} that determine B and we have to settle for an upper bound or "worst-case" value instead. If M is any common upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then B will be less than or equal to M and we will know that

$$|E(x, y)| \le \frac{1}{2} M\,(|x - x_0| + |y - y_0|)^2.$$

This is the inequality normally used in estimating E. When we need to make $|E(x, y)|$ small for a given M, we just make $|x - x_0|$ and $|y - y_0|$ small.

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \le \frac{1}{2} M\,(|x - x_0| + |y - y_0|)^2. \qquad (5)$$

EXAMPLE 2 In Example 1, we found the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at (3, 2) to be

$$L(x, y) = 4x - y - 2.$$

Find an upper bound for the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle

$$R:\quad |x - 3| \le 0.1, \quad |y - 2| \le 0.1.$$

Express the upper bound as a percentage of $f(3, 2)$, the value of f at the center of the rectangle.

Solution We use the inequality

$$|E(x, y)| \le \frac{1}{2} M\,(|x - x_0| + |y - y_0|)^2. \qquad \text{Eq. (5)}$$

To find a suitable value for M, we calculate f_{xx}, f_{xy}, and f_{yy}, finding, after a routine differentiation, that all three derivatives are constant, with values

$$|f_{xx}| = |2| = 2, \qquad |f_{xy}| = |-1| = 1, \qquad |f_{yy}| = |1| = 1.$$

The largest of these is 2, so we may safely take M to be 2. With $(x_0, y_0) = (3, 2)$, we then know that, throughout R,

$$|E(x, y)| \le \frac{1}{2}(2)\left(|x-3| + |y-2|\right)^2 = \left(|x-3| + |y-2|\right)^2.$$

Finally, since $|x - 3| \le 0.1$ and $|y - 2| \le 0.1$ on R, we have

$$|E(x, y)| \le (0.1 + 0.1)^2 = 0.04.$$

As a percentage of $f(3, 2) = 8$, the error is no greater than

$$\frac{0.04}{8} \times 100 = 0.5\%.$$

As long as (x, y) stays in R, the approximation $f(x, y) \approx L(x, y)$ will be in error by no more than 0.04, which is 1/2% of the value of f at the center of R. ❑

Predicting Change with Differentials

Suppose we know the values of a differentiable function $f(x, y)$ and its first partial derivatives at a point (x_0, y_0) and we want to predict how much the value of f will change if we move to a point $(x_0 + \Delta x, y_0 + \Delta y)$ nearby. If Δx and Δy are small, f and its linearization at (x_0, y_0) will change by nearly the same amount, so the change in L will give a practical estimate of the change in f.

The change in f is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

A straightforward calculation with Eq. (4), using the notation $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, shows that the corresponding change in L is

$$\begin{aligned} \Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0) \\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \end{aligned}$$

The formula for Δf is usually as hard to work with as the formula for f. The change in L, however, is just a known constant times Δx plus a known constant times Δy.

The change ΔL is usually described in the more suggestive notation

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy,$$

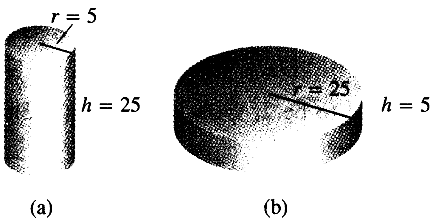
in which df denotes the change in the linearization that results from the changes dx and dy in x and y. As usual, we call dx and dy differentials of x and y, and call df the corresponding differential of f.

Definition

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting differential in f is

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy. \tag{6}$$

This change in the linearization of f is called the **total differential of f.**



12.22 The volume of cylinder (a) is more sensitive to a small change in r than it is to an equally small change in h. The volume of cylinder (b) is more sensitive to small changes in h than it is to small changes in r.

EXAMPLE 3 *Sensitivity to change*

Your company manufactures right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

Solution As a function of radius r and height h, the typical tank's volume is

$$V = \pi r^2 h.$$

The change in volume caused by small changes dr and dh in radius and height is approximately

$$\begin{aligned} dV &= V_r(5, 25)\,dr + V_h(5, 25)\,dh \qquad \text{Eq. (6) with } f = V \text{ and } (x_0, y_0) = (5, 25) \\ &= (2\pi r h)_{(5,25)}\,dr + (\pi r^2)_{(5,25)}\,dh \\ &= 250\pi\,dr + 25\pi\,dh. \end{aligned}$$

Thus, a 1-unit change in r will change V by about 250π units. A 1-unit change in h will change V by about 25π units. The tank's volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h. As a quality control engineer concerned with being sure the tanks have the correct volume, you would want to pay special attention to their radii.

In contrast, if the values of r and h are reversed to make $r = 25$ and $h = 5$, then the total differential in V becomes

$$dV = (2\pi r h)_{(25,5)}\,dr + (\pi r^2)_{(25,5)}\,dh = 250\pi\,dr + 625\pi\,dh.$$

Now the volume is more sensitive to changes in h than to changes in r (Fig. 12.22).

The general rule to be learned from this example is that functions are most sensitive to small changes in the variables that generate the largest partial derivatives. ❑

Absolute change vs. relative change

If you measure a 20-volt potential with an error of 10 volts, your reading is probably too crude to be useful. You are off by 50%. But if you measure a 200,000-volt potential with an error of 10 volts, your reading is within 0.005% of the true value. An absolute error of 10 volts is significant in the first case but of no consequence in the second because the relative error is so small.

In other cases, a small relative error—say, traveling a few meters too far in a journey of hundreds of thousands of meters—can have spectacular consequences.

Absolute, Relative, and Percentage Change

When we move from (x_0, y_0) to a point nearby, we can describe the corresponding change in the value of a function $f(x, y)$ in three different ways.

	True	**Estimate**
Absolute change:	Δf	df
Relative change:	$\dfrac{\Delta f}{f(x_0, y_0)}$	$\dfrac{df}{f(x_0, y_0)}$
Percentage change:	$\dfrac{\Delta f}{f(x_0, y_0)} \times 100$	$\dfrac{df}{f(x_0, y_0)} \times 100$

EXAMPLE 4 Suppose that the variables r and h change from the initial values of $(r_0, h_0) = (1, 5)$ by the amounts $dr = 0.03$ and $dh = -0.1$. Estimate the resulting absolute, relative, and percentage changes in the values of the function $V = \pi r^2 h$.

Solution To estimate the absolute change in V, we evaluate

$$dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh$$

to get

$$\begin{aligned} dV &= 2\pi r_0 h_0\,dr + \pi r_0^{\,2}\,dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.3\pi - 0.1\pi = 0.2\pi. \end{aligned}$$

We divide this by $V(r_0, h_0)$ to estimate the relative change:

$$\frac{dV}{V(r_0, h_0)} = \frac{0.2\pi}{\pi {r_0}^2 h_0} = \frac{0.2\pi}{\pi (1)^2(5)} = 0.04.$$

We multiply this by 100 to estimate the percentage change:

$$\frac{dV}{V(r_0, h_0)} \times 100 = 0.04 \times 100 = 4\%.$$

❑

EXAMPLE 5 The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Solution We are told that

$$\left|\frac{dr}{r} \times 100\right| \le 2 \qquad \text{and} \qquad \left|\frac{dh}{h} \times 100\right| \le 0.5.$$

Since

$$\frac{dV}{V} = \frac{2\pi r h\, dr + \pi r^2 dh}{\pi r^2 h} = \frac{2\,dr}{r} + \frac{dh}{h},$$

we have

$$\left|\frac{dV}{V} \times 100\right| = \left|2\frac{dr}{r} \times 100 + \frac{dh}{h} \times 100\right|$$

$$\le 2\left|\frac{dr}{r} \times 100\right| + \left|\frac{dh}{h} \times 100\right| \le 2(2) + 0.5 = 4.5.$$

We estimate the error in the volume calculation to be at most 4.5%. ❑

How accurately do we have to measure r and h to have a reasonable chance of calculating $V = \pi r^2 h$ with an error, say, of less than 2%? Questions like this are hard to answer because there is usually no single right answer. Since

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h},$$

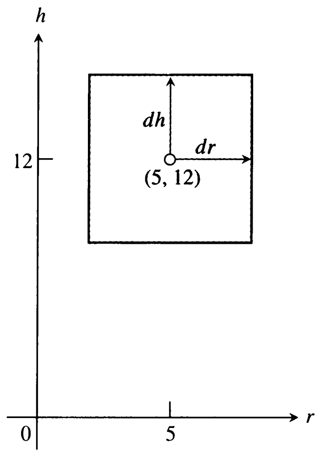
we see that dV/V is controlled by a combination of dr/r and dh/h. If we can measure h with great accuracy, we might come out all right even if we are sloppy about measuring r. On the other hand, our measurement of h might have so large a dh that the resulting dV/V would be too crude an estimate of $\Delta V/V$ to be useful even if dr were zero.

What we do in such cases is look for a reasonable square about the measured values (r_0, h_0) in which V will not vary by more than the allowed amount from $V_0 = \pi {r_0}^2 h_0$.

EXAMPLE 6 Find a reasonable square about the point $(r_0, h_0) = (5, 12)$ in which the value of $V = \pi r^2 h$ will not vary by more than ± 0.1.

Solution We approximate the variation ΔV by the differential

$$dV = 2\pi r_0 h_0\, dr + \pi {r_0}^2 dh = 2\pi(5)(12)\, dr + \pi(5)^2 dh = 120\pi\, dr + 25\pi\, dh.$$



12.23 A small square about the point (5, 12) in the *rh*-plane (Example 6).

Since the region to which we are restricting our attention is a square (Fig. 12.23), we may set $dh = dr$ to get

$$dV = 120\pi\, dr + 25\pi\, dr = 145\pi\, dr.$$

We then ask, How small must we take dr to be sure that $|dV|$ is no larger than 0.1? To answer, we start with the inequality

$$|dV| \le 0.1,$$

express dV in terms of dr,

$$|145\pi\, dr| \le 0.1,$$

and find a corresponding upper bound for dr:

$$|dr| \le \frac{0.1}{145\pi} \approx 2.1 \times 10^{-4}.$$

Rounding down to make sure dr won't accidentally be too big

With $dh = dr$, then, the square we want is described by the inequalities

$$|r - 5| \le 2.1 \times 10^{-4}, \qquad |h - 12| \le 2.1 \times 10^{-4}.$$

As long as (r, h) stays in this square, we may expect $|dV|$ to be less than or equal to 0.1 and we may expect $|\Delta V|$ to be approximately the same size. ❑

Functions of More Than Two Variables

Analogous results hold for differentiable functions of more than two variables.

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0). \quad (7)$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \le \frac{1}{2} M \left(|x - x_0| + |y - y_0| + |z - z_0|\right)^2. \quad (8)$$

3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**

$$df = f_x(P_0)\, dx + f_y(P_0)\, dy + f_z(P_0)\, dz$$

gives a good approximation of the resulting change in f.

EXAMPLE 7 Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3 \sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangle

$$R: \quad |x - 2| \le 0.01, \quad |y - 1| \le 0.02, \quad |z| \le 0.01.$$

Solution A routine evaluation gives

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

With these values, Eq. (7) becomes

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

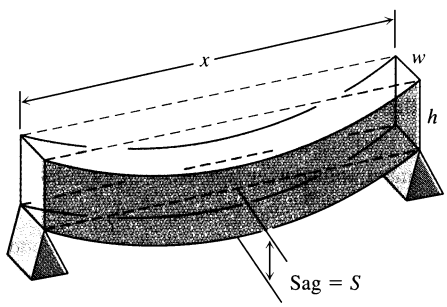
Equation (8) gives an upper bound for the error incurred by replacing f by L on R. Since

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3 \sin z,$$
$$f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

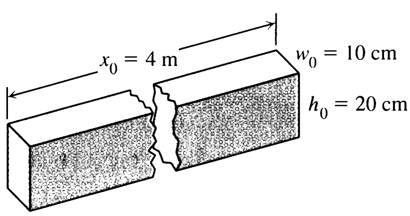
we may safely take M to be $\max|-3 \sin z| = 3$. Hence

$$|E| \le \frac{1}{2}(3)(0.01 + 0.02 + 0.01)^2 = 0.0024.$$

The error will be no greater than 0.0024. ❑



12.24 A beam supported at its two ends before and after loading. Example 8 shows how the sag S is related to the weight of the load and the dimensions of the beam.



12.25 The dimensions of the beam in Example 8.

EXAMPLE 8 *Controlling sag in uniformly loaded beams*

A horizontal rectangular beam, supported at both ends, will sag when subjected to a uniform load (constant weight per linear foot). The amount S of sag (Fig. 12.24) is calculated with the formula

$$S = C\,\frac{px^4}{wh^3}.$$

In this equation,

$p =$ the load (newtons per meter of beam length),
$x =$ the length between supports (m),
$w =$ the width of the beam (m),
$h =$ the height of the beam (m),
$C =$ a constant that depends on the units of measurement and on the material from which the beam is made.

Find dS for a beam 4 m long, 10 cm wide, and 20 cm high that is subjected to a load of 100 N/m (Fig. 12.25). What conclusions can be drawn about the beam from the expression for dS?

Solution Since S is a function of the four independent variables p, x, w, and h, its total differential dS is given by the equation

$$dS = S_p\,dp + S_x\,dx + S_w\,dw + S_h\,dh.$$

When we write this out for a particular set of values p_0, x_0, w_0, and h_0 and simplify the result, we find that

$$dS = S_0\left(\frac{dp}{p_0} + \frac{4dx}{x_0} - \frac{dw}{w_0} - \frac{3dh}{h_0}\right),$$

where $S_0 = S(p_0, x_0, w_0, h_0) = Cp_0{x_0}^4/(w_0{h_0}^3)$.

If $p_0 = 100$ N/m, $x_0 = 4$ m, $w_0 = 0.1$ m, and $h_0 = 0.2$ m, then

$$dS = S_0\left(\frac{dp}{100} + dx - 10dw - 15dh\right). \tag{9}$$

Here is what we can learn from Eq. (9). Since dp and dx appear with positive coefficients, increases in p and x will increase the sag. But dw and dh appear with negative coefficients, so increases in w and h will decrease the sag (make the beam stiffer). The sag is not very sensitive to changes in load because the coefficient of dp is 1/100. The magnitude of the coefficient of dh is greater than the magnitude of the coefficient of dw. Making the beam 1 cm higher will therefore decrease the sag more than making the beam 1 cm wider.

Exercises 12.4

Finding Linearizations

In Exercises 1–6, find the linearization $L(x, y)$ of the function at each point.

1. $f(x, y) = x^2 + y^2 + 1$ at (a) $(0, 0)$, (b) $(1, 1)$
2. $f(x, y) = (x + y + 2)^2$ at (a) $(0, 0)$, (b) $(1, 2)$
3. $f(x, y) = 3x - 4y + 5$ at (a) $(0, 0)$, (b) $(1, 1)$
4. $f(x, y) = x^3y^4$ at (a) $(1, 1)$, (b) $(0, 0)$
5. $f(x, y) = e^x \cos y$ at (a) $(0, 0)$, (b) $(0, \pi/2)$
6. $f(x, y) = e^{2y-x}$ at (a) $(0, 0)$, (b) $(1, 2)$

Upper Bounds for Errors in Linear Approximations

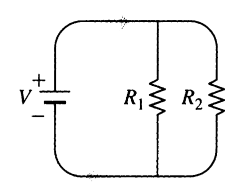
In Exercises 7–12, find the linearization $L(x, y)$ of the function $f(x, y)$ at P_0. Then use inequality (5) to find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

7. $f(x, y) = x^2 - 3xy + 5$ at $P_0(2, 1)$,
   R: $|x - 2| \le 0.1$, $|y - 1| \le 0.1$
8. $f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$ at $P_0(2, 2)$,
   R: $|x - 2| \le 0.1$, $|y - 2| \le 0.1$
9. $f(x, y) = 1 + y + x \cos y$ at $P_0(0, 0)$,
   R: $|x| \le 0.2$, $|y| \le 0.2$
   (Use $|\cos y| \le 1$ and $|\sin y| \le 1$ in estimating E.)
10. $f(x, y) = xy^2 + y \cos (x - 1)$ at $P_0(1, 2)$,
    R: $|x - 1| \le 0.1$, $|y - 2| \le 0.1$
11. $f(x, y) = e^x \cos y$ at $P_0(0, 0)$,
    R: $|x| \le 0.1$, $|y| \le 0.1$
    (Use $e^x \le 1.11$ and $|\cos y| \le 1$ in estimating E.)
12. $f(x, y) = \ln x + \ln y$ at $P_0(1, 1)$,
    R: $|x - 1| \le 0.2$, $|y - 1| \le 0.2$

Sensitivity to Change. Estimates

13. You plan to calculate the area of a long, thin rectangle from measurements of its length and width. Which dimension should you measure more carefully? Give reasons for your answer.
14. a) Around the point $(1, 0)$, is $f(x, y) = x^2(y + 1)$ more sensitive to changes in x, or to changes in y? Give reasons for your answer.
    b) What ratio of dx to dy will make df equal zero at $(1, 0)$?
15. Suppose T is to be found from the formula $T = x(e^y + e^{-y})$ where x and y are found to be 2 and ln 2 with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.
16. About how accurately may $V = \pi r^2 h$ be calculated from measurements of r and h that are in error by 1%?
17. If $r = 5.0$ cm and $h = 12.0$ cm to the nearest millimeter, what should we expect the maximum percentage error in calculating $V = \pi r^2 h$ to be?
18. To estimate the volume of a cylinder of radius about 2 m and height about 3 m, about how accurately should the radius and height be measured so that the error in the volume estimate will not exceed 0.1 m³? Assume that the possible error dr in measuring r is equal to the possible error dh in measuring h.
19. Give a reasonable square centered at $(1, 1)$ over which the value of $f(x, y) = x^3y^4$ will not vary by more than ± 0.1.
20. *Variation in electrical resistance.* The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel (Fig. 12.26) can be calculated from the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



12.26 The circuit in Exercises 20 and 21.

a) Show that

$$dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2.$$

b) You have designed a two-resistor circuit like the one in Fig. 12.26 to have resistances of $R_1 = 100$ ohms and $R_2 = 400$

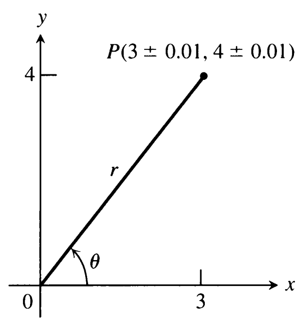
ohms, but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values. Will the value of R be more sensitive to variation in R_1, or to variation in R_2? Give reasons for your answer.

21. (*Continuation of Exercise 20.*) In another circuit like the one in Fig. 12.26, you plan to change R_1 from 20 to 20.1 ohms and R_2 from 25 to 24.9 ohms. By about what percentage will this change R?

22. *Error carry-over in coordinate changes*

a) If $x = 3 \pm 0.01$ and $y = 4 \pm 0.01$, as shown here, with approximately what accuracy can you calculate the polar coordinates r and θ of the point $P(x, y)$ from the formulas $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$? Express your estimates as percentage changes of the values that r and θ have at the point $(x_0, y_0) = (3, 4)$.

b) At the point $(x_0, y_0) = (3, 4)$, are the values of r and θ more sensitive to changes in x, or to changes in y? Give reasons for your answer.



Functions of Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 23–28 at the given points.

23. $f(x, y, z) = xy + yz + xz$ at

a) $(1, 1, 1)$ **b)** $(1, 0, 0)$ **c)** $(0, 0, 0)$

24. $f(x, y, z) = x^2 + y^2 + z^2$ at

a) $(1, 1, 1)$ **b)** $(0, 1, 0)$ **c)** $(1, 0, 0)$

25. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at

a) $(1, 0, 0)$ **b)** $(1, 1, 0)$ **c)** $(1, 2, 2)$

26. $f(x, y, z) = (\sin xy)/z$ at

a) $(\pi/2, 1, 1)$ **b)** $(2, 0, 1)$

27. $f(x, y, z) = e^x + \cos(y + z)$ at

a) $(0, 0, 0)$ **b)** $\left(0, \frac{\pi}{2}, 0\right)$ **c)** $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

28. $f(x, y, z) = \tan^{-1}(xyz)$ at

a) $(1, 0, 0)$ **b)** $(1, 1, 0)$ **c)** $(1, 1, 1)$

In Exercises 29–32, find the linearization $L(x, y, z)$ of the function $f(x, y, z)$ at P_0. Then use inequality (8) to find an upper bound for the magnitude of the error E in the approximation $f(x, y, z) \approx L(x, y, z)$ over the region R.

29. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2)$
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z - 2| \le 0.02$

30. $f(x, y, z) = x^2 + xy + yz + (1/4)z^2$ at $P_0(1, 1, 2)$
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z - 2| \le 0.08$

31. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0)$
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z| \le 0.01$

32. $f(x, y, z) = \sqrt{2}\cos x \sin(y + z)$ at $P_0(0, 0, \pi/4)$
R: $|x| \le 0.01$, $|y| \le 0.01$, $|z - \pi/4| \le 0.01$

Theory and Examples

33. The beam of Example 8 is tipped on its side so that $h = 0.1$ m and $w = 0.2$ m.

a) What is the value of dS now?

b) Compare the sensitivity of the newly positioned beam to a small change in height with its sensitivity to an equally small change in width.

34. A standard 12-fl oz can of soda is essentially a cylinder of radius $r = 1$ in. and height $h = 5$ in.

a) At these dimensions, how sensitive is the can's volume to a small change in the radius versus a small change in the height?

b) Could you design a soda can that *appears* to hold more soda but in fact holds the same 12 fl oz? What might its dimensions be? (There is more than one correct answer.)

