1. Outline to the Presentation of Results

The following report is concerned with the explanation of the results obtained in solving a problem set, consisting of two exercises, using the MATLAB software package. The overriding theme of this problem set is the convergence of the binomial pricing approach as a discrete time period model, following a probabilistic binomial tree, with the continuous-time Black-Scholes-Merton model (BSM model). Cox, Ross and Rubinstein (1979) base the derivation of their valuation method for options on the same idea, proofing a convergence of results obtained applying a discrete time period model with the result obtained in the continuous-time BSM model, for the limiting case of an ever increasing amount of n sub periods within a given time to maturity T.

Exercise 1 will be concerned with a succinct presentation of the basic ideas underlying the CRR model's convergence to the BSM model, illustrated for a European put option, whilst additionally comparing the outcome to the results obtained for a corresponding American put option. In Exercise 2 the same methodology is applied to an exotic Down-and-out Call barrier option (DOC). Here the comparison of discrete binomial pricing with continuous closed form solutions will first focus on a DOC with static barrier and subsequently the convergence characteristics for a DOC with moving barrier, following the closed form model by Kunitomo and Ikeda, in the same setting, will be assessed.

2. Exercise 1: The Cox-Ross-Rubinstein option valuation model

Input Data Set #7:

Put Option; S=32; X=35; T=1.25; r=0.02; q=0.04; σ=0.35

Interpreting the input data it becomes inherent that the European put option is currently in-the-money (ITM), with 15 months left until expiration. The option is written on an underlying paying a constant dividend yield of 4% and exhibiting a volatility of 0.35, as indicated by the standard deviation σ. The risk free rate is given at a level of 2%.

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a. Creation of a stock price distribution following the CRR binomial tree

In this first part of Exercise 1 a function (CRR_Stock.m) is created, mirroring the forward stock price evolution procedure underlying a binomial tree following the properties described by Cox, Ross and Rubinstein (1979). For explanatory purposes we briefly refer to the mechanics and assumptions underlying the stock price evolution procedure. Given a risk-neutral environment, meaning (1) the expected return from all traded assets is the risk-free interest rate *r* and consequently (2) expected values of the option's pay-off may be calculated by discounting at the risk-free interest rate, the time to maturity T is separated into smaller time periods of $\Delta t = \frac{T-t}{n}$. Within each time interval the price of the underlying asset may increase to a value of $S * u$, with a probability of *p*, representing an "up move", or decrease to a value of $S * d$, with a probability of *(1-p)*, thereby *u>1* and *d<1* (Hull 2012*)*.

b. Comparing Binomial and BSM prices for n=100

The results obtained in the last sub period N of T=1.25 years, divided into 100 sub periods of length Δt, applying both, the binomial pricing model and the BSM model, were 8.4364 and 8.4171, respectively. The binomial option price P was determined for each node within n time intervals by

$$P_{Bin} = e^{-r\Delta t}(p * P_u + (1-p)P_d) = 8.4171, \qquad (1)$$

with $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$, P_u and P_d thereby being the next periods option prices for an up- or down move, respectively. Additionally, the assumption of investors' risk neutrality will lead to an equilibrium of $p = q$, with q being the subjective risk probability of investors. Likewise, the value for the Put option P was calculated by the means of the BSM model stated as

