پاسخ سوالات تشریحی - تمرین 5 یادگیری ماشین - علی حزمی 810102129

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پاسخ سوال ۱)

$X_1 = (x_1, x_2) = \{(4,1), (2,4), (2,3), (3,6), (4,4)\}$

$X_2 = (x_1, x_2) = \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

پاسخ قسمت الف) ماتریس درون گروهی $= S_{within}$

$$S_W = S_1^2 + S_2^2$$

$$\text{Scatter} = \tilde{S}_1^2 = \sum_{i=1}^{n_1} (\tilde{x}_i - \tilde{m}_1)^2 = \sum_{i=1}^{n_1} (w^T x_i - w^T m_1)^2 = \sum_{i=1}^{n_1} \left(w^T (x_i - m_1)\right)\left((x_i - m_1)^T w\right)$$

$$= w^T \Big(\underbrace{\sum_{i=1}^{n_1} (x_i - m_1)(x_i - m_1)^T}_{S_1^2}\Big) w = w^T S_1^2 w$$

پس $S_1^2 = \sum_{i=1}^{n_1} (x_i - m_1)(x_i - m_2)^T$

و همین ترتیب برای S_2^2

$$S_2^2 = \sum_{i=1}^{n_2} (x_i - m_2)(x_i - m_2)^T$$

پس ابتدا باید میانگین هرکدام از دسته های X_1، X_2 را حساب کنیم

$\frac{1}{n}\sum x_i$

$$m_1 = \left(\frac{4+2+2+3+4}{5}, \frac{1+4+3+6+4}{5}\right) = (3, 3.6)$$

$$m_2 = \left(\frac{9+6+9+8+10}{5}, \frac{10+8+5+7+8}{5}\right) = (8.4, 7.6)$$

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$$S_1^2 = \sum_{i=1}^{5} (x_i - m_1)(x_i - m_1)^T = \begin{bmatrix} 1 \\ -2{,}6 \end{bmatrix} \begin{bmatrix} 1 & -2{,}6 \end{bmatrix} + \begin{bmatrix} -1 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} -1 & 0{,}4 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 \\ -0{,}6 \end{bmatrix} \begin{bmatrix} -1 & -0{,}6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} 1 & 0{,}4 \end{bmatrix} = \begin{bmatrix} 1+1+1+0+1 & -2{,}6-0{,}4+0{,}6+0+0{,}4 \\ -2{,}6-0{,}4+\cancel{0}+0{,}4 & 6{,}76+0{,}16+0{,}36+5{,}76+0{,}16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ -2 & 13{,}2 \end{bmatrix}$$

$$S_2^2 = \sum_{i=1}^{5} (x_i - m_2)(x_i - m_2)^T = \begin{bmatrix} 0{,}6 \\ 2{,}4 \end{bmatrix} \begin{bmatrix} 0{,}6 & 2{,}4 \end{bmatrix} + \begin{bmatrix} -2{,}4 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} -2{,}4 & 0{,}4 \end{bmatrix}$$

$$+ \begin{bmatrix} 0{,}6 \\ -2{,}6 \end{bmatrix} \begin{bmatrix} 0{,}6 & -2{,}6 \end{bmatrix} + \begin{bmatrix} -0{,}4 \\ -0{,}6 \end{bmatrix} \begin{bmatrix} -0{,}4 & -0{,}6 \end{bmatrix} + \begin{bmatrix} 1{,}6 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} 1{,}6 & 0{,}4 \end{bmatrix}$$

$$= \begin{bmatrix} 9{,}2 & -0{,}2 \\ -0{,}2 & 13{,}2 \end{bmatrix}$$

$$S_W = S_1^2 + S_2^2 = \begin{bmatrix} 13{,}2 & -2{,}2 \\ -2{,}2 & 26{,}4 \end{bmatrix}$$

$$S_B = (m_1 - m_2)(m_1 - m_2)^T$$

$$m_1 - m_2 = \begin{bmatrix} 3 - 8{,}4 \\ 3{,}6 - 7{,}6 \end{bmatrix} = \begin{bmatrix} -5{,}4 \\ -4 \end{bmatrix}$$

$$S_B = \begin{bmatrix} -5{,}4 \\ -4 \end{bmatrix} \begin{bmatrix} -5{,}4 & -4 \end{bmatrix} = \begin{bmatrix} 29{,}16 & 21{,}6 \\ 21{,}6 & 16 \end{bmatrix}$$