35. If $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, to which of a, b, c, and d is the value of the determinant

$$f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

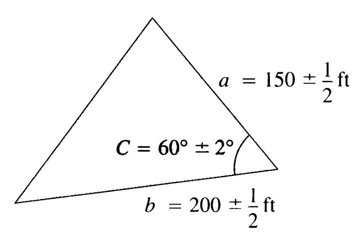
most sensitive? Give reasons for your answer.

36. Estimate how strongly simultaneous errors of 2% in a, b, and c might affect the calculation of the product

$$p(a, b, c) = abc.$$

37. Estimate how much wood it takes to make a hollow rectangular box whose inside measurements are 5 ft long by 3 ft wide by 2 ft deep if the box is made of lumber 1/2-in. thick and the box has no top.

38. The area of a triangle is $(1/2)ab \sin C$, where a and b are the lengths of two sides of the triangle and C is the measure of the included angle. In surveying a triangular plot, you have measured



a, b, and C to be 150 ft, 200 ft, and 60°, respectively. By about how much could your area calculation be in error if your values of a and b are off by half a foot each and your measurement of C is off by 2°? See the figure. Remember to use radians.

39. Suppose that $u = xe^y + y \sin z$ and that x, y, and z can be measured with maximum possible errors of $\pm 0.2, \pm 0.6$, and $\pm \pi/180$, respectively. Estimate the maximum possible error in calculating u from the measured values $x = 2, y = \ln 3, z = \pi/2$.

40. *The Wilson lot size formula.* The Wilson lot size formula in economics says that the most economical quantity Q of goods (radios, shoes, brooms, whatever) for a store to order is given by the formula $Q = \sqrt{2KM/h}$, where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding cost for each item (cost of space, utilities, security, and so on). To which of the variables K, M, and h is Q most sensitive near the point $(K_0, M_0, h_0) = (2, 20, 0.05)$? Give reasons for your answer.

41. Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R? Give reasons for your answer.

42. If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R, must the first order partial derivatives of f be continuous on R? Give reasons for your answer.

12.5 The Chain Rule

When we are interested in the temperature $w = f(x, y, z)$ at points along a curve $x = g(t)$, $y = h(t)$, $z = k(t)$ in space, or in the pressure or density along a path through a gas or fluid, we may think of f as a function of the single variable t. For each value of t, the temperature at the point $(g(t), h(t), k(t))$ is the value of the composite function $f(g(t), h(t), k(t))$. If we then wish to know the rate at which f changes with respect to t along the path, we have only to differentiate this composite with respect to t, provided, of course, the derivative exists.

Sometimes we can find the derivative by substituting the formulas for g, h, and k into the formula for f and differentiating directly with respect to t. But we often have to work with functions whose formulas are too complicated for convenient substitution or for which formulas are not readily available. To find a function's derivatives under circumstances like these, we use the Chain Rule. The form the Chain Rule takes depends on how many variables are involved but, except for the presence of additional variables, it works just like the Chain Rule in Section 2.5.

The Chain Rule for Functions of Two Variables

In Section 2.5, we used the Chain Rule when $w = f(x)$ was a differentiable function of x and $x = g(t)$ was a differentiable function of t. This made w a differentiable function of t and the Chain Rule said that dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

The analogous formula for a function $w = f(x, y)$ is given in Theorem 5.

Theorem 5

Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ is differentiable and x and y are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \qquad (1)$$

Proof The proof consists of showing that if x and y are differentiable at $t = t_0$, then w is differentiable at t_0 and

$$\left(\frac{dw}{dt}\right)_{t_0} = \left(\frac{\partial w}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0}, \tag{2}$$

where $P_0 = (x(t_0), y(t_0))$.

Let Δx, Δy, and Δw be the increments that result from changing t from t_0 to $t_0 + \Delta t$. Since f is differentiable (remember the definition in Section 12.4),

$$\Delta w = \left(\frac{\partial w}{\partial x}\right)_{P_0}\Delta x + \left(\frac{\partial w}{\partial y}\right)_{P_0}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \tag{3}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. To find dw/dt, we divide Eq. (3) through by Δt and let Δt approach zero. The division gives

$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_{P_0}\frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t},$$

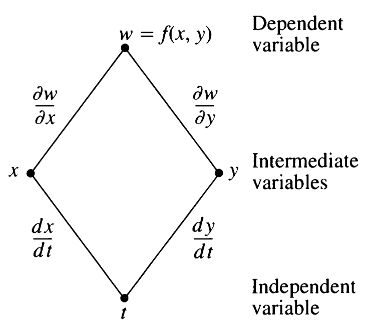
and letting Δt approach zero gives

$$\begin{aligned}\left(\frac{dw}{dt}\right)_{t_0} &= \lim_{\Delta t\to 0}\frac{\Delta w}{\Delta t}\\ &= \left(\frac{\partial w}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0} + 0\cdot\left(\frac{dx}{dt}\right)_{t_0} + 0\cdot\left(\frac{dy}{dt}\right)_{t_0}.\end{aligned}$$

This establishes Eq. (2) and completes the proof. ❑

The way to remember the Chain Rule is to picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

The **tree diagram** in the margin provides a convenient way to remember the Chain Rule. From the diagram you see that when $t = t_0$ the derivatives dx/dt and dy/dt are evaluated at t_0. The value of t_0 then determines the value x_0 for the differentiable function x and the value y_0 for the differentiable function y. The partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$ (which are themselves functions of x and y) are evaluated at the point $P_0(x_0, y_0)$ corresponding to t_0. The "true" independent variable is t, while x and y are *intermediate variables* (controlled by t) and w is the dependent variable.

A more precise notation for the Chain Rule shows how the various derivatives in Eq. (1) are evaluated:

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\cdot\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cdot\frac{dy}{dt}(t_0).$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?

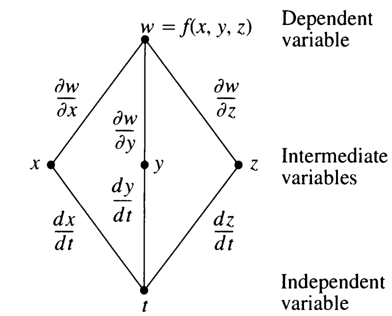
Solution We evaluate the right-hand side of Eq. (1) with $w = xy$, $x = \cos t$, and $y = \sin t$:

$$\frac{\partial w}{\partial x} = y = \sin t, \qquad \frac{\partial w}{\partial y} = x = \cos t, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (\sin t)(-\sin t) + (\cos t)(\cos t) \qquad \text{Eq. (1) with values from above}\\ &= -\sin^2 t + \cos^2 t = \cos 2t.\end{aligned}$$

Here we have three routes from w to t instead of two. But finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Notice in our calculation that we have substituted the functional expressions $x = \cos t$ and $y = \sin t$ in the partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$. The resulting derivative dw/dt is then expressed in terms of the independent variable t (so the intermediate variables x and y do not appear).

In this example we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos 2t = \cos 2t.$$

In either case,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1.$$

❑

The Chain Rule for Functions of Three Variables

To get the Chain Rule for functions of three variables, we add a term to Eq. (1).

> **Chain Rule for Functions of Three Independent Variables**
>
> If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and
>
> $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \qquad (4)$$

The derivation is identical with the derivation of Eq. (1) except that there are now three intermediate variables instead of two. The diagram we use for remembering the new equation is similar as well.

EXAMPLE 2 *Changes in a function's values along a helix*

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t$$

(Fig. 12.27). What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \qquad \text{Eq. (4)}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

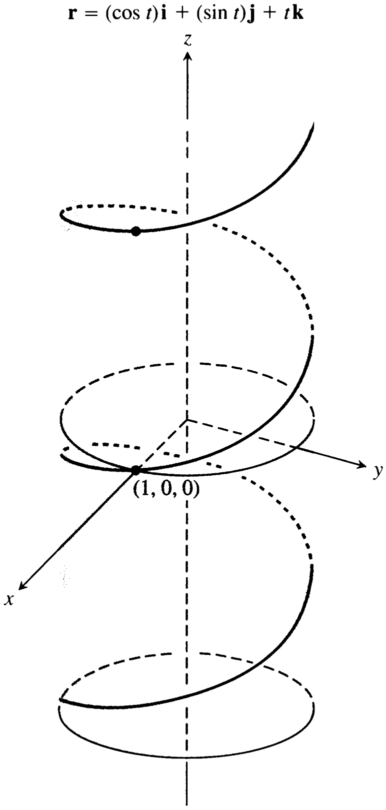
$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \qquad \text{Substitute for the intermediate variables.}$$

$$= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t$$

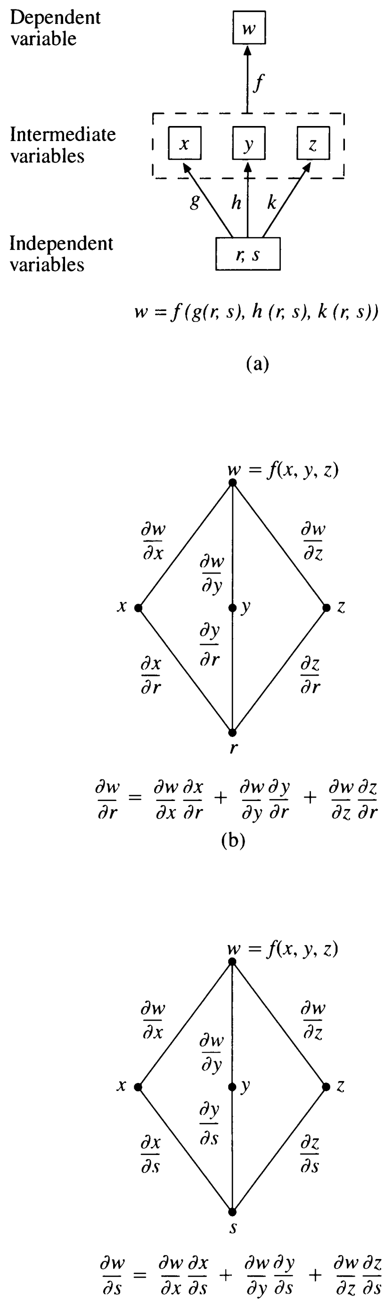
$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

❑

The helix
$\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$



12.27 Example 2 shows how the values of $w = xy + z$ vary with t along this helix.



12.28 Composite function and tree diagrams for Eqs. (5) and (6).

The Chain Rule for Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on a globe in space, we might prefer to think of x, y, and z as functions of the variables r and s that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$

Under the right conditions, w would have partial derivatives with respect to both r and s that could be calculated in the following way.

Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}, \tag{5}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}. \tag{6}$$

Equation (5) can be derived from Eq. (4) by holding s fixed and setting r equal to t. Similarly, Eq. (6) can be derived by holding r fixed and setting s equal to t. The tree diagrams for Eqs. (5) and (6) are shown in Fig. 12.28.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \text{Eq. (5)}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \qquad \text{Substitute for intermediate variable } z.$$

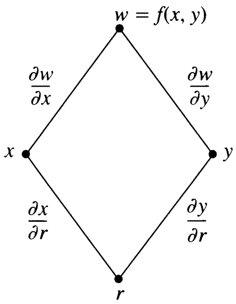
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \qquad \text{Eq. (6)}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}$$

❑

If f is a function of two variables instead of three, Eqs. (5) and (6) become one term shorter, because the intermediate variable z doesn't appear.

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

12.29 Tree diagram for the first of Eqs. (7).

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}. \tag{7}$$

Figure 12.29 shows the tree diagram for the first of Eqs. (7). The diagram for the second equation is similar—just replace r with s.

EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution We use Eqs. (7):

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r \end{aligned} \qquad \begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s \end{aligned}$$

Substitute for the intermediate variables.

❑

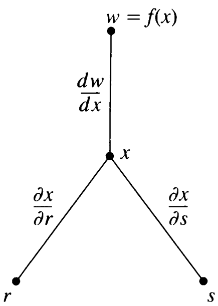
If f is a function of x alone, Eqs. (5) and (6) simplify still further.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}. \tag{8}$$

Here dw/dx is the ordinary (single-variable) derivative (Fig. 12.30).

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

12.30 Tree diagram for Eqs. (8).

Implicit Differentiation (Continued from Chapter 2)

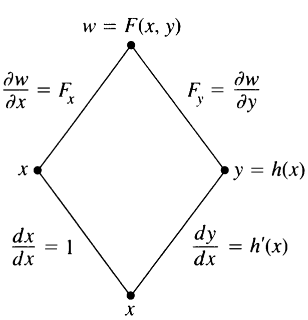
Believe it or not, the two-variable Chain Rule in Eq. (1) leads to a formula that takes most of the work out of implicit differentiation. Suppose:

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Since $w = F(x, y) = 0$, the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (tree diagram in Fig. 12.31), we find

$$\begin{aligned} 0 = \frac{dw}{dx} &= F_x \frac{dx}{dx} + F_y \frac{dy}{dx} \\ &= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}. \end{aligned} \tag{9}$$

Eq. (1) with $t = x$ and $f = F$



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

12.31 Tree diagram for Eq. (9).

If $F_y = \partial w/\partial y \neq 0$, we can solve Eq. (9) for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then, at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{10}$$

EXAMPLE 5 Find dy/dx if $x^2 + \sin y - 2y = 0$.

Solution Take $F(x, y) = x^2 + \sin y - 2y$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{\cos y - 2}. \qquad \text{Eq. (10)}$$

This calculation is significantly shorter than the single-variable calculation with which we found dy/dx in Section 2.6, Example 3. ❑

Remembering the Different Forms of the Chain Rule

How are we to remember all the different forms of the Chain Rule? The answer is that there is no need to remember them all. The best thing to do is to draw the appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each branch of the tree to the independent variable, calculating and multiplying the derivatives along the branch. Then add the products you found for the different branches. Let us summarize.

The Chain Rule for Functions of Many Variables

Suppose $w = f(x, y, \ldots, v)$ is a differentiable function of the variables $x, y, \ldots, v$ (a finite set) and the $x, y, \ldots, v$ are differentiable functions of $p, q, \ldots, t$ (another finite set). Then w is a differentiable function of the variables p through t and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}. \tag{11}$$

The other equations are obtained by replacing p by $q, \ldots, t$, one at a time.

One way to remember Eq. (11) is to think of the right-hand side as the dot product of two vectors with components

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \ldots, \frac{\partial w}{\partial v}\right)}_{\substack{\text{Derivatives of } w \text{ with}\\ \text{respect to the}\\ \text{intermediate variables}}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \ldots, \frac{\partial v}{\partial p}\right)}_{\substack{\text{Derivatives of the intermediate}\\ \text{variables with respect to the}\\ \text{selected independent variable}}}.$$

Exercises 12.5

Chain Rule: One Independent Variable

In Exercises 1–6, (a) express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then (b) evaluate dw/dt at the given value of t.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$

3. $w = \dfrac{x}{z} + \dfrac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$

4. $w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t,$
$z = 4\sqrt{t}; \quad t = 3$

5. $w = 2ye^x - \ln z, \quad x = \ln(t^2 + 1), \quad y = \tan^{-1} t,$
$z = e^t; \quad t = 1$

6. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}; \quad t = 1$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, (a) express $\partial z/\partial r$ and $\partial z/\partial \theta$ as functions of r and θ both by using the Chain Rule and by expressing z directly in terms of r and θ before differentiating. Then (b) evaluate $\partial z/\partial r$ and $\partial z/\partial \theta$ at the given point (r, θ).

7. $z = 4e^x \ln y, \quad x = \ln(r\cos\theta), \quad y = r\sin\theta;$
$(r, \theta) = (2, \pi/4)$

8. $z = \tan^{-1}(x/y), \quad x = r\cos\theta, \quad y = r\sin\theta;$
$(r, \theta) = (1.3, \pi/6)$

In Exercises 9 and 10, (a) express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then (b) evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

9. $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

10. $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

In Exercises 11 and 12, (a) express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then (b) evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

11. $u = \dfrac{p-q}{q-r}, \quad p = x + y + z, \quad q = x - y + z,$
$r = x + y - z; \quad (x, y, z) = (\sqrt{3}, 2, 1)$

12. $u = e^{qr} \sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = 1/z;$
$(x, y, z) = (\pi/4, 1/2, -1/2)$

Using a Tree Diagram

In Exercises 13–24, draw a tree diagram and write a Chain Rule formula for each derivative.

13. $\dfrac{dz}{dt}$ for $z = f(x, y), \quad x = g(t), \quad y = h(t)$

14. $\dfrac{dz}{dt}$ for $z = f(u, v, w), \quad u = g(t), \quad v = h(t), \quad w = k(t)$

15. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v),$
$z = k(u, v)$

16. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = f(r, s, t), \quad r = g(x, y), \quad s = h(x, y),$
$t = k(x, y)$

17. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = g(x, y), \quad x = h(u, v), \quad y = k(u, v)$

18. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = g(u, v), \quad u = h(x, y), \quad v = k(x, y)$

19. $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$ for $z = f(x, y), \quad x = g(t, s), \quad y = h(t, s)$

20. $\dfrac{\partial y}{\partial r}$ for $y = f(u), \quad u = g(r, s)$

21. $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ for $w = g(u), \quad u = h(s, t)$

22. $\dfrac{\partial w}{\partial p}$ for $w = f(x, y, z, v), \quad x = g(p, q), \quad y = h(p, q),$
$z = j(p, q), \quad v = k(p, q)$

23. $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ for $w = f(x, y), \quad x = g(r), \quad y = h(s)$

24. $\dfrac{\partial w}{\partial s}$ for $w = g(x, y), \quad x = h(r, s, t), \quad y = k(r, s, t)$

Implicit Differentiation

Assuming that the equations in Exercises 25–28 define y as a differentiable function of x, use Eq. (10) to find the value of dy/dx at the given point.

25. $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

26. $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

27. $x^2 + xy + y^2 - 7 = 0, \quad (1, 2)$

28. $xe^y + \sin xy + y - \ln 2 = 0, \quad (0, \ln 2)$

Equation (10) can be generalized to functions of three variables and even more. The three-variable version goes like this:

If the equation $F(x, y, z) = 0$ determines z as a differentiable function of x and y, then, at points where $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \tag{12}$$

Use these equations to find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.

29. $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

30. $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}-1=0, \quad (2, 3, 6)$

31. $\sin(x+y)+\sin(y+z)+\sin(x+z)=0, \quad (\pi, \pi, \pi)$

32. $xe^y+ye^z+2\ln x-2-3\ln 2=0, \quad (1, \ln 2, \ln 3)$

Finding Specified Partial Derivatives

33. Find $\partial w/\partial r$ when $r=1, \quad s=-1$ if $w=(x+y+z)^2$, $x=r-s, \quad y=\cos(r+s), \quad z=\sin(r+s)$.

34. Find $\partial w/\partial v$ when $u=-1, \quad v=2$ if $w=xy+\ln z$, $x=v^2/u, \quad y=u+v, \quad z=\cos u$.

35. Find $\partial w/\partial v$ when $u=0, \quad v=0$ if $w=x^2+(y/x)$, $x=u-2v+1, \quad y=2u+v-2$.

36. Find $\partial z/\partial u$ when $u=0, \quad v=1$ if $z=\sin xy+x\sin y$, $x=u^2+v^2, \quad y=uv$.

37. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u=\ln 2, \quad v=1$ if $z=5\tan^{-1}x$ and $x=e^u+\ln v$.

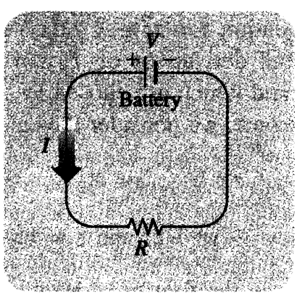
38. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u=1$ and $v=-2$ if $z=\ln q$ and $q=\sqrt{v+3}\tan^{-1}u$.

Theory and Examples

39. *Changing voltage in a circuit.* The voltage V in a circuit that satisfies the law $V=IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$$

to find how the current is changing at the instant when $R=600$ ohms, $I=0.04$ amp, $dR/dt=0.5$ ohm/sec, and $dV/dt=-0.01$ volt/sec.



40. *Changing dimensions in a box.* The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a=1$ m, $b=2$ m, $c=3$ m, $da/dt=db/dt=1$ m/sec, and $dc/dt=-3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length, or decreasing?

41. If $f(u, v, w)$ is differentiable and $u=x-y, v=y-z$, and $w=z-x$, show that

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0.$$

42. a) Show that if we substitute polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$ in a differentiable function $w=f(x, y)$, then

$$\frac{\partial w}{\partial r}=f_x\cos\theta+f_y\sin\theta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \theta}=-f_x\sin\theta+f_y\cos\theta.$$

b) Solve the equations in (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

c) Show that

$$(f_x)^2+(f_y)^2=\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2.$$

43. Show that if $w=f(u, v)$ satisfies the Laplace equation $f_{uu}+f_{vv}=0$, and if $u=(x^2-y^2)/2$ and $v=xy$, then w satisfies the Laplace equation $w_{xx}+w_{yy}=0$.

44. Let $w=f(u)+g(v)$, where $u=x+iy$ and $v=x-iy$ and $i=\sqrt{-1}$. Show that w satisfies the Laplace equation $w_{xx}+w_{yy}=0$ if all the necessary functions are differentiable.