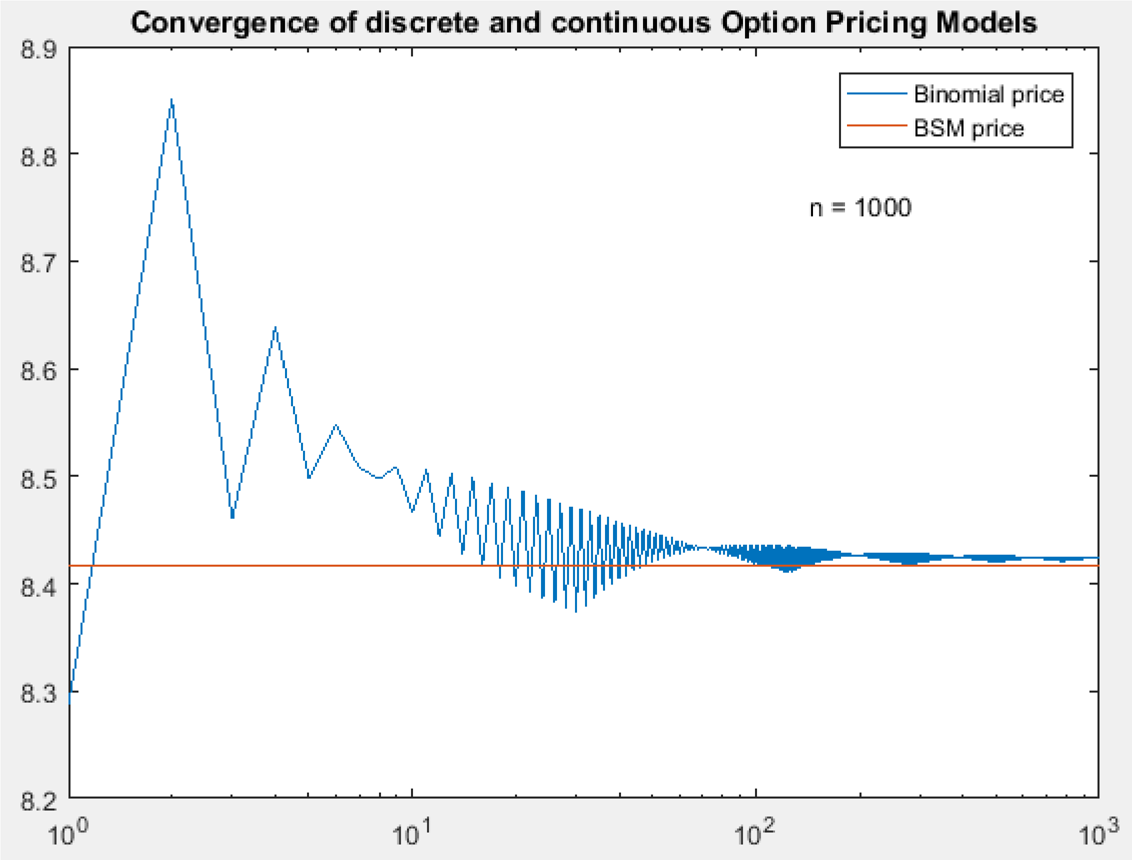
$$P_{BSM} = Xe^{-r_f(T-t)}N(-d_1) - Se^{-r_f(T-t)}N(-d_2) = 8.4364. \qquad (2)$$

When comparing the results obtained, the difference in prices of 0.0193 seems to be rather minor, and considering the solution matrix of P_{Bin}, converging towards the closed form solution. This observation will be assessed in more detail in the following part c), by increasing the number of sub periods further to n=1000, approaching the scenario of a limiting case where $\Delta t \rightarrow 0$ even closer.

c. Plotting Binomial and BSM price function relative to n=1000 sub periods

Figure 1 shows the graph obtained when subdividing the time to maturity into 1000 subintervals of length Δt. The graph shows a pattern of convergence between the binomial price and the closed form BSM price, being constant for all periods n. When referring to the last period the binomial price declines marginally to $P_{Bin} = 8.4234$, converging closer to the closed form solution sated earlier. Thus, the remaining price difference is 0.0121.

Figure 1: Convergence of Binomial and BSM Option Price



In order to find an analytical explanation for the observed phenomenon we refer to the binomial model established by Cox, Ross and Rubinstein, which essentially proves the convergence of a discrete, single period binomial pricing model to the results obtained in the continuous closed form BSM model (Cox, Ross & Rubinstein, 1979). In the following we will first outline the key assumptions made in the Cox-Ross-Rubinstein model (CRR model) to subsequently obtain the discrete period pricing formula, to then discuss the criteria required to be met in the process of convergence, meaning $\Delta t \to 0$.

When referring to the CRR model we establish the following assumptions:

- The price of the underlying follows a multiplicative binomial process over n discrete periods. Consequently, at the end of each period, the price of the underlying may rise to Su or decline to Sd.
- The risk free rate r, thereby represents the expected return in a risk-neutral world and follows u>1+r>d, or else there would be feasible opportunities for risk free arbitrage.
- The creation of a duplicating portfolio, or "hedging portfolio" (Cox et al., 1979) consisting of Δ amount of the underlying's assets and an amount of B in riskless one period bonds, is always possible.

