HW1 Answer Key

Problem1.

D: dangerous conditions D^c : normal conditions

T: alarms indicated dangerous T^c : alarm indicates normal conditions

a) $P(\,D^c \mid T\,) = 0.5116$
b) $P(\,D \mid T^c\,) = 0.0002525$
c) Number of false alarms = $P(\,D^c \mid T\,) \times P(T) \times 365 \times 10 = 18$

Problem2.

a. $E[Y_1] = E[X_1] + E[X_2] = 2m$
   $Var(Y_1) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2) = 2\sigma^2$
b. $E[Y_2] = 2E[X_1] = 2m$
   $Var(Y_2) = 2^2 Var(X_1) = 4\sigma^2$
c. $Cov(Y_1, Y_2) = 2\sigma^2$

Problem3.

a. There are 365^n sequences of n birthdays. Since they are all equally likely, $P(\omega) = \frac{1}{365^n}$ for every sequence ω.
b. Description
c. $P(A) = 1 - P(A^c) = 1 - \frac{364^n}{365^n} = 0.5 \rightarrow n \approx 253$
d. Description
e. Code
f. $P(B) = 1 - P(B^c) = 1 - \frac{365!}{(365-n)! \times 365^n}$
g. Code
h. Description

Problem4.

a. $\frac{1}{2}$

b. $\frac{1}{3}$

Problem5.

This is a question of 'inverting' conditional probability. We know

$$P(\text{the witness sees blue}|\text{the car is blue})$$

but we'd like to know

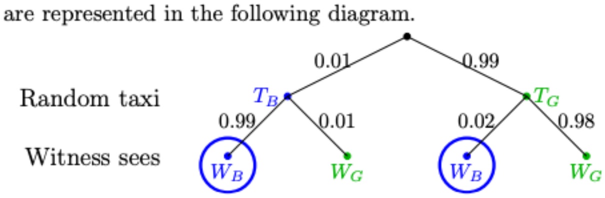
$$P(\text{the car is blue}|\text{the witness sees blue}).$$

Our first job is to translate this to symbols.

Let W_b = 'witness sees a blue taxi' and let W_g = 'witness sees a green car'. Further, let T_b = 'taxi is blue' and let T_g = 'taxi is green'. With this notation we want to find $P(T_b|W_b)$. We will compute this using Bayes' formula

$$P(T_b|W_b) = \frac{P(W_b|T_b) \cdot P(T_b)}{P(W_b)}.$$

All the pieces are represented in the following diagram.



We can determine each factor in the right side of Bayes' formula:
We are given $P(T_b) = .01$ (and $P(T_g) = .99$).
We are given, $P(W_b|T_b) = .99$ and $P(W_b|T_g) = .02$.
We compute $P(W_b)$ using the law of total probability:

$$P(W_b) = P(W_b|T_b)P(T_b) + P(W_b|T_g)P(T_g) = .99 \times .01 + .02 \times .99 = .99 \times .03.$$

Putting all this in Bayes' formula we get

$$P(T_b|W_b) = \frac{.99 \times .01}{.99 \times .03} = \boxed{\frac{1}{3}}$$

Ladies and gentlemen of the jury. The prosecutor tells you that the witness is nearly flawless in his ability to distinguish whether a taxi is green or blue. He claims that this

[redacted]

implies that beyond a reasonable doubt the taxi involved in the hit and run was blue. However probability theory shows without any doubt that the probability a random taxi seen by the witness as blue is actually blue is only 1/3. This is considerably more than a reasonable doubt. In fact it is more probable than not that the taxi involved in the accident was green. If the probability doesn't fit you must acquit!