Changes in Functions along Curves

45. Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x=\cos t$, $y=\sin t$, $z=t$ are

$$f_x=\cos t, \quad f_y=\sin t, \quad f_z=t^2+t-2.$$

At what points on the curve, if any, can f take on extreme values?

46. Let $w=x^2e^{2y}\cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x=\cos t$, $y=\ln(t+2)$, $z=t$.

47. Let $T=f(x, y)$ be the temperature at the point (x, y) on the circle $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$, and suppose that

$$\frac{\partial T}{\partial x}=8x-4y, \quad \frac{\partial T}{\partial y}=8y-4x.$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2.

b) Suppose $T=4x^2-4xy+4y^2$. Find the maximum and minimum values of T on the circle.

48. Let $T=g(x, y)$ be the temperature at the point (x, y) on the ellipse

$$x=2\sqrt{2}\cos t, \quad y=\sqrt{2}\sin t, \quad 0\le t\le 2\pi,$$

and suppose that

$$\frac{\partial T}{\partial x}=y, \quad \frac{\partial T}{\partial y}=x.$$

a) Locate the maximum and minimum temperatures on the ellipse by examining dT/dt and d^2T/dt^2.

b) Suppose that $T=xy-2$. Find the maximum and minimum values of T on the ellipse.

Differentiating Integrals

Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\, dt,$$

then $F'(x) = \int_a^b g_x(t, x)\, dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\, dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\, dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 49 and 50.

49. $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\, dt$

50. $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2}\, dt$

12.6 *Partial Derivatives with Constrained Variables

In finding partial derivatives of functions like $w = f(x, y)$, we have assumed x and y to be independent. But in many applications this is not the case. For example, the internal energy U of a gas may be expressed as a function $U = f(P, V, T)$ of pressure P, volume V, and temperature T. If the individual molecules of the gas do not interact, however, P, V, and T obey the ideal gas law

$$PV = nRT \qquad (n \text{ and } R \text{ constant})$$

and so fail to be independent. Finding partial derivatives in situations like these can be complicated. But it is better to face the complication now than to meet it for the first time while you are also trying to learn economics, engineering, or physics.

Decide Which Variables Are Dependent and Which Are Independent

If the variables in a function $w = f(x, y, z)$ are constrained by a relation like the one imposed on x, y, and z by the equation $z = x^2 + y^2$, the geometric meanings and the numerical values of the partial derivatives of f will depend on which variables are chosen to be dependent and which are chosen to be independent. To see how this choice can affect the outcome, we consider the calculation of $\partial w/\partial x$ when $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

EXAMPLE 1 Find $\partial w/\partial x$ if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

Solution We are given two equations in the four unknowns x, y, z, and w. Like many such systems, this one can be solved for two of the unknowns (the dependent variables) in terms of the others (the independent variables). In being asked for $\partial w/\partial x$, we are told that w is to be a dependent variable and x an independent variable. The possible choices for the other variables come down to

Dependent	*Independent*
w, z	x, y
w, y	x, z

In either case, we can express w explicitly in terms of the selected independent

*This section is based on notes written for MIT by Arthur P. Mattuck.

variables. We do this by using the second equation to eliminate the remaining dependent variable in the first equation.

In the first case, the remaining dependent variable is z. We eliminate it from the first equation by replacing it by $x^2 + y^2$. The resulting expression for w is

$$\begin{aligned} w &= x^2 + y^2 + z^2 = x^2 + y^2 + (x^2 + y^2)^2 \\ &= x^2 + y^2 + x^4 + 2x^2y^2 + y^4 \end{aligned}$$

and

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2. \tag{1}$$

This is the formula for $\partial w/\partial x$ when x and y are the independent variables.

In the second case, where the independent variables are x and z and the remaining dependent variable is y, we eliminate the dependent variable y in the expression for w by replacing y^2 by $z - x^2$. This gives

$$w = x^2 + y^2 + z^2 = x^2 + (z - x^2) + z^2 = z + z^2$$

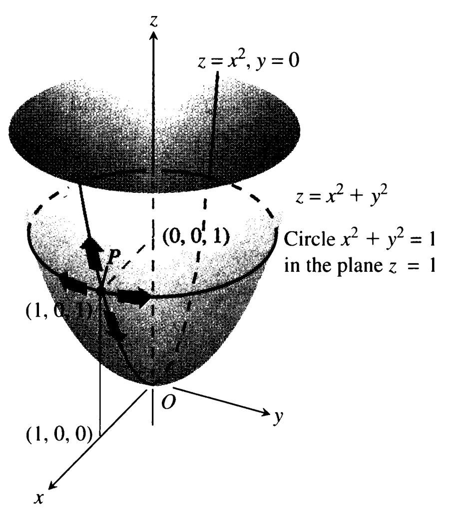
and

$$\frac{\partial w}{\partial x} = 0. \tag{2}$$

This is the formula for $\partial w/\partial x$ when x and z are the independent variables.

The formulas for $\partial w/\partial x$ in Eqs. (1) and (2) are genuinely different. We cannot change either formula into the other by using the relation $z = x^2 + y^2$. There is not just one $\partial w/\partial x$, there are two, and we see that the original instruction to find $\partial w/\partial x$ was incomplete. *Which* $\partial w/\partial x$? we ask.

The geometric interpretations of Eqs. (1) and (2) help to explain why the equations differ. The function $w = x^2 + y^2 + z^2$ measures the square of the distance from the point (x, y, z) to the origin. The condition $z = x^2 + y^2$ says that the point (x, y, z) lies on the paraboloid of revolution shown in Fig. 12.32. What does it mean to calculate $\partial w/\partial x$ at a point $P(x, y, z)$ that can move only on this surface? What is the value of $\partial w/\partial x$ when the coordinates of P are, say, $(1, 0, 1)$?



12.32 If P is constrained to lie on the paraboloid $z = x^2 + y^2$, the value of the partial derivative of $w = x^2 + y^2 + z^2$ with respect to x at P depends on the direction of motion (Example 1). (a) As x changes, with $y = 0$, P moves up or down the surface on the parabola $z = x^2$ in the xz-plane with $\partial w/\partial x = 2x + 4x^3$. (b) As x changes, with $z = 1$, P moves on the circle $x^2 + y^2 = 1, z = 1$, and $\partial w/\partial x = 0$.

If we take x and y to be independent, then we find $\partial w/\partial x$ by holding y fixed (at $y = 0$ in this case) and letting x vary. This means that P moves along the parabola $z = x^2$ in the xz-plane. As P moves on this parabola, w, which is the square of the distance from P to the origin, changes. We calculate $\partial w/\partial x$ in this case (our first solution above) to be

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2.$$

At the point $P(1, 0, 1)$, the value of this derivative is

$$\frac{\partial w}{\partial x} = 2 + 4 + 0 = 6.$$

If we take x and z to be independent, then we find $\partial w/\partial x$ by holding z fixed while x varies. Since the z-coordinate of P is 1, varying x moves P along a circle in the plane $z = 1$. As P moves along this circle, its distance from the origin remains constant, and w, being the square of this distance, does not change. That is,

$$\frac{\partial w}{\partial x} = 0,$$

as we found in our second solution. ❑

How to Find $\partial w/\partial x$ When the Variables in $w = f(x, y, z)$ Are Constrained by Another Equation

As we saw in Example 1, a typical routine for finding $\partial w/\partial x$ when the variables in the function $w = f(x, y, z)$ are related by another equation has three steps. These steps apply to finding $\partial w/\partial y$ and $\partial w/\partial z$ as well.

Step 1 *Decide* which variables are to be dependent and which are to be independent. (In practice, the decision is based on the physical or theoretical context of our work. In the exercises at the end of this section, we say which variables are which.)

Step 2 *Eliminate* the other dependent variable(s) in the expression for w.

Step 3 *Differentiate* as usual.

If we cannot carry out step 2 after deciding which variables are dependent, we differentiate the equations as they are and try to solve for $\partial w/\partial x$ afterward. The next example shows how this is done.

EXAMPLE 2 Find $\partial w/\partial x$ at the point $(x, y, z) = (2, -1, 1)$ if

$$w = x^2 + y^2 + z^2, \qquad z^3 - xy + yz + y^3 = 1,$$

and x and y are the independent variables.

Solution It is not convenient to eliminate z in the expression for w. We therefore differentiate both equations implicitly with respect to x, treating x and y as independent variables and w and z as dependent variables. This gives

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \tag{3}$$

and

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} + 0 = 0. \tag{4}$$

These equations may now be combined to express $\partial w/\partial x$ in terms of x, y, and z. We solve Eq. (4) for $\partial z/\partial x$ to get

$$\frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

and substitute into Eq. (3) to get

$$\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y + 3z^2}.$$

The value of this derivative at $(x, y, z) = (2, -1, 1)$ is

$$\left(\frac{\partial w}{\partial x}\right)_{(2,-1,1)} = 2(2) + \frac{2(-1)(1)}{-1 + 3(1)^2} = 4 + \frac{-2}{2} = 3.$$

❑

To show what variables are assumed to be independent in calculating a derivative, we can use the following notation:

$$\left(\frac{\partial w}{\partial x}\right)_y \qquad \partial w/\partial x \text{ with } x \text{ and } y \text{ independent}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,t} \qquad \partial f/\partial y \text{ with } y,\ x, \text{ and } t \text{ independent.}$$

EXAMPLE 3 Find $\left(\dfrac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2 + y - z + \sin t$ and $x + y = t$.

Solution

With x, y, z independent, we have

$$t = x + y, \qquad w = x^2 + y - z + \sin(x + y)$$

$$\begin{aligned}\left(\frac{\partial w}{\partial x}\right)_{y,z} &= 2x + 0 - 0 + \cos(x + y)\frac{\partial}{\partial x}(x + y)\\ &= 2x + \cos(x + y).\end{aligned}$$

❑

Arrow Diagrams

In solving problems like the one in Example 3, it often helps to start with an arrow diagram that shows how the variables and functions are related. If

$$w = x^2 + y - z + \sin t \qquad \text{and} \qquad x + y = t$$

and we are asked to find $\partial w/\partial x$ when x, y, and z are independent, the appropriate diagram is one like this:

$$\begin{array}{ccccc}\begin{pmatrix} x \\ y \\ z \end{pmatrix} & \to & \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} & \to & w \\ \text{independent} & & \text{intermediate} & & \text{dependent} \\ \text{variables} & & \text{variables and} & & \text{variable} \\ & & \text{relations} & & \\ & & x = x & & \\ & & y = y & & \\ & & z = z & & \\ & & t = x + y & & \end{array} \tag{5}$$

The diagram shows the independent variables on the left, the intermediate variables and their relation to the independent variables in the middle, and the dependent variable on the right.

To find $\partial w/\partial x$, we first apply the four-variable form of the Chain Rule to w, getting

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}. \tag{6}$$

We then use the formula for $w = x^2 + y - z + \sin t$ to evaluate the partial derivatives of w that appear on the right-hand side of Eq. (6). This gives

$$\begin{aligned}\frac{\partial w}{\partial x} &= 2x\frac{\partial x}{\partial x} + (1)\frac{\partial y}{\partial x} + (-1)\frac{\partial z}{\partial x} + \cos t\frac{\partial t}{\partial x}\\ &= 2x\frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} - \frac{\partial z}{\partial x} + \cos t\frac{\partial t}{\partial x}.\end{aligned} \tag{7}$$

To calculate the remaining partial derivatives, we apply what we know about the dependence and independence of the variables involved. As shown in the diagram (5), the variables x, y, and z are independent and $t = x + y$. Hence,

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial y}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = 0, \quad \frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(x + y) = (1 + 0) = 1.$$

We substitute these values into Eq. (7) to find $\partial w/\partial x$:

$$\begin{aligned}\left(\frac{\partial w}{\partial x}\right)_{y,z} &= 2x(1) + 0 - 0 + (\cos t)(1)\\ &= 2x + \cos t\\ &= 2x + \cos(x + y). \qquad \text{In terms of the independent variables}\end{aligned}$$

Exercises 12.6

Finding Partial Derivatives with Constrained Variables

In Exercises 1–3, begin by drawing a diagram that shows the relations among the variables.

1. If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, find

a) $\left(\frac{\partial w}{\partial y}\right)_z$ b) $\left(\frac{\partial w}{\partial z}\right)_x$ c) $\left(\frac{\partial w}{\partial z}\right)_y$

2. If $w = x^2 + y - z + \sin t$ and $x + y = t$, find

a) $\left(\frac{\partial w}{\partial y}\right)_{x,z}$ b) $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ c) $\left(\frac{\partial w}{\partial z}\right)_{x,y}$

d) $\left(\frac{\partial w}{\partial z}\right)_{y,t}$ e) $\left(\frac{\partial w}{\partial t}\right)_{x,z}$ f) $\left(\frac{\partial w}{\partial t}\right)_{y,z}$

3. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a) $\left(\frac{\partial U}{\partial P}\right)_V$ b) $\left(\frac{\partial U}{\partial T}\right)_V$

4. Find

a) $\left(\frac{\partial w}{\partial x}\right)_y$ b) $\left(\frac{\partial w}{\partial z}\right)_y$

at the point $(x, y, z) = (0, 1, \pi)$ if

$$w = x^2 + y^2 + z^2 \quad \text{and} \quad y\sin z + z\sin x = 0.$$

5. Find

a) $\left(\frac{\partial w}{\partial y}\right)_x$ b) $\left(\frac{\partial w}{\partial y}\right)_z$

at the point $(w, x, y, z) = (4, 2, 1, -1)$ if

$$w = x^2y^2 + yz - z^3 \quad \text{and} \quad x^2 + y^2 + z^2 = 6.$$

6. Find $\left(\frac{\partial u}{\partial y}\right)_x$ at the point $(u, v) = (\sqrt{2}, 1)$ if $x = u^2 + v^2$ and $y = uv$.

7. Suppose that $x^2 + y^2 = r^2$ and $x = r\cos\theta$, as in polar coordinates. Find

$$\left(\frac{\partial x}{\partial r}\right)_\theta \quad \text{and} \quad \left(\frac{\partial r}{\partial x}\right)_y.$$

8. Suppose that

$$w = x^2 - y^2 + 4z + t \quad \text{and} \quad x + 2z + t = 25.$$

Show that the equations

$$\frac{\partial w}{\partial x} = 2x - 1 \quad \text{and} \quad \frac{\partial w}{\partial x} = 2x - 2$$

each give $\partial w/\partial x$, depending on which variables are chosen to be dependent and which variables are chosen to be independent. Identify the independent variables in each case.

Partial Derivatives without Specific Formulas

9. Establish the fact, widely used in hydrodynamics, that if $f(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

(*Hint:* Express all the derivatives in terms of the formal partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.)

10. If $z = x + f(u)$, where $u = xy$, show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x.$$

11. Suppose that the equation $g(x, y, z) = 0$ determines z as a differentiable function of the independent variables x and y and that $g_z \neq 0$. Show that

$$\left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}.$$

12. Suppose that $f(x, y, z, w) = 0$ and $g(x, y, z, w) = 0$ determine z and w as differentiable functions of the independent variables

x and y, and suppose that

$$\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} \neq 0.$$

Show that

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial x}}{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial z}}$$

and

$$\left(\frac{\partial w}{\partial y}\right)_x = -\frac{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z}}{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial z}}.$$

12.7 Directional Derivatives, Gradient Vectors, and Tangent Planes

We know from Section 12.5 that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

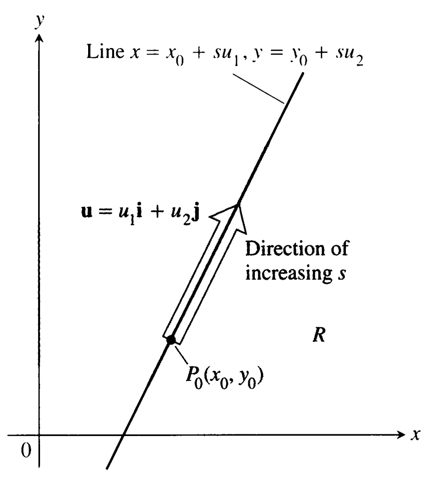
At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of motion along the curve. This observation is particularly important when the curve is a straight line and t is the arc length parameter along the line measured from P_0 in the direction of a given unit vector **u.** For then df/dt is the rate of change of f with respect to distance in its domain in the direction of **u.** By varying **u,** we find the rates at which f changes with respect to distance as we move through P_0 in different directions. These "directional derivatives" have useful interpretations in science and engineering as well as in mathematics. This section develops a formula for calculating them and proceeds from there to find equations for tangent planes and normal lines on surfaces in space.

Directional Derivatives in the Plane

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_0 + su_1, \qquad y = y_0 + su_2$$

parametrize the line through P_0 parallel to **u.** The parameter s measures arc length from P_0 in the direction of **u.** We find the rate of change of f at P_0 in the direction of **u** by calculating df/ds at P_0 (Fig. 12.33):



12.33 The rate of change of f in the direction of **u** at a point P_0 is the rate at which f changes along this line at P_0.

Definition

The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

The directional derivative is also denoted by

$$(D_{\mathbf{u}}f)_{P_0}. \qquad \text{"The derivative of } f \text{ at } P_0 \text{ in the direction of } \mathbf{u}\text{"}$$

EXAMPLE 1 Find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}.$

Solution

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \qquad \text{Eq. (1)}$$

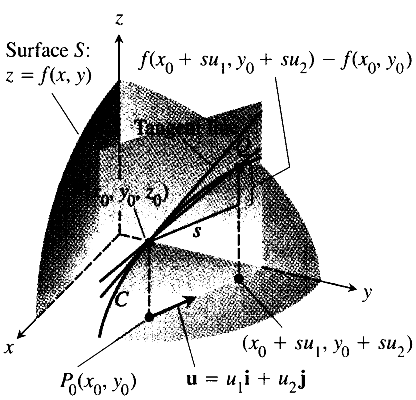
$$= \lim_{s\to 0} \frac{f\left(1 + s\cdot\frac{1}{\sqrt{2}}, 2 + s\cdot\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \left(\frac{5}{\sqrt{2}} + 0\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$ is $5/\sqrt{2}$. ❑



12.34 The slope of curve C at P_0 is

$$\lim_{Q\to P} \text{slope}(PQ) = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} = \left(\frac{df}{ds}\right)_{\mathbf{u}, P_0}$$

Geometric Interpretation of the Directional Derivative

The equation $z = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\mathbf{u}$ intersects S in a curve C (Fig. 12.34). The rate of change of f in the direction of $\mathbf{u}$ is the slope of the tangent to C at P.

Notice that when $\mathbf{u} = \mathbf{i}$ the directional derivative at P_0 is $\partial f/\partial x$ evaluated at (x_0, y_0). When $\mathbf{u} = \mathbf{j}$ the directional derivative at P_0 is $\partial f/\partial y$ evaluated at (x_0, y_0). The directional derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction $\mathbf{u}$, not just the directions $\mathbf{i}$ and $\mathbf{j}$.

Calculation

As you know, it is rarely convenient to calculate a derivative directly from its definition as a limit, and the directional derivative is no exception. We can develop a more efficient formula in the following way. We begin with the line

$$x = x_0 + su_1, \qquad y = y_0 + su_2, \tag{2}$$

through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. Then

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} && \text{Chain Rule}\\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0}\cdot u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}\cdot u_2 && \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2\\
&= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]}_{\textit{gradient} \text{ of } f \text{ at } P_0}\cdot\underbrace{\Big[u_1\,\mathbf{i} + u_2\,\mathbf{j}\Big]}_{\text{direction } \mathbf{u}}. && (3)
\end{aligned}$$

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f, read the way it is written.

Definition

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Equation (3) says that the derivative of f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with the gradient of f at P_0.

Theorem 6

If the partial derivatives of $f(x, y)$ are defined at $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0}\cdot\mathbf{u}, \tag{4}$$

the scalar product of the gradient f at P_0 and $\mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{A} = 3\,\mathbf{i} - 4\,\mathbf{j}$.

Solution The direction of $\mathbf{A}$ is obtained by dividing $\mathbf{A}$ by its length:

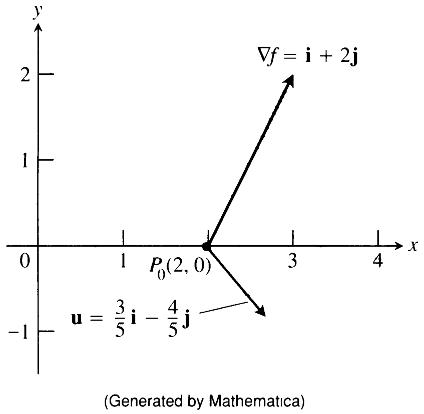
$$\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f at $(2, 0)$ are

$$\begin{aligned}
f_x(2, 0) &= (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1\\
f_y(2, 0) &= (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2.
\end{aligned}$$

The gradient of f at $(2, 0)$ is

$$\nabla f\big|_{(2,0)} = f_x(2, 0)\,\mathbf{i} + f_y(2, 0)\,\mathbf{j} = \mathbf{i} + 2\,\mathbf{j}$$



12.35 It is customary to picture ∇f as a vector in the domain of f. In the case of $f(x, y) = xe^y + \cos(xy)$, the domain is the entire plane. The rate at which f changes in the direction $\mathbf{u} = (3/5)\,\mathbf{i} - (4/5)\,\mathbf{j}$ is $\nabla f \cdot \mathbf{u} = -1$ (Example 2).

(Fig. 12.35). The derivative of f at (2, 0) in the direction of **A** is therefore

$$\begin{aligned}(D_{\mathbf{u}}f)\big|_{(2,0)} &= \nabla f\big|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4)}\\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.\end{aligned}$$

❑

Properties of Directional Derivatives

Evaluating the dot product in the formula

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, or when $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f|\cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to the gradient is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

As we will discuss later, these properties hold in three dimensions as well as two.