Considering a one period binomial tree the possible outcomes of a European put option are as follows:

$P_u = \max[0, K - Su]$ with probability p

P

$P_d = \max[0, K - Sd]$ with probability 1-p

Forming a portfolio duplicating the put option by shorting Δ amount of the underlying asset and borrowing an amount of B riskless one period bonds yields a duplicating portfolio of (Cox et al., 1979)

$$-\Delta uS + rB = P_u$$

$$-\Delta dS + rB = P_d$$

Solving this system of equations for Δ and B allows rewriting the initial definition of the hedging portfolio from $P = -\Delta S + B$ to

$$P = [pP_u + (1 - P)P_d]r^{-1}, \qquad (3)$$

with $p = \frac{r-d}{u-d}$ and $1 - p = \frac{u-r}{u-d}$

When analysing equation (3) Cox, Ross and Rubinstein refer to the absence of the variable q representing individual investors’ perceptions of risk. Hence the attitude towards risk does not have direct influence on the relationship of *P* with *S*, *u*, *d* and *r* (Cox et al., 1979). Using this single period model the extension to a multi period model involves the definition of a reiterating procedure calculating the option price at each node within its respective period. Cox, Ross and Rubinstein apply binomial coefficients to obtain the general valuation formula, consequently separating the same into two expressions containing complementary binomial distribution functions, eventually yielding

$$P = Kr^{-n}\phi[a; n, p] - S\phi[a; n, p'] \qquad (4)$$

Where *a* is the minimum number of upward moves that the stock must make over the next n periods for the call to finish in-the-money (Cox et al., 1979). For the further explanation we refer to the findings of Feng and Kwan, applying a simplified explanation of the convergence described above, applicable to introductory classes in secondary education.

"[From an analytical perspective the BSM model] is a continuous-time model and the binomial model may be viewed as its discrete time version" (Feng & Kwan, 2012). This becomes inherent when focusing on the difference in discounting factors in equations (2) and (4). The BSM model discounts the option pay-off on a continuous basis over the entire time to maturity of T years using $e^{-r_f(T-t)}$ as discount factor; in the CRR model the discount factor $r^{-n}$is of discrete nature. Thus the equation