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پاسخ قسمت ج)

$$S_W^{-1} S_B w = \lambda w$$

برای بدست آوردن مقادیر ویژه، det ماتریس برابر صفر می‌گذاریم:

$$|S_W^{-1} S_B - \lambda I| = 0$$

ابتدا S_W^{-1} را حساب می‌کنیم. فرمول معکوس ماتریس 2x2

$$S_W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S_W^{-1} = \frac{1}{det(S_W)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$det(S_W) = (13,2 \cdot 26,4) - (-2,2 \cdot -2,2) = (348,48 - (4,84)) = 343,64$$

$$S_W^{-1} = \frac{1}{343,64} \begin{bmatrix} 26,4 & 2,2 \\ 2,2 & 13,2 \end{bmatrix} = \begin{bmatrix} \frac{26,4}{343} & \frac{2,2}{343} \\ \frac{2,2}{343} & \frac{13,2}{343} \end{bmatrix} = \begin{bmatrix} 0,0768 & 0,0064 \\ 0,0064 & 0,0384 \end{bmatrix}$$

حاصل S_W^{-1} را در S_B ضرب می‌کنیم

$$S = S_W^{-1} S_B = \begin{bmatrix} 0,0768 & 0,0064 \\ 0,0064 & 0,0384 \end{bmatrix} \begin{bmatrix} 29,16 & 21,6 \\ 21,6 & 16 \end{bmatrix}$$

$$S = \begin{bmatrix} 2,2394 + 0,1382 & 1,6568 + 0,1024 \\ 0,1866 + 0,8294 & 0,13824 + 0,6144 \end{bmatrix} = \begin{bmatrix} 2,3777 & 1,76128 \\ 1,01606 & 0,75264 \end{bmatrix}$$

حال مقادیر ویژه‌ی S را محاسبه می‌کنیم

$$|S - \lambda I| = 0$$

$$\det\left(\begin{bmatrix} 2,3777 - \lambda & 1,76128 \\ 1,01606 & 0,75264 - \lambda \end{bmatrix} \right) = 0$$

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$$\lambda^2 - (2,3777 + 0,75264)\lambda + ((2,3777 \cdot 0,75264) - 1,76928 \cdot 1,01644)) = 0$$

$$\Rightarrow \lambda^2 - 3,1303\lambda - 0,0027 = 0$$

$$\frac{b \pm \sqrt{\Delta}}{2a}$$

$$b^2 - 4ac = (3,1303)^2 - 4 \times 1 \times 0,0027) = 9,8 - 0,0108 = \boxed{9,79}$$

$$\lambda = \frac{3,1303 \pm \sqrt{9,79}}{2}$$

$$\lambda = \frac{3,13 + 3,1281}{2} = \boxed{3,1292} \checkmark$$

$$\lambda = \frac{3,13 - 3,1281}{2} = \boxed{0,001184}$$

بیشترین مقدار eigen Value برابر $\boxed{3,1292}$

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پاسخ سوال ۳)

$$P(x) = \alpha \lambda_1 e^{-\lambda_1 x} + (1-\alpha)\lambda_2 e^{-\lambda_2 x}$$