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$ (a) increases most rapidly and (b) decreases most rapidly at the point (1, 1). (c) What are the directions of zero change in f at (1, 1)?

Solution

a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient is

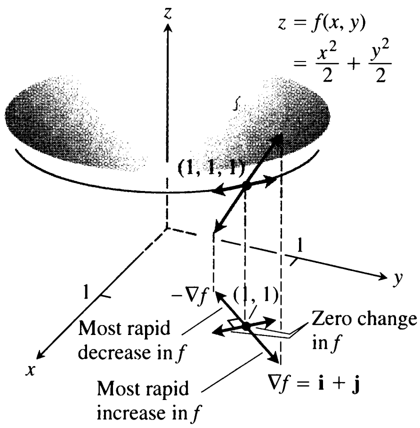
$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$



12.36 The direction in which $f(x, y) = (x^2/2) + (y^2/2)$ increases most rapidly at (1, 1) is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at (1, 1, 1).

c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Fig. 12.36. ❑

Gradients and Tangents to Level Curves

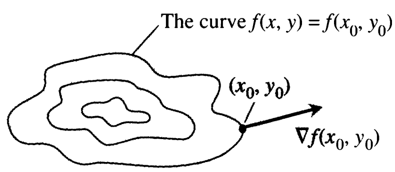
If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt}f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0. \tag{5}$$

Equation (5) says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve.



12.37 The gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

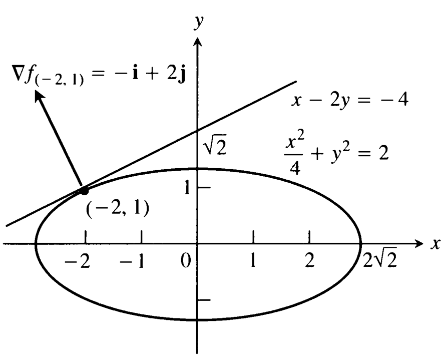
> At every point (x_0, y_0) in the domain of $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Fig. 12.37).

This observation enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

(Exercise 59). If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_0, y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, the equation becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0. \tag{6}$$



12.38 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Fig. 12.38) at the point $(-2, 1)$.

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent is the line

$$(-1)(x+2)+(2)(y-1)=0 \qquad \text{Eq. (6)}$$
$$x-2y=-4.$$

❑

Functions of Three Variables

We obtain three-variable formulas by adding the z-terms to the two-variable formulas. For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u}=u_1\,\mathbf{i}+u_2\,\mathbf{j}+u_3\,\mathbf{k}$ in space, we have

$$\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=\frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2+\frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=|\nabla f||u|\cos\theta=|\nabla f|\cos\theta,$$

so the properties listed earlier for functions of two variables continue to hold. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

EXAMPLE 5

a) Find the derivative of $f(x, y, z)=x^3-xy^2-z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{A}=2\,\mathbf{i}-3\,\mathbf{j}+6\,\mathbf{k}$.

b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

a) The direction of $\mathbf{A}$ is obtained by dividing $\mathbf{A}$ by its length:

$$|\mathbf{A}|=\sqrt{(2)^2+(-3)^2+(6)^2}=\sqrt{49}=7$$
$$\mathbf{u}=\frac{\mathbf{A}}{|\mathbf{A}|}=\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x=3x^2-y^2\Big|_{(1,1,0)}=2,$$
$$f_y=-2xy\Big|_{(1,1,0)}=-2, \qquad f_z=-1\Big|_{(1,1,0)}=-1.$$

The gradient of f at P_0 is

$$\nabla f\Big|_{(1,1,0)}=2\,\mathbf{i}-2\,\mathbf{j}-\mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{A}$ is therefore

$$(D_{\mathbf{u}}f)\Big|_{(1,1,0)}=\nabla f\Big|_{(1,1,0)}\cdot\mathbf{u}=(2\,\mathbf{i}-2\,\mathbf{j}-\mathbf{k})\cdot\left(\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}\right)$$
$$=\frac{4}{7}+\frac{6}{7}-\frac{6}{7}=\frac{4}{7}.$$

b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

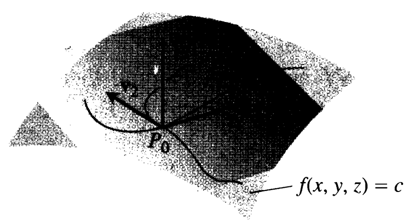
Equations for Tangent Planes and Normal Lines

If $\mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j} + \frac{dk}{dt}\mathbf{k}\right)}_{d\mathbf{r}/dt} = 0. \tag{7}$$



12.39 ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

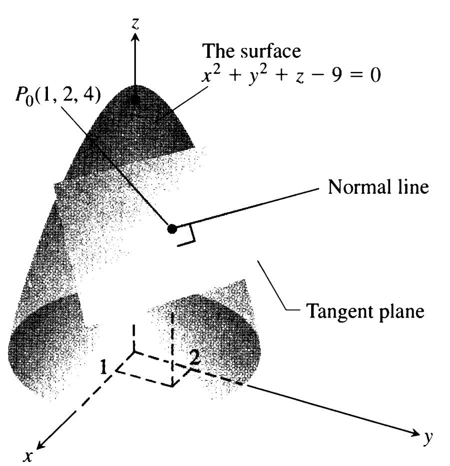
At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

Now let us restrict our attention to the curves that pass through P_0 (Fig. 12.39). All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to ∇f. We call this plane the tangent plane of the surface at P_0. The line through P_0 perpendicular to the plane is the surface's normal line at P_0.

Definitions

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.



12.40 The tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ at $P_0(1, 2, 4)$ (Example 6).

Thus, from Section 10.5, the tangent plane and normal line, respectively, have the following equations:

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \tag{8}$$

$$x = x_0 + f_x(P_0)\,t, \quad y = y_0 + f_y(P_0)\,t, \quad z = z_0 + f_z(P_0)\,t. \tag{9}$$

EXAMPLE 6 Find the tangent plane and normal line of the surface

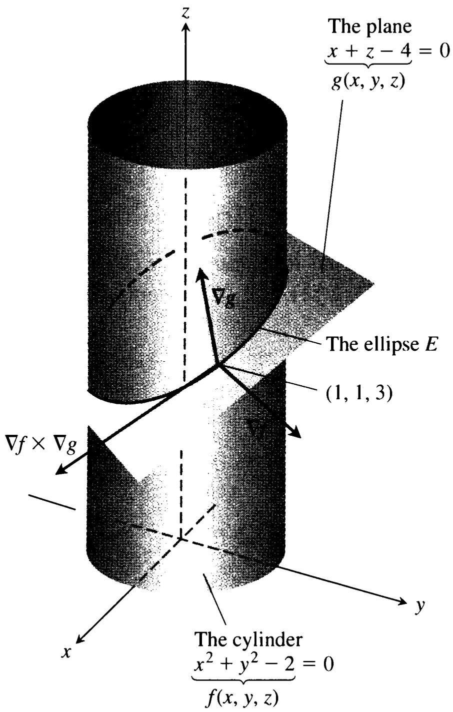
$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Fig. 12.40.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$



12.41 The cylinder $f(x, y, z) = x^2 + y^2 - 2 = 0$ and the plane $g(x, y, z) = x + z - 4 = 0$ intersect in an ellipse E (Example 7).

The plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t. \qquad \square$$

EXAMPLE 7 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Fig. 12.41). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f_{(1,1,3)} = (2x\,\mathbf{i} + 2y\,\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The line is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t. \qquad \square$$

Planes Tangent to a Surface $z = f(x, y)$

To find an equation for the plane tangent to a surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

The formula

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0 \qquad \text{Eq. (8) restated for } F(x, y, z)$$

for the plane tangent to the level surface at P_0 therefore reduces to

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

> The plane tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is
> $$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \tag{10}$$

EXAMPLE 8 Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Eq. (10):

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$
$$f_y(0, 0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (10)}$$

or

$$x - y - z = 0.$$

Increments and Distance

The directional derivative plays the role of an ordinary derivative when we want to estimate how much a function f changes if we move a small distance ds from a point P_0 to another point nearby. If f were a function of a single variable, we would have

$$df = f'(P_0)\,ds. \qquad \text{Ordinary derivative} \times \text{increment}$$

For a function of two or more variables, we use the formula

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})\,ds, \qquad \text{Directional derivative} \times \text{increment}$$

where $\mathbf{u}$ is the direction of the motion away from P_0.

> **Estimating the Change in f in a Direction u**
>
> To estimate the change in the value of a function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula
> $$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{directional}\\ \text{derivative}}} \cdot \underbrace{ds}_{\substack{\text{distance}\\ \text{increment}}}$$

EXAMPLE 9 Estimate how much the value of

$$f(x, y, z) = xe^y + yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(2, 0, 0)$ straight toward $P_1(4, 1, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector

$$\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(2,0,0)} = (e^y\mathbf{i} + (xe^y + z)\mathbf{j} + y\mathbf{k})|_{(2,0,0)} = \mathbf{i} + 2\mathbf{j}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(\frac{4}{3}\right)(0.1) \approx 0.13.$$

❑

Algebra Rules for Gradients

If we know the gradients of two functions f and g, we automatically know the gradients of their constant multiples, sum, difference, product, and quotient.

These rules have the same form as the corresponding rules for derivatives, as they should (Exercise 65).

Algebra Rules for Gradients

1. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
2. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
3. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE 10 We illustrate the rules with

$$f(x, y, z) = x - y \qquad g(x, y, z) = z$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = \mathbf{k}.$$

We have:

1. $\nabla(2f) = \nabla(2x - 2y) = 2\mathbf{i} - 2\mathbf{j} = 2\nabla f$
2. $\nabla(f + g) = \nabla(x - y + z) = \mathbf{i} - \mathbf{j} + \mathbf{k} = \nabla f + \nabla g$
3. $\nabla(f - g) = \nabla(x - y - z) = \mathbf{i} - \mathbf{j} - \mathbf{k} = \nabla f - \nabla g$
4. $\nabla(fg) = \nabla(xz - yz) = z\mathbf{i} - z\mathbf{j} + (x - y)\mathbf{k} = g\nabla f + f\nabla g$

5. $$\nabla\left(\frac{f}{g}\right)=\nabla\left(\frac{x-y}{z}\right)=\frac{\partial}{\partial x}\left(\frac{x-y}{z}\right)\mathbf{i}+\frac{\partial}{\partial y}\left(\frac{x-y}{z}\right)\mathbf{j}+\frac{\partial}{\partial z}\left(\frac{x-y}{z}\right)\mathbf{k}$$
$$=\frac{1}{z}\mathbf{i}-\frac{1}{z}\mathbf{j}+\frac{z\cdot 0-(x-y)\cdot 1}{z^2}\mathbf{k}$$
$$=\frac{z\,\mathbf{i}-z\,\mathbf{j}-(x-y)\,\mathbf{k}}{z^2}=\frac{g\nabla f-f\nabla g}{g^2}$$ ❑

Exercises 12.7

Calculating Gradients at Points

In Exercises 1–4, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x$, $(2, 1)$
2. $f(x, y) = \ln(x^2 + y^2)$, $(1, 1)$
3. $g(x, y) = y - x^2$, $(-1, 0)$
4. $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2}, 1)$

In Exercises 5–8, find ∇f at the given point.

5. $f(x, y, z) = x^2 + y^2 - 2z^2 + z\ln x$, $(1, 1, 1)$
6. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz$, $(1, 1, 1)$
7. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz)$, $(-1, 2, -2)$
8. $f(x, y, z) = e^{x+y}\cos z + (y + 1)\sin^{-1} x$, $(0, 0, \pi/6)$

Finding Directional Derivatives in the *xy*-Plane

In Exercises 9–16, find the derivative of the function at P_0 in the direction of $\mathbf{A}$.

9. $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathbf{A} = 4\mathbf{i} + 3\mathbf{j}$
10. $f(x, y) = 2x^2 + y^2$, $P_0(-1, 1)$, $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$
11. $g(x, y) = x - (y^2/x) + \sqrt{3}\sec^{-1}(2xy)$, $P_0(1, 1)$, $\mathbf{A} = 12\mathbf{i} + 5\mathbf{j}$
12. $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$, $P_0(1, 1)$, $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j}$
13. $f(x, y, z) = xy + yz + zx$, $P_0(1, -1, 2)$, $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
14. $f(x, y, z) = x^2 + 2y^2 - 3z^2$, $P_0(1, 1, 1)$, $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
15. $g(x, y, z) = 3e^x\cos yz$, $P_0(0, 0, 0)$, $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
16. $h(x, y, z) = \cos xy + e^{yz} + \ln zx$, $P_0(1, 0, 1/2)$, $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Directions of Most Rapid Increase and Decrease

In Exercises 17–22, find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

17. $f(x, y) = x^2 + xy + y^2$, $P_0(-1, 1)$
18. $f(x, y) = x^2y + e^{xy}\sin y$, $P_0(1, 0)$
19. $f(x, y, z) = (x/y) - yz$, $P_0(4, 1, 1)$
20. $g(x, y, z) = xe^y + z^2$, $P_0(1, \ln 2, 1/2)$
21. $f(x, y, z) = \ln xy + \ln yz + \ln xz$, $P_0(1, 1, 1)$
22. $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z$, $P_0(1, 1, 0)$

Estimating Change

23. By about how much will
$$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$$
change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ units in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?
24. By about how much will
$$f(x, y, z) = e^x\cos yz$$
change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ units in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?
25. By about how much will
$$g(x, y, z) = x + x\cos z - y\sin z + y$$
change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ units toward the point $P_1(0, 1, 2)$?
26. By about how much will
$$h(x, y, z) = \cos(\pi xy) + xz^2$$
change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ units toward the origin?

Tangent Planes and Normal Lines to Surfaces

In Exercises 27–34, find equations for the (a) tangent plane and (b) normal line at the point P_0 on the given surface.

27. $x^2 + y^2 + z^2 = 3$, $P_0(1, 1, 1)$

28. $x^2 + y^2 - z^2 = 18, \quad P_0(3, 5, -4)$

29. $2z - x^2 = 0, \quad P_0(2, 0, 2)$

30. $x^2 + 2xy - y^2 + z^2 = 7, \quad P_0(1, -1, 3)$

31. $\cos \pi x - x^2 y + e^{xz} + yz = 4, \quad P_0(0, 1, 2)$

32. $x^2 - xy - y^2 - z = 0, \quad P_0(1, 1, -1)$

33. $x + y + z = 1, \quad P_0(0, 1, 0)$

34. $x^2 + y^2 - 2xy - x + 3y - z = -4, \quad P_0(2, -3, 18)$

In Exercises 35–38, find an equation for the plane that is tangent to the given surface at the given point.

35. $z = \ln(x^2 + y^2), \quad (1, 0, 0)$ **36.** $z = e^{-(x^2+y^2)}, \quad (0, 0, 1)$

37. $z = \sqrt{y - x}, \quad (1, 2, 1)$ **38.** $z = 4x^2 + y^2, \quad (1, 1, 5)$

Tangent Lines to Curves

In Exercises 39–42, sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.

39. $x^2 + y^2 = 4, \quad (\sqrt{2}, \sqrt{2})$

40. $x^2 - y = 1, \quad (\sqrt{2}, 1)$

41. $xy = -4, \quad (2, -2)$

42. $x^2 - xy + y^2 = 7, \quad (-1, 2)$ (This is the curve in Section 2.6, Example 4.)

In Exercises 43–48, find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

43. Surfaces: $x + y^2 + 2z = 4, \quad x = 1$
Point: $(1, 1, 1)$

44. Surfaces: $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$
Point: $(1, 1, 1)$

45. Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$
Point: $(1, 1, 1/2)$

46. Surfaces: $x + y^2 + z = 2, \quad y = 1$
Point: $(1/2, 1, 1/2)$

47. Surfaces: $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$,
$x^2 + y^2 + z^2 = 11$
Point: $(1, 1, 3)$

48. Surfaces: $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$
Point: $\left(\sqrt{2}, \sqrt{2}, 4\right)$

Theory and Examples

49. In what directions is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?

50. In what two directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?

51. Is there a direction $\mathbf{A}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

52. Is there a direction $\mathbf{A}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3°\text{C/ft}$? Give reasons for your answer.

53. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3. What is the derivative of f in the direction of $-\mathbf{i} - 2\mathbf{j}$? Give reasons for your answer.

54. The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction the value of the derivative is $2\sqrt{3}$.

k) What is ∇f at P? Give reasons for your answer.
l) What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?

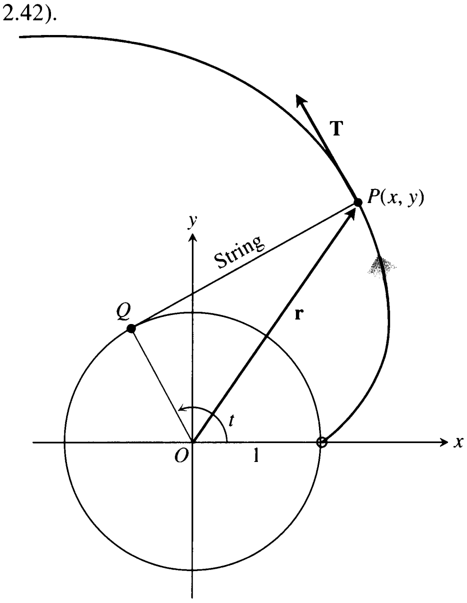
55. *Temperature change along a circle.* Suppose that the Celsius temperature at the point (x, y) in the xy-plane is $T(x, y) = x \sin 2y$ and that distance in the xy-plane is measured in meters. A particle is moving *clockwise* around the circle of radius 1 m centered at the origin at the constant rate of 2 m/sec.

a) How fast is the temperature experienced by the particle changing in °C/m at the point $P(1/2, \sqrt{3}/2)$?
b) How fast is the temperature experienced by the particle changing in °C/sec at P?

56. *Change along the involute of a circle.* Find the derivative of $f(x, y) = x^2 + y^2$ in the direction of the unit tangent vector of the curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\,\mathbf{i} + (\sin t - t \cos t)\,\mathbf{j}, \quad t > 0$$

(Fig. 12.42).



12.42 The involute of the unit circle from Section 11.3, Example 5. If you move out along the involute, covering distance along the curve at a constant rate, your distance from the origin will increase at a constant rate as well. (This is how to interpret the result of your calculation in Exercise 56.)

57. *Change along a helix.* Find the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in the direction of the unit tangent vector of the helix

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$$

at the points where $t = -\pi/4, 0$, and $\pi/4$. The function f gives the square of the distance from a point $P(x, y, z)$ on the helix to the origin. The derivatives calculated here give the rates at which the square of the distance is changing with respect to t as P moves through the points where $t = -\pi/4, 0$, and $\pi/4$.

58. The Celsius temperature in a region in space is given by $T(x, y, z) = 2x^2 - xyz$. A particle is moving in this region and its position at time t is given by $x = 2t^2$, $y = 3t$, $z = -t^2$, where time is measured in seconds and distance in meters.

a) How fast is the temperature experienced by the particle changing in °C/m when the particle is at the point $P(8, 6, -4)$?

b) How fast is the temperature experienced by the particle changing in °C/sec at P?

59. Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy-plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\,\mathbf{i} + B\,\mathbf{j}$.

60. *Normal curves and tangent curves.* A curve is *normal* to a surface $f(x, y, z) = c$ at a point of intersection if the curve's velocity vector is a scalar multiple of ∇f at the point. The curve is *tangent* to the surface at a point of intersection if its velocity vector is orthogonal to ∇f there.

a) Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t}\,\mathbf{j} - \frac{1}{4}(t + 3)\,\mathbf{k}$$

is normal to the surface $x^2 + y^2 - z = 3$ when $t = 1$.

b) Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + (2t - 1)\,\mathbf{k}$$

is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$.

61. *Another way to see why gradients are normal to level curves.* Suppose that a differentiable function $f(x, y)$ has a constant value c along the differentiable curve $x = g(t)$, $y = h(t)$ for all values of t. Differentiate both sides of the equation $f(g(t), h(t)) = c$ with respect to t to show that ∇f is normal to the curve's tangent vector at every point.

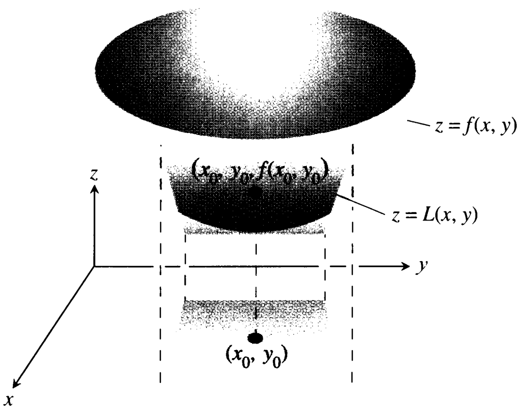
62. *The linearization of f(x, y) is a tangent-plane approximation.* Show that the tangent plane at the point $P_0(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ defined by a differentiable function f is the plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus the tangent plane at P_0 is the graph of the linearization of f at P_0 (Fig. 12.43).



12.43 The graph of a function $z = f(x, y)$ and its linearization at a point (x_0, y_0). The plane defined by L is tangent to the surface at the point above the point (x_0, y_0). This furnishes a geometric explanation of why the values of L lie close to those of f in the immediate neighborhood of (x_0, y_0) (Exercise 62).