$$e^{-r_f(T-t)} = r^{-n}$$

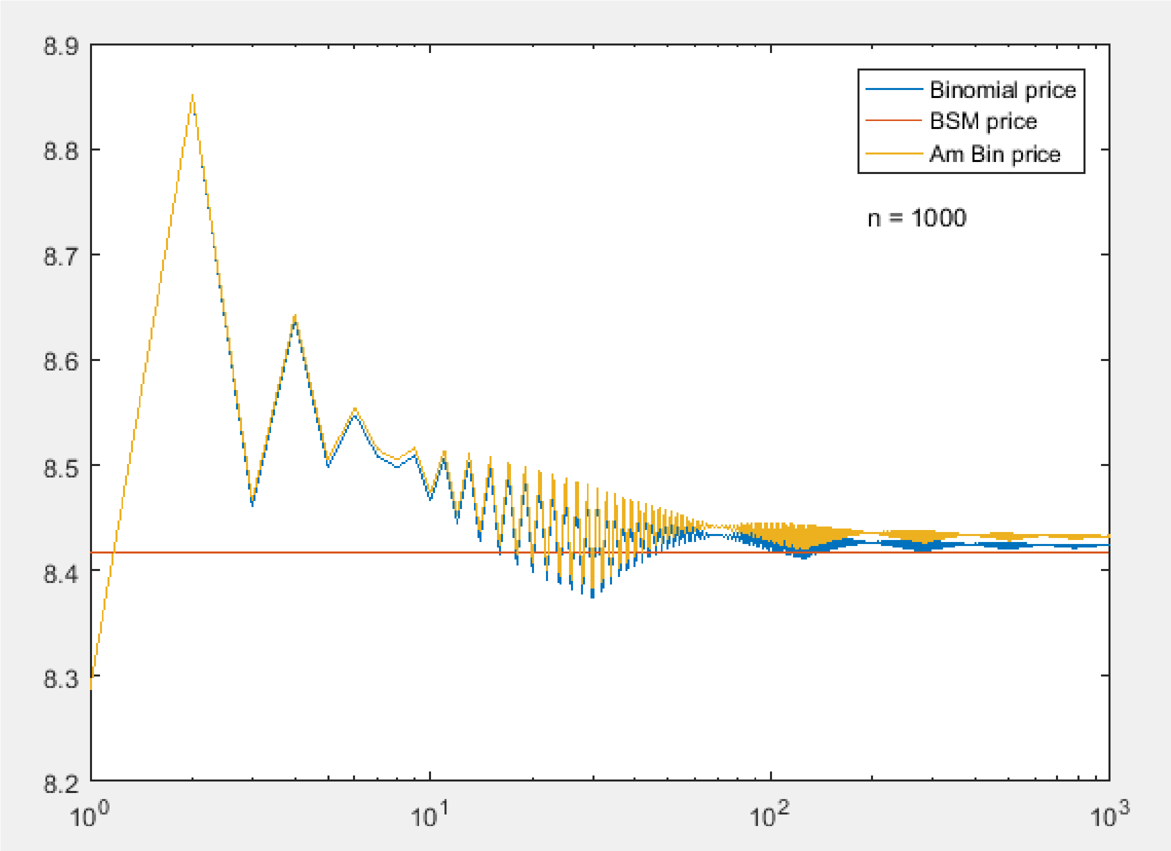
describes the relation of the risk-free rate over the period of T years r_f and the risk-free rate of each sub period n r. Hence for the two pricing formulas to match we furthermore require the equivalence of probability distributions, meaning the convergence of $[a; n, p']$ and $[a; n, p]$ to $N(d_1)$ and $N(d_2)$, respectively (Feng & Kwan, 2012). A binomial distribution essentially describes the summation of n individual trials and results within a Bernoulli experiment. According to the central limit theorem the results obtained for an increasing number n trials may be approximated by a normal distribution function (Feng & Kwan, 2012). Besides this theorem however, the interval dependent individual parameters u, d and r require adjustments to remain constant irrespective of the number of sub periods n (Cox et al., 1979). Cox, Ross and Rubinstein show the risk-free rate r within each period n to be dependent on n for the total return within the time period T, in order to be independent of n within a time interval Δt. Modelling jump stochastic processes for price movements in the underlying the parameters u and d may be derived as being $u = e^{\sigma\sqrt{T/n}}$ and $d = e^{-\sigma\sqrt{T/n}}$ in the limiting case as $\Delta t \rightarrow 0$ or $n \rightarrow \infty$.

Thus far no attempt has been made to formally prove the convergence of the two models, since doing so would require advanced statistical knowledge beyond the scope of this analysis. Therefore the graphs obtained will serve as illustrative explanation for the limiting case modelling the discrete-time binomial model into a nearly continuous-time setting.

d. Extending the model by a corresponding American put option

Figure 2 depicts the plot obtained when additionally considering the value of a corresponding American put option computed by the discrete-time binomial model.

Figure 2: Convergence of European and American binomial to the BSM Option Price



When valuing an American Put option the formula applied corresponds to equation (1), extending the same by the conditional selection of the higher of both, discounted option pay-off or early exercise value. Hence,

$$P_{BinAm} \;= \max[e^{-r\Delta t}(p * P_u + (1-p)P_d); X - S] = 8.4322$$

In general, the early exercise of a put option becomes more attractive as S decreases, as r increases, and as the volatility decreases. Early exercise of a put will always be optimal (Hull, 2012), since at maturity T the option pay-off $Xe^{-r(T-t)} - S_t$. will then X$-S_t$. For an American put option exercising the option early will allow to realize the undiscounted value of X in the current pay off. Hence the exact pay off may be realized prior to expiration. In other words, the put option holder in most cases will lose money by holding the American option until maturity. The additional benefit of early exercise is embodied by a higher premium paid to the put option's writer. We refer to this additional increment of the option price as the exercise premium e_P. Denoting the last sub period of T as N, the price obtained for the American put option is $P_{BinAm} \;= 8.4322$. Though more sophisticated approaches exist in determining the potential exercise premium of American options, we refer the simplest version of deducting the European binomial price from the corresponding price of the American put option; $e_P = P_{BinAm} - P_{Bin} = 0.0088$. We refrain from calculating the exercise premium as the difference between the binomial price for the American put option and the closed form BSM result, since n=1000 does, even if close not yet present a liming case of $n \to \infty$. Hence smaller deviations may occur as n increases further. Given the concurrent, iterative calculation procedure underlying and P_{Bin} and P_{BinAm} we deem the calculation stated above superior regarding internal consistency.

