QCL2010
Special Assignment

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1. Quantum Teleportation

Quantum teleportation is a technique for moving quantum states around [1].

Let's assume that Alice wants to send an arbitrary quantum state $|\psi\rangle$ to her friend Bob,

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

The No-Cloning Theorem states that it's impossible to create an exact copy of an arbitrary quantum state. It was derived by Wootters and Zurek in 1982 [2].

As a result, Alice can't simply create a copy of $|\psi\rangle$ and give it to Bob.

1.1. Step 1 : Alice and Bob share an entangled pair of qubits

Let the shared entangled pair of qubits be denoted by $|\beta_{00}\rangle$,

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Let the first member of the pair belong to Alice and the second member of the pair belong to Bob,

$$|\beta_{00}\rangle = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}$$

This will lead to a three qubit quantum system $|\phi\rangle$ where Alice has the first two qubits and Bob has the last qubit,

$$|\phi\rangle = |\psi\rangle \otimes |\beta_{00}\rangle$$

$$|\phi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes \left(\frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}\right)$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

1.2. Step 2 : Alice applies a CNOT Gate

Alice will now apply a CNOT Gate on the first two qubits. In a CNOT operation, if the control qubit is 0, then nothing happens, but if the control qubit is 1, then the target qubit is flipped.

Alice sets the arbitrary quantum state $|\psi\rangle$ as the control qubit and her member of the entangled pair $|\beta_{00}\rangle$ as the target qubit.

$$|\phi^{'}\rangle = U_{CNOT}|\phi\rangle$$

$$|\phi^{'}\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

1.3. Step 3 : Alice applies a Hadamard Gate

Alice will now apply a Hadamard Gate on the first qubit. In a Hadamard operation, the computational basis states turn into superpositions.

On rewriting $|\phi^{'}\rangle$, we get

$$|\phi^{'}\rangle = \frac{\alpha|0\rangle(|00\rangle + |11\rangle)}{\sqrt{2}} + \frac{\beta|1\rangle(|10\rangle + |01\rangle)}{\sqrt{2}}$$

After applying a Hadamard Gate on the first qubit, we get

$$|\phi^{''}\rangle = H|\phi^{'}\rangle$$

$$|\phi^{''}\rangle = \alpha(H|0\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \beta(H|1\rangle)\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)$$

$$|\phi^{''}\rangle = \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)$$

$$|\phi^{''}\rangle = \frac{1}{2}(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle))$$

1.4. Step 4 : Alice measures her pair

Alice will now make a measurement on the two qubits in her possession. The possible measurement results are $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ with equal probabilities.

(i) Case 1

If Alice measures $|00\rangle$, then the state collapses and Bob has $|\psi_{00}\rangle$,

$$|\psi_{00}\rangle = \alpha |0\rangle + \beta |1\rangle$$

Bob can obtain the original state $|\psi\rangle$ by applying the Identity operator on $|\psi_{00}\rangle$.

Therefore,

$$I |\psi_{00}\rangle = \alpha(I |0\rangle) + \beta(I |1\rangle)$$

$$I |\psi_{00}\rangle = \alpha |0\rangle + \beta |1\rangle$$

(ii) Case II

If Alice measures $|01\rangle$, then the state collapses and Bob has $|\psi_{01}\rangle$,

$$|\psi_{01}\rangle = \alpha |1\rangle + \beta |0\rangle$$

Bob can obtain the original state $|\psi\rangle$ by applying the X operator on $|\psi_{01}\rangle$.

Therefore,

$$X |\psi_{01}\rangle = \alpha(X |1\rangle) + \beta(X |0\rangle)$$

$$X |\psi_{01}\rangle = \alpha |0\rangle + \beta |1\rangle$$

(iii) Case III

If Alice measures $|10\rangle$, then the state collapses and Bob has $|\psi_{10}\rangle$,

$$|\psi_{10}\rangle = \alpha |0\rangle - \beta |1\rangle$$

Bob can obtain the original state $|\psi\rangle$ by applying the Z operator on $|\psi_{10}\rangle$.

Therefore,

$$Z |\psi_{10}\rangle = \alpha(Z |0\rangle) - \beta(Z |1\rangle)$$

$$Z |\psi_{10}\rangle = \alpha |0\rangle + \beta |1\rangle$$

(iv) Case IV

If Alice measures $|11\rangle$, then the state collapses and Bob has $|\psi_{11}\rangle$,

$$|\psi_{11}\rangle = \alpha\,|1\rangle - \beta\,|0\rangle$$

Bob can obtain the original state $|\psi\rangle$ by applying the X operator followed by a Z operator on $|\psi_{11}\rangle$.

Therefore,

$$ZX\,|\psi_{11}\rangle = \alpha(ZX\,|1\rangle) - \beta(ZX\,|0\rangle)$$

$$ZX\,|\psi_{11}\rangle = \alpha(Z\,|0\rangle) - \beta(Z\,|1\rangle)$$

$$ZX\,|\psi_{11}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Alice will send two classical bits to Bob through a classical communication channel (Note : this step of the process ensures that the communication cannot happen faster than the speed of light). Bob will use this information to apply appropriate Gate(s) and obtain the original state $|\psi\rangle$.