63. *Directional derivatives and scalar components.* How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$ related to the scalar component of $(\nabla f)_{P_0}$ in the direction of $\mathbf{u}$? Give reasons for your answer.

64. *Directional derivatives and partial derivatives.* Assuming that the necessary derivatives of $f(x, y, z)$ are defined, how are $D_{\mathbf{i}}f$, $D_{\mathbf{j}}f$, and $D_{\mathbf{k}}f$ related to f_x, f_y, and f_z? Give reasons for your answer.

65. *The algebra rules for gradients.* Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$\nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k},$$

use the scalar equations

$$\frac{\partial}{\partial x}(kf) = k\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x},$$

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}}{g^2},$$

and so on, to establish the following rules:

a) $\nabla(kf) = k\nabla f$

b) $\nabla(f + g) = \nabla f + \nabla g$

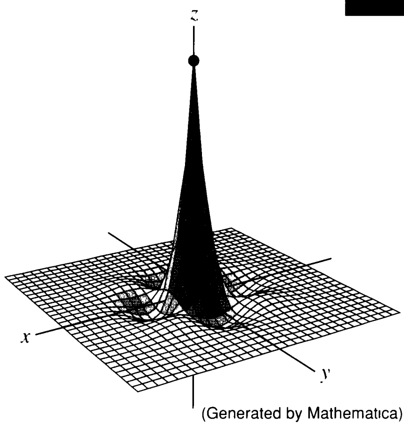
c) $\nabla(f - g) = \nabla f - \nabla g$

d) $\nabla(fg) = f\nabla g + g\nabla f$

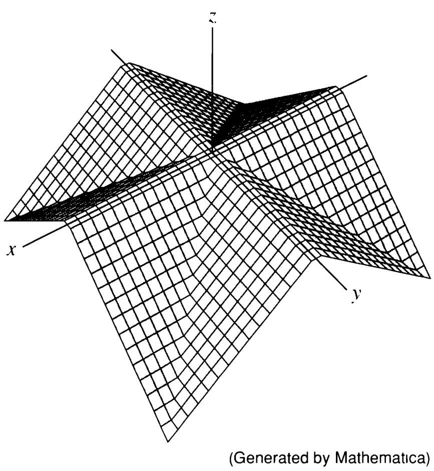
e) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

12.8 Extreme Values and Saddle Points

Continuous functions defined on closed bounded regions in the xy-plane take on absolute maximum and minimum values on these domains (Figs. 12.44 and 12.45). It is important to be able to find these values and to know where they occur. We can often accomplish this by examining partial derivatives.



12.44 The function
$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$
has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2$, $|y| \leq 3\pi/2$.



12.45 The "roof surface"
$$z = \frac{1}{2}(||x| - |y|| - |x| - |y|)$$
viewed from the point (10, 15, 20). The defining function has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \leq a$, $|y| \leq a$.

The Derivative Tests

To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points we then look for local maxima, local minima, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent *plane.* At such points we then look for local maxima, local minima, and saddle points (more about saddle points in a moment).

Definitions

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

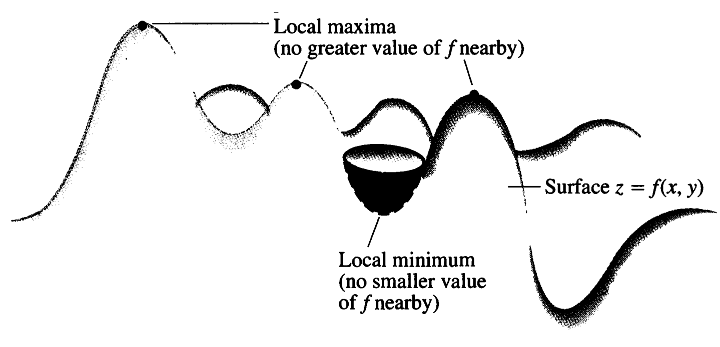
1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms (Fig. 12.46). At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema.**

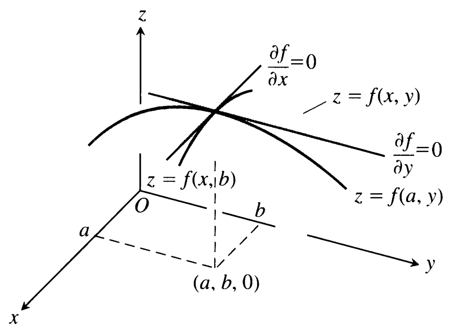
As with functions of a single variable, the key to identifying the local extrema is a first derivative test.

Theorem 7
First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain, and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.



12.46 A local maximum is a mountain peak and a local minimum is a valley low.



12.47 The maximum of f occurs at $x = a, y = b$.

Proof Suppose that f has a local maximum value at an interior point (a, b) of its domain. Then

1. $x = a$ is an interior point of the domain of the curve $z = f(x, b)$ in which the plane $y = b$ cuts the surface $z = f(x, y)$ (Fig. 12.47).
2. The function $z = f(x, b)$ is a differentiable function of x at $x = a$ (the derivative is $f_x(a, b)$).
3. The function $z = f(x, b)$ has a local maximum value at $x = a$.
4. The value of the derivative of $z = f(x, b)$ at $x = a$ is therefore zero (Theorem 2, Section 3.1). Since this derivative is $f_x(a, b)$, we conclude that $f_x(a, b) = 0$.

A similar argument with the function $z = f(a, y)$ shows that $f_y(a, b) = 0$.

This proves the theorem for local maximum values. The proof for local minimum values is left as Exercise 48. ❑

If we substitute the values $f_x(a, b) = 0$ and $f_y(a, b) = 0$ into the equation

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

for the tangent plane to the surface $z = f(x, y)$ at (a, b), the equation reduces to

$$0 \cdot (x - a) + 0 \cdot (y - b) - z + f(a, b) = 0$$

or

$$z = f(a, b).$$

Thus, Theorem 7 says that the surface does indeed have a horizontal tangent plane at a local extremum, provided there is a tangent plane there.

As in the single-variable case, Theorem 7 says that the only places a function $f(x, y)$ can ever have an extreme value are

1. Interior points where $f_x = f_y = 0$,
2. Interior points where one or both of f_x and f_y do not exist,
3. Boundary points of the function's domain.

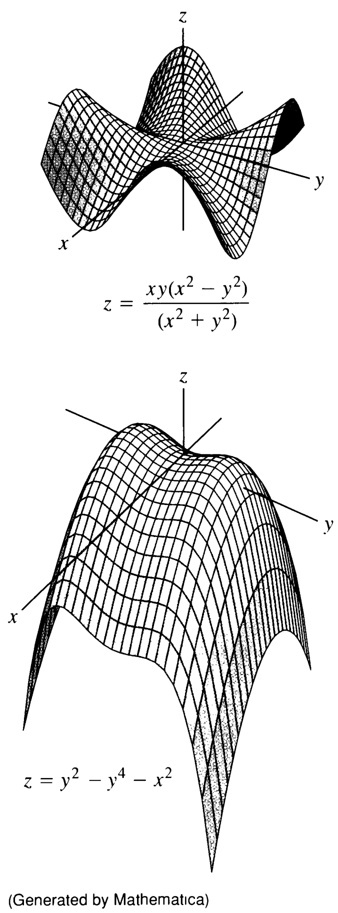
Definition

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

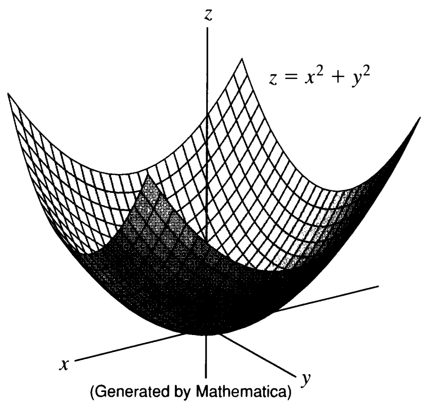
Thus, the only points where a function $f(x, y)$ can assume extreme values are critical points and boundary points. As with differentiable functions of a single variable, not every critical point gives rise to a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variables might have a saddle point.

Definition

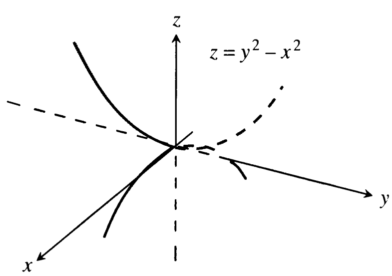
A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Fig. 12.48).



12.48 Saddle points at the origin.



12.49 The graph of the function $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$. The function has only one critical point, the origin, which gives rise to a local minimum value of 0 (Example 1).



12.50 The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values (Example 2).

EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum (Fig. 12.49). ❑

EXAMPLE 2 Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$. However, along the positive x-axis f has the value $f(x, 0) = -x^2 < 0$; along the positive y-axis f has the value $f(0, y) = y^2 > 0$. Therefore every open disk in the xy-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin (Fig. 12.50) instead of a local extreme value. We conclude that the function has no local extreme values. ❑

The fact that $f_x = f_y = 0$ at an interior point (a, b) of R does not tell us enough to be sure f has a local extreme value there. However, if f and its first and second partial derivatives are continuous on R, we may be able to learn the rest from the following theorem, proved in Section 12.10.

Theorem 8
Second Derivative Test for Local Extreme Values

Suppose $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a, b);
ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a, b);
iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at (a, b).
iv) **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - {f_{xy}}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - {f_{xy}}^2$ is called the **discriminant** of f. It is sometimes easier to remember the determinant form,

$$f_{xx}f_{yy} - {f_{xy}}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Theorem 8 says that if the discriminant is positive at the point (a, b), then the surface curves the same way in all directions: downwards if $f_{xx} < 0$, giving rise to a local maximum, and upwards if $f_{xx} > 0$, giving a local minimum. On the other hand, if the discriminant is negative at (a, b), then the surface curves up in some directions and down in others, so we have a saddle point.

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

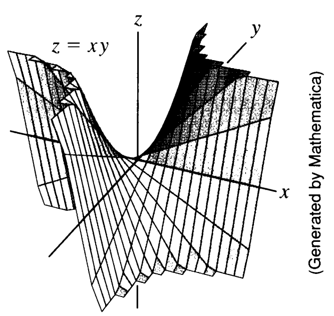
The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.



12.51 The surface $z = xy$ has a saddle point at the origin (Example 4).

EXAMPLE 4 Find the local extreme values of $f(x, y) = xy$.

Solution Since f is differentiable everywhere (Fig. 12.51), it can assume extreme values only where

$$f_x = y = 0 \qquad \text{and} \qquad f_y = x = 0.$$

Thus, the origin is the only point where f might have an extreme value. To see what happens there, we calculate

$$f_{xx} = 0, \qquad f_{yy} = 0, \qquad f_{xy} = 1.$$

The discriminant,

$$f_{xx}f_{yy} - f_{xy}^2 = -1,$$

is negative. Therefore the function has a saddle point at (0, 0). We conclude that $f(x, y) = xy$ has no local extreme values.

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

Step 1: *List the interior points* of R where f may have local maxima and minima and evaluate f at these points. These are the points where $f_x = f_y = 0$ or where one or both of f_x and f_y fail to exist (the critical points of f).

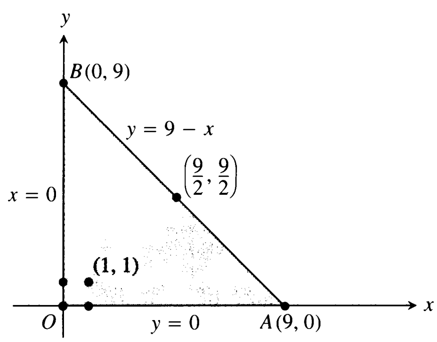
Step 2: *List the boundary points* of R where f has local maxima and minima and evaluate f at these points. We will show how to do this shortly.

Step 3: *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f already appear somewhere in the lists made in steps 1 and 2. We have only to glance at the lists to see what they are.

EXAMPLE 5 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.



12.52 This triangular plate is the domain of the function in Example 5.

Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle (Fig. 12.52) where $f_x = f_y = 0$ and points on the boundary.

Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 2 - 2y = 0,$$

yielding the single point $(x, y) = (1, 1)$. The value of f there is

$$f(1, 1) = 4.$$

Boundary points. We take the triangle one side at a time:

1. On the segment OA, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \le x \le 9$. Its extreme values (we know from Chapter 3) may occur at the endpoints

$$x = 0 \qquad \text{where} \quad f(0, 0) = 2$$

$$x = 9 \qquad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

2. On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 2y - y^2.$$

We know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -61, \qquad f(0, 1) = 3.$$

3. We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}.$$

At this value of x,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \qquad \text{and} \qquad f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary. We list all the candidates: 4, 2, -61, 3, $-(41/2)$. The maximum is 4, which f assumes at (1, 1). The minimum is -61, which f assumes at (0, 9) and (9, 0). ❑

Conclusion

Despite the power of Theorem 7, we urge you to remember its limitations. It does not apply to boundary points of a function's domain, where it is possible for a function to have extreme values along with nonzero derivatives. And it does not apply to points where either f_x or f_y fails to exist.

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

(i) **boundary points** of the domain of f,
(ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first and second order partial derivatives of f are continuous throughout a disk centered at a point (a, b), and $f_x(a, b) = f_y(a, b) = 0$, you may be able to classify $f(a, b)$ with the ***second derivative test:***

(i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum,**
(ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum,**
(iii) $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point,**
(iv) $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive.**

Exercises 12.8

Finding Local Extrema

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$
3. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
4. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$
5. $f(x, y) = x^2 + xy + 3x + 2y + 5$
6. $f(x, y) = y^2 + xy - 2x - 2y + 2$
7. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$
8. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
9. $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$
10. $f(x, y) = 3x^2 + 6xy + 7y^2 - 2x + 4y$
11. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$
12. $f(x, y) = 4x^2 - 6xy + 5y^2 - 20x + 26y$
13. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$
14. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$
15. $f(x, y) = x^2 + 2xy$
16. $f(x, y) = 3 + 2x + 2y - 2x^2 - 2xy - y^2$
17. $f(x, y) = x^3 - y^3 - 2xy + 6$
18. $f(x, y) = x^3 + 3xy + y^3$
19. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
20. $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

21. $f(x, y) = 9x^3 + y^3/3 - 4xy$

22. $f(x, y) = 8x^3 + y^3 + 6xy$

23. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

24. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

25. $f(x, y) = 4xy - x^4 - y^4$

26. $f(x, y) = x^4 + y^4 + 4xy$

27. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$ **28.** $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$

29. $f(x, y) = y \sin x$ **30.** $f(x, y) = e^{2x} \cos y$

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant

32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$

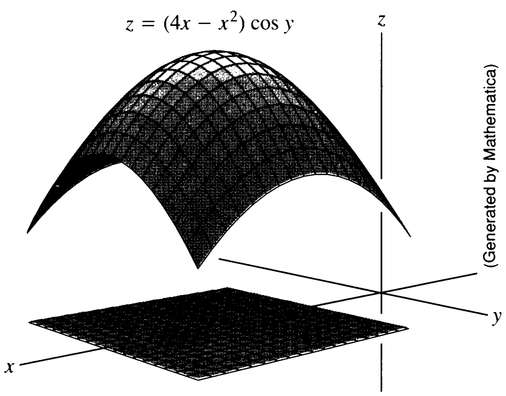
33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant

34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 3$

35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$

36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1$, $0 \le y \le 1$

37. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (Fig. 12.53)



12.53 The function and domain in Exercise 37.

38. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the first quadrant

39. Find two numbers a and b with $a \le b$ such that

$$\int_a^b (6 - x - x^2)\, dx$$

has its largest value.

40. Find two numbers a and b with $a \le b$ such that

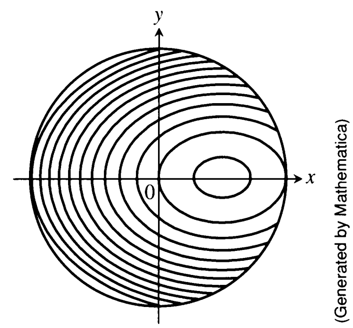
$$\int_a^b (24 - 2x - x^2)^{1/3} dx$$

has its largest value.

41. *Temperatures.* The flat circular plate in Fig. 12.54 has the shape of the region $x^2 + y^2 \le 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is

$$T(x, y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

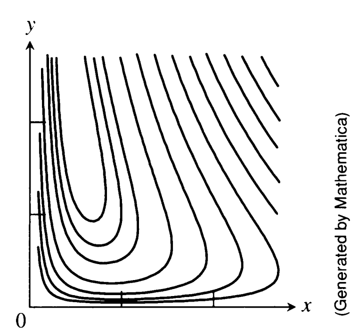


12.54 Curves of constant temperature are called isotherms. The figure shows isotherms of the temperature function $T(x, y) = x^2 + 2y^2 - x$ on the disk $x^2 + y^2 \le 1$ in the xy-plane. Exercise 41 asks you to locate the extreme temperatures.

42. Find the critical point of

$$f(x, y) = xy + 2x - \ln x^2 y$$

in the open first quadrant ($x > 0$, $y > 0$) and show that f takes on a minimum there (Fig. 12.55).



12.55 The function $f(x, y) = xy + 2x - \ln x^2 y$ (selected level curves shown here) takes on a minimum value somewhere in the open first quadrant $x > 0, y > 0$ (Exercise 42).

Theory and Examples

43. Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that

a) $f_x = 2x - 4y$ and $f_y = 2y - 4x$
b) $f_x = 2x - 2$ and $f_y = 2y - 4$
c) $f_x = 9x^2 - 9$ and $f_y = 2y + 4$

Describe your reasoning in each case.

44. The discriminant $f_{xx} f_{yy} - f_{xy}{}^2$ is zero at the origin for each of the following functions, so the second derivative test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.

a) $f(x, y) = x^2y^2$ **b)** $f(x, y) = 1 - x^2y^2$
c) $f(x, y) = xy^2$ **d)** $f(x, y) = x^3y^2$
e) $f(x, y) = x^3y^3$ **f)** $f(x, y) = x^4y^4$

45. Show that (0, 0) is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (*Hint:* Consider two cases: $k = 0$ and $k \neq 0$.)

46. For what values of the constant k does the second derivative test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at (0, 0)? a local minimum at (0, 0)? For what values of k is the second derivative test inconclusive? Give reasons for your answers.

47. a) If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.
b) Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.

48. Using the proof of Theorem 7 given in the text for the case in which f has a local maximum at (a, b), prove the theorem for the case in which f has a local minimum at (a, b).

49. Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

50. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. The function $f(x, y) = x + y$ fails to have an absolute maximum value in the closed first quadrant $x \geq 0$ and $y \geq 0$. Does this contradict the discussion on finding absolute extrema given in the text? Give reasons for your answer.

52. Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

a) Show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What *is* the absolute minimum value?
b) Find the absolute maximum value of f over the square.

Extreme Values on Parametrized Curves

To find the extreme values of a function $f(x, y)$ on a curve $x = x(t)$, $y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

a) critical points (points where df/dt is zero or fails to exist), and
b) endpoints of the parameter domain.

Find the absolute maximum and minimum values of the following functions on the given curves.

53. Functions:

a) $f(x, y) = x + y$ **b)** $g(x, y) = xy$
c) $h(x, y) = 2x^2 + y^2$

Curves:

i) The semicircle $x^2 + y^2 = 4$, $y \geq 0$
ii) The quarter circle $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$

Use the parametric equations $x = 2 \cos t$, $y = 2 \sin t$.

54. Functions:

a) $f(x, y) = 2x + 3y$ **b)** $g(x, y) = xy$
c) $h(x, y) = x^2 + 3y^2$

Curves:

i) The semi-ellipse $(x^2/9) + (y^2/4) = 1$, $y \geq 0$
ii) The quarter ellipse $(x^2/9) + (y^2/4) = 1$, $x \geq 0$, $y \geq 0$

Use the parametric equations $x = 3 \cos t$, $y = 2 \sin t$.

55. Function: $f(x, y) = xy$
Curves:

i) The line $x = 2t$, $y = t + 1$
ii) The line segment $x = 2t$, $y = t + 1$, $-1 \leq t \leq 0$
iii) The line segment $x = 2t$, $y = t + 1$, $0 \leq t \leq 1$

56. Functions:

a) $f(x, y) = x^2 + y^2$ **b)** $g(x, y) = 1/(x^2 + y^2)$

Curves:

i) The line $x = t$, $y = 2 - 2t$
ii) The line segment $x = t$, $y = 2 - 2t$, $0 \leq t \leq 1$

Least Squares and Regression Lines

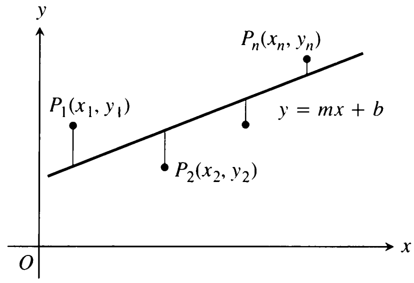
When we try to fit a line $y = mx + b$ to a set of numerical data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ (Fig. 12.56, on the following page), we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \tag{1}$$

The values of m and b that do this are found with the first and second derivative tests to be

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n \sum x_k y_k}{\left(\sum x_k\right)^2 - n \sum x_k^2}, \tag{2}$$

$$b = \frac{1}{n}\left(\sum y_k - m \sum x_k\right), \tag{3}$$



12.56 To fit a line to noncollinear points, we choose the line that minimizes the sum of the squares of the deviations.

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few key presses after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line, regression line,** or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x,
3. handle data analytically.