3. Exercise 2: Barrier Options

Input Data:

European Down-and-Out call option; S=1000; X=1000; T=0.25; r=0.05; H=800; σ=0.4; n=1000

Interpreting the input data, it becomes inherent that the European DOC call option is currently at-the-money (ATM), with 3 months left until expiration. The static barrier is fixed at a value of 800. The price will be assessed for n=1000 subintervals for T.

In the financial market, European and American call and put options, also known as plain vanilla options, are widely distributed. However, more complex financial products exist. In the over-the-counter derivatives market many nonstandard products have been created by financial engineers that are called exotic options. Usually those types of options build only small parts of a portfolio but are highly important to derivative dealers as profitability can be much higher in comparison to plain vanilla options.

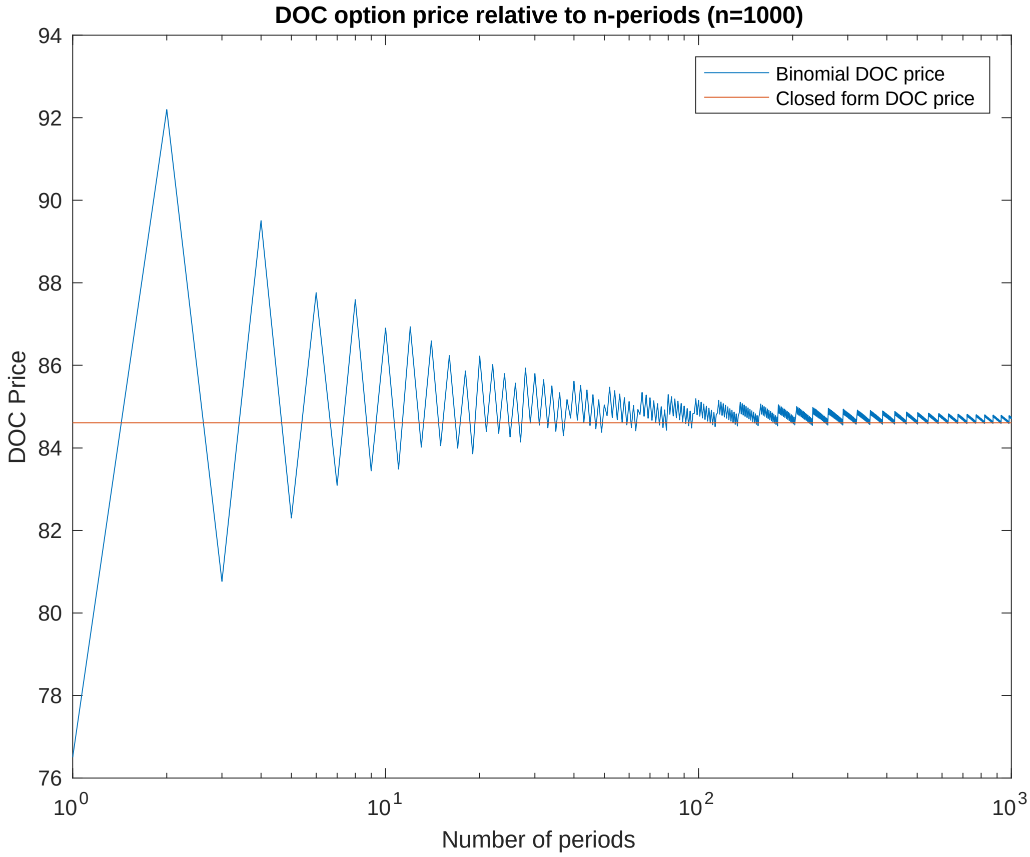
Exotic options exist for several reasons. A general hedging need exists in the market as dealers would like to back up their trades to minimize risk. Moreover, not only tax, accounting, legal and regulatory reasons make them more attractive but they also reflect a view on potential future movements of particular market variables.

Many different types of exotic options exist in the market. Some examples are forward start options that start at some point of time in the future, cliquet options, compound options or barrier options. The latter one will be further discussed when solving the exercise. Barrier options are attractive as they are less expensive than corresponding regular options. Knock-out options stop to exist when the price of the underlying asset reaches a certain barrier whereas knock-in options start existing when a certain barrier is reached (Hull, 2012).

a. Determination of DOC price with static barrier

In exercise 2a), a down-and-out call which is a knock-out option, is used. It stops to exist when the asset price decreases so much that it reaches the barrier.