تجربه داده: $\{x_1, x_2, x_3, \dots x_n\}$

بدست آوردن (log Complete likelihood)

$$\log P(x_1, z_1, \dots, x_n, z_n \mid \lambda_1, \lambda_2, \alpha) \overset{\text{iid}}{\underset{\text{با فرض}}{=}} \sum_{i=1}^{n} \log P(x_i, z_i \mid \lambda_1, \lambda_2, \alpha)$$

$$P(x_i, z_i \mid \lambda_1, \lambda_2, \alpha) = \left(\alpha \lambda_1 e^{-\lambda_1 x_i}\right)^{1-z_i} \left((1-\alpha)\lambda_2 e^{-\lambda_2 x_i}\right)^{z_i}$$

که در رابطه بالا z_i اگر x_i از دسته اول باشد ۰ و در غیر این صورت ۱ خواهد بود.

$$\Rightarrow \log P(x_i, z_i \mid \lambda_1, \lambda_2, \alpha) = (1-z_i)\left(\log \alpha \lambda_1 e^{-\lambda_1 x_i}\right) + z_i \log\left(\lambda_2 (1-\alpha) e^{-\lambda_2 x_i}\right)$$

$$\Rightarrow \log \text{Complete likelihood} = \sum_{i=1}^{n} (1-z_i)(\log \alpha + \log \lambda_1 - \lambda_1 x_i) + z_i(\log(1-\alpha) + \log \lambda_2 - \lambda_2 x_i)$$

E Part

$$Q(\theta, \theta^t) = E\left[\log P(x_1, z_1, \dots, x_n, z_n \mid \lambda_1, \lambda_2, \alpha)\right]$$

$$= \sum_{i=1}^{n} (1 - E[z_i])(\log \alpha + \log \lambda_1 - \lambda_1 x_i) + E[z_i](\log(1-\alpha) + \log \lambda_2 - \lambda_2 x_i)$$

$$E[z_i] = P(z_i = 1 \mid x_i, \alpha^t, \lambda_1^t, \lambda_2^t) = \frac{P(x_i \mid z_i = 1, \alpha^t, \lambda_1^t, \lambda_2^t) P(z_i = 1 \mid \alpha^t, \lambda_1^t, \lambda_2^t)}{P(x_i \mid \alpha^t, \lambda_1^t, \lambda_2^t)}$$

$$= \frac{\lambda_2^t e^{-\lambda_2^t x_i}(1-\alpha^t)}{\alpha^t \lambda^t e^{-\lambda_1^t x_i} + (1-\alpha^t)\lambda_2^t e^{-\lambda_2^t x_i}} = \gamma_i^t$$

برای سادهنویسی مراحل بعد

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$$\frac{\partial Q}{\partial \alpha} = \sum_{i=1}^{n} \left(1-\gamma_i^{t}\right)\frac{1}{\alpha} + \gamma_i^{t}\,\frac{-1}{1-\alpha} = 0$$

(MPoint

$$\Rightarrow (1-\alpha)\sum_{i=1}^{n}\left(1-\gamma_i^{t}\right) = \alpha\sum_{i=1}^{n}\gamma_i^{t} \Rightarrow \boxed{\alpha = 1 - \frac{\sum_{i=1}^{n}\gamma_i^{t}}{n}}$$

$$\frac{\partial Q}{\partial \lambda} = \sum_{i=1}^{n}\left(1-\gamma_i^{t}\right)\left(\frac{1}{\lambda_1} - x_i\right) = 0 \qquad \boxed{\lambda_1 = \frac{n - \sum_{i=1}^{n}\gamma_i^{t}}{\sum_{i=1}^{n}\left(1-\gamma_i^{t}\right)x_i}}$$

$$\frac{\partial Q}{\partial \lambda_2} = \sum_{i=1}^{n}\gamma_i^{t}\left(\frac{1}{\lambda_2} - x_i\right) = 0 \qquad \boxed{\lambda_2 = \frac{\sum_{i=1}^{n}\gamma_i^{t}}{\sum_{i=1}^{n}\gamma_i^{t}x_i}}$$