EXAMPLE Find the least squares line for the points (0, 1), (1, 3), (2, 2), (3, 4), (4, 5).

Solution We organize the calculations in a table:

k	x_k	y_k	x_k^2	$x_k y_k$
1	0	1	0	0
2	1	3	1	3
3	2	2	4	4
4	3	4	9	12
5	4	5	16	20
Σ	10	15	30	39

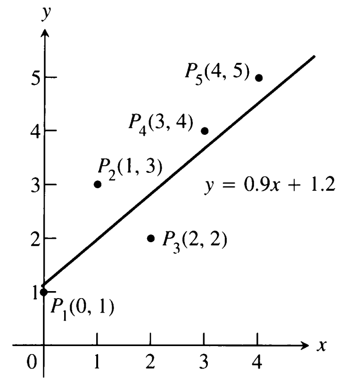
Then we find

$$m = \frac{(10)(15) - 5(39)}{(10)^2 - 5(30)} = 0.9 \qquad \text{Eq. (2) with } n = 5 \text{ and data from the table}$$

and use the value of m to find

$$b = \frac{1}{5}(15 - (0.9)(10)) = 1.2. \qquad \text{Eq. (3) with } n = 5,\ m = 0.9$$

The least squares line is $y = 0.9x + 1.2$ (Fig. 12.57). ❑



12.57 The least squares line for the data in the example.

In Exercises 57–60, use Eqs. (2) and (3) to find the least squares line for each set of data points. Then use the linear equation you obtain to predict the value of y that would correspond to $x = 4$.

57. $(-1, 2)$, $(0, 1)$, $(3, -4)$ **58.** $(-2, 0)$, $(0, 2)$, $(2, 3)$

59. $(0, 0)$, $(1, 2)$, $(2, 3)$ **60.** $(0, 1)$, $(2, 2)$, $(3, 2)$

61. Write a linear equation for the effect of irrigation on the yield of alfalfa by fitting a least squares line to the data in Table 12.1 (from the University of California Experimental Station, *Bulletin* No. 450, p. 8). Plot the data and draw the line.

Table 12.1 Growth of alfalfa

x (total seasonal depth of water applied, in.)	y (average alfalfa yield, tons/acre)
12	5.27
18	5.68
24	6.25
30	7.21
36	8.20
42	8.71

62. *Craters of Mars.* One theory of crater formation suggests that the frequency of large craters should fall off as the square of the diameter (Marcus, *Science,* June 21, 1968, p. 1334). Pictures from *Mariner IV* show the frequencies listed in Table 12.2. Fit a line of the form $F = m(1/D^2) + b$ to the data. Plot the data and draw the line.

Table 12.2 Crater sizes on Mars

Diameter in km, D	$1/D^2$ (for left value of class interval)	Frequency, F
32–45	0.001	51
45–64	0.0005	22
64–90	0.00024	14
90–128	0.000123	4

Table 12.3 Compositions by Mozart

Köchel number, K	Year composed, y
1	1761
75	1771
155	1772
219	1775
271	1777
351	1780
425	1783
503	1786
575	1789
626	1791

63. *Köchel numbers.* In 1862, the German musicologist Ludwig von Köchel made a chronological list of the musical works of Wolfgang Amadeus Mozart. This list is the source of the Köchel numbers, or "K numbers," that now accompany the titles of Mozart's pieces (Sinfonia Concertante in E-flat major, K.364, for example). Table 12.3 gives the Köchel numbers and composition dates (y) of ten of Mozart's works.

a) Plot y vs. K to show that y is close to being a linear function of K.

b) Find a least squares line $y = m\text{K} + b$ for the data and add the line to your plot in (a).

c) K.364 was composed in 1779. What date is predicted by the least squares line?

64. *Submarine sinkings.* The data in Table 12.4 show the results of a historical study of German submarines sunk by the U.S. Navy during 16 consecutive months of World War II. The data given for each month are the number of reported sinkings and the number of actual sinkings. The number of submarines sunk was slightly greater than the Navy's reports implied. Find a least squares line for estimating the number of actual sinkings from the number of reported sinkings.

Table 12.4 Sinkings of German submarines by U.S. during 16 consecutive months of WWII

Month	Guesses by U.S. (reported sinkings) x	Actual number y
1	3	3
2	2	2
3	4	6
4	2	3
5	5	4
6	5	3
7	9	11
8	12	9
9	8	10
10	13	16
11	14	13
12	3	5
13	4	6
14	13	19
15	10	15
16	16	15
	123	140

CAS Explorations and Projects

In Exercises 65–70, you will explore functions to identify their local extrema. Use a CAS to perform the following steps:

a) Plot the function over the given rectangle.

b) Plot some level curves in the rectangle.

c) Calculate the function's first partial derivatives and use the CAS equation solver to find the critical points. How do the critical points relate to the level curves plotted in (b)? Which critical points, if any, appear to give a saddle point? Give reasons for your answer.

d) Calculate the function's second partial derivatives and find the discriminant $f_{xx}f_{yy} - f_{xy}{}^2$.

e) Using the max-min tests, classify the critical points found in (c). Are your findings consistent with your discussion in (c)?

65. $f(x, y) = x^2 + y^3 - 3xy, \quad -5 \le x \le 5, \quad -5 \le y \le 5$

66. $f(x, y) = x^3 - 3xy^2 + y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

67. $f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16, \quad -3 \le x \le 3, \quad -6 \le y \le 6$

68. $f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2 + 3, \quad -3/2 \le x \le 3/2, \quad -3/2 \le y \le 3/2$

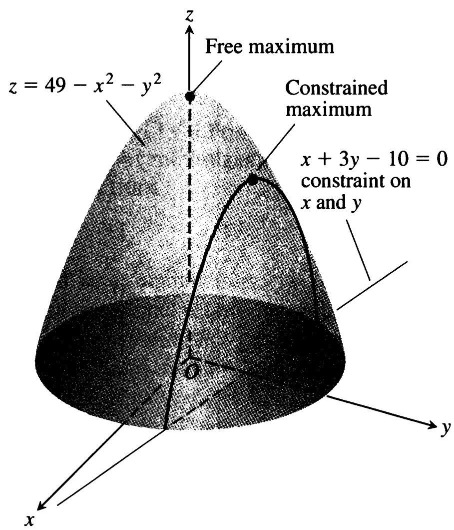
69. $f(x, y) = 5x^6 + 18x^5 - 30x^4 + 30xy^2 - 120x^3, \quad -4 \le x \le 3, \quad -2 \le y \le 2$

70. $f(x, y) = \begin{cases} x^5 \ln(x^2 + y^2), & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases},$

$-2 \le x \le 2, \quad -2 \le y \le 2$

12.9

Lagrange Multipliers



12.58 The function $f(x, y) = 49 - x^2 - y^2$, subject to the constraint $g(x, y) = x + 3y - 10 = 0$.

As we saw in Section 12.8, we sometimes need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane—a disk, for example, or a closed triangular region. But, as Fig. 12.58 suggests, a function may be subject to other kinds of constraints as well.

In this section, we explore a powerful method for finding extreme values of constrained functions: the method of *Lagrange multipliers.* Lagrange developed the method in 1755 to solve max-min problems in geometry. Today the method is important in economics, in engineering (where it is used in designing multistage rockets, for example), and in mathematics.

Constrained Maxima and Minima

EXAMPLE 1 Find the point $P(x, y, z)$ closest to the origin on the plane $2x + y - z - 5 = 0$.

Solution The problem asks us to find the minimum value of the function

$$\begin{aligned} |\overrightarrow{OP}| &= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \\ &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Since $|\overrightarrow{OP}|$ has a minimum value wherever the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

has a minimum value, we may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint $2x + y - z - 5 = 0$. If we regard x and y as the independent variables in this equation and write z as

$$z = 2x + y - 5,$$

our problem reduces to one of finding the points (x, y) at which the function

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

has its minimum value or values. Since the domain of h is the entire xy-plane, the first derivative test of Section 12.8 tells us that any minima that h might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \qquad h_y = 2y + 2(2x + y - 5) = 0.$$

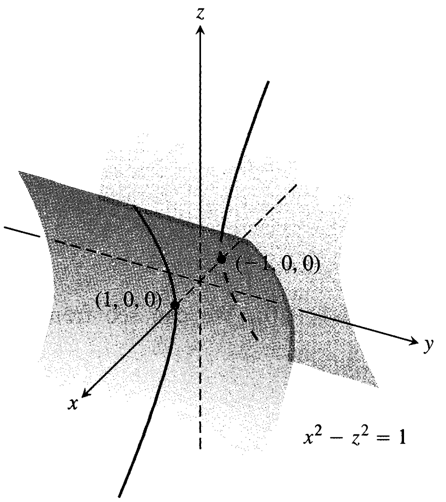
This leads to

$$10x + 4y = 20, \qquad 4x + 4y = 10,$$

and the solution

$$x = \frac{5}{3}, \qquad y = \frac{5}{6}.$$

We may apply a geometric argument together with the second derivative test to show that these values minimize h. The z-coordinate of the corresponding point on



12.59 The hyperbolic cylinder $x^2 - z^2 - 1 = 0$ in Example 2.

the plane $z = 2x + y - 5$ is

$$z = 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}.$$

Therefore, the point we seek is

$$\text{Closest point:} \qquad P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The distance from P to the origin is $5/\sqrt{6} \approx 2.04$. ❑

Attempts to solve a constrained maximum or minimum problem by substitution, as we might call the method of Example 1, do not always go smoothly. This is one of the reasons for learning the new method of this section.

EXAMPLE 2 Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

Solution 1 The cylinder is shown in Fig. 12.59. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Square of the distance}$$

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard x and y as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

and the values of $f(x, y, z) = x^2 + y^2 + z^2$ on the cylinder are given by the function

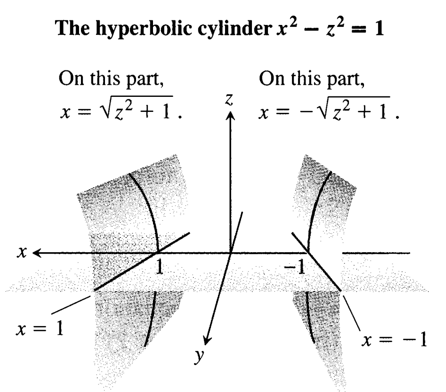
$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

To find the points on the cylinder whose coordinates minimize f, we look for the points in the xy-plane whose coordinates minimize h. The only extreme value of h occurs where

$$h_x = 4x = 0 \qquad \text{and} \qquad h_y = 2y = 0,$$

that is, at the point $(0, 0)$. But now we're in trouble—there are no points on the cylinder where both x and y are zero. What went wrong?

What happened was that the first derivative test found (as it should have) the point *in the domain of h* where h has a minimum value. We, on the other hand, want the points *on the cylinder* where h has a minimum value. While the domain of h is the entire xy-plane, the domain from which we can select the first two coordinates of the points (x, y, z) on the cylinder is restricted to the "shadow" of the cylinder on the xy-plane; it does not include the band between the lines $x = -1$ and $x = 1$ (Fig. 12.60).



12.60 The region in the xy-plane from which the first two coordinates of the points (x, y, z) on the hyperbolic cylinder $x^2 - z^2 = 1$ are selected excludes the band $-1 < x < 1$ in the xy-plane.

We can avoid this problem if we treat y and z as independent variables (instead of x and y) and express x in terms of y and z as

$$x^2 = z^2 + 1.$$

With this substitution, $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

and we look for the points where k takes on its smallest value. The domain of

k in the yz-plane now matches the domain from which we select the y- and z-coordinates of the points (x, y, z) on the cylinder. Hence, the points that minimize k in the plane will have corresponding points on the cylinder. The smallest values of k occur where

$$k_y = 2y = 0 \qquad \text{and} \qquad k_z = 4z = 0,$$

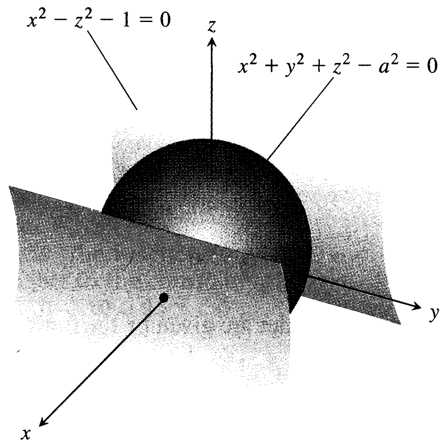
or where $y = z = 0$. This leads to

$$x^2 = z^2 + 1 = 1, \qquad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$. We can see from the inequality

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

that the points $(\pm 1, 0, 0)$ give a minimum value for k. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.



12.61 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder

$$x^2 - z^2 - 1 = 0.$$

See Solution 2 of Example 2.

Solution 2 Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Fig. 12.61). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \qquad \text{and} \qquad g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact we should therefore be able to find a scalar λ ("lambda") such that

$$\nabla f = \lambda \nabla g,$$

or

$$2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = \lambda(2x\,\mathbf{i} - 2z\,\mathbf{k}).$$

Thus, the coordinates x, y, and z of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z. \tag{1}$$

For what values of λ will a point (x, y, z) whose coordinates satisfy the equations in (1) also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use the fact that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$ in the first equation in (1). This means that $2x = 2\lambda x$ only if

$$2 = 2\lambda, \qquad \text{or} \qquad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, z must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \qquad x^2 = 1, \qquad \text{or} \qquad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ❑

The Method of Lagrange Multipliers

In Solution 2 of Example 2, we solved the problem by the **method of Lagrange multipliers.** In general terms, the method says that the extreme values of a function $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are to be found on the surface $g = 0$ at the points where

$$\nabla f = \lambda \nabla g$$

for some scalar λ (called a **Lagrange multiplier**).

To explore the method further and see why it works, we first make the following observation, which we state as a theorem.

Theorem 9

The Orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof We show that ∇f is orthogonal to the curve's velocity vector at P_0. The values of f on C are given by the composite $f(g(t), h(t), k(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = \nabla f \cdot \mathbf{v}.$$

At any point P_0 where f has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{v} = 0. \qquad \square$$

By dropping the z-terms in Theorem 9, we obtain a similar result for functions of two variables.

Corollary of Theorem 9

At the points on a smooth curve $\mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{v} = 0$.

Theorem 9 is the key to the method of Lagrange multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface. Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$. Therefore, ∇f is orthogonal to the velocity vector of every such differentiable curve through P_0. But so is ∇g (because ∇g is orthogonal to the level surface $g = 0$, as we saw in Section 12.7). Therefore, at P_0, ∇f is some scalar multiple λ of ∇g.

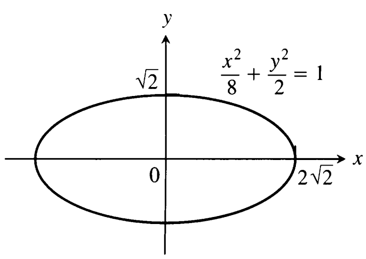
The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0.$$

For functions of two independent variables, the appropriate equations are

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$



12.62 Example 3 shows how to find the largest and smallest values of the product xy on this ellipse.

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Fig. 12.62)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solution We want the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation gives

$$y\,\mathbf{i} + x\,\mathbf{j} = \frac{\lambda}{4} x\,\mathbf{i} + \lambda y\,\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

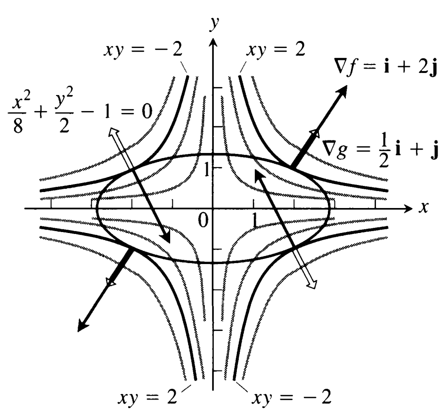
so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8, \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.



12.63 When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue) (Example 3).

The Geometry of the Solution The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$ (Fig. 12.63). The farther the hyperbolas lie from the origin, the larger the absolute value of f. We want to find the extreme values of $f(x, y)$, given that the point (x, y) also lies on the ellipse $x^2 + 4y^2 = 8$. Which hyperbolas intersecting the ellipse lie farthest from the origin? The hyperbolas that just graze the ellipse, the ones that are tangent to it. At these points, any vector normal to the hyperbola is normal to the ellipse, so $\nabla f = y\,\mathbf{i} + x\,\mathbf{j}$ is a multiple ($\lambda = \pm 2$) of $\nabla g = (x/4)\,\mathbf{i} + y\,\mathbf{j}$. At the point $(2, 1)$, for example,

$$\nabla f = \mathbf{i} + 2\,\mathbf{j}, \qquad \nabla g = \frac{1}{2}\,\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = 2\nabla g.$$

At the point $(-2, 1)$,

$$\nabla f = \mathbf{i} - 2\,\mathbf{j}, \qquad \nabla g = -\frac{1}{2}\,\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = -2\nabla g. \qquad \square$$

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\begin{aligned} \nabla f = \lambda \nabla g: &\quad 3\,\mathbf{i} + 4\,\mathbf{j} = 2x\lambda\,\mathbf{i} + 2y\lambda\,\mathbf{j}, \\ g(x, y) = 0: &\quad x^2 + y^2 - 1 = 0. \end{aligned}$$

The gradient equation implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so $$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

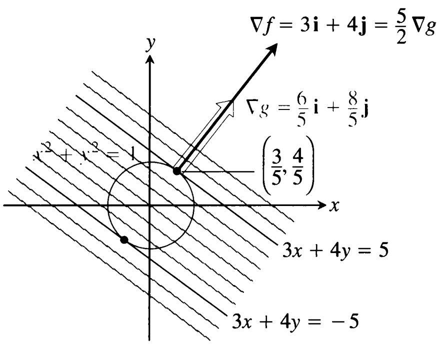
Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

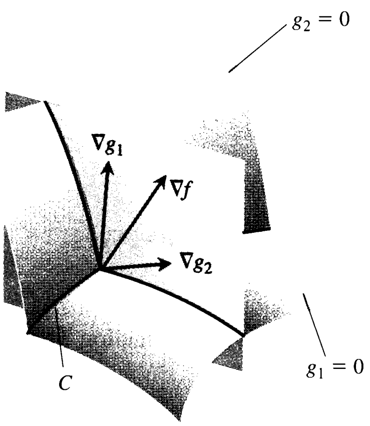
$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$



12.64 The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$ (Example 4). At each of these points, ∇f is a scalar multiple of ∇g. The figure shows the gradients at the first point but not the second.

The Geometry of the Solution (Fig. 12.64) The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$. The farther the lines lie from the origin, the larger the absolute value of f. We want to find the extreme values of $f(x, y)$ given that the point (x, y) also lies on the circle $x^2 + y^2 = 1$. Which lines intersecting the circle lie farthest from the origin? The lines tangent to the circle. At the points of tangency, any vector normal to the line is normal to the circle, so the gradient $\nabla f = 3\,\mathbf{i} + 4\,\mathbf{j}$ is a multiple ($\lambda = \pm 5/2$) of the gradient $\nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. At the point $(3/5, 4/5)$, for example,

$$\nabla f = 3\,\mathbf{i} + 4\,\mathbf{j}, \qquad \nabla g = \frac{6}{5}\,\mathbf{i} + \frac{8}{5}\,\mathbf{j}, \qquad \text{and} \qquad \nabla f = \frac{5}{2}\nabla g. \qquad \square$$



12.65 The vectors ∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1 = 0$ and ∇g_2 is normal to the surface $g_2 = 0$.

Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

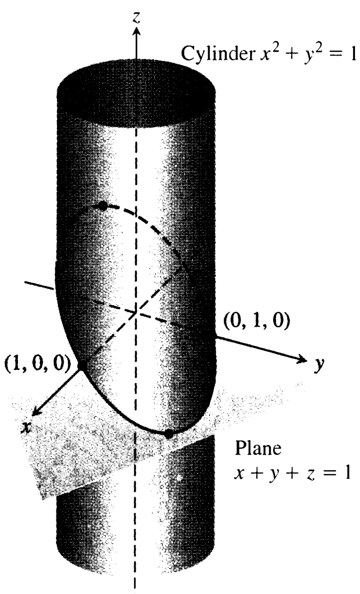
$$g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0$$

and g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2, we find the constrained local maxima and minima of f by introducing two Lagrange multipliers λ and μ (mu, pronounced "mew"). That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0. \tag{2}$$

The equations in (2) have a nice geometric interpretation. The surfaces $g_1 = 0$ and $g_2 = 0$ (usually) intersect in a smooth curve, say C (Fig. 12.65), and along this curve we seek the points where f has local maximum and minimum values relative to its other values on the curve. These are the points where ∇f is normal to C, as we saw in Theorem 9. But ∇g_1 and ∇g_2 are also normal to C at these points because C lies in the surfaces $g_1 = 0$ and $g_2 = 0$. Therefore ∇f lies in the plane determined by ∇g_1 and ∇g_2, which means that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ for some λ and μ. Since the points we seek also lie in both surfaces, their coordinates must satisfy the equations $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, which are the remaining requirements in Eqs. (2).

EXAMPLE 5 The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse (Fig. 12.66). Find the points on the ellipse that lie closest to and farthest from the origin.



12.66 On the ellipse where the plane and cylinder meet, what are the points closest to and farthest from the origin (Example 5)?