Figure 3: DOC option price relative to n-periods (n=1000)



It can be observed that the binomial DOC price converges into the closed-form DOC price over time with decreasing fluctuation (Fig.3). The reason for those fluctuations is that after every single period, the price and the probabilities are calculated again. As the methodological approach is already explained in depth in the chapter 2c, the focus lies more on interpretation in this part. Since the risk of touching the barrier is much higher in the beginning as more time to maturity exists and therefore a negative development would be possible, the volatility is higher in comparison to a point of time closer to the end. Also, as the binomial tree gets bigger from period to period, probabilities become further distributed and with increasing branches the volatility becomes smaller. Since each successive stock price is almost always close to the previous price, but occasionally, with low but continuing probability, significantly different, also the option price changes decline. This occurs due to path-dependency within the model. Moreover, as investors wish to neutralize their portfolio against risk, the closer the final period gets, the more likely they can successfully hedge their portfolio against risk and therefore volatility decreases as more precise predictions on the outcome can be made (Cox et al., 1979).

Figure 4: DOC option price relative to changes in S

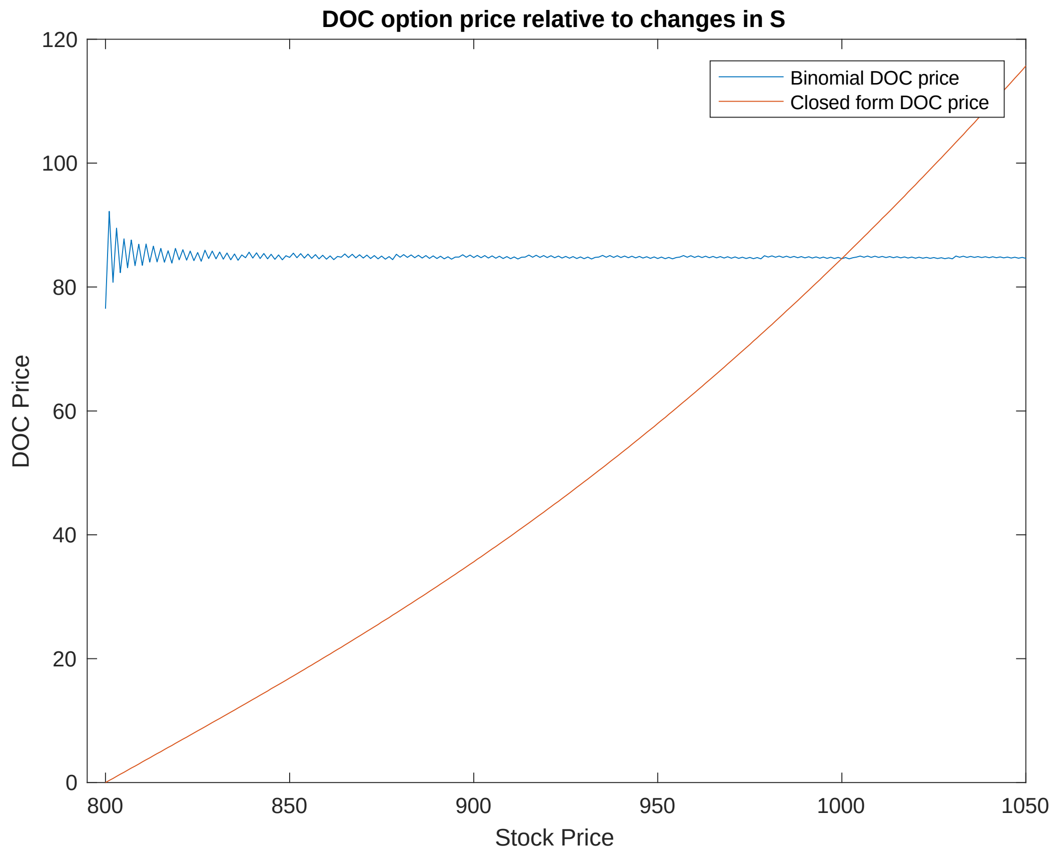


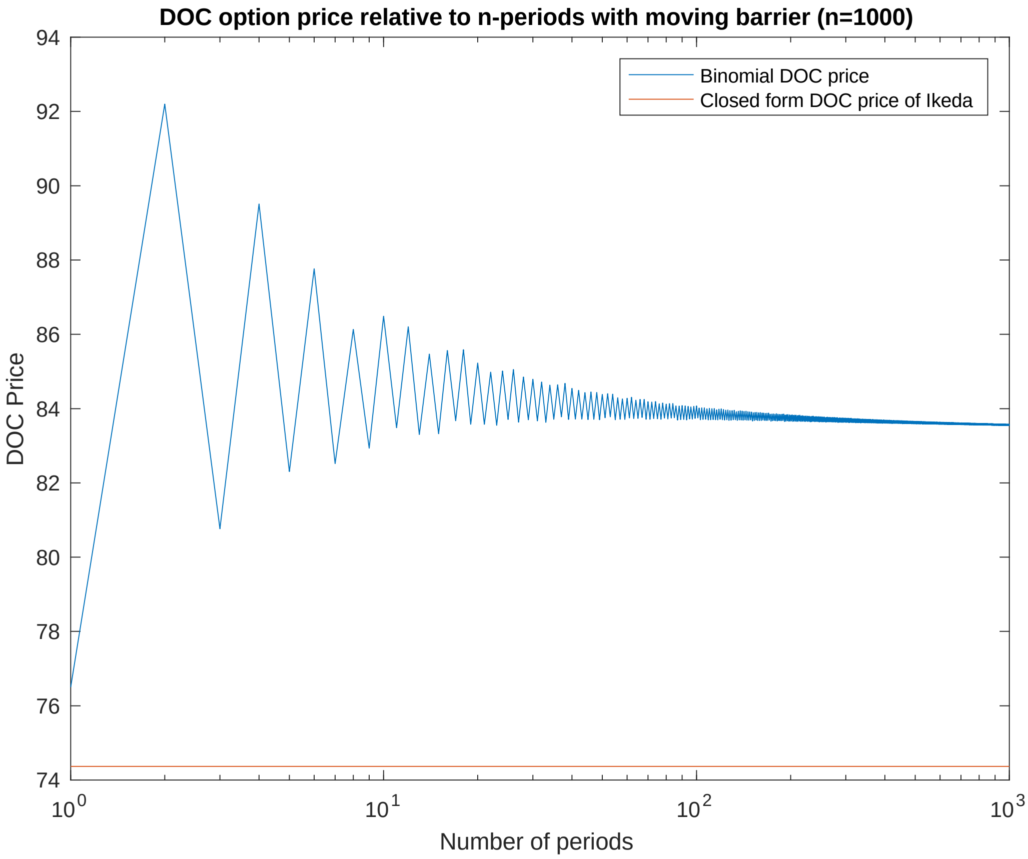
Figure 4 shows that at a stock price close to 800 the closed-form DOC price is close to 0 as the probability of hitting the barrier and therefore losing the option is extremely high. With an increasing stock price the closed-form DOC price increases as the probability of ending up in-the-money inclines. At the strike price of 1000 intersects with the binomial DOC price curve as those two curves converge due to the reasons explained above. The binomial DOC price fluctuates around 84.6072, which is the value of the closed-form DOC price at S=1000.