Problem6.

a.

$$P(X) = \begin{cases} \frac{1}{4} & X = 4 \\ \frac{1}{4} & X = 6 \\ \frac{1}{2} & X = 8 \end{cases}$$

b. $P(\,S = 4 \mid R = 3\,) = \frac{3}{8}$

$P(\,S = 6 \mid R = 3\,) = \frac{1}{4}$

$P(\,S = 8 \mid R = 3\,) = \frac{3}{8}$

c. $P(\,S = 4 \mid R = 6\,) = 0$

$P(\,S = 6 \mid R = 6\,) = \frac{2}{5}$

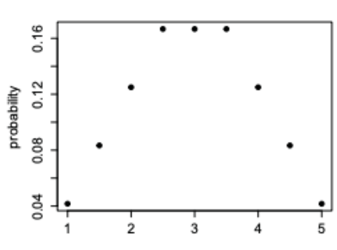
$P(\,S = 8 \mid R = 6\,) = \frac{3}{5}$

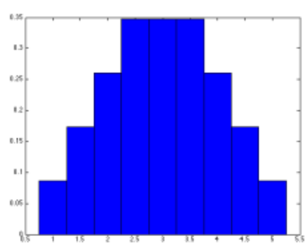
d. Just octahedral die.

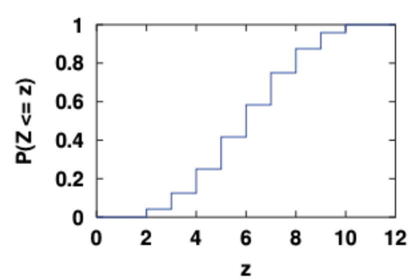
Problem7.

a. $Var(X) = \frac{5}{4}, \quad Var(Y) = \frac{35}{12}, \quad Var(Z) = \frac{1}{4}\big(Var(X) + Var(Y)\big) = \frac{25}{24}$

b.







c.

(c) We see that the only pairs of (X, Y) which satisfy $X > Y$ are $\{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$. So $P(X > Y) = \frac{6}{24}$. Moreover, we have

$$P(X > Y|X = 2) = \frac{1}{6} \qquad P(X > Y|X = 3) = \frac{2}{6} \qquad P(X > Y|X = 4) = \frac{3}{6}$$

If W is our winnings for one game, we find
$E(W) = (-1)P(Y \geq X) + 2(2P(X > Y|X = 2)P(X = 2) + 3P(X > Y|X = 3)P(X = 3) + 4P(X > Y|X = 3)P(X = 4))$

$$= -\frac{18}{24} + \frac{40}{24}$$
$$= \frac{11}{12}$$

Now if played the game 60 times, and received winnings $W_1, \ldots, W_{60}$, (with $E(W_i) = \frac{11}{12}$), our expected total gain is

$$E(W_1 + \cdots + W_{60}) = E(W_1) + \cdots + E(W_{60}) = 55.$$

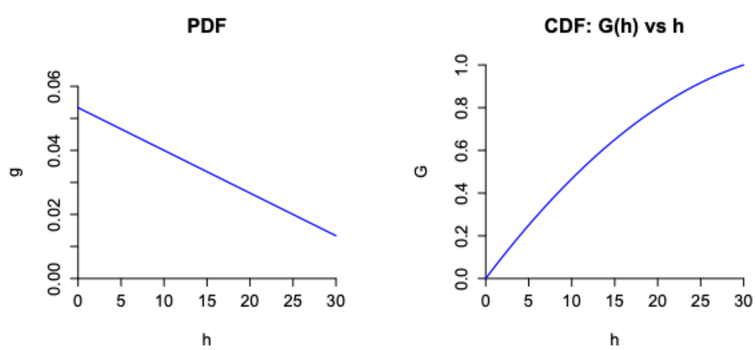
Problem8.

a. The number of raisins is

$$\int_0^{30} f(h)dh = \int_0^{30} (40 - h)dh = 750$$

b. $g(h) = \frac{1}{750}(40 - h)$

c. For $0 \leq h \leq 30$ we have $G(h) = \int_0^h g(x)dx = \frac{40h}{750} - \frac{h^2}{1500}$



d. $P(H \leq 10) = \frac{7}{15}$

Problem9.

a.

$$Cov(X,Y) = 4c - 1$$
$$Cor(X,Y) = 4c - 1$$

b. $c = 0, c = 0.5 \rightarrow fully\ correlated\ (anti - correlated)$

$c = \frac{1}{4} \rightarrow independant$

Problem11.

Sample size, Sample mean, Variance of the sample mean, The largest value in the sample