Solution We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from (x, y, z) to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \tag{3}$$

$$g_2(x, y, z) = x + y + z - 1 = 0. \tag{4}$$

The gradient equation in (2) then gives

$$\begin{aligned}
\nabla f &= \lambda \nabla g_1 + \mu \nabla g_2 && \text{Eq. (2)}\\
2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} &= \lambda(2x\,\mathbf{i} + 2y\,\mathbf{j}) + \mu\,(\mathbf{i} + \mathbf{j} + \mathbf{k})\\
2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} &= (2\lambda x + \mu)\,\mathbf{i} + (2\lambda y + \mu)\,\mathbf{j} + \mu\,\mathbf{k}
\end{aligned}$$

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in (5) yield

$$\begin{aligned}
2x &= 2\lambda x + 2z \quad \Rightarrow \quad (1 - \lambda)x = z,\\
2y &= 2\lambda y + 2z \quad \Rightarrow \quad (1 - \lambda)y = z.
\end{aligned} \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Eqs. (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Fig. 12.66.

If $x = y$, then Eqs. (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

But here we need to be careful. While P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. ❑

Exercises 12.9

Two Independent Variables with One Constraint

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.
2. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.
3. Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ (Fig. 12.58).
4. Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.
5. Find the points on the curve $xy^2 = 54$ nearest the origin.
6. Find the points on the curve $x^2y = 2$ nearest the origin.
7. Use the method of Lagrange multipliers to find
   a) the minimum value of $x + y$, subject to the constraints $xy = 16, x > 0, y > 0$;
   b) the maximum value of xy, subject to the constraint $x + y = 16$.

   Comment on the geometry of each solution.
8. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.
9. Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.
10. Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What *is* the largest surface area?
11. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.
12. Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?
13. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.
14. Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.
15. The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?
16. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

17. Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.
18. Find the point on the sphere $x^2 + y^2 + z^2 = 4$ which is farthest from the point $(1, -1, 1)$.

19. Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.

20. Find the point on the surface $z = xy + 1$ nearest the origin.

21. Find the points on the surface $z^2 = xy + 4$ closest to the origin.

22. Find the point(s) on the surface $xyz = 1$ closest to the origin.

23. Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

25. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

26. Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

27. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

28. Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0, b > 0$, and $c > 0$.

29. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

31. *An example from economics.* In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

32. You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Lagrange Multipliers with Two Constraints

33. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

34. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

35. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.

36. Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

37. Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.

38. **a)** Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.

b) Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w.

39. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

40. Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

41. *The condition $\nabla f = \lambda \nabla g$ is not sufficient.* While $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the condition $g(x, y) = 0$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

42. *A least squares plane.* The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k):

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 1, 1), \quad (1, 0, -1).$$

Find the values of A, B, and C that minimize the sum

$$\sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

43. a) Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

b) Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a+b+c}{3}.$$

That is, the *geometric mean* of three numbers is less than or equal to the *arithmetic mean.*

44. Let $a_1, a_2, \ldots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^{n} a_i x_i$ subject to the constraint $\sum_{i=1}^{n} x_i{}^2 = 1$.

CAS Explorations and Projects

In Exercises 45–50, use a CAS to perform the following steps implementing the method of Lagrange multipliers for finding constrained extrema:

a) Form the function $h = f - \lambda_1 g_1 - \lambda_2 g_2$, where f is the function to optimize subject to the constraints $g_1 = 0$ and $g_2 = 0$.

b) Determine all the first partial derivatives of h, including the partials with respect to λ_1 and λ_2, and set them equal to 0.

c) Solve the system of equations found in (b) for all the unknowns, including λ_1 and λ_2.

d) Evaluate f at each of the solution points found in (c) and select the extreme value subject to the constraints asked for in the exercise.

45. Minimize $f(x, y, z) = xy + yz$ subject to the constraints $x^2 + y^2 - 2 = 0$ and $x^2 + z^2 - 2 = 0$.

46. Minimize $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 - 1 = 0$ and $x - z = 0$.

47. Maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + 4z - 5 = 0$ and $4x^2 + 4y^2 - z^2 = 0$.

48. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$.

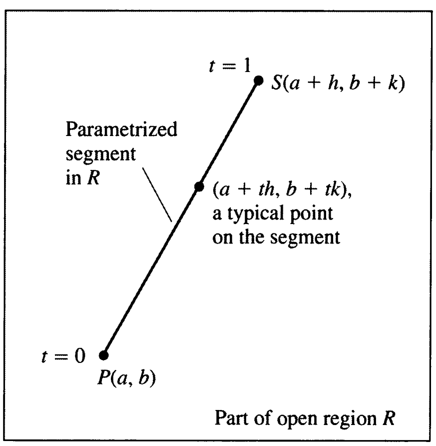
49. Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the constraints $2x - y + z - w - 1 = 0$ and $x + y - z + w - 1 = 0$.

50. Determine the distance from the line $y = x + 1$ to the parabola $y^2 = x$. (*Hint:* Let (x, y) be a point on the line and (w, z) a point on the parabola. You want to minimize $(x - w)^2 + (y - z)^2$.)

12.10 Taylor's Formula

This section uses Taylor's formula (Section 8.10) to derive the second derivative test for local extreme values (Section 12.8) and the error formula for linearizations of functions of two independent variables (Section 12.4, Eq. 5). The use of Taylor's formula in these derivations leads to an extension of the formula that provides polynomial approximations of all orders for functions of two independent variables.

The Derivation of the Second Derivative Test



12.67 We begin the derivation of the second derivative test at $P(a, b)$ by parametrizing a typical line segment from P to a point S nearby.

Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$ (Fig. 12.67). Let h and k be increments small enough to put the point $S(a + h, b + k)$ and the line segment joining it to P inside R. We parametrize the segment PS as

$$x = a + th, \qquad y = b + tk, \qquad 0 \leq t \leq 1.$$

If $F(t) = f(a + th, b + tk)$, the Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable (they have continuous partial derivatives), F' is a differentiable function of t and

$$\begin{aligned} F'' &= \frac{\partial F'}{\partial x}\frac{dx}{dt} + \frac{\partial F'}{\partial y}\frac{dy}{dt} = \frac{\partial}{\partial x}\left(hf_x + kf_y\right) \cdot h + \frac{\partial}{\partial y}\left(hf_x + kf_y\right) \cdot k \\ &= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}. \qquad f_{xy} = f_{yx} \end{aligned}$$

Since F and F' are continuous on [0, 1] and F' is differentiable on (0, 1), we can apply Taylor's formula with $n = 2$ and $a = 0$ to obtain

$$F(1) = F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2}$$

$$F(1) = F(0) + F'(0) + \frac{1}{2}F''(c) \tag{1}$$

for some c between 0 and 1. Writing Eq. (1) in terms of f gives

$$\begin{aligned} f(a+h, b+k) &= f(a,b) + hf_x(a,b) + kf_y(a,b) \\ &\quad + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,b+ck)}. \end{aligned} \tag{2}$$

Since $f_x(a,b) = f_y(a,b) = 0$, this reduces to

$$f(a+h, b+k) - f(a,b) = \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,b+ck)}. \tag{3}$$

The presence of an extremum of f at (a,b) is determined by the sign of $f(a+h, b+k) - f(a,b)$. By Eq. (3), this is the same as the sign of

$$Q(c) = (h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})|_{(a+ch,b+ck)}.$$

Now, if $Q(0) \neq 0$, the sign of $Q(c)$ will be the same as the sign of $Q(0)$ for sufficiently small values of h and k. We can predict the sign of

$$Q(0) = h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b) \tag{4}$$

from the signs of f_{xx} and $f_{xx}f_{yy} - f_{xy}{}^2$ at (a,b). Multiply both sides of Eq. (3) by f_{xx} and rearrange the right-hand side to get

$$f_{xx}\,Q(0) = (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - f_{xy}{}^2)k^2. \tag{5}$$

From Eq. (5) we see that

1. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a,b), then $Q(0) < 0$ for all sufficiently small nonzero values of h and k, and f has a *local maximum* value at (a,b).
2. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a,b), then $Q(0) > 0$ for all sufficiently small nonzero values of h and k, and f has a *local minimum* value at (a,b).
3. If $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a,b), there are combinations of arbitrarily small nonzero values of h and k for which $Q(0) > 0$, and other values for which $Q(0) < 0$. Arbitrarily close to the point $P_0(a, b, f(a,b))$ on the surface $z = f(x,y)$ there are points above P_0 and points below P_0, so f has a *saddle point* at (a,b).
4. If $f_{xx}f_{yy} - f_{xy}{}^2 = 0$, another test is needed. The possibility that $Q(0)$ equals zero prevents us from drawing conclusions about the sign of $Q(c)$.

The Error Formula for Linear Approximations

We want to show that the difference $E(x,y)$ between the values of a function $f(x,y)$ and its linearization $L(x,y)$ at (x_0, y_0) satisfies the inequality

$$|E(x,y)| \leq \frac{1}{2}B\big(|x - x_0| + |y - y_0|\big)^2.$$

The function f is assumed to have continuous second partial derivatives throughout an open set containing a closed rectangular region R centered at (x_0, y_0).

The number B is the largest value that any of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ take on R.

The inequality we want comes from Eq. (2). We substitute x_0 and y_0 for a and b, and $x - x_0$ and $y - y_0$ for h and k, respectively, and rearrange the result as

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{\text{linearization } L(x,y)}$$

$$\underbrace{+\ \frac{1}{2}\left((x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}\right)\Big|_{(x_0+c(x-x_0),\,y_0+c(y-y_0))}}_{\text{error } E(x,y)}.$$

This remarkable equation reveals that

$$|E| \le \frac{1}{2}\left(|x-x_0|^2|f_{xx}| + 2|x-x_0||y-y_0||f_{xy}| + |y-y_0|^2|f_{yy}|\right).$$

Hence, if B is an upper bound for the values of $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$ on R,

$$\begin{aligned} |E| &\le \frac{1}{2}\left(|x-x_0|^2 B + 2|x-x_0||y-y_0|B + |y-y_0|^2 B\right) \\ &\le \frac{1}{2}B\left(|x-x_0| + |y-y_0|\right)^2. \end{aligned}$$

Taylor's Formula for Functions of Two Variables

The formulas derived earlier for F' and F'' can be obtained by applying to $f(x,y)$ the operators

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) \quad \text{and} \quad \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\,\partial y} + k^2\frac{\partial^2}{\partial y^2}.$$

These are the first two instances of a more general formula,

$$F^{(n)}(t) = \frac{d^n}{dt^n}F(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y), \tag{6}$$

which says that applying d^n/dt^n to $F(t)$ gives the same result as applying the operator

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

to $f(x,y)$ after expanding it by the binomial theorem.

If partial derivatives of f through order $n+1$ are continuous throughout a rectangular region centered at (a,b), we may extend the Taylor formula for $F(t)$ to

$$F(t) = F(0) + F'(0)\,t + \frac{F''(0)}{2!}t^2 + \cdots + \frac{F^{(n)}(0)}{n!}t^n + \text{remainder},$$

and take $t = 1$ to obtain

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \text{remainder}.$$

When we replace the first n derivatives on the right of this last series by their equivalent expressions from Eq. (6) evaluated at $t = 0$ and add the appropriate remainder term, we arrive at the following formula.

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})|_{(a,b)}$$
$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 kf_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)}$$
$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,b+ck)}. \tag{7}$$

The first n derivative terms are evaluated at (a, b). The last term is evaluated at some point $(a+ch, b+ck)$ on the line segment joining (a, b) and $(a+h, b+k)$.

If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Eq. (7) assumes the following simpler form.

Taylor's Formula for $f(x, y)$ at the Origin

$$f(x, y) = f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+ \frac{1}{3!}(x^3 f_{xxx} + 3x^2 yf_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f$$
$$+ \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(cx,cy)} \tag{8}$$

The first n derivative terms are evaluated at $(0, 0)$. The last term is evaluated at a point on the line segment joining the origin and (x, y).

Taylor's formula provides polynomial approximations of two-variable functions. The first n derivative terms give the polynomial; the last term gives the approximation error. The first three terms of Taylor's formula give the function's linearization. To improve on the linearization, we add higher power terms.

EXAMPLE 1 Find a quadratic $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

Solution We take $n = 2$ in Eq. (8):

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+\frac{1}{6}(x^3 f_{xxx} + 3x^2 yf_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})_{(cx,cy)}$$

with

$$\begin{aligned} f(0,0) &= \sin x \sin y|_{(0,0)} = 0, & f_{xx}(0,0) &= -\sin x \sin y|_{(0,0)} = 0,\\ f_x(0,0) &= \cos x \sin y|_{(0,0)} = 0, & f_{xy}(0,0) &= \cos x \cos y|_{(0,0)} = 1,\\ f_y(0,0) &= \sin x \cos y|_{(0,0)} = 0, & f_{yy}(0,0) &= -\sin x \sin y|_{(0,0)} = 0, \end{aligned}$$

we have

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)),$$

$$\sin x \sin y \approx xy.$$

The error in the approximation is

$$E(x, y) = \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})|_{(cx,cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \le 0.1$ and $|y| \le 0.1$. Hence

$$|E(x, y)| \le \frac{1}{6}((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) \le \frac{8}{6}(0.1)^3 \le 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \le 0.1$ and $|y| \le 0.1$. ❑

Exercises 12.10

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

1. $f(x, y) = x e^y$

2. $f(x, y) = e^x \cos y$

3. $f(x, y) = y \sin x$

4. $f(x, y) = \sin x \cos y$

5. $f(x, y) = e^x \ln(1 + y)$

6. $f(x, y) = \ln(2x + y + 1)$

7. $f(x, y) = \sin(x^2 + y^2)$

8. $f(x, y) = \cos(x^2 + y^2)$

9. $f(x, y) = \dfrac{1}{1 - x - y}$

10. $f(x, y) = \dfrac{1}{1 - x - y + xy}$

11. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

12. Use Taylor's formula to find a quadratic approximation of $e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

CHAPTER 12 QUESTIONS TO GUIDE YOUR REVIEW

1. What is a real-valued function of two independent variables? three independent variables? Give examples.

2. What does it mean for sets in the plane or in space to be open? closed? Give examples. Give examples of sets that are neither open nor closed.

3. How can you display the values of a function $f(x, y)$ of two independent variables graphically? How do you do the same for a function $f(x, y, z)$ of three independent variables?

4. What does it mean for a function $f(x, y)$ to have limit L as $(x, y) \to (x_0, y_0)$? What are the basic properties of limits of functions of two independent variables?

5. When is a function of two (three) independent variables continuous at a point in its domain? Give examples of functions that are continuous at some points but not others.

6. What can be said about algebraic combinations and composites of continuous functions?

7. Explain the two-path test for nonexistence of limits.

8. How are the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of a function $f(x, y)$ defined? How are they interpreted and calculated?

9. How does the relation between first partial derivatives and continuity of functions of two independent variables differ from the relation between first derivatives and continuity for real-valued functions of a single independent variable? Give an example.

10. What is Euler's theorem for mixed second order partial derivatives? How can it help in calculating partial derivatives of second and higher orders? Give examples.

11. What does it mean for a function $f(x, y)$ to be differentiable? What does the Increment Theorem say about differentiability?

12. How can you sometimes decide from examining f_x and f_y that a function $f(x, y)$ is differentiable? What is the relation between the differentiability of f and the continuity of f at a point?

13. How do you linearize a function $f(x, y)$ of two independent variables at a point (x_0, y_0)? Why might you want to do this? How do you linearize a function of three independent variables?

14. What can you say about the accuracy of linear approximations of functions of two (three) independent variables?

15. If (x, y) moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, how can you estimate the resulting change in the value of a differentiable function $f(x, y)$? Give an example.

16. What is the Chain Rule? What form does it take for functions of two independent variables? three independent variables? functions defined on surfaces? How do you diagram these different forms? Give examples. What pattern enables one to remember all the different forms?

17. What is the derivative of a function $f(x, y)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$? What rate does it describe? What geometric interpretation does it have? Give examples.

18. What is the gradient vector of a function $f(x, y)$? How is it related to the function's directional derivatives? State the analogous results for functions of three independent variables.

19. How do you find the tangent line at a point on a level curve of a differentiable function $f(x, y)$? How do you find the tangent plane and normal line at a point on a level surface of a differentiable function $f(x, y, z)$? Give examples.

20. How can you use directional derivatives to estimate change?

21. How do you define local maxima, local minima, and saddle points for a differentiable function $f(x, y)$? Give examples.

22. What derivative tests are available for determining the local extreme values of a function $f(x, y)$? How do they enable you to narrow your search for these values? Give examples.

23. How do you find the extrema of a continuous function $f(x, y)$ on a closed bounded region of the xy-plane? Give an example.

24. Describe the method of Lagrange multipliers and give examples.

25. How does Taylor's formula for a function $f(x, y)$ generate polynomial approximations and error estimates?

CHAPTER 12 PRACTICE EXERCISES

Domain, Range, and Level Curves

In Exercises 1–4, find the domain and range of the given function and identify its level curves. Sketch a typical level curve.

1. $f(x, y) = 9x^2 + y^2$
2. $f(x, y) = e^{x+y}$
3. $g(x, y) = 1/xy$
4. $g(x, y) = \sqrt{x^2 - y}$

In Exercise 5–8, find the domain and range of the given function and identify its level surfaces. Sketch a typical level surface.

5. $f(x, y, z) = x^2 + y^2 - z$
6. $g(x, y, z) = x^2 + 4y^2 + 9z^2$
7. $h(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$
8. $k(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2 + 1}$

Evaluating Limits

Find the limits in Exercises 9–14.

9. $\displaystyle\lim_{(x,y)\to(\pi,\ln 2)} e^y \cos x$
10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2 + y}{x + \cos y}$
11. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x - y}{x^2 - y^2}$
12. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3y^3 - 1}{xy - 1}$
13. $\displaystyle\lim_{P\to(1,-1,e)} \ln|x + y + z|$
14. $\displaystyle\lim_{P\to(1,-1,-1)} \tan^{-1}(x + y + z)$

By considering different paths of approach, show that the limits in Exercises 15 and 16 do not exist.

15. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y\neq x^2}} \frac{y}{x^2 - y}$
16. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2 + y^2}{xy}$

17. **a)** Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?

b) Let

$$f(x, y) = \begin{cases} \dfrac{\sin(x - y)}{|x| + |y|}, & |x| + |y| \neq 0, \\ 0, & (x, y) = (0, 0). \end{cases}$$

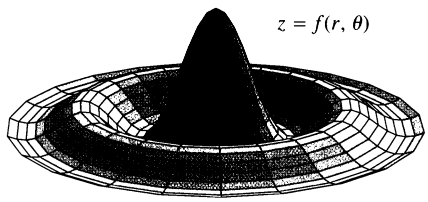
Is f continuous at the origin? Why?

18. Let

$$f(r, \theta) = \begin{cases} \dfrac{\sin 6r}{6r}, & r \neq 0, \\ 1, & r = 0, \end{cases}$$

where r and θ are polar coordinates. Find

a) $\displaystyle\lim_{r\to 0} f(r, \theta)$ **b)** $f_r(0, 0)$ **c)** $f_\theta(r, \theta), \quad r \neq 0$



(Generated by Mathematica)

Partial Derivatives

In Exercises 19–24, find the partial derivative of the function with respect to each variable.

19. $g(r, \theta) = r \cos \theta + r \sin \theta$

20. $f(x, y) = \dfrac{1}{2} \ln (x^2 + y^2) + \tan^{-1} \dfrac{y}{x}$

21. $f(R_1, R_2, R_3) = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$

22. $h(x, y, z) = \sin (2\pi x + y - 3z)$

23. $P(n, R, T, V) = \dfrac{nRT}{V}$ (the Ideal Gas Law)

24. $f(r, l, T, w) = \dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi w}}$

Second Order Partials

Find the second order partial derivatives of the functions in Exercises 25–28.

25. $g(x, y) = y + \dfrac{x}{y}$

26. $g(x, y) = e^x + y \sin x$

27. $f(x, y) = x + xy - 5x^3 + \ln (x^2 + 1)$

28. $f(x, y) = y^2 - 3xy + \cos y + 7e^y$

Linearizations

In Exercises 29 and 30, find the linearization $L(x, y)$ of the function $f(x, y)$ at the point P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

29. $f(x, y) = \sin x \cos y, \quad P_0(\pi/4, \pi/4)$

R: $\left|x - \dfrac{\pi}{4}\right| \le 0.1, \quad \left|y - \dfrac{\pi}{4}\right| \le 0.1$

30. $f(x, y) = xy - 3y^2 + 2, \quad P_0(1, 1)$

R: $|x - 1| \le 0.1, \quad |y - 1| \le 0.2$

Find the linearizations of the functions in Exercises 31 and 32 at the given points.

31. $f(x, y, z) = xy + 2yz - 3xz$ at $(1, 0, 0)$ and $(1, 1, 0)$

32. $f(x, y, z) = \sqrt{2} \cos x \sin (y + z)$ at $(0, 0, \pi/4)$ and $(\pi/4, \pi/4, 0)$

Estimates and Sensitivity to Change

33. You plan to calculate the volume inside a stretch of pipeline that is about 36 in. in diameter and 1 mi long. With which measurement should you be more careful—the length, or the diameter? Why?

34. Near the point $(1, 2)$, is $f(x, y) = x^2 - xy + y^2 - 3$ more sensitive to changes in x, or to changes in y? How do you know?

35. Suppose that the current I (amperes) in an electrical circuit is related to the voltage V (volts) and the resistance R (ohms) by the equation $I = V/R$. If the voltage drops from 24 to 23 volts and the resistance drops from 100 to 80 ohms, will I increase, or decrease? By about how much? Express the changes in V and R and the estimated change in I as percentages of their original values.