Bob's States	**Bits Received**	**Gate(s) Applied**
$\alpha\,|0\rangle + \beta\,|1\rangle$	00	I
$\alpha\,|1\rangle + \beta\,|0\rangle$	01	X
$\alpha\,|0\rangle - \beta\,|1\rangle$	10	Z
$\alpha\,|1\rangle - \beta\,|0\rangle$	11	ZX

Table 1: Quantum Teleportation

2. An example with Qiskit code

Let's teleport a qubit in the state $|\psi\rangle$ using Qiskit [3] [4],

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

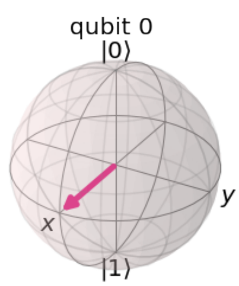
[10]:
```
# Importing the numpy and qiskit modules

import numpy as np
from qiskit import *
from math import *
from qiskit.extensions import Initialize
from qiskit.visualization import plot_bloch_multivector,␣
↪array_to_latex
```

[11]:
```
# Creating two registers for storing classical bits␣
↪generated after the measurement

b1 = ClassicalRegister(1,"b1")
b2 = ClassicalRegister(1,"b2")

# Creating an instance of a Quantum Circuit with 3 qubits

C = QuantumCircuit(QuantumRegister(3,"qubit"),b1,b2)
```

[12]:
```
# Generating phi

psi = [1/sqrt(2) +0.0j, 1/sqrt(2)+0.0j]

display(array_to_latex(psi, prefix="|\\psi\\rangle ="))
plot_bloch_multivector(psi)
```

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

[12]:



[13]:
```python
# Initializing psi

psi_Initialize = Initialize(psi)
C.append(psi_Initialize,[0])

C.barrier()
```

[13]: <qiskit.circuit.instructionset.InstructionSet at 0x7f2ddc6e6b80>

[14]:
```python
# Creating a bell state

C.h(1)
C.cx(1,2)

C.barrier()
```

[14]: <qiskit.circuit.instructionset.InstructionSet at 0x7f2de40dee20>

[15]:
```python
# Adding a CNOT gate with the 1st qubit as the control qubit and the 2nd qubit as target qubit
# And then adding a Hadamard gate to the 1st qubit

C.cx(0,1)
C.h(0)

C.barrier()
```

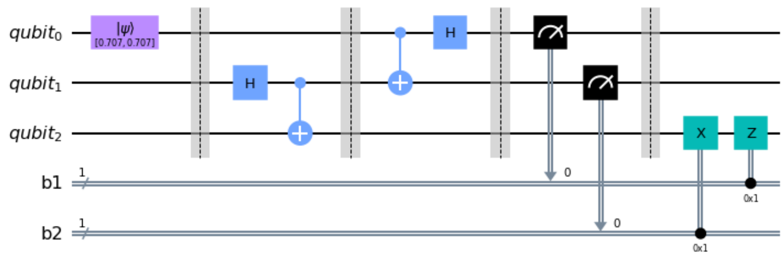
[15]: <qiskit.circuit.instructionset.InstructionSet at 0x7f2de40de370>

[16]:
```python
# Measuring the 1st and 2nd qubit

C.measure(0,b1)
C.measure(1,b2)

C.barrier()
```

[16]: <qiskit.circuit.instructionset.InstructionSet at
↪0x7f2ddc6e69a0>

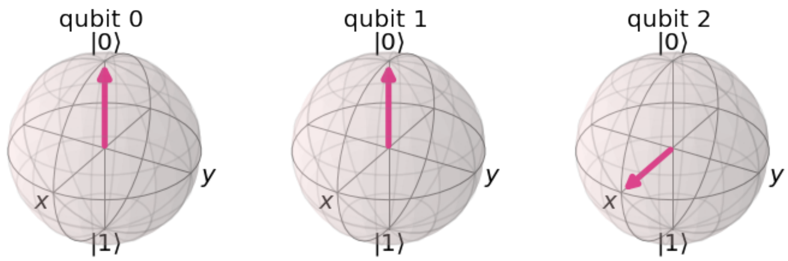
[17]:
```
# Applying appropriate gates to get back psi

C.x(2).c_if(b2, 1)
C.z(2).c_if(b1, 1)

C.draw()
```

[17]:



[18]:
```
# Simulating the circuit

Simulator = Aer.get_backend('aer_simulator')

C.save_statevector()

Sim_Instance = Simulator.run(C)
Sim_Result = Sim_Instance.result()

output = Sim_Result.get_statevector()

plot_bloch_multivector(output)
```

[18]:



From the above figure, we can observe that the 3^{rd} qubit is exactly the same as $|\psi\rangle$. Therefore, the results are in agreement with the quantum teleportation protocol.

References

[1] David McMahon. "Quantum computing explained". In: Wiley-Interscience : IEEE Computer Society, Hoboken, N.J., 2008. Chap. 10.

[2] W. K. Wootters and W. H. Zurek. "A single quantum cannot be cloned". In: *Nature* 299.5886 (1982), pp. 802–803. URL: `https://doi.org/10.1038/299802a0`.

[3] Qiskit. *Quantum Teleportation*. URL: `https://qiskit.org/textbook/ch-algorithms/teleportation.html`. (accessed: 20.11.2022).

[4] IBM. *IBM Quantum Lab*. URL: `https://quantum-computing.ibm.com/`. (accessed: 20.11.2022).