b. Determination of the DOC price with moving barrier

Additional data for moving barrier: A = 800; δ = 0.3; $B = Ae^{\delta u}$

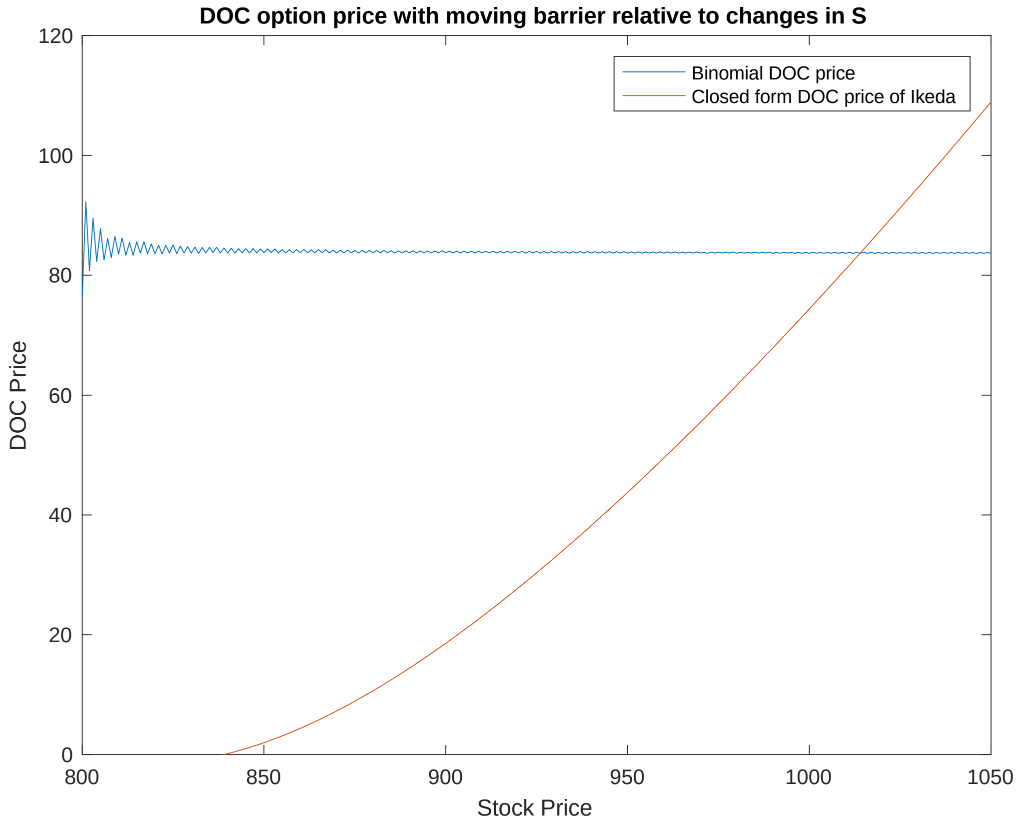
In 1992, Kunitomo and Ikeda published a method to value European options whose payoff is restricted by curved boundaries. As a basis, they used the yet existing closed form solution and extended it by certain factors. In the following, the closed form of Ikeda is set in comparison to the binomial model.

Figure 5: DOC option price relative to n-periods with moving barrier (n=1000)



In Figure 5, it can be observed that in contrast to the case with a fixed barrier the binomial DOC price decreases and is lower in the end. This can be explained that with elapsing time, the barrier moves upwards since u becomes bigger whereas all other variables stay constant. Since the barrier gets closer to the stock price. Therefore, the risk of elimination increases and the option price decreases. Further explanations are given in the parts before. The closed-form DOC price of Ikeda stays constant. In addition, in comparison to the fixed barrier case, the price is lower due to increased risk and a higher likelihood of an extinction. The difference could be called as a price risk discount that is due to the increased elimination risk when comparing the fixed barrier to the moving barrier. By looking at the binomial DOC price and the closed form DOC price of Ikeda, it can be seen that the latter one is smaller. Moreover, since the binomial DOC price decreases, it comes closer to the closed form DOC price of Ikeda.

Figure 6: DOC option price with moving barrier relative to changes in S



In Figure 6, the risk discount in price due to the change from a fixed to a moving barrier can be observed very nicely. The closed form price curve shifts to the right which leads to a price decrease. This price decrease reflects the additional risk of being eliminated by touching the barrier.

References

Cox, J., Ross, S., & Rubinstein, M. (1979). *Option pricing: A simplified approach*. Journal of Financial Economics, 7(3), 229-263. doi:https://doi.org/10.1016/0304-405X(79)90015-1

Feng, Yi and Kwan, Clarence C. Y. (2012) *Connecting Binomial and Black-Scholes Option Pricing Models: A Spreadsheet-Based Illustration*, Spreadsheets in Education (eJSiE): Vol. 5: Iss. 3, Article 2.

Hull, C. (2012). *Options, Futures, and Other Derivatives, 8th Edition* (8th ed.). Boston, United States of America: Prentice Hall.

Kunitomo, N., & Ikeda, M. (1992). *Pricing Options With Curved Boundaries*. Mathematical Finance, 2(4), 275-298. doi:10.1111/j.1467-9965.1992.tb00033.x