36. If $a = 10$ cm and $b = 16$ cm to the nearest millimeter, what should you expect the maximum percentage error to be in the calculated area $A = \pi ab$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$?

37. Let $y = uv$ and $z = u + v$, where u and v are positive independent variables.

a) If u is measured with an error of 2% and v with an error of 3%, about what is the percentage error in the calculated value of y?

b) Show that the percentage error in the calculated value of z is less than the percentage error in the value of y.

38. *Cardiac index.* To make different people comparable in studies of cardiac output (Section 2.7, Exercise 25), researchers divide the measured cardiac output by the body surface area to find the *cardiac index C:*

$$C = \frac{\text{cardiac output}}{\text{body surface area}}.$$

The body surface area B is calculated with the formula

$$B = 71.84 w^{0.425} h^{0.725},$$

which gives B in square centimeters when w is measured in kilograms and h in centimeters. You are about to calculate the cardiac index of a person with the following measurements:

Cardiac output:	7 L/min
Weight:	70 kg
Height:	180 cm

Which will have a greater effect on the calculation, a 1-kg error in measuring the weight, or a 1-cm error in measuring the height?

Chain Rule Calculations

39. Find dw/dt at $t = 0$ if $w = \sin (xy + \pi)$, $x = e^t$, and $y = \ln (t + 1)$.

40. Find dw/dt at $t = 1$ if $w = xe^y + y \sin z - \cos z$, $x = 2\sqrt{t}$, $y = t - 1 + \ln t$, $z = \pi t$.

41. Find $\partial w/\partial r$ and $\partial w/\partial s$ when $r = \pi$ and $s = 0$ if $w = \sin(2x - y)$, $x = r + \sin s$, $y = rs$.

42. Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln\sqrt{1+x^2} - \tan^{-1} x$ and $x = 2e^u \cos v$.

43. Find the value of the derivative of $f(x, y, z) = xy + yz + xz$ with respect to t on the curve $x = \cos t$, $y = \sin t$, $z = \cos 2t$ at $t = 1$.

44. Show that if $w = f(s)$ is any differentiable function of s and if $s = y + 5x$, then

$$\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 0.$$

Implicit Differentiation

Assuming that the equations in Exercises 45 and 46 define y as a differentiable function of x, find the value of dy/dx at point P.

45. $1 - x - y^2 - \sin xy = 0, \quad P(0, 1)$

46. $2xy + e^{x+y} - 2 = 0, \quad P(0, \ln 2)$

Partial Derivatives with Constrained Variables

In Exercises 47 and 48, begin by drawing a diagram that shows the relations among the variables.

47. If $w = x^2 e^{yz}$ and $z = x^2 - y^2$, find

a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ b) $\left(\dfrac{\partial w}{\partial z}\right)_x$ c) $\left(\dfrac{\partial w}{\partial z}\right)_y$

48. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a) $\left(\dfrac{\partial U}{\partial T}\right)_P$ b) $\left(\dfrac{\partial U}{\partial V}\right)_T$

Directional Derivatives

In Exercises 49–52, find the directions in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\mathbf{A}$.

49. $f(x, y) = \cos x \cos y, \quad P_0(\pi/4, \pi/4), \quad \mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$

50. $f(x, y) = x^2 e^{-2y}, \quad P_0(1, 0), \quad \mathbf{A} = \mathbf{i} + \mathbf{j}$

51. $f(x, y, z) = \ln(2x + 3y + 6z), \quad P_0(-1, -1, 1),$ $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

52. $f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4,$ $P_0(0, 0, 0)$, $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

53. Find the derivative of $f(x, y, z) = xyz$ in the direction of the velocity vector of the helix

$$\mathbf{r}(t) = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\,\mathbf{k}$$

at $t = \pi/3$.

54. What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?

55. At the point $(1, 2)$ the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and derivative of -2 in the direction toward $(1, 1)$.

a) Find $f_x(1, 2)$ and $f_y(1, 2)$.
b) Find the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.

56. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)?

a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j}) \cdot \mathbf{u}$.
b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.
d) At (x_0, y_0), vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.

Gradients, Tangent Planes, and Normal Lines

In Exercises 57 and 58, sketch the surface $f(x, y, z) = c$ together with ∇f at the given points.

57. $x^2 + y + z^2 = 0; \quad (0, -1, \pm 1), \quad (0, 0, 0)$

58. $y^2 + z^2 = 4; \quad (2, \pm 2, 0), \quad (2, 0, \pm 2)$

In Exercises 59 and 60, find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0. Also, find parametric equations for the line that is normal to the surface at P_0.

59. $x^2 - y - 5z = 0, \quad P_0(2, -1, 1)$

60. $x^2 + y^2 + z = 4, \quad P_0(1, 1, 2)$

In Exercises 61 and 62, find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point

61. $z = \ln(x^2 + y^2), \quad (0, 1, 0)$

62. $z = 1/(x^2 + y^2), \quad (1, 1, 1/2)$

In Exercises 63 and 64, find equations for the lines that are tangent and normal to the level curve $f(x, y) = c$ at the point P_0. Then sketch the lines and level curve together with ∇f at P_0.

63. $y - \sin x = 1, \quad P_0(\pi, 1)$

64. $\dfrac{y^2}{2} - \dfrac{x^2}{2} = \dfrac{3}{2}, \quad P_0(1, 2)$

Tangent Lines to Curves

In Exercises 65 and 66, find parametric equations for the line that is tangent to the curve of intersection of the surfaces at the given point.

65. Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$
Point: $(1, 1, 1/2)$

66. Surfaces: $x + y^2 + z = 2, \quad y = 1$
Point: $(1/2, 1, 1/2)$

Local Extrema

Test the functions in Exercises 67–72 for local maxima and minima and saddle points. Find each function's values at these points.

67. $f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$

68. $f(x, y) = 5x^2 + 4xy - 2y^2 + 4x - 4y$

69. $f(x, y) = 2x^3 + 3xy + 2y^3$

70. $f(x, y) = x^3 + y^3 - 3xy + 15$

71. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$

72. $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$

Absolute Extrema

In Exercises 73–80, find the absolute maximum and minimum values of f on the region R.

73. $f(x, y) = x^2 + xy + y^2 - 3x + 3y$
R: The triangular region cut from the first quadrant by the line $x + y = 4$

74. $f(x, y) = x^2 - y^2 - 2x + 4y + 1$
R: The rectangular region in the first quadrant bounded by the coordinate axes and the lines $x = 4$ and $y = 2$

75. $f(x, y) = y^2 - xy - 3y + 2x$
R: The square region enclosed by the lines $x = \pm 2$ and $y = \pm 2$

76. $f(x, y) = 2x + 2y - x^2 - y^2$
R: The square bounded by the coordinate axes and the lines $x = 2$, $y = 2$ in the first quadrant

77. $f(x, y) = x^2 - y^2 - 2x + 4y$
R: The triangular region bounded below by the x-axis, above by the line $y = x + 2$, and on the right by the line $x = 2$

78. $f(x, y) = 4xy - x^4 - y^4 + 16$
R: The triangular region bounded below by the line $y = -2$, above by the line $y = x$, and on the right by the line $x = 2$

79. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$
R: The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

80. $f(x, y) = x^3 + 3xy + y^3 + 1$
R: The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

Lagrange Multipliers

81. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

82. Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

83. Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$.

84. Find the extreme values of $f(x, y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \leq 9$.

85. Find the extreme values of $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

86. Find the points on the surface $z^2 - xy = 4$ closest to the origin.

87. A closed rectangular box is to have volume V cm^3. The cost of the material used in the box is a cents/cm^2 for top and bottom, b cents/cm^2 for front and back, and c cents/cm^2 for the remaining sides. What dimensions minimize the total cost of materials?

88. Find the plane $x/a + y/b + z/c = 1$ that passes through the point $(2, 1, 2)$ and cuts off the least volume from the first octant.

89. Find the extreme values of $f(x, y, z) = x(y + z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.

90. Find the point closest to the origin on the curve of intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$.

Theory and Examples

91. Let $w = f(r, \theta)$, $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$. Find $\partial w/\partial x$ and $\partial w/\partial y$ and express your answers in terms of r and θ.

92. Let $z = f(u, v)$, $u = ax + by$, and $v = ax - by$. Express z_x and z_y in terms of f_u, f_v, and the constants a and b.

93. If a and b are constants, $w = u^3 + \tanh u + \cos u$, and $u = ax + by$, show that

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}.$$

94. If $w = \ln(x^2 + y^2 + 2z)$, $x = r + s$, $y = r - s$, and $z = 2rs$, find w_r and w_s by the Chain Rule. Then check your answer another way.

95. The equations $e^u \cos v - x = 0$ and $e^u \sin v - y = 0$ define u and v as differentiable functions of x and y. Show that the angle between the vectors

$$\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} \quad \text{and} \quad \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}$$

is constant.

96. Introducing polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ changes $f(x, y)$ to $g(r, \theta)$. Find the value of $\partial^2 g/\partial\theta^2$ at the point $(r, \theta) = (2, \pi/2)$, given that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 1$$

at that point.

97. Find the points on the surface

$$(y + z)^2 + (z - x)^2 = 16$$

where the normal line is parallel to the yz-plane.

98. Find the points on the surface

$$xy + yz + zx - x - z^2 = 0$$

where the tangent plane is parallel to the xy-plane.

99. Suppose that $\nabla f(x, y, z)$ is always parallel to the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $f(0, 0, a) = f(0, 0, -a)$ for any a.

100. Show that the directional derivative of

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

at the origin equals 1 in any direction but that f has no gradient vector at the origin.

101. Show that the line normal to the surface $xy + z = 2$ at the point $(1, 1, 1)$ passes through the origin.

102. **a)** Sketch the surface $x^2 - y^2 + z^2 = 4$.
b) Find a vector normal to the surface at $(2, -3, 3)$. Add the vector to your sketch.
c) Find the equations for the tangent plane and normal line at $(2, -3, 3)$.

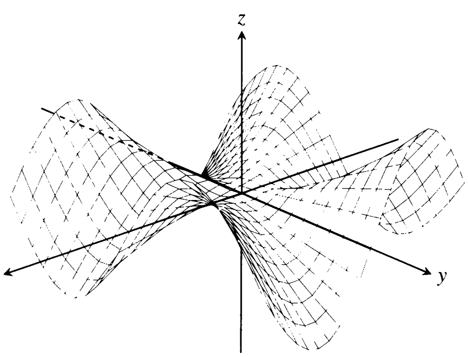
CHAPTER 12 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Partial Derivatives

1. If you did Exercise 50 in Section 12.2, you know that the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(see the accompanying figure) is continuous at $(0, 0)$. Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.



(Generated by Mathematica)

2. Find a function $w = f(x, y)$ whose first partial derivatives are $\partial w/\partial x = 1 + e^x \cos y$ and $\partial w/\partial y = 2y - e^x \sin y$, and whose value at the point $(\ln 2, 0)$ is $\ln 2$.

3. *A proof of Leibniz's rule.* Leibniz's rule says that if f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Prove the rule by setting

$$g(u, v) = \int_u^v f(t)\,dt, \quad u = u(x), \quad v = v(x)$$

and calculating dg/dx with the Chain Rule.

4. Suppose that f is a twice-differentiable function of r, that $r = \sqrt{x^2 + y^2 + z^2}$, and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants a and b,

$$f(r) = \frac{a}{r} + b.$$

5. *Homogeneous functions.* A function $f(x, y)$ is *homogeneous of degree n* (n a nonnegative integer) if $f(tx, ty) = t^n f(x, y)$ for all t, x, and y. For such a function (sufficiently differentiable), prove that

a) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

b) $x^2\left(\dfrac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\dfrac{\partial^2 f}{\partial x \partial y}\right) + y^2\left(\dfrac{\partial^2 f}{\partial y^2}\right) = n(n-1)f.$

6. *Spherical coordinates.* Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Express x, y, and z as functions of the spherical coordinates ρ, ϕ, and θ and calculate $\partial\mathbf{r}/\partial\rho$, $\partial\mathbf{r}/\partial\phi$, and $\partial\mathbf{r}/\partial\theta$. Then express these derivatives in terms of the unit vectors

$$\mathbf{u}_\rho = (\sin\phi\cos\theta)\,\mathbf{i} + (\sin\phi\sin\theta)\,\mathbf{j} + (\cos\phi)\,\mathbf{k}$$

$$\mathbf{u}_\phi = (\cos\phi\cos\theta)\,\mathbf{i} + (\cos\phi\sin\theta)\,\mathbf{j} - (\sin\phi)\,\mathbf{k}$$

$$\mathbf{u}_\theta = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}.$$

Gradients and Tangents

7. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r = |\mathbf{r}|$.

a) Show that $\nabla r = \mathbf{r}/r$.
b) Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$.
c) Find a function whose gradient equals $\mathbf{r}$.
d) Show that $\mathbf{r} \cdot d\mathbf{r} = r\,dr$.
e) Show that $\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$ for any constant vector $\mathbf{A}$.

8. Suppose that a differentiable function $f(x, y)$ has the constant value c along the differentiable curve $x = g(t)$, $y = h(t)$; that is,

$$f(g(t), h(t)) = c$$

for all values of t. Differentiate both sides of this equation with respect to t to show that ∇f is orthogonal to the curve's tangent vector at every point on the curve.

9. Show that the curve

$$\mathbf{r}(t) = (\ln t)\,\mathbf{i} + (t\ln t)\,\mathbf{j} + t\,\mathbf{k}$$

is tangent to the surface

$$xz^2 - yz + \cos xy = 1$$

at (0, 0, 1).

10. Show that the curve

$$\mathbf{r}(t) = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + \cos(t-2)\,\mathbf{k}$$

is tangent to the surface

$$x^3 + y^3 + z^3 - xyz = 0$$

at (0, −1, 1).

11. *The gradient in cylindrical coordinates.* Suppose cylindrical coordinates r, θ, z are introduced into a function $w = f(x, y, z)$ to yield $w = F(r, \theta, z)$. Show that

$$\nabla w = \frac{\partial w}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial w}{\partial \theta}\mathbf{u}_\theta + \frac{\partial w}{\partial z}\mathbf{k}, \qquad (1)$$

where

$$\mathbf{u}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}$$

$$\mathbf{u}_\theta = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}.$$

(*Hint:* Express the right-hand side of Eq. (1) in terms of **i, j,** and **k** and use the Chain Rule to express the components of **i, j,** and **k** in rectangular coordinates.)

12. *The gradient in spherical coordinates.* Suppose spherical coordinates ρ, ϕ, θ are introduced into a function $w = f(x, y, z)$ to yield a function $w = F(\rho, \phi, \theta)$. Show that

$$\nabla w = \frac{\partial w}{\partial \rho}\mathbf{u}_\rho + \frac{1}{\rho}\frac{\partial w}{\partial \phi}\mathbf{u}_\phi + \frac{1}{\rho\sin\phi}\frac{\partial w}{\partial \theta}\mathbf{u}_\theta, \qquad (2)$$

where

$$\mathbf{u}_\rho = (\sin\phi\cos\theta)\,\mathbf{i} + (\sin\phi\sin\theta)\,\mathbf{j} + (\cos\phi)\,\mathbf{k}$$

$$\mathbf{u}_\phi = (\cos\phi\cos\theta)\,\mathbf{i} + (\cos\phi\sin\theta)\,\mathbf{j} - (\sin\phi)\,\mathbf{k}$$

$$\mathbf{u}_\theta = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}.$$

(*Hint:* Express the right-hand side of Eq. (2) in terms of **i, j,** and **k** and use the Chain Rule to express the components of **i, j,** and **k** in rectangular coordinates.)

Extreme Values

13. Show that the only possible maxima and minima of z on the surface $z = x^3 + y^3 - 9xy + 27$ occur at (0, 0) and (3, 3). Show that neither a maximum nor a minimum occurs at (0, 0). Determine whether z has a maximum or a minimum at (3, 3).

14. Find the maximum value of $f(x, y) = 6xye^{-(2x+3y)}$ in the closed first quadrant (includes the nonnegative axes).

15. Find the minimum volume for a region bounded by the planes $x = 0, y = 0, z = 0$ and a plane tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the first octant.

16. By minimizing the function $f(x, y, u, v) = (x-u)^2 + (y-v)^2$ subject to the constraints $y = x + 1$ and $u = v^2$, find the minimum distance in the xy-plane from the line $y = x + 1$ to the parabola $y^2 = x$.

Theory and Examples

17. Prove the following theorem: If $f(x, y)$ is defined in an open region R of the xy-plane, and if f_x and f_y are bounded on R, then $f(x, y)$ is continuous on R. (The assumption of boundedness is essential.)

18. Suppose $\mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$ is a smooth curve in the domain of a differentiable function $f(x, y, z)$. Describe the relation between df/dt, ∇f, and $\mathbf{v} = d\mathbf{r}/dt$. What can be said about ∇f and $\mathbf{v}$ at interior points of the curve where f has extreme values relative to its other values on the curve? Give reasons for your answer.

19. Suppose that f and g are functions of x and y such that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

and suppose that

$$\frac{\partial f}{\partial x} = 0, \quad f(1,2) = g(1,2) = 5, \quad \text{and} \quad f(0,0) = 4.$$

Find $f(x, y)$ and $g(x, y)$.

20. We know that if $f(x, y)$ is a function of two variables and if $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is a unit vector, then $D_\mathbf{u} f(x, y) = f_x(x, y)a + f_y(x, y)b$ is the rate of change of $f(x, y)$ at (x, y) in the direction of **u.** Give a similar formula for the rate of change *of the rate of change of* $f(x, y)$ at (x, y) in the direction **u.**

21. *Path of a heat-seeking particle.* A heat-seeking particle has the property that at any point (x, y) in the plane it moves in the direction of maximum temperature increase. If the temperature at (x, y) is $T(x, y) = -e^{-2y}\cos x$, find an equation $y = f(x)$ for the path of a heat-seeking particle at the point $(\pi/4, 0)$.

22. A particle traveling in a straight line with constant velocity $\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ passes through the point (0, 0, 30) and hits the surface $z = 2x^2 + 3y^2$. The particle ricochets off the surface, the angle of reflection being equal to the angle of incidence. Assuming no loss of speed, what is the velocity of the particle after the ricochet? Simplify your answer.

23. Let S be the surface that is the graph of $f(x, y) = 10 - x^2 - y^2$. Suppose the temperature in space at each point (x, y, z) is $T(x, y, z) = x^2y + y^2z + 4x + 14y + z$.

a) Among all of the possible directions tangential to the surface S at the point (0, 0, 10), which direction will make the rate of change of temperature at (0, 0, 10) a maximum?

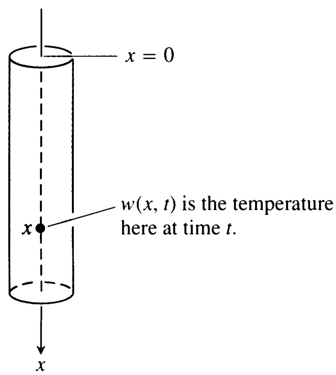
b) Which direction tangential to S at the point (1, 1, 8) will make the rate of change of temperature a maximum?

24. On a flat surface of land, geologists drilled a borehole straight down and hit a mineral deposit at 1000 ft. They drilled a second borehole 100 ft to the north of the first and hit the mineral deposit at 950 ft. A third borehole 100 ft east of the first borehole struck the mineral deposit at 1025 ft. The geologists have reasons to believe that the mineral deposit is in the shape of a dome and for the sake of economy they would like to find where the deposit is closest to the surface. Assuming the surface to be the xy-plane, in what direction from the first borehole would you suggest the geologists drill their fourth borehole?

The One-Dimensional Heat Equation

If $w(x, t)$ represents the temperature at position x at time t in a uniform conducting rod with perfectly insulated sides (see the accompanying figure), then the partial derivatives w_{xx} and w_t satisfy a differential equation of the form

$$w_{xx} = \frac{1}{c^2} w_t. \tag{3}$$



This equation is called the **one-dimensional heat equation.** The value of the positive constant c^2 is determined by the material from which the rod is made. It has been determined experimentally for a broad range of materials. For a given application one finds the appropriate value in a table. For dry soil, for example, $c^2 = 0.19$ ft^2/day.

In chemistry and biochemistry, the heat equation is known as the **diffusion equation.** In this context, $w(x, t)$ represents the concentration of a dissolved substance, a salt for instance, diffusing along a tube filled with liquid. The value of $w(x, t)$ is the concentration at point x at time t. In other applications, $w(x, t)$ represents the diffusion of a gas down a long, thin pipe.

In electrical engineering, the heat equation appears in the forms

$$v_{xx} = RCv_t \tag{4}$$

and

$$i_{xx} = RCi_t, \tag{5}$$

which are known as the **telegraph equations.** These equations describe the voltage v and the flow of current i in a coaxial cable or in any other cable in which leakage and inductance are negligible. The functions and constants in these equations are

$$\begin{aligned} v(x, t) &= \text{voltage at point } x \text{ at time } t \\ R &= \text{resistance per unit length} \\ C &= \text{capacitance to ground per unit of cable length} \\ i(x, t) &= \text{current at point } x \text{ at time } t. \end{aligned}$$

25. Find all solutions of the one-dimensional heat equation of the form $w = e^{rt} \sin \pi x$, where r is a constant.

26. Find all solutions of the one-dimensional heat equation that have the form $w = e^{rt} \sin kx$ and satisfy the conditions that $w(0, t) = 0$ and $w(L, t) = 0$. What happens to these solutions as $t \to \infty$?